REMARKS ON ERGODIC THEORY OF STOCHASTIC FLOWS AND CONTROL FLOWS

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ABSTRACT. Stochastic systems and control systems with values in a state space M can be considered as dynamical systems on the space $\mathcal{U} \times M$, where $\mathcal{U}$ denotes the space of admissible control functions for control systems, and the trajectory space of an underlying noise process for stochastic systems. Invariant probability measures for these flows are the main topic of this paper: We show that their support is contained in sets $\mathcal{D} \subset \mathcal{U} \times M$, which are the lifts of so-called control sets $D \subset M$ to invariant sets in $\mathcal{U} \times M$. Several results on the characterization of control sets D are given, together with criteria for the existence of invariant measures μ on $\mathcal{U} \times M$ with supp $\mu \subset \mathcal{D}$. The case of Markovian stochastic systems is treated in some detail. Because of the importance in applications, we prove rather complete results for two classes of systems: linearized systems, which play a crucial role in the theory of Lyapunov exponents for stochastic and control flows, and general nonlinear systems with one dimensional state space, which are important in stochastic bifurcation theory.

1. INTRODUCTION: COMMON TECHNIQUES FOR STOCHASTIC, CONTROL AND DYNAMICAL SYSTEMS

Common ideas and approaches in the theories of continuous time dynamical systems and of (Markovian) stochastic systems go back at least to the 1930's, when Kolmogorov introduced rigorously the generator of certain diffusion processes as a second order (elliptic) operator. With Itō's formulation of stochastic calculus, and Dynkin's characterization of those operators that are generators of diffusion processes, many techniques that had been developed for ordinary differential equations and their flows could be carried over to stochastic differential equations, yielding in particular results on invariant sets, recurrence and transience, invariant measures, ergodicity, stability, and stochastic perturbations of

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(deterministic) differential equations. (It should be noted that corresponding results for discrete time systems have been discovered at the same time or even earlier (see e.g. Kolmogorov (1931)), and that this correspondence also works for the newer developments in our area, but the discrete time situation is not the topic under consideration in this paper.)

The first important connection between (optimal) control theory and stochastic systems was probably the celebrated duality between the linear quadratic control problem and the Kalman-Bucy filter (1961), which led to an almost parallel development of linear optimal control for deterministic and stochastic control problems (compare e.g. the books of Kwakernaak and Sivan (1972) or Hijab (1987)). New ideas in nonlinear control theory, in particular the geometric approach of Brockett, Lobry, and Sussmann, and the so called support theorem of Stroock and Varadhan led since the 1970's to control theoretic descriptions of the supports of transition probabilities and invariant measures, and to characterizations of ergodicity, recurrence and transience. Some general uniqueness results on invariant measures for stochastic systems were then obtained using Hörmander's theorem (1967) on the characterization of hypoelliptic operators. (Again, similar approaches with corresponding results also hold for the discrete time case.)

Recently, the common viewpoint of flows of dynamical systems, stochastic systems and control systems is leading to a further unification and cross fertilization of these areas. While the theory of smooth flows for (time independent) vector fields is classical (see e.g. the textbook of Nemytskii and Stepanov (1949)), the theory for stochastic flows has been developed in the 1980's by Kunita, Elworthy, Baxendale and others. The corresponding concept of control flows was introduced by Colonius and Kliemann (1990^a). Besides questions about invariant measures and (smooth or measurable) ergodicity, which will be treated later on in this paper, it is in particular linearization techniques that have led to common developments, based on Oseledeč's multiplicative ergodic theory (1968), which can be interpreted as a result for dynamical systems, stochastic systems, or even control systems. In particular the areas of Lyapunov exponents, bifurcation theory, chaos, hyperbolicity and strange attractors are active research fields at the moment for systems with noise and for systems with control inputs, as these and other conference proceedings (e.g. Arnold et al. (1991)) demonstrate.

In this paper we will utilize ideas from ergodic theory of dynamical systems (see e.g. Mañé (1987)) to discuss existence and possible supports of invariant measures for control and stochastic flows, where it is assumed that the stochastic flows are defined over a probability space Ω, which is the trajectory space of an underlying (stationary) driving process. Over the same space, now viewed as the space of admissible control functions, control flows can be defined, and their interplay yields the results in Section 4, mainly with the Krylov-Bogolyubov construction of invariant probability measures. In Section 2, the precise concepts of stochastic and control flows are defined, and some examples are given. Since it turns out that the so-called 'control sets' and 'chain control sets' play a crucial role in the analysis of invariant measures, Section 3 is devoted to the study of dynamical properties of control flows, in particular to lifts of control and chain control sets. Two classes of examples, which are central in linearization techniques and in the study of bifurcation behavior, are treated in detail in Sections 3 and 4. An application of the theory developed here to linearized systems and their Lyapunov spectrum will appear elsewhere (Colonius and Kliemann (1991[a])).

But first of all it seems useful to describe a little bit more precisely the current state of common techniques in ergodic theory of dynamical, stochastic and control systems, and so the remainder of this introductory section is devoted to this topic.

Dynamical systems and Markovian stochastic systems

Ordinary differential equations with generator X, a smooth vector field, on a smooth manifold M, give rise to (local) flows and (local) one parameter groups of diffeomorphisms, describing the solutions of the differential equation. (From now on we will assume that all systems are complete, i.e. the explosion time of all trajectories is $\pm\infty$.) Generators of (Markov) diffusion processes given by stochastic differential equations $dx = X_0(x)dt + \sum_{i=1}^{m} X_i(x) \circ dW_i$ on M (where $(W_1 \dots W_m)$ is a vector of independent standard Wiener processes, and "$\circ$" denotes the symmetric or Stratonovič stochastic differential) are second order (elliptic) operators $\mathcal{A} = X_0 + \frac{1}{2} \sum_{i=1}^{m} X_i^2$, acting on the space of bounded measurable (or C^∞-) functions. An associated one parameter semigroup (for $t \geq 0$, i.e. forward in time) is given by the transition probabilities $P(t, x, B)$, see e.g. Ethier and Kurtz (1986) for these facts in a much more general context. Qualitative theory, in particular ergodic theory, for dynamical and stochastic systems analyzes the long term behavior (i.e. $t \to +\infty$) of these (semi-)groups, without solving the equations explicitly. For problems like invariant sets, recurrence and transience, stability, etc. Lyapunov functions are one appropriate tool. The stochastic version of this theory is described e.g. in some detail in the books by Hasminskii (1969) or Friedman (1975), where it becomes clear that stochastic Lyapunov functions are a convenient tool for the generator $\mathcal{A}$ as are ordinary Lyapunov functions for the vector field X. For the existence of invariant probability measures of the above semigroups, a Krylov-Bogolyubov construction works in both cases: If the time averages of the semigroups yield a tight family of probability measures, then any accumulation point is an invariant measure (see e.g. Nemytskii and Stepanov (1949) for the deterministic, and Hasminskii (1969) for the stochastic case). Compactness of the state space M is in any case sufficient for the existence of at least one invariant measure (compare Kunita (1971) for the stochastic case). Ergodic theorems can now be developed in complete analogy, because for the stochastic case there is a one-to-one correspondence of invariant sets in the trajectory space and the state space for Markov processes. Lyapunov functions are again a convenient tool to obtain criteria for the existence of invariant measures etc. in terms of the vector field(s), see e.g. Bhatia and Szegö (1970) and Hasminskii (1969). For the problem of analyzing stochastic perturbations of (deterministic) vector fields via Markov theory, large deviation approaches have been developed (see e.g. Freidlin and Wentzell (1984) or also Zeeman (1988) for related ideas), combining qualitative theory for ordinary differential equations with that for Markov diffusion processes. Finally it should be mentioned that for non-flat manifolds the behavior of solutions of ordinary differential equations and stochastic differential equations depend on the (global) geometry of M; for many we mention only Emery (1990).

Control systems and Markovian stochastic systems

Besides the connections via optimal control theory (see the remarks above) it is in particular the support theorems that allow the use of control theoretic results for the analysis of stochastic systems: If we replace the Wiener processes in a stochastic differential equation by admissible control functions with values in $\mathbb{R}^m$, we arrive at the control system $\dot{x} = X_0(x) + \sum_{i=1}^{m} u_i(t)X_i(x)$. The support theorem says that the closure of the trajectory space of this control system is the support of the diffusion measure induced by the stochastic differential equation on this space (see e.g. Wong and Zakai (1969), Stroock and Varadhan (1972), Kunita (1974), Ikeda and Watanabe (1981)). In particular, if the distribution $\Delta_{\mathcal{L}}$, generated by the Lie algebra of vector fields $\mathcal{L} = \mathcal{LA}\{X_0, \ldots, X_m\}$ in the tangent bundle TM is integrable, then the control system and the stochastic system live on the maximal integral manifolds of $\Delta_{\mathcal{L}}$ and the support of the transition probabilities $P(t, x, B)$ and of invariant measures are described by accessible and control sets of the control system (see e.g. Brockett (1973), Sussmann and Jurdjevic (1972), Clark (1973), Elliott (1973), Kunita (1978), Kliemann (1987)). Together with Hörmander's characterization of hypoelliptic operators (Hörmander (1967)), one obtains uniqueness results for invariant measures on control sets and corresponding ergodic theorems (Kliemann (1987), Arnold and Kliemann (1987[a])). This is also the starting point for various results on the controllability of stochastic systems with input (see e.g. Zabczyk (1981), Ehrhardt and Kliemann (1982), Varsan (1982)). It should be mentioned that, among others, also the theory of large deviations for stochastic differential equations has an interpretation in terms of control theoretic concepts, which becomes particularly clear from Azencott's (1980) formulation, see also Arnold and Kliemann (1987[b]).

Dynamical, stochastic, and control flows

So far we have considered questions in the qualitative theory of stochastic systems that could be analyzed using the Markov semigroup (i.e. the generator) of a stochastic differential equation. Problems that are concerned with the long term behavior of different trajectories relative to each other (e.g. convergence or divergence of trajectories), i.e. properties of the multipoint motion, do not only depend on the generator $\mathcal{A}$, but on the stochastic flow induced by a stochastic differential equation (see e.g. Baxendale (1986[b]) for example). The theory of flows for stochastic differential equations was developed by Kunita (1984), Elworthy (1978), Baxendale (1980), and others. In particular, Baxendale (1986[b]) has shown that the stochastic differential equation for the 2-point motion (and its generator) are sufficient to construct the corresponding stochastic flow. Once a stochastic flow is given, it is of particular importance to characterize those flows that are associated to Markov processes, and to determine those invariant measures of the flow that are also invariant under the corresponding Markov semigroup (in general, a Markov stochastic flow can have invariant measures that are not Markovian). These are questions of appropriate measurability, and they are treated e.g. in Crauel (1987, 1990). Problems concerning invariant probability measures of stochastic flows (see e.g. LeJan (1986)) and associated control flows are treated in Section 4 of this paper. The unified formulation of systems as flows does not only allow a common approach to basic problems of ergodic theory, but also, via linearization techniques, to the analysis of local behavior of nonlinear systems: Given the linearization with respect to a stationary situation (e.g. a rest point or a stationary solution), Oseledeč's multiplicative ergodic theory (1968) describes the Lyapunov exponents and the corresponding invariant subspaces of the linearized system, and these subspaces can be projected down to the state space M as stable, unstable, or center manifolds (see e.g. Boxler (1989) or Dahlke (1989)). For linear stochastic systems (i.e. linearizations around rest points) the theory of Lyapunov exponents is fairly complete (see e.g. Arnold and Wihstutz (1986), Arnold and

Kliemann (1987^b)). (Note that for linear systems all trajectories are 'compared' with the unique steady state 0, and hence the theory of stochastic flows is not really needed.) For nonlinear stochastic systems basic properties have been proved e.g. by Carverhill (1985^a, 1985^b), Kifer (1986), Carverhill et al. (1986) (see also Arnold et al. (1991)), connections with stability, the existence of Lyapunov functions and invariant measures are discussed in Baxendale and Stroock (1988), and Baxendale (1990), and the relation to fiber entropy is analyzed e.g. in Ledrappier and Young (1985, 1988) and Crauel (1991). Based on the concept of Lyapunov exponents, Arnold and Boxler (1989, 1990) have defined a concept of stochastic bifurcations. Similar developments for control flows can be found e.g. for linear systems in Colonius and Kliemann (1990^c, 1991^b), and some basic properties for the nonlinear situation in Colonius and Kliemann (1990^a). Connections between the stochastic and the controlled situation are given in Arnold and Kliemann (1987^b), Colonius and Kliemann (1990^d), see also Baxendale and Stroock (1988). These connections lead e.g. to a common (deterministic and stochastic) concept for the analysis and stabilization of uncertain linear systems (see e.g. Willems and Willems (1983), Colonius and Kliemann (1990^d, 1990^e)).

