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Localization Effects in Disordered Systems

Dieter Vollhardt

Max-Planck-Institut für Physik und Astrophysik – Werner-Heisenberg-Institut für Physik – D-8000 München 40, Federal Republic of Germany

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Summary: The basic concepts involved in the physics of localization in disordered systems are discussed on an elementary level. In the case of weak disorder localization effects may be understood as a coherent wave phenomenon. Initially developed to describe electronic transport in disordered metals, localization theory has now found wide application in other areas related to disordered systems. The article is intended to explain how and why localization has recently experienced such an explosive growth. We discuss localization effects in the propagation of various wave-like quantities, quantum oscillations in different geometries (hollow cylinders, networks, rings), the developments concerning mesoscopic systems, as well as the effects of universal fluctuations in such systems. An extensive list of references is given.

1 Introduction

The concept of "localization" due to disorder originates from the work of Anderson in 1958 [1]. He investigated the motion of a quantum mechanical particle on a three-dimensional lattice with randomly varying site-energies. The "disorder" in the problem was thus given by the magnitude of the energy fluctuations from site to site on the lattice. Assuming a particle on a site j at time $t = 0$ he calculated the return probability P for the particle in the limit $t \to \infty$. Below a critical value of the disorder he found $P = 0$, i.e. the particle had diffused away and had disappeared in the system. The particle is thus described by an *extended* state. For larger disorder one finds $P > 0$, indicating that the particle did not disappear but remained within a certain region around the site j. This corresponds to a *localized* state with a certain spatial extent (localization length). There is then a critical strength of the disorder where a sharp transition ("Anderson-transition") distinguishes an extended and a localized regime. In other words: if the energy E of the particle lies below a certain critical energy E_c it is localized, while for $E > E_c$ the energy fluctuations of the system will not be able to dominate the particle such that it is described by an extended state. In the first case one deals with an insulator, in the second one with a metal.

We note that the localization is caused by fluctuations imposed on the wave function and does *not* mean some kind of trapping or local binding to a particular site. Hence it is the coherence of the wave function which is important in this problem.

The disorder discussed above is due to the randomness of the on-site energies of the lattice. Alternatively and equivalently the disorder may enter via a random *spatial* distribution of scattering centers, off which a quantum mechanical particle (with constant energy E) is scattered elastically [2]. The latter approach ("Edwards model") is particularly suited for a perturbation-theoretical approach studying weak disorder.

Then, in the end of the seventies, the investigation of localization in two-dimensional systems, i.e. of the question whether very thin films may have a metallic conductivity at zero temperature or not, led to an explosive development in this field of condensed matter physics [3]. New approaches to the problem of Anderson localization and the metal-insulator transition were devised [4], which – together with perturbational methods – clarified the situation and demonstrated the absence of true metallic conductivity (or of a minimal metallic conductivity [5], for that matter) in two dimensions. In particular, the perturbational treatment of the effects of weak disorder by diagrammatic means [6 ... 8] ("maximally crossed diagrams" [9]) and the subsequent interpretation of the underlying physics [10 ... 12] in the beginning of the eighties led to a significantly new understanding of transport in disordered systems. It was soon realized that the localization effects known from weakly disordered electronic systems ("weak localization") were not peculiar to quantum mechanical particles as such but rather were due to the *wave-nature* of quantum mechanical particles and were thus a general phenomenon common to any wave-propagation. The basic physics of weak localization, namely the coherent backscattering of electrons, is therefore shared by all wave-like transport and is not connected to quantum mechanics or particle statistics. This notion will be discussed in Sect. 2 in the context of the diffusion of electrons in a weakly disordered metal. Consequently, localization effects are in principle also observable in the propagation of light (i.e. electromagnetic radiation) and any kind of sound (phonons). This is indeed the case and will be discussed in Sects. 3 and 4, respectively. Yet different systems and geometries discussed in the context of localization will be mentioned in Sect. 5.

The magnetic field dependence of weak localization led to the prediction of *macroscopic quantum oscillations* in multiply connected, normal-conducting geometries with a periodicity of half the flux quantum known from the Aharonov-Bohm effect. This will be discussed in Sect. 6.

The subsequent experimental investigation of these effects involved the study of *metallic networks*, composed of a macroscopic number of small loops, where quantum oscillations could be well observed (Sect. 7).

On the other hand, the investigation of single rings, i.e. *mesoscopic systems*, led to the discovery of quantum oscillations of the flux both with period $\frac{hc}{e}$ and $\frac{hc}{2e}$, which involve very different physics (Sect. 8).

In the course of the investigation of such mesoscopic systems unexpected *universal fluctuations* were discovered, which will be addressed in Sect. 9.

2 Localization in Disordered Electronic Systems

We will first give a rather detailed discussion of the physics of "weak localization"* in disordered electronic systems. (An introduction to this topic in the general context of the metal-insulator transition in disordered systems can be found in [13].) The basic ideas [14] may then easily be applied to comprehend localization-effects involving other wave-like quantities.

Weak Disorder and Weak Localization

Concentrating on the case of weak disorder, we consider (i) non-interacting, quantum mechanical particles, which (ii) are scattered by point-like randomly distributed scattering centers of equal strength. The scattering in turn leads to a diffusive motion of the particles. We are then interested in the conductivity σ or the diffusion coefficient D of such a disordered system. The disorder is measured by a dimensionless parameter γ with $\gamma \sim n_i V_0^2$, i.e. γ is essentially given by the impurity concentration n_i and the scattering strength V_0^2 of the scatterers. The case of very weak disorder corresponds to $\gamma \ll 1$. The starting point is the *metallic* regime, which is characterized by a finite dc-conductivity

$$\sigma_0 = \frac{e^2 n}{m} \tau, \tag{1}$$

where e and m are the charge and the mass of the particles (e.g. non-interacting electrons), respectively, n is the density and τ is an average collision time between successive scatterings; τ is related to the mean free path ℓ by $\ell = v_F \tau (v_F = \hbar k_F/m$ is the Fermi velocity). The quantity σ_0 is often called "Boltzmann-conductivity", because Eq. (1) is a simple result of the Boltzmann transport theory.

In the following we want to understand how a small concentration of impurities affects the metallic behavior.

