# OPTIMAL PERIODIC CONTROL OF QUASILINEAR SYSTEMS IN HILBERT SPACES

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Abstract. This paper discusses optimal periodic control problems for quasilinear systems in Hilbert spaces. Using Ekeland's variational principle, a global maximum principle is proven.
Then the question of local properness is discussed, i.e. if an optimal steady state can be improved by allowing proper periodic controls and trajectories. This discussion is based on second order necessary optimality conditions obtained from general optimization theory in Banach spaces.

## 1. INTRODUCTION

This paper deals with optimal periodic control problems for quasilinear systems of the following form

Minimize

(1.1) $\quad 1/\tau \int_0^\tau g(y(t),u(t))dt \qquad$ s.t.

(1.2) $\quad y'(t) = Ay(t) + f(y(t),u(t)), \ t \in T := [0,\tau]$

(1.3) $\quad y(0) = y(\tau)$

(1.4) $\quad u \in \mathcal{U}_{ad} := \{u: T \to \Omega, \text{ strongly measurable}\}$

where A is the infinitesimal generator of a strongly continuous semigroup $\{T(t), t \geq 0\}$ in a Hilbert space Y, $\Omega$ is a bounded subset of a Banach space U, $g: Y \times U \to \mathbb{R}$ and $f: Y \times U \to Y$ have continuous Frêchet derivatives $\partial_y g(y,u)$, $\partial_y f(y,u)$, with respect to y and f,g(resp. $\partial_y f$, $\partial_y g$) are continuous (resp. strongly continuous) and bounded on bounded subsets of $Y \times U$.

By definition, a solution of (1.2) is a continuous solution of the integrated version

$$y(t) = T(t)y(0) + \int_0^t T(t-s)f(y(s),u(s)ds, \quad t \in T$$

We assume that T(t) is compact for t > 0.

For ordinary differential systems and for functional differential systems, optimal periodic control problems have been studied for quite a while (cp. [3] for recent results and references). This paper aims at developing a similar theory for the quasilinear system described above (with parabolic partial differential equations in mind).
In section 2, a global maximum principle is given. The proof relies on Ekeland's Variational Principle and is only sketched; it uses similar arguments as [4,5,3]. Section 3 discusses the problem of local properness, i.e., the problem how one can decide if near an optimal steady state the average system behaviour can be improved by introducing oscillations. In particular, second order necessary optimality conditions for weak optima are derived and applied.

## 2. A GLOBAL MAXIMUM PRINCIPLE

This section gives necessary optimality conditions for a strong minimum.

The first proposition assures existence and uniqueness of solutions of (1.2).

Proposition 2.1 Assume

(2.1) $A - \omega I$ is dissipative for some $\omega \in \mathbb{R}$, that is

$$\langle y^*, Ay\rangle \leq \omega \|y\|^2$$

for $y \in \mathcal{D}(A)$, $y^* \in \theta(y) := \{y^* \in Y^*: \|y^*\|^2 = \|y\|^2 = \langle y^*, y\rangle\}$.

(2.2) There is $C > 0$ such that

$$\langle y^*, f(y,u)\rangle \leq C(1 + \|y\|^2)$$

for $y \in Y$, $y^* \in \theta(y)$, $u \in \Omega$.

Then for every initial condition $y(0) = y_0 \in Y$, and every $u \in \mathcal{U}_{ad}$, equation (1.2) has a unique solution $y(\cdot) = y(\cdot, u, y_0)$.

Proof: cp. [6, Section 5].

The linearized system corresponding to (1.2) is

$$(2.3) \quad \begin{aligned} z'(t) &= Az(t) + \partial_y f(y(t),u(t))z(t), \quad t \in [s,\tau]. \\ z(s) &= z_0 \end{aligned}$$

The solutions of (2.3) are, by definition, solutions of the integrated version

$$z(t) = T(t-s)z_0 + \int_s^t T(t-\sigma)\partial_y f(y(\sigma),u(\sigma))z(\sigma)d\sigma, \quad 0 \le s \le t \le \tau.$$

The corresponding solution operator is denoted by $\Phi(t,s,y,u)$.

<u>Lemma 2.2</u> Let $0 \le s \le t \le \tau$, $u \in \mathcal{U}_{ad}$. Then the operator $\Phi(t,s,y,u) : Y \to Y$ is compact.

<u>Proof</u>: Follows as [6, Theorem 6.4].

The next theorem presents a global maximum principle for Problem (1.1) - (1.4).