This is, of course, only a very brief review, neglecting in particular the recent developments in the theory of dynamical systems, of discrete time systems (like products of random matrices, and iterated function systems), and of infinite dimensional systems. But it shows, how ideas from these three fields penetrate into the other areas, creating a common toolbox, and new applications and examples.

2. Systems and Associated Flows

The unifying point of view, which enables us to use common concepts and techniques for dynamical systems, stochastic systems and control systems, is the concept of topological flows:

Let S be a complete metric space, T a time set (e.g. $T = \mathbb{R}, \mathbb{R}^+, \mathbb{Z}, \mathbb{N}$), and $\Psi: T \times S \to S$ a continuous map, then (S, Ψ) is a flow (or continuous dynamical system) if $\Psi_{t+s} = \Psi_t \circ \Psi_s$, where $\Psi_t: S \to S$ denotes the map $\Psi_t(x) = \Psi(t,x)$ for all $t \in T$.

The most prominent examples of flows are those generated by (ordinary) differential equations: Let M be a smooth manifold and X a smooth complete vectorfield on M. Denote by $\psi(t,x)$ the solution of $\dot{x} = X(x)$ at time t with $\psi(0,x) = x$, then $\Psi: T \times M \to M$, $\Psi(t,x) = \psi(t,x)$ is a flow on M (with $T = \mathbb{R}$). In this case, the Ψ_t, $t \in \mathbb{R}$ are even diffeomorphisms. If X is not complete, then one obtains a local flow on M, and local flows are in $1-1$ correspondence with the vectorfields on M, see e.g. Boothby (1975), Chapter IV.4. If M is compact, then all vector fields are complete. If the vectorfield is time dependent, then the solutions of $\dot{x} = X(t,x)$ will, in general, not generate a flow on M, because $\psi(t+s,x,X(\tau,\cdot)) = \psi(t,\psi(s,x,X(\tau,\cdot)),X(\tau+s,\cdot))$, where $\psi(t,x,X(\tau,\cdot))$ denotes the solution of $\dot{x} = X(\tau,x)$ at time t with $\psi(0,x,X(\tau,\cdot)) = x$. For time dependent differential equations one therefore has to take the time shift into account, when formulating an associated flow. This will become important, when we discuss control systems and stochastic systems.

Next consider the nonlinear control system on a smooth manifold M

$$(1) \qquad \dot{x} = X_0(x) + \sum_{i=1}^{m} u_i(t) X_i(x)$$

where $X_0, \dots, X_m$ are given smooth vectorfields on M, and $(u_i)_{i=1\dots m} =: u \in \mathcal{U} := \{u\colon \mathbf{R} \to U \subset \mathbf{R}^m, \text{ locally integrable}\}$. Typical questions asked in control theory are e.g.: Given $x, y \in M$, does there exist a time $t \geq 0$ and a control $u \in \mathcal{U}$ such that $\varphi(t,x,u) = y$ (controllability); here $\varphi(t,x,u)$ is the solution of (1) at time t using the control function u such that $\varphi(0,x,u) = x$. Or given a rest point $x^0 \in M$ of $\dot{x} = X_0(x)$, does there exist a control function u such that x^0 becomes stable for (1), etc. (see e.g. Wonham (1979), Isidori (1989) or Nijmeijer and van der Schaft (1990) for a discussion of control theoretic problems).

If we want to formulate control systems as flows, we have to take the dependence of the solutions of (1) on the functions $u \in \mathcal{U}$ into account:

Denote by $\theta\colon \mathbf{R} \times \mathcal{U} \to \mathcal{U}$, $\theta_t u(\cdot) = u(t+\cdot)$ the usual time shift, and define

$$(2) \qquad \phi\colon \mathbf{R} \times \mathcal{U} \times M \to \mathcal{U} \times M, \quad \phi(t,u,x) = (\theta_t u, \varphi(t,x,u)).$$

Then $\theta_{t+s} = \theta_t \circ \theta_s$, $\varphi(t+s,x,u) = \varphi(t,\varphi(s,x,u),\theta_s u)$, and therefore $\phi_{t+s} = \phi_t \circ \phi_s$. In particular we obtain (compare Colonius and Kliemann (1990^a, Lemma 3.3)): If $U \subset \mathbf{R}^m$ is compact and convex, equip $\mathcal{U}$ with the weak *-topology of $L^\infty(\mathbf{R},\mathbf{R}^m) = \left(L^1(\mathbf{R},\mathbf{R}^m)\right)^*$. Then (2) is a continuous dynamical system on the separable, complete metric space $\mathcal{U} \times M$, called the control flow associated with (1).

Now the typical objects of control theory for (1), like reachable sets etc., are projections of similar objects for (2), e.g. denote by $\mathcal{O}^+(x) = \{y \in M;$ there exist $t \geq 0$ and $u \in \mathcal{U}$ with $\varphi(t,x,u) = y\}$ the reachable set (or forward orbit) of $x \in M$ for (1), then $\mathcal{O}^+(x) = \bigcup_{u \in \mathcal{U}} \pi_M \phi(t \geq 0, x, u)$, where $\pi_M\colon \mathcal{U} \times M \to M$ is the projection onto the second component, and similarly for $\mathcal{O}^-(x)$, the set of points, from which x can be reached. In Section 3. we will discuss in more detail several connections between control theoretic concepts for (1) and notions from the theory of dynamical systems for (2).

Stochastic flows are abstractly defined in the following way: Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\theta_t \colon \Omega \to \Omega$, $t \in T$, a family of measurable maps such that $\theta_{t+s} = \theta_t \circ \theta_s$ and $\theta_t P = P$ for all $t \in T$, i.e. P is a θ-invariant probability measure. Let M be a Polish space (separable, complete, metric) and $\mathcal{B}$ its Borel σ-algebra. A stochastic flow is then a measurable map

$$(3) \qquad \phi \colon T \times \Omega \times M \to \Omega \times M, \quad \phi(t, w, x) = (\theta_t w, \varphi(t, w, x))$$

such that $\varphi(t, \cdot, w) \colon M \to M$ is a homeomorphism for all (t, w),

and $\varphi(t+s, x, w) = \varphi(t, \varphi(s, x, w), \theta_t w)$ for all $s, t \in T$.

Again, this implies $\phi_{t+s} = \phi_t \circ \phi_s$.

In the present context, two types of stochastic flows are of particular interest:
Random differential equations: Let $\{\xi_t,\ t \in \mathbf{R}\}$ be a stationary (ergodic) stochastic process taking values in some Polish space $(N, \mathcal{N})$, where $\mathcal{N}$ denotes the Borel σ-algebra of N. Let Ω be the trajectory space of $\{\xi_t,\ t \in \mathbf{R}\}$, and construct the measure P from the finite dimensional distributions of $\{\xi_t,\ t \in \mathbf{R}\}$ via the Kolmogorov construction. Denote by $\zeta \colon \mathbf{R} \times \Omega \to N$ the evaluation map $\zeta(t, w) = w(t)$, then $\{\zeta_t,\ t \in \mathbf{R}\}$ is a stationary (ergodic) stochastic process with the same finite dimensional distributions as $\{\xi_t,\ t \in \mathbf{R}\}$, and the probability measure P is θ-invariant, where θ is the shift on Ω (compare e.g. Rozanov (1967, Chapter 4.)). Now let X be a measurable map from N into the smooth vectorfields on a smooth manifold M, and consider the random differential equation $\dot{x} = X(x, \zeta_t(w))$ on M, and assume that all its solutions $\varphi(t, x, w)$ do not explode in finite time (this holds e.g. if M is compact). Then $\phi = (\theta, \varphi)$ defines a stochastic flow.

Stochastic differential equations: Let M be a smooth manifold and $X_0 \dots X_m$ smooth vector fields on M. Consider the stochastic differential equation $dx = X_0(x)dt + \sum_{i=1}^{m} X_i(x) \circ dW_i$, where the W_i are independent standard Wiener processes and "$\circ$" denotes the symmetric or Stratonovič stochastic integral. Denote by $(\Omega, \mathcal{F}, P)$ the Wiener space of continuous functions into $\mathbf{R}^m$, vanishing at zero, with the Wiener measure P. P is invariant (and ergodic) with respect to the Wiener shift on Ω, defined by $\theta_t w(\cdot) = w(t + \cdot) - w(t)$. Under certain regularity conditions on the vector fields $X_0 \dots X_m$, $\phi = (\theta, \varphi)$ defines a stochastic flow (of diffeomorphisms on M) for $t \geq 0$, where $\varphi(t, x, w)$ denote the (pathwise) solution of the stochastic differential equation (see e.g. Kunita (1984, Chapter II)). If we extend the Wiener process backwards in time for $t \leq 0$ with an independent copy, we obtain with the same construction a stochastic flow for $t \in \mathbf{R}$.

We have seen that stochastic flows and control flows are flows of homeomorphisms (or even diffeomorphisms) over a shift space of trajectories (the trajectories of an underlying stochastic process, or the admissible control functions, respectively) i.e. skew product flows. If they are defined over the same function space (and with the same dynamics on M), then the difference is basically that for stochastic flows the trajectory space carries an additional shift invariant probability measure. In this sense one can talk of the control system associated with a random or stochastic differential equation (in the latter case one has to consider admissible controls, which vanish at zero), and vice versa each control system has associated with it a class of stochastic flows, determined by all θ-invariant probability measures on $(\mathcal{U}, \theta)$. The goal of this paper is to discuss ergodic properties of such stochastic flows, using control theoretic concepts and aspects of the theory of dynamical systems, applied to control flows. This point of view will be discussed in the next section.

3. Dynamical Properties of Control Flows

In this section we will characterize several properties of control systems using concepts from the theory of dynamical systems. This will enable us to analyze invariant measures and their supports in Section 4. While most of the theory developed here also works for the discrete time case (i.e. $T = \mathbf{Z}$ or $\mathbf{N}$), we will restrict ourselves to $T = \mathbf{R}$.

We will need the following concepts (see e.g. Mañé (1987) and Conley (1978)):

Definition 3.1. Let (S, Ψ) be a continuous dynamical system. For $x \in S$ the *limit set* $\omega(x)$ is defined as $\omega(x) = \{y \in S;$ there exists $t_k \to \infty$ with $\Psi(t_k, x) \to y\}$. (S, Ψ) is *topologically transitive*, if there exists $x \in S$ with $\omega(x) = S$, and *topologically mixing*, if for any two open sets $V_1, V_2 \subset S$ there exist $T_0 \in \mathbf{R}$, $T_1 > 0$ such that for all $n \in \mathbf{N}$ $\Psi(-nT_1 + T_0, V_1) \cap V_2 \neq \phi$.

A closed Ψ-invariant subset $W \subset S$ is called a *maximal topologically mixing set* if $(W, \Psi|_W)$ is topologically mixing and every closed Ψ-invariant set $W' \supset W$, for which $(W', \Psi|_{W'})$ is topologically mixing, satisfies $W' = W$. Analogously maximal topologically transitive sets are defined.

Consider the nonlinear control system on a paracompact, C^∞ Riemannian manifold M

$$\dot{x} = X_0(x) + \sum_{i=1}^{m} u_i(t) X_i(x) \tag{1}$$

where $X_0, \dots, X_m$ are C^∞ vector fields, $(u_i) = u \in \{u \colon \mathbf{R} \to U$, locally integrable$\}$, $U \subset \mathbf{R}^m$ is compact and convex. We define for such a control system:

Definition 3.2. The *positive orbit* of (1) from $x \in M$ is given by

$$\mathcal{O}^+(x) = \{y \in M;\ \text{there is } t \geq 0 \text{ and } u \in \mathcal{U} \text{ such that } y = \varphi(t, x, u)\}.$$

$D \subset M$ is called a *control set* of (1) if (i) $D \subseteq \overline{\mathcal{O}^+(x)}$ for all $x \in D$, (ii) for all $x \in D$ there exists $u \in \mathcal{U}$ such that $\varphi(t, x, u) \in D$ for all $t \geq 0$, and (iii) D is maximal with respect to these properties.

The system (1) is *completely controllable* if $\mathcal{O}^+(x) = M$ for all $x \in M$.

