

A global maximum principle for periodic control of functional differential equations

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Angaben zur Veröffentlichung / Publication details:

Colonius, Fritz. 1986. "A global maximum principle for periodic control of functional differential equations." In *1986 25th IEEE Conference on Decision and Control, 10-12 December 1986, Athens, Greece*, edited by Anthony Ephremides, Spyros Tzafestas, and Sergio Verdu, 1424–27. Piscataway, NJ: IEEE. <https://doi.org/10.1109/cdc.1986.267105>.

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Summary

The paper presents a global maximum principle for optimal periodic control of retarded functional differential systems. The proof is based on Ekeland's Variational Principle, and a structural theory of linear functional differential equations.

1. Introduction

Consider the following optimal periodic control problem:

$$\text{Minimize (1.1) } 1/(t_2 - t_1) \int_{t_1}^{t_2} g(x(t), u(t), t) dt \text{ s.t.}$$

$$(1.2) \quad \dot{x}(t) = f(x_t, u(t), t) \text{ a.a. } t \in T := [t_1, t_2]$$

$$(1.3) \quad x_{t_1} = x_{t_2}$$

$$(1.4) \quad u \in U_{ad} := \{u: T \rightarrow \Omega, \text{ measurable}\}$$

where $x_t(s) := x(t+s) \in \mathbb{R}^n$, $s \in [-r, 0]$, $r > 0$, $\partial_x \subset \mathbb{R}^n$, $\partial_u \subset \mathbb{R}^m$, $\partial_\varphi \subset C(-r, 0; \mathbb{R}^n)$ are open sets, $g: \partial_x \times \partial_u \rightarrow \mathbb{R}$, $f: \partial \subset \partial_\varphi \times \partial_u \times T \rightarrow \mathbb{R}^n$ and $\Omega \subset \partial_u$, closed.

Optimal control problems for retarded equations have been studied since a long time, initially with targets in \mathbb{R}^n . Problems with fixed targets $x_{t_1} = \psi \in C(-r, 0; \mathbb{R}^n)$

were studied when it became clear, that an appropriate notion of state is the function segment x_t . However, for this latter problem, a maximum principle is valid only under very restrictive assumptions. In contrast, the periodic boundary condition (1.3) allows to develop a theory of necessary optimality conditions under similarly weak conditions as in the finite dimensional case. The present paper gives a global maximum principle for this problem.

Our hypotheses are:

$$(1.5) \quad \text{The functions } g(x, u, t) \text{ and } f(\varphi, u, t) \text{ are measurable in } t \text{ and jointly continuous in } (x, u) \text{ and } (\varphi, u), \text{ resp., and continuously Fréchet differentiable in } x \text{ and } \varphi, \text{ resp.}$$

$$(1.6) \quad \text{There exists a monotonically increasing function } q: \mathbb{R}_+ \rightarrow \mathbb{R}_+ \text{ such that for all } x \in \partial_x, \varphi \in \partial_\varphi, u \in \Omega \text{ and a.a. } t \in T$$

$$|g(x, \omega, t)| + |g_x(x, \omega, t)| \leq q(|x|)$$

$$|f(\varphi, \omega, t)| + |D_1 f(\varphi, \omega, t)| \leq q(|\varphi|)$$

$$(1.7) \quad \text{For every } (u, \varphi) \in U_{ad} \times \partial_\varphi \text{ the initial value problem}$$

$$x_{t_1} = \varphi, \dot{x}(t) = f(x_t, u(t), t), \text{ a.a. } t \in T$$

has a solution $x(u, \varphi)$, and $x(u, \varphi)_t \in \partial_\varphi$ is uniformly bounded for $u \in U_{ad} := \{u: T \rightarrow \Omega, \text{ measurable}\}$.

Under these assumptions we can reformulate problem

$$(1.1) - (1.4) \text{ as}$$

$$\text{Minimize } J(u, \varphi) = 1/(t_2 - t_1) \int_{t_1}^{t_2} g(x(t, u, \varphi), u(t)) dt$$

$$\text{s.t. } \varphi = x(u, \varphi)_{t_2}$$

$$u \in U_{ad}.$$

One can show that for fixed u the map

$$\varphi \rightarrow x(u, \varphi): W_2^1(-r, 0; \mathbb{R}^n) \rightarrow W_2^1(T; \mathbb{R}^n)$$

is continuously Fréchet differentiable.

