On problems of periodic homogenisation of highly heterogeneous media

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Abstract

This thesis focuses on problems of periodic homogenisation of families of elliptic and parabolic problems that are constituted by non-linear, monotone operators. A key difficulty in these problems is the simultaneous presence of both fast and slow domains which characterise highly heterogeneous materials. In analytical terms, one finds that classical compactness theorems are ruled out from the very outset making compensation techniques necessary. This works suggests monotonicity as a method of compensation; however, this requires not only on having a refined machinery of two-scale convergence available but also the availability of certain recovery sequences. We sketch a framework which allows to understand the compensation technique as a modified completion procedure provided the existence of recovery sequences can be established. Unfortunately, this latter question could not be resolved in this work, besides its vital importance to our approach.

Diese Dissertation behandelt Probleme der periodischen Homogenisierung welche durch elliptische und parabolische, nicht-lineare, monotone Operatoren gegeben sind. Eine Hauptschwierigkeit besteht in der gleichzeitigen Gegenwart von schnellen und langsamen Gebietsbereichen welche charakteristisch ist für stark heterogene Materialien. Aus analytischer Sicht zeigt sich, dass klassische Kompaktheitsresultate hierbei nicht anwendbar sind, so dass Kompensationstechniken grundlegende Bedeutung zukommt. Diese Arbeit erarbeitet Monotonie als eine solche Technik, jedoch benötigt diese eine ausgereifte Maschinerie an Zweiskalenkonvergenz und sogenannter Wiederherstellungsfolgen. Wir skizzieren einen Rahmen in welchem unsere Kompensationstechnik als eine modifizierte Vervollständigung aufgefasst werden kann, sofern die Existenz von Wiederherstellungsfolgen gesichert ist. Leider bleibt deren Existenz im Rahmen dieser Arbeit trotz deren Wichtigkeit offen.

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Fifth, I want to express my high appreciation for MOPS, my friends and colleagues at the Institute for Mathematics at the University of Augsburg who accompanied me through my undergraduate, graduate and PhD studies. I came to cherish the stimulating discussions with Ingo Blechschmidt, Ferdinand Eitler, Harsha Hutridurga, Simon Kapfer, Sina Reichelt, Jonas Tölle and David Wiedemann who impressed me deeply, personalising true scholarship and setting shining examples what it means to strife for knowledge. Moreover, I want to thank all my other co-workers and my students in particular for sharing their time, their valuable insights and their inspiration with me.

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Notation and conventions

- $\mathbb{J} \subset (0, \infty)$ is a non-void, countable set fulfilling $0 \in cl(\mathbb{J})$. Notably, $\mathbb{J}$ and its closure are directed sets by the ordering $\leq$.
- $\varepsilon \to 0$ stands for $\varepsilon_n \overset{n\to\infty}{\longrightarrow} 0$ with $(\varepsilon_n)_{n\in\mathbb{N}} \subset \mathbb{J}$ being a null sequence which contains all elements of $\mathbb{J}$.
- Throughout, $(u_\varepsilon)_{\varepsilon\in\mathbb{J}} \subset S$, stands for a sequence indexed by $\mathbb{J}$ possessing elements in some given set S. Of course, one could equivalently write $(u_{\varepsilon_n})_{n\in\mathbb{N}}$; however, large parts of the existing literature on two-scale convergence prefer the first way of notation.
- Domain always means a connected and open subset of $\mathbb{R}^d$ in the Euclidean topology.
- (Strong) Lipschitz regularity or $C^{0,1}$-regularity of a domain in $\mathbb{R}^d$ mean that its boundary $\partial\Omega$ is a $C^{0,1}$-regular hypersurface. Note that for bounded domains Ω this condition is strictly stronger than $\overline{\Omega}$ being bi-Lipschitz homeomorphic to a $[0,1]^d$.
- $\mathcal{Y} := \mathbb{R}^d/\mathbb{Z}^d$ is the flat torus, tacitly equipped with the quotient topology.
- $\mathcal{L}(\mathbb{R}^d)$ denotes the σ-algebra of Lebesgue measurable sets in $\mathbb{R}^d$. Likewise, $\mathcal{B}(\mathbb{R}^d)$ is the Borel measurable counterpart.
- $\mathcal{L}(\Omega)$ is the σ-algebra of Lebesgue measurable sets in Ω which stems from $\mathcal{L}(\mathbb{R}^d)$ by restriction. One obtains $\mathcal{B}(\Omega)$ from $\mathcal{B}(\mathbb{R}^d)$ analogously.
- $\mathcal{L}(\mathcal{Y})$ and $\mathcal{B}(\mathcal{Y})$ stem from their $\mathbb{R}^d$-counterparts via pushforward along the projection $\pi_{\mathcal{Y}} : \mathbb{R}^d \longrightarrow \mathcal{Y}$.
- $\Omega_\varepsilon^+, \Omega_\varepsilon^-$ are auxiliary domains of a given domain. They can be defined for arbitrary sets in $\mathbb{R}^d$ though. $\Omega_\varepsilon^\pm$ denotes both families of domains.
- $\mathcal{T}_\varepsilon^*$ is the unfolding operator which stems from the domain decomposition map $\mathcal{T}_\varepsilon$. The latter is successively defined on $\mathbb{R}^d$ first and extended afterwards to auxiliary domains of domains Ω with compact, Lipschitz-regular boundary.
- $s \overset{\text{a.e.}}{\in} S$ denotes 'for almost all $s \in S$'.
- For $S \subset \mathbb{R}^d$ and $t \in \mathbb{R}$, $S_t := (0,t) \times \mathbb{R}$, in particular, $\Omega_t = (0,t) \times \Omega$.
- $u_\varepsilon \overset{2w}{\rightharpoonup} u, u_\varepsilon \overset{2s}{\longrightarrow} u_0$ stand for weak and strong two-scale convergence as $\varepsilon \to 0$.
  Important: In this context, notations like '$u_\varepsilon(x) \overset{2w}{\rightharpoonup} u_0(x)+v(x,y)$' indicate dependencies of variables of the functions involved. Yet, such convergence statements refer to convergence in $L^p(\Omega\times\mathcal{Y})$ and not to be understood in a point-wise sense. In the given instance, the fact that u_0 is only x-dependent and independent of y is emphasised.

Foreword

This thesis focuses on the theory and application of *periodic homogenisation*, one of the techniques which constitute *multi scale analysis*, the puzzle of understanding the interplay of processes of differing length scales.

As an illustration, consider ferromagnetism: by the Bohr–van Leeuwen theorem, ferromagnetism is quantum mechanical in origin, yet its effects are visible even in astrophysics. In other words, it is a phenomenon which originates on a length-scale well below 10^{-9} metres and is felt on scales well above 10^6 metres. The plain question is: what is happening on the quantum mechanical scale, and how would this process transit fifteen magnitudes?

Tremendous obstacles must be expected as soon as elementary explanations fail. Besides other shortcomings, considering molecules as magnetic dipoles yields magnetic energies of 10^{-23} Joules, disqualifying any explanation attempt for an everyday magnet at room temperature. Consequently, another interaction must be responsible and most probably, it is very subtle. Moreover, the abundance of roughly $N_A \approx 10^{23}$ molecules imposes another problem: not only must one specify a process on the molecular scale with promising effects on the macroscale, but in addition, a machinery of scale transition which can transit scales of magnitudes credibly. Such machineries are roughly known as *averaging*, however, averaging *suitably* turns out to be a immensely difficult.

To the author's best knowledge, there is neither a unified theory nor a supreme theorem available at present which grasp the matter in its entirety. Rather, different problems require different approaches to explanations, and periodic homogenisation is one of them.

In this thesis periodic homogenisation is presented in two-fold fashion. On the one hand, we will track its development in geometric and functional analytical terms. On the other hand, applications stemming from non-linear partial differential equations would require a machinery helping to disclose the actual mechanics at work. As one may easily anticipate, high contrast media underline this need for a more refined framework. To this end, we will first revise the conventional machinery of periodic homogenisation. Second, and as an application, we will turn to a family of stationary quasi-linear problems generated by monotone operators. The corresponding limiting procedure will exhibit a certain loss of monotonicity yielding pseudo-monotonic behaviour. As a third step, we will sketch an extension of the limiting machinery to a more abstract set-up. Assuming the existence of so-called recovery sequences, this machinery is even be flexible enough to handle the parabolic analogue, as well, a task that is done as a forth step.

For completeness's sake it should be mentioned, that this thesis was originally presented in April 2019 but the existence proofs for recovery sequences turned out to be wrong. Since this grave error affected a major step in this thesis's progress, a refurbishing was in order to adapt and weaken the original results. Unfortunately, the author's attempts to fix the wrong proofs were unsuccessful.

I. An introduction to periodic homogenisation and two-scale convergence

This chapter serves to introduce fundamentals of two-scales converges. Note that the first section §1 is heuristic in nature; we will give rigorous definitions later on.

§1. Motivation: the origins of periodic homogenisation

Let us introduce periodic homogenisation, two-scale convergence and some related questions by considering a model problem. On a open and bounded set $\Omega \subset \mathbb{R}^d$ one considers a family of problems, namely

$$-\nabla \cdot \left[A\left(x, \frac{x}{\varepsilon}\right) : \nabla u_\varepsilon\right] = f \qquad \text{with} \quad u_{\varepsilon|\partial\Omega} \equiv 0 \tag{1.1a}$$

for every $\varepsilon \in \mathbb{J}$ with $\mathbb{J} \subset (0, \infty)$ being a countable set with $0 \in \overline{\mathbb{J}}$. In structural terms, we require that

$$(x, y) \longmapsto A(x, y) \in \mathbb{R}^{d\times d}_{spd} \quad \text{is periodic in its } y\text{-variable.} \tag{1.1b}$$

In addition, certain regularity assumptions are imposed on f and A. For the moment, let us assume that both A and f are continuous and bounded. Furthermore, the values of A are required to be symmetric, positive definite matrices which are uniformly bounded from above and uniformly bounded away from zero.

§1.1. Related real-world problems

There are numerous problems which fall into the framework of (1.1). In general, the periodic argument of A allows to model a bulk Ω which is made up by a periodic and very fine mixture of different materials. As it turns out, such configurations are not difficult to find; in fact, they are ubiquitous. Let us describe several instances.

Bone of vertebrates

Cancellous bone, also known as *spongy bone* or *trabecular*, is the internal tissue of skeletal bone of vertebrate, and thus a very common and widespread structure. Roughly, it is a lattice structure that is largely made of calcium and phosphate, forming very fine cavities. These cavities host haematopoiesis, the generation of blood cells. Moreover, they are moderating fundamental metabolic activities, in particular those which are linked to calcium or phosphate, a fact which is made possible by the very fine structure of the spongy bone. Its very large surfaces allows considerable extents of blood vessels to interact with the surrounding bone structure, a process which is indispensable for maintaining precise levels of calcium, which in turn is vital for the cardiovascular system in total. More drastically speaking, severe disruptions of one's calcium levels will lead to cardiac arrest, a condition not deemed desirable.

Returning to spongy bone itself, note its mediating character. On the one hand, bone has obvious mechanical duties, literally including the backbone itself. On the other hand, it has an important metabolic task to fulfil. Both aspects are structurally opposing each other, the first calls for solid structures, the second favours loose tissue. Consequently, a good compromise is necessary and obviously, evolution has managed to provide a powerful solution by employing a very clever geometry which yields both surface and stability.

For much more information on the genesis and functions of bone, see [BRM08], a standard references on bone biology. In this context, monographs on histology like [LF99; Küh14] can be very educational. Finally, let us note that trabecular bone and its functions are a matter of ongoing research as is witnessed by [Lee+07] and the reference therein.

Concrete and its reinforcement

Historically, the idea of mixing different materials for construction purposes can be traced back to the late neolithic age, at least. Whereas clay bricks have been in use for roughly 10000 years from now on, the concept of reinforcing primitive air-dried bricks with straw is obviously present in the second book of Moses. There, Israeli slaves are required to produce reinforced bricks for the Egyptians; For more information on bricks, their history and their present day use, see [Min06; Min09].

Later, air-dried bricks were mostly superseded by fire dried bricks which themselves lost their leading role to the family of concrete. Of course, all sorts of concrete are composite materials: in practice, the individual ingredients are not only very diverse, but ironically only of very limited use to construction purposes by themselves. In simple terms, the resulting mixtures are always made of aggregate bound by cement and further optional additives. However, the precise composition of the ingredients is vital for the resulting concrete's properties. In fact, concrete is indispensable for construction purposes nowadays and thus, it is subject to extreme wear, severe strain and ever-growing demands for security precautions like fire prevention, noise prevention, prevention of chemical degradation, thermal protection and so on. Evidently, there has always been an interest in enhancing concrete for numerous, specific purposes: steel suggests itself for increasing concrete's tensile strength, and though it is widely used nowadays, the resulting *reinforced concrete* is a delicate matter. Cement contains water which may corrode steel and is subject to degradation by chemical reactions with air itself. Is it possible to guarantee the stability of the resulting mixture over a long period of time, like 50 years and more? Moreover, steel, cement and aggregate vary significantly in relevant physical processes such as heat conduction: how can one manage to arrive at a feasible and reliable mathematical description which is fit for carrying out simulations of large constructions subject to significant security demands? For an engineer's presentation, we refer to [Gjo09]. Mathematically rigorous work can be found in [Pet07; Pet+08; PB09] and the references therein.

Flows through soil

A third instance is formed by problems which stem from flows through soil. Besides rather innocent examples as water flow through soil, its most relevant motivation stems from ecological and economical questions. A first application stems from oil production: many oil reservoirs are naturally fractured, making a sufficiently good understanding the flow in such fractured containers necessary. Such matters motivated a lot of research initially, we name [Hor97] as a classical resource which gathers and refers to significant amounts of work. A more recent and highly recommendable exposition on flows through porous media is H. Hutridurga's thesis [Hut13], which considers very sophisticated related questions and includes numerical computations as well, thus giving a full presentation of the subject.

Besides its economical importance, ecological questions arose, too. Over time economic pressure to exhaust highly fractured oil reservoirs has increased considerably since more conventional reservoirs are either depleted or do not offer rising revenue. Thus, more refined technical means are requested to handle such delicate reservoirs economically. However, one of the most widespread techniques for such tasks, *hydraulic fracturing*, which is often referred to as *fracking*, has stirred up a considerable debate on its possible contamination of ground water through the fracking fluid injected into the soil. Even without political controversies, developing both feasible and sensible computational models of fluid flow in soil is quite difficult. Yet, the presence of health concerns drastically increases the demands on the credibility of the simulation in use; in particular, the policy of sitting things out is hardly satisfying nor is it helping anyone. Of course, it is hard to underestimate the importance of simulations in the presence of a heated political debate garnished by economical interests that affect all of society.

Another example of fluid flow, especially on a longer time scale, comes from nuclear waste management and its assessment of security. For instance, let us consider the deep geological repository for high-level nuclear waste near Zheleznogorsk in the Nizhnekansky rock massif in central Russia's Krasnoyarsk region. Very involved simulations are trying to evaluate the risk of the high-level wastes diffusing into the nearby Yenisei river; we refer to [GPK18; SK12] and the references therein. Evidently, a profound understanding of flow in soil is desirable here, too. As far as the author knows, conclusive simulations in this context are still lacking, though.

A closing word in mathematical terms

Let us emphasise that (1.1) *relates* to the problems just described – it does not offer a complete, perfect and exhaustive description. Naturally, this holds for other problems motivated by different contexts, as well. In general, mathematical modelling rather contributes to existing knowledge by refining ideas and concepts or yielding sensible simulations; rarely is it possible to describe real-world problems without simplifications.

In this context, let us stress that (1.1) relies on the idealisation that the problem at hand is spatially aligned in periodic fashion. We shall accept this assumptions for two reasons: first, the x-dependence of $A(x, y)$ allows to weaken the periodicity assumption considerably, such that a *locally periodic* configuration is at hand.[1] Secondly, local periodicity assumptions themselves are widespread and have brought about tremendous success. As a shining example consider solid state physics where key results like Bloch's theorem rest on the assumption that the crystal at hand is made up of atoms which are arranged in periodical manner or at least in near-perfect periodic manner, see [AM76] for a thorough presentation.

[1] Of course, periodic media are locally periodic, as well.

§1.2. Job description: homogenisation and periodic homogenisation

Let us return to more mathematical matters. Recall that (1.1) resembles a family of problems indexed by $\varepsilon \in \mathbb{J}$. The following questions on this family constitute the task of *homogenisation.*

a) Are there reasonable technical assumptions guaranteeing the solvability of (1.1) in a meaningful way?

b) If so, are there suitable conditions such that a (sub-)sequence of solutions $(u_\varepsilon)_{\varepsilon\in\mathbb{J}}$ converges to some limit function u_0 for $\mathbb{J} \ni \varepsilon \longrightarrow 0$ in a sensible topological vector space?

c) If so, is it possible to relate u_0 to the sequence of original problems? Ideally, u_0 would solve a 'limit problem' which can be linked to (1.1); ideally, by quantitative estimates.

Periodic homogenisation is a specialisation of homogenisation: it relies on a separation-of-scales and a periodicity assumption. More specifically, one requires that the problem at hand can be described well by macroscopic and microscopic variables, the latter being periodic. In this context, the notions macroscopic and microscopic express the optimism that a sufficiently accurate model can be obtained by using two families of variables, e.g. the *macro-variables* x and the *micro-variables* y, which are required to differ in length scale significantly. For instance, a bone sample of some centimetres may be considered macroscopic, i.e. x suffices to describe the sample. In contrast, the aforementioned air cavities inhabit the sub-millimetre scale and thus are deemed microscopic, i.e. y provides us with a sensible description. All this requires a gap of significant intermediate length scales.

A key motivation of homogenisation is the fact that the interplay of processes on a microscopic level, may have a strong influence on the macroscale and vice versa. Yet, to find a precise and appropriate description of this interplay of scales is extremely challenging.

Besides numerical simulations becoming increasingly realisable, the interplay of macroscale and microscale mechanisms is often eluding brute force simulations quite subtly. As a very simple example, consider simulating heat conduction of a bone tissue with its cavities being filled with water. Since we will return to this example later on, let us suffice to say that even for very small samples such simulations are very demanding. However, even if the computational costs were moderate, and that is effectively never the case, then the resulting output would still yield the problem of handling an overflow of information. Even a full simulation would *still* lead to the problem of isolating the relevant information reliably. Paradoxically, the portions of information that are eventually disposable are most expansive to obtain.

In this state of affairs, it is precisely homogenisation which steps up to avoid generating 'too much' information by reducing the problems at hand to suitable limit problems which hopefully carry the relevant information and not much more at reasonable costs.

Homogenisation and the importance of highly heterogeneous media

Homogenisation is receiving significant attention since the 1970s and has made remarkable progress so far. Referring to the literature below, we quote two key results whose precise formulations will be given later on:

a) For $\varepsilon \longrightarrow 0$, $u_\varepsilon \longrightarrow u_0$ holds in $H^1_0(\Omega)$ and u_0 solves a limit problem which is connected (1.1). However, establishing this link does require more refined tools than merely weak convergence. It is among the merits of two-scale convergence to identify so-called correctors, which are indispensable for formulating the limit problem.

b) Depending on the original problem, two cases need to be distinguished: standard media and highly heterogeneous media. The latter implies severe losses of compactness.

Let us address the matter of standard media and highly heterogeneous media first. There are no conclusive criteria to decide whether a given composite is highly heterogeneous or not. Rather, it is a question of material parameters and the amplitude of their oscillations. Let us return to the example of heat conduction in bone marrow: roughly, the calcium salts forming the collagen cavities conduct heat fairly well, with calcium's thermal conductivity being roughly $c_C \approx 201W/(mK)$. In contrast, water's thermal conductivity does not exceed $c_W \approx 0.5W/(mK)$. The ratio of $c_C/c_W \approx 10^2$ can be considered as a high contrast which increases the difficulties of a brute force simulation. To be more specific, besides the challenge of handling a macroscopic sample with microscopically oscillating properties, one must take care of the oscillations' considerable amplitudes. In fact, the latter are quite an obstacle for the (numerical) condition numbers of the corresponding numerical quadrature will be severely flawed. Moreover, this aspect cannot be ignored for the resulting stiffness matrices inherit coefficients of ill-conditioned origin, a state that is hardly changed when considering non-linear problems and their linearisations. Coming to the point, it is the very matter of condition that neither more numerous nor more accurate calculations are a remedy. Instead, one has to look for a reformulation of the problem that is better behaved.

Let us note that the treatment of highly heterogeneous media is very appealing for modelling purposes, though. A great lot of composite materials exploits the very fact that the ingredients themselves are behaving very differently from each other. However, in mathematical terms, highly heterogeneous materials – also known as the the *high contrast regime* – suffer from compactness defects caused by very weak a priori bounds, a fact leading to drastic convergence defects forming the fundamental difficulty of our work. Besides significant applications remaining poorly understood, a refinement of the existing methods seems necessary in this respect.

Homogenisation versus periodic homogenisation

Naturally, the periodicity assumption is a significant restriction such that periodic homogenisation is far from being a universal tool. However, there is little risk in claiming that presently there is not a single homogenisation technique which can handle all problems alone. Periodic homogenisation does have its drawbacks and other techniques are more fit to handle certain problems; as alternatives, let us name *stochastic homogenisation*, we refer to [GNO14] and the related works and *numerical homogenisation*, referring to [PVV18; Pet16] and the literature named therein. A third alternative originally stemming from the calculus of variations is homogenisation by the means of Young measures and *H-measures* in particular; the monograph [Tar10] is an extensive resource on this matter but as far as the author knows H-measures still await their numerical realisation.

G. Nguetseng not only contributed greatly to periodic homogenisation, see theorem §7.1 below, but also provided immense extensions in [Ngu03] and [Ngu04]. There, homogenisation procedures are formulated as *homogenisation structures* in highly abstract form by the help of C^*-algebras, yielding the notion of Σ*-convergence*. It must be mentioned, that Σ-convergence forms an umbrella for both deterministic and stochastic methods, thus providing an impressive unification of the subject. Unfortunately, G. Nguetseng's work seems to be little-noticed at the time of this writing as related numerical implementations also await their realisation. Nevertheless, analytical works employing homogenisation structures have been published, for instance, we refer to [HHR17] where Σ-convergence is used for homogenisation along flows of given vector fields. Most admirably, Nguetseng's school in Cameroon has flourished against great odds, we cite the collaborations [NW07; NSW10; Wou10; DW15; DT17] as representatives for the impressive theoretic works developed around related questions like *almost periodic homogenisation.*

Coming back to periodic homogenisation, one encounters an ironic situation: the prerequisite of periodic homogenisation to have a geometric description of the problem at hand in both macroscopic and microscopic terms is boon and bane: it allows tackling problems whose geometry is both critical and known a priori. Returning to bone marrow again, it is well-known that the lattice around the cavities is aligned along sensible axes of physical stress such that geometry directs stability requirements, thus contributing to the femur being stable. Another instance that makes critical use of geometry are reactions on interfaces. We do not elaborate on this matter but refer to the research work of M. Neuss-Radu, M. Gahn and their collaborators, naming [MN10; Neu14; GNK16; GNK18] as related instances.

§1.3. Outlook

This thesis aims at applying a well-established technique of non-linear analysis, the method of monotone operators, to quasi-linear problems made up by periodic or locally periodic high contrast media. More specifically, we start with the examination of an elliptic problem in the framework of monotone operators. Realising that significant amounts of compactness results turn invalid, in particular the Rellich–Kondrachov embedding, we are in need of an alternative method in order to carry out certain limiting procedures successfully. A close examination of the latter underlines the importance of an improved understanding of the two-scale method. We will start to consider parts of the two-scale method as a completion procedure similar to the well-known Cauchy completion of metric or uniform spaces. It will turn out, that the availability of recovery sequences will be a most decisive aspect. This work will outline certain peculiarities in establishing the existence of such sequences that could not be overcome by the author. Consequently, we will need to assume the existence of recovery sequences.

Assuming to have sufficiently many recovery sequences at hand, we will turn to investigate the natural evolution problem counterpart of the foregoing elliptic case. Again, high contrast media defects foil the use of classical methods, most notably the Aubin–Lions–Simon lemma. In addition, two-scale convergence is intolerant to lacking spatial regularity and therefore, time derivatives are basically unusable, a state of affairs which is hardly helpful for limiting procedures. In sum, the transfer of elliptic methods to a parabolic set-up is not straightforward but there are conventional means for compensation, in particular Steklov averages.

Literature on periodic homogenisation and two-scale convergence

Let us complement our overview on the literature a little more. Homogenisation has received considerable attention for decades. Important results were accomplished by the end of the 1970s, see [LBP78] as a classical monograph, mainly on linear problems. However, also non-linear examples were under consideration, see [MB82]. Note that the highly famous and now classical device of *compensated compactness* originated from this context in [Mur78]. Throughout the 1980s, research on homogenisation thrived leading to [Ngu89] and [All92] as foundational works on two-scale analysis.

At the same time and throughout the 1990s, homogenisation flourished remarkably with numerous works being published on a multitude of problems. Let us name the monographs [OSY92; ZKO94; Hor97; CP99; CD99] which gather large amounts of research on homogenisation and modelling problems and similar questions. In addition, the 1990s witnessed the emergence of periodic unfolding, culminating in [CDG02] and [CDG08].

Excluding elasticity theory for a moment, one may claim that by the end of the 2000s works like [Pet07; PB08] had complemented the understanding of linear equations and systems, including the highly heterogeneous type. As a result, large portions of the elliptic and parabolic linear cases were understood sufficiently well, a fact which intensified the treatment of non-linear problems. This endeavour was advanced significantly by [Vis06] and [Vis07b] which enhanced two-scale techniques with classical tools from non-linear analysis such as monotonicity methods and compensated compactness methods. Among the pinnacles of two-scale convergence and periodic homogenisation is [Vis07a] which successfully carried out the periodic homogenisation of a full Stefan problem in standard media. Thus, one may consider standard media to be dealt with very successfully, even in the non-linear frame.

Of course, other non-linear problems received considerable attention, as well: [Ols08] focused on monotone operators in connection with a multitude of differing homogenisation problems. Very important contributions to elasticity theory were established in [Neu10]. [Sch13] is directed towards shape optimisation, whereas biological models were at the focus of [Gra13].

However, the treatment of non-linear, highly heterogeneous problems did not receive too much attention. Very successful in the treatment of highly heterogeneous media was S. Reichelt, see [Rei15] which contains a very thorough overview of the foregoing two-scale literature, too.

Let us conclude with referring to the works of A.-L. Dalibard whose works [Dal06; Dal09] concentrate on scalar conservation laws, a matter that can be deemed to possess maximal difficulty without hesitation.

§2. Periodic homogenisation and two-scale convergence

Periodic homogenisation of (1.1) requires the notion of two-scale convergence, let us recall the original definition from [Ngu89] and [All92] based on two-scale test functions. Throughout, we will write $\mathcal{Y} := \mathbb{R}^d/\mathbb{Z}^d$ for the flat torus such that a $\mathbb{Z}^d$-periodic map $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ is in one-to-one correspondence with a map $\widetilde{f} : \mathcal{Y} \longrightarrow \mathbb{R}$. Nevertheless, periodicity and regularity are subtly linked to each other and we will return to this aspect later. Throughout, $\Omega \subset \mathbb{R}^d$ is assumed to be a domain, that is an open and connected subset.

Definition §2.1: Two-scale convergence (original form)

A function $\psi \in L^2(\Omega \times \mathcal{Y})$ is an admissible two-scale test function if it suffices

$$\int_{\Omega} |\psi(x, x/\varepsilon)|^2 \, dx \xrightarrow{\varepsilon \to 0} \iint_{\Omega \times \mathcal{Y}} \psi(x, y) \, dy dx. \tag{2.1}$$

A sequence $(u_\varepsilon)_{\varepsilon \in \mathbb{J}} \subset L^2(\Omega)$ is two-scale convergent to $u_0 \in L^2(\Omega \times \mathcal{Y})$ if

$$\int_{\Omega} u_\varepsilon(x) \psi(x, x/\varepsilon) \, dx \xrightarrow{\varepsilon \to 0} \iint_{\Omega \times \mathcal{Y}} u_0(x, y) \psi(x, y) \, d(x, y) \tag{2.2}$$

holds for all admissible two-scale test functions $\psi \in L^2(\Omega \times \mathcal{Y})$. In this case we write '$u_\varepsilon \xrightarrow{2w} u_0$ in $L^2(\Omega \times \mathcal{Y})$'.

The above definition easily adapts to $p \in [1, \infty]$ as it is not restricted to Hilbert spaces. Nevertheless, the question which $\psi \in L^2(\Omega \times \mathcal{Y})$ are actually admissible is subtle. It was shown in [All92, Ch. 5] that one cannot expect all $\psi \in L^2(\Omega \times \mathcal{Y})$ to be suitable by constructing a pathological counter-example.

