

On persistence in discounted optimal control problems

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Summary

This paper considers asymptotic properties of optimal control systems defined on the positive cone of \mathbb{R}^n . A result is proven which connects persistence of optimal solutions to properties of the induced control system on the boundary of this cone. This generalizes recent results for uncontrolled ordinary differential systems.

1. Introduction

This paper studies asymptotic properties of solutions of the following optimal control system (F):

Minimize

$$(1) \quad V(x_0, u) = \int_0^\infty e^{-\delta t} \{g_0(x) + \sum_{i=1}^m u_i g_i(x)\} dt \quad \text{s.t.}$$

$$(2) \quad \dot{x}_j = x_j \{f_0(x) + \sum_{i=1}^m u_i f_i(x)\}, \quad j = 1, \dots, n$$

$$(3) \quad x(0) = (x_j(0)) = x_0 \in \mathbb{R}_+^n$$

$$(4) \quad u \in U_{ad} = \{u: \mathbb{R}_+ \rightarrow \Omega, \text{ measurable}\};$$

here g_i, f_i , $i = 0, 1, \dots, m$, are continuous functions, locally Lipschitzian with respect to x and $\Omega \subset \mathbb{R}^m$ is convex and compact.

We assume, that for every $x_0 \in \mathbb{R}_+^n$, $u \in U_{ad}$ the corresponding (unique) trajectory $\varphi(\cdot, x_0, u)$ of (2), (3) exists on \mathbb{R}_+ and is bounded.

A pair $(x_0, u) \in \mathbb{R}_+^n \times U_{ad}$ is called optimal, if for all $v \in U_{ad}$ one has $V(x_0, u) \geq V(x_0, v)$. We assume that for every $x_0 \in \mathbb{R}_+^n$ there is $u \in U_{ad}$ with (x_0, u) optimal. For optimal (x_0, u) we write $V(x_0) = V(x_0, u)$.

Asymptotic properties of optimal control systems as the one described above have found some interest in the literature, mainly motivated by economic and bioeconomic problems [3,4,6,9]. In particular, it turned out that the assumption, often made in applications, that an optimal trajectory converges to an equilibrium is not in general feasible (e.g. [1]). The paper [5], in particular, sheds some light on the asymptotic behaviour of optimal trajectories for two dimensional systems by deriving a certain analogue of classical Poincaré-Bendixson theory. At the other hand, there has recently been reported considerable progress in the analysis of

the asymptotic behaviour of ordinary differential systems [2,7,8]. It is the purpose of this note to show that some of these results (and hopefully more in the future) can also be proven for optimal control problems (or "optimal harvesting problems", in the terminology of bioeconomics). More precisely, the following property, which is of great interest for bioeconomic applications (cp. [3,4]) will be studied here:

Definition 1 The optimal control system (F) is called persistent if for all optimal pairs $(x, u) \in \text{int } \mathbb{R}_+^n \times U_{ad}$

$$\liminf d(\varphi(t, x, u), \partial \mathbb{R}_+^n) > 0.$$

Other questions related to persistence (or "extinction"), which are specific for optimal control systems (e.g. dependence on δ) will not be considered here.

We introduce some additional notation:

$U_{ad}(\mathbb{R}) := \{u: \mathbb{R} \rightarrow \Omega, \text{ measurable}\}$. The restriction of (F) to the boundary $\partial \mathbb{R}_+^n$ is denoted by (∂F) . Note that on the faces of $\partial \mathbb{R}_+^n$ one obtains an optimal control problem of the same form as (F).

2. Results

Before the main result can be stated, some preparations are required.

Definition 2 For $(x, u) \in \mathbb{R}_+^n \times U_{ad}$ define the omega limit set $\Delta^+(x, u)$ by

$$\begin{aligned} \Delta^+(x, u) &= \{y \in \mathbb{R}_+^n \mid \text{there exists } t_k \in \mathbb{R}_+ \text{ such that} \\ &\quad t_k \rightarrow \infty \text{ and } \varphi(t_k, x, u) \rightarrow y\} \\ &= \bigcap_{n \in \mathbb{N}} \text{cl} \{\varphi(t, x, u) : t \geq n\} \end{aligned}$$

