REGULARIZATION OF LAGRANGE MULTIPLIERS FOR TIME DELAY SYSTEMS WITH FIXED FINAL STATE

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0. Introduction

This paper deals with necessary optimality conditions for time delay systems with fixed final state and pointwise control restrictions. Since the state of a time delay system is given by a function segment, the end condition is infinite dimensional. This causes particular difficulties (see the surveys [1,2] and also the more recent paper [7]). In the presence of pointwise control restrictions, as considered here, only Bien and Chyung [3] have established the existence of non-trivial Lagrange multipliers for systems with a single constant delay. However, they have to impose a very strict a-priori condition on the optimal solution. In particular, the number m of control inputs must be not less than the dimension n of the phase space. This latter condition appears also in non-linear problems with energy constrained controls [7].

We shall deal with the relaxed problem in the sense of Warga. In terms of the original problem this means that the end condition has to be satisfied with arbitrary accuracy (see [12,4]). This relaxation of the problem allows to treat a much broader class of systems where the condition $m \geq n$ may not be satisfied.

For a detailed study of relaxed controls we refer to Warga's book [12], in particular to the heuristic discussion in Chapter III and to the exact definition and characterization of the set $\mathcal{S}$ of relaxed controls in sections IV. 1 and 2.

This paper is built up as follows: In section 1 first the existence of non-trivial Lagrange multipliers $(l_o,l) \in R_+ \times (W^{n,\infty}[-h,0])^*$ is established. Then it is shown, that in the case of <u>regular reach-</u>

ability, l can be identified with an element of $W^{n,\infty}[-h,0]$. This regularization of the Lagrange multiplier is the key for the proof of a global, pointwise maximum principle and constitutes the main result of this paper. The maximum principle is formulated without proof (compare [6]).

In section 2, regular reachability is characterized for linear relaxed systems. Regularity turns out to be a generic property of those trajectories leading to interior points of the reachable set. The consequences for the validity of the maximum principle are discussed.

Notation and Conventions

$C^n[a,b]$ is the Banach space of continuous functions on $[a,b]$ with values in R^n. For $1 \le p \le \infty$, $W^{n,p}[a,b]$ is the Sobolev space of absolutely continuous functions $x:[a,b] \longrightarrow R^n$ with derivative $\dot{x} \in L_p^n[a,b]$, that is with p-integrable, resp. essentially bounded derivative. The norm in the Banach space $W^{n,p}[a,b]$ is given by $\|x\| := |(x(a), \|\dot{x}\|_{L_p})|$, where $|\cdot|$ denotes the Euclidean norm in R^{n+1}. $W^{n,p}[a,b]$ is identified in the canonical way with $R^n x L_p^n[a,b]$. The topological dual of a Banach space X is denoted by X^*. For a subset $A \subset R^n$ χ_A is the characteristic function of A, int A is its interior and coA its convex hull; for $\delta > 0$, $\mathrm{int}_\delta A$ is the set of all points in intA having at least distance δ to the boundary of A. For the compact subset $\Omega \subset R^m$ of control values, $\mathcal{S}$ denotes the set of <u>relaxed controls</u> v defined on the fixed time interval $T:=[t_0,t_1]$ with values in the set $\mathrm{rpm}(\Omega)$ of Radon probability measures on Ω (see [12]). Relaxed controls v satisfy the following weak measurability requirement:

$$t \longmapsto c(v(t)) := \int_\Omega c(\omega)v(t)(d\omega)$$

is measurable for each continuous function $c:\Omega \longrightarrow R$. For f satisfying assumption (a) of Theorem 1 below, we define

$$f(x_t,v(t),t) := \int_\Omega f(x_t,\omega,t)v(t)(d\omega).$$

1. Regularization of Lagrange Multipliers

We treat the following problem

(P)

$$\text{Minimize} \quad G(x,v) := \int_T g(x(t),v(t),t)\,dt$$

s.t.

