

# A NEGATIVE RESULT ON POSITIVE CONTROLLABILITY OF DELAY SYSTEMS

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This paper is concerned with complete controllability of linear retarded systems to the Sobolev space  $W_2^{(1)}(-h, 0; X)$  under a positivity constraint for the control; here  $X$  denotes a separable Hilbert space. I call the result "negative", since this positive controllability property can only be achieved under a very strong and obviously sufficient condition. For systems governed by ordinary differential equations the situation is - as is well known - entirely different. This paper is a modified and shortened version of a paper submitted to the SIAM J. Control Optim.

Notation:  $W_2^{(1)}(-h, 0; X)$  denotes the space of all absolutely continuous functions with values in  $X$  having a derivative in  $L_2(-h, 0; X)$ .

We consider the following general linear retarded system

$$\dot{x}(t) = L(t)x_t + B(t)u(t), \quad t \in [t_0, t_1] \quad (1)$$

where  $h$  is the length of the retardation,  $t_1 - h > t_0$ ,  $x_t(s) := x(t+s) \in X$  for  $s \in [-h, 0]$ ,  $u(t) \in U$ ,  $U$  and  $X$  are separable Hilbert spaces,  $\|B(t)\|$  and  $\|L(t)\|$  are essentially bounded on  $[t_0, t_1]$  and  $\|L(t)\|Y$ ,  $\|B(t)u\|$  are measurable for all  $Y \in C(-h, 0; X)$  and  $u \in U$ , resp. We impose a positivity constraint on the admissible controls  $u$  by requiring throughout that they are elements of

$$\{u \in L_2(t_0, t_1; U) : u(t) \in P(t) \text{ a.e.}\}$$

where  $P(t)$  is a closed convex cone in  $U$  with vertex at 0 and  $t \mapsto P(t)$  is measurable. A trajectory  $x$  of (1) corresponding to an admissible control  $u$  and the initial condition  $x_t = 0$  is called admissible.

Define the controllability set  $\mathcal{R}$  at time  $t_1$  by

$$\mathcal{R} := \{x_{t_1} : x \text{ is an admissible trajectory}\}$$

Certainly

$$\mathcal{R} \subset W_2^{(1)}(-h, 0; X).$$

System (1) is called completely controllable if  $\mathcal{R} = W_2^{(1)}(-h, 0; X)$ .

Define the operator  $\tilde{B} : L_2(t_1 - h, t_1; U) \rightarrow L_2(t_1 - h, t_1; X)$  by

$$(\tilde{B}u)(t) := B(t)u(t), \quad t \in [t_1 - h, t_1],$$

and the closed convex cone  $\tilde{P} \subset L_2(t_1 - h, t_1; U)$  by

$$\tilde{P} := \{u \in L_2(t_1 - h, t_1; U) : u(t) \in P(t) \text{ a.e.}\}$$

We assume that  $B(t)$  has closed range in  $U$ .

Lemma (i) The norm of the generalized inverse  $B(t)^+$  of  $B(t)$  is essentially bounded on  $[t_1 - h, t_1]$  iff  $\tilde{B}$  has a closed range.

(ii) Suppose that  $\tilde{B}$  has a closed range. Then  $\tilde{B}\tilde{P} = L_2(t_1 - h, t_1; X)$  iff  $B(t)P(t) = X$  for a.a.  $t \in [t_1 - h, t_1]$ .

Proof. Assertion (i) follows as Lemma 3 in [2]. In (ii), one direction is trivial. Conversely, it follows that  $\tilde{B}\tilde{P}$  is a closed linear subspace, since  $\tilde{P}$  is closed and  $\tilde{B}$  has a closed range. Suppose that  $e$  is orthogonal to  $\tilde{B}\tilde{P}$ , i.e.

$$\int_{t_1 - h}^{t_1} e(t)B(t)u(t)dt = 0 \text{ for all } u \in \tilde{P}.$$

Then by standard arguments used in the proof of the maximum principle,  $e(t)B(t)u(t) = 0$  a.e. Hence by the assumption  $e = 0$ .

Now we can give the following characterization of complete controllability.

Theorem. Suppose that the norm of the generalized inverse  $B(t)^+$  of  $B(t)$  is essentially bounded on  $[t_1 - h, t_1]$ . Then system (1) is completely controllable to the Sobolev space  $W_2^{(1)}(-h, 0; X)$  iff the following two conditions are satisfied:

- (i)  $\{x(t_1 - h) : x \text{ is an admissible trajectory}\} = X$ ;
- (ii)  $B(t)P(t) = X$  for a.a.  $t \in [t_1 - h, t_1]$ .

Proof: By the second part of the lemma, (i) and (ii) imply complete controllability. Conversely, observe that (i) follows trivially. The proof of (ii) uses smoothness arguments, compare [1].

Remark 1. Property (ii) implies in particular that  $\text{rank } B(t) = n$  for a.a.  $t \in [t_1 - h, t_1]$ . This rank condition is well known for complete controllability without control restrictions.

Remark 2. The theorem above has applications to optimal control theory, since it allows to relate the notion of "regular reachability" to complete controllability (compare [1]).

## References

- [1] F. Colonius, Positive controllability to  $W_2^{(1)}$  and regular reachability of hereditary systems, submitted to SIAM J. Control Optim.
- [2] S. Kurcyusz and A.W. Olbrot, On the closure in  $W_1^q$  of the attainable subspace of linear time lag systems, J. Differential Eqns., 24 (1977), 29-50.