§2.1. Two-scale convergence and standard media

The following result is from [All92, Thm. 2.3] and employs a subspace of functions with constants factored out: $L^p\left(\Omega; W^{1,p}(\mathcal{Y}_1)/\mathbb{R}\right) := \left\{ f \in L^p(\Omega; W^{1,p}(\mathcal{Y}_1)) : \forall x \overset{\text{a.e.}}{\in} \Omega : \int_{\mathcal{Y}} \widetilde{f}(x, y) \, dy = 0 \right\}$.

Theorem §2.1: Allaire's periodic homogenisation theorem for standard media

Let $f \in L^2(\Omega)$ and assume $A : \Omega \times \mathcal{Y} \longrightarrow \mathbb{R}^{d\times d}$ to yield matrices which are symmetric, uniformly positive definite and uniformly bounded. Then, for every $\varepsilon \in \mathbb{J}$ (1.1) is uniquely solvable by a function $u_\varepsilon \in W_0^{1,2}(\Omega)$ and the sequence of solutions $(u_\varepsilon)_{\varepsilon\in\mathbb{J}} \subset W_0^{1,2}(\Omega)$ is bounded independently of $\varepsilon \in \mathbb{J}$ such that for a subsequence again denoted by $\varepsilon \in \mathbb{J}$ we have

$$u_\varepsilon \longrightarrow u_0 \qquad \text{in } W_0^{1,2}(\Omega), \text{ and} \tag{2.3a}$$

$$\nabla u_\varepsilon \xrightarrow{2w} \nabla_x u_0 + \nabla_y u_1 \qquad \text{in } L^2(\Omega \times \mathcal{Y}; \mathbb{R}^d) \tag{2.3b}$$

for a y-independent function $u_0 \in W_0^{1,2}(\Omega)$ and $u_1 \in L^2(\Omega; W^{1,2}(\mathcal{Y})/\mathbb{R})$. Finally, if the coefficients of A are admissible two-scale test functions, too, that is they fulfil

$$\lim_{\varepsilon\to 0} \int_\Omega \left\| A\left(x, \frac{x}{\varepsilon}\right)\right\|^2 dx = \iint_{\Omega\times\mathcal{Y}} \|A(x,y)\|^2 \, d(x,y), \tag{2.4}$$

then the following limit problem is solved by (u_0, u_1):

$$-\nabla_x \cdot \left[\int_{\mathcal{Y}} A(x,y) : \left[\nabla_x u_0 + \nabla_y u_1\right] dy\right] = f \qquad \text{in } \Omega, \tag{2.5a}$$

$$-\nabla_y \cdot \left[A(x,y) : \left[\nabla_x u_0 + \nabla_y u_1\right]\right] = 0 \qquad \text{in } \Omega \times \mathcal{Y}. \tag{2.5b}$$

Proof. To begin with, solvability of (1.1) for all $\varepsilon \in \mathbb{J}$ is clear due to Lax–Milgram. A bound on $\|u_\varepsilon\|_{W_0^{1,2}(\Omega)}$ follows from

$$1/c_A \|u_\varepsilon\|^2_{W_0^{1,2}(\Omega)} \le \|A\nabla u_\varepsilon \cdot \nabla u_\varepsilon\|_{L^1(\Omega)} = \|f u_\varepsilon\|_{L^1(\Omega)} \le c_P \|f\|_{L^2(\Omega)} \|u_\varepsilon\|_{W_0^{1,2}(\Omega)} \tag{2.6}$$

employing the coercivity constant $c_A > 0$ and the Poincaré lemma's constant $0 < c_p < \infty$ being independent of $\varepsilon \in \mathbb{J}$. Thanks to the bound, (2.3a) follows immediately. (2.3b) is a two-scale refinement of the former and due to [Ngu89; All92] which we will proof later on in theorem §7.1 and theorem §7.2. Thanks to (2.4), one can pass to the limit $\varepsilon \longrightarrow 0$ in the weak form of (1.1) leading to

$$\iint_{\Omega\times\mathcal{Y}} A(x,y) \left[\nabla_x u_0 + \nabla_y u_1\right] \cdot \left[\nabla_x \varphi_0 + \nabla_y \varphi_1\right] dydx = \iint_{\Omega\times\mathcal{Y}} f\varphi_0 dydx \tag{2.7}$$

for all $(\varphi_0, \varphi_1) \in W_0^{1,2}(\Omega) \times L^2(\Omega; W^{1,2}(\mathcal{Y})/\mathbb{R})$. Integration by parts yields (2.5a) if one chooses $\varphi_1 = 0$ and (2.5b) for $\varphi_0 = 0$. ∎

Let us digress on (2.5) briefly, which is often referred to as the decoupled problem. First, two-scale convergence allows for the rigorous introduction of the micro-variable $y \in \mathcal{Y}$ which is strongly linked to the corrector function u_1 which in turn is determined by (2.5b). Finally, this corrector function is contributing to the macroscopic problem in (2.5a).

§2.2. Two-scale convergence and highly heterogeneous media

From a modelling perspective, (1.1) is suitable for composites made of materials having comparable properties of interest, e.g. comparable heat conductivity of two metals. As pointed out in [PB08], significant parameter differences suggest modifying (1.1) in order to treat the encountered heterogeneities suitably. In fact, this is appealing for both applications' and theoretical reasons. As a prototype, one might think of a composite of metal and silicate. Such problems are formulated in terms of periodic homogenisation as a problem on a domain Ω in periodic fashion.

More specifically, assume that a decomposition $\mathcal{Y}_1 \cup \mathcal{Y}_2 \cup \Gamma = \mathcal{Y}$ is given such that $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are open in $\mathcal{Y}$ and $\partial\mathcal{Y}_1 \cap \partial\mathcal{Y}_2 = \Gamma$ is Lipschitz-regular hypersurface. Denote the characteristic functions of $\mathcal{Y}_i$ by χ_i for $i = 1, 2$ such that $\Omega^i_\varepsilon := \{x \in \Omega : \chi_i(x/\varepsilon) = 1\}$ defines subdomains $\Omega^1_\varepsilon, \Omega^2_\varepsilon$ that are assumed to have a sufficiently regular common interface Γ_ε. A prototypical boundary value problem reads

$$-\left[\chi_1\left(\frac{x}{\varepsilon}\right) + \chi_2\left(\frac{x}{\varepsilon}\right)\varepsilon^2\right]\Delta u_\varepsilon + \alpha u_\varepsilon = f \qquad \text{with} \quad u_{\varepsilon|\partial\Omega} \equiv 0, \tag{2.8}$$

for some $\alpha \in [0, \infty)$. (2.8) can be interpreted as a heat diffusion process in Ω which is a composite of a 'fast domain' in Ω^1_ε and a 'slow domain' in Ω^2_ε.

Theorem §2.2: Allaire's theorem for high contrast media

Provided $f \in L^2(\Omega)$ holds, (2.8) is uniquely solvable by $u_\varepsilon \in W^{1,2}_0(\Omega)$ for every $\varepsilon \in \mathbb{J}$ but the sequence $(u_\varepsilon)_{\varepsilon\in\mathbb{J}} \subset W^{1,2}_0(\Omega)$ is no longer bounded independently of $\varepsilon \in \mathbb{J}$, we have

$$\exists C \geq 0 : \forall \varepsilon \in \mathbb{J} : \quad \|u_\varepsilon\|_{L^2(\Omega)} + \|\nabla u_\varepsilon\|_{L^2(\Omega^1_\varepsilon)} + \varepsilon\|\nabla u_\varepsilon\|_{L^2(\Omega^2_\varepsilon)} \leq C. \tag{2.9}$$

If $\mathcal{Y}_1$ is connected for all $\varepsilon \in \mathbb{J}$, and if Γ is Lipschitz-regular then there exist $u_0 \in W^{1,2}_0(\Omega)$, $u_1 \in L^2(\Omega; W^{1,2}(\mathcal{Y})/\mathbb{R})$ and $v \in L^2(\Omega; W^{1,2}_0(\mathcal{Y}_2))$ with

$$u_\varepsilon \xrightarrow{2w} u_0 + \chi_2 v \qquad \text{in } L^2(\Omega), \text{ and} \tag{2.10a}$$

$$[\chi_1 + \varepsilon\chi_2]\nabla u_\varepsilon \xrightarrow{2w} \chi_1\left[\nabla_x u_0 + \nabla_y u_1\right] + \chi_2\nabla_y v \qquad \text{in } L^2(\Omega\times\mathcal{Y}; \mathbb{R}^d). \tag{2.10b}$$

Moreover, (u_0, u_1, v) solve a limit problem that can be decoupled to yield:

$$-\nabla_x \cdot \left[\int_{\mathcal{Y}_1} \nabla_x u_0 + \nabla_y u_1 \, dy\right] + \alpha\left[u_0 + \int_{\mathcal{Y}_2} v(x, y)\, dy\right] = f \qquad \text{in } \Omega, \tag{2.11a}$$

$$-\nabla_y \cdot \left[\nabla_x u_0 + \nabla_y u_1\right] = 0 \qquad \text{in } \Omega\times\mathcal{Y}_1, \tag{2.11b}$$

$$-\Delta_{yy} v + \alpha(u + v) = f \qquad \text{in } \Omega\times\mathcal{Y}_2, \tag{2.11c}$$

$$\vec{n}_\Gamma \cdot \left[\nabla_x u_0 + \nabla_y u_1\right] = 0 \qquad \text{on } \Gamma. \tag{2.11d}$$

Observe that both standard and high contrast media need two-scale convergence to formulate the convergence properties of the gradients in (2.3b) and (2.10b) sensibly. In contrast to the standard case where (2.3a) works with weak convergence only, the use of two-scale convergence is inevitable in (2.10a).

Proof. We follow the original proof from [All92, Sec. 4], restricting to $\alpha = 0$ without loss of generality. First, the the Lax–Milgram lemma yields the solvability of the weak form of (2.8) which reads

$$\forall \varphi \in W_0^{1,2}(\Omega): \quad \int_\Omega [\chi_1 + \varepsilon\chi_2]\, \nabla u_\varepsilon \cdot [\chi_1 + \varepsilon\chi_2]\, \nabla\varphi \, dx = \int_\Omega f\varphi \, dx \tag{2.12}$$

for every $\varepsilon \in \mathbb{J}$. Taking $\varphi = u_\varepsilon$ one defers $\|u_\varepsilon\|_{L^2(\Omega)} + \|\,[\chi_1 + \varepsilon\chi_2]\,\nabla u_\varepsilon\|_{L^2(\Omega;\mathbb{R}^d)} \le \|f\|_{L^2(\Omega)}$ which is precisely the claimed a priori bound. Next, one rewrites u_ε by $u_\varepsilon = \chi_1(x/\varepsilon)u_\varepsilon + \chi_2(x/\varepsilon)u_\varepsilon =: u_\varepsilon^{(1)} + u_\varepsilon^{(2)}$ and extends $u_\varepsilon^{(i)}$ to all of Ω by extension with zero, writing $\widetilde{u}_\varepsilon^{(i)}$. This yields the weak form

$$\forall \varphi \in W_0^{1,2}(\Omega): \quad \int_\Omega \nabla\widetilde{u}_\varepsilon^{(1)} \cdot \nabla\varphi \, dx + \int_\Omega \varepsilon\nabla\widetilde{u}_\varepsilon^{(2)} \cdot \varepsilon\nabla\varphi \, dx = \int_\Omega f\varphi \, dx. \tag{2.13}$$

Next, we claim that the a priori bounds yield the convergence statements (2.10a) and (2.10b) by theorem §7.2 which we establish in the next section: there exists a subsequence again denoted $\mathbb{J} \ni \varepsilon \longrightarrow 0$ such that

$$\left(\widetilde{u}_\varepsilon^{(1)}, \widetilde{u}_\varepsilon^{(2)}\right) \xrightarrow{2w} \left(u_{0|\Omega\times\mathcal{Y}_1}(x), u_{0|\Omega\times\mathcal{Y}_2}(x) + v(x,y)\right) \quad \text{in } L^2(\Omega\times\mathcal{Y})^2 \tag{2.14a}$$

(2.14b)
$$\left(\nabla\widetilde{u}_\varepsilon^{(1)}, \varepsilon\nabla\widetilde{u}_\varepsilon^{(2)}\right) \xrightarrow{2w} \left(\chi_1(y)\left[\nabla_x u_0(x) + \nabla_y u_1(x,y)\right], \chi_2(y)\nabla_y v(x,y)\right) \quad \text{in } L^2(\Omega\times\mathcal{Y})^{2d}.$$

Next, we intent to insert an admissible two-scale test function of the form $\varphi(x,y) = \varphi_0(x) + \varepsilon\varphi_1(x,y) + \psi(x,y)$ in the above weak formulation. To this end, we choose $\varphi_0 \in C_0^\infty(\Omega)$, $\varphi_1 \in C^\infty(\Omega; C^\infty(\mathcal{Y}_1)/\mathbb{R})$ and $\psi \in C^\infty(\Omega; C_0^\infty(\mathcal{Y}_2))$. Then, the original definition of two-scale convergence can be used to arrive at an integral inequality for $\varepsilon \longrightarrow 0$, which can be considered as the weak formulation of (2.11), namely

$$\begin{cases} \displaystyle\iint_{\Omega\times\mathcal{Y}_1} \left[\nabla_x u_0(x) + \nabla_y u_1(x,y)\right] \cdot \left[\nabla_x\varphi_0(x) + \nabla_y\varphi_1(x,y)\right] \, dx \\ \displaystyle\qquad + \iint_{\Omega\times\mathcal{Y}_2} \nabla_y v(x,y) \cdot \nabla_y\psi(x,y)\, dx = \int_{\Omega\times\mathcal{Y}} f\,[\varphi_0 + \psi]\, dx. \end{cases} \tag{2.15}$$

Now, one argues that by density of smooth functions, this weak form holds for all $\varphi_0 \in W_0^{1,2}(\Omega)$, $\varphi_1 \in L^2(\Omega; W^{1,2}(\mathcal{Y}_1)/\mathbb{R})$ and $\psi \in L^2(\Omega; W_0^{1,2}(\mathcal{Y}_2))$. Then, one can deduce (2.11), by choosing test functions which are either independent of x or y or which vanish on $\mathcal{Y}_1$ or $\mathcal{Y}_2$, respectively. ■

§2.3. A roadmap for compactness theorems

Both theorem §2.1 and theorem §2.2 rest on ominous compactness theorems used to obtain (2.3) and (2.10), respectively. Hardly as a surprise, such statements are among the most crucial tools for the identification of limit problems. Classically, both statements were given proven using suitable two-scale test functions. As far as the author is aware of, (2.3) was established in [Ngu89], with (2.10) being presented in [All92, Sec. 4]. Notice though that homogenisation and compactness results relying on skilfully chosen test functions were around before, a very prominent instance being L. Tartar's *method of oscillating test functions* which is extensively described in [CD99, Ch. 8] and [Tar10, Ch. 10], the latter carrying out the homogenisation of non-linear, monotone operators in standard media.

Returning to the results (2.3) and (2.10), one needs two ingredients to arrive at a suitable proof. First, the two-scale convergence machinery itself must be developed, a task which we refer to as the *convergence machinery*. We will prefer a functional analytical of two-scale convergence which will be done by periodic unfolding.

Independently but linked to the convergence machinery is the geometric set-up of the domain Ω in periodic terms. Again and for clarity's sake, neither traditional two-scale convergence nor periodic unfolding are necessary to decompose a domain suitably. Nevertheless, both are linked in applications by the compactness theorems theorem §7.1 and theorem §7.2 we wish to develop.

To this end, we will proceed in three steps, starting with the introduction of periodic unfolding which is followed by supplying more refined notions of two-scale convergence. As a third step, we will tend geometric issues related to the boundary value problems under consideration.

§3. Convergence machinery I: definition of periodic unfolding

By 1990 an interest to develop alternative descriptions of two-scale convergence had been sparked by problems involving reiterated periodic homogenisation and functional analytical considerations. [ADH90] can be considered as the forerunner of an effort which culminated in the development of periodic unfolding. Besides its more geometric intuition, periodic unfolding also allowed to formulate two-scale convergence in well-known functional analytical terms, making it our method of choice.

Let us fix some notation: again, the flat d-torus is given by $\mathcal{Y} := \mathbb{R}^d/\mathbb{Z}^d$ and $\pi := \pi_{\mathcal{Y}} : \mathbb{R}^d \longrightarrow \mathcal{Y}$ denotes the corresponding quotient map. A set $A \subset \mathcal{Y}$ is *open in the quotient topology on* $\mathcal{Y}$ if its pre-image $\pi^{-1}(A) \subset \mathbb{R}^d$ is open in the standard Euclidian topology on $\mathbb{R}^d$. The quotient map is a local C^∞-diffeomorphism and surjective since $\pi(Y) = \mathcal{Y}$ holds for $Y := [0,1)^d$. To sum up, $\mathcal{Y}$ is a compact C^∞-manifold carrying the quotient topology induced by π.

§3.1. Two-scale decomposition of the whole space

The presentation of periodic unfolding given here follows [CDG08; MT06] and [Vis06]. Let us begin by noticing that for every $\varepsilon > 0$ every $x \in \mathbb{R}^d$ can be uniquely decomposed as $x = \varepsilon\left([x/\varepsilon]_{\mathbb{Z}^d} + \{x/\varepsilon\}_{\mathbb{Z}^d}\right)$ with $[x]_{\mathbb{Z}^d} := (\lfloor x_i \rfloor)_{i=1}^d \in \mathbb{Z}^d$ and $\{x/\varepsilon\}_{\mathbb{Z}^d} := x/\varepsilon - [x/\varepsilon]_{\mathbb{Z}^d} \in Y$. Heuristically, one interprets $\varepsilon\,[x/\varepsilon]_{\mathbb{Z}^d}$ as the macroscopic portion of x and $\varepsilon\{x/\varepsilon\}_{\mathbb{Z}^d}$ as the microscopic contribution; roughly, this is motivated by $|x - \varepsilon\,[x/\varepsilon]_{\mathbb{Z}^d}|_{\mathbb{R}^d} \leq \varepsilon$.

Definition §3.1: Periodic unfolding in $\mathbb{R}^d$

The two-scale decomposition map $\mathcal{T}_{\varepsilon,\mathbb{R}^d}$, in short $\mathcal{T}_\varepsilon$, is given by

$$\mathcal{T}_{\varepsilon,\mathbb{R}^d} : \mathbb{R}^d \times \mathcal{Y} \longrightarrow \mathbb{R}^d \qquad (x,y) \longmapsto \mathcal{T}_{\varepsilon,\mathbb{R}^d}(x,y) := \varepsilon\left([x/\varepsilon]_{\mathbb{Z}^d} + y\right). \tag{3.1}$$

The periodic unfolding operator $\mathcal{T}_\varepsilon^*$ is given by pulling back with $\mathcal{T}_\varepsilon$, more specifically

$$\begin{cases} \mathcal{T}_\varepsilon^* : \left\{u : \mathbb{R}^d \longrightarrow S\right\} =: S^{\mathbb{R}^d} \longrightarrow S^{\mathbb{R}^d \times Y} \\ [x \mapsto u(x)] \longmapsto \left[(x,y) \mapsto \mathcal{T}_\varepsilon^*(u)(x,y) := u \circ \mathcal{T}_\varepsilon(x,y)\right] \end{cases} \tag{3.2}$$

for some set S.

Let us consider a standard example: $u_\varepsilon(x) = \sin(2\pi x \varepsilon^{-\kappa})$ for $x \in \mathbb{R}$ and $\kappa \in \mathbb{R}$. Unfolding leads to $\mathcal{T}_\varepsilon^*(u_\varepsilon)(x,y) = \sin\left(2\pi\varepsilon^{1-\kappa}(\lfloor x/\varepsilon \rfloor + y)\right)$ which turns into $\mathcal{T}_\varepsilon^*(u_\varepsilon)(x,y) = \sin(2\pi y)$ if $\kappa = 1$. Thus, periodic unfolding manages to simplify u_ε considerably provided the oscillations are correctly captured by the microvariable y. In contrast, for $\kappa \neq 1$ periodic unfolding is essentially ineffective.

Next, we will consider some technical measurability properties. To fix notation, we write $\mathcal{L}(\mathbb{R}^d)$ for the σ-algebra of Lebesgue measurable sets in $\mathbb{R}^d$, which stems from the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$ by completion, i.e. by setting $\lambda^d(Z_0) = 0$ for all $Z_0 \subset Z \in \mathcal{B}(R^d)$ with $\lambda^d(Z) = 0$ such that $Z_0 \in \mathcal{L}(\mathbb{R}^d)$. One can obtain a similar constructs for $\mathcal{Y}$ by pushing forward the Lebesgue measure from $\mathbb{R}^d$ to $\mathcal{Y}$ via $\lambda_Y^d(C) := \lambda_{\mathbb{R}^d}^d(\pi^{-1}(C) \cap Y)$ for $C \in \mathcal{L}(\mathcal{Y}) := \{M \subset \mathcal{Y} : \pi^{-1}(M) \cap Y \in \mathcal{L}(\mathbb{R}^d)\}$. For additional background material and more sophisticated approaches, most notably the Haar integral on compact Lie groups, consider [AE09; Bou64; Edw65; HS65]. Finally, let two measurable spaces (S_i, Σ_i) with $i = 1,2$ be given. $\Sigma_1 \otimes \Sigma_2$ is the smallest σ-algebra of subsets of $S_1 \times S_2$ which is generated by sets of the form $M_1 \times M_2$ with $M_i \in \Sigma_i$ for $i = 1,2$. Do note that even if the M_i are complete, $\Sigma_1 \otimes \Sigma_2$ does not have to be so.

Lemma §3.1: Measurability properties on $\mathbb{R}^d$

For all $C \in \mathcal{L}(\mathbb{R}^d)$ we have $\mathcal{T}_\varepsilon^{-1}(C) \in \mathcal{L}(\mathbb{R}^d) \otimes \mathcal{L}(\mathcal{Y})$. Therefore, the periodic unfolding of a Lebesgue measurable map is Lebesgue measurable. In addition, periodic unfolding respects zero sets, that is, if $Z_1 \subset \mathbb{R}^d$ with $\lambda^d(Z_1) = 0$, then $\lambda^{2d}(\mathcal{T}_\varepsilon^{-1}(Z_1)) = 0$ holds, too.

Proof. The measurability statement is due to $\mathcal{T}_\varepsilon^{-1}(C)$ being the union of a product of ε-cubes covering C with the preimage of a Lebesgue measurable set under the local diffeomorphism π, namely

$$\mathcal{T}_\varepsilon^{-1}(C) = \bigcup_{z \in [C/\varepsilon]} \left[[\varepsilon(z+Y)] \times \pi^{-1}(C \cap [\varepsilon(z+Y)])\right]. \tag{3.3}$$

The zero set statement now follows, too: since π^{-1} maps Z_1 on a product of a union of cubes with zero sets in $\mathcal{Y}$, which is a zero set due to Fubini's theorem. ∎

§3.2. Two-scale decomposition of a domain's auxiliary domains

The problem

Applying periodic unfolding to subsets of $\mathbb{R}^d$ is not straightforward due to the lack of match encountered near the boundary. More explicitly, $\mathcal{T}_{\epsilon,\mathbb{R}^d}^{-1}(\Omega) = \mathcal{T}_\varepsilon^{-1}(\Omega) \supsetneq \Omega \times \mathcal{Y}$ holds in general, even for bounded C^∞-domains. For instance, let $\Omega = B_1(0) \subset \mathbb{R}^d$ and $x \in B_\varepsilon(\partial\Omega) \cap \Omega$ with $(\widetilde{x}, y) \in \mathcal{T}_\varepsilon^{-1}(x)$. As one may figure out by a brief sketch, one encounters cases like $\widetilde{x} \notin \Omega$ or $\mathcal{T}_\varepsilon(\widetilde{x}, y) \notin \Omega$ for some $y \in \mathcal{Y}$, see figure I.1 for an illustration.

$\mathcal{T}_\varepsilon^{-1}(\Omega) \supsetneq \Omega \times \mathcal{Y}$ would not be too bad if it would not actually cause us trouble; of course, it does, the situation is as follows. Say we suspect a sequence $(u_\varepsilon)_{\varepsilon\in\mathbb{J}} \subset L^1(\Omega)$ to possess oscillations that can be resolved nicely with periodic unfolding. Writing $w_\varepsilon = \mathcal{T}_\varepsilon^*(u_\varepsilon)$ for the moment, it is clear by the foregoing considerations that $\operatorname{supp}(w_\varepsilon) \not\subset \Omega \times \mathcal{Y}$ holds in general, as $\mathcal{T}_\varepsilon^{-1}(\Omega)$ is strictly greater than $\Omega \times \mathcal{Y}$. Since $\mathcal{T}_\varepsilon^{-1}(\Omega)$ is rather awkward, it is tempting to stick with $w_{\varepsilon|\Omega\times\mathcal{Y}}$ instead. However, as pointed out in [MT06, Ex. 2.3], this is not a good idea.

Explicit construction of an defective sequence

Following [MT06], one can construct a sequence $(u_\varepsilon)_{\varepsilon\in\mathbb{J}} \subset L^1(\Omega)$ explicitly whose periodic unfolding restricted to $\Omega \times \mathcal{Y}$ cuts off considerable parts of the function. The idea is to consider an approximation to the identity, centred at $x_\varepsilon \in \Omega$ with $\operatorname{dist}(x_\varepsilon, \partial\Omega) \to 0$ such that $\mathcal{T}_\varepsilon^{-1}(x_\varepsilon) \notin \Omega \times \mathcal{Y}$. More explicitly, set

$$\eta_\delta(y) := \delta^{-d}\eta(y/\delta) \text{ and } \eta(y) := \chi_{B_1(0)}(y) \exp\left(\frac{-1}{1-|x|^2}\right) \in C_0^\infty(\mathbb{R}^d) \tag{3.4}$$

such that $\operatorname{supp}(\eta_\delta) = \overline{B}_\delta(0)$ and $\|\eta_\delta\|_{L^1(\mathbb{R}^d)} = 1$ for $\delta > 0$. Then, a defective sequence can be made from

$$u_\varepsilon := a_\varepsilon \eta_\delta(x - x_\varepsilon) \tag{3.5}$$

with $\delta = \varepsilon^{12345}$, your favourite sequence $(a_\varepsilon)_{\varepsilon\in\mathbb{J}} \subset \mathbb{R}$, and some $(x_\varepsilon)_{\varepsilon\in\mathbb{J}} \subset \Omega$. Returning to figure I.1 again, take $\Omega = B_1(0) \subset \mathbb{R}^d$ and $x_\varepsilon = (1-\varepsilon)(-\sqrt{1/d}, \dots, -\sqrt{1/d})$ to arrive at a sequence $(u_\varepsilon)_{\varepsilon\in\mathbb{J}}$ that vanishes under periodic unfolding restricted to $\Omega \times \mathcal{Y}$. Thus, $\|\mathcal{T}_\varepsilon^*(u_\varepsilon)\|_{L^1(\Omega\times\mathcal{Y})} = 0$ and at the same time $\|u_\varepsilon\|_{L^1(\Omega)} = |a_\varepsilon|$ hold, with $(a_\varepsilon)_{\varepsilon\in\mathbb{J}}$ being arbitrary.

A possible resolution

A compensation of the lacking boundary match is due to [CDG02; CDG08; Cio+12; MT06] where Ω is approximated by two open domains Ω_ε^- and Ω_ε^+, we write $\Omega_\varepsilon^\pm$ in short, which we call *auxiliary domains of* Ω. The latter are tailor-made to fulfil $\mathcal{T}_\varepsilon(\Omega_\varepsilon^\pm \times \mathcal{Y}) = \Omega_\varepsilon^\pm$ such that unambiguous meaning can be given to $\mathcal{T}_\varepsilon : L^0(\Omega_\varepsilon^\pm) \longrightarrow L^0(\Omega_\varepsilon^\pm \times \mathcal{Y})$. By $\Omega_\varepsilon^- \subset \Omega \subset \Omega_\varepsilon^+$ one aims at transferring a suitable periodic unfolding concept onto Ω, as well.

To begin, let us write $Z := [0,1]^d$ and $\operatorname{int}[A]$ for the interior of a given set $A \subset \mathbb{R}^d$, i.e. the union of all subsets of A which are open in $\mathbb{R}^d$. The following, somewhat technical definition is exemplified in figure I.1 below.