(1.1) $\dot{x}(t) = f(x_t, v(t), t)$ a.e. $t \in T$,

(1.2) $x_{t_o} = \varphi_o$,

(1.3) $v \in \mathcal{Y}$,

(1.4) $x_{t_1} = \varphi_1$,

where $x_t(s) := x(t+s) \in R^n$ for $s \in [-h,o]$ and $o < h < \infty$, $t_1 - h > t_o$,

$g: R^n x R^m x T \longrightarrow R$, $f: C^n[-h,o] x R^m x T \longrightarrow R^n$, and φ_o, $\varphi_1 \in C^n[-h,o]$ are given.

h is the length of the time delay.

The following theorem contains conditions on the problem data implying that this problem is well-defined. It establishes necessary optimality conditions.

Theorem 1 Let (x^o, v^o) be an optimal solution of Problem (P), where we assume that the following assumptions are satisfied:

(a) f and g are jointly continuous in the first two arguments and measurable in the third; the Fréchet derivatives $D_1 f(\varphi, \omega, t)$ and $D_1 g(y, \omega, t)$ with respect to the first argument exist and are continuous in (φ, ω, t) and (y, ω), respectively;

(b) for each relaxed control $v \in \mathcal{Y}$, there is a unique solution $x(v)$ of (1.1) and (1.2) with $x(v)|T \in W^{n,\infty}(T)$ depending in a continuously Fréchet differentiable way on $v \in \mathcal{Y}$;

(c) consider the linearized relaxed system

$$(1.5) \quad \begin{aligned} \dot{z}(t) &= D_1 f(x_t^o, v^o(t), t) z_t + f(x_t^o, v(t) - v^o(t), t) \quad \text{a.e.} t \in T \\ z_{t_o} &= o \end{aligned}$$

The attainable set $\mathcal{A}$ defined by

$$\mathcal{A} := \{\varphi \in W^{n,\infty}[-h,o]: \text{ there is } v \in \mathcal{Y} \text{ s.t. the trajectory } z(v) \text{ of (1.5) satisfies } z(v)_{t_1} = \varphi\}$$

has a non-empty interior.

Under these assumptions, there are non-trivial Lagrange multipliers $(l_o, l) \in R_+ x (W^{n,\infty}[-h,o])^*$ s.t.

(1.6) $l_o D_1 G(x^o, v^o) z(v) + l_o G(x^o, v-v^o) + l(z(v)_{t_1}) \geq o$ for all $v \in \mathcal{Y}$.

This theorem is a consequence of [5, Theorem 1.3] and the chain rule.

Remark 1: See [8] for results on the existence of unique solutions for time delay equations on closed intervals. Differentiability of the trajectory $x(v)$ with respect to relaxed controls v can be analyzed using the results in [12, section II.3].

Remark 2: Observe that for the linearization of the relaxed system no differentiation with respect to $\omega \in \Omega$ is needed.

Theorem 1 is only a preliminary result. The optimality condition (1.6) involves the Lagrange multiplier $l = (l_1, l_2) \in (W^{n,\infty}[-h,o])^* = R^n x (L^n_\infty[-h,o])^*$. The dual space of L_∞ is very complicated and l_2 may not be identifiable with a real function. Thus further analysis and a certain regularity assumption are required in order to show that l_2 can be identified with an element of $L^n_\infty[-h,o] \subset (L^n_\infty[-h,o])^*$.

The following notion will be crucial:

Definition: Suppose x^o is a trajectory satisfying (1.1)-(1.3). Then $\varphi_1 \in W^{n,\infty}[-h,o]$ is called regularly reachable with x^o iff $\varphi_1 = x^o_{t_1}$ and there is a neighbourhood V of $o \in R^n$ s.t.

(1.7) $V \subset -\dot{\varphi}_1(t-t_1) + \text{co}\{f(x^o_t, \omega, t) : \omega \in \Omega\}$ a.e. $t \in [t_1-h, t_1]$.

x^o is called a regular trajectory, iff $x^o_{t_1}$ is reached regularly with x^o.