Weak disorder means that the mean free path ℓ is much greater than the average particle distance $a \approx k_F^{-1}$, i.e. $k_F \ell \gg 1$. We will therefore choose

$$\gamma = \frac{1}{\pi k_F \ell} \tag{2}$$

as our (small) perturbation parameter. Starting from the metallic regime we intend to consider the precursor effects of localization, i.e. the correction $\delta\sigma$ to the metallic conductivity

$$\sigma = \sigma_0 + \delta\sigma, \quad |\delta\sigma| \ll \sigma_0. \tag{3}$$

These perturbational effects are commonly called "weak localization". The correction $\delta\sigma$ depends on external parameters like the system's size L, the frequency ω, the temperature T, or the magnetic field H.

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* The discussion follows the presentation of Altshuler, Aronov, Khmelnitskii, and Larkin [10]; see also [11].

As already mentioned, the dc-conductivity $\sigma_0 \sim 1/\gamma$ is a direct consequence of Boltzmann transport theory. In this theory consecutive collisions of particles are assumed to be independent of each other, i.e. collisions are uncorrelated. This implies that multiple scattering of a particle at a particular scattering center is not taken into account. Consequently, if there is a finite probability of the repeated occurrence of such multiple scatterings, the basic assumption of the independence of scattering events breaks down and, the validity of the result for σ in Eq. (1) becomes, at least, questionable.

To investigate this fundamental point we consider the diffusive behavior of a particle in a d-dimensional disordered system.

If at $t = 0$ a particle is located at some point $\vec{r}_0$ then, after some time $t \gg \tau$, the solution of the diffusion equation implies that the particle will have diffused into a smooth volume $V_{diff} \approx (D_0\, t)^{d/2}$ around $\vec{r}_0$, where $D_0 = v_F^2\,\tau/d$ is the diffusion constant. This is an entirely classical result. To understand the differences in the diffusive behavior of classical and quantum ("wave") mechanical particles, we take a look at the path of a particle diffusing from point A to point B (Fig. 1). This transport can take place via different trajectories (in Fig. 1 four examples are shown). The trajectories, or "tubes", have a typical width given by the Fermi wavelength

$$\lambda_F = \frac{h}{v_F m}. \tag{4}$$

In the classical case ($h = 0$) these paths are arbitrarily sharp ($\lambda_F = 0$) – in the quantum mechanical case, however, one has $\lambda_F = 2\pi/k_F \approx a$, i.e. the tubes have a finite diameter due to the wave nature of the electrons. We assume that the temperature is low enough such that inelastic processes, characterized by an inelastic scattering time τ_{in}, occur only very rarely ($\tau_{in} \gg \tau$).

Since the transport from A to B may take place along different trajectories, there is a probability amplitude A_i connected to every path i. The total probability W to reach point B from A is then given by the square of the magnitude of the sum of all amplitudes:

$$W = |\sum_i A_i|^2 \tag{5a}$$

$$= \sum_i |A_i|^2 + \sum_{i \neq j} A_i A_j^*. \tag{5b}$$

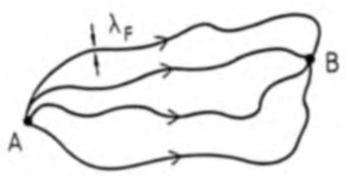

Fig. 1
Typical paths for an electron (or wave) diffusing from A to B.

The first term in Eq. (5b) describes separate, i.e. non-interfering paths – this is the classical result, in which the tubes are infinitely sharp. On the other hand, the second term represents the contribution due to *interference* of the path-amplitudes. It originates from the wave nature of the electron and is therefore an exclusively quantum mechanical effect. In Boltzmann theory these interference terms are neglected. In most cases this is in fact justified: since the trajectories have different lengths, the amplitudes A_i carry different phases. On the average this leads to destructive interference. Hence the quantum mechanical interferences in Fig. 1 are generally unimportant.

There is, however, *one* particular exception to this conclusion, namely, if point A and B coincide (Fig. 2). In this case starting-point and end-point are identical, such that the path in between can be traversed in two opposite directions: forward and backward. The probability to go from A to B is then nothing but the *return*-probability to the starting-point. If paths 1 and 2 in Fig. 2 are indeed equal (this is only the case if time reversal invariance of single particle states, i.e. the equivalence of states with $\vec{k}$ and $-\vec{k}$, is valid) the amplitudes A_1 and A_2 have a coherent phase relation. This leads to *constructive* interference, such that the contribution to W from interference becomes very important. Eq. (5b) then tells us that for $A_1 = A_2 \equiv A$ the classical return probability (due to the neglect of the interference terms) is given by $W_{class} = 2|A|^2$, while the quantum mechanical case yields $W_{qm} = 2|A|^2 + 2A_1A_2^* = 4|A|^2$. Hence one obtains

$$W_{qm} = 2 \cdot W_{class}. \tag{6}$$

The probability for a quantum mechanical particle to return to its starting point is hence seen to be *twice* that of a classical particle. One might say "quantum-diffusion" is slower than classical diffusion because in the first case, where the electron is described by a wave, there exists a constructive interference for backscattering. In other words: quantum mechanical particles in a disordered medium are (at low temperatures) less mobile than classical particles. This in turn leads to a correspondingly lower conductivity σ.

It should be stressed that the factor of 2 in (6) is simply a consequence of constructive wave interference of the two time reversed paths in Fig. 1. In the case of electrons its origin is quantum mechanical only because the wave nature of electrons is an inherently quantum mechanical effect. In general, any wave propagation in a disordered medium will lead to a qualitatively identical result. Any wave

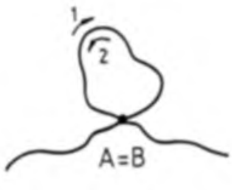

Fig. 2
Time reversed paths on a closed loop.

will do. For example, shouting into a forest* (we assume a naturally grown forest, where trees are irregularly spaced ...) will yield the same kind of enhancement ("echo") into the backward direction as will result from shining light into white paint [15]. Localization involving classical wave propagation has been discussed by Anderson [15], who also gave a number of examples for related electromagnetic and acoustic phenomena.

