ON PERSISTENCE IN DISCOUNTED OPTIMAL CONTROL PROBLEMS

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Summary

This paper considers asymptotic properties of optimal control systems defined on the positive cone of $\mathbb{R}^n$. A result is proven which connects persistence of optimal solutions to properties of the induced control system on the boundary of this cone. This generalizes recent results for uncontrolled ordinary differential systems.

1. Introduction

This paper studies asymptotic properties of solutions of the following optimal control system (F):

Minimize

(1) $V(x_o,u) = \int_0^\infty e^{-\delta t}\{g_o(x)+\sum_{i=1}^m u_i g_i(x)\}dt$ s.t.

(2) $\dot{x}_j = x_j\{f_o(x)+\sum_{i=1}^m u_i f_i(x)\}$, $j = 1, \dots n$

(3) $x(0) = (x_j(0)) = x_o \in \mathbb{R}^n_+$

(4) $u \in \mathcal{U}_{ad} = \{u: \mathbb{R}_+ \to \Omega, \text{ measurable}\}$;

here g_i, f_i, $i = 0,1,\dots m$, are continuous functions, locally Lipschitzean with respect to x and $\Omega \subset \mathbb{R}^m$ is convex and compact.

We assume, that for every $x_o \in \mathbb{R}^n_+$, $u \in \mathcal{U}_{ad}$ the corresponding (unique) trajectory $\varphi(\cdot,x_o,u)$ of (2), (3) exists on $\mathbb{R}_+$ and is bounded.

A pair $(x_o,u) \in \mathbb{R}^n_+ \times \mathcal{U}_{ad}$ is called optimal, if for all $v \in \mathcal{U}_{ad}$ one has $V(x_o,u) \geq V(x_o,v)$. We assume that for every $x_o \in \mathbb{R}^n_+$ there is $u \in \mathcal{U}_{ad}$ with (x_o,u) optimal. For optimal (x_o,u) we write $V(x_o) = V(x_o,u)$.

Asymptotic properties of optimal control systems as the one described above have found some interest in the literature, mainly motivated by economic and bioeconomic problems [3,4,6,9]. In particular, it turned out that the assumption, often made in applications, that an optimal trajectory converges to an equilibrium is not in general feasible (e.g. [1]). The paper [5], in particular, sheds some light on the asymptotic behaviour of optimal trajectories for two dimensional systems by deriving a certain analogue of classical Poincaré-Bendixson theory. At the other hand, there has recently been reported considerable progress in the analysis of the asymptotic behaviour of ordinary differential systems [2,7,8]. It is the purpose of this note to show that some of these results (and hopefully more in the future) can also be proven for optimal control problems (or "optimal harvesting problems", in the terminology of bioeconomics). More precisely, the following property, which is of great interest for bioeconomic applications (cp. [3,4]) will be studied here:

Definition 1 The optimal control system (F) is called persistent if for all optimal pairs $(x,u) \in \text{int}\,\mathbb{R}^n_+ \times \mathcal{U}_{ad}$

$$\liminf d(\varphi(t,x,u),\partial\,\mathbb{R}^n_+) > 0.$$

Other questions related to persistence (or "extinction"), which are specific for optimal control systems (e.g. dependence on δ) will not be considered here.

We introduce some additional notation:

$\mathcal{U}_{ad}(\mathbb{R}) := \{u: \mathbb{R} \to \Omega, \text{ measurable}\}$. The restriction of (F) to the boundary $\partial\,\mathbb{R}^n_+$ is denoted by (∂F). Note that on the faces of $\partial\,\mathbb{R}^n_+$ one obtains an optimal control problem of the same form as (F).

2. Results

Before the main result can be stated, some preparations are required.

Definition 2 For $(x,u) \in \mathbb{R}^n_+ \times \mathcal{U}_{ad}$ define the omega limit set $\Lambda^+(x,u)$ by

$$\Lambda^+(x,u) = \{y \in \mathbb{R}^n_+ \text{ there exists } t_k \in \mathbb{R}_+ \text{ such that } t_k \to \infty \text{ and } \varphi(t_k,x,u) \to y\}$$
$$= \bigcap_{n\in\mathbb{N}} \text{cl}\{\varphi(t,x,u): t \geq n\}$$