Observe that $\dot{\varphi}_1(t-t_1) = f(x^o_t, v^o(t), t)$ a.e. $t \in [t_1-h, t_1]$ for a relaxed control $v^o \in \mathcal{Y}$. Furthermore

$$\{f(x^o_t, v(t), t) : v \in \mathcal{Y}\} = \text{co}\{f(x^o_t, \omega, t) : \omega \in \Omega\} \text{ a.e. } t \in T,$$

Thus regular reachability means that φ_1 is reachable with x^o and that a uniform neighbourhood of $\dot{\varphi}_1(t-t_1)$ is contained in the set of relaxed velocity vectors, if the system at time t is in the state x^o_t.

Regular reachability is investigated in section 2.

Now we can derive the result on regularization of Lagrange multipliers.

Theorem 2: If x^o is a regular trajectory, the assertion of Theorem 1 holds with $(l_o,l) \in R_+ x W^{n,\infty}[-h,o]$.

Proof: Let $l_o \in R_+$ and $l=(l_1,l_2) \in (W^{n,\infty}[-h,o])^* = R^n x (L^n_\infty[-h,o])^*$ be the Lagrange multipliers existing by Theorem 1. We show that there is a dense subspace E_∞ of $L^n_\infty[-h,o]$ such that $l_2|E_\infty$ is continuous with L_1-norm on E_∞. Then $l_2|E_\infty$ can be extended to a continuous linear functional l_2' on $L^n_1[-h,o]$ which by duality of L_1 and L_∞ can be identified with an element of $L^n_\infty[-h,o]$. Since l_2 and l_2' are continuous on $L^n_\infty[-h,o]$ and coincide on the dense subspace E_∞, they coincide on $L^n_\infty[-h,o]$.

Thus

$$l(z(v)_{t_1}) = l_1 z(v,t_1-h) + l_2(\dot{z}(v)_{t_1}) = l_1 z(v,t_1-h) + l_2'(\dot{z}(v)_{t_1})$$

$$= (l_1,l_2')(z(v)_{t_1}),$$

and the theorem is proven.

We first construct E_∞.

Consider the subspace $S \subset L^n_\infty[t_1-h,t_1]$ of simple functions. By [9, Theorem 11.35], S is dense in $L^n_\infty[t_1-h,t_1]$ and hence also in $L^n_1[t_1-h,t_1]$.

For $p=1,\infty$ define

$$\xi_p : L^n_p[t_1-h,t_1] \longrightarrow L^n_p[-h,o]$$

as the continuous linear map associating with each $y \in L^n_p$ $\dot{x}_{t_1}$, where x is the (unique) solution of

$$\dot{x}(t) = D_1 f(x^o_t, v^o(t),t) x_t + y(t) \quad \text{a.e. } t \in [t_1-h,t_1], \quad x_{t_1-h}=o.$$

Then ξ_p is an isomorphism and it follows that

$$E_p := \xi_p(S)$$

is dense in $L_p^n[-h,o]$, $p=1,\infty$.

For $e \in E_\infty$, there is a unique $s \in S$ with

$$e = \xi_\infty(s) = \xi_1(s). \tag{1.8}$$

We can write s as

$$s(t) = \sum_{i=1}^{k} \sum_{j=1}^{n} s_{ij} \, \chi_{A_i}(t) y_j(t), \quad t \in [t_1-h, t_1],$$

where $\{A_i\}$ is a measurable decomposition of $[t_1-h,t_1]$, $s_{ij} \in R$ and $y_j : [t_1-h,t_1] \longrightarrow R^n$ are constant functions having value 0 in all components y_{jl} for $j \neq l$ and $y_{jj} > o$.

We can choose y_j such that $\pm\, y_j(t) \in V$, where V is a neighbourhood of $O \in R^n$ satisfying (1.7).