Correction to the conductivity

To estimate this effect on σ we consider the change $\delta\sigma$ relative to the metallic conductivity σ_0, i.e. $\delta\sigma/\sigma_0$. Because of the expected lowering of σ, the sign of $\delta\sigma/\sigma_0$ will be negative. Furthermore, the change will be proportional to the probability to find a particle in a closed tube, i.e. for the trajectory to intersect itself during the diffusion. Let us therefore have a look at a d-dimensional tube (Fig. 3) with diameter λ_F, i.e. cross-section λ_F^{d-1}. During the time interval dt the particle moves a distance $d\ell = v_F\,dt$, such that the corresponding volume element of the tube is given by $dV = v_F\,dt\,\lambda_F^{d-1}$. On the other hand, the maximally attainable volume for the diffusing particle is given by $V_{diff} \approx (D_0\,t)^{d/2}$. The above mentioned probability for a particle to be in a closed tube is therefore given by the ratio of these two volumes. We find

$$W = \int_{\tau}^{\tau_{in}} \frac{dV}{V_{diff}} = v_F\,\lambda_F^{d-1} \int_{\tau}^{\tau_{in}} \frac{dt}{(D_0\,t)^{d/2}} \tag{7}$$

where we have integrated over all times $\tau \leqslant t \leqslant \tau_{in}$; τ is the microscopic time for a single elastic collision, while τ_{in} is the shortest inelastic relaxation time in the system; it determines the maximal time during which coherent interference of the path-amplitudes is possible. Because of $D_0 \sim 1/\gamma$ and $\lambda_F \sim \hbar$ we obtain

$$\frac{\delta\sigma}{\sigma_0} \sim -\gamma \cdot \begin{cases} (\tau_{in}/\tau)^{1/2}, & d = 1 \\ \hbar \ln(\tau_{in}/\tau), & d = 2 \\ \hbar^2 (\tau_{in}/\tau)^{-1/2}, & d = 3. \end{cases} \tag{8}$$

If we assume that for $T \to 0$ the inelastic relaxation rate vanishes with some power of T, i.e. $1/\tau_{in} \sim T^p$, where p is a constant,

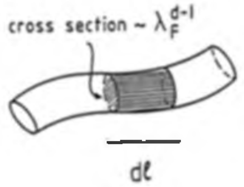

Fig. 3
Part of a trajectory of a quantum mechanical particle.

* This example was mentioned to me by G. Bergmann.

Eq. (8) is given by

$$\frac{\delta\sigma}{\sigma_0} \sim -\gamma \cdot \begin{cases} T^{-p/2}, & d = 1 \\ \hbar \frac{p}{2} \ln\left(\frac{\hbar/\tau}{k_B T}\right), & d = 2 \\ \hbar^2 T^{p/2}, & d = 3. \end{cases} \tag{9}$$

We observe the following: (i) the conductivity decreases for decreasing temperature, (ii) the relative correction $\delta\sigma/\sigma_0$ is linear in the disorder parameter $\gamma \ll 1$ (lowest order in γ), (iii) except for $d = 1$, these corrections are of quantum mechanical origin, i.e. they disappear for $\hbar \to 0$. (In the case $d = 1$ the "tube" in Fig. 3 has no finite diameter – just as in the classical situation; furthermore, since in $d = 1$ there is only forward and backward scattering all paths are trivially "closed".)

In $d = 2$ one therefore obtains a *logarithmic* temperature dependence of the conductivity correction $\delta\sigma$, which has been confirmed in many experiments (Fig. 4). We note that the elastic scattering due to the disorder in principle leads to a *divergent* temperature behavior of $\delta\sigma$ in $d \leqslant 2$. For the initial assumption $|\delta\sigma| \ll \sigma_0$ to remain valid, the results in Eq. (9) for $d \leqslant 2$ may therefore not be used at too low temperatures. In particular, Eq. (9) does not allow to draw conclusions about $\delta\sigma$ at exactly $T = 0$.

We should like to stress once more that Eqs. (8) and (9) are based on the explicit consideration of backscattering effects, i.e. multiple scattering and the correlation of consecutive collisions. Thus they cannot be obtained within the framework of the Boltzmann transport theory. Also CPA ("coherent potential approximation" [17]), is not able to obtain these results because it makes similar assumptions as the Boltzmann theory ("single site approximation", etc.).

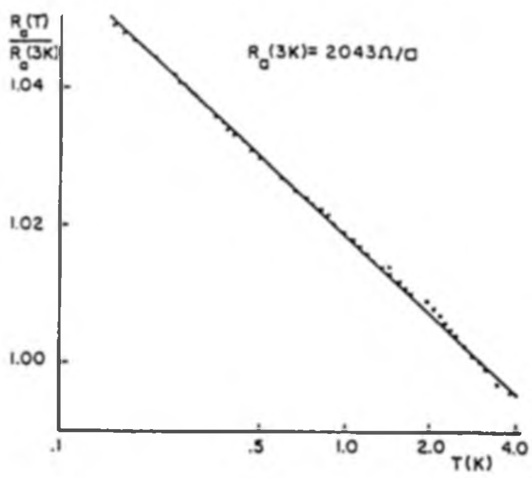

Fig. 4
Logarithmic temperature dependence of the resistivity of a thin palladium film [16].

The preceding discussion was limited to the so-called "normal" scattering, i.e. scattering by non-magnetic impurities. Therefore the spin of the particles was unimportant. However, in the case that the impurities carry a magnetic moment, spin scattering will occur, causing the spin of the particles to flip. Therefore the particles experience something similar to a fluctuating magnetic field. Time reversal invariance is then destroyed, and the weak-localization picture is no longer valid. Field theoretical investigations [18, 19] have shown that even in this new situation the conductivity acquires a logarithmic correction in $d = 2$. However, now the prefactor goes like γ^2 instead of γ, i.e. the correction is even smaller than in the case of normal scattering. It has not yet been possible to understand this result by means of the simple probability arguments used before in the case of normal scattering.

Impurities with a heavy nucleus lead to yet another type of scattering, namely to spin-orbit scattering of the particles. Theoretical investigations [19, 20] have again predicted a logarithmic correction for σ – but this time with a *positive* sign. The conductivity therefore *increases* with decreasing temperature. A simple quantum mechanical explanation of this effect in terms of multiple scattering (time reversal invariance holds in this case) and experimental results fully supporting these findings have been given by Bergmann [12].

Beyond weak localization

The intuitive picture of constructive interference of waves, propagating on time reversed paths, only allows for an estimation of the lowest order correction to the conductivity or the diffusion coefficient of the metallic regime. Higher order corrections or the Anderson transition itself cannot be studied in this way. For this purpose more powerful theoretical methods have to be employed.

