TESTS FOR PROPERNESS IN PERIODIC CONTROL OF FUNCTIONAL DIFFERENTIAL SYSTEMS

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1. Summary

A fundamental problem in optimal periodic control may be formulated as follows: Suppose one has an optimal steady state x^0 corresponding to a constant control u^0. Can performance be improved by allowing for trajectories x and controls u being periodic with some common period $\tau > 0$? If this is the case, the problem is called proper. For systems governed by ordinary differential equations the so called Π-criterion is a second order variational test for (local) properness. It has been proposed by Bittanti, Fronza, and Guarbadassi [1] and proven by Bernstein and Gilbert [3]; the most general version can be found in Bernstein [2]. Watanabe, Nishimura and Matsubara [12] gave a variant of the Π-criterion ('singular control test' or 'infinite frequency Π-criterion') which tests properness for sufficiently high frequencies and requires significantly less computational effort.

The Π-criterion is of some relevance in chemical engineering and aircraft flight performance optimization (cp. Sincic and Bailey [9], Speyer [11] and the survey papers by Matsubara, Nishimura, Watanabe, Onogi [7] and Noldus [8]).

This paper presents a generalization to functional differential systems of the Π-criterion and its "high-frequency" variant.

2. Problem Formulation

We consider the following optimal periodic control problem:

(OPC) Minimize $\frac{1}{\tau}\int_0^\tau g(x(t),u(t))dt$

s.t. (2.1) $\dot{x}(t) = f(x(t),u(t))$ a.a. $t \in [0,\tau]$

(2.2) $x_0 = x_\tau$

where $\tau > 0$ is fixed, $x_t(s) := x(t+s) \in \mathbb{R}^n$, $s \in [-h,0]$, $u(t) \in \mathbb{R}^m$, and $h > 0$ is the length of the delay. The maps $f : C(-h,0;\mathbb{R}^n) \times \mathbb{R}^m \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ are assumed to be twice continuously Fréchet differentiable. The controls u are taken in $L_\infty(0,\tau;\mathbb{R}^m)$.

In this problem formulation, the periodicity condition for x is incorporated into (2.2). Observe that the finite dimensional condition

$$x(0) = x(\tau)$$

does not guarantee periodic extendability of x to a solution of (2.1) for $t \geq 0$ (with periodic extension of u). Instead we have to consider the constraint (2.2) involving the states x_0 and x_τ. We impose the following

Assumption: For every initial function $x_0 = \psi \in C(-h,0;\mathbb{R}^n)$ and every control $u \in L_\infty(0,T;\mathbb{R}^m)$, equation (2.1) has a unique absolutely continuous solution x.

The optimal steady state problem corresponding to (OPC) has the following form.

(OSS) $\underset{x\in\mathbb{R}^n, u\in\mathbb{R}^m}{\text{Minimize}}$ $g(x,u)$

s.t. (2.3) $0 = f(\bar{x},u)$

where $\bar{x} \in C(-h,0;\mathbb{R}^n)$ is the constant function $\bar{x}(s) := x$.

We are interested in the property specified by the following definition.

Definition: Let $(x^0,u^0) \in \mathbb{R}^n \times \mathbb{R}^m$ be an optimal solution of (OSS). Then (x^0,u^0) is called locally proper if for all $\varepsilon > 0$ there exist x and u satisfying (2.1) and (2.2) with $\|x - \bar{x}^0\|_\infty < \varepsilon$, $\|u - \bar{u}^0\|_\infty < \varepsilon$ and

$$\frac{1}{\tau}\int_0^\tau g(x(t),u(t))dt < g(x^0,u^0).$$

3. Tests for Properness

Let $(x^0,u^0) \in \mathbb{R}^n \times \mathbb{R}^m$ be a steady state, i.e., satisfy (2.3). Then we can linearize the system equation (2.1) around $(\bar{x}^0,\bar{u}^0)$ and find

$$\dot{x}(t) = Lx_t + Bu(t) \text{ , a.a. } t \in [0,\tau]$$

where

$$L := \mathcal{D}_1 f(\bar{x}^0,u^0) : C(-h,0;\mathbb{R}^n) \to \mathbb{R}^n$$
$$B := \mathcal{D}_2 f(\bar{x}^0,u^0) \in \mathbb{R}^{n\times m}.$$

The corresponding characteristic matrix $\Delta(z)$ is given by

(2.6) $\Delta(z) := zI - L(e^{z\cdot}) \quad z \in \mathbb{C}$,

where $e^{z\cdot}$ denotes the function $\exp(z\theta)$, $\theta \in [-h,0]$, and I is the $n \times n$ unit matrix. Introduce the function $H: C(-h,0;\mathbb{R}^n) \times \mathbb{R}^m \times R^n \to \mathbb{R}$

$$H(\psi,u,\lambda) := g(\psi(0),u) + \lambda^t f(\psi,u).$$

Then the following expressions exist (here $j := \sqrt{-1}$).