Thus there are $v_j^{\pm} \in \mathcal{S}$ s.t. for a.e. $t \in [t_1-h, t_1]$

$$\begin{aligned} y_j(t) &= f(x_t^o, v_j^+(t) - v^o(t), t) \\ -y_j(t) &= f(x_t^o, v_j^-(t) - v^o(t), t) \,. \end{aligned} \tag{1.9}$$

Let $s_{ij}^{\pm} := \max(o, \pm s_{ij})$. Then for $t \in [t_1-h, t_1]$

$$s(t) = \sum_{i=1}^{k} \sum_{j=1}^{n} \chi_{A_i}(t) \left[s_{ij}^+ y_j(t) - s_{ij}^-(t) \, y_j(t) \right], \tag{1.1o}$$

and since ξ_1 is an isomorphism,

$$\|e\|_{L_1} \longrightarrow o \quad \text{implies for} \quad j=1,\dots n$$

$$\Big\| \sum_{i=1}^{k} \chi_{A_i} (s_{ij}^+ + s_{ij}^-) \Big\|_{L_1} = \sum_{i=1}^{k} \lambda(A_i)(s_{ij}^+ + s_{ij}^-) \longrightarrow o \tag{1.11}$$

Define for $i=1,\dots,k$, $j=1,\dots n$, $w_{ij}^{\pm} \in \mathcal{S}$ by

$$w_{ij}^{\pm} := \begin{cases} v_j^{\pm}(t) & \text{for } t \in A_i \\ v^o(t) & \phantom{\text{for }} t \in T \setminus A_i . \end{cases} \tag{1.12}$$

Taking together (1.8)-(1-1o) and (1.12), we find

$$l_2(e) = l_2(\xi_1(s))$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{k} s_{ij}^{+} (l_2 \circ \xi_1)(f(x_t^o, w_{ij}^{+}(t) - v^o(t), t), t \in [t_1-h, t_1])$$

$$+ \sum_{j=1}^{n} \sum_{i=1}^{k} s_{ij}^{-} (l_2 \circ \xi_1)((f(x_t^o, w_{ij}^{-}(t) - v^o(t), t), t \in [t_1-h, t_1])).$$

By definition of ξ_1

$$\xi_1((f(x_t^o, w_{ij}^{\pm}(t) - v^o(t), t), t \in [t_1-h, t_1]))$$

$$= \dot{z}(w_{ij}^{\pm})_{t_1} .$$

where z is the solution of the linearized system (1.5).
The variation of constants formula [8, Chapter 6, Theorem 2.1] implies

(1.13) $$\| z(w_{ij}^{\pm}) \|_{\infty} \leq c_o \lambda(A_i)$$

for a constant $c_o > o$ which is independent of e.
Apply Theorem 1 2nk times in order to obtain

$$l_2(e) \geq - \sum_{j=1}^{n} \sum_{i=1}^{k} s_{ij}^{+} \{ l_o D_1 G(x^o, v^o) z(w_{ij}^{+}) + l_o G(x^o, w_{ij}^{+} - v^o) + l_1 z(w_{ij}^{+}, t_1 - h) \}$$

$$- \sum_{j=1}^{n} \sum_{i=1}^{k} s_{ij}^{-} \{ l_o D_1 G(x^o, v^o) z(w_{ij}^{-}) + l_o G(x^o, w_{ij}^{-} - v^o) + l_1 z(w_{ij}^{-}, t_1 - h) \}$$

$$\geq - c_1 \sum_{j=1}^{n} \sum_{i=1}^{k} (s_{ij}^{+} + s_{ij}^{-}) \lambda(A_i)$$

for a constant $c_1 > o$. This follows from (1.13) and the properties of G.

By (1.11) this last expression converges to 0 for $\|e\|_{L_1} \longrightarrow o$.

The same argument for $-e$ proves that $l_2(e) \longrightarrow o$ for $\|e\|_{L_1} \longrightarrow o$.
Thus Theorem 2 is proven.

□

Remark 3: The proof is based on an idea in [11].

Using this theorem, a pointwise global maximum principle for Problem (P) can be proven. It exploits the abstract optimality condition in Theorem 1. We restrict ourselves to its formulation, since the proof involves only standard, although lengthy arguments (Compare [6]).