It was Wegner [18, 21] who first realized that Anderson localization shared many properties with the problem of critical phenomena and who accomplished a mapping to a suitable field theoretical model. This approach [18, 22, 23] as well as other field theoretical methods [19, 24], allowed for conclusions about localization and the Anderson transition in various physical situations unrivaled by any other approach. This is particularly true for the case of localization in the presence of spin-flip scattering or magnetic fields, where localization in $d = 2$ was also predicted to occur as in the case of normal impurity scattering, although the underlying physics is necessarily quite different (see the discussion above).

A different approach to Anderson localization is due to Abrahams, Anderson, Licciardello, and Ramakrishnan [6]. They constructed a one-parameter scaling theory for the conductance g of a d-dimensional system in connection with the first diagrammatic, perturbative calculation of $\delta\sigma$ in Eq. (3). Assuming g to be the only relevant parameter these authors constructed a flow diagram which led to the conclusion that for $d \leqslant 2$ all states of a disordered system are localized, irrespective of the strength of disorder, while for $d > 2$ an Anderson transition occurs at a finite critical disorder γ_c. For $\gamma < \gamma_c$ the system is metallic, for $\gamma > \gamma_c$ it is insulating.

Yet another approach to the Anderson transition is based on a self-consistent calculation of the diffusion coefficient $D(\omega)$ or conductivity $\sigma(\omega)$. Within this concept, which was first introduced by Götze [25, 26], one attempts to express $\sigma(\omega)$ or $D(\omega)$ by means of a non-trivial, generally approximate relation which itself involves this quantity. So one wants to find an equation of the form

$$D(\omega) = \mathscr{F}[D(\omega)]. \tag{10}$$

whose "self-consistent" solution then yields $D(\omega)$ for all ω and all disorder-parameters γ. For this to be successful it is necessary to start from known limiting cases (e.g. the perturbation theory for $\gamma \ll 1$) such that the theory can be anchored to an exact result [27, 28]. The self-consistency is then used to go beyond perturbation theory, i.e. to the transition itself (and even further). It is therefore used as a substitute for an (untractable) perturbation theory to infinite order.

Since $D(\omega)$ vanishes at the transition, its inverse $D_0/D(\omega)$ correspondingly diverges at that point. Within a diagrammatic perturbation theory Vollhardt and Wölfle [27, 28] showed that a self-consistent calculation of the latter quantity can be performed by summing up the largest (i.e. most divergent) contributions of perturbation theory [29]. In this way a self-consistent equation is derived. It has the simple structure

$$\frac{D_0}{D(\omega)} = 1 + \frac{1}{\pi N_F} \int \frac{d\vec{k}}{(2\pi)^d} \frac{1}{-i\omega + D(\omega) k^2}, \tag{11}$$

where $D_0/D(\omega)$ is given by the integral over a diffusion pole involving the diffusion coefficient $D(\omega)$ rather than the diffusion constant D_0. (Eq. (11) actually involves the particle-particle diffusion pole obtained from the "maximally crossed diagrams" [6...8], which, in the case of time reversal invariance, can be related to the diffusion coefficient $D(\omega)$ of particle-hole diffusion.) This relation can also be derived by other methods [30...32]. Its solution can easily be obtained: One finds that for $d \leqslant 2$ the dc-conductivity $\sigma(0)$ is always zero, irrespective of how small the disorder is (insulating behavior). However, in dimension $d = 2$ the localization length ξ is exponentially large for $\gamma \ll 1$ [27, 28]: $\xi \sim \exp(1/2\gamma)$. For $d > 2$ there exists a critical value of the disorder below which $\sigma(0)$ is finite (metallic regime), while for larger values it vanishes (insulating regime). Since the limit $\omega \to 0$ can be explicitly performed within this theory, one obtains results which go beyond the range of applicability of the scaling theory described above. Besides that one obtains complete agreement [33] with the results of scaling theory.

A field theoretical analysis of the problem by Hikami [24], which involves the solution of the Callan-Simanzyk equation, yields exactly the same relation for $D(\omega)$ as in Eq. (11). Hence, if perturbation theory is valid at all, Eq. (11) is an exact relation at least close to two dimensions.

Most recently a full renormalization-group treatment of the density-density correlation function by Abrahams and Lee [34] yielded the scaling behavior of the diffusion coefficient $D(\omega, \vec{q})$. For $v_F |\vec{q}| < \omega$ the result is again identical to the self-consistent equation (11).

3 Weak Localization of Light (Photons)

By 1983 the interpretation of weak localization in disordered electronic systems as an interference phenomenon of waves had been widely recognized. On the other hand, a very similar effect had already been discussed much earlier by Watson [35] and de Wolf [36] with respect to scattering of electromagnetic waves from fluctuations in a plasma and general turbulent media, respectively. These investigations were related to questions concerning radar scattering from ionized or neutral gases, which, for example, arise in the remote probing of the atmosphere. Since they originated in a subject very different from condensed matter physics, these findings went unnoticed by the disorder community. In fact, in 1984 Kuga and Ishimaru [37] and Tsang and Ishimaru [38] presented experimental and theoretical results, respectively, for the scattering of electromagnetic waves from a random distribution of discrete scatters, which clearly showed an enhancement in backscattering. The authors [38] used second-order multiple scattering theory to explain their results. Therefore their interpretation was based on the same physical idea, i.e. constructive interference in the backward direction due to *multiple* scattering, which had already been identified as the cause for weak localization in disordered systems. A connection with electron localization was not made. The same interference effects involving light where simultaneously discussed by Golubentsev [39].

It was Anderson [15] who discussed the phenomenon of localization from a general point of view and who explicitly addressed the question of classical wave localization.

Inspired by the weak localization effects known from disordered electronic system van Albada and Lagendijk [40] and Wolf and Maret [41] convincingly showed that coherent backscattering equally applies to the propagation of light in a disordered medium. Shining light into a highly concentrated aqueous suspension of sub-micron size polystyrene spheres, (also used in [37]) these two groups measured the scattered intensity and found a striking enhancement in the backscattering direction within a narrow cone. This enhancement comes from the constructive interference of light-waves travelling on closed, time-reversed paths just as explained in the case of weak localization. Note that the explicit condition of *static* disorder necessary for weak localization is fulfilled even in these experiments, since the thermal motion in the liquid is much slower than the propagation of the light wave along any relevant closed path in the medium. Ideally, i.e. assuming isotropic scattering and scalar waves, the backscattered intensity should be enhanced by the factor of 2 in Eq. (6) relative to the incoherent background. This would require that the starting and endpoint of the loops really coincide (A = B in Fig. 2). Otherwise interference cannot be complete, resulting in a reduced enhancement. This effect

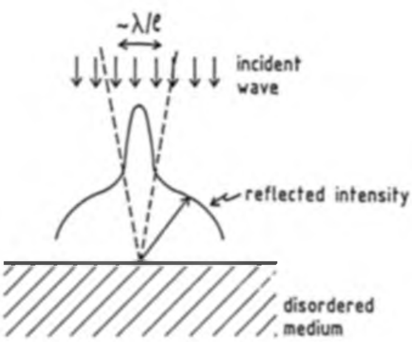

Fig. 5 Schematic view of the backscattering intensity of a (light) wave reflected from a disordered medium.