2. Structure Theory of Linear Retarded Equations

Linearizing the system equation (1.2) one obtains a retarded equation of the form

$$(2.1) \quad \dot{x}(t) = L(t)x_t,$$

where $L(t): C := C(-r, 0; \mathbb{R}^n) \rightarrow \mathbb{R}^n$ is linear and bounded and for all $\varphi \in C$, $\varphi \rightarrow L(t)\varphi$ is measurable, with $|L(t)|$ essentially bounded. For the formulation of the maximum principle an adjoint equation is employed. As is well-known, the functional-analytic adjoint and the so-called transposed equation do not coincide. This section sketches the required structural theory developed in [2]cp. also [6].

First observe that $L(\cdot)$ can be represented by a measurable $n \times n$ -matrix valued function $\eta(t, \tau)$ in the form

$$(2.2) \quad L(t)\varphi = \int_{-r}^0 [d_\tau \eta(t, \tau)] \varphi(\tau), \quad \varphi \in C,$$

where $\eta(t, \cdot) \in NBV(-r, 0; \mathbb{R}^{n \times n})$, i.e. $\eta(t, \cdot)$ is of bounded variation and left continuous for $-r < \tau < 0$ and $\eta(t, \tau) = 0$ for $\tau \geq 0$. The evolution of the state $x_t \in C$

is given by a family of strongly continuous evolution operators $\Phi(t, s)$ with $\Phi(t, s)$ compact for $t \geq s$.

Associated with (2.1) are the structural operators

$$F(t): C(-r, 0; \mathbb{R}^n) \rightarrow C(0, r; \mathbb{R}^n)$$

$$(2.3) \quad [F(t)\varphi](s) = \varphi(0) + \int_0^s \int_{[-r,-\sigma]} [d_t \eta(t+\sigma, \tau)] \varphi(\sigma+\tau) d\sigma$$

and $G(t) : C(0, r; \mathbb{R}^n) \rightarrow C(-r, 0; \mathbb{R}^n)$ given by

$$(2.4) \quad [G(t)^{-1}\varphi](s) = \varphi(s-r) - \int_0^s \int_{[-\sigma, 0]} [d_t \eta(t+\sigma, \tau)] \varphi(\sigma+\tau-r) d\sigma.$$

The operator $F(t_1)$ maps the initial function $x_{t_1} = \varphi$ into the corresponding "forcing term" f^{t_1} of the integrated version of (2.1); and the operator $G(t_1)$ maps this forcing term into the corresponding solution segment x_{t_1+r} . The system (2.1) can also be described by the evolution of the forcing terms $f^t = F(t)x_t$. The corresponding operators are denoted by $\Psi(t, s)$. Then for $t \geq s$

$$(2.5) \quad \Phi(t+r, t) = G(t)F(t), \quad \Psi(t+r, t) = F(t+r)G(t)$$

$$(2.6) \quad F(t)\Phi(t, s) = \Psi(t, s)F(s), \quad \Phi(t+r, s+r)G(s) = G(t)\Psi(s, t).$$

The adjoints of $F(t)$, $G(t)$ are given by

$$[F^*(t)\psi](\tau) = \psi(0) - \int_0^r [\eta^T(t+s, \tau-s) - \eta^T(t+s, -s)] \psi(s) ds,$$

$$[G^*(t)^{-1}\psi](\tau) = \psi(\tau+r) + \int_t^0 \eta^T(t+r+\sigma, \tau-\sigma) \psi(\sigma+r) d\sigma,$$

$$-r \leq \tau < 0, \quad \psi \in NBV(0, r; \mathbb{R}^n).$$

They are related to the transposed equation

$$(2.7) \quad z(t) - z(t_2) = - \int_t^{t_2+r} [\eta^T(\alpha, t-\alpha) - \eta^T(\alpha, t_2-\alpha)] z(\alpha) d\alpha$$

$$t \leq t_2.$$

The evolution of the states $z^t \in NBV(0, r; \mathbb{R}^n)$

$$(2.8) \quad z^t(s) = \begin{cases} z(t+s) & , \quad 0 \leq s < r \\ 0 & , \quad s = r \end{cases}$$

is described by $\Psi^*(t, s)$:

$$(2.9) \quad z^t = \Psi^*(t_2, t)\psi$$

for the solution of (2.8) with final condition $z^{t_2} = \psi$.