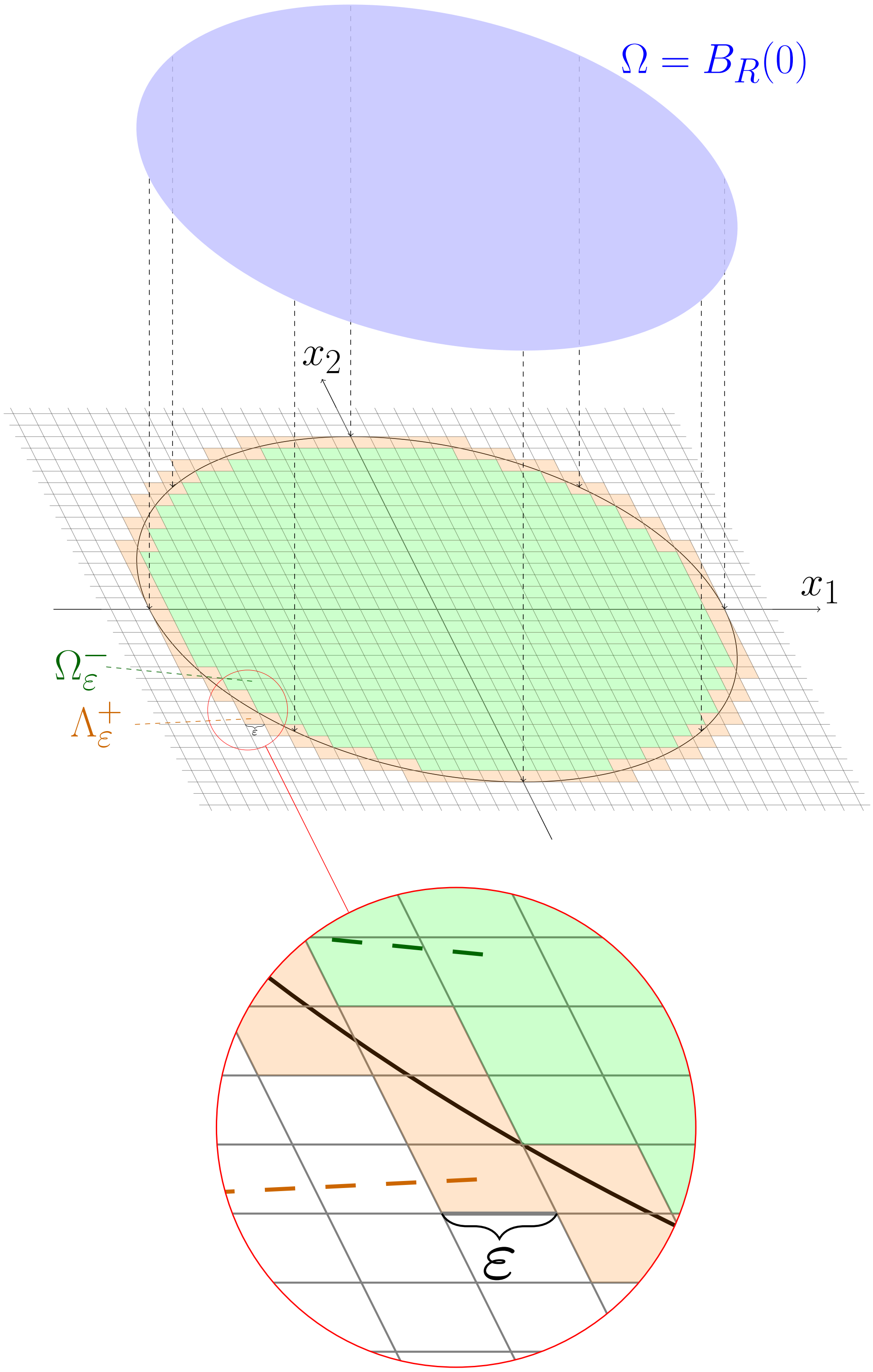


Figure I.1.: The domain $\Omega = B_R(0) \subset \mathbb{R}^2$ is approximated in terms of the grid $\varepsilon\mathbb{Z}^2$; only Ω_ε^- and Λ_ε^+ are drawn. Here, $\varepsilon = R/N$ for some $N \in \mathbb{N}$.

Definition §3.2: **Auxiliary domains**

The inner and outer (ε-dependent) auxiliary domains of an open set $\Omega \subset \mathbb{R}^d$ are denoted Ω_ε^- and Ω_ε^+ respectively, and are given through the following cascade of definitions:

$$R_\varepsilon^-(\Omega) := \left\{z \in \mathbb{Z}^d : \varepsilon(z+Z) \subset \overline{\Omega}\right\} \quad \& \quad R_\varepsilon^+(\Omega) := \left\{z \in \mathbb{Z}^d : \varepsilon(z+Z) \cap \Omega \neq \emptyset\right\}, \tag{3.6a}$$

$$\Omega_\varepsilon^- := \operatorname{int}\left[\bigcup_{z\in R_\varepsilon^-(\Omega)} \varepsilon(z+Z)\right] \quad \& \quad \Omega_\varepsilon^+ := \operatorname{int}\left[\bigcup_{z\in R_\varepsilon^+(\Omega)} \varepsilon(z+Z)\right], \tag{3.6b}$$

$$\Lambda_\varepsilon^- := \Omega \setminus \Omega_\varepsilon^- \quad \& \quad \Lambda_\varepsilon^+ := \Omega_\varepsilon^+ \setminus \Omega_\varepsilon^-. \tag{3.6c}$$

By construction, the sets $\Omega_\varepsilon^\pm$ are open, Lebesgue measurable and fulfil $\Omega_\varepsilon^- \subset \Omega \subset \Omega_\varepsilon^+$ for all $\varepsilon > 0$. So, $\Omega_\varepsilon^\pm$ form several sequences, namely

a) $\left(\Omega_\varepsilon^\pm\right)_{\varepsilon\in\mathbb{J}}$, a sequence of open C^∞-submanifolds of $\mathbb{R}^d$,

b) $\left(\partial\Omega_\varepsilon^\pm\right)_{\varepsilon\in\mathbb{J}}$, a sequence of closed subsets of $\mathbb{R}^d$ which are no manifolds in general, and

c) $\left(\overline{\Omega_\varepsilon^\pm}\right)_{\varepsilon\in\mathbb{J}}$, also a sequence of closed subsets of $\mathbb{R}^d$ which again are neither manifolds nor manifolds with corners in general.

Proposition §3.1: **Convergence of auxiliary domains**

Let $\Omega \subset \mathbb{R}^d$ be an open set with compact, $C^{0,1}$-regular boundary, then the aforementioned sequences can be considered to converge to subsets of $\overline{\Omega}$ in the following manner:

a) $(\Omega_\varepsilon^\pm)_{\varepsilon\in\mathbb{J}}$ Hausdorff-converges to Ω in $\mathbb{R}^d$, i.e. for cofinitely many $\varepsilon > 0$ we have $\Omega_\varepsilon^\pm \subset B_{d\varepsilon}(\Omega)$ and $\Omega \subset B_{d\varepsilon}(\Omega_\varepsilon^\pm)$ together with the vanishing of the corresponding Hausdorff-metric in $\mathbb{R}^d$:

$$d_H^{\mathbb{R}^d}(\Omega_\varepsilon^\pm, \Omega) := \inf\left\{\delta > 0 : \Omega_\varepsilon^\pm \subset B_\delta(\Omega) \ \wedge\ \Omega \subset B_\delta(\Omega_\varepsilon^\pm)\right\} \le d\varepsilon \longrightarrow 0. \tag{3.7a}$$

b) $\left(\partial\Omega_\varepsilon^\pm\right)_{\varepsilon\in\mathbb{J}}$ Hausdorff–converges to $\partial\Omega$ in $\mathbb{R}^d$, too: for cofinitely many $\varepsilon > 0$ $\partial\Omega_\varepsilon^\pm \subset B_{d\varepsilon}(\partial\Omega)$ and $\partial\Omega \subset B_{d\varepsilon}(\partial\Omega_\varepsilon^\pm)$ hold and the corresponding Hausdorff-metric vanishes as

$$d_H^{\mathbb{R}^d}(\partial\Omega_\varepsilon^\pm, \partial\Omega) \le d\varepsilon \longrightarrow 0 \quad \text{holds.} \tag{3.7b}$$

c) Thanks the regularity assumption on $\partial\Omega$ we infer

$$0 \le \lambda^d(\Lambda_\varepsilon^+) + \lambda^d(\Lambda_\varepsilon^-) \overset{\varepsilon\to 0}{\longrightarrow} 0. \tag{3.7c}$$

Furthermore, the domains $\Omega_\varepsilon^\pm$ satisfy the cone property but not the uniform cone property. In particular, $\Omega_\varepsilon^\pm$ generally fail to be strong Lipschitz domains. However, $\lambda^d\left(\partial\Omega_\varepsilon^\pm\right) = 0$ holds for all $\varepsilon > 0$ and $\partial\Omega_\varepsilon^\pm$ has an outward unit normal vector $\mathscr{H}^{d-1}$-almost everywhere.

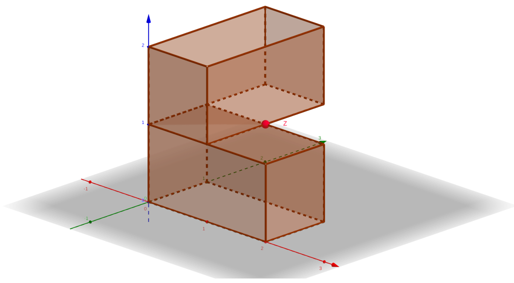


Figure I.2.: The two-bricks set fails to be a Lipschitz domain due to the point z.

Proof. Keeping $\sqrt{d} = \text{diam}(Z)$ in mind, the Hausdorff-convergence statements follow from the construction of the $\Omega_\varepsilon^\pm$ in a geometric manner: to show (3.7c) we focus on Λ_ε^+ as $\Lambda_\varepsilon^- \subset \Lambda_\varepsilon^+$. As $\partial\Omega$ is a Lipschitz-regular, compact manifold of codimension 1, there are finitely many, $\mathbb{R}^d$-open and bounded sets $U_1, \ldots, U_K$ which cover $\partial\Omega$ together with uniformly bi-Lipschitz continuous chart maps $\phi_1, \ldots, \phi_K$ which locally map $U_i \cap \Omega$ to $\phi_i(U_i \cap \Omega) = \{x \in V_i : x_n < 0\}$ for an open and bounded neighbourhood $V_i \subset \mathbb{R}^d$ of $0 \in \mathbb{R}^d$. Thus, the problem reduces to establishing that these rectified parts of the boundary vanish when covered with cubes of the form εZ that are aligned in some manner which can be arbitrary. Fortunately, it is quite obvious to see that is indeed the case: since $\phi_i(U_i \cap \partial\Omega) = \{x \in V_i : x_n = 0\}$ is a linear hypersurface and the cover of εZ-cubes has Lebesgue measure proportional to $\varepsilon^d \text{diam}(V_i)$. Also, ϕ_i^{-1} preserves this proportionality. Finally, this procedure can be done for all $U_1, \ldots, U_K$ along $\partial\Omega$ such that the statement follows.
Next, to see that $\Omega_\varepsilon^\pm$ satisfies the cone property, notice that $Z = \varepsilon[0,1]^d$ does and so do unions of translations along $\mathbb{Z}^d$ of Z and their interiors for every fixed $\varepsilon > 0$.
To see that the uniform cone property can fail, consider the *two-brick set* depicted in figure I.2 which is given by $Z \cup (e_1 + Z) \cup (e_2 + Z) \cup (e_2 + e_3 + Z) \subset \mathbb{R}^3$ for e_i being the i-th canonical unit vector. The two-brick set fails to be a manifold in $z = (1, 1, 1)$; a back-of-an-envelope sketch can be very useful in this matter, too. The moral is that as long as two-bricks configurations cannot be ruled out satisfactorily – at least the author knows not how to do so at present – and so the uniform cone property must be expected to collapse.
Concerning $\lambda^d\left(\partial\Omega_\varepsilon^\pm\right) = 0$, observe that $\lambda^d(\varepsilon\partial Z) = 0$ is obvious for every $\varepsilon > 0$. Moreover, since $\partial\Omega$ is assumed to be compact, so is $\partial\Omega_\varepsilon^\pm$ for every $\varepsilon > 0$. Thus, $\partial\Omega_\varepsilon^\pm$ consists of a finite union of ε-hypercubes isometric to $[0,\varepsilon]^{d-1} \times 0$. Now, the λ^d-claim is elementary. Finally, the ε-hypercubes touch at worst on an $(d-2)$-dimensional edge since $(d-1)$-dimensional faces are glued together by construction. However, every ε-hypercube has a outward normal unit vector $\mathscr{H}^{d-1}$-almost everywhere and this transfers to $\partial\Omega_\varepsilon^\pm$. ■

Presumably, the sets $\Omega_\varepsilon^\pm$ inherit both connectedness and the segment condition from their original domain Ω for $\varepsilon \in (0, \varepsilon_0]$ with $\varepsilon_0 = \varepsilon_0(\Omega) > 0$ being small enough. If so the sets $\Omega_\varepsilon^\pm$ behave very reasonable although not being Lipschitz-regular: following P. W. Jones [Jon81], the $\Omega_\varepsilon^\pm$ are (ϵ, δ)-domains which possess extension operators for Sobolev functions, c.f. [Jon81, Thm. 1] and enjoy the density of $C^\infty(\mathbb{R}^d)$-functions, c.f. [Jon81, Sec. 4]. Both properties lead to the conclusion that Sobolev spaces $W^{1,p}$ on domains $\Omega_\varepsilon^\pm$ are suitably linked to both $W^{1,p}(\Omega)$ and $W^{1,p}(\mathbb{R}^d)$. Unfortunately, [AF03] is not very extensive about (ϵ, δ)-domains, V. Maz'ya's [Maz11] being a more suitable resource. Concerning traces on $\partial\Omega_\varepsilon^\pm$, one may consider more recent works on traces, for instance traces for d-sets, see H. Triebel's [Tri01, Ch. 9] and [Tri06, Thm. 7.16], or traces of BV-functions, see [AFP00; Maz11].

Anyway, since we are not going to use the $\Omega_\varepsilon^\pm$ to define the oscillating coefficients, we refrain from elaborating on this matter. More important to our applications is the realisation that since each $\Omega_\varepsilon^\pm$ is made up by periodicity cells, we have the following results.

Lemma §3.2: Invariance of $\Omega_\varepsilon^\pm$ under $\mathcal{T}_\varepsilon$

Let $\Omega \subset \mathbb{R}^d$ be a domain with compact Lipschitz-regular boundary. Then the auxiliary domains are preserved under the two-scale decomposition map, which means that for every $\varepsilon > 0$ we have $\mathcal{T}_{\varepsilon,\mathbb{R}^d}^{-1}\left(\overline{\Omega_\varepsilon^\pm}\right) = \overline{\Omega_\varepsilon^\pm} \times \mathcal{Y}$.

Proof. Without loss of generality we only consider $\overline{\Omega_\varepsilon^+} \neq \emptyset$. To show $\mathcal{T}_{\varepsilon,\mathbb{R}^d}^{-1}\left(\overline{\Omega_\varepsilon^\pm}\right) \subset \overline{\Omega_\varepsilon^\pm} \times \mathcal{Y}$, decompose $x \in \Omega_\varepsilon^+$ as $x = \varepsilon[x/\varepsilon]_{\mathbb{Z}^d} + \varepsilon\{x/\varepsilon\}_{\mathbb{Z}^d} = \mathcal{T}_\varepsilon(x, \{x/\varepsilon\}_{\mathbb{Z}^d}) = \mathcal{T}_\varepsilon(\widetilde{x}, \{x/\varepsilon\}_{\mathbb{Z}^d})$ with $\widetilde{x} \in \mathbb{R}^d$ sufficing $[\widetilde{x}/\varepsilon]_{\mathbb{Z}^d} = [x/\varepsilon]_{\mathbb{Z}^d}$. By construction any such $\widetilde{x}$ are contained in Ω_ε^+ and for almost all $x \in \Omega$ one even has $x \neq \mathcal{T}_\varepsilon(\widetilde{x}, y)$ for all $y \in \mathcal{Y}$ if $[\widetilde{x}/\varepsilon]_{\mathbb{Z}^d} \neq [x/\varepsilon]_{\mathbb{Z}^d}$.
Conversely, let $(x, y) \in \overline{\Omega_\varepsilon^\pm} \times \mathcal{Y}$ then $\mathcal{T}_\varepsilon(x, y) \in \overline{\Omega_\varepsilon^\pm}$ holds by the very definition of $\overline{\Omega_\varepsilon^\pm}$ given in (3.6) and therefore the claim is established. ∎

Thanks to the foregoing lemma, restricting periodic unfolding to auxiliary domains is well-justified. We will write $L^0(\Omega_\varepsilon^\pm; \mathscr{B})$ for the vector space of all $\mathcal{L}(\Omega_\varepsilon^\pm) - \mathcal{B}(\mathscr{B})$-measurable maps from $\Omega_\varepsilon^\pm$ to $\mathscr{B}$.

Definition §3.3: Periodic unfolding on auxiliary domains

Let $\Omega \subset \mathbb{R}^d$ be a domain with compact Lipschitz boundary and $\varepsilon > 0$. The <u>periodic unfolding operator (on auxiliary domains)</u> $\Omega_\varepsilon^\pm$ is given via the restriction of the periodic unfolding operator on $\mathbb{R}^d$ as given in definition §3.1, that is

$$\mathcal{T}_{\varepsilon,\Omega_\varepsilon^\pm}^* : L^0(\Omega_\varepsilon^\pm) \longrightarrow L^0(\Omega_\varepsilon^\pm \times \mathcal{Y}) \qquad v \longmapsto \mathcal{T}_{\varepsilon,\Omega_\varepsilon^\pm}^*(v) := \left(\widetilde{v} \circ \mathcal{T}_{\varepsilon,\mathbb{R}^d}\right)(x, y)_{|\Omega_\varepsilon^\pm \times \mathcal{Y}}, \tag{3.8}$$

with $\widetilde{v}$ being the extension of v by zero outside of Ω. Note that $\lambda^d(\partial\Omega_\varepsilon^\pm) = 0$ is used here, too. Also, measurability issues are postponed to lemma §3.3. Besides, one abbreviates $\mathcal{T}_{\varepsilon,\Omega_\varepsilon^\pm}^*$ by $\mathcal{T}_\varepsilon^*$ in most instances.

Lemma §3.3: Banach valued periodic unfolding

Referring to definition §3.3, let $\mathscr{B}$ be a Banach space. One can extend $\mathcal{T}_{\varepsilon,\Omega_\varepsilon^\pm}^*$ to $\mathscr{B}$-valued functions in order to get

$$\mathcal{T}_{\varepsilon,\Omega_\varepsilon^\pm,\mathscr{B}}^* : L^0(\Omega_\varepsilon^\pm; \mathscr{B}) \longrightarrow L^0(\Omega_\varepsilon^\pm \times \mathcal{Y}; \mathscr{B}) \tag{3.9}$$

which we denote by $\mathcal{T}_\varepsilon^*$, again, provided there is no ambiguity.

Proof. Recall lemma §3.1 and the fact that $\mathcal{L}(\Omega_\varepsilon^\pm)$, stems from $\mathcal{L}(\mathbb{R}^d)$ by restriction, i.e. $A \in \mathcal{L}(\Omega_\varepsilon^\pm) \iff \exists \widetilde{A} \in \mathcal{L}(\mathbb{R}^d) : A = \widetilde{A} \cap \Omega_\varepsilon^\pm$. Now, let $f : \Omega_\varepsilon^\pm \longrightarrow \mathscr{B}$ be given and extend it to all of $\mathbb{R}^d$ by zero, making a $\mathcal{L}(\Omega_\varepsilon^\pm) - \mathcal{B}(\mathscr{B})$-measurable function $\mathcal{L}(\mathbb{R}^d) - \mathcal{B}(\mathscr{B})$-measurable. By lemma §3.1 we have $\mathcal{T}_\varepsilon^{-1} : \mathcal{L}(\mathbb{R}^d) \longrightarrow \mathcal{L}(\mathbb{R}^d) \otimes \mathcal{L}(\mathcal{Y})$ and for every $B \in \mathcal{B}(\mathscr{B})$ $(\mathcal{T}_\varepsilon^*(f))^{-1}(B) = \mathcal{T}_\varepsilon^{-1}(f^{-1}(B))$ holds. As f is sufficiently measurable $f^{-1}(B) \in \mathcal{L}(\mathbb{R}^d)$ holds,

using the extended function already. Alas, this shows that $\mathcal{T}_\varepsilon^*(f)$ is $\mathcal{L}(\mathbb{R}^d) \otimes \mathcal{L}(\mathcal{Y}) - \mathcal{B}(\mathscr{B})$-measurable.
Moreover, since $\mathcal{T}_\varepsilon^{-1}$ inherits the invariance of zero sets, $\mathcal{T}_\varepsilon^{-1}(\Omega_\varepsilon^\pm) = \Omega_\varepsilon^\pm \times \mathcal{Y}$ holds up to zero sets, making $(x, y) \longmapsto \mathcal{T}_\varepsilon^*(f)(x, y)$ a well-defined map on $\Omega_\varepsilon^\pm \times \mathcal{Y}$ as claimed. ■

Proposition §3.2: **Isometry of $\mathcal{T}^*_{\varepsilon,\Omega_\varepsilon^\pm}$**

Referring to definition §3.3, $\mathcal{T}^*_{\varepsilon,\Omega_\varepsilon^\pm}$ has the following invariance property:

$$\forall u \in L^1(\Omega_\varepsilon^\pm; \mathscr{B}) : \quad \int_{\Omega_\varepsilon^\pm} u(x)\, dx = \iint_{\Omega_\varepsilon^\pm \times \mathcal{Y}} \mathcal{T}^*_{\varepsilon,\Omega_\varepsilon^\pm}(u)(x, y)\, dy dx. \tag{3.10}$$

As a consequence, $\mathcal{T}_\varepsilon^* : L^p(\Omega_\varepsilon^\pm; \mathscr{B}) \longrightarrow L^p(\Omega_\varepsilon^\pm \times \mathcal{Y}; \mathscr{B})$ is an isometry for $p \in [1, \infty]$, i.e.

$$\forall u \in L^p(\Omega_\varepsilon^\pm; \mathscr{B}) : \quad \|u\|_{L^p(\Omega_\varepsilon^\pm;\mathscr{B})} = \|\mathcal{T}_\varepsilon^*(u)\|_{L^p(\Omega_\varepsilon^\pm \times \mathcal{Y};\mathscr{B})}. \tag{3.11}$$

Proof. First, observe that for $p \in [1, \infty)$, (3.11) follows from (3.10), so let us address the latter by choosing $u \in L^1(\Omega_\varepsilon^\pm; \mathscr{B})$. Since our measurability issues have been settled in lemma §3.3, we obtain the following transition formula by calculation:

$$\begin{cases} \displaystyle\int_{\Omega_\varepsilon^\pm} u(x)\, dx &= \displaystyle\sum_{z \in R_\varepsilon^\pm(\Omega)} \int_{\varepsilon(z+Y)} u(z + y)\, dy \\ &= \varepsilon^d \displaystyle\sum_{z \in R_\varepsilon^\pm(\Omega)} \int_{z+Y} u(\varepsilon z + \varepsilon y)\, dy = \iint_{\Omega_\varepsilon^\pm \times \mathcal{Y}} \mathcal{T}_\varepsilon^*(f)(x, y)\, dy dx \end{cases} \tag{3.12}$$

which relies on $\mathcal{T}_\varepsilon(\Omega_\varepsilon^\pm \times \mathcal{Y}) = \Omega_\varepsilon^\pm$ from lemma §3.2 and $x \longmapsto z = [x]_{\mathbb{Z}^d}$ being constant inside of every $\varepsilon(z + Y)$ so that $x \longmapsto \mathcal{T}_\varepsilon(x, y)$ is piecewise constant.
For the case $p = \infty$, observe that $|u|_{\mathscr{B}}$ coincides with a bounded, $\mathbb{R}$-valued function $\widetilde{u}$ on all of $\Omega_\varepsilon^\pm \setminus Z$ for a zero set $Z \subset \Omega$. Additionally, unfolding respects zero sets, i.e. $\lambda^{2d}(\mathcal{T}_\varepsilon^{-1}(Z)) = 0$, and is surjective onto $\Omega_\varepsilon^\pm$. Thus, the pullback of $\widetilde{u}$ and $|u|_{\mathscr{B}}$ by $\mathcal{T}_\varepsilon$ neither changes the match of two functions almost everywhere, nor does it change the bound of $\widetilde{u}$. ■

§3.3. Two-scale decomposition of a domain

We are now in place to define periodic unfolding for suitable domains, namely open connected sets with compact, Lipschitz-regular boundary. The plan is simple: unfolding $L^p(\Omega)$ yields spaces $\mathcal{T}_\varepsilon^{-1}(L^p(\Omega)) \subset L^p(\Omega_\varepsilon^+ \times \mathcal{Y})$ and the latter 'converges' suitably to $L^p(\Omega \times \mathcal{Y})$.

To avoid confusion we distinguish two tasks. First, using a suitable convergence notion that we specify later, one may take $\Omega_\varepsilon^- \nearrow \Omega$ to establish $L^p(\Omega_\varepsilon^-) \nearrow L^p(\Omega)$, a procedure that carries over to $\Omega_\varepsilon^- \times \mathcal{Y} \nearrow \Omega \times \mathcal{Y}$ and $L^p(\Omega_\varepsilon^- \times \mathcal{Y}) \nearrow L^p(\Omega \times \mathcal{Y})$, as well. Likewise, one can do the same for $\Omega_\varepsilon^+ \searrow \Omega$ with $L^p(\Omega_\varepsilon^+) \searrow L^p(\Omega)$ and $\Omega_\varepsilon^+ \times \mathcal{Y} \searrow \Omega \times \mathcal{Y}$ with $L^p(\Omega_\varepsilon^+ \times \mathcal{Y}) \searrow L^p(\Omega \times \mathcal{Y})$, respectively.

One the other hand, one may consider sequences of the form $(w_\varepsilon)_{\varepsilon \in \mathbb{J}}$ with $w_\varepsilon \in \mathcal{T}_\varepsilon^*(L^p(\Omega))$ for every $\varepsilon \in \mathbb{J}$. An example from [MT06] can be interpreted as showing that the space of sequences obtained by periodic unfolding is strictly greater than just taking $w_\varepsilon \in L^p(\Omega \times \mathcal{Y})$. Thus, $L^p(\Omega \times$

$\mathcal{Y})^{\mathbb{N}}$ is the wrong space for this endeavour, with $L^p(\mathbb{R}^d \times \mathcal{Y})^{\mathbb{N}}$ being a more promising candidate. We will return to this matter shortly, defining periodic unfolding on Ω first. Seeming rather unorthodox at first glance, we will employ the notion of a bundle to do so.

Definition §3.4: **Two-scale decomposition bundle of Ω**

For $\Omega \subset \mathbb{R}^d$ open with a compact, Lipschitz-regular boundary, the two-scale decomposition (bundle) of Ω is given by

$$\left(\prod_{\varepsilon\in\mathbb{J}} \Omega_\varepsilon^+ \times \mathcal{Y}, \prod_{\varepsilon\in\mathbb{J}} \Omega_\varepsilon^+, (\mathcal{T}_\varepsilon)_{\varepsilon\in\mathbb{J}}\right) \tag{3.13}$$

with the Cartesian product of sets $\prod_{\varepsilon\in\mathbb{J}} \Omega_\varepsilon^+ \times \mathcal{Y}$ being the total space over the base space $\prod_{\varepsilon\in\mathbb{J}} \Omega_\varepsilon^+$ together with the following $\mathbb{J}$-indexed family of two-scale decomposition maps that constitute the corresponding projections:

$$\begin{cases} \mathcal{T}_{\varepsilon,\Omega_\varepsilon^+} : \Omega_\varepsilon^+ \times \mathcal{Y} \longrightarrow \Omega_\varepsilon^+ \\ (x, y) \longmapsto \mathcal{T}_{\varepsilon,\Omega_\varepsilon^+}(x, y) := \varepsilon \left[\frac{x}{\varepsilon}\right] + \varepsilon y. \end{cases} \tag{3.14}$$

The induced periodic unfolding operators (on Ω) $(\mathcal{T}_\varepsilon^*)_{\varepsilon\in\mathbb{J}}$ read

$$\begin{cases} \mathcal{T}_{\varepsilon,\Omega}^* : L^0(\Omega) \longrightarrow L^0(\Omega_\varepsilon^+ \times \mathcal{Y}) \\ [x \mapsto v(x)] \longmapsto \left[(x, y) \mapsto \mathcal{T}_{\varepsilon,\Omega}^*(v)(x, y) := \widetilde{v}\left(\mathcal{T}_{\varepsilon,\Omega_\varepsilon^+}(x, y)\right)\right], \end{cases} \tag{3.15}$$

denoting by $\widetilde{v}$ the extension of v by zero. As the domain is usually clear from the context, we shall only write $\mathcal{T}_\varepsilon$ and $\mathcal{T}_\varepsilon^*$ for simplicity's sake.

Naturally, the above definition carries over to $\varepsilon > 0$ instead of $\varepsilon \in \mathbb{J}$.