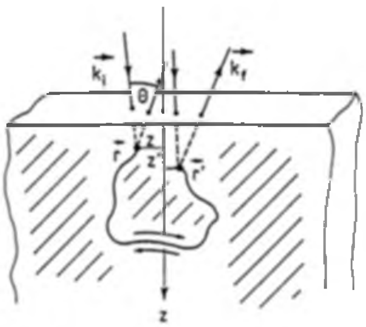

Fig. 6 Interference of light waves propagating on time reversed paths in a disordered medium (after [43]).

leads to a limitation of the angular width within which the enhancement is visible (see below), as indicated in Fig. 5. A detailed theoretical analysis of the effect [42] and of the experiments, in particular of the observed peak line shape, was given by Akkermans, Wolf, and Maynard [43]. Fig. 6 illustrates the wave propagation in the disordered system. Here $\vec{k}_i$ and $\vec{k}_f$ are the wave vectors of the incident and the emerging light wave (with wave length λ) and θ is their relative angle. The figure makes clear that coherent backscattering is only possible within an angular width $\delta\theta \approx \lambda/|\vec{r}_i - \vec{r}_f|$, where $\vec{r}_i$, $\vec{r}_f$ are the positions of first and last scattering in the medium, respectively. Since $|\vec{r}_i - \vec{r}_f| \gtrsim \ell$, the enhancement is only visible in a narrow cone of width $\delta\theta = \lambda/\ell \ll 1$.

The intensity of the reflected relative to the incident light, the so-called albedo $\alpha(\vec{k}_i, \vec{k}_f)$, is given by [43]

$$\alpha(\hat{k}_i, \hat{k}_f) = \frac{c}{4\pi\ell^2}\int dz\, dz'\, d^2\rho\, \exp\left(-\frac{z}{p_i \ell}\right) \cdot \\ \cdot \{1 + \cos[\vec{q}\cdot(\vec{r}_i - \vec{r}_f)]\}\, Q(\vec{r}, \vec{r}')\, \exp\left(-\frac{z'}{p_f \ell}\right) \tag{12}$$

where c is the wave velocity, z and z′ are the z-components of $\vec{r}$ and $\vec{r}'$ (see Fig. 6), $\vec{q} = \vec{k}_i + \vec{k}_f$, $\vec{\rho}$ is the projection onto the surface, and $p_{i,f} = \hat{k}_{i,f}\cdot\hat{z}$. The two exponentials in Eq. (12) are attenuation factors associated with the scattering events which take place at distances z and z′ from the interface inside the disordered medium. The factor $1 + \cos[\vec{q}\cdot(\vec{r}_i - \vec{r}_f)]$ corresponds to the usual $(1 - \cos\vartheta)$ term in transport theory, e.g. appearing in the expression for the relaxation time, which favors *large*-angle scattering (note, that $\vartheta = \pi - \vec{q}\cdot(\vec{r} - \vec{r}')$). The Green function $Q(\vec{r}, \vec{r}')$ describes the wave propagation from $\vec{r}$ to $\vec{r}'$. Owing to the disorder this transport is diffusive in nature, such that $Q(\vec{r} - \vec{r}')$ is determined by the three-dimensional diffusion equation with proper boundary conditions. For isotropic scattering Eq. (12) may be evaluated to yield the angular shape of the relative scattered flux [43]

$$\alpha(\theta) = \frac{3}{8\pi}\left\{1 + \frac{2z_0}{\ell} + \frac{1}{(1+q_\perp \ell)^2}\left[1 + \frac{1-\exp(-2q_\perp z_0)}{q_\perp \ell}\right]\right\} \tag{13}$$

where $q_\perp$ is the magnitude of $\vec{q}$ normal to the z-axis, with $q_\perp \approx 2\pi|\theta|/\lambda$, and z_0 is a length which enters in the boundary condition for $Q(\vec{r})$ (for pointlike scatterers $z_0 \approx 0.7\ell$). The backscattering is indeed found to be enhanced in a small regime $\delta\theta \sim \lambda/\ell$, the enhancement being a factor of two in the exact backward direction ($q_\perp$, $\theta = 0$) as compared with the incoherent intensity outside the width $\delta\theta$. $\alpha(\theta)$ has a peculiar shape: for small angles it varies linearly with θ, i.e. has a triangular shape with a sharp tip at $\theta = 0$. Using this result Akkermans, Wolf, and Maynard [43] were able to describe the experimental curves for $\alpha(\theta)$ of [41] without any adjustable parameter. Convoluting their theoretical result for $\alpha(\theta)$ with the given instrumental profil they found a truely remarkable quantitative agreement (Fig. 7).

These authors also pointed out that for small angles θ all diffusional trajectories of size $L \lesssim \lambda^2/(\ell\theta^2)$ contribute to the coherent backscattering; hence, the smaller θ is, the larger the size of the maximal loops are. The quantity $\lambda^2(D\cdot\theta^2)$ therefore corresponds to the inelastic scattering time τ_{in} acting as a cut-off in Eqs. (8), (9). The sharp tip of $\alpha(\theta)$ will always be rounded unless loops of arbitrary size are included.

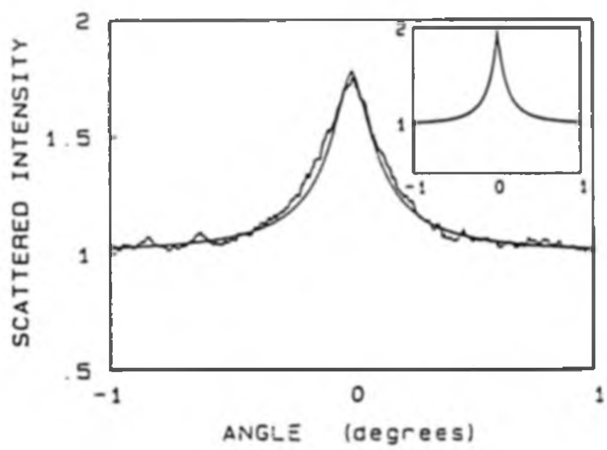

Fig. 7 Line shape of the intensity of coherently backscattering light [43]. The measured curve by Wolf and Maret [41] is compared with the theoretical results of Akkermans, Wolf, and Maynard [43] after folding with the experimental profil. Insert: bare theoretical curve.