$$P(\omega) := \mathcal{D}_1\mathcal{D}_1 H(\overline{x}^0, u^0, \lambda)(e^{j\omega\cdot}, e^{-j\omega\cdot})$$

$$Q(\omega) := \mathcal{D}_2\mathcal{D}_1 H(\overline{x}^0, u^0, \lambda)(e^{j\omega\cdot})$$

$$R := \mathcal{D}_2\mathcal{D}_2 H(\overline{x}^0, u^0, \lambda).$$

We identify $P(\omega)$, $Q(\omega)$, and R with elements in $\mathbb{C}^{n\times n}$, $\mathbb{C}^{n\times m}$, and $\mathbb{R}^{m\times m}$, respectively. Define for $\omega \in \mathbb{R}$ the complex $m\times m$ - matrix $\Pi(\omega)$ by

$$(2.10)\quad \Pi(\omega) := B^t\Delta^{-1}(-j\omega)^t P(\omega)\Delta^{-1}(j\omega)B + Q(-\omega)^t\Delta^{-1}(j\omega)B + B^t\Delta^{-1}(-j\omega)^t Q(\omega) + R .$$

The matrix $\Pi(\omega)$ is Hermitian. We assume that the following normality condition for (OSS) is satisfied:

$$\mathbb{R}^n = \{\mathcal{D}_1 f(\overline{x}^0, u^0)\overline{x} \mid x \in \mathbb{R}^n\} + \{\mathcal{D}_2 f(\overline{x}^0, u^0)u \mid u \in \mathbb{R}^m\}.$$

Then the following Π - Criterion is valid:

Theorem 1: Suppose that $(x^0, u^0) \in \mathbb{R}^n \times \mathbb{R}^m$ is an optimal solution of (OSS) and that $jk\omega$, $k \in \mathbb{Z}$, is not a zero of $\Delta(z)$, for $\omega = 2\pi/\tau$. .

(i) Then there exists $\lambda \in \mathbb{R}^n$ such that

$$\begin{cases} 0 = \mathcal{D}_1 H(\overline{x}^0, u^0, \lambda\tau) \\ 0 = \mathcal{D}_2 H(\overline{x}^0, u^0, \lambda\tau). \end{cases}$$

(ii) Let $\lambda \in \mathbb{R}^n$ satisfy (i) and suppose that there is $\eta \in \mathbb{R}^m$ with

$$\eta^t \Pi(\omega)\, \eta < 0.$$

Then (x^0, u^0) is locally proper. Suppose that $\Delta(z)$ has no zeros in the closed right half plane $\{z \in C : \operatorname{Re} z \geq 0\}$. Then a high frequency variant of this Π-criterion can be obtained through the following series expansion of $\Pi(\omega)$: Let

$$A(\omega) := L(e^{j\omega\cdot})$$

and define

$$R_0 := R$$

$$R_k := [Q^t(-\omega)\ B^t]\begin{bmatrix} A(\omega) & 0 \\ -P(\omega)(j\omega) & -A^t(-\omega) \end{bmatrix}^{k-1}\begin{bmatrix} B \\ -Q(\omega) \end{bmatrix}.$$

Then

$$\Pi(\omega) = \sum_{k=0}^{\infty} (j\omega)^{-k} R_k(\omega).$$

and one can prove the following high-frequency Π-Criterion.

Theorem 2: There exists $\omega_0 > 0$ such that for all $\omega \geq \omega_0$ either of the following conditions implies that the optimal steady state (x^0, u^0) is locally proper:

(i) For all $k = 0,1, \ldots 2\ell-1$ one has $R_k(\omega) = 0$ and there exists $\eta \in \mathbb{R}^m$ such that $(-1)^\ell \eta^t R_{2\ell}(\omega)\eta < 0$.

(ii) For all $k = 0,1,\ldots,2\ell$ one has $R_k(\omega) = 0$ and there exists $\eta \in \mathbb{R}^m$ such that $(-1)^{\ell+1}\, j\, \eta^t\ R_{2\ell+1}(\omega)\eta < 0$.

Remark: The system equation (2.1) also allows the delays to depend on state and time. Manitius [6] computed the corresponding Frechet derivatives. Sincic, Bailey [10] use the same formulae for the derivatives, and indicate the formulae for the second derivatives. They give a (formal) proof of the Π-Criterion in this case.

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