The restriction of $\Phi(t, s)$ to $W_2^1(-r, 0; \mathbb{R}^n)$ is bounded and denoted by $\Phi_w(t, s)$; note that $\Phi_w(t+r, t)$ is compact. The dual space of W_2^1 is in this context appropriately identified with $\mathbb{R}^n \times L_2$ and for $\varphi \in W_2^1$ and $\psi = (\psi^0, \psi^1) \in (W_2^1)^* = \mathbb{R}^n \times L_2$ the duality is given by

$$\langle \varphi, \psi \rangle = \varphi(0)^T \psi^0 + \int \dot{\varphi} \psi \, d\theta.$$

cp. [2].

Note that $\Psi^*(t+r, t)$ can be continuously extended to a map from $\mathbb{R}^n \times L^2(0, r; \mathbb{R}^n) (= W_2^1(0, r; \mathbb{R}^n)^*)$ into $C(0, r; \mathbb{R}^n)^* = NBV(0, r; \mathbb{R}^n)$, replacing the final con-

dition $z^{t_2} = \psi$ by $(z(t_2), z^{t_2}) = (\psi^0, \psi^1) \in \mathbb{R}^n \times L^2$.

Finally let for $\alpha \in \mathbb{R}^n$

$$(X_0 \alpha)(s) := \begin{cases} 0 & -r \leq s < 0 \\ \alpha & s = 0 \end{cases}.$$

3. The Global Maximum Principle

The proof of the following theorem is based on Ekeland's Variational Principle, and uses arguments similar to those in [4, 5].

Theorem 3.1 Suppose that (x^0, u^0) is a strong local minimum with (1.5) - (1.7) holding. Then there exist $\lambda_0 \geq 0$ and $y^* \in W_2^1(-r, 0; \mathbb{R}^n)^*$ such that

$$(i) \quad y^*(s) := \Phi_w^*(t_2, s)y^* +$$

$$\lambda_0 \int_s^{t_2} \Phi_w^*(t, s) X_0 g_x(x^0(t), u^0(t), t) dt \quad s \in T$$

satisfies

$$(ii) \quad y^*(t_1) = y^*(t_2) = y^*$$

and

$$(iii) \quad \lambda_0 g(x^0(s), u^0(s), s) + \langle y^*(s), X_0 f(x_s^0, u^0(s), s) \rangle$$

$$\leq \lambda_0 g(x^0(s), \omega, s) + \langle y^*(s), X_0 f(x_s^0, \omega, s) \rangle$$

for a.a. $s \in [t_1, t_2-r]$ and all $\omega \in \Omega$.

Here $\Phi(s, t)$ is the evolution operator associated with

$$\dot{x}(t) = D_1 f(x_t^0, u^0(t), t) x_t, \quad t \in T.$$

Proof: The proof uses strong variations of the optimal control u^0 of the following form:

$$(3.1) \quad u_{s, \omega}^0(t) := \begin{cases} u^0(t) & t_1 \leq t < s-p \text{ and } s < t \leq t_2 \\ \omega & \text{for } s-p \leq t < s; \end{cases}$$

here $p > 0$, $\omega \in \Omega$ and $s \in (t_1, t_2 - r]$.

Where no confusion should possibly arise the shorthand u^0 for $u_{s, \omega}^0$ is used. Note that for p small enough,

$u^0 \in U_{ad}$ and let x^0 be the corresponding trajectory with initial state $x_{t_1}^0 := x_{t_1}^0 := \varphi^0$.

First two lemmata are cited from [1, Lemma III.2.2, 2.3] (slightly strengthened here). Let $\omega \in \Omega$, $s \in [t_1, t_2]$.