In the case of light the backscattered intensity is polarization dependent [37, 40, 41], i.e. it depends on the relative orientation between incident and reflected polarization (parallel or perpendicular). Depending on the instrumental profil of the apparatus an enhancement in the former case of up to 1.96 ± 0.02 [44] has been observed. For perpendicular orientation only smaller enhancements (≈ 1.3) have been found. In [43] a maximal enhancement of 2 and ≈ 1.5, respectively, has been calculated for the two different cases. A detailed theoretical investigation of the polarization dependence of the backscattering was given by Stephen [45] and Stephen and Cwilich [46].

Localization effects in light scattering from a disordered *solid* have also been observed [47, 48]. In contrast to a liquid, a rigid disordered medium leads to large-amplitude fluctuations in the scattered intensity which has to be subtracted (by ensemble averaging) to reveal the backscattering peak. (In the liquid this is automatically done by the thermal motion of the atoms.) Localization effects caused by the reflection of light from random gratings [49] or random layered systems [50] have also been considered. The critical behavior of electromagnetic absorption, i.e. the "photon mobility edge" was studied within a renormalization group theory by John [51].

4 Localization of Acoustic Waves (Phonons) and other Sound

The wave nature of acoustic phenomena may in principle lead to similar localization effects in a disordered medium as discussed in the case of electronic transport or light propagation. On the other hand, as pointed out by Anderson [15], there will be fewer systems available than in the case of light where localization effects can actually be expected to be seen. (This is due to the problem of having an inhomogeneous mixture of two propagating media, where different waves will be excited within the system; see, however, Sect. 5.) He suggested to use expanded silica gel filled with a denser liquid to study acoustic localization. Most recently the observability of acoustical and optical localization was analyzed by Condat and Kirkpatrick [52]. Using the self-consistent equation (11) of Vollhardt and Wölfle [27, 28] to investigate the transition to the localized state, they conclude that in $d = 3$ acoustic localization will not be observed unless the scatterers are more efficient than hard spheres. On the other hand the optical localization transition is found to be not far from the conditions used in the weak localization experiments [37, 40, 41, 47].

A different kind of "acoustic" localization experiment in a one-dimensional system (where weak localization can never really be observed because there is no true extended wave behavior) has been reported by He and Maynard [53]. Disorder was introduced in a wire by either varying the size of periodically positioned small masses along a wire (alloy-type disorder) or their position itself (liquid-type disorder). The frequency response of the system, i.e. of a transverse wave generated in the wire, was then measured, where an additional electron-phonon interaction was simulated by means of a longitudinal strain in the wire. Studying the eigenfrequencies of the wire extended and localized states were clearly observed.

Concerning theoretical work the study of phonon localization was the first to follow the respective investigations of electronic systems [54, 55]. Using a field-theoretical formulation John, Sompolinsky, and Stephen [55] showed that a "phonon mobility edge" should occur above $d = 2$. A diagrammatic theory, based on a weak-localization, i.e. perturbational, approach to the localization of phonons, was presented by Akkermans and Maynard [56]. Although in spirit and technique very similar to the corresponding diagrammatic theory of the localization of electrons, there are important differences due to the absence of Fermi-Dirac statistics and because of the strong frequency dependence of characteristic quantities.

Localization of acoustic (sound) waves in a random array of hard scatterers in $d = 1$ [57] and in $d = 2, 3$ [58], has also been investigated within the self-consistent theory, Eq. (11). Agreement with the field-theoretical results [55] (which were derived for $d = 2 + \epsilon$) are obtained even in $d = 3$ where $\epsilon = 1$.

Localization of sound modes different from the conventional acoustical sound ("first" sound), namely, of "third" sound (i.e. surface modes in a superfluid such as ^4He) was also studied [59...61] using the self-consistent theory. It appears that in such a system the localization length can be continuously varied from practically infinity to a few millimeters. This would allow for investigations of localization in $d = 2$, unfeasible in electronic systems.

5 Other Localizing Media and Waves

Studying localization one usually considers the random scatterer to be uncorrelated. The more complicated situation where wave propagation and localization takes place in a random potential having a long range correlation has been investigated by John and Stephen [62]. Using an appropriate field-theoretical model they again find that for $d \leqslant 2$ all states are localized and that the mobility edge in $d = 2 + \epsilon$ is characterized by the same critical exponents as for spatially uncorrelated disorder.

Localization of waves in a fluctuating plasma was studied by Escande and Souillard [63]. They find that, in the absence of dissipation, density fluctuations in a plasma may lead to exponential localization of electron plasma waves. The corresponding localization transition is expected to be easily observable since the strength of the disorder can be readily varied.

Localization caused by surface roughness has been discussed in the context of electrons in thin films [64]. A conceptually similar but nonetheless different effect, namely, localization of surface plasmon polaritons (SPP) and its role in surface enhanced optical phenomena, was addressed by Arya, Su, and Birman [65]. After having been excited by a photon the SPP can propagate parallel to the metal surface but will be scattered elastically by spatial fluctuations in the dielectric function near the otherwise smooth metal-vacuum interface. Using the self-consistent theory of Vollhardt and Wölfle [27, 28] an equation for the renormalized diffusion coefficient is derived, showing that localization effects occur over a certain frequency range, in particular, if radiation losses are small.

The same approach has been employed to study scalar wave localization in a two-component composite [66] (see the remarks made in the context of acoustic effects). Localization is predicted if the impedance contrast of the medium (i.e. the ratio of the respective indices of refraction) exceeds a certain minimum value.