We call $(x, u) \in \mathbb{R}_+^n \times U_{ad}(\mathbb{R})$ an optimal \mathbb{R} -solution if the corresponding trajectory $\varphi(\cdot, x, u)$ exists on \mathbb{R} and for all $t \in \mathbb{R}$

$$V(\varphi(t, x, u), u(t+\cdot)) = V(\varphi(t, x, u)).$$

For an optimal \mathbb{R} -solution (x, u) define the alpha limit set $\Delta^-(x, u)$ by

$$\Delta^-(x, u) := \bigcap_{n \in \mathbb{N}} \text{cl} \{\varphi(t, x, u) : t \leq -n\}$$

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Furthermore

$\Delta^+(x) := \bigcup_u \Delta^+(x,u)$ where the union is taken over all $u \in U_{ad}$ with (x,u) optimal; similarly for $\Delta^-(x,u)$.

By the definitions it is clear that limit sets of optimal control systems may be at least as complicated as those of ordinary differential systems (take $f_i = 0$, $i = 1, \dots, m$).

Definition 3 A nonvoid subset L of \mathbb{R}_+^n is called invariant if for all $y \in L$ there exists $v \in U_{ad}(\mathbb{R})$ such that (y,v) is an optimal IR-solution and $\varphi(t,y,v) \in L$ for all $t \in \mathbb{R}$.

The following result is proven in [5].

Proposition 1 Let $(x,u) \in \mathbb{R}_+^n \times U_{ad}$ be optimal. Then $\Delta^+(x,u)$ is nonvoid, connected and compact. For every $y = \lim_{t_k \rightarrow \infty} \varphi(t_k, x, u) \in \Delta^+(x,u)$ there exists $v \in U_{ad}(\mathbb{R})$ such that (y,v) is an optimal IR-solution, $\varphi(t,y,v) \in \Delta^+(x,u)$ for all $t \in \mathbb{R}$ and for a subsequence (t_{k_l}) $\varphi(t_{k_l}, x, u) \rightarrow \varphi(\cdot, y, v)$ locally uniformly as $l \rightarrow \infty$; in particular, $\Delta^+(x,u)$ is invariant. Analogous statements hold for $\Delta^-(x,u)$ if (x,u) is an optimal IR-solution.

The following definitions are adapted - with appropriate changes - from [2].

Definition 4 The optimal control system (F) is dissipative if $\Omega(F) := \bigcup_{(x,u) \text{ optimal}} \Delta^+(x,u)$ has compact closure.

Definition 5 A nonvoid subset M of \mathbb{R}_+^n is an isolated invariant set for (F) if it has the following properties:

- (i) it is the maximal invariant set in some neighbourhood V of itself,
- (ii) suppose that $(y,v) \in M \times U_{ad}$ is an optimal IR-solution and there exist optimal (x,u) with $x \in M$ such that for a sequence $t_k \rightarrow \infty$ $\lim \varphi(t_k, x, u) = \varphi(\cdot, y, v)$ locally uniformly. Then $\varphi(t, y, v) \in M$ for all $t \in \mathbb{R}$.

The neighbourhood V is called an isolating neighbourhood. If such a set is compact, a compact isolating neighbourhood exists.

Given that (F) is dissipative, $\text{cl } \Omega(\partial F)$ is a compact isolated invariant set for (∂F) .

Definition 6 The stable set $W^+(M)$ of an isolated invariant set M is defined to be

$$\{x \in \mathbb{R}_+^n : \text{there exists } u \in U_{ad} \text{ with } (x,u) \text{ optimal and } \Delta^+(x,u) \subset M\}$$

and the unstable set $W^-(M)$ is defined to be

$$\{x \in \mathbb{R}_+^n : \text{there exists } u \in U_{ad}(\mathbb{R}) \text{ such that } (x,u) \text{ is an optimal IR-solution and } \Delta^-(x,u) \subset M\}$$

Definition 7 The weakly stable set $W_w^+(M)$ of an isolated invariant set M is defined to be

$$\{x \in \mathbb{R}_+^n : \text{there exists } u \in U_{ad} \text{ with } (x,u) \text{ optimal and } \Delta^+(x,u) \cap M \neq \emptyset\};$$

analogously for $W_w^-(M)$.

Definition 8 Let M, N be isolated invariant sets. M is chained to N , written $M \rightarrow N$, if there exists $x \in M \cap W^-(N)$ such that $x \in W^-(M) \cap W^+(N)$.

Definition 9 A finite sequence M_1, M_2, \dots, M_k of isolated invariant sets will be called a chain, if $M_1 \rightarrow M_2 \rightarrow \dots \rightarrow M_k$. The chain will be called a cycle if $M_k = M_1$.