In order to avoid degenerate situations we will assume that $\mathcal{O}^+(x)$ (and also the negative orbit $\mathcal{O}^-(x) = \{y \in M;$ there is $t \geq 0$ and $u \in \mathcal{U}$ such that $x = \varphi(t, y, u)\})$ have nonvoid interior in M. To ensure this property, it is convenient to assume

(H) $$\dim \mathcal{LA}\{X_0 + \sum u_i X_i,\ (u_i) \in U\}(x) = \dim M \text{ for all } x \in M,$$

where for a set $\mathcal{X}$ of vector fields $\mathcal{LA}\{\mathcal{X}\}$ denotes the Lie algebra generated by $\mathcal{X}$, and $\mathcal{LA}\{\mathcal{X}\}(x)$ is the linear subspace of T_xM, the tangent space at x, which is spanned by $\mathcal{LA}\{\mathcal{X}\}$, compare Isidori (1989) and Nijmeijer and van der Schaft (1990) for a detailed discussion of (H). Here it suffices to note the following consequences:

- If $\mathcal{V} \subset \mathcal{U}$ is a dense subset, then the control sets defined via $\mathcal{V}$ are the same as those defined via $\mathcal{U}$. This applies in particular to the continuous, piecewise constant, or periodic control functions in $\mathcal{U}$.
- If $D \subset M$ is a control set with int $D \neq \phi$, then for all $x \in D$, all $y \in \text{int } D$ there exist $t \geq 0$ and $u \in \mathcal{U}$ with $\varphi(t, x, u) = y$, i.e. we have precise controllability in int D. In particular $\overline{\mathcal{O}^+(x)} = M$ for all $x \in M$ implies $\mathcal{O}^+(x) = M$ for all $x \in M$, i.e. complete controllability.

Consider now the control flow induced by (1) on $\mathcal{U} \times M$

(2) $$\phi\colon \mathbf{R} \times \mathcal{U} \times M \to \mathcal{U} \times M, \quad \phi = (\theta, \varphi).$$

We lift the control sets $D \subset M$ with nonvoid interior to ϕ-invariant sets on $\mathcal{U} \times M$ via

(3) $$\mathcal{D} = c\ell\{(u, x) \in \mathcal{U} \times M;\ \varphi(t, x, u) \in \text{int } D \text{ for all } t \in \mathbf{R}\},$$

where the closure is taken with respect to the weak*-topology on $\mathcal{U}$ and the manifold topology on M. If we are looking for properties of ϕ that are related to the control structure of (1), then these must be properties of $\mathcal{D}$. And, since the shift θ is not affected by the dynamics of (1), θ has to enjoy these properties as well. It turns out that topological mixing (and transitivity) are the appropriate concepts:

Theorem 3.3. (i) *The shift* $(\mathcal{U}, \theta)$ *is topologically transitive and mixing.*

(ii) *Let* $D \subset M$ *be a control set with int* $D \neq \phi$. *Then* $\mathcal{D}$, *defined by* (3), *is a maximal topologically mixing (and a maximal topologically transitive) set with*

$$\text{int } D = \text{int } \pi_M \mathcal{D}, \text{ and } \overline{D} = \pi_M \mathcal{D}. \tag{4}$$

(iii) *If* $\mathcal{D} \subset \mathcal{U} \times M$ *is a maximal topologically mixing (or transitive) set of* $(\mathcal{U} \times M, \phi)$ *with int* $\pi_M \mathcal{D} \neq \phi$, *then there exists a control set* $D \subset M$, *which satisfies* (4).

The proof is given in Colonius and Kliemann (1990[a], Proposition 2.6, Theorem 3.8, and Corollary 3.9). Note, in particular, that the lifted control sets $\mathcal{D} \subset \mathcal{U} \times M$ are topologically mixing and transitive. This leads to the following characterization of complete controllability:

Corollary 3.4. *Under the assumptions above, the following statements are equivalent:*

(i) *The control system* (1) *is completely controllable on* M.
(ii) *The dynamical system* $(\mathcal{U} \times M, \phi)$ *is topologically mixing.*
(iii) *The dynamical system* $(\mathcal{U} \times M, \phi)$ *is topologically transitive.*

The ergodic theory of stochastic flows and control flows deals with invariant measures of $(\mathcal{U} \times M, \phi)$ and their properties. In particular, the supports of invariant measures and therefore the ω-limit sets have to be characterized. It was shown in Colonius and Kliemann (1989, Proposition 3.6) that $\pi_M \omega(u, x) \subset M$ always has nonvoid intersection with some control set D. But it does not follow that $\pi_M \omega(u, x) \subset D$, nor that D is unique. This leads to the concept of chain control sets $E \subset M$ and their lifts to $\mathcal{E} \subset \mathcal{U} \times M$, which always contain entire ω-limit sets. (For the connection of chain control sets with subbundle decompositions of linear flows on vector bundles see Colonius and Kliemann (1990[b]).)

Definition 3.5. Let (S, Ψ) be a continuous dynamical system on a metric space (S, d). For $\varepsilon > 0$ and $T > 0$ an (ε, T)-*chain* from $x \in S$ to $y \in S$ consists of a sequence $x_0, \ldots, x_n \in S$ and a sequence $t_0, \ldots, t_{n-1}$ in $\mathbb{R}$ such that $x_0 = x$, $x_n = y$, $t_j \geq T$ and $d(\Psi(t_j, x_j), x_{j+1}) \leq \varepsilon$ for $j = 0, \ldots, k-1$.

For $A \subset S$ define the *chain limit set* by

$$\Omega(A) = \{y \in S;\ \text{for all } \varepsilon > 0,\ T > 0 \text{ there exists } x \in A \text{ such that there is an } (\varepsilon, T) - \text{chain from } x \text{ to } y\},$$

and the *chain recurrent set* as $\mathcal{CR} = \{x \in S;\ x \in \Omega(x)\}$.

The system (S, Ψ) is called *chain recurrent*, if $S = \mathcal{CR}$, and *chain transitive*, if $y \in \Omega(x)$ for all $x, y \in S$.

Recall that (S, Ψ) is chain transitive iff it is chain recurrent and S is connected, and for A closed, $\Omega(A)$ is closed, invariant and contains $\omega(x)$ for all $x \in A$, compare Conley (1978).

For the control system (1) we define the corresponding concept using chain control sets:

Definition 3.6. A set $E \subset M$ is called a *chain control set* of (1), if

(i) for all $x, y \in E$ and all $\varepsilon > 0$, $T > 0$ there are $n \in \mathbf{N}$, $x_0, \ldots, x_n \in M$, $u_0, \ldots, u_{n-1} \in \mathcal{U}$, and $t_0, \ldots, t_{n-1} \geq T$ with $x_0 = x$, $x_n = y$ and $d(\varphi(t_j, x_j, u_j), x_{j+1}) < \varepsilon$ for $j = 0, \ldots, n-1$,

(ii) for all $x \in E$ there exists $u \in \mathcal{U}$ such that $\varphi(t, x, u) \in E$ for all $t \in \mathbf{R}$, and

(iii) E is maximal with respect to these properties.

The system (1) is *completely chain controllable*, if M is the chain control set.

For the control flow $(\mathcal{U} \times M, \phi)$ we again lift the chain control sets $E \subset M$ to $\mathcal{U} \times M$ via

$$\mathcal{E} = \{(u, x) \in \mathcal{U} \times M;\ \varphi(t, x, u) \in E \text{ for all } t \in \mathbf{R}\}. \tag{5}$$

The analogue of Theorem 3.3 for chain control sets reads:

Theorem 3.7. (i) *The shift $(\mathcal{U}, \theta)$ is chain transitive.*

(ii) *$E \subset M$ is a chain control set of (1) iff $\mathcal{E}$ is a maximal invariant chain transitive set of $(\mathcal{U} \times M, \phi)$.*

(iii) *(1) is completely chain controllable iff $(\mathcal{U} \times M, \phi)$ is chain transitive iff $(\mathcal{U} \times M, \phi)$ is chain recurrent.*

For a proof of (i) and (ii) see Colonius and Kliemann (1990[a], Lemma 4.6 and Theorem 4.9), (iii) is an easy consequence of (ii).

We now have two control structures on M that are related to ergodic properties of $(\mathcal{U} \times M, \phi)$. For the remainder of this section we discuss these control structures and their relations in some more detail.

Lemma 3.8. (i) *Chain control sets are pairwise disjoint, closed, and connected.*

(ii) *Control sets are pairwise disjoint and connected. They are closed if they are invariant, i.e. $\overline{D} = \overline{\mathcal{O}^+(x)}$ for all $x \in D$.*

(iii) *Invariant control sets have nonvoid interior.*

(iv) *If* int $D \neq \phi$ *for some control set* D, *then* $\mathcal{O}^+(x) \supset$ int D *for all* $x \in D$, *i.e. we have precise controllability in* int D.

Remark 3.9. (i) For each control set D there exists a (unique) chain control set $E \subset M$ with $D \subset E$.

(ii) Several control sets may be contained in one chain control set.

(iii) There may be points in a chain control set that are in no control set.

While (i) is obvious from the definitions, (ii) and (iii) can be seen from the following example.

Example 3.10. Consider the control system on the circle $\mathbf{S}^1$

$$\dot{x} = -\sin^2 x + a\cos^2 x - u\cos^2 x, \quad x \in \mathbf{R} \bmod 2\pi, \quad a > 0,$$

with $U = [A, a] \subset \mathbf{R}$. There are four control sets $D_1 = [0, \arctan(a - A)^{1/2}]$, $D_2 = (\pi - \arctan(a - A)^{1/2}, \pi)$, $D_3 = D_1 + \pi$, $D_4 = D_2 + \pi$, compare Arnold and Kliemann (1983, Theorem 4.8) for a general technique to compute control sets for systems with one-dimensional state space. There exists, however, only one chain control set $E = \mathbf{S}^1$. Furthermore, the points in $\mathbf{S}^1 \setminus \bigcup_{i=1}^{4} D_i$ are in no control set, but in E. It is true in general that two control sets D_1 and D_2 with $\overline{D_1} \cap \overline{D_2} \neq \phi$ are in one chain control set. Control sets are chain control sets, if they have a certain isolation property, compare Colonius and Kliemann (1990[a], Section 4).

Control sets and chain control sets are ordered in the following way: Let D_1 and D_2 be control sets, then we define

(6) $\qquad D_1 \prec D_2$ if there exist $x \in D_1$ and $y \in D_2$ with $y \in \overline{\mathcal{O}^+(x)}$.

Lemma 3.11. (i) $\prec$ *defined by* (6) *is an order on the control sets of* (1).

(ii) *The closed (i.e. invariant) control sets are maximal elements of* $\prec$, *the open control sets are minimal elements.*

(iii) *If* M *is compact, then* (1) *has at least one closed and one open control set. In this case the maximal (minimal) elements are exactly the closed (open) control sets.*

Proof. (i) For a control set D, $D \prec D$ is obvious from the definition. $D_1 \prec D_2$ and $D_2 \prec D_1$ means that there exist $x_1 \in D_1$, $x_2 \in D_2$ with $x_2 \in \overline{\mathcal{O}^+(x)}$, and also $y_2 \in D_2$, $y_1 \in D_1$ with $y_1 \in \overline{\mathcal{O}^+(y_2)}$. Since $x_i, y_i \in D_i$ for $i = 1, 2$, we have $x_i \in \overline{\mathcal{O}^+(y_i)}$ and $y_i \in \overline{\mathcal{O}^+(x_i)}$. Therefore $y_2 \in \overline{\mathcal{O}^+(x_1)}$ and $x_1 \in \overline{\mathcal{O}^+(y_2)}$, and hence x_1 and y_2 are in the same control set by maximality. Finally, if $D_1 \prec D_2$

and $D_2 \prec D_3$, then there are $x \in D_1$, $y_1 \in D_2$ with $y_1 \in \overline{\mathcal{O}^+(x)}$, and $y_2 \in D_2$, $z \in D_3$ with $z \in \overline{\mathcal{O}^+(y_2)}$. Since $y_1, y_2 \in D_2$, we know that $y_2 \in \overline{\mathcal{O}^+(y_1)}$, and hence, using continuous dependence on initial values $z \in \overline{\mathcal{O}^+(x)}$, i.e. $D_1 \prec D_3$.

(ii) For all $x \in D$, an invariant control set, we have $\overline{\mathcal{O}^+(x)} \subset \overline{D}$, and hence invariant control sets are maximal elements. Now consider the time reversed system associated with (1):

$$\dot{x}^* = -X_0(x^*) - \sum_{i=1}^{m} u_i(t) X_i(x^*). \tag{7}$$

The positive orbits $\mathcal{O}^{*+}(x)$ of (7) are exactly the negative orbits $\mathcal{O}^-(x)$ of (1). Hence the interior of the closed (i.e. invariant) control sets of (7) are the open control sets of (1), and this proves the second assertion of (ii).

(iii) It was proved in Colonius and Kliemann (1989) that under our assumptions the control system (1) on a compact manifold M has at least one invariant control set D with int $D \neq \phi$. Furthermore for each $x \in M$ there exists an invariant control set $D \subset \overline{\mathcal{O}^+(x)}$. Hence in this case the invariant control sets are exactly the maximal elements of $\prec$. Using time reversal, one sees that the open control sets are exactly the minimal elements of $\prec$. □

The next example shows that for noncompact M there need not exist invariant control sets, and maximal elements of $\prec$ need not be closed.