6 Localization and Magnetic Fields

In the case of normal impurity scattering weak localization is due to the constructive interference of waves on time-reversed paths. Therefore this effect is very sensitive to any kind of disturbance of time reversal invariance of the momentum states $\vec{k}$ and $-\vec{k}$. Such a perturbation is, for example, caused by a magnetic field. In its presence a state is no longer characterized by a momentum $\vec{k}$, but rather by the electromagnetic momentum $\vec{k} - 2e\vec{A}$. Here, $\vec{A}$ is the vector potential and the factor 2e (instead of simply e) is due to the correlation of *two* particles just as in superconductivity. If we now let $\vec{k}$ go into $-\vec{k}$, the momentum states, i.e. the paths 1 and 2 in Fig. 2, are no longer equivalent. Mathematically speaking this is a consequence of the fact that now the amplitudes A_1 and A_2 carry field dependent phase factors [10, 11], determined by the magnetic flux $\Phi = \oint d\vec{\ell} \cdot \vec{A}$, such that $A_1 \to Ae^{i\varphi}$, $A_2 \to Ae^{-i\varphi}$, where

$$\varphi = 2\pi \frac{\Phi}{hc/e}. \tag{14}$$

The magnetic flux is given by $\Phi = H \cdot S$, where H is the magnetic field and S is the area of the closed path in Fig. 2 (c = velocity of light). Since the motion of the particles is diffusive, S is given by $S = D_0 t$. The return probability W_H of a particle to its starting point in the presence of a magnetic field is again given by Eq. (5). One therefore obtains

$$W_H = 2|A|^2 \left[1 + \cos \frac{2eHD_0 t}{\hbar c}\right]. \tag{15}$$

The conductivity correction in the presence of a magnetic field, $\delta\sigma(H)$, is determined by the return probability W_H. The total change of the conductivity due to a magnetic field, $\Delta\sigma(H) = \delta\sigma(H) - \delta\sigma(0)$, therefore depends on the probability difference $W = W_H - W_{H=0}$, such that

$$\Delta\sigma(H) = -\int_{\tau}^{\tau_{in}} dt \frac{v_F \lambda_F^{d-1}}{(D_0 t)^{d/2}} \left[\cos \frac{2eHD_0 t}{\hbar c} - 1\right]. \tag{16}$$

In d = 2 Eq. (16) can be written as $\Delta\sigma(H) = e^2 F(x)$, where $x = 2eHD_0\tau_{in}/\hbar c$. The function F (x) has the limits

$$F(x) = \begin{cases} x^2, & x \ll 1 \\ \ln x, & x \gg 1. \end{cases} \tag{17}$$

For weak magnetic fields, $x \ll 1$, one therefore finds

$$\Delta\sigma(H) \sim H^2 \tau_{in}^2 \tag{18}$$

while stronger fields, $x \gg 1$, give rise to a logarithmic field dependence

$$\Delta\sigma(H) \sim \ln(H\tau_{in}). \tag{19}$$

In any case, $\Delta\sigma$ is always positive ($\Delta R < 0$), so the resistance *decreases* with increasing magnetic field ("anomalous magnetoresistance") [19]. The reason lies in the disturbance of the phase coherence by the magnetic field, leading to a weakening of the localization effects. The "critical" field H_c, determined by $x = 1$, at which the change from the H^2 to the ln H behavior occurs, depends on τ_{in} and thereby on temperature. At temperatures commonly used in experiments, H_c is of the order $H_c \approx 100 \dots 500$ Gauss ($\hat{=} 10 \dots 50$ mT). This should be contrasted with the classical result $\Delta\sigma(H)/\sigma_0 \approx -(\omega_L\tau)^2$, which is not only many orders of magnitude smaller but also has a different sign (ω_L is the Larmor frequency)! So we see that even very small magnetic fields have a drastic influence on localization.

Oscillation effects

As first observed by Altshuler, Aronov, and Spivak [67] the phase dependence of the electron wave function leads to a novel kind of quantum oscillation in the magnetoresistance of a multiply connected geometry, e.g. a torus made by wrapping up a thin, disordered metallic film (Fig. 8). The total change in phase, $\Delta\varphi_{loc}$, of the two oppositely traversed paths in the cylinder (Fig. 9(b)), is given by (14)

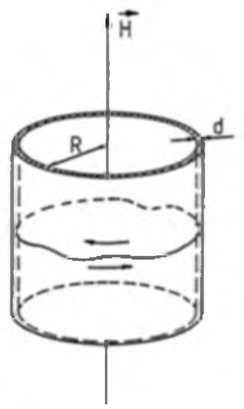

Fig. 8
Geometry used for detecting quantum oscillations in the magnetoresistance with period hc/2e.

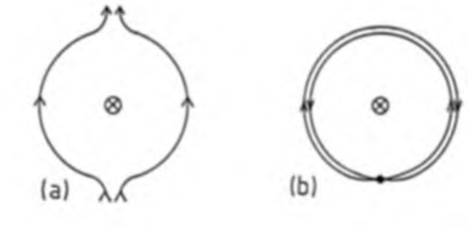

Fig. 9
Paths of interfering waves around screened magnetic field (⊗); (a) Usual Aharonov-Bohm effect, (b) geometry of Altshuler, Aharonov, and Spivak [67].

$$\Delta\varphi_{loc} = 2\varphi = 2\pi \frac{\Phi}{\Phi_0}, \tag{20}$$

where $\Phi_0 = hc/2e$ is the flux quantum known from superconductivity (although we are here in a normal-conducting situation!). The weak localization correction to the conductivity is thus given by a straightforward extension of Eq. (16), i.e. [10, 11],

$$\Delta\sigma(H) = \int_{\tau}^{\tau_{in}} dt \left[W_0 + 2 \sum_n W_n(t) \cos \left(2\pi n \frac{\Phi}{\Phi_0} \right) \right], \tag{21}$$

where the W_n are the return-probabilities for an electron after having traversed the loop n times. Clearly, $\Delta\sigma$ is an oscillatory function of the flux with periodicity Φ_0. This finding must be contrasted with the well-known Aharonov-Bohm effect (Fig. 9(a)) where electrons passing a coil enclosing a magnetic field, will only acquire a phase change of *half* the change given by Eq. (14). This is so because in the Aharonov-Bohm effect each electron only samples half the flux Φ, because it only passes through half the loop (Fig. 9(a)). This yields a total phase change of

$$\Delta\varphi_{AB} = 2 \cdot \frac{\varphi}{2} = 2\pi \frac{\Phi}{hc/e} \tag{22}$$

leading to oscillations of the flux with period $hc/e = 2\Phi_0$. While the Aharonov-Bohm effect in a cylinder geometry is similar to Dingle-oscillations [68], the effect predicted in [67] with period Φ_0 is reminiscent of Parks-Little oscillations [69] in a superconducting geometry.
To observe the effect it is important that the inelastic diffusion length $L_{in} = \sqrt{D_0 \tau_{in}}$ be larger than the circumference $2\pi R$ of the cylinder (Fig. 8), because otherwise the coherence is destroyed. (Note that both L_{in} and R can be much larger than the mean free path!).
The predicted oscillations were first measured by Sharvin and Sharvin [70] (Fig. 10), who found full agreement with theory. These experiments were done on thin Mg-films; other experiments used Li (where spin-orbit scattering is negligible) and also yielded very good agreement with theory [71].