We need a functional representation for D_1f. By the Riesz theorem, there is a measurable nxn-matrixfunction η defined on $T \times [t_o-h, t_1]$ s.t. for all $x \in C^n[t_o-h, t_1]$

$$D_1 f(x_s^o, v^o(s), s)x_s = \int_{t_o-h}^{s} d_t \eta(s,t)x(t), \qquad s \in T,$$

and $\eta(\cdot, s)$ of bounded variation, left continuous on (t_o-h, s) and $\eta(s,t) = o$ for $t_o \leq s \leq t \leq t_1$.

Corollary (Maximum Principle) Under the assumptions of Theorem 1, let (x^o, v^o) be an optimal solution. If x^o is a regular trajectory, there are non-trivial Lagrange multipliers $(l_o, l_1, l_2) \in R_+ \times R^n \times L_\infty^n[-h, o]$, such that the adjoint variable $\psi \in L_\infty^n(T)$ defined by

$$\psi(t) = -l_o \int_t^{t_1} D_1 g(x^o(s), v^o(s), s)ds - \int_t^{t_1} \eta(s,t)^* \psi(s)ds - \begin{cases} l_1, & t \in [t_o, t_1-h] \\ l_2(t-t_1), & t \in (t_1-h, t_1] \end{cases}$$

satisfies the maximum condition

$$-l_o g(x^o(t), v^o(t), t) + \psi(t) f(x_t^o, v^o(t), t)$$

$$\geqslant -l_o g(x^o(t), \omega, t) + \psi(t) f(x_t^o, \omega, t) \quad \text{for all } \omega \in \Omega, \text{ a.e. } t \in T.$$

Remark 4: For $h=o$, the assertion reduces to Pontryagin's maximum principle for ordinary differential equations.

Remark 5: In special cases, one can easily construct the functional representation η.

Remark 6: With respect to the adjoint variable Ψ, the non-triviality condition reads as follows:

$$(o,o,o)\neq(l_o,\ \Psi(t_1-h),\ \Psi|[t_1-h,t_1])\in R_+ x R^n x L^n_\infty[t_1-h,t_1].$$

Remark 7: On $[t_o,t_1-h]$, Ψ can be identified with a function of bounded variation [7,Remark 3.3]. In the case of constant delays, Ψ is even absolutely continuous on $[t_o,t_1-h]$.

2. Regular Reachability

The maximum principle holds if φ_1 is reached regularly with the optimal trajectory x^o. However, we do not know when this assumption is satisfied. In fact, [5] contains an example of a scalar optimal control problem where φ_1 is not reached regularly with the optimal trajectory x^o and the maximum principle is not satisfied. Thus the assumption of regularity is crucial.

In this section, we investigate regular reachability for the following class of linear relaxed systems (with performance index as in Problem (P)):

(2.1) $\dot{x}(t)=L(t)x_t+b(v(t))$ a.e. $t\in T$,

(2.2) $x_{t_o}=\varphi_o$,

(2.3) $v\in \mathcal{V}$,

(2.4) $x_{t_1}=\varphi_1$,

where φ_o, φ_1 and $\mathcal{V}$ are as in (1.2)-(1.4), L is a measurable map from T into the space of bounded linear maps from $C^n[-h,o]$ into R^n with $\operatorname{ess\,sup}_{t\in T}\|L(t)\|<\infty$, and $b:\Omega\longrightarrow R^n$ is continuous.

Remark 8: The set of trajectories of the relaxed system (2.1), (2.3) coincides with the set of trajectories of the following system with ordinary controls:

$$\dot{x}(t)=L(t)x_t+u(t) \quad \text{a.e. } t\in T,$$

where $u:T\longrightarrow \text{co } b(\Omega)$ is measurable (compare [12, Theorem IV.3.2] and [4, Satz 2.5]). Thus the reachability theories for this system and (2.1), (2.3) are equivalent. However, the associated control problems of type (P) will in general have different optimal trajectories.