Example 3.12. Consider the control system in $\mathbb{R}^1$

$$\dot{x} = X_0(x) + uX_1(x) = 2 + u(x^2 - 1) \tag{8}$$

with $U = [A, B] \subset [0, \infty)$. If $B < 2$, then (8) has no control set and $\lim_{t \to \infty} \varphi(t, x, u) = \infty$ for all $x \in \mathbb{R}$, all $u \in \mathcal{U}$.

If $A < 2$, $B \geq 2$, then (8) has a unique control set $D = \left[-\sqrt{1 - \frac{2}{B}}, \sqrt{1 - \frac{2}{B}}\right)$, which is neither open nor closed.

If $A \geq 2$, then (8) has two control sets $D_1 = \left(\sqrt{1 - \frac{2}{A}}, \sqrt{1 - \frac{2}{B}}\right)$, and $D_2 = \left[-\sqrt{1 - \frac{2}{B}}, -\sqrt{1 - \frac{2}{A}}\right]$. We have $D_1 \prec D_2$, and D_1 is open, D_2 is closed.

For chain control sets it is convenient to define the order via the lifts to $\mathcal{U} \times M$: Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two maximal invariant chain transitive sets of $(\mathcal{U} \times M, \phi)$, define

$$\begin{gathered} \mathcal{E}_1 \prec \mathcal{E}_2 \text{ if there exists } (u, x) \in \mathcal{U} \times M \text{ sucht that} \\ \omega^*(u, x) \subset \mathcal{E}_1 \text{ and } \omega(u, x) \subset \mathcal{E}_2. \end{gathered} \tag{9}$$

where $\omega^*(u, x) = \{(v, y) \in \mathcal{U} \times M;\ \text{there exists } t_k \to -\infty \text{ with } \phi(t_k, u, x) \to (v, y)\}$ is the α-limit set of (u, x).

Lemma 3.13. (i) $\prec$ *defined in* (9) *is an order on the chain recurrent components of*
$(\mathcal{U} \times M, \phi)$*, and hence on the chain control sets of* (1).

(ii) *Any finite collection* $\{\mathcal{E}_1, \ldots, \mathcal{E}_n\}$ *of chain recurrent components with the order* $\prec$ *defines a Morse decomposition of* $(\mathcal{U} \times M, \phi)$.

Proof. (i) According to Conley (1978, Section II.6.2), the chain recurrent set $\mathcal{CR}$ can be written as $\mathcal{CR} = \cap\{A \cup A^*,\ A$ is an attractor of $(\mathcal{U} \times M, \phi)$ and A^* its complementary repeller$\}$, and hence (9) defines an order on the components of the chain recurrent set.

(ii) Again using Conley (1978, Section II.7), any finite decomposition into disjoint invariant sets with the order $\prec$ defines a Morse decomposition. □

At this moment, we have primarily two areas of application in mind for the ergodic theory of stochastic and control flows: the theory of Lyapunov exponents and stochastic bifurcation theory. We will now characterize more precisely the control sets and chain control sets that come up in these areas.

Lyapunov exponents are the exponential growth rates of the linearized system. We consider here only the simple case, where the control system (1) has a rest point $x^0 \in M$, i.e. $X_j(x^0) = 0$ for $j = 0, \ldots, m$. Linearization around x^0 yields locally in a neighborhood of x^0

$$\dot{y} = A_0(x^0)y + \sum_{i=1}^{m} u_i(t)A_i(x^0)y \qquad \text{in } \mathbf{R}^d \quad (d = \dim M) \tag{10}$$

with $A_j(x^0) := X_{j*}(x^0)$, the linearization of X_j at x^0 for $j = 0, \ldots, m$. The Lyapunov exponents of (1) at x^0 are then defined by

$$\lambda(y_0, u) = \limsup_{t \to \infty} \frac{1}{t} \log |\varphi(t, y_0, u)|, \qquad y_0 \neq 0$$

where $\varphi(t, y_0, u)$ denotes the solution of (10). Since $\lambda(\alpha y_0, u) = \lambda(y_0, u)$ for all $\alpha \in \mathbf{R}$, $\alpha \neq 0$, it suffices to consider (10) on the projective space $\mathbf{P}^{d-1}$ in $\mathbf{R}^d$:

$$\dot{s} = h_0(s) + \sum_{i=1}^{m} u_i(t)h_i(s) \tag{11}$$

with $h_j(s) := \big(A_j(x^0) + s^T A_j(x^0)s \cdot Id\big)\, s$, the projected vector field on $\mathbf{P}^{d-1}$ for $j = 0, \ldots, m$. We will assume again the nondegeneracy condition for (11) on $\mathbf{P}^{d-1}$, i.e.

$$\dim \mathcal{LA}\left\{h_0 + \sum_{i=1}^{m} u_i h_i,\ (u_i) \in U\right\}(s) = d - 1 \quad \text{for all } s \in \mathbf{P}^{d-1}.$$

The system (10) is a bilinear control system in $\mathbf{R}^d$, its associated semigroup is given by

$$\mathcal{S} = \{\exp t_n B_n \cdot \ldots \cdot \exp t_1 B_1,\ B_i \in N,\ t_i \geq 0,\ i = 1, \ldots, n \in \mathbf{N}\} \subset G\ell(d, \mathbf{R}),$$

with $N := \left\{ A_0(x^0) + \sum_{i=1}^{m} u_i A_i(x^0),\ (u_i) \in U \right\}$, the possible right hand sides of (10) for piecewise constant controls. $\mathcal{S}$ acts on $\mathbf{P}^{d-1}$ in a natural way via $s \mapsto \frac{1}{|gs|} gs$ for $s \in \mathbf{P}^{d-1}$, $g \in \mathcal{S}$, and the differential equation (11) corresponds to this action. (Compare Colonius and Kliemann (1990^b) for details of the entire set up.) For the control sets of (11) under the assumption (H) we obtain:

Theorem 3.14. (i) *There are k control sets D_i with int $D_i \neq \phi$, $i = 1, \ldots, k$, $1 \leq k \leq d$, called the main control sets.*

(ii) *The order, defined by (6) on the main control sets, is linear. We enumerate these sets by $D_1 \prec D_2 \prec \cdots \prec D_k$.*

(iii) *D_k is closed and $D_k = \bigcap_{s \in \mathbf{P}} \overline{\mathcal{O}^+(s)}$, D_1 is open and $\overline{D}_1 = \bigcap_{s \in \mathbf{P}} \overline{\mathcal{O}^-(s)}$.*

(iv) *For every $g \in$ int $\mathcal{S}$ and every $\lambda \in$ spec g there is a main control set D_i such that the corresponding generalized eigenspace $E(g, \lambda)$ satisfies $\mathbf{P}E(g, \lambda) \subset$ int D_i, where $\mathbf{P}E$ is the projection of $E \subset \mathbf{R}^d$ onto $\mathbf{P}^{d-1}$. Vice versa, the interior of the main control sets consists exactly of those elements $s \in \mathbf{P}$, which are eigenvectors for a (real) eigenvalue of some $g \in$ int $\mathcal{S}$.*

(v) *For every $g \in \mathcal{S}$ and every $\lambda \in$ spec g there is a main control set D_i with $\mathbf{P}E(g, \lambda) \cap \overline{D}_i \neq \phi$, and vice versa for every main control set D_i and every $g \in \mathcal{S}$ there exists a $\lambda \in$ spec g with $\mathbf{P}E(g, \lambda) \cap \overline{D}_i \neq \phi$.*

The proof of this result is given in Colonius and Kliemann (1990^b, Theorem 3.10).

For the chain control sets of (11) the corresponding result is

Theorem 3.15. (i) *There are ℓ chain control sets E_j, $j = 1, \ldots, \ell$, $1 \leq \ell \leq d$.*

(ii) *Every chain control set contains a main control set, in particular int $E_j \neq \phi$ for $j = 1, \ldots, \ell$ and $1 \leq \ell \leq k \leq d$.*

(iii) *The order defined by (9) on the (lifted) chain control sets is linear, and we write $E_1 < E_2 < \cdots < E_\ell$.*

(Compare Colonius and Kliemann (1990^b, Theorem 5.5) for a proof.) Note that the situations, described in Remark 3.9(ii) and (iii) can occur for systems of the type (11) as well. This can be seen from Example 3.10, which is the projection of the following bilinear system in $\mathbf{R}^2$ onto $\mathbf{P}^1$:

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & -2b \end{pmatrix} x + u(t) \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} x$$

has a projection onto $\mathbf{P}^1$ in polar coordinates $s = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$, $\varphi \in \mathbf{R} \bmod \pi$

$$\dot{\varphi} = -\sin^2\varphi + (b^2 - 1)\cos^2\varphi - u(t)\cos^2\varphi.$$

For $b^2 - 1 = a$ and $u \in [A, a]$ this is Example 3.10. In particular we have in this case $k = 2$, $\ell = 1$, and there are points $s \in \mathbf{P}^1 \setminus \bigcup_{i=1}^{2} \overline{D}_i$ that are not in the (generalized) eigenspace of any $g \in \mathcal{S}$, but in int $E = \mathbf{P}^1$.

The next class of examples deals with situations arising in (stochastic) codimension one bifurcations, compare Arnold and Boxler (1990) for the general set up. Here one considers a family of one-dimensional systems (replacing the noise by controls)

$$\dot{x} = X_\alpha(x) + u(t)Y_\alpha(x), \tag{12}$$

where X_α, Y_α are smooth vector fields on $\mathbf{R}$ or $\mathbf{S}^1$ and $\alpha \in I$ is the bifurcation parameter. We will analyze the case with compact $U \subset \mathbf{R}$, i.e. with bounded noise. The problem is to determine the invariant probability measures of (12), hence, first of all, the control sets and chain control sets need to be determined for each $\alpha \in I$. We will therefore consider the following system

$$\dot{x} = X(x) + u(t)Y(x) \qquad \text{on } M = \mathbf{R}^1 \text{ or } \mathbf{S}^1 \tag{13}$$

under the following assumptions:

(14)
- (i) X and Y are smooth vector fields on M,
- (ii) $u(t) \in U \subset \mathbf{R}$, U a compact interval,
- (iii) for each $u \in U$ there exists at most a finite number of zeros of $X + uY$.

These assumptions are typical for codimension one bifurcation diagrams with bounded noise, and cover also the case, where the bifurcation parameter α itself is noisy and appears linearly in (12), i.e. systems of the form $\dot{x} = X(x) + \alpha_t Y(x)$, with α_t a stochastic process with values in $U \subset \mathbf{R}$.

A general procedure for finding the control sets of one dimensional control systems was described in Kliemann (1980, Section II.6). Here we summarize these results and extend them to chain control sets. For each $u \in U$ define $S(u) = \{x \in M;\ X(x) + uY(x) = 0\}$, the rest points of (13) corresponding to

u. Denote $S = \bigcup_{u \in U} S(u)$ and

$$SA = \{s \in S;\ X(s) + uY(s) = 0 \text{ for all } u \in U\},$$
$$SB = \{s \in S;\ X(s) + uY(s) \geq 0 \text{ for all } u \in U, \text{ and there exist } u_1, u_2 \in U \text{ with } X(s) + u_1Y(s) = 0 \text{ and } X(s) + u_2Y(s) > 0\},$$
$$SC = \{s \in S;\ X(s) + uY(s) \leq 0 \text{ for all } u \in U, \text{ and there exist } u_1, u_2 \in U \text{ with } X(s) + u_1Y(s) = 0 \text{ and } X(s) + u_2Y(s) < 0\},$$
$$SD = \{s \in S;\ \text{there exist } u_1, u_2 \in U \text{ with } X(s) + u_1Y(s) > 0 \text{ and } X(s) + u_2Y(s) < 0\}.$$

The control sets of (13) are intervals, possibly consisting of only one point. The points in $SA \cup SB \cup SC$ are the boundary points of these intervals, the elements of SD are the interior points. More precisely, all control sets for $M = \mathbb{R}$ can be found as follows:

Let $s_0 \in SA$, then four cases are possible:

(a) There exists an interval $(s_0 - \varepsilon,\ s_0 + \varepsilon)$ for some $\varepsilon > 0$, such that for all $p \in (s_0 - \varepsilon,\ s_0 + \varepsilon) \setminus \{s_0\}$ we have $p \notin S$, then $\{s_0\}$ is a one point invariant control set, and no other control set intersects this interval.