<u>Theorem 2.3</u> Let conditions (2.1) and (2.2) be satisfied and assume that $(y^o,u^o)$ is a strong minimum for Problem (1.1) - (1.4), i.e.

$$(2.4) \quad 1/\tau \int_0^\tau g(y^o(t),u^o(t)dt \le 1/\tau \int_0^\tau g(y(t),u(t))dt$$

for all $(y,u)$ satisfying (1.2) - (1.4).

Then there are $\lambda \ge 0$ and a solution $y^*$ of the adjoint equation

$$(2.5) \quad y^{*\prime}(s) = A^*y^*(s) + \partial_y f(y^o(s),u^o(s))^*y^*(s) + \lambda\partial_y g(y^o(s),u^o(s)), \quad s \in T$$

$$(2.6) \quad y^*(0) = y^*(\tau)$$

such that $(\lambda,y)$ is not identically zero and

$$(2.7) \quad \lambda\, g(y^o(t),u^o(t)) + \langle y^*(t),f(y^o(t),u^o(t))\rangle = \min_{u\in\Omega} \{ \lambda\, g(y^o(t),u) + \langle y^*(t),f(y^o(t),u)\rangle \quad \text{for a.a. } t \in T.$$

<u>Proof</u>: The proof is a slight variant of the one given in [5] and will only be sketched.

The space $\mathcal{U}_{ad}$ becomes a complete metric space under the Ekeland distance defined by

$$d(u,v) = \text{meas } \{t: u(t) \neq v(t)\}.$$

Consider a sequence $\delta_n \to 0$, $\delta_n > 0$, and define functionals $F_n$ on $\mathcal{U}_{ad} \times Y$ by

$$(2.8) \qquad F_n(u,y_0) := [|y(\tau,u,y_0)-y_0|^2 + |J(u,y_0) - (m-\delta_n)|^2]^{1/2}$$

where

$$J(u,y_0) := 1/\tau \int_0^\tau g(y(t,u,y_0),u(t))dt$$

and

$$m := \inf \{J(u,y_0): u \in \mathcal{U}_{ad}, y_0 \in Y\} .$$

Then $F_n(u,y_0) > 0$ for all $(u,y_0)$. Using Ekeland's variational principle one finds $(u^n,y_0^n)$ approximating $(u^0,y^0(0))$ and $\lambda^n \geq 0$, $z^n \in Y^*$ such that $|(\lambda^n,z^n)| = 1$ and

$$[\Phi(\tau,0,y^n,u^n) - Id]x = -\lambda^n/\tau \int_0^\tau \partial_y g(y^n(\sigma),u^n(\sigma))\Phi(\sigma,0,y^n,u^n)x d\sigma \quad \text{for}$$

all $x \in Y$ and

$$-\varepsilon_n^{1/2} \leq z^n \, \Phi(\tau,s,y^n,u^n) \, [f(y^n(s),u) - f(y^n(s),u^n(s))]$$
$$+ \lambda^n \int_0^\tau \partial_y(y^n(\sigma),u^n(\sigma)) \, \Phi(\sigma,s,y^n,u^n)[f(y^n(s),u)-f(y^n(s),u^n(s))]d\sigma$$
$$+ \lambda^n[g(y^n(s),u) - g(y^n(s),u^n(s))]$$

for all $u \in \Omega$ and almost all $s \in T$.

Now compactness of $\Phi(\tau,0,y^n,u^n)$ and convergence of $u^n \to u^0$, $y^n \to y^0$ can be used to establish the assertion.

For details cp. [5] and also [3] for similar arguments applied to functional differential equations.

Remark

Observe that no constraint qualification (comparable to Fattorini's "fat cone condition") is required in the proof of the maximum principle above. This is due to the consideration of periodic boundary conditions.

## 3. LOCAL PROPERNESS AND SECOND ORDER CONDITIONS

Associated with the periodic problem (1.1) - (1.4) is the following steady state problem:

(3.1) Minimize $g(y,u)$ over $(y,u) \in \mathcal{D}(A) \times U$ s.t.

(3.2) $0 = Ay + f(y,u)$

(3.3) $u \in \Omega$

Let $(y^o,u^o)$ be an optimal solution of this problem. Then a natural question is to ask whether the constant functions $y^o(\cdot) \equiv y^o$, $u^o(\cdot) \equiv u^o$, which obviously satisfy the constraints (1.2) - (1.4) of the periodic problem, are an optimal solution of the latter problem. If this is not the case, $(y^o,u^o)$ are called locally proper. In order to decide whether an optimal solution of (3.1) - (3.3) is locally proper, it is convenient to check if it satisfies necessary optimality conditions for (1.1) - (1.4).