Lemma 3.2 There exists a sequence (ρ_i) tending to zero such that for all $t \in T$

$$\bar{x}(t) := \lim 1/\rho_i [x^{\rho_i}(t) - x^0(t)]$$

exists, vanishes on $[t_1, s)$ and coincides on $[s, t_2]$

with the unique solution of

$$(3.2) \quad \dot{x}(t) = 0, \quad t < s, \quad x(s) = f(x_s^0, \omega, s) - f(x_s^0, u^0(s), s)$$

$$(3.3) \quad \dot{x}(t) = D_1 f(x_t^0, u^0(t), t) x_t, \quad \text{a.a. } t \in [s, t_2].$$

Furthermore

$$\lim |\bar{x} - 1/\rho_i [x^{\rho_i} - x^0]|_{W_2^1[s, t_2]; \mathbb{R}^n} = 0.$$

Lemma 3.3 For a subsequence (ρ^i)

$$\begin{aligned} & \lim 1/\rho_i [J(u^{\rho_i}, \varphi^0) - J(u^0, \varphi^0)] \\ &= g(x^0(s), \omega, s) - g(x^0(s), u^0(s), s) + \int_s^{t_2} g_x(x_\sigma^0, u^0(\sigma), \sigma) \\ & \quad \{ \Phi(\sigma, s) X_0 [f_\sigma(x_\sigma^0, \omega, s) - f(x_\sigma^0, u^0(s), s)] \} (0) d\sigma. \end{aligned}$$

Next we set the stage for an application of Ekeland's variational principle. Consider U_{ad} in the Ekeland metric

$$d(u, v) := \text{meas } \{t: u(t) \neq v(t)\}.$$

Then $V := U_{ad} \times Q_\varphi^1$ is a complete metric space where $Q_\varphi^1 \subset Q_\varphi$ is closed, endowed with the metric induced by the W_2^1 -norm. Take a sequence $\delta_n \rightarrow 0$, $\delta_n > 0$, and define functionals $F_n: V \rightarrow \mathbb{R}$ by

$$F_n(u, \varphi) := [|x(u, \varphi)_{t_2} - \varphi |_2^2 + |J(u, \varphi) - (m - \delta_n)|^2]^{1/2}$$

where $m := J(u^0, \varphi^0)$, and the norm in the first summand is taken in $W_2^1(-r, 0; \mathbb{R}^n)$. Then F_n is continuous on V and $F_n(u, \varphi) > 0$ for all $(u, \varphi) \in V$. The sequence (ε_n) with $\varepsilon_n := F_n(u^0, \varphi^0)$ converges to zero and

$$F_n(u^0, \varphi^0) \leq \inf \{F_n(u, \varphi) : (u, \varphi) \in V\} + \varepsilon_n.$$

Thus Ekeland's Variational Principle [4, p.456] implies that there exist $(u^n, \varphi^n) \in V$ with the following properties:

$$(3.4) \quad 0 \leq F_n(u^n, \varphi^n) \leq \varepsilon_n$$

$$(3.5) \quad d(u^0, u^n) \leq \varepsilon_n^{1/2}, \quad |\varphi^0 - \varphi^n| \leq \varepsilon_n^{1/2}$$

$$(3.6) \quad F_n(u, \varphi) \geq F_n(u^n, \varphi^n) - \varepsilon_n^{1/2} [d(u, u^n) + |\varphi - \varphi^n|] \text{ for all } (u, \varphi) \in V.$$

Let $s \in T$, $\omega \in \Omega$ and $\rho > 0$, small enough. We shall use the relations above for $(u, \varphi) = (u_s^n, \rho, \varphi^n)$ and for $(u, \varphi) = (u^n, \varphi^0 + \rho\psi)$.