(b) There exists a (maximal) interval of the form $(s_0,\ s_0 + \varepsilon)$ (or $(s_0 - \varepsilon,\ s_0)$) for some $\varepsilon > 0$, such that all points in this interval are in SB (or in SC), then each point p in the interval is a one point control set $\{p\}$ (not invariant), and $\{s_0\}$ is an invariant control set.

(c) There exists an interval $(s_0,\ s_0 + \varepsilon)$ for some $\varepsilon > 0$, such that for all $p \in (s_0,\ s_0 + \varepsilon)$ one has $p \in SD$. Take $(s_0,\ s_0 + \varepsilon)$ as the maximal interval with: $p \in (s_0,\ s_0 + \varepsilon)$ implies $p \in SD$, and define $s_1 = s_0 + \varepsilon$. (Set $s_1 = +\infty$ if $p \in SD$ for all $p > s_0$.) Then there exists a control set of the form

$$\begin{array}{lll} (s_0, s_1), & \text{if } s_1 \in SA, & \text{(invariant)}, \\ (s_0, s_1), & \text{if } s_1 \in SB, & \text{(not invariant)}, \\ (s_0, s_1], & \text{if } s_1 \in SC, & \text{(invariant)}, \\ (s_0, \infty), & \text{if } s_1 = \infty, & \text{(invariant)}, \end{array}$$

and $\{s_0\}$ is a one point invariant control set.

(d) There exists an interval $(s_0 - \varepsilon,\ s_0)$ for some $\varepsilon > 0$, such that for all $p \in (s_0 - \varepsilon,\ s_0)$ one has $p \in SD$. Take $(s_0 - \varepsilon,\ s_0)$ as the maximal interval with: $p \in (s_0 - \varepsilon,\ s_0)$ implies $p \in SD$, and define $s_1 = s_0 - \varepsilon$.

(Set $s_1 = -\infty$ if $p \in SD$ for all $p < s_0$.) Then there is a control set of the form

$$\begin{array}{lll} (s_1, s_0), & \text{if } s_1 \in SA, & \text{(invariant)}, \\ [s_1, s_0), & \text{if } s_1 \in SB, & \text{(invariant)}, \\ (s_1, s_0), & \text{if } s_1 \in SC, & \text{(not invariant)}, \\ (-\infty, s_0), & \text{if } s_1 = -\infty, & \text{(invariant)}, \end{array}$$

and $\{s_0\}$ is a one point invariant control set.

(Note that $SA = \phi$, if Assumption (H) holds for (13).)

If $s_0 \in SB$, then the following cases can occur:

(e) If there exists an interval of the form $(s_0 - \varepsilon,\ s_0 + \varepsilon)$ for some $\varepsilon > 0$, such that for all $p \in (s_0 - \varepsilon,\ s_0 + \varepsilon) \setminus \{s_0\}$ one has $p \notin S$, then $\{s_0\}$ is a one point (not invariant) control set.

(f) If there exists a (maximal) interval of the form $[s_0,\ s_0 + \varepsilon]$ or $[s_0 - \varepsilon,\ s_0]$ for some $\varepsilon > 0$, such that all points in the interval are in SB, then each point p in the interval is a one point (not invariant) control set $\{p\}$.

(g) If there exists an interval $(s_0,\ s_0 + \varepsilon)$ for some $\varepsilon > 0$, such that all points $p \in (s_0,\ s_0 + \varepsilon)$ are in SD, let $s_1 = s_0 + \varepsilon$ with $(s_0,\ s_0 + \varepsilon)$ the maximal interval as above. Then there exists a control set of the form

$$\begin{array}{lll} [s_0, s_1), & \text{if } s_1 \in SA \cup SB, & \text{(invariant if } s_1 \in SA\text{, not invariant otherwise)}, \\ [s_0, s_1], & \text{if } s_1 \in SC, & \text{(invariant)}, \\ [s_0, \infty), & \text{if } s_1 = \infty, & \text{(invariant)}. \end{array}$$

Similarly, if an interval $(s_0 - \varepsilon,\ s_0)$ exists with points in SD.

If $s_0 \in SC$, the cases are completely analogous to (e)–(g) above.

Finally, if $SD = M$, then M is the (unique, invariant) control set. Using Assumption (14(iii)) we see, that SA cannot contain infinitely many points. Hence after finitely many steps of the type (a)–(g) (and similarly for SC), all control sets of (13) in $\mathbb{R}$ are described. The general principle is: points in SA and intervals of points in SB and SC lead to one-point control sets, which are invariant iff the point is in SA. All other control sets are intervals with nonvoid interior, where the lower boundary belongs to the set, if the point is in SB, and similarly for the upper boundary, if this point belongs to SC. These intervals are invariant, iff the boundary points belong to the control sets or are in SA.

If M is compact, the same principles as above apply, except that, letting $M \simeq \mathbb{S}^1$, one has to consider the intervals mod 2π, when parametrizing $\mathbb{S}^1$ through the angle in $[0, 2\pi)$.

For the chain control sets of the control system (13) we obtain the following characterization: Define $\widehat{D} = \cup\{\overline{D},\ D$ is a control set of (13)$\}$, then by (14(iii)), $\widehat{D} = I_1 \dot{\cup} \ldots \dot{\cup} I_n$, the finite disjoint union of closed 'intervals', constructed as above.

We call an interval I of $\widehat{D}$ isolated, if there exists an open neighborhood N of I such that for all $y \in N \setminus I$ we have for all $u \in U$: $X(y) + uY(y) < 0$ for $y < I$, and $X(y) + uY(y) > 0$ for $y > I$, (or $X(y) + uY(y) > 0$ for $y < I$, and $X(y) + uY(y) < 0$ for $y > I$).

Theorem 3.16. (i) *If $M = \mathbf{R}$, then the I_i, $i = 1, \ldots, n$ are exactly the chain control sets of* (13).

(ii) *If M is compact, then*

(a) *if $\widehat{D} = I_1$, then M is the chain control set,*

(b) *if $n \geq 2$, then the I_i, $i = 1, \ldots, n$ are exactly the chain control sets of* (13) *iff there is at least one isolated interval in $\hat{D}$, otherwise M is the chain control set.*

(iii) *If $M = \mathbf{R}$ or if the system* (13) *has more than one chain control set, then for a control set D the set $\overline{D}$ is a chain control set iff there exists an open neighborhood N of $\overline{D}$, which intersects with no other control set. If M is compact and* (13) *has only one chain control set, then the closure of a control set D is a chain control set iff SA consists of at most one point and* (13) *is completely controllable in $M \setminus SA$.*

Proof. Note first of all that, by the construction of control sets above $\widehat{D} = S$, i.e. $x \in M \setminus \widehat{D}$ means for all $u \in U$ either $X(x) + uY(x) > 0$ or $X(x) + uY(x) < 0$. Recall also that by Lemma 3.8 chain control sets are pairwise disjoint, connected and closed.

(i) It is clear from the definition of chain control sets and from Lemma 3.8 that the intervals I_i, $i = 1, \ldots, n$ are contained in chain control sets. Let I be such an interval, and assume that there exists a chain control set $E \supsetneq I$. Then there is a point $x \in E \setminus I$, say w.l.o.g. $x < p$ for all $p \in I$, and $x \notin \widehat{D}$. Then, by the remark above, there is an open neighborhood $N(x)$ of x such that $N(x) \cap \widehat{D} = \phi$ and for all $y \in N(x)$, all $u \in U$ we have either (a) $X(y) + uY(y) > 0$ or (b) $X(y) + uY(y) < 0$. In case (a) there cannot exist an (ε, T)-chain from I to y for ε small enough, and T large enough, in case (b) there is no (ε, T)-chain from y to I for small ε and large T. I.e. $E = I$ and this proves (i).

(ii) If M is compact, then the system (13) has at least one closed, invariant control set (this part of Lemma 3.11 holds without Assumption (H)), i.e. $\widehat{D} \neq \phi$, and there exists at least one chain control set, compare Remark 3.9(i).

(a) If $\widehat{D}$ is one 'interval', then there is again one chain control set $E \supset \widehat{D}$. But for all $u \in U$, all $y \in M \setminus \widehat{D}$ we have either $X(y) + uY(y) > 0$ or $X(y)+uY(y) < 0$, and hence there is $y_0 \in M\setminus\widehat{D}$ and $u_0 \in U$ with $\lim\limits_{t\to-\infty} \varphi(t, y_0, u_0) \subset \widehat{D}$, and $\lim\limits_{t\to+\infty} \varphi(t, y_0, u_0) \subset \widehat{D}$, where $\varphi(t, y_0, u_0)$ denotes the solution of (13) corresponding to the constant control $u(t) \equiv u_0$. Thus $E = M$.

(b) If all intervals $I_1 \dots I_n$ are not isolated, then it is easy to see that there exist $y_1, \dots, y_n \in M\setminus\widehat{D}$, and $u_1, \dots, u_n \in U$ such that y_i lies in the gap between I_i and I_{i+1} for $i = 1, \dots, n$, y_n lies in between I_n and I_1, and $\lim\limits_{t\to-\infty} \varphi(t, y_i, u_i) \subset I_i$, $\lim\limits_{t\to+\infty} \varphi(t, y_i, u_i) \subset I_{i+1}$, $\lim\limits_{t\to-\infty} \varphi(t, y_n, u_n) \subset I_n$, and $\lim\limits_{t\to+\infty} \varphi(t, y_n, u_n) \subset I_1$. Hence M is the chain control set.

If one interval I of $\widehat{D}$ is isolated, then there is an open neighborhood N of I such that either for all $y \in N \setminus I$ and all $u \in \mathcal{U}$ $\lim\limits_{t\to-\infty} \varphi(t, y, u) \subset I$ and $\lim\limits_{t\to+\infty} \varphi(t, y, u) \subset \widehat{D} \setminus I$, or for all such y and u $\lim\limits_{t\to-\infty} \varphi(t, y, u) \subset \widehat{D} \setminus I$ and $\lim\limits_{t\to+\infty} \varphi(t, y, u) \subset I$. Furthermore, because the right hand side of (13) depends continuously on u, it cannot happen that for some $z \in M \setminus \widehat{D}$ there exist $u_1 \in \mathcal{U}$, $u_2 \in \mathcal{U}$ with $\lim\limits_{t\to-\infty} \varphi(t, z, u_1) \subset I$, $\lim\limits_{t\to+\infty} \varphi(t, z, u_2) \subset I$. Hence, arguing as in (i), in this case the $I_1 \dots I_n$ are exactly the chain control sets of (13).

(iii) If the $I_1 \dots I_n$ are the chain control sets of (13), then the result follows directly from the definition. If this is not the case, then according to (i) and (ii), M is compact and the chain control set. Hence for $\overline{D} = M$ we need that (13) is completely controllable with the possible exception of one point in SA.

$\square$

Remark 3.17. We have not assumed Hypothesis (H) for the system (13), because the situation, where $x \in M$ is a rest point for all $u \in U$, occurs frequently in bifurcation diagrams. If, however, we can assume (H), then the following simplifications hold: $SA = \phi$, and for the determination of the control sets we only have to go through the steps (e)–(g) for points in SB and SC. Theorem 3.16(iii) reads in this case: If M is compact and (13) has only one chain control set, then the closure of a control set D is a chain control set iff $D = M$.

Remark 3.18. Consider the projected linear system (11) on $\mathbb{P}^1$, i.e. $d = 2$, under Assumption (H). There are at most two main control sets, one closed, say C, and one open, denoted by C^-. In this situation we obtain for the chain control sets: $\mathbb{P}^1$ is the chain control set iff either

(i) the system is completely controllable, i.e. $C \cap C^- \neq \phi$, or
(ii) $C \cap \overline{C^-} \neq \phi$, or
(iii) there exists $u \in U$ such that $h_0(s) + \sum_{i=1}^{m} u_i h_i(s) = 0$ for all $s \in \mathbb{P}^1$.

In all other cases there are exactly two chain control sets, namely, C and $\overline{C^-}$.

4. Invariant Measures of Control Flows and Stochastic Flows

In this section we discuss existence and supports of invariant probability measures of control flows and stochastic flows. The situation for stochastic flows is distinguished by the following two facts: first of all, here an invariant measure P on the underlying shift space Ω is given a priori, and we are looking for invariant measures of the flow, whose marginal on Ω is P. Secondly one has to take measurability questions into account. We will first study the case of control flows, and then specialize the results to stochastic flows.

Definition 4.1. Let (S, Ψ) be a dynamical system. A probability measure μ on S is called *Ψ-invariant*, if $\Psi_t \mu = \mu$ for all $t \in \mathbb{R}$. We denote this set by M_Ψ.