Lemma §3.4: **Banach values**

Complementing definition §3.4, let $\mathcal{B}$ be a Banach space such that the periodic unfolding operator $\mathcal{T}_{\varepsilon,\Omega}^*$ can be extended to a map $\mathcal{T}_{\varepsilon,\Omega,\mathcal{B}}^* : L^0(\Omega; \mathcal{B}) \longrightarrow L^0(\Omega_\varepsilon^+ \times \mathcal{Y}; \mathcal{B})$ which, again, is usually shortened to $\mathcal{T}_\varepsilon^*$.

Proof. Considering functions extended by zero, the statement is entirely due to lemma §3.3. ∎

Proposition §3.3: **Isometry of $\mathcal{T}^*_{\epsilon,\Omega,\mathcal{B}}$**

Referring to definition §3.4 and lemma §3.4,

$$L^p(\Omega;\mathcal{B}) \ni u \longmapsto \mathcal{T}^*_\varepsilon(u) = \mathcal{T}^*_{\varepsilon,\Omega,\mathcal{B}} \in L^p(\Omega^+_\varepsilon \times \mathcal{Y};\mathcal{B}) \qquad \text{is linear, and} \tag{3.16}$$

$$\forall u \in L^1(\Omega;\mathcal{B}): \qquad \int_\Omega u(x)\,dx = \int_{\Omega^+_\varepsilon} \widetilde{u}(x)\,dx = \iint_{\Omega^\pm_\varepsilon\times\mathcal{Y}} \mathcal{T}^*_{\varepsilon,\Omega^+_\varepsilon}(u)(x,y)\,dydx \tag{3.17}$$

hold for every $\varepsilon > 0$ with $\widetilde{u}$ being the extension of u by zero outside of Ω. Consequently, the following isometry property holds for $\mathcal{T}^*_{\epsilon,\Omega,\mathcal{B}} : L^p(\Omega;\mathcal{B}) \longrightarrow L^p(\Omega^\pm_\varepsilon \times \mathcal{Y};\mathcal{B})$ with $p \in [1,\infty]$:

$$\forall u \in L^p(\Omega;\mathcal{B}): \qquad \|u\|_{L^p(\Omega;\mathcal{B})} = \|\mathcal{T}^*_\varepsilon(u)\|_{L^p(\Omega^\pm_\varepsilon\times\mathcal{Y};\mathcal{B})}. \tag{3.18}$$

Proof. Again, the measurability issues are settled already, and (3.17) follows directly from (3.10) as $u \mapsto \widetilde{u}$ does not alter the integral, being an isometry itself and (3.17) infers (3.18). ∎

Remark §3.1: **Warning**

An statement fully analogous to proposition §3.2 with $\Omega^\pm_\varepsilon$ being replaced by Ω in both (3.17) and (3.18) is wrong! Unfortunately, (3.5) shows that there do exist $u \in L^p(\Omega;\mathcal{B})$ with $\|u\|_{L^p(\Omega;\mathcal{B})} = \|\mathcal{T}^*_\varepsilon(\widetilde{u})\|_{L^p(\Omega^\pm_\varepsilon\times\mathcal{Y};\mathcal{B})} \neq \|\mathcal{T}^*_\varepsilon(\widetilde{u})_{|\Omega\times\mathcal{Y}}\|_{L^p(\Omega\times\mathcal{Y};\mathcal{B})}$.

Corollary §3.1:

Isometric images of closed sets are closed and so,

$$\mathcal{T}^*_\varepsilon(L^p(\Omega;\mathcal{B})) := \left\{ f \in L^p(\Omega^+_\varepsilon \times \mathcal{Y};\mathcal{B}) : \exists u \in L^p(\Omega;\mathcal{B}) : \mathcal{T}^*_\varepsilon(u) = f \right\}. \tag{3.19}$$

is a closed subspace of $L^p(\Omega^+_\varepsilon \times \mathcal{Y};\mathcal{B}) \subset L^p(\mathbb{R}^d \times \mathcal{Y};\mathcal{B})$ for every $p \in [1,\infty]$.

Proof. Every converging sequence $(\mathcal{T}^*_\varepsilon(u_n))_{n\in\mathbb{N}} \subset \mathcal{T}^*_\varepsilon(L^p(\Omega;\mathcal{B}))$ is a Cauchy sequence, such that $\|\mathcal{T}^*_\varepsilon(u_n) - \mathcal{T}^*_\varepsilon(u_m)\|_{L^p(\mathbb{R}^d\times\mathcal{Y};\mathcal{B})} \longrightarrow 0$ for $n, m \longrightarrow \infty$. Since $\mathcal{T}^*_\varepsilon$ is an isometry this implies that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\Omega;\mathcal{B})$ already. Since the latter is complete, $\exists u \in L^p(\Omega;\mathcal{B}) : \|u_n - u\|_{L^p(\Omega;\mathcal{B})} = \|\mathcal{T}^*_\varepsilon(u_n) - \mathcal{T}^*_\varepsilon(u)\|_{L^p(\mathbb{R}^d\times\mathcal{Y};\mathcal{B})} \longrightarrow 0$ for $n \longrightarrow \infty$. ∎

§4. Convergence machinery II: definition of two-scale convergence

This section will supersede two-scale convergence's original definition §2.1. We shall reformulate two-scale convergence in terms of periodic unfolding. The main reason to do so is the resulting functional analytic conciseness which allows us to retrieve the classical two-scale compactness theorems of G. Allaire and G. Nguetseng in a very natural manner.

Since periodicity is linked to smoothness properties of functions, we shall begin with the definition of periodicity notions in the context of varying regularity. Throughout, $(\mathcal{B}, |\cdot|_{\mathcal{B}})$ will stand for an arbitrary Banach space. Of course, in all relevant applications $\mathcal{B}$ will be required to be separable in order to have 'sufficiently many' non-trivial, measurable functions at hand.

§4.1. Periodic distributions and functions

A function $u : \mathbb{R}^d \longrightarrow \mathscr{B}$ is $\mathbb{Z}^d$-periodic if it is invariant under translation by vectors from $\mathbb{Z}^d$. Writing $\tau_y(u) = u(\,\cdot\, + y)$ for $y \in \mathbb{R}^d$, this can be formulated as

$$(4.1) \qquad u : \mathbb{R}^d \longrightarrow \mathscr{B} \text{ is } \mathbb{Z}^d\text{-periodic} \Longleftrightarrow \forall z \in \mathbb{Z}^d : \ \tau_z(u) = u.$$

Likewise, one may consider equivalence classes of $\mathbb{Z}^d$-periodic functions which are invariant under τ_z almost everywhere in $\mathbb{R}^d$. Moreover, any $\mathbb{Z}^d$-periodic function is in one-to-one correspondence with a function $\widetilde{u} : \mathcal{Y} \longrightarrow \mathscr{B}$. Naturally, periodicity with respect to grids different from $\mathbb{Z}^d$ make sense as well; these are linked by a simple change of coordinates.

Now, observe that a non-trivial, infinitely differentiable, $\mathbb{Z}^d$-periodic function u is not a test functions in $\mathbb{R}^d$ – test functions being elements of $C_0^\infty(\mathbb{R}^d;\mathscr{B})$ – due non-compact support in $\mathbb{R}^d$. In principle, this inconvenience can be overcome in two-fold fashion: either one realigns the notion of test functions or one rearranges the domain of definition. In essence, both approaches are equivalent in the sense that the resulting function spaces are isomorphic in a canonical and topological way, we refer to [Tri83, Ch. 9] for a full discussion. Since we have been working with the flat torus $\mathcal{Y} := \mathbb{R}^d/\mathbb{Z}^d$ already, we shall stick with the approach of changing the domain when defining distributions that are periodic an a specific variable, our reference on distributions being [Sch55; Sch57; Sch58].

Definition §4.1: Periodic test functions

Let $\Omega \subset \mathbb{R}^d$ be an open set. The (vector) space of $\mathscr{B}$-valued test functions on $\Omega \times \mathcal{Y}$, is given by

$$(4.2) \quad \mathscr{D}(\Omega\times\mathcal{Y};\mathscr{B}) := C_0^\infty(\Omega\times\mathcal{Y};\mathscr{B}) := \{u \in C^\infty(\Omega\times\mathcal{Y};\mathscr{B}) : \operatorname{supp}(u) \subset\subset \Omega\times\mathcal{Y}\}.$$

Clearly, one can consider $u \in \mathscr{D}(\Omega\times\mathcal{Y};\mathscr{B})$ as a $\mathscr{B}$-valued, smooth function on $\Omega\times\mathbb{R}^d$, $\mathbb{Z}^d$-periodic in its second argument. Realise that for $\Omega = \emptyset$ $\mathscr{D}(\mathcal{Y};\mathscr{B}) = C^\infty(\mathcal{Y};\mathscr{B})$ due to $\mathcal{Y}$ being compact.

So far, $\mathscr{D}(\Omega\times\mathcal{Y};\mathscr{B})$ is a vector space without topology. Let us specify the inductive (locally convex) limit topology which we recall for convenience's sake. One starts with an ascending sequence $(\Omega_k)_{k\in\mathbb{N}}$ of compact subsets of Ω with $\cup_{k\in\mathbb{N}}\Omega_k = \Omega$. $(\Omega_k)_{k\in\mathbb{N}}$ induces a countable family of semi-norms on $C^\infty(\Omega\times\mathcal{Y};\mathscr{B})$, namely

$$(4.3) \qquad \rho_{k,\ell}(u) := \max\{|(D^\alpha u)(x,y)|_{\mathscr{B}} : (x,y) \in \Omega_k\times\mathcal{Y}, |\alpha| \le \ell\} \quad \text{for } k,\ell \in \mathbb{N}.$$

As shown in lemma §4.1 below, $\mathscr{E}(\Omega\times\mathcal{Y};\mathscr{B}) := (C^\infty(\Omega\times\mathcal{Y};\mathscr{B}), \{\rho_{k,\ell}\}_{k,\ell\in\mathbb{N}})$ is not only locally convex, but even a Fréchet space in which the spaces $\mathscr{D}(\Omega_k\times\mathcal{Y};\mathscr{B})$ form an ascending sequence of closed subspaces. The inductive limit topology is defined as the final topology of the inclusion maps $\iota_k : \mathscr{D}(\Omega_k\times\mathcal{Y};\mathscr{B}) \longhookrightarrow \mathscr{D}(\Omega\times\mathcal{Y};\mathscr{B})$. This means that $U \subset \mathscr{D}(\Omega\times\mathcal{Y};\mathscr{B})$ is open if and only if $\iota_k^{-1}(U)$ is open in $\mathscr{D}(\Omega_k\times\mathcal{Y};\mathscr{B})$ for every $k \in \mathbb{N}$.

In fact, the semi-norms $\{\rho_{k,\ell}\}_{k,\ell\in\mathbb{N}}$ also yield a uniform structure which is transferred by the inductive limit uniform structure. In hands-on terms, one finds that a sequence $(\varphi_k)_{k\in\mathbb{N}} \subset \mathscr{D}(\Omega\times\mathcal{Y};\mathscr{B})$ is Cauchy if and only if

$$(4.4) \qquad \Big\{ \ \exists m \in \mathbb{N} : \forall k \in \mathbb{N} : \quad \operatorname{supp}(\varphi_k) \subset \Omega_m \quad \wedge \quad \forall \ell \in \mathbb{N} : \rho_{m,\ell}(\varphi_i - \varphi_j) \overset{i,j\to\infty}{\longrightarrow} 0.$$

For $\mathscr{B} = \mathbb{R}$, endowing $\mathscr{D}(\Omega \times \mathcal{Y}; \mathbb{R})$ with the locally convex topology and the uniform structure that stem from the inductive limit construction yields a complete Montel space. In particular, it has non-trivial, continuous linear forms and is reflexive, being no Fréchet-space though.

Definition §4.2: **Periodic distributions**

The space of $\mathscr{B}$-valued distributions on $\Omega \times \mathcal{Y}$ is given by

$$\mathscr{D}'(\Omega \times \mathcal{Y}; \mathscr{B}) := \mathscr{L}(\mathscr{D}(\Omega \times \mathcal{Y}; \mathbb{R}); \mathscr{B}) \tag{4.5}$$

equipped with the topology of uniform convergence on bounded sets. Throughout, one usually drops the codomain if $\mathscr{B} = \mathbb{R}$, writing only $\mathscr{D}'(\Omega \times \mathcal{Y})$ instead of $\mathscr{D}'(\Omega \times \mathcal{Y}; \mathscr{B})$.

As before, one can consider $\mathscr{D}'(\Omega \times \mathcal{Y}; \mathscr{B})$ as $\mathscr{B}$-valued distributions on $\Omega \times \mathcal{Y}$ that are '$\mathbb{Z}^d$-periodic in their second argument', requiring to say what a periodic distribution should be in the first place. Yet, we do not elaborate on this matter any further, see [Tri83, Ch. 9] for further reference. The following lemma complements the foregoing definition.

Lemma §4.1: **Metrisability of $\mathscr{E}$**

Writing $\rho_k := \rho_{k,k}$, $d(u, w) := \sum_{k \in \mathbb{N}} 2^{-k} \arctan(\rho_k(u - w))$ is a metric on $\mathscr{E}(\Omega \times \mathcal{Y}; \mathscr{B})$ whose induced topology coincides with the topology given by the family of semi-norms $(\rho_{k,\ell})_{k,\ell \in \mathbb{N}}$. Moreover, $\mathscr{E}(\Omega \times \mathcal{Y}; \mathscr{B})$ is a Fréchet space then.

Proof. First, observe that both families $(\rho_k)_{k \in \mathbb{N}}$ and $(\rho_{k,\ell})_{k,\ell \in \mathbb{N}}$ are countable, so, it is sufficient to consider sequences. Secondly, the two families of metrics are equivalent on $\mathscr{E}$ in the sense that they yield the same convergent sequences, a fact made possible by dropping the requirement of compactness of support, see (4.4). Thirdly, let $d(u_n, u) \xrightarrow{n \to 0} 0$, then we have $0 \leq \arctan(\rho_k(u_n - u)) \leq 2^k d(u_n, u) \xrightarrow{n \to 0} 0$ for every $k \in \mathbb{N}$, from which $\rho_k(u - u_n) \xrightarrow{n \to 0} 0$ follows.
Conversely, let $\forall k \in \mathbb{N} : \rho_k(u_n - u) \xrightarrow{n \to 0} 0$ hold, and let $\delta_0 > 0$ be given. Since $0 \leq \arctan(y) \leq \pi/2$ for all $y \geq 0$, there exists $M = M(\delta_0) \in \mathbb{N}_0$ such that $\sum_{k \geq M} 2^{-k} \arctan(\rho_k(u_n - u)) \leq \delta_0$. Thus, the metric's series can be truncated as $d(u_n, u) \leq \sum_{k \leq M} 2^{-k} \arctan(\rho_k(u_n - u)) + \delta_0$. Now, for $n \geq N(\delta_0)$ we have $\arctan(\rho_k(u_n - u)) \leq \delta_0/2$, for $k = 0, \ldots, M$ which infers $d(u_n, u) \leq 5\delta_0$ for $n \geq N(\delta_0) + M(\delta_0)$.
For $\mathscr{E}(\Omega \times \mathcal{Y}; \mathscr{B})$ to be a Frécht space, one needs to verify local convexity of the topology – which is obvious due to metrisibility –, translation invariance of the metric itself – obvious as well – and completeness. For the last assertion, it suffices to consider a Cauchy sequence $(u_n)_{n \in \mathbb{N}} \subset \mathscr{E}(\Omega \times \mathcal{Y}; \mathscr{B})$ which can even be reduced to $(|u_n|_{\mathscr{B}})_{n \in \mathbb{N}} \subset \mathscr{E}(\Omega \times \mathcal{Y}; \mathbb{R})$. As customary for completeness proofs, one resorts completeness of $\mathbb{R}$ by pointwise convergence. Thus, a candidate function $|u|$ is in reach and differentiability of u can be verified without pain thanks to the very strong convergence properties available from the semi-norms ρ_k. ■

§4.2. At last: definition of two-scale convergence

Here, we present the definition of two-scale convergence based on periodic unfolding for L^p-functions. We do stress though, that periodic unfolding does require L^1_{loc}-regular functions. For the purpose of less regular objects, one has to resort to an adaptation of the original definition. Let us start with distributions.

Definition §4.3: Two-scale convergence of distributions

A sequence $(T_\varepsilon)_{\varepsilon\in\mathbb{J}} \subset \mathscr{D}'(\Omega;\mathscr{B})$ is strongly two-scale convergent in the sense of distributions to $T \in \mathscr{D}'(\Omega\times\mathcal{Y},\mathscr{B})$ for $\mathbb{J} \ni \varepsilon \longrightarrow 0$ if we have

$$\forall\varphi \in \mathscr{D}(\Omega\times\mathcal{Y}): \quad |T_\varepsilon(\varphi(x,x/\varepsilon)) - T(\varphi(x,y))|_{\mathscr{B}} \overset{\varepsilon\to 0}{\longrightarrow} 0, \tag{4.6}$$

and (weakly) two-scale convergent in the sense of distributions to $T \in \mathscr{D}'(\Omega\times\mathcal{Y},\mathscr{B})$

$$\forall\varphi \in \mathscr{D}(\Omega\times\mathcal{Y}), b' \in \mathscr{B}': \quad b'\left(T_\varepsilon(\varphi(x,x/\varepsilon)) - T(\varphi(x,y))\right) \overset{\varepsilon\to 0}{\longrightarrow} 0, \tag{4.7}$$

writing simply $T_\varepsilon \overset{2s}{\longrightarrow} T$ in $\mathscr{D}'(\Omega\times\mathcal{Y};\mathscr{B})$ and $T_\varepsilon \overset{2w}{\rightharpoonup} T$ in $\mathscr{D}'(\Omega\times\mathcal{Y};\mathscr{B})$, respectively.

Since $\mathcal{T}_\varepsilon^*(\varphi) \notin \mathscr{D}(\mathbb{R}^d\times\mathcal{Y})$ holds for any non-trivial $\varphi \in \mathscr{D}(\Omega\times\mathcal{Y})$, periodic unfolding is rather misfit for the foregoing definition. Alas, this classically flavoured definition circumvents auxiliary domains altogether.

Contrasting the classical definition, we will use periodic unfolding for sufficiently regular functions as it clearly indicates where the two-scale topology actually comes from, namely from $L^p(\mathbb{R}^d\times\mathcal{Y};\mathscr{B})$ together with periodic unfolding. This information is not retrievable in the original definition §2.1.

Definition §4.4: Two-scale convergence of functions

Let $\Omega \subset \mathbb{R}^d$ be an open set with compact, Lipschitz-regular boundary. For $p \in [1,\infty]$, a sequence $(w_\varepsilon)_{\varepsilon\in\mathbb{J}} \subset L^p(\Omega;\mathscr{B})$ is strongly two-scale convergent (in L^p) to a function $w \in L^p(\Omega\times\mathcal{Y},\mathscr{B})$ for $\mathbb{J} \ni \varepsilon \longrightarrow 0$ if we have

$$\mathcal{T}_\varepsilon^*(w_\varepsilon) = \mathcal{T}_{\varepsilon,\Omega}^*(w_\varepsilon) \longrightarrow w \text{ in } L^p(\mathbb{R}^d\times\mathcal{Y};\mathscr{B}), \tag{4.8}$$

and (weakly) two-scale convergent (in L^p) to $w \in L^p(\Omega\times\mathcal{Y},\mathscr{B})$ if

$$\mathcal{T}_\varepsilon^*(w_\varepsilon) = \mathcal{T}_{\varepsilon,\Omega}^*(w_\varepsilon) \rightharpoonup w \text{ in } L^p(\mathbb{R}^d\times\mathcal{Y};\mathscr{B}) \tag{4.9}$$

holds, with 'weakly' being replaced by 'weakly*' if $p = \infty$. Finally, we will mostly write $w_\varepsilon \overset{2s}{\longrightarrow} w$ in $L^p(\mathbb{R}^d\times\mathcal{Y};\mathscr{B})$ and $w_\varepsilon \overset{2w}{\rightharpoonup} w$ in $L^p(\mathbb{R}^d\times\mathcal{Y};\mathscr{B})$, respectively.

Again, we stress the concept of separating convergence structures from the actual function space under consideration. For instance, for $w_\varepsilon \in L^p(\Omega;\mathscr{B})$ one has $\mathcal{T}_\varepsilon^*(w_\varepsilon) \in L^p(\Omega_\varepsilon^+\times\mathcal{Y};\mathscr{B})$ with $L^p(\Omega_\varepsilon^+\times\mathcal{Y};\mathscr{B}) \hookrightarrow L^p(\mathbb{R}^d\times\mathcal{Y};\mathscr{B})$ by extension by zero. Thus, $L^p(\mathbb{R}^d\times\mathcal{Y};\mathscr{B})$'s primary role is to supply a topology. It does not mean that w_ε was assumed to be some element from $L^p(\mathbb{R}^d;\mathscr{B})$ in the first place – this is true if and only if w_ε is extended by zero. Instead, it is our very aim to establish that a two-scale limit is indeed an element of $L^p(\Omega\times\mathcal{Y};\mathscr{B})$, a fact which is not entirely obvious when abandoning the sets Ω and $\Omega\times\mathcal{Y}$ when turning to auxiliary domains where periodic unfolding works nicely.

Corollary §4.1: **Equivalent notions**

Let $p \in (1, \infty)$ and $\mathscr{B}$ both reflexive and separable. Then, classical two-scale convergence and two-scale convergence via periodic unfolding coincide.

Proof. We revisit the original definition §2.1 with $\psi \in C^0(\overline{\Omega} \times \mathcal{Y}; \mathscr{B})$ and use (3.17) to get

$$\int_{\Omega} w_\varepsilon(x)\psi(x, x/\varepsilon)\, dx = \iint_{\mathbb{R}^d \times \mathcal{Y}} \mathcal{T}_\varepsilon^*(w_\varepsilon)(x, y)\widetilde{\psi}(\mathcal{T}_\varepsilon(x, y), y)\, dydx \tag{4.10}$$

which exploits the multiplicative property $\mathcal{T}_\varepsilon^*(ab) = \mathcal{T}_\varepsilon^*(a)\mathcal{T}_\varepsilon^*(b)$ of periodic unfolding. Now, if $(w_\varepsilon)_{\varepsilon\in\mathbb{J}}$ is two-scale convergent in the original sense to $w_0 \in L^p(\Omega \times \mathcal{Y})$ then (4.10) converges to $\iint_{\Omega\times\mathcal{Y}} w_0(x, y)\psi(x, y)\, dydx$. At the same time, $\psi(\mathcal{T}_\varepsilon(x, y), y) \overset{\varepsilon\to0}{\longrightarrow} \psi(x, y)$ in $L^{p'}(\mathbb{R}^d \times \mathcal{Y}; \mathscr{B})$ due to its uniform continuity, a fact that is revisited in section §5 later on. Ruling out $p = 1$, the subspace $\left\{\mathcal{T}_\varepsilon^*(\psi) : \psi \in C^0(\overline{\Omega} \times \mathcal{Y}; \mathscr{B})\right\}$ is dense for every $\varepsilon > 0$ in $\mathcal{T}_\varepsilon^*(L^{p'}(\Omega; \mathscr{B}))$ given in corollary §3.1. Foiling the use of $p = \infty$, one can rewrite $\mathcal{T}_\varepsilon^*(w_\varepsilon) \rightharpoonup w_0 \in L^p(\Omega \times \mathcal{Y}; \mathscr{B})$ as

$$\forall \varphi \in L^{p'}(\mathbb{R}^d \times \mathcal{Y}; \mathscr{B}') : \quad \iint_{\mathbb{R}^d\times\mathcal{Y}} \left(\mathcal{T}_\varepsilon^*(w_\varepsilon) - w_0\right) \varphi\, dydx \longrightarrow 0 \tag{4.11}$$

such that equivalence is shown. ∎

Remark §4.1: **Periodicity defect and smoothness**

Periodicity is linked to a minimal degree of smoothness, for illustration consider $Y = [0, 1)^d$ and $\mathcal{Y} = \mathbb{R}^d/\mathbb{Z}^d$ where $L^p(\mathcal{Y}) \simeq L^p(Y)$ and $W^{1,p}(\mathcal{Y}) \subsetneq W^{1,p}(Y)$ holds in general. Similarly, $\mathcal{T}_\varepsilon^*(W^{1,p}_{loc}(\mathbb{R}^d)) \overset{?}{\subset} L^p_{loc}(\mathbb{R}^d; W^{1,p}(\mathcal{Y}))$ needs to be adjusted to

$$\forall p \in [1, \infty) : \quad \mathcal{T}_\varepsilon^*(W^{1,p}_{loc}(\mathbb{R}^d)) \subset L^p_{loc}(\mathbb{R}^d; W^{1,p}(Y))$$

Thus, sensible periodicity cannot be expected by mere unfolding, a circumstance dubbed *periodicity defect*, for a deeper discussion see [Rei15].

§4.3. A description in categorical terms

Here, we want formalise the exhaustion procedure underlying periodic unfolding with the help of category theory. Though the latter is not particularly esteemed among analysts at present, we assert that it does help in providing decent and handy formulations of the procedures and structures under consideration, making things a little more well-arranged. We will keep the necessary machinery at an absolute minimum, referring to appendix A for elementary definitions and examples.

The underlying directed sets

Recall that we assumed to be given an open set $\Omega \subset \mathbb{R}^d$ with compact, $C^{0,1}$-boundary. The auxiliary domains provide the configurations $\Omega_\varepsilon^+ \searrow \Omega$ and $\Omega_\varepsilon^- \nearrow \Omega$ in the sense that $\cap_{\varepsilon\in\mathbb{J}}\Omega_\varepsilon^+ =$

$\cup_{\varepsilon\in\mathbb{J}}\Omega_\varepsilon^- = \Omega$ holds. In fact, this isotone families of set provide the following property

$$\forall \varepsilon_1,\varepsilon_2 \in \mathbb{J} : \exists \varepsilon_3 \in \mathbb{J} : \varepsilon_3 \le \varepsilon_1,\varepsilon_2 \quad \wedge \quad \begin{cases} \Omega^+_{\varepsilon_1} \cap \Omega^+_{\varepsilon_2} \longleftarrow \Omega^+_{\varepsilon_3} \overset{\text{Def.}}{\Longleftrightarrow} \Omega^+_{\varepsilon_3} \lhd_+ \Omega^+_{\varepsilon_1}, \Omega^+_{\varepsilon_2} \\ \Omega^-_{\varepsilon_1} \cup \Omega^-_{\varepsilon_2} \hookrightarrow \Omega^-_{\varepsilon_3} \overset{\text{Def.}}{\Longleftrightarrow} \Omega^+_{\varepsilon_1}, \Omega^+_{\varepsilon_2} \lhd_- \Omega^+_{\varepsilon_3}. \end{cases} \tag{4.12}$$

Thus, both families of sets $(\Omega_\varepsilon^\pm)_{\varepsilon\in\mathbb{J}}$ are directed sets when endowed with the preorders $\lhd_\pm$ induced by set inclusions. For $\left[\{\Omega\} \cup (\Omega_\varepsilon^+)_{\varepsilon\in\mathbb{J}}, \lhd_+\right]$ one can consider Ω as an least element. Likewise, Ω is the maximum element of $\left[\{\Omega\} \cup (\Omega_\varepsilon^-)_{\varepsilon\in\mathbb{J}}, \lhd_-\right]$.

However, since $\{\Omega\}$ is not contained in $(\Omega_\varepsilon^\pm)_{\varepsilon\in\mathbb{J}}$ it rather plays the role of an element in a suitable completion. In fact, one can follow this direction by perceiving $\left[\{\Omega\} \cup (\Omega_\varepsilon^\pm)_{\varepsilon\in\mathbb{J}}, \lhd_\pm\right]$ as *semi-lattices* that can be completed via the *Dedekind–MacNeille completion*, see [AHS04, 21H] for a full exposition (in categorical terms).

In a more simple manner, one can identify Ω as *inductive* and *projective limit* of families of sets. This notion will be suitable to be carried over the corresponding function spaces which is why prefer it, next to its simplicity.

The auxiliary domains' categories and their limits

Now, both $\left[(\Omega_\varepsilon^\pm)_{\varepsilon\in\mathbb{J}}, \lhd_\pm\right]$ can be thought of as categories $\mathscr{C}_\pm$ with $Ob(\mathscr{C}_\pm) = (\Omega_\varepsilon^\pm)_{\varepsilon\in\mathbb{J}}$ and morphism given by set inclusions or orderings:

$$\mathrm{Mor}_{\mathscr{C}_\pm}\left(\Omega^\pm_{\varepsilon_1}, \Omega^\pm_{\varepsilon_2}\right) := \begin{cases} \Omega^+_{\varepsilon_1} \longleftarrow \Omega^+_{\varepsilon_2} & \text{for } \Omega^+_{\varepsilon_1} \lhd_+ \Omega^+_{\varepsilon_2}, \\ \Omega^-_{\varepsilon_1} \hookrightarrow \Omega^-_{\varepsilon_2} & \text{for } \Omega^-_{\varepsilon_1} \lhd_- \Omega^-_{\varepsilon_2}, \\ \emptyset & \text{else.} \end{cases} \tag{4.13}$$

Again, this is motivated by supplying morphisms if and only if set inclusions hold, thus rephrasing orderings as morphisms. Obviously, $\mathscr{C}_\pm$ are small categories and thus they can be embedded into **Set** (via the forgetful functor) with $\mathscr{C}_+$ forming a projective system and $\mathscr{C}_-$ yielding an inductive system there.