Such "tests" for local properness in ordinary differential systems based on the Pontryagin maximum principle are given in [7].

However, frequently, the maximum principle is equivalent to the first order necessary optimality conditions obtained by weak variations, and it is well-known that these conditions do not give a test for local properness since they are always satisfied by solutions of the steady state problem. A similar result is true for the infinite dimensional problem considered here:

<u>Proposition 3.1</u> Suppose that f and g are continuously Fréchet differentiable and let $(y^o,u^o)$ be optimal for (3.1) - (3.3).

(3.4) Suppose that the cone $\{Ay+\partial_y f(y^o,u^o)y + \partial_u f(y^o,u^o)u : y \in \mathcal{D}(A) u \in \mathbb{R}_+(\Omega-u^o)\}$ contains a subspace of finite codimension in Y, $\Omega$ is closed and convex.
Then there are $(0,0) \neq (\lambda,y^*) \in \mathbb{R}_+ \times Y^*$ with

(3.5) $0 = \lambda\, \partial_y\, g(y^o,u^o) + y^*A + y^*\partial_y f(y^o,u^o)$

(3.6) $[\lambda\, \partial_u g(y^o,u^o) + y^*\, \partial_u f(y^o,u^o)]\, [u-u^o] \geq 0$ for all $u \in \Omega$.

This follows as standard Lagrange multiplier theorems. This result

shows in particular, that if (3.4) holds and f and g are affine in u, an optimal solution of the steady state problem satisfies the maximum principle, Theorem 2.3, for the periodic problem. Hence in order to decide the question of local properness, we have to recur to second order optimality conditions for the periodic problem.
The following result shows that only Lagrange multipliers of the steady state problem have to be considered.

Proposition 3.2 Suppose that f and g are continuously Frêchet differentiable in a neighbourhood of an optimal solution $(y^0,u^0)$ of Problem (1.1) - (1.4).
Assume

(3.7) the homogeneous linearized equation

$$z'(t) = [A + \partial_y f(y^0,u^0)]z(t), \quad t \geq 0$$

has only the trivial $\tau$-periodic solution.

Then (2.5) - (2.7) can only hold with $\lambda \neq 0$ and there exists a unique $\tau$-periodic solution $y^*$ of (2.5) with $\lambda = 1$.
In particular, if $(y^0,u^0)$ is constant, $y^*$ is constant and given by (3.5).

Proof: A Standard Lagrange multiplier theorem implies $\lambda \neq 0$. Uniqueness of the solution of (2.5) follows from compactness of the solution operator of the linearized system equation, Lemma 2.2, and hence of its adjoint. Furthermore, the constant solution $y^*$ of (3.5) is the unique solution of (2.5). □

We turn to the second order necessary optimality conditions and consider for simplicity an optimal steady state solution $(y^0,u^0)$.
Our analysis is based on the following general 2nd order conditions in Banach spaces.

Theorem 3.3 Let $x^0$ be a local optimum of the problem.

Minimize $G(x)$ s.t.
$F(x) = 0, \ x \in C,$

where $G: X \to \mathbb{R}$, $F: X \to Z$, $X,Z$ Banach spaces, are twice continuously Frêchet differentiable and $C \subset X$ is closed and convex. Assume

(3.8) $F'(x^o)(\mathbb{R}_+(C-x^o))$ contains a subspace of finite codimension in Z.

Then for every $h \in X$ with $G'(x^o)h \le 0$, $F'(x^o)h = 0$, $x^o + h \in \text{int } C$ there exists $(0,0) \neq (\lambda, z^*) \in \mathbb{R}_+ \times Z^*$ with

(3.9) $\lambda_o \, G'(x^o) - z^* \, F'(x^o) = 0;$

(3.10) $[\lambda_o \, G''(x^o) - z^* \, F''(x^o)](h,h) \ge 0.$

If $F'(x^o) \, (\mathbb{R}_+(C-x^o)) = Z$ and there exists $\tilde{x} \in \mathbb{R}_+(C-x^o) \cap \text{Ker } F'(x^o)$, then $\lambda \neq 0$.

Proof: [3, Corollary II. 2.15].

In the following we assume

(3.11) The maps f and g are twice continuously Frêchet differentiable and together with their derivatives bounded in a neighbourhood of $(y^o, u^o)$, the set $\Omega$ is closed and convex.