Lemma 4.2. *Consider the control flow $(\mathcal{U} \times M, \phi)$ defined in (2.2). A probability measure μ on $\mathcal{U} \times M$ is ϕ-invariant iff μ is of the form $\mu(du, dx) = \mu_u(dx)\rho(du)$, where ρ is a θ- invariant measure on $\mathcal{U}$, and $\varphi(t, \cdot, u)\mu_u = \mu_{\theta_t u}$ for all $t \in \mathbb{R}$, where $\varphi(t, x, u)$ denotes again the solution of the control equation.*

For a proof see Crauel (1986, Lemma 2) or Colonius and Kliemann (1990[a], Proposition 5.2).

If a measure μ is Ψ-invariant for $t \geq 0$, then invertibility of Ψ_t implies $\mu = (\Psi_t)^{-1}\Psi_t\mu = \Psi_{-t}\mu$, i.e. μ is Ψ-invariant for all $t \in \mathbb{R}$.

Invariant measures of the control flow can be constructed via the Krylov-Bogolyubov device, i.e. for $(u, x) \in \mathcal{U} \times M$ consider the Cesaro limits for sequences $t_k \to \infty$

$$(1) \qquad \lim_{t_k \to \infty} \frac{1}{t_k} \int_0^{t_k} F(\theta_\tau(u), \varphi(\tau, x, u))\, d\tau = \int_{\mathcal{U} \times M} F(v, y) d\mu_{u,x}$$

for all $F \in C(\mathcal{U} \times M, \mathbb{R})$, the continuous functions from $\mathcal{U} \times M$ into $\mathbb{R}$. Note that in general $\mu_{u,x}$ is not unique for $(u, x) \in \mathcal{U} \times M$. The following properties of the probability measures $\mu_{u,x}$ are well known, see e.g. Mañé (1987, Chapter II.6):

(a) M_θ and M_ϕ are nonempty, if U and M are compact.

(b) Define

$$\Sigma_\phi^e = \{(u,x) \in \mathcal{U} \times M;\ \text{the measure } \mu_{u,x} \text{ defined in (1) is independent of the sequence } t_k \text{ and ergodic}\}$$

$$\Sigma_\phi^s = \{(u,x) \in \Sigma_\phi^e;\ (u,x) \in \operatorname{supp} \mu_{u,x}\},$$

then for M, U compact, $\Sigma_\phi^s \neq \phi$ and Σ_ϕ^s has total measure with respect to M_ϕ, i.e. $\mu\left(\Sigma_\phi^s\right)^c = 0$ for all $\mu \in M_\phi$. (Here supp μ denotes the support of μ, and A^c denotes the complement of the set A.)

(c) Each $\mu \in M_\phi$ has an ergodic decomposition: Every $F \in L^1(\mathcal{U} \times M,\ \mu)$ is $\mu_{u,x}$-integrable for μ-almost all $(u,x) \in \Sigma_\phi^s$ and $\int \left(\int F\, d\mu_{u,x}\right) d\mu = \int F\, d\mu$.

Hence, if we want to characterize the possible support of some $\mu \in M_\phi$, it suffices to characterize the set Σ_ϕ^s.

Define for a control set $D \subset M$

$$\mathcal{D}^+ = c\ell\{(u,x) \in \mathcal{U} \times M;\ \varphi(t,x,u) \in D \text{ for all } t \in \mathbb{R}\}.$$

If int $D \neq \phi$, then $\mathcal{D}^+ \supset \mathcal{D}$, with $\mathcal{D}$ as defined in (3) from Section 3. However, the lift $\mathcal{D}^+$ is nonvoid even if int $D = \phi$, as can be seen from Assertion (ii) in the following theorem.

Theorem 4.3.

(i) *For all $(u,x) \in \mathcal{U} \times M$ and all $\mu_{u,x}$ as in (1) we have supp $\mu_{u,x} \subset \omega(u,x) \subset \mathcal{E}$, the lift of some chain control set $E \subset M$ of (2.1) to $\mathcal{U} \times M$.*

(ii) *For all $(u,x) \in \mathcal{U} \times M$ and all $\mu_{u,x}$ as in (1) there exists $\Gamma \subset$ supp $\mu_{u,x}$ with $\mu_{u,x}\Gamma = 1$, such that for all $(v,y) \in \Gamma$ there is a control set $D \subset M$ with $\varphi(t,y,v) \in D \cap \pi_M\omega(u,x)$, for all $t \in \mathbb{R}$.*

(iii) *If $(u,x) \in \Sigma_\phi^e$, then there exists a (unique) control set $D \subset M$ such that for $\mu_{u,x}$-almost all (v,y) we have $\varphi(t,y,v) \in D$ for all $t \in \mathbb{R}$, i.e. supp $\mu_{u,x} \subset \mathcal{D}^+$, with $\mathcal{D}^+$ defined above. Also, if $\mu \in M_\phi$ is ergodic, then supp $\mu \subset \mathcal{D}^+$ for some control set D.*

(iv) *Vice versa, if $D \subset M$ is a control set with int $D \neq \phi$, then for each $x \in$ int D there exists $u \in \mathcal{U}$ such that $(u,x) \in \Sigma_\phi^s$.*

(v) *Assume that int $D \neq \phi$ for all control sets $D \subset M$. Then*

$$c\ell \cup \{D;\ D \text{ is a control set}\} = \pi_M c\ell \cup \{\operatorname{supp} \mu_{u,x};\ (u,x) \in \Sigma_\phi^s\}$$
$$= \pi_M c\ell \cup \{\operatorname{supp} \mu;\ \mu \in M_\phi\}.$$

The proof of this theorem can be found in Colonius and Kliemann (1990[a], Lemma 5.3, Proof of Theorem 5.5, and Corollary 5.7).

This theorem says in particular that for all $\mu \in M_\phi$ one has supp $\mu \subset \mathcal{E}$, where E is some chain control set, and supp $\mu \subset \{\mathcal{D}^+;\ D \subset E$ is a control set$\}$. If μ is ergodic, then supp $\mu \subset \mathcal{D}^+$ for some control set D. Note that an analogue of (iv) for chain control sets is not true, compare e.g. Example 3.10.

We now turn to the existence of invariant probability measures for control flows. Starting from $x \in M$, we know by Theorem 4.3 that for $x \notin \cup\{\overline{D};\ D$ is a control set$\}$ there is no Krylov-Bogolyubov measure for x and any $u \in \mathcal{U}$, (i.e. $(u,x) \in$ supp $\mu_{u,x}$ cannot hold). On the other hand, for $x \in$ int D, D some control set, there always exists a $u \in \mathcal{U}$ such that $(u,x) \in \Sigma_\phi^s$: Just take a periodic u, which leads to a periodic trajectory in int D through x. Furthermore, it is easy to construct examples for dim $M \geq 2$, such that for some $x \in \partial D$ we have that $x \notin \pi_M \Sigma_\phi^e$.

What is more important, however, by the characterization in Lemma 4.2, is to construct the measures μ_u on M for $u \in \mathcal{U}$. We proceed in the following way: Define for a control set $D \subset M$ and a control function $u \in \mathcal{U}$

$$D_u^+ = \{x \in \overline{D};\ \varphi(t,x,u) \in \overline{D} \text{ for all } t \geq 0\}. \tag{2}$$

Note that $D_u^+ \neq \phi$ iff $D_u = \{y \in \overline{D};\ \varphi(t,y,u) \in \overline{D} \text{ for all } t \in \mathbf{R}\} \neq \phi$.

Theorem 4.4. *Assume that M is compact, or more generally that $\widehat{D} = \cup\{D;\ D$ is a control set$\}$ is bounded. Then the following holds:*

(i) *Let $u \in \mathcal{U}$, and let $D \subset M$ be some control set. Then there exists an invariant measure $\mu_{u,x}$ of the form (1) with supp $\mu_{u,x} \subset \mathcal{D}^+$ iff $D_u^+ \neq \phi$.*

(ii) *Let μ be a Φ-invariant measure with decomposition $\mu = \mu_u\rho$ according to* Lemma 4.2. *Then for all control sets D with supp $\mu \cap \mathcal{D}^+ \neq \phi$ there exists $u \in \mathcal{U}$ such that $D_u^+ \neq \phi$. Conversely, if $\rho \in M_\theta$ and D is a control set with $D_u^+ \neq \phi$ for ρ-almost all $u \in \mathcal{U}$, then there is $\mu \in M_\phi$ with $\mu = \mu_u\rho$ and supp $\mu \subset \mathcal{D}^+$.*

(iii) *If $\mu \in M_\phi$ in* (ii) *is ergodic, then the control set D with supp $\mu \cap \mathcal{D}^+ \neq \phi$ is unique and $D_u^+ \neq \phi$ for ρ-almost all $u \in \mathcal{U}$. If $\rho \in M_\theta$ in* (ii) *is ergodic, and if $D_u^+ \neq \phi$ for ρ-almost all $u \in \mathcal{U}$, then $\mu \in M_\phi$.*

(iv) *Suppose $\rho \in M_\theta$ is ergodic and $\mu \in M_\phi$ can be desintegrated as $\mu = \mu_u\rho$. Then for any control set D with $\mu(\mathcal{D}^+) > 0$ we have that $D_u^+ \neq \phi$ for ρ-almost all $u \in \mathcal{U}$.*

Proof.

(i) One direction is obvious, the other one follows from Theorem 4.3(i), because $\mathcal{D}^+$ is ϕ-invariant.

(ii) Let $\mu \in M_\phi$ with $\mu = \mu_u \rho$, and let supp $\mu \cap \mathcal{D}^+ \neq \phi$. Then there exists $(u, x) \in \mathcal{D}^+$ and hence $x \in D_u^+ \neq \phi$.

Conversely, let $\rho \in M_\theta$ and $D_u^+ \neq \phi$ for ρ-almost all $u \in \mathcal{U}$. The map $u \mapsto D_u^+$ from $\mathcal{U}$ into M is a set valued map with compact values. It is measurable, because the set $\{u \in \mathcal{U};\ D_u^+ \cap A \neq \phi\}$ is closed for every closed set A. Hence by a selection theorem due to von Neumann-Aumann-Castaing (see e.g. Warga (1972, Theorems I.7.4 and I.7.7)) there exists a measurable selection $u \mapsto x(u) \in D_u^+$. Consider the measure $\hat{\mu} = \delta_{x(u)}\rho$ on $\mathcal{U} \times M$, where $\delta_{x(u)}$ denotes the Dirac measure at the point $x(u)$. Apply the Krylov-Bogolyubov construction to this measure $\hat{\mu}$ (instead of $\delta_{u,x}$ as in (1) above) in order to obtain a ϕ-invariant measure μ as

$$\lim_{t_k \to \infty} \frac{1}{t_k} \int_0^{t_k} \int_{\mathcal{U} \times M} F(\theta_\tau(u), \varphi(\tau, x, u))\, \hat{\mu}\,(d(u, x))\, d\tau$$
$$= \int_{\mathcal{U} \times M} F(v, u) d\mu$$

for all $F \in C(\mathcal{U} \times M, \mathbb{R})$. Clearly supp $\mu \subset \mathcal{D}^+$ and $\mu = \mu_u \rho$, because the $\mathcal{U}$- component of $\hat{\mu}$ coincides with the θ-invariant measure ρ.

(iii) If μ is ergodic, then supp $\mu \subset \mathcal{D}^+$ for some control set D by Colonius and Kliemann (1990[a], Theorem 5.5(iii)). Hence D is unique and $D_u^+ \neq \phi$ for ρ-almost all $u \in \mathcal{U}$.

If $\rho \in M_\theta$ is ergodic, and if $D_u^+ \neq \phi$ for ρ-almost all $u \in \mathcal{U}$, then (ii) yields a ϕ-invariant measure $\tilde{\mu}$ with supp $\tilde{\mu} \subset \mathcal{D}^+$. Now any measure $\mu_{u,x}$ appearing in an ergodic decomposition of $\tilde{\mu}$ (see e.g. Mañé (1987, Theorem II.6.4)) is ϕ-invariant and ergodic with supp $\mu_{u,x} \subset \mathcal{D}^+$.

(iv) Suppose $\mu(\mathcal{D}^+) > 0$. Then the set $\{u \in \mathcal{U};\ D_u^+ \neq \phi\}$ is θ-invariant and has positive ρ-measure. By ergodicity of ρ, this set has ρ- measure 1.

□

Remark. If $\mu \in M_\phi$ is ergodic, then the proof of (iii) shows in particular that supp $\mu \subset \mathcal{D}^+$ for some control set D, compare Theorem 4.3(iii).

The crucial question for the existence of ϕ-invariant measures is therefore: Is $D_u^+ \neq \phi$. We will next present a general result for this problem and then analyze the systems (3.11) and (3.13) in more detail.