Definition 10 The optimal control system (∂F) on $\partial \mathbb{R}_+^n$ is called isolated if there exists a covering M of $\Omega(\partial F) = \bigcup_{(x,u)} \Delta^+(x,u)$ where the union is taken over all optimal $(x,u) \in \partial \mathbb{R}_+^n \times U_{ad}$, by pairwise disjoint, compact, isolated, invariant sets M_1, M_2, \dots, M_k for (∂F) such that each M_i is also isolated invariant for F . M is then called an isolated covering.

Definition 11 (∂F) will be called acyclic if there exists some isolated covering $M = \bigcup_{i=1}^k M_i$ of $\Omega(\partial F)$ such that no subset of the $\{M_i\}$ forms a cycle.

The next theorem presents the main result of this paper. It generalizes [2, Theorem 3.1] to optimal control systems (defined on \mathbb{R}_+^n). The proof is very similar to the one given in [2]. However, there are some significant changes mainly due to the fact that optimal solutions need not be unique.

Theorem Assume that the optimal control system (F) is dissipative and the boundary $\partial \mathbb{R}_+^n$ is isolated and has an acyclic covering M . Then F is persistent if and only if (H) for each $M_i \in M$: $W^+(M_i) \cap \text{int } \mathbb{R}_+^n = \emptyset$.

The proof of this theorem is based on the following two auxiliary results, generalizing [2, Lemma 2.1 and Theorem 4.1, resp.].

Proposition 2 Let M be a compact isolated invariant set. Suppose that $W_w^+(M) \setminus M \neq \emptyset$. Then $W^+(M) \setminus M \neq \emptyset$.

Proof: Let $x \in W_w^+(M) \setminus M$. If $x \in W^+(M)$ we are done. Otherwise there exists $u \in U_{ad}$ such that (x,u) is optimal and

$$\emptyset \neq \Delta^+(x,u) \cap M, \text{ but } \Delta^+(x,u) \not\subset M.$$

Hence we may choose a compact isolating neighbourhood V of M which $\varphi(\cdot, x, u)$ enters and leaves infinitely many often as $t \rightarrow \infty$. We may also choose a sequence (s_k) of positive times with $s_k \rightarrow \infty$ and a sequence (t_k) of negative times such that $s_k + t_k \rightarrow \infty$ and for $x_k := \varphi(s_k, x, u)$

$$\lim_{k \rightarrow \infty} d(x_k, M) = 0,$$

$$\varphi([s_k + t_k, s_k], x, u) \subset V, \varphi(s_k + t_k, x, u) \in \partial V.$$

Suppose without loss of generality that $x_k \rightarrow y \in M$. If (t_k) is bounded, we may assume that $t_k \rightarrow \bar{t}$. Since M is isolated invariant, Proposition 1 implies that there is $v \in U_{ad}(\mathbb{R})$ such that (y, v) is IR-optimal, $\varphi(t, y, v) \in M$ for all $t \in \mathbb{R}$ and for a subsequence (t_{k_ℓ})

$$\partial V \ni \varphi(s_{k_\ell} + t_{k_\ell}, x, u) \rightarrow \varphi(\bar{t}, y, v),$$

and hence $\varphi(\bar{t}, y, v) \in \partial V$. Contradiction!

Hence (t_k) is unbounded and we may assume $t_k \rightarrow -\infty$. We may assume that $\varphi(s_k + t_k, x, u)$ converges to $z \in \partial V \cap \Delta^+(x, u)$. By Proposition 1 there is $w \in U_{ad}(\mathbb{R})$ such that (z, w) is an optimal IR-solution and for all $t \in \mathbb{R}$

$$\varphi(t, z, w) = \lim_{k \rightarrow \infty} \varphi(s_k + t_k + t, x, u)$$

Choose k large enough such that $t_k + t < 0$. Then $\varphi(s_k + (t_k + t), x, u) \in V$ and so $\varphi(t, z, w) \in V$ for all $t > 0$. Thus $\Delta^+(z, w)$ is an invariant subset of V . But the isolating property of V implies $\Delta^+(z, w) \subset M$ i.e. $W^+(M) \setminus M \neq \emptyset$.