Define the reachable set $\mathcal{R}$ by

$$\mathcal{R}:=\{\varphi\in W^{n,\infty}[-h,o]\text{: there is a trajectory } x \text{ satisfying (2.1)-(2.3) with } x_{t_1}=\varphi\}$$

Observe that

$$\mathcal{R}=\mathcal{A}+\varphi_1 ,$$

for $\mathcal{A}$ defined as in assumption (c) of Theorem 1. Hence $\text{int}\,\mathcal{R}=\emptyset$ iff $\text{int}\,\mathcal{A}=\emptyset$. Then the following proposition holds:

Proposition: $\{\varphi\in\mathcal{R}$: there is $o\neq l\in W^{n,1}[-h,o]$ s.t. (o,l) are Lagrange multipliers satisfying (1.6) $\}$ is norm-dense in the norm-boundary of $\mathcal{R}$.

Proof: The assertion (1.6) for $l_o=o$ can be rewritten as

$$l(x(v)_{t_1}-x(v^o)_{t_1})\geq o \quad \text{f.a. } v\in\mathcal{V},$$

where $x(v)$ is the trajectory of (2.1) with initial condition (2.2) corresponding to v.

Thus (O,l) satisfies (1.6) iff l is a support functional to $\mathcal{R}$ in φ_1. Since $\mathcal{R}$ is a convex and weakly* closed subset of $W^{n,\infty}[-h,o]$, the proposition follows by [1o, Theorem 1].

□

Remark 9: [4, Satz 4.7] gives an explicit characterization of those final states φ_1 for which there are non-trivial Lagrange multipliers $(o,l)\in R_+\times W^{n,2}[-h,o]$.

The proposition shows that one can obtain the existence of $\theta \neq 1 \in W^{n,1}[-h,o]$ such that (0,1) are Lagrange multipliers after a slight perturbation of φ_1 in the boundary of $\mathcal{R}$.

If $\operatorname{int} \mathcal{R} \neq \emptyset$, then for <u>all</u> φ_1 in the boundary of $\mathcal{R}$ there are non-trivial Lagrange multipliers $(o,1) \in \mathbb{R}_+ \times (W^{n,\infty}[-h,o])^*$. In the following we exclude this abnormal case and restrict our attention to the case where $\varphi_1 \in \operatorname{int} \mathcal{R}$. First, we prove the following simple, but important

<u>Lemma 1:</u> Suppose that $\psi^o \in \mathcal{R}$ is reached with x^o and $\psi^1 \in \mathcal{R}$ is reached regularly with x^1. Then for all $o < \varepsilon \le 1$, $\psi^\varepsilon := (1-\varepsilon)\psi^o + \varepsilon \psi^1 \in \mathcal{R}$ is reached regularly with $x^\varepsilon := (1-\varepsilon)x^o + \varepsilon x^1$.

<u>Proof:</u> x^ε is a trajectory satisfying (2.1)-(2.3), since $\mathcal{G}$ is convex and the system equation is linear. Obviously, $x^\varepsilon_{t_1} = \psi^\varepsilon$.

By regularity of x^1 there is $\delta > o$ s.t.

$$\dot{x}^1(t) - L(t)x^1_t \in \operatorname{int}_\delta \operatorname{cob}(\Omega), \quad \text{a.e. } t \in [t_1-h, t_1].$$

Since $\dot{x}^o(t) - L(t)x^o_t \in \operatorname{cob}(\Omega)$ and $\operatorname{cob}(\Omega)$ is convex, this implies for $o < \varepsilon \le 1$:

$$\begin{aligned} &\dot{x}^\varepsilon(t) - L(t)x^\varepsilon_t \\ &= (1-\varepsilon)(\dot{x}^o(t) - L(t)x^o_t) + \varepsilon(\dot{x}^1(t) - L(t)x^1_t) \\ &\in \operatorname{int}_{\varepsilon\delta} \operatorname{cob}(\Omega) \qquad \text{a.e. } t \in [t_1-h, t_1]. \end{aligned}$$

This shows regularity of x^ε.