Lemma §4.2: **Auxiliary domains and limits**

Considering $\mathscr{C}_\pm$ as a diagram in **Set** we obtain

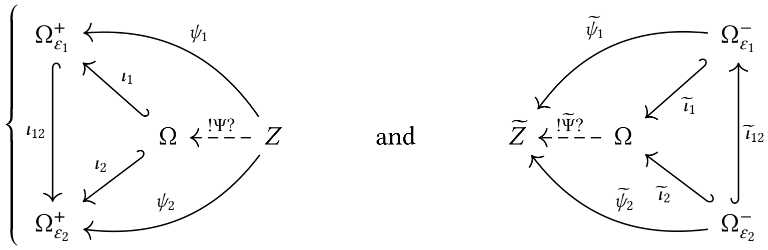
$$\lim_{\varepsilon\to 0}\mathrm{proj}\, \Omega_\varepsilon^+ = \Omega = \lim_{\varepsilon\to 0}\mathrm{ind}\, \Omega_\varepsilon^-. \tag{4.14}$$

Those who are familiar with category theory may argue the statement is trivial by common folklore. Nevertheless, we deem such proofs non-standard in the given context so we do give a proof. Proofs of limit and co-limit constructions always need to verify two aspects: first, the *cone property*[2] also known as *source property* or its counterpart the *co-cone* or *target property* and second, *universality* which states that the desired object is unique up to isomorphisms of the category under consideration. Note that a **Set**-isomorphism is a bijection of sets, whereas a **Diff**-isomorphism is a diffeomorphism, for instance.

[2] This notion is purely categorical. It has no relation at all to the cone property of Euclidean domains $\Omega \subset \mathbb{R}^d$.

Proof. We assume $\Omega^{\pm}_{\varepsilon_2} \lhd_{\pm} \Omega^{\pm}_{\varepsilon_1}$ and write $\iota_{1,2}, \iota_1, \iota_2\ \widetilde{\iota}_{1,2}, \widetilde{\iota}_1$ and $\widetilde{\iota}_2$ for inclusion maps between $\Omega, \Omega^{\pm}_{\varepsilon_1}$ and $\Omega^{\pm}_{\varepsilon_2}$. Consider the following two diagrams.

(4.15)



The sets $Z, \widetilde{Z}$ and their morphisms $\psi_1, \psi_2, \widetilde{\psi}_1, \widetilde{\psi}_2$ are relevant only to universality and can be ignored when addressing the cone and co-cone property.
So, to show the cone property in the left diagram, one needs to verify that $\iota_{1,2} \circ \iota_1 = \iota_2$ holds. This is trivially the case. The co-cone property on the right is treated in very same manner.
Second, let us turn to universality. Here, one assumes the existence of another cone or co-cone intending to show that unique maps Ψ or $\widetilde{\Psi}$ exist that factor the maps as shown in the diagrams.
Let us deal with the diagram on the right. For every $x \in \Omega_{\varepsilon_2}$ we have $\widetilde{\iota}_1 \circ \widetilde{\iota}_{1,2}(x) = \widetilde{\iota}_2(x)$ and $\widetilde{\psi}_2(x) = \widetilde{\psi}_1 \circ \widetilde{\iota}_{1,2}(x)$ by assumption. Now, one tries to define $\widetilde{\Psi}$ by the relation

$$\Omega \ni \widehat{x} \longmapsto \widetilde{\Psi}(\widehat{x}) := \widetilde{\psi}(x) \tag{4.16}$$

assuming $\varepsilon \in \mathbb{J}$ to be such that $\exists x \in \Omega^-_{\varepsilon} : \widehat{x} = \widetilde{\iota}(x)$ together with a morphism $\widetilde{\psi} : \Omega^-_{\varepsilon} \longrightarrow \widetilde{Z}$. Of course, the resulting relation does not need to be a well-defined function this way. Fortunately, it precisely the commutativity of the embedding maps $\widetilde{\iota}_1, \widetilde{\iota}_2, \widetilde{\iota}_{1,2}$ that yields this desirable property and in addition, the universality property.
We conclude with arguing that the diagram on the left hand side can be handled in the very same fashion. ∎

Having limits in **Set** is of fundamental nature: though it is not the category we are actually after, it will serve to provide suitable candidates for limits in more appropriate categories. Indeed, this is no coincidence from an abstract point of view: vast amounts of analysis take place in the *topos of sets*. Other branches of mathematics are lead to alternative topoi for their own reasons, most prominently algebraic topology, see [Ble17]. We do not digress on this matter any further, but refer to [Gol84] for a classical and very decent overview instead.

Function spaces regarded as functors

Functors describe correspondences among categories and their morphisms. It will be convenient to consider L^p-spaces as functors from a modified category of measure spaces to the category of Banach spaces. In more detail, the standard category of measure spaces **Measure** having measure spaces $(\omega, \Sigma_\omega, \mu_\omega)$ as objects and measure preserving maps as morphisms, is rather badly behaved. One encounters several technical issues which are irrelevant to our endeavour making it misfit for us. Rather, we intend to augment $\mathscr{C}_{\pm}$ as a directed systems of measure spaces in a very straightforward and elementary fashion that suits our purpose.

To this end, let **Measure**$_*$ be the category of measure spaces (ω, Σ, μ) as objects with morphisms $f : (\omega_1, \Sigma_1, \mu_1) \longrightarrow (\omega_2, \Sigma_2, \mu_2)$ that satisfy

$$f^{-1}(\Sigma_2) \subset \Sigma_1\ ,\ f(\Sigma_1) \subset \Sigma_2 \ \text{ and } \ \forall E \in \Sigma_1 : \mu_1(E) = \mu_2(f(E)). \tag{4.17}$$

$\mathscr{C}_\pm$ can be embedded into **Measure**$_*$ canonically: one attaches to each $\Omega_\varepsilon^\pm \in \mathscr{C}_\pm$ the σ-algebra $\mathcal{L}(\Omega_\varepsilon^\pm)$ stemming from $\mathcal{L}(\mathbb{R}^d)$ by restriction to $\Omega_\varepsilon^\pm$ and the measure $\lambda^d_{|\Omega_\varepsilon^\pm}$. One verifies effortlessly that embedding maps $\iota_{1,2} : \Omega_{\varepsilon_1}^\pm \hookrightarrow \Omega_{\varepsilon_2}^\pm$ fulfil (4.17).

Lemma §4.3

$\mathscr{C}_\pm$ form diagrams in **Measure**$_*$ whose projective and direct limit is $(\Omega, \mathcal{L}(\Omega), \lambda^d_{|\Omega})$.

Proof. Verifying $(\Omega, \mathcal{L}(\Omega), \lambda^d_{|\Omega})$ as cone or co-cone, respectively, is trivial due to the imbedding maps being restrictions of the identity map.
To show universality of the co-cone, we reconsider (4.15) as a diagram in **Measure**$_*$. It is sufficient to show that the factorisation map defined via (4.16) is a **Measure**$_*$-morphism,taking into account that the maps $\widetilde{\psi}_1$ and $\widetilde{\psi}_2$ are assumed **Measure**$_*$-morphism. However, this is easily verified.
Treating universality of the cone accordingly, one obtains the desired result. ∎

Next, we can specify a twofold association: assume $\Omega_{\varepsilon_1}^\pm \lhd_\pm \Omega_{\varepsilon_2}^\pm$ so that there is an embedding $\iota_{1,2}^\pm : \Omega_{\varepsilon_1}^\pm \hookrightarrow \Omega_{\varepsilon_2}^\pm$ and $\varepsilon_1 \leq \varepsilon_2$ holds. Clearly, the map

$$\left(\Omega_\varepsilon^\pm, \mathcal{L}(\Omega_\varepsilon^\pm), \lambda^d_{|\Omega_\varepsilon^\pm}\right) \longmapsto L^p\left(\left(\Omega_\varepsilon^\pm, \mathcal{L}(\Omega_\varepsilon^\pm), \lambda^d_{|\Omega_\varepsilon^\pm}\right); \mathscr{B}\right) = L^p(\Omega_\varepsilon^\pm, \mathscr{B}) \tag{4.18}$$

makes sense and yields a Banach space for $p \in [1, \infty]$ and a given Banach space $\mathscr{B}$. Of course, the foregoing holds with Ω in place of $\Omega_\varepsilon^\pm$. Complementing the map of measure spaces to Banach spaces, we need to associate a map $\kappa_{1,2}^\pm$ to every $\iota_{1,2}^\pm$. Of course, this is provided by extending functions by zero outside of their domain, yielding

$$\iota_{1,2}^\pm \longmapsto \left[L^p(\Omega_{\varepsilon_1}^\pm, \mathscr{B}) \overset{\kappa_{1,2}^\pm}{\hookrightarrow} L^p(\Omega_{\varepsilon_2}^\pm, \mathscr{B})\right] \quad u \longmapsto \kappa_{1,2}^\pm(u) := \begin{cases} u & \text{on } \Omega_{\varepsilon_1}^\pm \\ 0 & \text{else,} \end{cases} \tag{4.19}$$

if and only if $\iota_{1,2}^\pm : \Omega_{\varepsilon_1}^\pm \hookrightarrow \Omega_{\varepsilon_2}^\pm$ exists in the first place. Again, one can replace some $\Omega_\varepsilon^\pm$ with Ω itself since the latter constructions rely on the presence of embeddings and $\Omega_\varepsilon^- \lhd_- \Omega \lhd_+ \Omega_\varepsilon^+$ holds for all $\varepsilon > 0$. Furthermore, the preceding aspects apply to the functor $L^p(\,\cdot \times \mathcal{Y}, \mathscr{B})$ instead of $L^p(\,\cdot\,, \mathscr{B})$ as well. We have the following result.

Proposition §4.1

$L^p(\,\cdot\,;\mathscr{B})$ and $L^p(\,\cdot\,\times\mathcal{Y};\mathscr{B})$ can be understood as functors mapping **Measure**$_*$ into **Ban**$_1$. The respective images of $\mathscr{C}_\pm$ under both functors yield diagrams in **Ban**$_1$ whose limits from **Measure**$_*$ are preserved continuously and co-continuously which means that we have

$$\lim\text{proj}\, L^p(\Omega_\varepsilon^+;\mathscr{B}) = L^p(\Omega;\mathscr{B}) = \lim\text{ind}\, L^p(\Omega_\varepsilon^-;\mathscr{B}) \tag{4.20}$$

in **Ban**$_1$, together with

$$\lim\text{proj}\, L^p(\Omega_\varepsilon^+\times\mathcal{Y};\mathscr{B}) = L^p(\Omega\times\mathcal{Y};\mathscr{B}) = \lim\text{ind}\, L^p(\Omega_\varepsilon^-\times\mathcal{Y};\mathscr{B}). \tag{4.21}$$

Proof. The statements can be shown just like in the foregoing proofs. Following [Cas10, Sec. 4.2], one can use a direct description of limits and co-limits in **Ban**$_1$, for instance

$$\begin{cases} \lim\text{proj}\, L^p(\Omega_\varepsilon^+\times\mathcal{Y};\mathscr{B}) = & \big\{(u_\varepsilon)_{\varepsilon\in\mathbb{J}} : \forall\varepsilon\in\mathbb{J} : u_\varepsilon\in L^p(\Omega_\varepsilon^+\times\mathcal{Y};\mathscr{B}) \\ & \quad\wedge\ \sup_{\varepsilon\in\mathbb{J}}\|u_\varepsilon\| < \infty \\ & \quad\wedge\ \forall\varepsilon_1,\varepsilon_2\in\mathbb{J} : \Omega_{\varepsilon_1}^+ \lhd_+ \Omega_{\varepsilon_2}^+ \Longrightarrow \kappa_{1,2}^+(u_{\varepsilon_1}) = u_{\varepsilon_2}\big\} \end{cases} \tag{4.22}$$

$$\begin{cases} \lim\text{ind}\, L^p(\Omega_\varepsilon^-\times\mathcal{Y};\mathscr{B}) & = \big\{(u_\varepsilon)_{\varepsilon\in\mathbb{J}} : \forall\varepsilon\in\mathbb{J} : u_\varepsilon\in L^p(\Omega_\varepsilon^-\times\mathcal{Y};\mathscr{B}) \\ & \quad\wedge\ \sup_{\varepsilon\in\mathbb{J}}\|u_\varepsilon\| < \infty \\ & \quad\wedge\ \exists\varepsilon_1\in\mathbb{J} : \forall\varepsilon_2\in\mathbb{J} : \Omega_{\varepsilon_1}^- \lhd_- \Omega_{\varepsilon_2}^- \Longrightarrow \kappa_{1,2}^-(u_{\varepsilon_1}) = u_{\varepsilon_2}\big\} \end{cases} \tag{4.23}$$

for the case of (4.21), and similarly for (4.20). The descriptions in (4.22) and (4.23) yield Banach spaces as well and it is straightforward to verify that the resulting spaces are **Ban**$_1$-isomorphic, i.e. there is a isometric, linear homeomorphism to $L^p(\Omega\times\mathcal{Y};\mathscr{B})$ and $L^p(\Omega;\mathscr{B})$, respectively. Notice the pivotal role played by proposition §3.1 and by (3.7c) in particular. ■

Interlude: a warning on extensions

It should not come as a big surprise, that $W_0^{1,p}(\,\cdot\,,\mathscr{B})$ can be considered as a functor in likewise fashion, though we shall only work with $\mathscr{B}=\mathbb{R}^n$ when encountering Sobolev spaces. In stark contrast, the readers' will certainly anticipate that the situation is extremely involved for $W^{1,p}(\,\cdot\,,\mathscr{B})$. Ω_ε^- and Ω need to verify geometric restrictions in order to secure the existence of suitable extension operators: Unless one works with $W_0^{1,p}(\,\cdot\,,\mathscr{B})$ one cannot simply extend functions by zero without sacrificing weak differentiability.

In this respect, recall that E. Stein's famous construction of an (total) extension operator $E : W^{k,p}(U) \longrightarrow W^{k,p}(\mathbb{R}^d)$ for all $k\in\mathbb{N}$ and $p\in[1,\infty)$ holds under the assumption that $U\subset\mathbb{R}^d$ is a strong local Lipschitz domain, c.f. [Ste70]. Whereas such regularity conditions on Ω are very well acceptable, the Ω_ε^- simply not regular enough for Stein's extension operator.

So, category theory mirrors this subtle peculiarity of Sobolev space theory; a state of affairs that suits our paradigma of category theory being a tool to formulate preexisting knowledge conveniently. As mentioned before, one may resort to extension operators for (ϵ,δ)-domains from [Jon81] instead.

Conclusion – a sandwich lemma interpretation

We have seen in (3.5) that a direct application of $\mathcal{T}_\varepsilon^*$ to Ω is problematic. Fortunately, the situation is redeemed by proposition §4.1 which essentially resembles a sandwich lemma: starting from $\Omega_\varepsilon^- \subset \Omega \subset \Omega_\varepsilon^+$ we have seen that this monotone arrangement is preserved by the L^p-functors such that

$$\begin{array}{llll} L^p(\Omega_\varepsilon^-, \mathscr{B}) & \subset L^p(\Omega, \mathscr{B}) & \subset L^p(\Omega_\varepsilon^+, \mathscr{B}) & \text{and} \\ L^p(\Omega_\varepsilon^- \times \mathcal{Y}, \mathscr{B}) & \subset L^p(\Omega \times \mathcal{Y}, \mathscr{B}) & \subset L^p(\Omega_\varepsilon^+ \times \mathcal{Y}, \mathscr{B}) & \end{array}$$

hold. Proposition §4.1 then yields that the spaces $L^p(\Omega_\varepsilon^\pm \times \mathcal{Y}, \mathscr{B})$ converge to $L^p(\Omega \times \mathcal{Y}, \mathscr{B})$ in a rigorous sense, namely as limits and co-limits in $\mathbf{Ban_1}$ and besides, the same is true for $L^p(\Omega_\varepsilon^\pm, \mathscr{B})$ converging to $L^p(\Omega, \mathscr{B})$.

Quite relevant to us is the circumstance that $\mathcal{T}_\varepsilon^{-1}(\Omega)$ is inconvenient to describe but it certainly is a subset of $\mathcal{T}_\varepsilon^{-1}(\Omega_\varepsilon^+)$ by construction. Lemma §4.3 formalises the procedure of squeezing Ω in between the auxiliary domains $\Omega_\varepsilon^\pm$ and proposition §4.1 carries squeezing over to the resulting L^p-spaces, as well. Let us fix this result in a new subsection.

§4.4. Asymptotic convergence properties of periodic unfolding

The trivial embedding map $E : L^p(\Omega; \mathscr{B}) \longrightarrow L^p(\Omega \times \mathcal{Y}; \mathscr{B})$ is given by $E(u)(x, y) := u(x)$ and as customary, $\widetilde{E(u)} = E(\widetilde{u}) \in L^p(\mathbb{R}^d \times \mathcal{Y}; \mathscr{B})$ is its extension by zero outside of $\Omega \times \mathcal{Y}$ or Ω, respectively. Seeming rather unimpressive at first glance, the next result will turn out to be pivotal.

Proposition §4.2: **Asymptotic convergence of $\mathcal{T}_\varepsilon^*$**

Let $\Omega \subset \mathbb{R}^d$ be open with compact, Lipschitz-regular boundary and $p \in [1, \infty)$. $\mathcal{T}_\varepsilon^* = \mathcal{T}_{\varepsilon,\Omega,\mathscr{B}}^*$ converges to E in the following sense:

$$\forall u \in L^p(\Omega; \mathscr{B}) : \quad \left\| \mathcal{T}_\varepsilon^*(u) - \widetilde{E(u)} \right\|_{L^p(\mathbb{R}^d \times \mathcal{Y}; \mathscr{B})} \overset{\varepsilon \to 0}{\longrightarrow} 0. \tag{4.24}$$

For $p = \infty$ the statement is wrong in general.

Proof. To address (4.24) for $p < \infty$, let $u \in L^p(\Omega \times \mathcal{Y}; \mathscr{B})$ be given. To every $\delta > 0$ there exists $u_\delta \in \mathscr{D}(\Omega \times \mathcal{Y}; \mathscr{B})$ with $\|u - u_\delta\|_{L^p(\Omega \times \mathcal{Y}; \mathscr{B})} \leq \delta$, c.f. [Ama03, 1.3.6 Thm]. If we can show (4.24) for u_δ, then the statement follows for u due to

$$\left\{ \begin{array}{lcl} \|\widetilde{E(u)} - \mathcal{T}_\varepsilon^*(u)\|_{L^p(\mathbb{R}^d \times \mathcal{Y}; \mathscr{B})} & \leq & \left\| \widetilde{E(u - u_\delta)} \right\|_{L^p(\mathbb{R}^d \times \mathcal{Y}; \mathscr{B})} \\ + \left\| \widetilde{E(u_\delta)} - \mathcal{T}_\varepsilon^*(u_\delta) \right\|_{L^p(\mathbb{R}^d \times \mathcal{Y}; \mathscr{B})} & + & \left\| \mathcal{T}_\varepsilon^*(u_\delta - u) \right\|_{L^p(\mathbb{R}^d \times \mathcal{Y}; \mathscr{B})} \\ = 2\,\|u - u_\delta\|_{L^p(\Omega \times \mathcal{Y}; \mathscr{B})} & + & \left\| \widetilde{E(u_\delta)} - \mathcal{T}_\varepsilon^*(u_\delta) \right\|_{L^p(\mathbb{R}^d \times \mathcal{Y}; \mathscr{B})} \overset{\varepsilon \to 0}{\longrightarrow} 2\delta \end{array} \right. \tag{4.25}$$

which employs the isometry property of periodic unfolding (3.18) and extension by zero.

Turning to u_δ itself, let us employ Fubini's theorem and $\mathbb{R}^d = \sum_{z\in\mathbb{Z}^d}(\varepsilon z + \varepsilon[0,1)^d)$ to expand the integral under consideration.

$$(4.26)\quad \begin{cases} \|\widetilde{E(u_\delta)} - \mathcal{T}_\varepsilon^*(u_\delta)\|_{L^p(\mathbb{R}^d\times\mathcal{Y};\mathscr{B})} = \iint_{\mathbb{R}^d\times\mathcal{Y}} |\widetilde{u}_\delta(x) - \widetilde{u}_\delta(\mathcal{T}_\varepsilon(x,y))|^p_{\mathscr{B}}\, dydx \\ = \int_Y \sum_{z\in\mathbb{Z}^d}\int_{\varepsilon z+\varepsilon Y} |\widetilde{u}_\delta(\widehat{y}) - \widetilde{u}_\delta(\mathcal{T}_\varepsilon(\widehat{y},y))|^p_{\mathscr{B}}\, d\widehat{y}dy \\ = \int_Y \sum_{z\in\mathbb{Z}^d}\int_{\varepsilon Y} |\widetilde{u}_\delta(\varepsilon z + \widehat{y}) - \widetilde{u}_\delta(\mathcal{T}_\varepsilon(\varepsilon z,y))|^p_{\mathscr{B}}\, d\widehat{y}dy \\ = \int_Y \varepsilon^d \sum_{z\in\mathbb{Z}^d}\int_{Y} |\widetilde{u}_\delta(\varepsilon z + \varepsilon\widehat{y}) - \widetilde{u}_\delta(\varepsilon z + \varepsilon y)|^p_{\mathscr{B}}\, d\widehat{y}dy \\ \leq \varepsilon^d \sum_{z\in\mathbb{Z}^d}\int_Y\int_Y \left|\sup_{\varepsilon(z+Y)}\widetilde{u}_\delta - \inf_{\varepsilon(z+Y)}\widetilde{u}_\delta\right|^p dy\, d\widehat{y} \\ \leq \varepsilon^d \sum_{z\in\mathbb{Z}^d} \left|\sup_{\varepsilon(z+Y)}\widetilde{u}_\delta - \inf_{\varepsilon(z+Y)}\widetilde{u}_\delta\right|^p \\ \stackrel{\varepsilon\to 0}{\longrightarrow} 0 \end{cases}$$

Convergence to 0 for $\varepsilon \longrightarrow 0$ is only due to uniform continuity of $\widetilde{u}$ as there are $\sim \varepsilon^{-d}$ cubes $\varepsilon z+\varepsilon[0,1)^d$ to sum over. Using Hölder's inequality for sums, one can estimate the last term by $\sim C\sup_{z\in\mathbb{Z}^d}|\sup_{\varepsilon(z+Y)}\widetilde{u}_\delta - \inf_{\varepsilon(z+Y)}\widetilde{u}_\delta|^p$ for some fixed constant $C \geq 0$ which depends upon $R > 0$ such that $\operatorname{supp}(u_\delta) \subset\subset B_R(0)$. The second term measures the maximum oscillations on all cubes which vanish for $\varepsilon \longrightarrow 0$.
Finally, for the failure of $\mathcal{T}_\varepsilon \longrightarrow E$ on L^∞, consider $\Omega = (-1,1) \subset \mathbb{R} = \mathscr{B}$ and $u(x) = \sin(1/x)$ which clearly suffices $u \in L^\infty(\Omega)$, just as $\mathcal{T}_\varepsilon^*(u) \in L^\infty(\Omega\times\mathcal{Y})$. Let $\varepsilon \ll 1$ be fixed for the moment to consider $x_n = \varepsilon/2n$ such that $[x_n/\varepsilon] = [1/2n] = 0$ for $n \geq 1$ and insert x_n into $w_\varepsilon(x,y) := |u(x) - \mathcal{T}_\varepsilon^*(u)(x,y)|$ yielding

$$w_\varepsilon(x_n,y) = |\sin(2n/\varepsilon) - \sin(1/(\varepsilon y))|.$$

Keeping $y \in \mathcal{Y} = \pi([0,1))$ in mind, it is straightforward to see that $0 \leq \operatorname{ess\,sup}_{\Omega\times\mathcal{Y}} w_\varepsilon \neq 0$ must hold. Finally, neither a smaller ε nor removing λ^1-zero sets around 0 resolves this defect which relies on the essential singularity of u at $x = 0$. ∎

§5. Admissible two-scale test functions

The partial differential equations we encounter will invoke coefficient functions of the form $x \longmapsto f_\varepsilon(x) := f(x, x/\varepsilon, r, \vec{s})$ for a given function $f : \Omega\times\mathcal{Y}\times\mathbb{R}\times\mathbb{R}^d \longrightarrow \mathscr{B}$. For several reasons such coefficient functions f_ε are related to the following problem. Assume $L^p(\Omega;\mathbb{R}^{d+1}) \ni u_\varepsilon \stackrel{2s}{\longrightarrow} u$ in $L^p(\mathbb{R}^d\times\mathcal{Y};\mathbb{R}^{d+1})$ with $u_\varepsilon = (u_{0,\varepsilon}, u_{1,\varepsilon}, \dots, u_{d,\varepsilon})$ and $u = (u_0, u_1, \dots, u_d) \in L^p(\Omega\times\mathcal{Y};\mathbb{R}^{d+1})$, under which conditions on f can we infer

$$(5.1)\quad \left\{ w_\varepsilon(x) \stackrel{2s}{\longrightarrow} f(x,y,u_0(x,y),u_1(x,y),\dots,u_d(x,y)) \quad \text{in } L^p(\mathbb{R}^d\times\mathcal{Y};\mathscr{B}) \right.$$

for $x \longmapsto w_\varepsilon(x) := f(x, x/\varepsilon, u_{0,\varepsilon}(x), u_{1,\varepsilon}(x), \dots, u_{d,\varepsilon}(x))$? Notice that the notation used in (5.1) is sloppy, indeed. We do mean actual functions and not their values. This notation serves to illustrate which variables are present.

It turns out that $\mathscr{B}$ is not really the problem. Severe defects already occur for $\mathscr{B} = \mathbb{R}$. In fact, the question raised in (5.1) is inherently linked to the admissibility of two-scale test functions encountered in definition §2.1. Let us recall G. Allaire's famous counter-example function given in [All92, Prop. 5.8] which serves as a defective prototype violating (5.1).

Proposition §5.1: **Allaire's counter-example**

There exist a C^∞-domain $\Omega \subset \mathbb{R}^d$, a function $f : \Omega \times \mathcal{Y} \longrightarrow \mathbb{R}$ and a null sequence $\mathbb{J}$ such that (5.1) does not hold. Consequently, this function is not an admissible two-scale test function, i.e. $\lim_{\varepsilon\to 0} \int_\Omega f(x, x/\varepsilon)\, dx \neq \iint_{\Omega\times\mathcal{Y}} f(x, y)\, dydx$.

Proof. Choose $\Omega = (0, 1)$, $\mathcal{Y} = \mathbb{R}/\mathbb{Z}$, and set $f(x, y) := \chi_{\mathcal{A}}(x, y)$ with $\mathcal{A}$ being Allaire's defect set given by

$$\mathcal{A} := \{(x, y) \in \Omega \times \mathcal{Y} : y = \{x/\varepsilon\}_{\mathbb{Z}} = x/\varepsilon - \lfloor x/\varepsilon \rfloor \text{ for } \varepsilon = 1/n \text{ and } n \in \mathbb{N}\}\,. \tag{5.2}$$

Clearly, we have $\lambda^2(\mathcal{A}) = 0$ as $\mathcal{A}$ is the countable union of straight lines which are λ^2-zero sets in $\Omega \times \mathcal{Y}$. Therefore, we have $f(x, y) = 0$ for λ^2-almost all $(x, y) \in \Omega \times \mathcal{Y}$ but $f_\varepsilon(x) := f(x, x/\varepsilon) = 1$ for every $x \in \Omega$. Consequently, we have $\|f\|_{L^1(\Omega\times\mathcal{Y})} = 0$ and $\|f_\varepsilon\|_{L^1(\Omega)} = \|\mathcal{T}_\varepsilon^*(f_\varepsilon)\|_{L^1(\Omega_\varepsilon^+\times\mathcal{Y})} = 1$ for every $\varepsilon = 1/n$. ■

Several aspects of Allaire's counter-example are worthwhile to be discussed. First, the defectiveness of $\chi_{\mathcal{A}}$ does depend on the specific choice of $\varepsilon = 1/n$. If one constructs $\mathcal{A}$ in (5.2) using $\varepsilon = 1/n$ but inserting another null sequence $\eta_n \to 0$ into $\chi_{\mathcal{A}}(x, x/\eta)$, one can derive almost arbitrary results ranging from avoiding the defect altogether ($\chi_{\mathcal{A}}(x, x/\eta) = 0$ almost everywhere in Ω) to matching it perfectly again ($\chi_{\mathcal{A}}(x, x/\eta) = 1$ almost everywhere in Ω).