Proposition 4.5. *Consider the control flow* (2.2) *under* Assumption (H). *Then a control set $C \subset M$ is invariant iff $C_u^+ = C$ for all $u \in \mathcal{U}$.*

Proof. This result follows directly from the definitions, because for invariant control sets C we have $\overline{\mathcal{O}^+(x)} = \overline{C}$ for all $x \in C$, and under (H) invariant control sets are closed.

□

For variant control sets $D \subset M$ the situation is different: We may have $D_u^+ \neq \phi$ for all $u \in \mathcal{U}$, or $D_u^+ = \phi$ for an open subset of $\mathcal{U}$, as the following analysis shows.

Consider the projected bilinear control system (3.11) on $\mathbb{P}^{d-1}$, always under the Assumption (H).

Proposition 4.6.

(i) *Let $D \subset \mathbb{P}^{d-1}$ be a main control set, i.e. int $D \neq \phi$, and let $g \in$ int $\mathcal{S}$. Denote by $u(g)$ a control corresponding to g. Then*

$$D_{u(g)} := \{x \in \overline{D};\ \varphi(t,x,u) \in \overline{D} \text{ for all } t \in \mathbb{R}\} = \mathbb{P}\bigoplus_{\lambda} E(g,\lambda),$$

where the sum is taken over all $\lambda \in$ spec g with $\mathbb{P}E(g,\lambda) \subset$ int D. (Recall that $E(g,\lambda)$ is the (generalized) eigenspace of g for the eigenvalue λ, and $\mathbb{P}E(g,\lambda)$ denotes its projection onto $\mathbb{P}^{d-1}$.)

(ii) *For a chain control set $E \subset \mathbb{P}^{d-1}$ of (3.11) denote by $\mathcal{E}$ its lift to $\mathcal{U} \times \mathbb{R}^d$. Then*

$$\mathcal{E} = c\ell\{(u(g),x) \subset \mathcal{U} \times \mathbb{R}^d;\ g \in \text{int } \mathcal{S} \text{ and } \mathbb{P}x \in \oplus D_{u(g)}, \text{ where the sum is taken over all main control sets } D \subset E\}.$$

(iii) *Let D be a main control set, then for all $x \in \overline{D}$ there exists $u \in \mathcal{U}$ with $\varphi(t,x,u) \in \overline{D}$ for all $t \geq 0$.*

(iv) *Let D be a main control set, then $D_u^+ \neq \phi$ for all $u \in \mathcal{U}$.*

Proof.

(i) Theorem 3.13 in Colonius and Kliemann (1990^b).

(ii) Theorem 5.6 in Colonius and Kliemann (1990^b).

(iii) The assertion is valid for all $x \in$ int D, and hence for all $x \in \overline{D}$ by compactness of $\mathcal{U}$.

(iv) By (i) this result is true for all $u(g)$ with $g \in$ int $\mathcal{S}$. But these controls are dense in $\mathcal{U}$ (see Colonius and Kliemann (1990^a, Lemma 2.2)), and the result follows from the compactness of $\overline{D}$.

□

Corollary 4.7. *Given a θ-invariant measure ρ on $\mathcal{U}$, then for each main control set D of (3.11) there exists a ϕ-invariant measure $\mu = \mu_u\rho$ with supp $\mu \subset \mathcal{D}^+$.*

Proof. This follows from Proposition 4.6(iv) and Theorem 4.4(ii).

□

The situation for the general one-dimensional nonlinear system (3.13) is more complicated. We will discuss this case again without Assumption (H).

Proposition 4.8. *Assume that the system* (3.13) *satisfies the hypothesis* (3.14) *and that* $\cup\{D;\ D$ *is a control set*$\}$ *is bounded. Then*

(i) C *is an invariant control set of* (3.13) *iff* $C_u^+ = \overline{C}$ *for all* $u \in \mathcal{U}$.

(ii) *Let* D *be a variant control set with int* $D \neq \phi$. *Assume: For all* $u \in U$ *there exists* $x \in \overline{D}$ *with* $X(x)+uY(x) = 0$. *Then* $D_u^+ \neq \phi$ *for all* $u \in \mathcal{U}$.

(iii) *For all other control sets* $D \subset M$ *there exists an open set* $\mathcal{V} \subset \mathcal{U}$ *such that* $D_u^+ = \phi$ *for all* $u \in \mathcal{V}$.

Proof. We will use again the notations introduced in Section 3 following (14).

(i) By definition of invariant control sets C we have $\overline{\mathcal{O}^+(x)} = \overline{C}$ for all $x \in C$. Without Assumption (H) C need not be closed and need not have nonvoid interior. If $C = \{x\}$ is a one point set, then $x \in SA$ and the result is obvious. If int $C \neq \phi$, then the boundary points are either in SA, or in SB (for the lower boundary), or in SC (for the upper boundary). Hence by inspection of the vector fields on the boundary, one sees that the assertion holds.

(ii) We will prove only the case, where for each $u \in U$ there exists a unique $x_u \in \overline{D}$ such that $X(x_u)+uY(x_u) = 0$, and x_u is an unstable rest point for this vector field. All other cases are similar, because each variant control set with nonvoid interior contains an interval of unstable rest points and by Assumption 14(iii) the construction below can be carried out in this interval.

Let $u \in \mathcal{U}$ be constant, then using the assumptions $A_u(t) := \{\varphi(-t, y, u),\ y \in \overline{D}\} \subset \overline{D}$ is, for all $t > 0$, a closed interval in $\overline{D}$ containing x_u. Consider now $u \in \mathcal{U}$, piecewise constant and periodic, i.e. there exists a time interval $[0, T]$ and a partition $0 = t_0 < t_1 < \cdots < t_n = T$ such that $u(\tau) = u_i$ if $\tau \in [t_{i-1}, t_i)$. For $i = 1, \ldots, n$ consider the solution map $\varphi(-t_i, \cdot, u_i)\colon \overline{D} \to \overline{D}$, and note that im $\varphi(-t_i, \cdot, u_i) = A_{u_i}(t_i) \subset \overline{D}$. Then the map $\varphi(-T, \cdot, u) = \varphi(-t_1, \cdot, u_1) \circ \cdots \circ \varphi(-t_n, \cdot, u_n)$ is continuous (even a diffeomorphism) and maps the compact interval $\overline{D}$ into itself. Therefore $\varphi(-T, \cdot, u)$ has a fixed point $x_p \in \overline{D}$. By construction the solution $\varphi(t, x_p, u)$ is periodic and contained in $\overline{D}$ for all $t \geq 0$, hence $x_p \in D_u^+$. To complete the proof, note that the set of piecewise constant, periodic controls is dense in $\mathcal{U}$ (Colonius and Kliemann (1990[a], Lemma 2.2), and compactness yields $D_u^+ \neq \phi$ for all $u \in \mathcal{U}$.

(iii) By assumption there exists $u \in U$ such that $D_u^+ = \phi$, i.e. for all $x \in \overline{D}$ there is $T_x > 0$ with $\varphi(T_x, x, u) \notin \overline{D}$. Using compactness of $\overline{D}$, one sees that there is a universal $T > 0$ such that $\varphi(T, x, u) \notin \overline{D}$ for all $x \in \overline{D}$. Now continuity of the map $v \mapsto \varphi(t, x, v)$ from $\mathcal{U}$ into M implies the assertion.

□

Corollary 4.9. *Suppose that system* (3.13) *satisfies* (3.14) *and that* $\cup \{D;\ D$ *is a control set*$\}$ *is bounded. Then:*

(i) *Given a* θ*-invariant measure* ρ *on* $\mathcal{U}$ *and a control set* $D \subset M$ *of the kind described in* Proposition 4.8(i) *or* (ii). *Then there exists a* ϕ*-invariant measure* $\mu = \mu_u \rho$ *with supp* $\mu \subset \mathcal{D}^+$.

(ii) *Let* $D \subset M$ *be a control set of the kind described in* Proposition 4.8(iii). *Then there exists an ergodic* θ*-invariant measure* ρ*, such that there is no* ϕ*-invariant measure* μ *with marginal* ρ *on* $\mathcal{U}$ *and supp* $\mu \subset \mathcal{D}^+$.

Proof.

(i) Follows directly from Proposition 4.8(i) and (ii).

(ii) By the proof of Proposition 4.8(iii), for this kind of control sets D there exists a $u_0 \in U$ such that $X(x) + uY(x) > 0$, or < 0, for all $x \in \overline{D}$. Consider the constant control function $u_0(t) \equiv u_0$, then the Dirac measure δ_{u_0} is θ-invariant and ergodic. But $D_{u_0}^+ = \phi$, and no ϕ-invariant measure with the desired properties can exist.

□

The remainder of this paper is devoted to the study of invariant measures of stochastic flows, as described in Section 2. In order to use the results above directly, we will restrict ourselves here to flows associated with random differential equations of the following form:

Let $\{\xi_t,\ t \in \mathbf{R}$ or $t \in \mathbf{R}^+\}$ be a stationary, ergodic process taking values in $U \subset \mathbf{R}^m$, where U is compact and convex, and let $\mathcal{B}$ be the Borel σ- algebra of U. Denote by (Ω, θ) the trajectory space of measurable functions with values in U, with the shift θ (for $t \in \mathbf{R}$ or for $t \in \mathbf{R}^+$). Then there is a θ-invariant, ergodic measure P on Ω. Let $\zeta \colon \mathbf{R} \times \Omega \to U$ be the evaluation map $\zeta(t, w) = w(t)$, then $\{\zeta_t,\ t \in \mathbf{R}$ or $t \in \mathbf{R}^+\}$ is a stationary, ergodic process over $(\Omega, \mathcal{F}, P, \theta)$ where $\mathcal{F}$

is generated by the cylinder sets. Consider the random differential equation

$$(3) \qquad \dot{x} = X_0(x) + \sum_{i=1}^{m} \zeta_i(t) X_i(x) \text{ on a smooth manifold } M,$$

where $X_0, \ldots, X_m$ are smooth vectorfields on M. Then (3) defines a stochastic flow $(\Omega \times M, \phi)$, as described in Section 2. We continue to assume condition (H).

Remark 4.10. (One and two-sided stochastic flows, compare Crauel (1990)) A flow, defined for $t \in \mathbf{R}$, is called two-sided, if $t \in \mathbf{R}^+$, it is called one-sided. A two-sided flow can always be restricted to a one-sided: E.g. let $\mathcal{F}' = \sigma\{\phi_t,\ t \geq 0\}$ be the σ-algebra generated by ϕ for positive time, then the restriction to $(\Omega \times M, \mathcal{F}', \widetilde{P})$ and $t \geq 0$ is a one sided flow, where $\widetilde{P}$ is the measure induced by ϕ on $\Omega \times M$. Vice versa, if $\phi = (\theta, \varphi)$ is a one-sided flow, and if θ is invertible, then $\varphi(-t, x, w) = \varphi^{-1}(t, x, \theta_{-t} w)$ is a two-sided extension. Note that in our situation θ is always invertible.

For stochastic flows we are interested in ϕ-invariant measures μ, whose marginal on Ω is the given measure P, i.e. $\mu = \mu_w P$. For a σ-algebra $\tilde{\mathcal{F}} \subset \mathcal{F}$ we denote by $\mathbf{E}\{\mu \,|\, \tilde{\mathcal{F}}\}(w)$ the desintegration of μ restricted to $\tilde{\mathcal{F}} \otimes \mathcal{B}$ with respect to $P|_{\tilde{\mathcal{F}}}$. The measure μ is ϕ- invariant iff $\mathbf{E}\{\varphi(t, \cdot, w)\mu_w \,|\, \theta_t^{-1}\mathcal{F}\}(w) = \mu_{\theta_t w}$, which for θ_t invertible (i.e. in particular for the two-sided situation) reduces to $\varphi(t, \cdot, w)\mu_w = \mu_{\theta_t w}$, see Lemma 4.2. We denote by $M_\phi(P)$ the two-sided ϕ-invariant measures with marginal P on Ω, and by $M_\phi^+(P)$ the one-sided ones with marginal $P|_{\mathcal{F}'}$. Note that measures in $M_\phi(P)$ can always be restricted to measures in $M_\phi^+(P)$, while one-sided invariant measures can be extended if θ is invertible: Let $\mu^+ \in M_\phi^+(P)$ with desintegration $\mu^+ = \mu_w^+ P$, then $\varphi^{-1}(-t, w)\mu^+_{\theta_{-t} w}$ converges P-almost surely for $t \to \infty$ to a measure μ_w, such that $\mu_w P$ is ϕ-invariant for all $t \in \mathbf{R}$, compare e.g. Crauel (1990, Remark 2.3).