□

<u>Theorem 3:</u> (i) $\operatorname{int} \mathcal{R} \neq \emptyset$ iff $\operatorname{int} \operatorname{cob}(\Omega) \neq \emptyset$;

(ii) If $\varphi_1 \in \operatorname{int} \mathcal{R}$, then φ_1 is regularly reachable;

(iii) If $\varphi_1 \in \operatorname{int} \mathcal{R}$, then

$\{x \in C^n[t_o-h, t_1]$: x is a regular trajectory satisfying (2.1)-(2.4)$\}$

is open and dense in

$\{x \in C^n[t_o-h, t_1]$: x satisfies (2.1) - (2.4)$\}$.

Proof: ad(i): Suppose that there is $y \in \operatorname{int} \operatorname{cob}(\Omega)$. Then there is $v^o \in \mathcal{V}$ s.t. $y = b(v^o(t))$, a.e. $t \in T$. We claim that the corresponding trajectory x^o satisfying (2.1) and (2.2) is in int $\mathcal{R}$.

We have

$$\dot{x}^o(t) - L(t)x_t^o = y \in \operatorname{int} \operatorname{cob}(\Omega) \qquad \text{a.e. } t \in T.$$

Thus there are $\delta > o$ and a neighbourhood U of $\dot{x}^o \in L_\infty^n(T)$ s.t. for all x with $\|x - x^o\|_\infty < \delta$ and all $z \in U$

$$z(t) - L(t)x_t \in \operatorname{cob}(\Omega) \qquad \text{a.e. } t \in T.$$

The set Z defined by

$$Z := \{\varphi \in W^{n,\infty}[-h,o] : \varphi = x_{t_1} \text{ for a } x \in C^n[t_o - h, t_1] \text{ with}$$

$$\|x - x^o\|_\infty < \delta,\ x_{t_o} = \varphi_o,\ \dot{x} \in U\}$$

forms a neighbourhood of $\varphi_1 \in W^{n,\infty}[-h,o]$. Furthermore, all elements of Z are reached by trajectories satisfying (2.1)-(2.3). Thus $Z \subset \mathcal{R}$. Conversely, let there be a neighbourhood Z of φ_1 with $Z \subset \mathcal{R}$, and assume that $\operatorname{int}\operatorname{cob}(\Omega) = \emptyset$. Then there are $e \in R^n$ and $c_o \in R$ s.t. $ye = c_o$ for all $y \in \operatorname{cob}(\Omega)$. Without loss of generality, we may assume that

$$\left(\varphi_1(s) + \int_{-r}^{s} \alpha(\tau + t_1) e \, d\tau,\ s \in [-h,o]\right) \in Z$$

for all $\alpha \in L_\infty^1[t_1 - h, t_1]$ with $\|\alpha\|_\infty \leq 1$.

Invoking a strong version of Lusin's theorem [12, Theorem I.5.26(2)], we find that there is a subset N of $[t_1 - h, t_1]$ of positive measure s.t. $\dot{\varphi}_1(t - t_1)$ and $L(t)x_t$ are for all trajectories x of (2.1)-(2.3) continuous functions of t on N.

For $\alpha \in L_\infty^1(N)$ with $\|\alpha\|_\infty \leq 1$ define

$$\alpha(t) := o, \qquad t \in [t_1 - h, t_1] \setminus N.$$

Then there are (x^α, v^α) satisfying (2.1)-(2.3) with

$$\dot{\varphi}_1(t - t_1) + \alpha(t)e = \dot{x}^\alpha(t) = L(t)x_t^\alpha + b(v^\alpha(t)), \qquad t \in N.$$

Scalar product with e in R^n yields

$$\alpha(t)ee = [L(t)x_t^\alpha - \dot{\varphi}_1(t - t_1)]e + b(v^\alpha(t))e$$

$$= [L(t)x_t^\alpha - \dot{\varphi}_1(t - t_1)]e + c_o.$$

Since the right hand side is continuous on N, $e e \neq o$ is a constant, and α is an arbitrary element of $L^1_\infty(N)$, this is a contradiction proving (i).

ad(ii) Let $\varphi_1 \in \operatorname{int} \mathcal{R}$ be reached with x^o.
We have to show that there is a trajectory reaching φ_1 regularly.