We call $(x,u) \in \mathbb{R}^n_+ \times \mathcal{U}_{ad}(\mathbb{R})$ an optimal $\mathbb{R}$-solution if the corresponding trajectory $\varphi(\cdot,x,u)$ exists on $\mathbb{R}$ and for all $t \in \mathbb{R}$

$$V(\varphi(t,x,u),u(t+\cdot)) = V(\varphi(t,x,u)).$$

For an optimal $\mathbb{R}$-solution (x,u) define the alpha limit set $\Lambda^-(x,u)$ by

$$\Lambda^-(x,u) := \bigcap_{n\in\mathbb{N}} \text{cl}\,\{\varphi(t,x,u): t \leq -n\}$$

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Furthermore

$\Lambda^+(x) := \bigcup_u \Lambda^+(x,u)$ where the union is taken over all $u \in \mathcal{U}_{ad}$ with (x,u) optimal; similarly for $\Lambda^-(x,u)$.

By the definitions if is clear that limit sets of optimal control systems may be at least as complicated as those of ordinary differential systems (take $f_i \equiv 0$, $i = 1,\dots m$).

Definition 3 A nonvoid subset L of $\mathbb{R}^n_+$ is called invariant if for all $y \in L$ there exists $v \in \mathcal{U}_{ad}(\mathbb{R})$ such that (y,v) is an optimal $\mathbb{R}$-solution and $\varphi(t,y,v) \in L$ for all $t \in \mathbb{R}$.

The following result is proven in [5].

Proposition 1 Let $(x,u) \in \mathbb{R}^n_+ \times \mathcal{U}_{ad}$ be optimal. Then $\Lambda^+(x,u)$ is nonvoid, connected and compact. For every $y = \lim_{t_k\to\infty} \varphi(t_k,x,u) \in \Lambda^+(x,u)$ there exists $v \in \mathcal{U}_{ad}(\mathbb{R})$ such that (y,v) is an optimal $\mathbb{R}$-solution, $\varphi(t,y,v) \in \Lambda^+(x,u)$ for all $t \in \mathbb{R}$ and for a subsequence (t_{k_ℓ}) $\varphi(t_{k_\ell}+\cdot,x,u) \to \varphi(\cdot,y,v)$ locally uniformly as $\ell \to \infty$; in particular, $\Lambda^+(x,u)$ is invariant. Analogous statements hold for $\Lambda^-(x,u)$ if (x,u) is an optimal $\mathbb{R}$-solution.

The following definitions are adapted - with appropriate changes - from [2].

Definition 4 The optimal control system (F) is dissipative if $\Omega(F) := \bigcup_{(x,u)\ \text{optimal}} \Lambda^+(x,u)$ has compact closure.

Definition 5 A nonvoid subset M of $\mathbb{R}^n_+$ is an isolated invariant set for (F) if it has the following properties:

(i) it is the maximal invariant set in some neighbourhood V of itself,

(ii) suppose that $(y,v) \in M \times \mathcal{U}_{ad}$ is an optimal $\mathbb{R}$-solution and there exist optimal (x,u) with $x \notin M$ such that for a sequence $t_k \to \infty$ $\lim \varphi(t_k+\cdot,x,u) = \varphi(\cdot,y,v)$ locally uniformly. Then $\varphi(t,y,v) \in M$ for all $t \in \mathbb{R}$.

The neighbourhood V is called an isolating neighbourhood. If such a set is compact, a compact isolating neighbourhood exists.

Given that (F) is dissipative, $c\ell\,\Omega(\partial F)$ is a compact isolated invariant set for (∂F).

Definition 6 The stable set $W^+(M)$ of an isolated invariant set M is defined to be

$$\{x \in \mathbb{R}^n_+ : \text{there exists } u \in \mathcal{U}_{ad} \text{ with } (x,u) \text{ optimal and } \Lambda^+(x,u) \subset M\}$$

and the unstable set $W^-(M)$ is defined to be

$$\{x \in \mathbb{R}^n_+: \text{there exists } u \in \mathcal{U}_{ad}(\mathbb{R}) \text{ such that } (x,u) \text{ is an optimal } \mathbb{R}\text{-solution and } \Lambda^-(x,u) \subset M\}$$

Definition 7 The weakly stable set $W^+_w(M)$ of an isolated invariant set M is defined to be

$$\{x \in \mathbb{R}^n_+ : \text{there exists } u \in \mathcal{U}_{ad} \text{ with } (x,u) \text{ optimal and } \Lambda^+(x,u) \cap M \neq \emptyset\};$$

analogously for $W^-_w(M)$.

Definition 8 Let M,N be isolated invariant sets. M is chained to N, written $M \to N$, if there exists $x \notin M \cup N$ such that $x \in W^-(M) \cap W^+(N)$.