Secondly, G. Allaire's original example given in [All92, Prop 5.8] actually differs slightly from our presentation. The difference is as follows: The set $\mathcal{A}$ is originally constructed by open stripes made by lines with non-zero width. In more detail, for every $\varepsilon \in \mathbb{J}$ the lines given by $y = \{x/\varepsilon\}$ is given a width proportional to ε^3. This is sufficient to arrive at an open, dense set in $\Omega \times \mathcal{Y}$ without full measure, i.e. $\lambda^2(\Omega \times \mathcal{Y} \setminus \mathcal{A}) > 0$. Although the original configuration produces the same defect, we prefer no width at all since we think it distracts from the actual problem. More specifically, by giving the lines no width at all one arrives at $\mathcal{A}$ being a zero set. Most importantly, this allows to modify *any* function $g \in L^1(\Omega \times \mathcal{Y})$ on a zero set to make it violate (5.1). In particular, proposition §5.2 below ensures that every C^0-function is admissible but modifying it on $\mathcal{A}$ turns it into defective function. Consequently, we find that the equivalence classes in L^p-spaces are neither suitable for admissible test functions nor for (5.1) at all! For further discussion, let us fix some nomenclature that is to supersede definition §2.1, too.

Definition §5.1: Two-scale test functions

Let $\Omega \subset \mathbb{R}^d$ be open, $p \in [1, \infty)$, $\mathscr{B}$ be a Banach space. For a function $f : \Omega \times \mathcal{Y} \longrightarrow \mathscr{B}$ with $[f] \in L^p(\Omega \times \mathcal{Y}; \mathscr{B})$ we set $f_\varepsilon(x) := f(x, x/\varepsilon)$ for all $x \in \Omega$ and $\varepsilon \in \mathbb{J}$. Let us define the following two conditions.

A.1) We call f a two-scale test function in the sense of Allaire or simply an Allaire function (in $L^p(\Omega \times \mathcal{Y}; \mathscr{B})$) if $(f_\varepsilon)_{\varepsilon \in \mathbb{J}}$ fulfils

$$\lim_{\varepsilon \longrightarrow \infty} \|f_\varepsilon\|_{L^p(\Omega;\mathscr{B})} = \|f\|_{L^p(\Omega \times \mathcal{Y};\mathscr{B})} \,. \tag{5.3}$$

A.2) f is an admissible two-scale test function (in $L^p(\Omega \times \mathcal{Y}; \mathscr{B})$) if independently of the positive null sequence $\varepsilon \in \mathbb{J}$ the following two requirements are met: $\mathcal{T}_\varepsilon^*(f_\varepsilon) \in L^p(\Omega \times \mathcal{Y}; \mathscr{B})$ holds for cofinitely many $\varepsilon \in \mathbb{J}$ and $f_\varepsilon \overset{2s}{\longrightarrow} f$ in $L^p(\Omega \times \mathcal{Y}; \mathscr{B})$ i.e.

$$\lim_{\varepsilon \to 0} \left\| \mathcal{T}_\varepsilon^*(f_\varepsilon) - \tilde{f} \right\|_{L^p(\Omega_-^+ \times \mathcal{Y};\mathscr{B})} = 0 \tag{5.4}$$

is in effect.

As readers will notice, A.1) is the original definition from [All92] whereas A.2) is a specialisation of the former. Again, the task of characterising Allaire functions is precarious, and only sufficient conditions are known, namely Caratheodory functions and products of Allaire functions, see [All92, Sec. 5] for the original discussion. Preferring to circumvent such tedious affairs, we claim that A.2) is actually what one should be after. Our reasons are threefold:

1. A.1) curls up certain measurability issues concerning f_ε. For instance, fixing a specific positive null sequence $\varepsilon \in \mathbb{J}$ and placing non-measurable Vitali sets from $[0, 1] \subset \mathbb{R}$ on the lines of Allaire's defective set $\mathcal{A}$ from (5.2) yields a set that is in $\mathcal{L}(\Omega \times \mathcal{Y})$ but not in $\mathcal{L}(\Omega) \otimes \mathcal{L}(\mathcal{Y})$[3]. Naturally, this carries over to $\chi_{\mathcal{A}}$ that becomes measurable with respect to $\mathcal{L}(\Omega \times \mathcal{Y}) - \mathcal{B}(\mathbb{R})$. but not with respect to $\mathcal{L}(\Omega) \otimes \mathcal{L}(\mathcal{Y}) - \mathcal{B}(\mathbb{R})$. Consequently, $\chi_\varepsilon(x) := \chi_{\mathcal{A}}(x, x/\varepsilon)$ is not $\mathcal{L}(\Omega) - \mathcal{B}(\mathbb{R})$-measurable for any $\varepsilon \in \mathbb{J}$.

2. As mentioned before, A.1) places decisive importance on the sequence $\varepsilon \in \mathbb{J}$ in use. Ignoring measurability issues for a moment, this behaviour is deemed pathological by us for modelling reasons. Referring to the use of two-scale test functions as coefficient functions in applications, c.f. theorem §2.1 and theorem §2.2, the asymptotic behaviour of a mathematical model should not decisively depend on the null sequence in use unless there is convincing reason to do so.

3. Allaire's original definition of two-scale convergence from definition §2.1 reads

$$\lim_{\varepsilon \to 0} \int\limits_{\Omega} u_\varepsilon(x) \psi(x, x/\varepsilon)\, dx = \iint\limits_{\Omega \times \mathcal{Y}} u(x, y) \psi(x, y)\, dy dx$$

for every Allaire function $\psi \in L^2(\Omega \times \mathcal{Y})$. Periodic unfolding interprets this definition as a weak convergence statement of $(\mathcal{T}_\varepsilon^*(u_\varepsilon))_{\varepsilon \in \mathbb{J}} \subset L^2(\mathbb{R}^d \times \mathcal{Y})$. Since $(\psi_\varepsilon)_{\varepsilon \in \mathbb{J}}$ is a non-trivial sequence itself, the above product can only be expected to converge sensibly if significant

[3] $\mathcal{L}(\Omega \times \mathcal{Y})$ is the smallest complete σ-algebra containing $\mathcal{L}(\Omega) \otimes \mathcal{L}(\mathcal{Y})$. The latter is the smallest σ-algebra containing product sets $A \times B \in \mathcal{L}(\Omega) \times \mathcal{L}(\mathcal{Y})$. $\mathcal{L}(\Omega)$ is the completion of $\mathcal{B}(\Omega)$, the σ-algebra of Borel sets, c.f. [HS65].

additional information is available on either both $(\mathcal{T}_\varepsilon^*(u_\varepsilon))_{\varepsilon\in\mathbb{J}}$ and $(\psi_\varepsilon)_{\varepsilon\in\mathbb{J}}$ or on $(\psi_\varepsilon)_{\varepsilon\in\mathbb{J}}$ alone. The first alternative is extremely involved with *compensated compactness* being its most famous prototype. On the other hand, if $(\psi_\varepsilon)_{\varepsilon\in\mathbb{J}}$ is strongly two-scale convergent, the product's limit is behaving just as it should, at least for $p \in (1,\infty)$: a weakly two-scale convergent sequence $(u_\varepsilon)_{\varepsilon\in\mathbb{J}}$ would yield convergence of the products.

To identify a suitable family of functions that suffice A.2), two task are in order. First, measurability issues need to be addressed. Second, the continuity condition (5.4) is under consideration.

So, for $\varepsilon > 0$ let $\mathfrak{w}_\varepsilon : \Omega \longrightarrow \Omega \times \mathcal{Y}$ be given by $\mathfrak{w}_\varepsilon(x) := (x, \pi_{\mathcal{Y}}(x/\varepsilon))$. Analogous to the final topology one can define the final σ-algebra or pushforward σ-algebra of $\mathcal{L}(\Omega)$ along $(\mathfrak{w}_\varepsilon)_{\varepsilon>0}$ as the finest σ-algebra $\Sigma \in P(P(\Omega\times\mathcal{Y}))$ (on $\Omega\times\mathcal{Y}$) such that every $\mathfrak{w}_\varepsilon$ is $\mathcal{L}(\Omega) - \Sigma$-measurable.

Lemma §5.1: **Description of Σ**

The final σ-algebra Σ of $\mathcal{L}(\Omega)$ along $(\mathfrak{w}_\varepsilon)_{\varepsilon>0}$ coincides with the final σ-algebra Σ_ε of $\mathcal{L}(\Omega)$ along $\mathfrak{w}_\varepsilon$ for every $\varepsilon > 0$. Moreover, for every fixed $\varepsilon_0 > 0$ we have:

$$(5.5)\quad \begin{cases} \Sigma &= \bigcap_{\varepsilon>0} \left\{A\times B \subset \Omega\times\mathcal{Y} : \mathfrak{w}_\varepsilon^{-1}(A\times B) \in \mathcal{L}(\Omega)\right\} \\ &= \bigcap_{\varepsilon>0} \left\{A\times B \subset \Omega\times\mathcal{Y} : B = \pi_Y(\varepsilon^{-1}A) \text{ with } A \in \mathcal{L}(\Omega)\right\} \\ &= \bigcap_{\varepsilon>0} \Sigma_\varepsilon \quad = \quad \Sigma_{\varepsilon_0} \quad = \quad \mathcal{L}(\Omega)\otimes\mathcal{L}(\mathcal{Y}), \end{cases}$$

writing $\varepsilon^{-1}A := \{x \in \mathbb{R}^d : \exists a \in A : x = a/\varepsilon\}$.

Proof. First, notice that $\mathfrak{w}_\varepsilon^{-1}$ commutes with set operations, and therefore $\Sigma_? := \left\{A\times B \subset \Omega\times\mathcal{Y} : \mathfrak{w}_\varepsilon^{-1}(A\times B) \in \mathcal{L}(\Omega)\right\} \in P(P(\Omega\times\mathcal{Y}))$ is a σ-algebra in $\Omega\times\mathcal{Y}$. Moreover, $\Sigma_?$ maximal by construction: any σ-algebra Σ_* with $\Sigma_? \subset \Sigma_*$ and w_ε being measurable with respect to $\mathcal{L}(\Omega) - \Sigma_*$ must be contained in $\Sigma_?$. Consequently, $\Sigma_? = \Sigma_\varepsilon$.
Secondly, let us fixed $\varepsilon > 0$ for the moment. Our claim reads $\Sigma_\varepsilon = \mathcal{L}(\Omega)\otimes\mathcal{L}(\mathcal{Y})$. Since $\Sigma_\varepsilon \subset \mathcal{L}(\Omega)\otimes\mathcal{L}(\mathcal{Y})$ is obviously the case, we must make sure that at least all product sets $A\times B$ are in Σ_ε for $A \in \mathcal{L}(\Omega), B \in \mathcal{L}(\mathcal{Y})$. Considering $w_\varepsilon^{-1}(A\times B) = A \cap \left(\varepsilon\pi_{\mathcal{Y}}^{-1}(B)\right)$, one clearly has a section of sets in $\mathcal{L}(\Omega)$ which is again in $\mathcal{L}(\Omega)$. This argumentation is independent of the specific $\varepsilon > 0$, and depends solely on Id being a diffeomorphism and $\pi_{\mathcal{Y}}$ being a local diffeomorphism which preserve Borel sets and Lebesgue null sets. Thus, we have $\Sigma_\varepsilon = \mathcal{L}(\Omega)\otimes\mathcal{L}(\mathcal{Y})$ for all $\varepsilon > 0$, from which it follows that $\cap_{\varepsilon>0}\Sigma_\varepsilon = \Sigma_{\varepsilon_0}$ is true for every fixed $\varepsilon_0 > 0$.
Finally, claiming that Σ can be characterised as the intersection of all Σ_ε stems from two arguments: arbitrary intersections of σ-algebras yield a σ-algebra, and Σ is maximal in the sense that there can be no larger final σ-algebra of the family $(\mathfrak{w}_\varepsilon)_{\varepsilon>0}$. ■

Lemma §5.1 concludes that a reasonable candidate for a two-scale test function must be $\mathcal{L}(\Omega)\otimes\mathcal{L}(\mathcal{Y}) - \mathcal{B}(\mathscr{B})$-measurable and concerning measurability, this condition is sharp, as can be seen by placing Vitali sets on the stripes of $\mathcal{A}$. Independently, Allaire's counter-example function $\chi_{\mathcal{A}}$ is even $\mathcal{B}(\Omega)\otimes\mathcal{B}(\mathcal{Y}) - \mathcal{B}(\mathbb{R})$-measurable, its defectiveness being no measurability issue.

To address (5.4), the superposition operators' slightly non-orthodox point of view will be helpful. Doubtlessly, [AZ90] is the single, authoritative resource of this field which is why we adhere to it, too.

The main idea is to regard (5.1) and its likes as *superposition operators*, also known as *Nemytskii operators*. Thanks to (3.17), one may rewrite (5.1) as

$$(\mathcal{T}_\varepsilon(x,y),u_\varepsilon) \longmapsto \mathcal{N}_f(\mathcal{T}_\varepsilon(x,y),u_\varepsilon) := f(\mathcal{T}_\varepsilon,y,u_\varepsilon) \tag{5.6}$$

with $\mathcal{N}_f$ being the Nemytskii operator induced by f. Now, (5.4) can be interpreted as asking for $\mathcal{N}_f$ to be continuous at $(id_{\Omega\times\mathcal{Y}},u)$ along the sequence $(\mathcal{T}_\varepsilon(x,y),u_\varepsilon)_{\varepsilon\in\mathbb{J}}$.

Referring to [AZ90, Ch. 1], there are two main issues of concern. First, $\mathcal{N}_f$ should yield measurable functions, a matter that is extremely subtle in general and not fully solved to date.[4] Fortunately, lemma §5.1 places us in an extremely favourable special case, though. Secondly, continuity conditions inherit the foregoing subtleties such that most conditions for pointwise continuity or continuity everywhere of a superposition operator only work within the function class of *Shragin functions*[5] which is quite large but not fully exhaustive. We refrain from giving a definition which can be found in [AZ90, Sec. 1.4] together with related discussions. Aiming to avoid several tedious technical definitions, let us cut a long story short by stating that lemma §5.1 places us in the context of Shragin functions from the very outset.

As a result, we may retreat to classical results concerning continuity. The following result is an adaptation of [AZ90, Thm. 1.5].

Proposition §5.2: Continuity of Nemytskii operators

Provided f is a Shragin function then its induced superposition operator $\mathcal{N}_f$ is continuous at $(id_{\Omega\times\mathcal{Y}},u)$ if and only if f is continuous on the subset $\{(id_{\Omega\times\mathcal{Y}},u)\,(x,y) : (x,y)\in\Omega\times\mathcal{Y}\}$.

For clarity's sake, two aspects need to be emphasised. First, proposition §5.2 restricts itself to the periodic unfolding perspective. As is known due to [All92, Sec. 5], f being continuous in its y-argument is sufficient as well, but this takes place in a related but different set-up. Second and to make matters even more confusing, Allaire's admissible test functions can form an algebra if $\mathscr{B}$ is one. Most prominently, one can show that $f(x)g(y)$ is an Allaire function. Heuristically, this stems from both functions being admissible by themselves and carrying this over to the product function.

Without going into too much detail, we will therefore restrict to functions that are Caratheodory functions with respect to both their x-arguments and their u-arguments.

§6. Periodic description of highly heterogeneous media

For the boundary value problems which we are going to encounter, a suitable description of highly heterogeneous, periodic materials is necessary. In general, this step can be problematic if additional boundary conditions on the interface are to be imposed. We will discuss this matter here in brief and as a second step, we will establish a convenient solution for C^1-regular domains by classical transversality theorems from differential geometry. In a third subsection we will

[4] To the author's best knowledge, of course. Lamentably, the research on superposition operators seems to have faded.
[5] I. V. Shragin's notion stems from 1971 and has not been outperformed to date.

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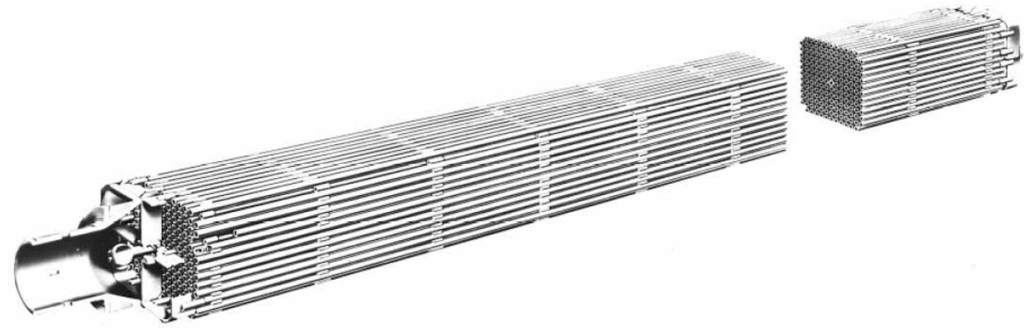

Figure I.3.: This photograph shows a single nuclear fuel element made of 41 fuel rods. Several elements are combined and inserted in a nuclear moderator medium.

return to our original of specifying geometric conditions necessary for a neat treatment of the problems in chapter II and chapter III. Thus, impatient readers may skip subsection §6.1 and subsection §6.2.

§6.1. Peculiarities of periodic decompositions

Let us fall back for a moment to a classical problem that is closely linked to homogenisation of strong heterogeneities, namely the homogenisation of media with periodicly distributed, tiny holes. This family of problems is in a way the twin of highly heterogeneous media and of course a worthwhile matter of its own. As references let us name the classical work [LBP78], [CP99] which is certainly among the most established and extensive resources of this field, and the very recent [Höp16] which covers a wide range of difficult problems, as well.

A digression on media with tiny holes

To begin with, the link of tiny holes and strong heterogeneities is rather simple: the slow medium is replaced by tiny holes which exhibit some prescribed behaviour, mostly either by Dirichlet or Neumann conditions. To give an impression, one of the original motivations came from *nuclear fuel elements* as depicted in figure I.3 and figure I.4. For rather obvious reasons, one wishes to predict heat conduction inside of such nuclear elements. These nuclear elements are made of *nuclear fuel rods*, long, slim rods which are clustered in large numbers, and inserted into some nuclear moderator, hopefully water or less favourably carbon. The rods are tailor-made to contain pre-manufactured nuclear fuel pellets whose heating properties are relatively easy to understand. So, the rods' thermal behaviour can be estimated well in the first place such that one uses the resulting data as prescribed boundary values on the tiny holes' boundaries.

Most crucially, both tiny holes and slow media are associated to a fast medium by a given interface. In technical terms, if one intends to work with Sobolev spaces $W^{1,p}(\Omega)$ with $p \in (1, \infty)$ then the interface's regularity becomes vital to the mathematical description: besides boundary conditions imposed on $\partial\Omega$, there are considerable applications with interplay on the interfaces of the media; for instance, recall the trabecular interface of bone and blood vessels. No matter whether differing materials or holes are present, one needs sensible notions of traces to give sense to such procedures.

The classical regularity condition for traces of $W^{1,p}$-functions is to ask for a Lipschitz hypersurface, c.f. [AF03, Thm. 5.36]. However, even the most elementary and smooth domains can

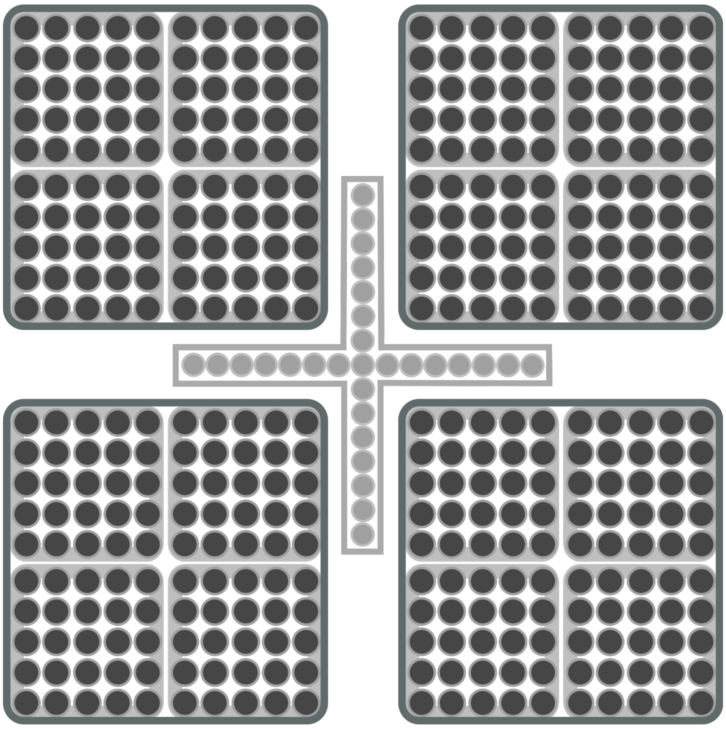

Figure I.4.: A single nuclear fuel element illustrated schematically.

easily yield strongly degenerated geometries. In what follows, we shall adopt the point of view of manifolds since it is a suitable notion of discussing the emerging issue. Recall that every open set of $\mathbb{R}^d$ is a C^∞-submanifold of $\mathbb{R}^d$ which is a smooth manifold itself. Moreover, every strong Lipschitz domain's boundary is a Lipschitz manifold by definition. As customary, smooth will always stand for C^∞-regular.

Degenerated intersections of manifolds

The decomposition of a given domain Ω into periodic potions Ω^1_ε and Ω^2_ε relies on intersections, however, this type of set operation does not preserve regularity properties particularly well.

Let us describe the problem more technically. Again, we invoke the *flat d-torus* $\mathcal{Y} := \mathbb{R}^d/\mathbb{Z}^d$ which is a compact, d-dimensional C^∞-manifold with a projection map $\pi_{\mathcal{Y}} : \mathbb{R}^d \longrightarrow \mathcal{Y}$ which is a local diffeomorphism.

Given a decomposition of $\mathcal{Y}$ into two disjoint, open subsets $\mathcal{Y}_1, \mathcal{Y}_2$ with a common Lipschitz-regular boundary Γ, one can expand each $\mathcal{Y}_i$ to all of $\mathbb{R}^d$ by periodicity and decompose $\mathbb{R}^d$ into $\mathbb{R}^d_{i,\varepsilon} := \{x \in \mathbb{R}^d : \pi_{\mathcal{Y}}(x/\varepsilon) \in \mathcal{Y}_i\}$ for $i = 1, 2$ for $\varepsilon > 0$ and $\Gamma_{\mathbb{R}^d,\varepsilon} := \partial\mathbb{R}^d_{1,\varepsilon} = \partial\mathbb{R}^d_{2,\varepsilon}$. So, the domain Ω can be decomposed into $\Omega_{i,\varepsilon} := \Omega \cap \mathbb{R}^d_{i,\varepsilon}$ for $i = 1, 2$. Alas, this can lead to $\Omega_{1,\varepsilon}$ or $\Omega_{2,\varepsilon}$ having irregular boundary, even for cofinitely many $\varepsilon > 0$.

Let us give an an elementary illustration for $\varepsilon = 1$: assume $\Omega = B_2(2e_1)$ and $\mathbb{R}^d_{2,\varepsilon} = B_\varepsilon(G)$ for $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^d$ and a grid $G = \varepsilon(e_1 + 3\mathbb{Z}^d)$. Figure I.5 shows this configuration for $d = 2, 3$ via a plot produced via *GeoGebra 6.0*, see [Hoh+18]. The decomposition of Ω into Ω^1_ε and Ω^2_ε now yields a boundary that is irregular near the origin: not only is Lipschitz-regularity itself lost, but even worse, neither does the resulting set fulfil the segment nor the weak cone condition. Finally, this defect holds for every $\varepsilon \neq 0$!

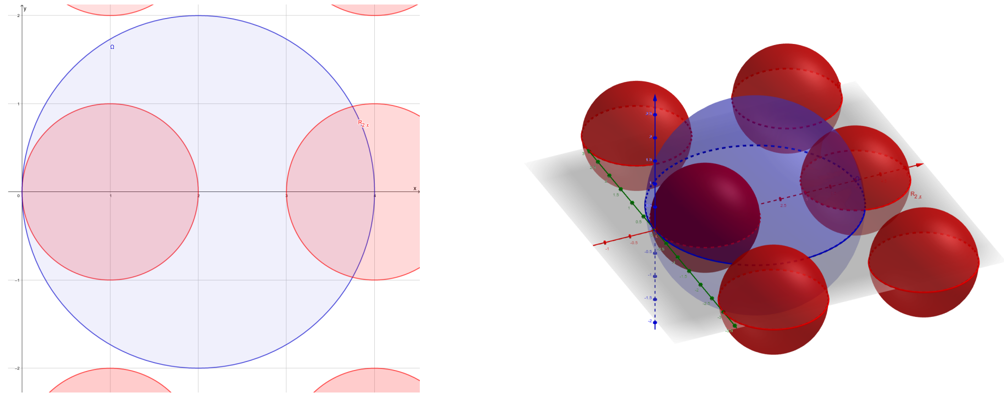

Figure I.5.: Intersections of manifolds do not preserve regularity in general.

In broader terms, the problem at hand is that **Diff**, the category whose objects are smooth manifolds and smooth maps as morphisms, is well-known to behave badly under set operations such as intersections and pull-backs. In addition, these defects are not only linked to smoothness issues, they are also connected to the concept of manifolds itself. Basically, at the time of writing and to the author's best knowledge, two possible remedies are conceivable: first, one may pass to more general objects that behave better under set operations. Alternatively, one may try to identify conditions under which the aforementioned defects do not occur. The first approach is currently being developed with *manifolds with corners*, *generalised smooth spaces* and *polyfolds* evolving as a suitable notions, we refer to [Mic80; Joy10], [BH09] and [HWZ17] as references. However, following this route is technically overburdening to the author. Instead, we arrive at the well-known sufficiency criterion of *transversality* which we consider next.

§6.2. Transversality of manifolds

This subsection will present the concept of transversality from conventional differential topology, together with a number of relevant results, all of which are purely classical. Afterwards, we will employ the machinery to justify that for bounded C^1-domains one can always resolve the irregularities described above.

Definitions and classical results

Let us recall the classical concept of transversality, which we take from [Bou07], [GP74, Ch. 2] and [Hir76, Sec. 3.2]. So, let three C^1-manifolds M_1, M_2, N be given such that only M_1 is allowed to be a manifold with boundary and M_2 is a submanifold of N. One calls a map $f \in C^1(M_1, N)$ transverse to M_2 if

$$df_m\,(T_m M_1) + T_{f(m)} M_2 = T_{f(m)} N \quad \text{holds for all } m \in f^{-1}(M_2) \subset M_1, \tag{6.1}$$

writing for instance $T_m M_1$ for the tangential space of M_1 at m and df_m being the total differential at m. The raison d'être of transversality is given in the following proposition.

Proposition §6.1: **Transversality theorems**

a) If both $f : M_1 \longrightarrow N$ and $f_{|\partial M_1} : \partial M_1 \longrightarrow N$ are transverse on M_2 then $f^{-1}(M_2)$ is a C^1-submanifold of M_1 with boundary $\partial M_1 \cap f^{-1}(M_2)$ and $\operatorname{codim}(f^{-1}(M_2)) = \operatorname{codim}(M_2)$.

b) R. Thom's transversality theorem:

$\{f \in C^1(M_1, N) : f \text{ is transverse to } M_2\}$ is comeagre and dense in $C^1(M_1, N)$ with respect to the compact-open topology and the Whitney topology of $C^1(M_1, N)$.

c) Parametric transversality theorem:

Let S be a d_s-dimensional C^1-manifold without boundary and let $F \in C^1(M_1 \times S, N)$ be such that F and $F_{|\partial M_1 \times S}$ are transverse to M_2 in N. Then for the family of C^1-maps given by

$$m \longmapsto f_s(m) := F(m, s) \quad \text{for} \quad s \in S$$

the following holds for λ^{d_S}-almost every $s \in S$: f_s and $f_{s|\partial M_1}$ are transverse to M_2.

Proof. For the first statement see [GP74, p. 60,] or [Hir76, Thm. 1.4.2], the second statement can be found in [Tho54] or [Hir76, p. 74]. Observe that parametric transversality theorem employs the identity $\partial(M_1 \times S) = (\partial M_1) \times S$ which holds only if $\partial S = \emptyset$. A proof the statement itself is given in [GP74, p. 68] ■

Remark §6.1

Thom's transversality theorem originally invokes jets and is therefore significantly more general. So, we presented a more simple form.
Secondly, much of the literature on differential geometry and transversality is written for C^∞-manifolds, a fact that is no severe obstacle for C^1-manifolds with boundary. These can be equipped with an equivalent C^∞-regular differential structure that is unique up to C^∞-diffeomorphisms. Since the boundary of a manifold is tacitly endowed with a corresponding differential structure, the same argument applies to it, too.
Unfortunately, all these arguments break down for Lipschitz-regular boundaries, c.f. [Gri85]. due to the break down of the implicit function theorem, an indispensable cornerstone of smooth differential geometry.