Proposition 3 Let M be a compact isolated invariant set for F . Then for every $x \in W^+(M) \setminus W^+(M)$ it follows that there exists $u \in U_{ad}$ with (x, u) optimal, $\Delta^+(x, u) \cap W^+(M) \setminus M \neq \emptyset$ and $\Delta^+(x, u) \cap W^-(M) \setminus M \neq \emptyset$ (a similar statement holds for Δ^-).

Proof: Let $x \in W^+(M) \setminus W^+(M)$. Then there exist $u \in U_{ad}$ with (x, u) optimal and a compact isolating neighbourhood V of M such that $\varphi(\cdot, x, u)$ enters and leaves V infinitely many often as $t \rightarrow \infty$. Without loss of generality we may assume that $x \in V$. Choose $s_k \rightarrow \infty$ such that $d(x_k, M) \rightarrow 0$ as $k \rightarrow \infty$, where $x_k := \varphi(s_k, x, u)$ and $t_k < 0$ so that $s_k + t_k \rightarrow \infty$, $\varphi([s_k + t_k, s_k], x, u) \subset V$, $\varphi(s_k + t_k, x, u) \in \partial V$. Since M is invariant and compact, it follows that $t_k \rightarrow -\infty$ as $k \rightarrow \infty$.

Since $W^+(M) \setminus M \neq \emptyset$, Proposition 2 shows that $W^+(M) \setminus M \neq \emptyset$. Clearly $W^+(M) \cap V \neq \emptyset$ and $W^+(M) \not\subset V$ otherwise the isolating property of V is violated by the invariant set $W^+(M) \cup M$.

Consider $\varphi(s_k + t_k, x, u) \in \partial V$. By compactness, we may as-

sume that $\lim_{k \rightarrow \infty} \varphi(s_k + t_k, x, u) = z \in \partial V \cap \Delta^+(x, u)$. The arguments used in the proof of Proposition 2 show that $z \in W^+(M)$. Thus $\Delta^+(x, u) \cap (W^+(M) \setminus M) \neq \emptyset$.

Now choose $\sigma_k > 0$ so that $\varphi([s_k, s_k + \sigma_k], x, u) \subset V$ and $z_k := \varphi(s_k + \sigma_k, x, u) \in \partial V$. Then by the arguments used in the proof of Proposition 2, $\lim_{k \rightarrow \infty} \sigma_k = \infty$. We may assume that $\lim_{k \rightarrow \infty} z_k = \bar{z} \in \partial V$. Again arguing as in the proof of Proposition 2, one sees that there is $v \in U_{ad}$ such that (\bar{z}, v) is an optimal IR-solution and $\varphi(t, \bar{z}, v) \in V$ for all $t < 0$. Hence $\bar{z} \in W^-(M) \setminus M$. Since $\bar{z} \in \Delta^+(x, u)$, it follows that $\Delta^+(x, u) \cap W^-(M) \setminus M \neq \emptyset$.

Proof of Theorem Obviously, (H) is necessary. Now suppose that (H) holds. If (F) were not persistent, there would exist $x \in \text{int } \mathbb{R}_+^n$ with $\Delta^+(x) \cap \partial \mathbb{R}_+^n \neq \emptyset$. Hence there exists $u \in U_{ad}$ with (x, u) optimal such that $\Delta^+(x, u) \cap \Omega(\partial F) \neq \emptyset$. Therefore we can select i_1 so that $\Delta^+(x, u) \cap M_{i_1} \neq \emptyset$. By (H), $W^+(M_{i_1}) \subset \partial \mathbb{R}_+^n$ and so $x \in W^+(M_{i_1}) \setminus W^+(M_{i_1})$. By Proposition 3 it follows that $\Delta^+(x, u) \cap W^+(M_{i_1}) \setminus M_{i_1} \neq \emptyset$. Let $p_{i_1} \in \Delta^+(x, u) \cap W^+(M_{i_1}) \setminus M_{i_1}$. Since the M_i are pairwise disjoint, we can ensure that $p_{i_1} \notin M_i$ for all M_i . By Proposition 1, there exists $w_{i_1} \in U_{ad}$ such that (p_{i_1}, w_{i_1}) is an optimal IR-solution and $\varphi(t, p_{i_1}, w_{i_1}) \in \Delta^+(x, u) \cap \partial \mathbb{R}_+^n$ for all $t \in \mathbb{R}$. Hence $\Delta^-(p_{i_1}, w_{i_1})$ is a nonempty, compact, connected subset of $\Delta^+(x, u) \cap \partial \mathbb{R}_+^n$. Hence $\Delta^-(\Delta^-(p_{i_1}, w_{i_1}))$ is a nonempty subset of $\Omega(\partial F)$. It follows that $\Delta^-(p_{i_1}, w_{i_1}) \cap \bigcup M_i \neq \emptyset$, since $\Omega(\partial F)$ is isolated invariant.