Definition 9 A finite sequence $M_1, M_2, \dots M_k$ of isolated invariant sets will be called a chain, if $M_1 \to M_2 \to \dots \to M_k$. The chain will be called a cycle if $M_k = M_1$.

Definition 10 The optimal control system (∂F) on $\partial\mathbb{R}^n_+$ is called isolated if there exists a covering $\mathcal{M}$ of $\Omega(\partial F) = \bigcup_{(x,u)} \Lambda^+(x,u)$ where the union is taken over all optimal $(x,u) \in \partial\mathbb{R}^n_+ \times \mathcal{U}_{ad}$, by pairwise disjoint, compact, isolated, invariant sets $M_1, M_2, \dots, M_k$ for (∂F) such that each M_i is also isolated invariant for F. $\mathcal{M}$ is then called an isolated covering.

Definition 11 (∂F) will be called acyclic if there exists some isolated covering $\mathcal{M} = \bigcup_{i=1}^k M_i$ of $\Omega(\partial F)$ such that no subset of the $\{M_i\}$ forms a cycle.

The next theorem presents the main result of this paper. It generalizes [2, Theorem 3.1] to optimal control systems (defined on $\mathbb{R}^n_+$). The proof is very similar to the one given in [2]. However, there are some significant changes mainly due to the fact that optimal solutions need not be unique.

Theorem Assume that the optimal control system (F) is dissipative and the boundary $\partial\mathbb{R}^n_+$ is isolated and has an acyclic covering $\mathcal{M}$. Then F is persistent if and only if

(H) for each $M_i \in \mathcal{M}$: $W^+(M_i) \cap \operatorname{int} \mathbb{R}^n_+ = \emptyset$.

The proof of this theorem is based on the following two anxiliary results, generalizing [2, Lemma 2.1 and Theorem 4.1, resp.].

Proposition 2 Let M be a compact isolated invariant set. Suppose that $W^+_w(M) \setminus M \neq \emptyset$. Then $W^+(M) \setminus M \neq \emptyset$.

Proof: Let $x \in W^+_w(M) \setminus M$. If $x \in W^+(M)$ we are done. Otherwise there exists $u \in \mathcal{U}_{ad}$ such that (x,u) is optimal and

$$\emptyset \neq \Lambda^+(x,u) \cap M, \text{ but } \Lambda^+(x,u) \not\subset M.$$

Hence we may choose a compact isolating neighbourhood V of M which $\varphi(\cdot,x,u)$ enters and leaves infinitely many often as $t\to\infty$. We may also choose a sequence (s_k) of positive times with $s_k\to\infty$ and a sequence (t_k) of negative times such that $s_k+t_k\to\infty$ and for $x_k:=\varphi(s_k,x,u)$

$$\lim_{k\to\infty} d(x_k,M)=0,$$

$$\varphi([s_k+t_k,s_k],x,u)\subset V,\ \varphi(s_k+t_k,x,u)\in\partial V.$$

Suppose without loss of generality that $x_k\to y\in M$. If (t_k) is bounded, we may assume that $t_k\to\bar t$. Since M is isolated invariant, Proposition 1 implies that there is $v\in \mathcal{U}_{ad}(\mathbb{R})$ such that (y,v) is $\mathbb{R}$-optimal, $\varphi(t,y,v)\in M$ for all $t\in\mathbb{R}$ and for a subsequence (t_{k_ℓ})

$$\partial V\ni\varphi(s_{k_\ell}+t_{k_\ell},x,u)\to\varphi(\bar t,y,v),$$

and hence $\varphi(\bar t,y,v)\in\partial V$. Contradiction!

Hence (t_k) is unbounded and we may assume $t_k\to-\infty$. We may assume that $\varphi(s_k+t_k,x,u)$ converges to $z\in\partial V\cap\Lambda^+(x,u)$. By Proposition 1 there is $w\in\mathcal{U}_{ad}(\mathbb{R})$ such that (z,w) is an optimal $\mathbb{R}$-solution and for all $t\in\mathbb{R}$

$$\varphi(t,z,w)=\lim_{k\to\infty}\varphi(s_k+t_k+t,x,u)$$

Choose k large enough such that $t_k+t<0$. Then $\varphi(s_k+(t_k+t),x,u)\in V$ and so $\varphi(t,z,w)\in V$ for all $t>0$. Thus $\Lambda^+(z,w)$ is an invariant subset of V. But the isolating property of V implies $\Lambda^+(z,w)\subset M$ i.e. $W^+(M)\setminus M\neq\emptyset$.