Before we can apply the theorem above, the solution operator of equation (1.2) has to be analysed:

Define

$$S: L^\infty(0,\tau;U) \times Y \to C(0,\tau;Y)$$

$$S(u,y_o) := y(\cdot,u,y_o).$$

Lemma 3.4 The map S is twice continuously Frêchet differentiable; the first derivative $z := \mathcal{D}S(u^o,y^o)(u,y)$ is the solution of

(3.12) $z(t) = \Phi(t,0)y + \int_0^t \Phi(t,s)\partial_u f(y^o(s),u^o(s))u(s)ds, \; t \in T;$
the second derivative $\xi := \mathcal{D}\mathcal{D}S(u^o,y^o)((u,y),(u,y))$ is the solution of

(3.13)
$$\xi(t) = \int_0^t \Phi(t,s)[\partial_y\partial_y f(y^o(s),u^o(s))(z(s),z(s)) + 2\partial_y\partial_u f(y^o(s),u^o(s))(z(s),u(s)) + \partial_u\partial_u f(y^o(s),u^o(s))(u(s),u(s))]ds, \; t \in T.$$

Proof: This follows by implicitly differentiating equation (1.2); use the second order chain rule (cp. e.g. [3, Lemma II.2.5]).

For an application of Theorem 3.3 let

$$X := L^{\infty}(0,\tau,U)\times Y,\ Z := Y,\ C := U_{ad} \times Y$$

$$F(u,y_0) := y(\tau,u,y_0) - y_0$$

$$G(u,y_0) := 1/\tau \int_0^{\tau} g(y(t,u,y_0),u(t))dt.$$

Using Lemma 3.4 and the chain rule one can show that F and G are twice continuously Frêchet differentiable.

Introduce a function

$$H : Y \times U \times Y^* \to \mathbb{R}$$

$$H(y,u,y^*) := g(y,u) + y^* f(y,u)$$

Definition 3.5 An optimal solution $(y^0,u^0) \in Y \times U$ of problem (3.1)-(3.3) is locally proper, if for every $\varepsilon > 0$ there are $(y,u)$ satisfying (1.2) - (1.4) with

$$\sup_{t\in T}|y(t)-y^0| \le \varepsilon,\ \operatorname{ess\,sup}_{t\in T}|u(t) - u^0| \le \varepsilon$$

$$1/\tau \int_0^{\tau} g(y(t),u(t))dt < g(y^0,u^0). \tag{3.14}$$

Theorem 3.6 Let $(y^0,u^0) \in Y \times U$ be an optimal solution of Problem (3.1) - (3.3) and assume that conditions (3.7) and (3.11) hold. Then $(y^0,u^0)$ is locally proper if there exists $(y,u)$ with

$$\int_0^{\tau} [\partial_y g(y^0,u^0)y(t) + \partial_u g(y^0,u^0)u(t)]dt \le 0 \tag{3.15}$$

$$y(0) = y(\tau),\ y'(t) = Ay(t) + \partial_y f(y^0,u^0)y(t) + \partial_u f(y^0,u^0)u(t),\quad t \in T \tag{3.16}$$

$$u^0 + u \in \operatorname{int} U_{ad} \tag{3.17}$$

$$\int_0^{\tau}[D_1D_1H(y^0,u^0,y^*)(y(t),y(t)) + 2D_1D_2H(y^0,u^0,y^*)(y(t),u(t)) + D_2D_2H(y^0,u^0,y^*)(u(t),u(t))]dt < 0 \tag{3.18}$$

where $y^* \in Y^*$ is given by

$$(3.19)\qquad \begin{aligned} 0 &= \partial_y g(y^o,u^o) + y^* \partial_y f(y^o,u^o) + y^*A \\ 0 &= \partial_u g(y^o,u^o) + y^* \partial_u f(y^o,u^o) \end{aligned}$$

Proof: Using Lemma 3.4 one shows that the assumptions of Theorem 3.3 are satisfied. Now suppose that $(y^o,u^o)$ is an optimal solution of Problem (1.1) - (1.4). Then Theorem 3.3 yields necessary optimality conditions. Using assumption (3.7) and Proposition 3.2 one sees that the corresponding Lagrange multiplier coincides with the Lagrange multiplier of the steady state problem (3.1) - (3.3). Thus it satisfies (3.5), (3.6).

Using Lemma 3.4, one computes the concrete form of the necessary optimality conditions. These must be violated if $(y^o,u^o)$ is locally proper.
This yields the assertion.

Remark: This result may be applied to parabolic partial differential systems along the lines of [6].

Remark: For ordinary differential systems this criterion for local properness was developed in [1], [2].

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