From (3.6) one obtains for $(u, \varphi) \neq (u^n, \varphi^n)$ the following important inequality:

$$\begin{aligned} (3.7) \quad -\varepsilon_n^{1/2} &\leq [|x(u, \varphi)_{t_2} - \varphi |_2^2 + |J(u, \varphi) - (m - \delta_n)|^2]^{1/2} \\ &\quad - [|x(u^n, \varphi^n)_{t_2} - \varphi^n |_2^2 + |J(u^n, \varphi^n) - (m - \delta_n)|^2]^{1/2} \\ &\quad / [d(u, u^n) + |\varphi - \varphi^n|]. \end{aligned}$$

Let $x^n := x(u^n, \varphi^n)$ and define $y^n \in W_2^1(-r, 0; \mathbb{R}^n)^* = \mathbb{R}^n \times L^2(-r, 0; \mathbb{R}^n)$ and $\lambda^n \in \mathbb{R}_+$ by

$$(3.8) \quad y^n := [|x_{t_2}^n - \varphi^n |^2 + |J(u^n, \varphi^n) - (m - \delta_n)|^2]^{1/2}$$

$$[(x^n(t_2), \dot{x}_{t_2}^n) - (\varphi^n(0), \dot{\varphi}^n)]$$

$$(3.9) \quad \lambda^n := [|x_{t_2}^n - \varphi^n |^2 + |J(u^n, \varphi^n) - (m - \delta_n)|^2]^{1/2} [J(u^n, \varphi^n) - (m - \delta_n)].$$

Let $\Phi^n(t, s)$, $n = 1, 2, \dots$ be the family of evolution operators associated with the linearized equations

$$(3.10) \quad \dot{x}(t) = D_1 f(x_t^n, u^n(t), t) x_t, \quad \text{a.a. } t \in T.$$

Finally, define bounded linear operators

$$T^n : W_2^1(-r, 0; \mathbb{R}^n) \rightarrow W_2^1(-r, 0; \mathbb{R}^n) \text{ by}$$

$$(3.11) \quad T^n \psi := \Phi_W^n(t_2, t_1) \psi - \psi.$$

Now take $(u, \varphi) = (u^n, \varphi^n + \rho\psi)$ in (3.7). The chain rule implies that F_n is continuously Fréchet differentiable with respect to φ . Thus in the limit for $\rho \rightarrow 0$ one gets, noting that ψ is arbitrary in W_2^1 ,

$$(3.12) \quad (T^{n*} y^n) \psi = -\lambda^n \int_{t_1}^{t_2} g_x(x^n(s), u^n(s), s) [\Phi_W^n(s, t_1) \psi] (0) ds$$

for all $\psi \in W_2^1$.

Next we consider the limit for $n \rightarrow \infty$. By (3.5), $|\varphi^n - \varphi^0| \rightarrow 0$. Hence by continuity, resp. continuous Fréchet differentiability we get

$|x^n - x^0| \rightarrow 0$, $|\Phi_W^n(s, t_1) - \Phi_W(s, t_1)| \rightarrow 0$ for all $s \in T$. $|T^n - T^0| \rightarrow 0$ and $|\Phi_W(\sigma, s)|$ is uniformly bounded. Recall that $|\lambda^n| \leq 1$ and $|y^n|_{(W_2^1)^*} \leq 1$. We have to exclude that both sequences (λ^n) and (y^n) converge to zero. Suppose first that the Fredholm operator $\Phi_W(t_2, t_1) - \text{Id}$ is surjective, hence an isomorphism on $W_2^1(-r, 0; \mathbb{R}^n)$. This implies that for sufficiently large n also $T^n = \Phi_W^n(t_2, t_1) - \text{Id}$ is an isomorphism cp. [3, Lemma VII.6.1] and hence also T^{n*} is an isomorphism. Suppose that there exists a subsequence of (λ^n) again denoted by (λ^n) converging to zero. This yields

$$|y^n|_{(W_2^1)^*} \rightarrow 1.$$

This implies existence of $\psi^n \in W_2^1(-r, 0; \mathbb{R}^n)$ with $|\psi^n| \leq C_0$ and $(T^{n*} y^n) \psi^n \geq 1/2$; but from (3.12) one obtains

$$|(T^{n*} y^n) \psi^n| \leq |\lambda^n| C_1$$

here C_0, C_1 are constants independent of n . This is a contradiction.