For convenience's sake, a basis for neighbourhoods of the compact-open topology and Whitney topology of $C^1(M_1, N)$ of a fixed $f \in C^1(M_1, N)$ can be described as follows.

a) The neighbourhoods of f in the compact-open topology $C^1(M_1, N)$ are generated by sets of the form

$$\left\{ g \in C^1(M_1, N) : g(K) \subset V \ \wedge \ \|\psi_V \circ f \circ - \psi_V \circ g \circ \phi_U^{-1}\|_{C^1(\phi_U(K);\mathbb{R}^{\dim N})} \leq \delta \right\} \tag{6.2}$$

for $\delta \in (0, \infty]$, chart maps $\phi_U : U \subset M_1 \longrightarrow \mathbb{R}^{\dim M_1}$, $\psi_V : V \subset N \longrightarrow \mathbb{R}^{\dim N}$ with U, V open and $K \subset U$ compact in M_1.

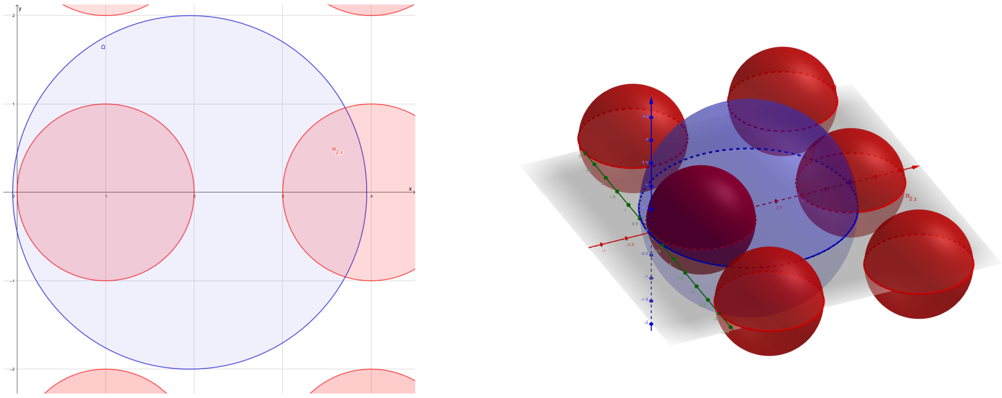

Figure I.6.: Just as R. Thom's transversality theorem claims, an arbitrarily 'small' perturbation suffices to resolve all transversality issues completely.

b) Similarly, the neighbourhoods of f in the Whitney topology $C^1(M_1, N)$ are generated by the sets

$$(6.3) \qquad \left\{ g \in C^1(M_1, N) : g(K_i) \subset V_i \;\wedge\; \|\psi_{V_i} \circ f \circ - \psi_{V_i} \circ g \circ \phi_{U_i}^{-1}\|_{C^1(\phi_{U_i}(K_i);\mathbb{R}^{\dim N})} \leq \delta_i \right\}$$

for a countable family of $\delta_i \in (0, \infty)$, chart maps $\phi_{U_i} : U_i \subset M_1 \longrightarrow \mathbb{R}^{\dim M_1}$, $\psi_{V_i} : V_i \subset N \longrightarrow \mathbb{R}^{\dim N}$ with U_i, V_i open and $K_i \subset U$ compact in M_1 such that $i \in \mathbb{N}$ and the respective charts are locally finite.

Finally, let us remark that both topologies coincide if M_1 is compact. The latter is responsible for asking for Ω to be bounded. An extension to unbounded domains is not entirely ruled out, though, our restriction is due to keeping things simple and brief.

Applying Thom's theorem to periodic decompositions

First, let us revisit figure I.5 with f being the embedding $\iota : \overline{\Omega} = M_1 \hookrightarrow N = \mathbb{R}^d$. Clearly, f fails to be transverse to $M_2 = \partial\mathbb{R}^d_{2,\varepsilon}$ in $0 \in \partial\Omega$ making the link of transversality to our problem quite apparent. Moreover, Thom's theorem suggests that only a 'tiny' modification of ι would be necessary to fix this situation. For instance, perturbing ι slightly to get $x \mapsto x - \delta e_1$ for some $0 < \delta \ll 1$ resolves everything in a deus ex machina manner, as depicted in figure I.6.

As we have seen, already a tiny translation was enough to rule out all problems, a fact that is paraphrased by the parametric transversality theorem with $F(m, s) = \iota(m) + s$ for $s \in \mathbb{R}^d$. So, let us apply the parametric transversality theorem to yield a sufficiently well-behaved periodic decomposition, at least for a bounded C^1-domain Ω. To this end, assume that for some $r \in \mathbb{N}$ both $M_1 = \overline{\Omega} \subset \mathbb{R}^d$ and $\Gamma \subset \mathcal{Y}$ are C^r-regular submanifolds, with $\Omega \subset \mathbb{R}^d$ being a bounded and open submanifold of $\mathbb{R}^d$ and Γ being a closed submanifold of $\mathcal{Y}$. Consequently, $\mathcal{Y}_1$ and $\mathcal{Y}_2$ inherit Γ's C^r-regularity just as $\mathbb{R}^d_{1,\varepsilon}$, $\mathbb{R}^d_{2,\varepsilon}$ and $M_2 = \Gamma_{\mathbb{R}^d,\varepsilon}$ do. In fact, by construction Ω, $\mathbb{R}^d_{1,\varepsilon}$ and $\mathbb{R}^d_{2,\varepsilon}$ are open submanifolds of $\mathbb{R}^d$ and thus even C^∞-submanifolds of codimension 0 since open submanifolds of a smooth manifold inherit its regularity. Moreover, $M_2 = \Gamma_{\mathbb{R}^d,\varepsilon}$ is obviously a closed C^r-submanifold of $\mathbb{R}^d$.

Next, since $\mathbb{R}^d_{1,\varepsilon}$ and $\mathbb{R}^d_{2,\varepsilon}$ are open submanifolds with codimension 0 the imbedding $f = \iota : M_1 = \overline{\Omega} \hookrightarrow N = \mathbb{R}^d$ is *always* transverse to both $\mathbb{R}^d_{1,\varepsilon}$ and $\mathbb{R}^d_{2,\varepsilon}$ regardless of $\varepsilon > 0$; this follows from openness since $T_x A = T_x\mathbb{R}^d$ holds for every $x \in A = \Omega, \mathbb{R}^d_{1,\varepsilon}, \mathbb{R}^d_{2,\varepsilon}$. Consequently, every open map $f \in C^r(\overline{\Omega};\mathbb{R}^d) = C^3(M_1, N)$ is transverse to $A = \Omega, \mathbb{R}^d_{1,\varepsilon}, \mathbb{R}^d_{2,\varepsilon}$, shifting the main issue to transversality of the boundary sets $\partial M_2 = \partial\Omega$ and $M_2 = \Gamma_{\mathbb{R}^d,\varepsilon}$. There, we proceed as follows: for $S = \mathbb{R}^d = N$ define

$$F : M_1 \times S \longrightarrow N \qquad (m, s) \longmapsto \iota(m) + s \tag{6.4}$$

which is clearly a submersion making F and $F_{\partial M_1 \times S}$ transverse to M_2. Then, the parametric transversality theorem from proposition §6.1 infers the following result.

Lemma §6.1: domain shifting lemma

Every bounded C^1-domain Ω can be translated by a vector $s \in \mathbb{R}^d$ such that its translation intersects with $\Gamma_{\mathbb{R}^d,\varepsilon}$ such that the resulting subdomains $\Omega^i_\varepsilon = (\Omega + s) \cap \mathbb{R}^d_{i,\varepsilon}$ are bounded C^1-manifolds for any $\varepsilon \neq 0$.

Proof. If $f = f_0$ and $f_{|\partial M_1}$ are transverse to M_2 there is nothing to show, so assuming this is not so, let us construct an embedding that is sufficiently close to f_0 in the Whitney topology. First, since $F = Id_{\mathbb{R}^d|M_1} + Id_{\mathbb{R}^d}$ we have $DF = 2Id_{\mathbb{R}^d}$ such that the image of DF is always $\mathbb{R}^d$. Consequently DF fulfils (6.1), making F and $F_{\partial M_1 \times S}$ transverse to M_2 as necessary for the parametric transversality theorem. Then, $f_s(m) := f(m) + s$ and $f_{s|\partial M_1}$ are transverse to M_2 for λ^d-almost every $s \in \mathbb{R}^d$. Fixing a suitable $s \in \mathbb{R}^d$ makes $f_s^{-1}(M_2)$ a submanifold of M_1. Since f_s is a translation in $\mathbb{R}^d$ of an embedding, it is an embedding itself. Likewise, one may apply f_s to M_1 and $f_s^{-1}(M_2)$ yielding that $f_s(\Omega) \cap M_2$ is a submanifold of $f_s(\Omega) = \Omega + s$. ■

By lemma §6.1 we are now in the nice situation that

a) either $\iota_{|\partial\Omega}$ is transverse to $\Gamma_{\mathbb{R}^d,\varepsilon}$ such that $\Gamma_{\mathbb{R}^d,\varepsilon} \cap \Omega$ is a submanifold of $\overline{\Omega}$ and so Ω^1_ε and Ω^2_ε are a C^1-manifolds with C^1-regular boundary.

b) Alternatively, we translate Ω by a vector $s \in \mathbb{R}^d$ such that $\iota_{|\partial\Omega+s}$ is transverse to $\Gamma_{\mathbb{R}^d,\varepsilon}$, making $\Gamma_{\mathbb{R}^d,\varepsilon}$, Ω^1_ε and Ω^2_ε C^1-manifolds, again. In fact, λ^d-almost every $s \in \mathbb{R}^d$ is suitable for such a translation.

Pitfalls of strong Lipschitz domains and manifolds with corners

To sum up, we have ensured that C^1-regular bounded domains can be decomposed conveniently in periodic fashion. The question whether this carries over to strong Lipschitz domains considered as the more genera *manifolds with corners*, c.f. P. Michor's work [Mic80], is notoriously subtle though. The breakdown of the implicit function theorem has far-reaching consequences, for instance, as L. Nielsen points out in [Nie81, Sec. 7], the first statement of proposition §6.1 fails, so that the preimage of a transverse map no longer needs to be a manifold at all. Rather, additional conditions must be imposed to ensure that the corresponding faces, edges, vertices and so on do not cause additional defects. [Nie81] seems to be the only work on this matter the author is aware of. As a rule of thumb it can be said that the implicit function theorem organises the connection between the strata of a manifold – that is the cascade of the interior of a manifold, the relative interior of its boundary, the relative boundary of the boundary, and so on – sensibly and its breakdown below C^1-regularity requires to organise the strata of the manifold manually.

Nevertheless, astounding generalisations of R. Thom's transversality theorem to manifolds with corners can be found in [Mic80, Sec. 6]. In any instance, expanding transversality theorems to manifolds with corners is a largely open field that receives insufficient attention, even besides the recent work of D. Joyce in [Joy10] which seems to revive the matter a little.

Returning to periodic decompositions of manifolds, let us conclude this subsection by pointing out that other works assume that the domains Ω_ε^i are sufficiently regular. M. Höpker gathers several related assumptions that are commonly used in [Höp16, Sec. 3.2], including a rich reference to related works and a wide range of applications. There, both perforated Lipschitz domains and high contrast media are considered, such that Ω_ε^2 either represents tiny holes or a slow medium. The working assumption is that Ω_ε^1 is assumed to be a Lipschitz domain for all ε in question, see [Höp16, 3.2.1.c)] for details. Realise that lemma §6.1 is precisely the justification of this assumption for C^1-regular domains, making M. Höpker's assumption completely sound for such domains. Moreover, this state of affairs encourages that a similar vindication should be obtainable for Lipschitz domains, too.

§6.3. Admissible domains – requirements and assumptions

Here, we return to our original endeavour, let us start with a definition.

Definition §6.1: **Admissible decompositions**

As customary, we write $\mathcal{Y} = \mathbb{R}^d/\mathbb{Z}^d$ for the flat torus. A decomposition of the form

$$\mathcal{Y}_1 \cup \mathcal{Y}_2 \cup \Gamma = \mathcal{Y} \tag{6.5}$$

is an <u>admissible decomposition of the periodic cell $\mathcal{Y}$</u> if $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are open sub-manifolds of $\mathcal{Y}$, with $\mathcal{Y}_1 \neq \emptyset$ and $\Gamma = \partial\mathcal{Y}_1 \cap \partial\mathcal{Y}_2$ being a Lipschitz sub-manifold of $\mathcal{Y}$ of codimension 1. Moreover, we will write $\chi_i := \chi_{\mathcal{Y}_i} : \mathcal{Y} \longrightarrow \{0,1\}$ for $i = 1,2$. Note that $\mathcal{Y}_2 = \emptyset$ is allowed.
Given an admissible decomposition of $\mathcal{Y}$, its <u>resulting periodic decomposition of a domain $\Omega \subset \mathbb{R}^d$</u> is given for $i = 1,2$ and $\varepsilon \in \mathbb{J}$ by

$$\Omega_\varepsilon^i := \{x \in \Omega : \chi_i(x/\varepsilon) = 1\} \text{ and } \Gamma_\varepsilon := \left(\partial\Omega_\varepsilon^1 \cap \partial\Omega_\varepsilon^2\right) \setminus \partial\Omega. \tag{6.6}$$

A given resulting periodic decomposition of a bounded domain $\Omega \subset \mathbb{R}^d$ is called an <u>admissible periodic decomposition of Ω</u> if $\mathcal{Y}_1$ is connected and if the asymptotic negligibility condition holds:

$$\lim_{\varepsilon\to 0} \lambda^d\left(\Lambda_\varepsilon^+\right) = 0. \tag{6.7}$$

Referring to definition §6.1, we have $\Omega = \Omega_\varepsilon^1 \cup \Omega_\varepsilon^2 \cup \Gamma_\varepsilon$ for every $\varepsilon > 0$ by construction and therefore independently of the specific choice of the null sequence in use.

Moreover, Γ being sufficiently regular is necessary to establish a compactness theorem §7.2. We will need trace operators to be available on Γ and Lipschitz-regularity is well-known to be sufficient in this respect. We do not exploit more refined methods of trace operators on less regular sets, c.f. [Tri06; AFP00]. Nevertheless, we can establish the following result.

Proposition §6.2

Let $\Omega \subset \mathbb{R}^d$ be a domain with C^1-regular boundary, then without loss of generality we may assume that Γ_ε is a C^1-regular hypersurface. If $\partial\Omega$ is compact then $\lim_{\varepsilon\to 0} \lambda^d \left(\Lambda_\varepsilon^+\right) = 0$ holds, too.

Proof. The regularity statement about Γ_ε follows from lemma §6.1 by translating Ω to $\Omega + s$ for some $s \in \mathbb{R}^d \setminus N$ and $\lambda^d(N) = 0$. The second statement was discussed in proposition §3.1. ∎

Let us comment on some aspects of definition §6.1 and related admissible results.

a) As encountered beforehand, Ω_ε^1 is considered as the standard or fast medium, and may not necessarily be fully homogeneous itself. Alas, it has to behave 'sufficiently homogeneous'. In particular, its properties are drastically different from the slow medium displayed by Ω_ε^2.

b) Of course, one can work with $\mathcal{Y}_2 = \emptyset$ altogether, thus working on a locally periodic medium which is 'relatively' homogeneous. However, in this instance much better results are attainable, for instance [Vis07a].

c) Non-trivial admissible periodic decompositions of domains do exist. For instance let $\Omega = (0,1)^d$ and set $\mathcal{Y}_2 := \pi(B_\delta([1/2,\dots,1/2]^T)) \subset \mathcal{Y}$ for some $\delta < 1/2$, together with $\mathcal{Y}_1 := \operatorname{int}[\mathcal{Y} \setminus \mathcal{Y}_2]$. See [Pet07, Sec. 2.2] for a thorough discussion.

d) The requirement of cofinitely many $\mathcal{Y}_1$ to be connected stems from gradient compactness theorems which we present below. This condition cannot be weakened.

e) Regularity requirements on Γ are indispensable. However, asking for Γ_ε may be necessary in other applications than ours where various reactions between the slow and the fast domain are to be modelled. For instance, see [Gra13; GP14] and the references therein.

f) Our approach is deterministic in nature. In this respect, let us point to *stochastic unfolding*, a merge of periodic unfolding with stochastic methods, developed in [HNV18; NV18].

§6.4. The two-scale gradient

Definition §6.1 asks for $\mathcal{Y}_1$ to be connected for cofinitely many $\varepsilon \in \mathbb{J}$. In fact, this requirement is necessary to identify two-scale limits of sequences of gradients appropriately, a task addressed in theorem §7.2. Concerning gradients under two-scale convergence, it will be convenient to define the following weight function

$$\mathfrak{g}_\varepsilon(x) := \chi_1(x/\varepsilon) + \varepsilon\chi_2(x/\varepsilon) \tag{6.8}$$

λ^d-almost everywhere in $\mathbb{R}^d$. The purpose of $\mathfrak{g}_\varepsilon$ is to keep account of whether or not $x \in \Omega$ is in Ω_ε^1 or in Ω_ε^2 when combined with ∇ yielding

$$\nabla_\varepsilon := \mathfrak{g}_\varepsilon(x)\nabla \tag{6.9}$$

which we call two-scale gradient (subordinate to the given periodic decomposition of Ω). For instance, for a function $x \longmapsto w_\varepsilon(x) := w(x, x/\varepsilon)$ given by a more regular function $(x, y) \longmapsto$

$w(x,y) \in C^1(\Omega \times \mathcal{Y}; \mathbb{R})$ we have

$$\nabla_\varepsilon w(x, x/\varepsilon) = \chi_1(x/\varepsilon)\left[\nabla_x w_\varepsilon + \frac{1}{\varepsilon}\nabla_y w_\varepsilon\right] + \chi_2(x/\varepsilon)\left[\varepsilon\nabla_x w_\varepsilon + \nabla_y w_\varepsilon\right]. \tag{6.10}$$

Observe that we have $\mathcal{T}_\varepsilon^*(\mathfrak{g}_\varepsilon)(x,y) \longrightarrow \chi_{\mathbb{R}^d \times \mathcal{Y}_1}$ in $L^\infty(\mathbb{R}^d \times \mathcal{Y})$ such that $\mathfrak{g}_\varepsilon$ can be considered as a multiplier.

§7. Two-scale convergence and derivatives – compactness

Convergence statements in some suitable sense are among the very essence of analysis and usually, both the statements and the precise sense itself are quite an issue. In the presence of the (full) axiom of choice, compactness is a very customary but non-constructive device to bring about convergence statements, though the author wonders whether sequential compactness can be attained with the countable axiom of choice, too. Anyway, since two-scale analysis extends conventional calculus by an additional microscale, one may hope to retrieve compactness results analogous to the single-scale case.

All of the results presented here are classical, though we include very mild extensions which are tailor-made for our purposes. Classical resources on two-scale compactness are [Ngu89], [All92, Sec. 1 & 4], and [CDG08, Sec. 3]. In addition, we refer to [PB08, Sec. 3] and [Vis06] as additional supplements. For our purposes we will distinguish two cases: first, we shall retrieve sequential weak compactness in the weak topology for L^p-spaces for $p \in (1,\infty)$. Secondly, we shall investigate sequences of gradients: there, the situation will be more subtle.

§7.1. Compactness without derivatives

The next result was originally established by G. Nguetseng in [Ngu89], refined by G. Allaire in [All92] and reformulated by D. Cioranescu, A. Damlamian and G. Griso [CDG02] via periodic unfolding.

Theorem §7.1: **Nguetseng's compactness theorem**

Let $p \in (1,\infty)$, $\mathcal{B}$ be reflexive and separable and Ω be a domain with compact, Lipschitz-regular boundary. Every bounded sequence $(u_\varepsilon)_{\varepsilon\in\mathbb{J}} \subset L^p(\Omega;\mathcal{B})$ has a subsequence that is weakly two-scale convergent:

$$\exists u_0 \in L^p(\Omega \times \mathcal{Y}; \mathcal{B}), \mathbb{J} \supset \mathbb{J}' \ni \varepsilon' \to 0 : \quad u_{\varepsilon'} \xrightarrow{2w} u_0 \quad \text{in } L^p(\mathbb{R}^d \times \mathcal{Y}; \mathcal{B}). \tag{7.1}$$

The result holds for $p = 1$ if $(\mathcal{T}_\varepsilon^*(u_\varepsilon))_{\varepsilon\in\mathbb{J}}$ is uniformly integrable and $\lambda^d(\Omega) < \infty$.

Recall that in (7.1) the actual limit is $\widetilde{u}_0$, our notation being slightly sloppy at this point.

Proof. Periodic unfolding is an isometric map and so the sequence $(\mathcal{T}_\varepsilon^*(u_\varepsilon))_{\varepsilon\in\mathbb{J}}$ is bounded in $L^p(\mathbb{R}^d \times \mathcal{Y}; \mathcal{B})$. As reflexivity of $\mathcal{B}$ carries over to $L^p(\mathbb{R}^d \times \mathcal{Y}; \mathcal{B})$, sequential compactness of bounded sets in the weak topology is due to the Banach–Alaoglu and the Eberlein–Šmulian theorems. Thus, $(\mathcal{T}_\varepsilon^*(u_\varepsilon))_{\varepsilon\in\mathbb{J}}$ has a subsequence weakly converging to some $v_0 \in L^p(\mathbb{R}^d \times \mathcal{Y}; \mathcal{B})$. To show $v_0 = \widetilde{u}_0$ for some $u_0 \in L^p(\Omega \times \mathcal{Y}; \mathcal{B})$ consider $\operatorname{supp}(v_0) \subset \overline{\Omega} \times \mathcal{Y}$ which follows from $\operatorname{supp}(\mathcal{T}_\varepsilon^*(u_\varepsilon)) \subset \overline{\Omega_\varepsilon^+} \times \mathcal{Y}$ and the latter converges to $\overline{\Omega} \times \mathcal{Y}$ by proposition §3.1.

Since $\lambda^d(\partial\Omega) = 0$ holds, one may infer $\exists u_0 \in L^p(\Omega \times \mathcal{Y}; \mathscr{B})$ such that $\widetilde{u}_0 = v_0$ which was to show.

Turning to $p = 1$, the boundedness of Ω implies that $\Omega \times \mathcal{Y}$ is bounded such that the measure space $(\Omega^+_{\varepsilon_0} \times \mathcal{Y}, \mathcal{L}(\Omega^+_{\varepsilon_0} \times \mathcal{Y}), \lambda^{2d})$ has the Dunford–Pettis property and contains cofinitely many Ω^+_ε, too. Moreover, since $\mathscr{B}$ is reflexive, the Dunford–Pettis theorem holds almost verbatim, c.f. [DJ77, p. 101]. Then, the dual space of $L^1(\Omega \times \mathcal{Y}; \mathscr{B})$ is canonically isomorphic to $L^\infty(\Omega \times \mathcal{Y}; \mathscr{B}')$. Combining the Dunford–Pettis and the Eberlein–Šmulian theorems, a sequence is sequentially compact in the $\sigma(L^1, L^\infty)$-topology if and only if it is uniformly integrable. However, periodic unfolding respects uniform integrability by (3.17). The statement then follows by arguing similarly as for $p > 1$ that the weak limit is supported only on $\overline{\Omega} \times \mathcal{Y}$. ■

§7.2. Two-scale compactness of derivatives – a guiding example

The considerations for gradients are significantly more involved that the foregoing result. Let us give a motivation for our rather specific setting. The applications in chapter II and chapter III will lead to the following situation: we are given a bounded $C^{0,1}$-regular domain Ω on which we consider boundary value problems or initial–boundary value problems that depend on $\varepsilon \in \mathbb{J}$. Under suitable conditions, one can guarantee solvability of the problems at hand for every $\varepsilon \in \mathbb{J}$, and so, there is a sequence of solutions $(u_\varepsilon)_{\varepsilon\in\mathbb{J}}$ made of elements of suitable function spaces. The elliptic setting will lead to $(u_\varepsilon)_{\varepsilon\in\mathbb{J}} \subset W^{1,p}(\Omega)$, the parabolic regime will yield mixed regularity spaces of the form $(u_\varepsilon)_{\varepsilon\in\mathbb{J}} \subset L^p([0,T]; W^{1,p}(\Omega)) \cap W^{1,p'}([0,T]; W^{1,p}(\Omega)')$ for $p \in (1,\infty)$. Since we want to investigate the behaviour of $(u_\varepsilon)_{\varepsilon\in\mathbb{J}}$ for $\varepsilon \to 0$, a priori estimates independent of ε play a key role. As it will turn out, the fast domain will behave quite conveniently, yielding bounds on u_ε and ∇u_ε on Ω^1_ε that are independent of ε. In stark contrast, the slow domain's bounds are ε-independent only for u_ε and $\varepsilon\nabla u_\varepsilon$ but *not* for ∇u_ε. Moreover, one can argue that this situation is sound from a modelling perspective and thus, one cannot hope to improve the given bounds.

Let us illustrate the matter of concern a little by considering two sequences on $\Omega = (0, 2\pi)$:

$$\left\{\begin{aligned} u_\varepsilon(x) &= \sin(x) + \varepsilon\sin(x/\varepsilon) + \varepsilon\sin(x) \\ w_\varepsilon(x) &= \sin(x) + \sin(x/\varepsilon) + \varepsilon\sin(x/\varepsilon) + \varepsilon\sin(x). \end{aligned}\right. \tag{7.2}$$

Letting $\varepsilon \to 0$ one find $u_\varepsilon \longrightarrow \sin(x)$ in $L^p(0,2\pi)$. In contrast, the same is impossible for w_ε, $\nabla u_\varepsilon = \cos(x) + \cos(x/\varepsilon) + \varepsilon\cos(x/\varepsilon)$ and $\nabla w_\varepsilon = \cos(x) + 1/\varepsilon\cos(x/\varepsilon) + \cos(x/\varepsilon) + \varepsilon\cos(x)$ due to the presence of terms that depend on x/ε. Now, periodic unfolding steps up to yield

$$\left\{\begin{aligned} \mathcal{T}^*_\varepsilon(u_\varepsilon)(x,y) &= \sin(\mathcal{T}_\varepsilon(x,y)) + \varepsilon\sin(y) + \varepsilon\sin(\mathcal{T}_\varepsilon(x,y)) && \longrightarrow \sin(x), \\ \mathcal{T}^*_\varepsilon(w_\varepsilon)(x,y) &= \sin(\mathcal{T}_\varepsilon(x,y)) + \sin(y) + \varepsilon\sin(y) + \varepsilon\sin(\mathcal{T}_\varepsilon(x,y)) && \longrightarrow \sin(x) + \sin(y), \\ \mathcal{T}^*_\varepsilon(\nabla u_\varepsilon)(x,y) &= \cos(\mathcal{T}_\varepsilon(x,y)) + \cos(y) + \varepsilon\cos(\mathcal{T}_\varepsilon(x,y)) && \longrightarrow \cos(x) + \cos(y), \end{aligned}\right.$$

for $\varepsilon \to 0$ in $L^p(0,2\pi)$. However, $\mathcal{T}^*_\varepsilon(\nabla w_\varepsilon) = \cos(\mathcal{T}_\varepsilon(x,y)) + 1/\varepsilon\cos(y) + \varepsilon\cos(y) + \varepsilon\cos(\mathcal{T}_\varepsilon(x,y))$ is clearly divergent. Fortunately and of fundamental importance, it was pointed out in [PB08] that applications actually impose a scaling on the gradient sequences of w_ε. For modelling reasons, one simply does not encounter ∇w_ε but rather $\varepsilon\nabla w_\varepsilon$ which behaves much better since

$$\mathcal{T}^*_\varepsilon(\varepsilon\nabla w_\varepsilon)(x,y) = \varepsilon\cos(\mathcal{T}_\varepsilon(x,y)) + \cos(y) + \varepsilon^2\cos(y) + \varepsilon^2\cos(\mathcal{T}_\varepsilon(x,y)) \longrightarrow \cos(y)$$

holds in $L^p(0,2\pi)$, stemming from the elementary identity $\nabla_y \mathcal{T}_\varepsilon^*(w_\varepsilon) = \mathcal{T}_\varepsilon^*(\varepsilon \nabla w_\varepsilon)$.

Summing up, $(u_\varepsilon)_{\varepsilon\in\mathbb{J}}$ can be considered to be asymptotically independent of the micro-variable y, whereas the gradient does contain information encoded exclusively by y. On the other hand, $(w_\varepsilon)_{\varepsilon\in\mathbb{J}}$ is asymptotically y-dependent itself and oscillations on the microscale foil macroscopic regularity. Of course, the very same oscillations can be sufficiently regular on the microscale.

§7.3. Two-scale compactness of gradients – unfolded spaces

Analogously to corollary §3.1 one can consider the space of periodically unfolded Sobolev functions which is again a Banach space itself.