We have the following general result about the support of ϕ- invariant measures:

Theorem 4.11. *Consider the two-sided stochastic flow induced by the random differential equation* (3). *Let $\mu \in M_\phi(P)$ or $\mu \in M_\phi^+(P)$ be given. Then* $\operatorname{supp} \mu \subset c\ell \cup \{\mathcal{D}^+;\ D \text{ is a control set}\}$, *and for every control set D with $\mu(\mathcal{D}^+) > 0$ it holds that $D_u^+ \neq \phi$ P-almost surely.*

Proof. This follows from ergodicity of P and Theorem 4.4(iv).

□

For the existence of invariant measures for a two-sided stochastic flow the results given before still hold, compare Theorem 4.4, Proposition 4.5, Corollary 4.7, and Corollary 4.9(i). From Theorem 4.4(iv) one obtains in this context:

Corollary 4.12. *Let $D \subset M$ be a control set of the kind described in* Proposition 4.8(iii). *Denote $V := \{u \in U;$ there is no $x \in \overline{D}$ with $X(x) + uY(x) = 0\}$, and by $\mathcal{V}$ the corresponding trajectory space. If $P\{\text{supp } P \cap \mathcal{V}\} > 0$, then there exists no $\mu \in M_\phi^+(P)$ (or $\mu \in M_\phi(P)$) with supp $\mu \subset \mathcal{D}^+$.*

For the existence of ϕ-invariant measures the sets D_w^+ play a crucial role. In general, these sets can only be determined, if the entire trajectory $\{w(t),\ t \geq 0\}$ is known. (An exception are the invariant control sets C, for which we have $C_w^+ = C$ for all $w \in \Omega$.) This fact reduces the possible invariant measures, if we require certain measurability conditions, as in the case of Markov processes. We will discuss the Markovian case next, and use the following set up, see Crauel (1990):

If the stochastic process $\{\zeta_t,\ t \in \mathbb{R}$ or $t \in \mathbb{R}^+\}$ in (3) is a (time homogeneous) Markov process, then we call $\{\phi_t,\ t \in \mathbb{R}$ or $t \in \mathbb{R}^+\}$ a Markovian stochastic flow. In order that the pair process $\{(\zeta_t, \varphi_t),\ t \geq 0\}$ becomes a Markov process, we need a condition for the initial variable of φ_t: For a random variable $\eta \colon \Omega \to M$ the process $\{(\zeta_t, \varphi(t, \eta, w)),\ t \geq 0\}$ is a Markov process iff $\{\zeta_t,\ t \geq 0\}$ is a Markov process with respect to the enlarged family of σ-algebras $\mathcal{F}^\eta_{\leq t} := \sigma\{\eta, \mathcal{F}_{\leq t}\}$ for $t \geq 0$, compare Arnold and Kliemann (1983, Lemma 2.1) and Crauel (1990, Lemma 3.4). Here $\mathcal{F}_{\leq t} := \sigma\{\zeta_\tau,\ 0 \leq \tau \leq t\}$ is the σ- algebra generated by the Markov process $\{\zeta_\tau,\ \tau \geq 0\}$ between 0 and t.

The problem of a stationary (and ergodic) Markov solution (ζ_t, φ_t) of (3) can be formulated in this context as follows: Let ϕ be a Markovian stochastic flow, then $Q(t, (p, x), A) := P\{(\zeta_t(\cdot), \varphi(t, x, \cdot)) \in A \mid \zeta_0 = p\}$ defines a family of Markov transition probabilities on $U \times M$ for $t \geq 0$, where $A \in \mathcal{B}_U \times \mathcal{B}_M$, compare Bunke (1972, Satz 6.1). A probability measure μ on $\Omega \times M$ (with marginal P on Ω) is called a Markov measure, if the pair process $\{(\zeta_t(\cdot), \varphi(t, \cdot, \cdot)),\ t \geq 0\}$ is a (time homogeneous) Markov process with transition probabilities Q, and initial distribution $\pi_M \mu$ on M. The relation between this approach for Markovian stochastic flows and the usual definition of initial distributions via the Markov semigroup $\{Q_t,\ t \geq 0\}$, defined by the Markov transition probabilities Q, is as follows: For a Markov measure μ on $\Omega \times M$ satisfying $\mu_\cdot = \mathbb{E}\{\mu_\cdot \mid \mathcal{F}_{<\infty}\}$ define a family of probability measures on U by $\nu_u = \nu_{\zeta_0(w)} = \mu_w$, which gives (the desintegration of) an initial distribution ν on $U \times M$ for $\{Q_t,\ t \geq 0\}$. Vice versa, given such a ν, define $\mu_w = \nu_{\zeta_0(w)}$, yielding a Markov measure μ on $\Omega \times M$. Hence this correspondence is one-to-one, see again Crauel (1990, Section 5.2.1).

The following result characterizes the Markov measures of (3) among all probability measures on $\Omega \times M$ with marginal P on Ω:

Proposition 4.13.

(i) *If $\mu = \mu.P$ satisfies $\mu. = \mathbf{E}\{\mu. \,|\, \mathcal{F}_{\leq 0}\}$, then μ is a Markov measure.*

(ii) *If μ is a Markov measure, then $\mathbf{E}\{\mu. \,|\, \mathcal{F}_{\leq \infty}\} = \mathbf{E}\{\mu. \,|\, \mathcal{F}_{\leq 0}\}$ P-a.s.*

For a proof see Crauel (1990, Theorem 4.4).

In this paper we are primarily interested in ϕ-invariant measures and their support. From the discussion above we obtain immediately for invariant Markov measures of (3):

(a) Every invariant measure of the Markov semigroup $\{Q_t,\ t \geq 0\}$ corresponds to a (unique) invariant Markov measure of (3) for $t \geq 0$, with $\mu. = \mathbf{E}\{\mu. \,|\, \mathcal{F}_{<\infty}\}$.

If $\{\xi_t,\ t \geq 0\}$ is a diffusion process with generator $\mathcal{L}$, then $\{(\xi_t, \varphi_t),\ t \geq 0\}$ is a diffusion process with generator $\mathcal{L} + Y$, where Y is the right hand side of (3). In this case, the Q_t invariant measures μ are the solutions of the Fokker-Planck equation $(\mathcal{L} + Y)^* \mu = 0$, where $*$ denotes the adjoint operator.

(b) If M is compact, then an invariant Markov measure exists.

(c) The invariant Markov measures form a convex, $\{\phi_t,\ t \geq 0\}$ invariant subset of $M_\phi(P)$ (see Crauel (1990, Lemma 5.1)).

For invariant Markov measures of product type we obtain from our previous results:

Proposition 4.14.

(i) *Let $\mu = P \times \lambda$ be an invariant Markov measure for the one-sided flow $\{\phi_t,\ t \geq 0\}$ of a 'colored noise system', i.e. the $(\zeta_i,\ i = 1, \ldots, m)$ are the solution of a (Stratonovič type) stochastic differential equation in $\mathbb{R}^m$. Assume that supp $P = \mathcal{U}$. Then supp $\lambda = C$ for some invariant control set C of the associated control system.*

(ii) *Suppose that either* (H) *holds and $C \subset M$ is compact, or for the system* (3.13) *that the assumptions of* Proposition 4.8 *are met. Then for every stationary Markov process $\{\zeta_t,\ t \geq 0\}$ there exists an invariant Markov measure μ of* (3) *with supp $\mu \subset C^+$, the lift of C to $\Omega \times M$.*

Proof.

(i) By the discussion after Remark 4.10, $\mu = P \times \lambda$ is invariant iff P-a.s. for all $t \geq 0$ $\mathbf{E}\{\varphi(t, \cdot, w)\lambda \,|\, \theta_t^{-1}\mathcal{F}_{\geq 0}\} = \lambda$. Now $\varphi(t, \cdot, \cdot)$ and $\theta_t^{-1}\mathcal{F}_{\geq 0} = \sigma\{\zeta_{t+s} - \zeta_t \,|\, s \geq 0\}$ are independent for $t \geq 0$, and hence μ is invariant iff $\lambda = \mathbf{E}\{\varphi(t, \cdot, \cdot)\lambda\} = \int \varphi(t, \cdot, w)dP$ for all $t \geq 0$. If D is a variant control set, then there exist $u \in \mathcal{U}$ and $x \in D$ such that $\varphi(t,x,u) \notin \overline{D}$ for some $t > 0$. Hence there are open neighborhoods $N(u)$ and $N(x)$ such that $\varphi(t,y,v) \notin \overline{D}$ for all $v \in N(u)$, all $y \in N(x) \cap \overline{D}$. Since, by assumption supp $P = \mathcal{U}$, we have that $\overline{D} \cap \text{supp } \lambda = \phi$, which proves (i).

(ii) follows directly from Proposition 4.5, and from Proposition 4.8(i) for systems of the type (3.13) (without Assumption (H)).

□

Remark 4.15.

(i) The assumption "supp $P = \mathcal{U}$" applies e.g. to stochastic processes $\{\zeta_t, t \geq 0\}$ with continuous or cadlag trajectories, if supp P covers all these functions. Hence Proposition 4.14 applies in particular to noise processes $\{\zeta_t,\ t \geq 0\}$, which are stationary, nondegenerate diffusion processes, compare Kunita (1978). Hence this proposition generalizes results from Arnold, Kliemann and Oeljeklaus (1986) and Kliemann (1987).

(ii) If "supp $P = \mathcal{U}$" does not hold, then the conclusion (i) need not be true, as Example 4.16 below shows. In other words , if $\{\zeta_t,\ t \geq 0\}$ takes values in $V \subset U$, one should use $\Omega = \mathcal{V}$, the trajectory space over V, and the corresponding control system to check for supports of invariant Markov measures of product type.

(iii) At this moment we do not know, whether the result of Proposition 4.14(i) holds for Markov measures, which are not of product type, nor whether in (ii) a product type measure always exists.

The next example shows that a stochastic flow, defined by (3), can have ϕ-invariant measures, which are not Markov measures, and that without the assumption "supp $P = \mathcal{U}$" Proposition 4.14(i) need not hold.

Example 4.16. Consider again the system from Example 3.10

$$\dot{x} = \sin^2 x + a\cos^2 x - \zeta_t \cos^2 x, \qquad x \in \mathbb{R} \bmod 2\pi,\ a > 0, \tag{4}$$

with $U = [A,a] \subset \mathbb{R}$, $A < a$. If $\{\zeta_t,\ t \geq 0\}$ is a stationary process in the set up as above, then there exist ϕ-invariant measures μ_1 and μ_2 with supp $\mu_i \subset \mathcal{D}_i^+$, $i = 1,2$ by Corollary 4.7. (Here D_i are the main control sets as in Example 3.10.)

By Proposition 4.14(i) the measure μ_2 cannot be an invariant Markov measure of product type. Furthermore, Remark 4.2 in Arnold and Kliemann (1983) shows that there need not be any invariant Markov measure for a Markovian stochastic system, which, nevertheless, may possess a ϕ-invariant probability.

Now fix $b \in (A, a)$, and consider the Markov process $\zeta_t \equiv b$. Then the system (4) has invariant Markov measures $\mu_i = P \times \delta_{x_i}$, $i = 1, \ldots, 4$, where x_i are the four rest points corresponding to $-\sin^2 x + a\cos^2 x - b\cos^2 x = 0$. More generally, if ζ_t has values in $V \subset U$, then the invariant control sets of the control system corresponding to V determine the possible supports of invariant Markov measures of product type.

Remark 4.17. (On uniqueness of invariant measures) In general the invariant measures in $\mathcal{D}^+$, the lift of some control set, need not be unique — this is also true for Markov measures and measures over invariant control sets. Consider e.g. the class of systems given by (3.11), with more than one main control set. Then, according to Corollary 4.7, for each θ-invariant measure P and over each main control set there exists a ϕ-invariant measure. In particular, for each $u \in \mathcal{U}$ there exist invariant Krylov-Bogolyubov measures in each $\mathcal{D}^+$. On the other hand, for each $x_0 \in \operatorname{int} D$, D a main control set, there exist different periodic functions $u \in \mathcal{U}$ and different periodic solutions $\varphi(t, x_0, u)$ of (3.11) in int D. Hence there are different ϕ- invariant measures, all of which have x_0 in their support projected onto $\mathbb{P}^{d-1}$.

Consider now the Markov situation of Proposition 4.14(i), i.e. supp $P = \mathcal{U}$ and $\mu = P \times \lambda$ is an invariant Markov measure for (3). Then supp $\lambda = \overline{C}$ for some invariant control set C, and λ is unique on C, if λ has a lower semicontinuous density with respect to the Lebesgue measure on M, compare Arnold and Kliemann (1987^a). This is true, in particular, if $\{(\zeta_t, \varphi_t),\ t \geq 0\}$ is a hypoelliptic diffusion process, compare Kliemann (1987) and Arnold and Kliemann (1987^a).

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