Let λ_0 be a clusterpoint of (λ^n) and y^* be a weak clusterpoint of (y^n) . Then (3.12) implies for all $\varphi \in W_2^1(-r, 0; \mathbb{R}^n)$

$$\begin{aligned} (T^{0*} y^*)\varphi &= [(\Phi_W(t_2, t_1)^* - \text{Id})y^*]\varphi \\ &= -\lambda_0 \int_{t_1}^{t_2} g_x(x^0(s), u^0(s), s) [\Phi(s, t)\varphi](0) ds. \end{aligned}$$

Thus (i) and (ii) hold.

The other case can be treated using the Hahn-Banach Theorem.

Now take $(u, \varphi) = (u^n, \varphi^n)$ in (3.7) in order to derive the maximum condition. Again take first the limit for $\rho \rightarrow 0$ and then for $n \rightarrow \infty$. Then use the adjoint equation in order to get (iii).

Theorem 3.1 has an abstract form. The structural theory of retarded equations exposed in section 2 allows to derive the concrete form of the optimality condition. The following lemma is crucial.

Lemma 3.4 Suppose that $t_2 \geq t_1 + r$. Then $y^* \in \text{NBV}(0, r; \mathbb{R}^n)$ and the equations (i) and (ii) imply that there exists $\psi \in \text{NBV}(0, r; \mathbb{R}^n)$ with $y^* = F^*(t_2)\psi$.

Proof: This follows by an analysis of

$$\begin{aligned} y^* &= y^*(t_1) = \Phi_W^*(t_2, t_1)y^* \\ &+ \lambda_0 \int_{t_1}^{t_2} \Phi_W^*(t, t_1) X_0 g_x(x^0(t), u^0(t), t) dt \end{aligned}$$

and (2.5). Observe that $y^*(t_1) \in \text{NBV}(0, r; \mathbb{R}^n)$.

Using (2.5), (2.6) one obtains the following global maximum principle.

Corollary 3.5 Let $t_2 \geq t_1 + r$ and suppose that (x^0, u^0) is a strong local minimum with (1.5) - (1.7) holding.

Then there exist $\lambda_0 \geq 0$ and a solution y of the transposed equation

$$\begin{aligned} \text{(i)} \quad \frac{d}{ds} \{y(s) + \int_s^{t_2+r} [\eta^T(\alpha, s-\alpha) - \eta^T(\alpha, t_2-\alpha)] y(\alpha) d\alpha\} \\ = \lambda_0 g_x(x^0(s), u^0(s), s), \quad s \in T \end{aligned}$$

such that

$$\text{(ii)} \quad y^{t_1} = y^{t_2} \quad \text{and} \quad (\lambda_0, y^{t_2}) \neq (0, 0) \quad \text{in} \quad \mathbb{R} \times \text{NBV}(0, r; \mathbb{R}^n).$$

$$\text{(iii)} \quad \lambda_0 g(x^0(s), u^0(s), s) + y(s)^T f(x_s^0, u^0(s), s)$$

$$\leq \lambda_0 g(x^0(s), \omega, s) + y(s)^T f(x_s^0, \omega, s)$$

for a.a. $s \in [t_1, t_2 - r]$ and all $\omega \in \Omega$;

here η is given by the representation (cp. (2.2))

$$\mathcal{D}_1 f(x_s^0, u^0(s), s) \varphi = \int_{-r}^0 [d_s \eta(t, s)] \varphi(s), \quad \varphi \in C(-r, 0; \mathbb{R}^n).$$

Remark In the theorems above, the minimum condition (iii) is obtained only on $[t_1, t_2 - r]$. Suppose, however, that $\|\Phi^n(t_2, t_1) \times\|_{\mathbb{R}^n \times L^2(-r, 0; \mathbb{R}^n)}$, $n = 1, 2, \dots$, is uniformly bounded for all $x \in C(-r, 0; \mathbb{R}^n)$ with $\|x\|_{\mathbb{R}^n \times L^2} \leq 1$ (this is e.g. the case for systems with constant delays, cp. [2]).

Then one can show that (iii) holds on the whole interval $T = [t_1, t_2]$. One has to modify the definition of F_n by taking the $\mathbb{R}^n \times L^2$ -norm instead of the W_2^1 -norm and then argue with the continuous extensions of T^n to $\mathbb{R}^n \times L^2$.

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