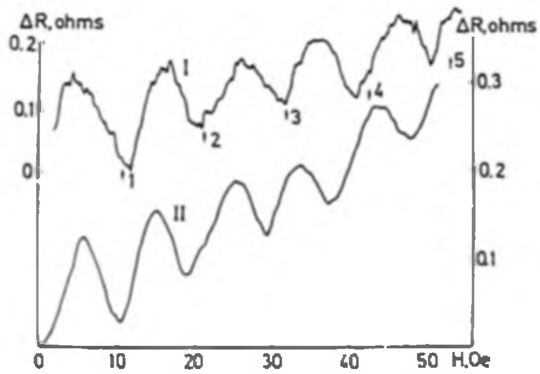

Fig. 10
Oscillations of the magnetoresistance of two different cylindrical Mg-films as measured by Sharvin and Sharvin [70].

7 Networks

In the attempt to reproduce the experiments of [70] other multiply-connected geometries than a single hollow cylinder where studied. This led to the investigation of networks of loops, e.g. of samples containing about $2.7 \cdot 10^6$ identical hexagonal loops forming a regular, two-dimensional honeycomb-network [72], or of ladders of 1000 little squares in series with 50 ladders in parallel [73]. The magnetoresistance of these new geometries were measured, and oscillations with period $\Phi_0 = hc/2e$ were found. A detailed theory of interference effects and quantum oscillations in the magnetoresistance of normal metal networks (loops, ladders, lassos, fractal networks, etc.) was worked out by Doucot and Rammal [74, 75] who also found remarkable agreement with the experiment (Fig. 11). This work is

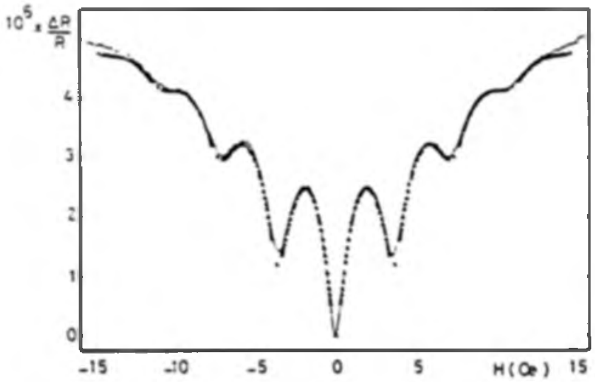

Fig. 11 Comparison between the theoretical results [74, 75] and experimental data [74] of Doucot and Rammal for magnetoresistance oscillations measured with a copper network with honey comb structure.

closely related to similar investigations of *superconducting* networks [76], where fascinating physics is known to occur (frustration, fractional number of flux quanta per unit cell of the network, fractal fine structure of the upper critical field line due to interference effects between adjacent loops, etc.). In contrast, static properties of normal-conducting networks do not show such a fine structure because of an inherent regularization of the otherwise complicated spectrum [74, 75].

8 hc/e Versus hc/2e Oscillations

Both from an experimental and theoretical point of view a single ring should be about the simplest geometry to observe the above-mentioned oscillations in the magnetoresistance. Experimentally, the opposite was true since, at first, the Φ_0-oscillations could not be found. Later, both $2\Phi_0 = hc/e$ and $\Phi_0 = hc/2e$-oscillations were detected in individual, micron size, normal metal rings [77]. The different temperature and field dependence clearly distinguishes between the two effects and their physical origin. At low fields the localization induced hc/2c-effect is seen, while at higher fields, when localization is suppressed, the hc/e-effect is visible. Most recently both types of oscillations were also measured in samples made up of N such rings in series [78]. It was found that, on averaging, the amplitude of the hc/e oscillations showed a $1/\sqrt{N}$ decrease, while the hc/2e-effect was independent of N. This has also been verified theoretically [79]. It clarifies the role of *ensemble averaging* in calculating corrections to the conductivity. This kind of averaging is canonically employed in the framework of weak localization (yielding hc/2e-oscillations) but *not* in the calculation of the transmission coefficient [80] in metal rings, where only the hc/e-effect is found. So, to obtain the hc/2e-effect, ensemble averaging is necessary.

9 Mesoscopic Systems and Universal Fluctuations

As an unexpected byproduct, the investigations of quantum oscillations in (sub-) micron structures ("mesoscopic" systems) led to the discovery of anomalously large, *universal* fluctuations [81 ... 83]. These fluctuations, known from experiment [84] and numerical simulations [85], are not due to time dependent noise or finite-size effects. They are a consequence of quantum interference, are reproducible and occur if the temperature is low enough such that the inelastic diffusion length L_{in} exceeds the sample dimension. In this case the conductance of a small sample shows fluctuations as a function of magnetic field, chemical potential, or impurity configuration whose r.m.s. value is approximately e^2/h, *independent* of sample size, i.e. is universal. The effect depends only weakly on dimensionality and the strength of (weak) disorder and is, of course, much larger than expected classically. It has clearly been observed in the experiment [86]. As discussed by Lee and Stone [82] this behavior is compatible with one-parameter scaling of Anderson localization [6] where in the scaling regime the conductance is essentially length independent and of order e^2/h such that its fluctuations must also be large and scale independent. The quantum interference of randomly diffusing electrons, which leads to the large fluctuations, implies an extraordinarily large sensitivity of the conductance on the impurity configuration. Indeed, the displacement of a *single* impurity by only λ_F (de Broglie wave length) affects essentially all quantum mechanical paths and hence changes the conductance by a universal, i.e. sample size independent amount [87, 88]. There are many other unusual phenomena (e.g. asymmetries [89] in the magnetoconductance [90], ability to rectify alternating currents [89]). A comprehensive discussion of the effects of finite temperatures, interactions, and magnetic fields on the universal conductance fluctuations, as well as of the physical assumptions underlying the ergodic hypothesis has been presented by Lee, Stone, and Fukuyama [91], who also relate theory to experiment.

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