There are two cases to consider.

Case (i). Suppose that $\Delta^-(p_{i_1}, w_{i_1})$ is not contained in any one of the M_i . Choose i_2 so that $\Delta^-(p_{i_1}, w_{i_1}) \cap M_{i_2} \neq \emptyset$. Then $p_{i_1} \in W^-(M_{i_2}) \setminus W^-(M_{i_2})$. By Proposition 3, there exist $w \in U_{ad}$ such that (p_{i_1}, w) is an optimal IR-solution and an element $q_{i_2} \in \Delta^-(p_{i_1}, w) \cap W^-(M_{i_2}) \setminus M_{i_2}$. Now $q_{i_2} \in \partial \mathbb{R}_+^n$ and so $\Delta^+(q_{i_2}) \subset \Omega(\partial F) \subset \bigcup M_i$. There exists $w_{i_2} \in U_{ad}$ such that (q_{i_2}, w_{i_2}) is an optimal IR-solution and $\varphi(t, q_{i_2}, w_{i_2}) \in \Delta^-(p_{i_1}, w)$ for all $t \in \mathbb{R}$. Since $\Delta^+(q_{i_2}, w_{i_2})$ is connected, there exists i_3 so that $\Delta^+(q_{i_2}, w_{i_2}) \subset M_{i_3}$. If we had $q_{i_2} \in M_{i_3}$, then $\Delta^-(q_{i_2}, w_{i_2}) \subset M_{i_3}$ by isolated invariance. Hence $M_{i_3} = M_{i_2}$ and $q_{i_2} \in M_{i_2}$, a contradiction. Therefore we have

$q_{i_2} \in W^-(M_{i_2}) \cap W^+(M_{i_3})$, $q_{i_2} \notin M_{i_2} \cup M_{i_3}$, i.e.
 $M_{i_2} \rightarrow M_{i_3}$. Now $q_{i_2} \in \Delta^-(p_{i_1}, w)$ and $\varphi(t, q_{i_2}, w_{i_2}) \in$
 $\Delta^-(p_{i_1}, w)$ for all $t \in \mathbb{R}$. Hence $\Delta^+(q_{i_2}, w_{i_2}) \subset \Delta^-(p_{i_1}, w)$
and recall $\Delta^+(q_{i_2}, w_{i_2}) \subset M_{i_3}$. Hence $p_{i_1} \in W_w^-(M_{i_3}) \setminus W^-(M_{i_3})$.

Repeating the above argument, we find q_{i_3} and M_{i_4} such
that $q_{i_3} \in W^-(M_{i_3}) \cap W^+(M_{i_4})$, $q_{i_3} \notin M_{i_3} \cup M_{i_4}$, i.e.
we have $M_{i_2} \rightarrow M_{i_3} \rightarrow M_{i_4}$.

Continuing with this argument, we must eventually
arrive at a cycle, since there are only finitely many
 M_i .

Case (ii). Suppose that $\Delta^-(p_{i_1}, w_{i_1}) \subset M_{j_1}$ for some j_1 .

Since p_{i_1} does not lie in any of the M_i , we have

$M_{j_1} \rightarrow M_{i_1}$. Recall $\Delta^-(p_{i_1}, w_{i_1}) \subset \Delta^+(x, u)$. So

$x \in W_w^+(M_{j_1}) \setminus W^+(M_{j_1})$. Appealing to the proof of Proposi-
tion 3, we find $p_{j_1} \in \Delta^+(x, u) \cap W^+(M_{j_1}) \setminus M_{j_1}$, where p_{j_1}
does not lie in any of the M_j .

Arguing as before with p_{i_1} replaced by p_{j_1} , we either
find ourselves back in case (i) or we remain in case
(ii) and find k_1 such that $M_{k_1} \rightarrow M_{j_1} \rightarrow M_{i_1}$. Repeating
the preceding arguments, we must eventually achieve
a cycle either by getting into case (i) or by remain-
ing in case (ii).

This contradicts the definition of M . Hence (F) must
be persistent.

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