<u>Proposition 3</u> Let M be a compact isolated invariant set for F. Then for every $x\in W_w^+(M)\setminus W^+(M)$ it follows that there exists $u\in\mathcal{U}_{ad}$ with (x,u) optimal, $\Lambda^+(x,u)\cap W^+(M)\setminus M\neq\emptyset$ and $\Lambda^+(x,u)\cap W^-(M)\setminus M\neq\emptyset$ (a similar statement holds for Λ^-).

<u>Proof</u>: Let $x\in W_w^+(M)\setminus W^+(M)$. Then there exist $u\in\mathcal{U}_{ad}$ with (x,u) optimal and a compact isolating neighbourhood V of M such that $\varphi(\cdot,x,u)$ enters and leaves V infinitely many often as $t\to\infty$. Without loss of generality we may assume that $x\in V$. Choose $s_k\to\infty$ such that $d(x_k,M)\to 0$ as $k\to\infty$, where $x_k:=\varphi(s_k,x,u)$ and $t_k<0$ so that $s_k+t_k\to\infty$, $\varphi([s_k+t_k,s_k],x,u)\subset V$, $\varphi(s_k+t_k,x,u)\in\partial V$. Since M is invariant and compact, it follows that $t_k\to-\infty$ as $k\to\infty$.

Since $W_w^+(M)\setminus M\neq\emptyset$, Proposition 2 shows that $W^+(M)\setminus M\neq\emptyset$. Clearly $W^+(M)\cap V\neq\emptyset$ and $W^+(M)\not\subset V$ otherwise the isolating property of V is violated by the invariant set $W^+(M)\cup M$.

Consider $\varphi(s_k+t_k,x,u)\in\partial V$. By compactness, we may assume that $\lim_{k\to\infty}\varphi(s_k+t_k,x,u)=z\in\partial V\cap\Lambda^+(x,u)$. The arguments used in the proof of Proposition 2 show that $z\in W^+(M)$. Thus $\Lambda^+(x,u)\cap(W^+(M)\setminus M)\neq\emptyset$.

Now choose $\sigma_k>0$ so that $\varphi([s_k,s_k+\sigma_k],x,u)\subset V$ and $z_k:=\varphi(s_k+\sigma_k,x,u)\in\partial V$. Then by the arguments used in the proof of Proposition 2, $\lim_{k\to\infty}\sigma_k=\infty$. We may assume that $\lim_{k\to\infty}z_k=\bar z\in\partial V$. Again arguing as in the proof of Proposition 2, one sees that there is $v\in\mathcal{U}_{ad}$ such that $(\bar z,v)$ is an optimal $\mathbb{R}$-solution and $\varphi(t,\bar z,v)\in V$ for all $t<0$. Hence $\bar z\in W^-(M)\setminus M$. Since $\bar z\in\Lambda^+(x,u)$, it follows that $\Lambda^+(x,u)\cap W^-(M)\setminus M\neq\emptyset$.

<u>Proof of Theorem</u> Obviously, (H) is necessary. Now suppose that (H) holds. If (F) were not persistent, there would exist $x\in\operatorname{int}\mathbb{R}_+^n$ with $\Lambda^+(x)\cap\partial\mathbb{R}_+^n\neq\emptyset$. Hence there exists $u\in\mathcal{U}_{ad}$ with (x,u) optimal such that $\Lambda^+(x,u)\cap\Omega(\partial F)\neq\emptyset$. Therefore we can select i_1 so that $\Lambda^+(x,u)\cap M_{i_1}\neq\emptyset$. By (H), $W^+(M_{i_1})\subset\partial\mathbb{R}_+^n$ and so $x\in W_w^+(M_{i_1})\setminus W^+(M_{i_1})$. By Proposition 3 it follows that $\Lambda^+(x,u)\cap W^+(M_{i_1})\setminus M_{i_1}\neq\emptyset$. Let $p_{i_1}\in\Lambda^+(x,u)\cap W^+(M_{i_1})\setminus M_{i_1}$

Since the M_i are pairwise disjoint, we can ensure that $p_{i_1}\notin M_i$ for all M_i. By Proposition 1, there exists $w_{i_1}\in\mathcal{U}_{ad}$ such that (p_{i_1},w_{i_1}) is an optimal $\mathbb{R}$-solution and $\varphi(t,p_{i_1},w_{i_1})\in\Lambda^+(x,u)\cap\partial\mathbb{R}_+^n$ for all $t\in\mathbb{R}$. Hence $\Lambda^-(p_{i_1},w_{i_1})$ is a nonempty, compact, connected subset of $\Lambda^+(x,u)\cap\partial\mathbb{R}_+^n$. Hence $\Lambda^+(\Lambda^-(p_{i_1},w_{i_1}))$ is a nonempty subset of $\Omega(\partial F)$. It follows that $\Lambda^-(p_{i_1},w_{i_1})\cap\bigcup M_i\neq\emptyset$, since $\Omega(\partial F)$ is isolated invariant.