By (i) there is $y \in \operatorname{int}_\delta \operatorname{cob}(\Omega)$ for a $\delta > o$. Then there is $v^1 \in \mathcal{V}$ s.t. $y=b(v^1(t))$. The corresponding trajectory x^1 satisfying (2.1) and (2.2) is regular. Application of Lemma 1 with $\psi^o := \varphi_1$, $\psi^1 := x^1_{t_1}$ yields that the set of regularly reachable φ is dense in $\operatorname{int} \mathcal{R}$. Thus for $\varphi_1 \in \operatorname{int} \mathcal{R}$ there is $\varphi \in W^{n,\infty}[-h,o]$ s.t.

$\varphi_1 + \varphi$ is regularly reachable, say with x^1,

$\varphi_1 - \varphi$ is reachable, say with x^2.

Then, by Lemma 1 again, φ_1 is reached regularly with $\frac{1}{2}x^1 + \frac{1}{2}x^2$, and (ii) is proven.

ad(iii) By (ii) there is x^1 reaching φ_1 regularly. Suppose that x^o is any trajectory reaching φ_1. Then apply Lemma 1 with $\psi^o := \psi^1 := \varphi_1$ in order to see density. Openness is clear. □

Remark 1o: Using Remark 8, one can deduce one direction in (i) from well-known results in the theory of unconstrained hereditary systems with ordinary controls. Let A be the affine subspace of R^n spanned by $\operatorname{cob}(\Omega)$. If $\operatorname{int} \mathcal{R} \neq \emptyset$, the system

$$\dot{x}(t) = L(t)x_t + u(t) \qquad \text{a.e. } t \in T$$
$$x_{t_o} = \varphi_o,$$

where the controls u take values in A, reaches each element of $W^{n,\infty}[-h,o]$. For unconstrained linear hereditary systems, complete reachability of $W^{n,p}[-h,o]$, $1 \leq p < \infty$, implies that the dimension of the control space is not less than the dimension of the phase space (cf.e.g. [7, Proposition 4.3]). This can easily be seen to remain true for systems with control values in an affine subspace of R^n and $p=\infty$. Thus $A=R^n$. Since the interior of the convex set $\operatorname{cob}(\Omega)$ in A is non-empty, it follows that $\operatorname{intcob}(\Omega) \neq \emptyset$.

Remark 11: [4] contains an example of a non-linear system, where all trajectories reaching a certain final state φ_1 are regular.

Remark 12: Suppose that Ω contains at least n + 1 points. Then the condition intco $b(\Omega) \neq \emptyset$ is generically satisfied for b in the Banach space of continuous functions defined on Ω with values in R^n i.e. the set of functions b satisfying this condition is open and dense. It does not presuppose a relation between the number m of control inputs and the dimension n of the phase space. Consider e.g. a n-dimensional system with scalar control where

$$\Omega := [o,1] , \quad b(\omega) := \begin{pmatrix} \omega \\ \omega^2 \\ \vdots \\ \omega^n \end{pmatrix}$$

Then intco $b(\Omega) \neq \emptyset$ and the assertions (i)-(iii) of Theorem 3 apply.

Only if we restrict ourselves to the non-generic class of linear functions $b:\Omega \longrightarrow R^n$, the condition $m \geqslant n$ becomes necessary again for regularity.

Theorem 3(iii) showsthat regularity is a generic property of trajectories reaching an element in the interior of $\mathcal{R}$. Though it is very difficult to decide in a particular optimal control problem, whether the (unknown) optimal trajectory is regular, we find that "almost all" trajectories are regular. Thus use of the necessary optimality condition in the maximum principle appears to be reasonable.

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