There are two cases to consider.

<u>Case (i)</u>. Suppose that $\Lambda^-(p_{i_1},w_{i_1})$ is not contained in any one of the M_i. Choose i_2 so that $\Lambda^-(p_{i_1},w_{i_1})\cap M_{i_2}\neq\emptyset$. Then $p_{i_1}\in W_w^-(M_{i_2})\setminus W^-(M_{i_2})$. By Proposition 3, there exist $w\in\mathcal{U}_{ad}$ such that (p_{i_1},w) is an optimal $\mathbb{R}$-solution and an element $q_{i_2}\in\Lambda^-(p_{i_1},w)\cap W^-(M_{i_2})\setminus M_{i_2}$. Now $q_{i_2}\in\partial\mathbb{R}_+^n$ and so $\Lambda^+(q_{i_2})\subset\Omega(\partial F)\subset\bigcup M_i$. There exists $w_{i_2}\in\mathcal{U}_{ad}$ such that (q_{i_2},w_{i_2}) is an optimal $\mathbb{R}$-solution and $\varphi(t,q_{i_2},w_{i_2})\in\Lambda^-(p_{i_2},w)$ for all $t\in\mathbb{R}$. Since $\Lambda^+(q_{i_2},w_{i_2})$ is connected, there exists i_3 so that $\Lambda^+(q_{i_2},w_{i_2})\subset M_{i_3}$. If we had $q_{i_2}\in M_{i_3}$, then $\underline{\Lambda}^-(q_{i_2},w_{i_2})\subset M_{i_3}$ by isolated invariance. Hence $M_{i_3}=M_{i_2}$ and $q_{i_2}\in M_{i_2}$, a contradiction. Therefore we have

$q_{i_2} \in W^-(M_{i_2}) \cap W^+(M_{i_3})$, $q_{i_2} \notin M_{i_2} \cup M_{i_3}$, i.e. $M_{i_2} \to M_{i_3}$. Now $q_{i_2} \in \Lambda^-(p_{i_1},w)$ and $\varphi(t,q_{i_2},w_{i_2}) \in \Lambda^-(p_{i_1},w)$ for all $t \in \mathbb{R}$. Hence $\Lambda^+(q_{i_2},w_{i_2}) \subset \Lambda^-(p_{i_1},w)$ and recall $\Lambda^+(q_{i_2},w_{i_2}) \subset M_{i_3}$. Hence $p_{i_1} \in W_w^-(M_{i_3}) \setminus W^-(M_{i_3})$.

Repeating the above argument, we find q_{i_3} and M_{i_4} such that $q_{i_3} \in W^-(M_{i_3}) \cap W^+(M_{i_4})$, $q_{i_3} \notin M_{i_3} \cup M_{i_4}$, i.e. we have $M_{i_2} \to M_{i_3} \to M_{i_4}$.

Continuing with this argument, we must eventually arrive at a cycle, since there are only finitely many M_i.

<u>Case (ii)</u>. Suppose that $\Lambda^-(p_{i_1},w_{i_1}) \subset M_{j_1}$ for some j_1. Since p_{i_1} does not ly in any of the M_i, we have $M_{j_1} \to M_{i_1}$. Recall $\Lambda^-(p_{i_1},w_{i_1}) \subset \Lambda^+(x,u)$. So $x \in W_w^+(M_{j_1}) \setminus W^+(M_{j_1})$. Appealing to the proof of Proposition 3, we find $p_{j_1} \in \Lambda^+(x,u) \cap W^+(M_{j_1}) \setminus M_{j_1}$, where p_{j_1} does not ly in any of the M_j.

Arguing as before with p_{i_1} replaced by p_{j_1}, we either find ourselves back in case (i) or we remain in case (ii) and find k_1 such that $M_{k_1} \to M_{j_1} \to M_{i_1}$. Repeating the preceding arguments, we must eventually achieve a cycle either by getting into case (i) or by remaining in case (ii).

This contradicts the definition of $\mathcal{M}$. Hence (F) must be persistent.

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