Proposition §7.1: **Unfolded $W^{1,p}$-spaces**

Let $\Omega \subset \mathbb{R}^d$ be a domain with compact and Lipschitz-regular boundary, $p \in [1,\infty)$ and $X \subset W^{1,p}(\Omega)$ be a closed subspace, e.g. $X = W_0^{1,p}(\Omega)$. In addition, we assume that a periodic decomposition of Ω is at hand yielding a two-scale gradient ∇_ε as defined in (6.9). Then, periodically unfolding $(u, \nabla_\varepsilon u)$ component-wisely for $u \in X$ yields

$$\mathcal{T}_\varepsilon^*(X) := \left\{ (w_0, w_1) \in L^p(\Omega_\varepsilon^+ \times \mathcal{Y})^{1+d} : w_0 = \mathcal{T}_\varepsilon^*(u_\varepsilon), w_1 = \mathcal{T}_\varepsilon^*(\nabla_\varepsilon u) \right\} \tag{7.3}$$

which is a closed subspace of $L^p(\Omega_\varepsilon^+ \times \mathcal{Y})^{1+d}$.

Proof. First, X is assumed to be closed and the isometric embedding $X \ni u \longmapsto (u, \nabla u) \in L^p(\Omega)^{1+d}$ yields a closed subspace X_1. $(u, \nabla u) \longmapsto (u, \nabla_\varepsilon u)$ is no isometry but but stems from the linear isomorphism on $L^p(\Omega)^{1+d}$ given by $(w_0, w_1) \longmapsto (w_0, \mathfrak{g}_\varepsilon w_1)$. This isomorphism preserves closed sets and therefore the image of its restriction to X_1 is a closed subspace denoted X_2. Finally, periodic unfolding is an isometry by (3.18). ∎

Remark §7.1: **Sobolev embeddings**

$u \longmapsto (u, \nabla u)$ embeds $W^{1,p}(\Omega)$ in $L^p(\Omega) \times L^p(\Omega)^d \cong L^p(\Omega)^{1+d}$ which is handy for showing completeness of Sobolev spaces. Of course, it is the holy grail of Sobolev space theory that even $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega) \times L^p(\Omega)^d$ works with p^* being the Sobolev exponent such that the Rellich–Kondrachov theorem can be inferred. Our applications will yield bounds on $\|\nabla_\varepsilon u_\varepsilon\|_{L^p(\Omega)}$ and these are strong enough to ensure $u_\varepsilon \in W^{1,p}(\Omega)$ for fixed $\varepsilon > 0$ but the situation degenerates completely for $\varepsilon \longrightarrow 0$. Unfortunately, the estimates in question cannot be improved such that in parts only distributional derivatives can be expected, foiling weak differentiability of possible (two-scale) limits. Consequently, the classical compactness machinery is unavailable and anisotropic Sobolev spaces are no simple remedy here for they do require minimal amounts of regularity of the derivatives to yield relevant embedding theorems, c.f. [Tri06, Ch. 5].

§7.4. Two-scale compactness of derivatives – a theorem

First, the aforementioned identity

$$\mathcal{T}_\varepsilon^*(\varepsilon \nabla_x u)(x,y) = \left(\nabla_y \mathcal{T}_\varepsilon^*(u_\varepsilon)\right)(x,y) \quad \text{almost everywhere in } \Omega_\varepsilon^+ \times \mathcal{Y} \tag{7.4}$$

effectively allows to ask for the derivative of the micro-variable. Keep in mind that periodic unfolding violates regularity: even for $w \in C_0^\infty(\Omega)$ only $\mathcal{T}_\varepsilon^*(w) \in L^p(\Omega; W^{1,p}(Y))$ with $Y = [0,1)^d$ can be expected and in general, $\mathcal{T}_\varepsilon^*(w) \notin L^p(\Omega; W^{1,p}(\mathcal{Y}))$, recalling that periodicity notions are linked to smoothness.

Second, our compactness theorems on gradients will only consider $\mathcal{B} = \mathbb{R}^m$ for $m \in \mathbb{N}$; the following result will only employ $\mathbb{R}^1$ but taking finite products yields the vectorial case, too. More refined results invoking infinite-dimensional spaces $\mathcal{B}$ are conceivable but are tied to very subtle technical discussions since integration by parts plays a key role such that a Stokes theorem for Banach valued functions must be considered; we want to avoid such inconveniences here.

Though working with gradients, no compactness theorems about the strong topology are going to be derived. All we can hope for are weak sequential compactness statements. For the next statement recall $L^p\left(\Omega; W^{1,p}(\mathcal{Y}_1)/\mathbb{R}\right) := \left\{ f \in L^p(\Omega; W^{1,p}(\mathcal{Y}_1)) : \forall x \overset{\text{a.e.}}{\in} \Omega : \int_{\mathcal{Y}} \widetilde{f}(x,y)\, dy = 0 \right\}$.

Theorem §7.2: Two-scale compactness of gradients

Let $p \in (1,\infty)$ and $\Omega \subset \mathbb{R}^d$ be a bounded domain with an admissible periodic decomposition and a $C^{0,1}$-regular boundary. If a sequence $(u_\varepsilon)_{\varepsilon\in\mathbb{J}} \subset W^{1,p}(\Omega)$ admits the bound

$$\left\{ \exists C > 0 : \forall \varepsilon \in \mathbb{J} : \quad \|u_\varepsilon\|_{L^p(\Omega)} + \|\nabla_\varepsilon u_\varepsilon\|_{L^p(\Omega)^d} \le C \right. \tag{7.5}$$

then there exist three functions u_0, u_1, v in the vector spaces

$$\left\{ \begin{aligned} u_0 &\in W^{1,p}(\Omega) \\ v &\in L^p\left(\Omega; W_0^{1,p}(\mathcal{Y}_2)\right) \\ u_1 &\in L^p\left(\Omega; W^{1,p}(\mathcal{Y}_1)/\mathbb{R}\right) \end{aligned} \right. \tag{7.6}$$

together with a subsequence $\varepsilon' \in \mathbb{J}' \subset \mathbb{J}$ such that for $\varepsilon' \longrightarrow 0$ the following convergence results hold.

$$\mathcal{T}_{\varepsilon'}^*(u_{\varepsilon'}) \longrightarrow u_0 + v \text{ in } L^p(\mathbb{R}^d \times \mathcal{Y}), \tag{7.7a}$$

$$\mathcal{T}_{\varepsilon'}^*(\nabla_{\varepsilon'} u_{\varepsilon'}) \longrightarrow \chi_1 \left[\nabla_x u_0 + \nabla_y u_1\right] + \chi_2 \nabla_y v \text{ in } L^p(\mathbb{R}^d \times \mathcal{Y})^d. \tag{7.7b}$$

Often, we will denote $\varepsilon' \in \mathbb{J}'$ by $\varepsilon \in \mathbb{J}$ again.

Proof. Let us depart from the decomposition $u_\varepsilon = u_{\varepsilon|\Omega_\varepsilon^1} + u_{\varepsilon|\Omega_\varepsilon^2}$ which is defined almost everywhere in Ω and extend both portions to all of Ω by zero to get $u_\varepsilon = \widetilde{u_{\varepsilon|\Omega_\varepsilon^1}} + \widetilde{u_{\varepsilon|\Omega_\varepsilon^2}}$ and $\nabla_\varepsilon u_\varepsilon = \widetilde{\nabla u_{\varepsilon|\Omega_\varepsilon^1}} + \widetilde{\varepsilon \nabla u_{\varepsilon|\Omega_\varepsilon^2}}$, accordingly. Now, (7.5) makes Theorem §7.1 applicable to each portion. This brings about the existence of a subsequence again denoted by $\varepsilon \in \mathbb{J}$ with

$$\left\{ \begin{aligned} &\exists v_0 \in L^p(\Omega \times \mathcal{Y}) : & v_\varepsilon &:= \widetilde{u_{\varepsilon|\Omega_\varepsilon^1}} \xrightarrow{2w} v_0 && \text{in } L^p(\mathbb{R}^d \times \mathcal{Y}), \\ &\exists w_0 \in L^p(\Omega \times \mathcal{Y}) : & w_\varepsilon &:= \widetilde{u_{\varepsilon|\Omega_\varepsilon^2}} \xrightarrow{2w} w_0 && \text{in } L^p(\mathbb{R}^d \times \mathcal{Y}), \\ &\exists a_0 \in L^p(\Omega \times \mathcal{Y})^d : & a_\varepsilon &:= \widetilde{\nabla u_{\varepsilon|\Omega_\varepsilon^1}} \xrightarrow{2w} a_0 && \text{in } L^p(\mathbb{R}^d \times \mathcal{Y})^d, \\ &\exists b_0 \in L^p(\Omega \times \mathcal{Y})^d : & b_\varepsilon &:= \widetilde{\varepsilon \nabla u_{\varepsilon|\Omega_\varepsilon^2}} \xrightarrow{2w} b_0 && \text{in } L^p(\mathbb{R}^d \times \mathcal{Y})^d. \end{aligned} \right. \tag{7.8}$$

From this point on, it remains to specify the limit functions and their relations as claimed. The general procedure is to set out from some weak convergence result like

$$\forall \Phi \in L^{p'}(\mathbb{R}^d \times \mathcal{Y}) : \quad \iint_{\mathbb{R}^d \times \mathcal{Y}} \mathcal{T}_\varepsilon^*(v_\varepsilon)\Phi \, dxdy \overset{\varepsilon \to 0}{\longrightarrow} \iint_{\mathbb{R}^d \times \mathcal{Y}} v_0 \Phi \, dxdy \tag{7.9}$$

and use suitable test functions to derive characterisation results, for instance let $\Phi = \mathcal{T}_\varepsilon^*(\varphi_\varepsilon)$ for some $\varphi \in \mathcal{D}(\Omega \times \mathcal{Y})$ – $(\varphi_\varepsilon(x) := \varphi(x, x/\varepsilon))_{\varepsilon \in \mathbb{J}}$ is strongly two-scale convergent to φ in $L^p(\mathbb{R}^d \times \mathcal{Y})$ – and use (3.17) and to arrive at

$$\lim_{\varepsilon \to 0} \int_\Omega v_\varepsilon(x)\varphi(x, x/\varepsilon) \, dx = \iint_{\Omega \times \mathcal{Y}} v_0(x,y)\varphi(x,y) \, dxdy \tag{7.10}$$

which is Allaire's original definition of two-scale convergence; the latter is often a little more convenient to manipulate when working with derivatives. Now, the idea is to show desirable properties by choosing $(\varphi_\varepsilon)_{\varepsilon \in \mathbb{J}}$ suitably and to argue by density to arrive at the final results.
Step 1: v_0 and a_0. First, both v_0 and a_0 vanish on $\Omega \times \mathcal{Y}_2$ since they are weak limits of sequences that vanish there, a property that can be carried over easily considering smooth test functions. To characterise the gradient structure of a_0 we take $\vec{\varphi} \in \mathcal{D}(\Omega \times \mathcal{Y}_1)^d$ and mime (7.10) to get

$$-\int_\Omega u_\varepsilon(x) \left[\nabla_x \cdot \vec{\varphi}(x, x/\varepsilon) + \frac{1}{\varepsilon} \nabla_y \cdot \vec{\varphi}(x, x/\varepsilon) \right] dx = \int_\Omega \nabla u_\varepsilon(x) \cdot \vec{\varphi}(x, x/\varepsilon) \, dx \tag{7.11}$$

by the chain rule and integration by parts. Recall, that $\nabla_x \cdot \vec{\varphi}(x,y) := \sum_{i=1}^d \partial_{x_i} \varphi(x,y)$ and its likes denote the divergence operators with respect to the specified variables. Multiplying (7.11) with ε and passing to the limit yields

$$0 = \iint_{\Omega \times \mathcal{Y}} v_0 \nabla_y \cdot \vec{\varphi} \, dxdy \tag{7.12}$$

for all $\vec{\varphi} \in \mathcal{D}(\Omega \times \mathcal{Y}_1)^d$ which infers $\nabla_y v_0 \equiv 0$ in $\Omega \times \mathcal{Y}_1$ by density of test functions in $L^p(\Omega \times \mathcal{Y}_1)^d$ such that $v_0(x,y) \equiv v_0(x)$, i.e. v_0 is constant in y. Reconsidering (7.11) with $\vec{\varphi} \in D_1 := \left\{ \vec{\varphi} \in \mathcal{D}(\Omega \times \mathcal{Y}_1)^d : \nabla_y \cdot \vec{\varphi} \equiv 0 \right\}$ and passing to the limit leads to

$$\forall \vec{\varphi} \in D_1 : \quad \iint_{\Omega \times \mathcal{Y}} v_0 \nabla_x \cdot \vec{\varphi} \, dxdy = \iint_{\Omega \times \mathcal{Y}} a_0 \cdot \vec{\varphi} \, dxdy \tag{7.13}$$

since $\nabla_x \cdot \vec{\varphi}(x, x/\varepsilon)$ is strongly two-scale convergent to $\nabla_x \cdot \vec{\varphi}(x,y) \in \mathcal{D}(\Omega \times \mathcal{Y}_1)$. Choosing $\vec{\varphi}(x,y) \equiv \vec{\varphi}(x)$, i.e. independent of its y-variable, quickly infers $\nabla v_0 = a_0$ in $\mathcal{D}'(\Omega)$ which can be specified to $\nabla v_0 = \int_{\mathcal{Y}} a_0 \, dy \in L^p(\Omega)$, so $v_0 \in W^{1,p}(\Omega)$. To introduce the two-scale corrector $\nabla_y u_1$ we proceed as follows. First, define a norm

$$\|\vec{\varphi}\|_* := \|\vec{\varphi}\|_{L^{p'}(\Omega \times \mathcal{Y}_1)^d} + \|\nabla_x \cdot \vec{\varphi}\|_{L^{p'}(\Omega \times \mathcal{Y}_1)} + \left\|\nabla_y \cdot \vec{\varphi}\right\|_{L^{p'}(\Omega \times \mathcal{Y}_1)} \tag{7.14}$$

and let D_0 denote the completion of $\mathcal{D}(\Omega \times \mathcal{Y}_1)^d$ under the resulting uniform structure. Clearly, the image of D_1 in D_0 has a closure in D_0 denoted by D_2 which is strictly smaller

than D_0. Furthermore, both divergence operators $\nabla_x\cdot, \nabla_y\cdot$ have an extension to all of D_0 as they are continuous and so, we have two operators $\nabla_x\cdot, \nabla_y\cdot : D_0 \longrightarrow L^{p'}(\Omega \times \mathcal{Y}_1)$.
By density, (7.13) infers $(\nabla_x \cdot)'(v_0) = a_0$ in D_2'. It is rather straightforward to see that D_2 must contain all y-gradients, i.e. the closure of the set $\{\vec{\varphi} \in \mathcal{D}(\Omega \times \mathcal{Y}_1)^d : \exists u \in \mathcal{D}(\Omega \times \mathcal{Y}_1) : \nabla_y u = \vec{\varphi}\}$, a result that makes crucial use of compact supports. Moreover, it is a quite a non-trivial result – see [Are+15, Thm. 13.9] – that if $\Omega \times \mathcal{Y}_1$ is connected, bounded and has Lipschitz-regular boundary then the remaining kernel is made up by constant functions only and thus isometrically isomorphic to $\mathbb{R}$. Consequently, we have

$$\exists u_1 \in L^p\left(\Omega; W^{1,p}(\mathcal{Y}_1)/\mathbb{R}\right) : \quad a_0 = \nabla_x v_0 + \nabla_y u_1 \quad \text{in } L^p(\Omega \times \mathcal{Y}_1)^d \tag{7.15}$$

and u_1 is well-defined up to a constant.
Step 2: w_0 and b_0. Take $\varphi \in \mathcal{D}(\Omega \times \mathcal{Y}_2)^d$ in (7.11) multiplied with ε to get

$$-\int_\Omega u_\varepsilon(x)\left[\varepsilon\nabla_x \cdot \vec{\varphi}(x, x/\varepsilon) + \nabla_y \cdot \vec{\varphi}(x, x/\varepsilon)\right] dx = \int_\Omega \varepsilon \nabla u_\varepsilon(x) \cdot \vec{\varphi}(x, x/\varepsilon)\, dx \tag{7.16}$$

which converges for $\varepsilon \longrightarrow 0$ to

$$-\iint_{\Omega\times\mathcal{Y}_2} w_0(x,y)\left[\nabla_y \cdot \vec{\varphi}(x,y)\right] dxdy = \iint_{\Omega\times\mathcal{Y}_2} b_0(x,y)\cdot\vec{\varphi}(x,y)\, dxdy \tag{7.17}$$

for all $\vec{\varphi} \in \mathcal{D}(\Omega \times \mathcal{Y}_2)^d$, inferring $\nabla_y w_0 = b_0$ with $w_0 \in L^p(\Omega; W^{1,p}(\mathcal{Y}_2))$.
Step 3: The interface Γ. Since $v_0 + w_0$ and $a_0 + b_0$ are defined λ^{2d}-almost everywhere in $\Omega \times \mathcal{Y}$, having disjoint supports outside of Γ we need to investigate the latter in order to arrive at a globally regular function. So, let us revisit (7.16) with $\vec{\varphi} \in \mathcal{D}(\Omega \times \mathcal{Y})^d$. For $\varepsilon \longrightarrow 0$ we retrieve

$$\begin{cases} \forall \vec{\varphi} \in \mathcal{D}(\Omega \times \mathcal{Y})^d : & -\displaystyle\iint_{\Omega\times\mathcal{Y}} \chi_2(y)\nabla_y w_0(x,y)\cdot\vec{\varphi}(x,y)\, dxdy \\ & = \displaystyle\iint_{\Omega\times\mathcal{Y}} \left[\chi_1(y)v_0(x) + \chi_2(y)w_0(x,y)\right]\left[\nabla_y\cdot\vec{\varphi}(x,y)\right] dxdy. \end{cases} \tag{7.18}$$

Again one integrates (7.18) by parts, invoking that inside of Ω for $i = 1, 2$ the outward normal unit vectors on $\mathcal{Y}_i$ – we denote these by $\vec{n}_i$ – fulfil $\vec{n}_1 = -\vec{n}_2$. We arrive at

$$\forall \vec{\varphi} \in \mathcal{D}(\Omega \times \mathcal{Y})^d : \; 0 = \int_\Omega \int_{\partial\mathcal{Y}_1} \vec{n}_1(y)\cdot\varphi(x,y)\left[w_0(x,y) - v_0(x)\right]_{|\partial\mathcal{Y}_1} d\mathcal{H}^{d-1}(y)\, dx \tag{7.19}$$

inferring $[w_0(x,y) - v_0(x)]_{|\partial\mathcal{Y}_1} = 0$ for $\mathcal{H}^{d-1}$-almost all $y \in \partial\mathcal{Y}_1 = \partial\mathcal{Y}_2$ (in the sense of traces on $\partial\mathcal{Y}_1$). Thus, for λ^d-almost all $x \in \Omega$ the support of $y \longmapsto w_0(x,y)$ must be contained in $\mathcal{Y}_2$ and therefore $w_0 \in L^p(\Omega; W_0^{1,p}(\mathcal{Y}_2))$ which was to show. ■

Remark §7.2: Time-dependency and compactness

Chapter III will consider parabolic initial–boundary value problems, so let us note that both theorem §7.1 and theorem §7.2 carry over almost verbatim to the time dependent case if (temporal) L^1-regularity is available. The latter is a minimal condition due to periodic unfolding. So, if no temporal oscillations of the coefficients are present then periodic unfolding only works on spatial variables. Under these conditions the very same proofs carry over with minimal changes, for example $\mathscr{D}(\Omega \times \mathcal{Y}_i)$ is replaced by $\mathscr{D}(I \times \Omega \times \mathcal{Y}_i)$ for some open and bounded time interval $I \subset \mathbb{R}$.

On regularity requirements

Let us comment on the regularity requirements of theorem §7.1 and theorem §7.2 about the boundaries $\partial\Omega$ and $\partial\mathcal{Y}_1$. Theorem §7.1's $C^{0,1}$-regularity of the boundary can be weakened since only $\lambda^d(\partial\Omega) = 0$ is actually necessary, a state of affairs that [MT06] already pointed out. The compactness of $\partial\Omega$ is necessary for the projective limit procedure to be convergent such that $\lambda^d(\Lambda_\varepsilon^+) \to 0$ holds for $\varepsilon \to 0$. If the latter aspect can be assured by alternative means it is conceivable to drop the compactness assumption, too.

Concerning Theorem §7.2, Γ was required to be $C^{0,1}$-regular. Its first purpose was to ensure $\Omega \times \mathcal{Y}_1$ would have a $C^{0,1}$-regular boundary enabling us to characterise the kernel of $\nabla_y \cdot$ as gradients plus constant functions, leading to (7.15). The author conjectures that this characterisation holds under weaker assumptions since the test functions under consideration have compact support in $\Omega \times \mathcal{Y}_1$ and $\Omega \times \mathcal{Y}_1$ is a bounded C^∞-manifold. More specifically, it is well-known, c.f. [BT82, Cor. 5.8], that (only) connectedness of $\Omega \times \mathcal{Y}_1$ implies $H_c^d(\Omega \times \mathcal{Y}_1) \cong \mathbb{R}$ to hold, i.e. the *d-th de Rham cohomology group* of smooth functions with compact support is isomorphic to $\mathbb{R}$ in terms of groups. The latter is strongly linked to divergence operators and the author plans to enquire this matter in future works. If the conjecture would prove true then this step would require no regularity from the boundary at all working only inside of the domain.

Secondly, establishing (7.18) required the foregoing characterisation to hold. Provided the latter holds, its consequence (7.19) is the actually important result and it relies on having Gauß–Green–Stokes theorems and suitable trace operators on $\partial\mathcal{Y}_1$ available. The common assumption of $C^{0,1}$-regular interfaces is of course only sufficient. Without intending any offence at all, it seems to the author that this condition is based on the state of knowledge attained around the time of the original release of [AF03] in 1975.[6] However, drastically refined versions are available nowadays, we refer to H. Triebel's trace theorems on d-sets presented in [Tri06, Sec. 7.1], to Gauß–Green theorems for BV-functions given in V. Maz'ya's [Maz11, Sec. 9.6] and – in particular – to [AFP00, Thm 3.77]. Presently, the latter result seems to be the most promising to the author asking for $\partial\mathcal{Y}_1$ to be rectifiable. Again, the author plans to address this matter in a future piece of work, thanking Dr. M. Santilli and Dr. J. Tölle for indicating to him the more recent works and results on traces and related matters.

A very defective example

Let us conclude with this section with underlining two rather simple but fundamental facts. First, theorem §7.2 is only a statement about weak topologies and secondly, estimates like (7.5) and its time-dependent analogue are way too weak to yield stronger compactness results. As an

[6] A claim based solely on the author's mere impression which may be just as well plain ignorance.

illustration consider the sequence

$$\left(z_\varepsilon(x) := \sum_{k \in \mathbb{N}} \frac{1}{k^2} \sin\left(x / \sqrt[k]{\varepsilon}\right) \right)_{\varepsilon \in \mathbb{J}} \tag{7.20}$$

which fulfils $\|\varepsilon \nabla z_\varepsilon\|_{L^\infty(\mathbb{R})}$ for all $\varepsilon > 0$. However, $(z_\varepsilon)_{\varepsilon \in \mathbb{J}}$ entirely defies the scale separation approach: neither two-scale convergence nor a finite number of additional scales can handle $(z_\varepsilon)_{\varepsilon \in \mathbb{J}}$'s convergence properties correctly to yield more than weak convergence results. Thus, $(z_\varepsilon)_{\varepsilon \in \mathbb{J}}$ must be considered as a worst case scenario that can occur inside the fast domain having only (7.5) and its likes at hand. Consequently, usable convergence results must stem from other means than compactness theorems.

§7.5. Two-scale compactness of gradients – a limit space

Theorem §7.2 and the set-up therein suggest the definition of a limit space containing the functions (u_0, u_1, v) from (7.6). As it turns out, one can regard these equivalently as pairs or as triplets.

Definition §7.1: **Subspace of two-scale limits**

Referring to theorem §7.1 and theorem §7.2 we define the corresponding two-scale limit space by

(7.21)
$$\begin{aligned} \Xi(\Omega) := \Big\{ (w_0, w_1) \in L^p(\Omega \times \mathcal{Y})^{1+d} : \exists u_0 \in W_0^{1,p}(\Omega), \quad & u_1 \in L^p(\Omega; W^{1,p}(\mathcal{Y}_1)/\mathbb{R}) \\ \text{and} \quad v \in L^p(\Omega; W_0^{1,p}(\mathcal{Y}_2)) : \forall (x,y) & \overset{\text{a.e.}}{\in} \Omega \times \mathcal{Y} : \\ w_0(x,y) = u_0(x) + v(x,y) \quad & \wedge \\ w_1(x,y) = \chi_1(y) \left[\nabla_x u_0(x) + \nabla_y u_1(x,y) \right] & + \nabla_y v(x,y) \Big\}. \end{aligned}$$

$\Xi := \Xi(\Omega)$ is a closed subspace in $L^p(\Omega \times \mathcal{Y})^{1+d}$. We will frequently identify $(w_0, w_1) \in \Xi$ with the triplet of functions $(u_0, u_1, v) = (w_0, w_1)$ which essentially represent (w_0, w_1). Moreover, taking advantage of these alternative representations it will be convenient to write

$$w_1 = \widetilde{\nabla} w_0(x,y) = \chi_1(y) \left[\nabla_x u_0 + \nabla_y u_1 \right] + \nabla_y v \tag{7.22}$$

such that $(w_0, w_1) = (w_0, \widetilde{\nabla} w_0) = (u_0, u_1, v) \in \Xi$, a notation the expresses the fact that $\widetilde{\nabla} w_0 = w_1$ is a gradient augmented by two-scale correctors.

We conclude the first chapter with the prediction that Ξ is to play a pivotal role on our analysis. It is quite apparent that solutions of limit problems will be elements of Ξ, though a rigorous justification for this claim is the purpose of the next chapters. In functional analytical terms, Ξ will be characterised as stemming from procedure similar to Cauchy completion. Perhaps, this is not trivial to anticipate at the current state of affairs.

II. A quasi-linear, elliptic problem

§8. Configuration of the problem

§8.1. Crude formulation of the problem

This chapter will introduce a sequence of quasi-linear, stationary boundary value problems and their abstract formulation. The domain $\Omega \subset \mathbb{R}^d$ is assumed to be non-void, bounded and with $C^{0,1}$-regular boundary throughout. As pointed out at the beginning of chapter I, our interest lies in materials that possess a fine locally periodic structure. For the purpose of clarity, let us distinguish periodic coefficients on interfaces such as $\partial\Omega$ or Γ and periodic coefficients on Ω itself. Whereas the former is a *very* worthwhile matter of investigation, we refrain from such considerations, placing our attention on the second item.

We will begin with a discussion of related modelling assumptions which will serve to guide the following mathematical treatment. In rather crude terms, we want to consider boundary value problems of the form

$$(8.1) \qquad \begin{cases} -\nabla \cdot \vec{a}_\varepsilon\,(x, u_\varepsilon, \nabla u_\varepsilon) + b_\varepsilon(x, u_\varepsilon) = f_\varepsilon & \text{in } \Omega \text{ and} \\ u_\varepsilon = g & \text{on } \partial\Omega \end{cases}$$

whose coefficients typically oscillate at scale ε with ε becoming increasingly small. To this end, let $0 < \varepsilon_0 \ll \operatorname{diam}(\Omega)$ be fixed and assume $\mathbb{J} \subset (0, \varepsilon_0)$ to be a given, countable, non-void set with $0 \in \overline{\mathbb{J}}$. $\mathbb{J}$ supplies us with a null sequence $(\varepsilon_n)_{n\in\mathbb{N}} = \mathbb{J}$ which we simply denote by $(\mathbb{J} \ni)\varepsilon \to 0$.

A roadmap for the problem: compensation of compactness by monotonicity

Naturally, we need to specify suitable conditions on (8.1) and its coefficient functions $\vec{a}_\varepsilon$ and b_ε such that our toolbox from chapter I is applicable. To this end, we ask that an admissible periodic decomposition of Ω is given in the sense of definition §6.1. Leaving the coefficients aside for a moment, we will impose conditions on the right hand side functions $(f_\varepsilon)_{\varepsilon\in\mathbb{J}}$ such that (8.1) can be reformulated as a problem that is weakly solvable for all, or at least cofinitely many $\varepsilon \in \mathbb{J}$, i.e. for all but finitely many $\varepsilon \in \mathbb{J}$. Recall that its preferable not to specialise the sequence $\varepsilon \in \mathbb{J}$ nor to use constructions that exploit specific null sequences.

Most crucially, we will encounter severely flawed a priori estimates making compactness methods basically unusable. In contrast, the method of monotonicity remains potentially unchallenged, a fact providing us with a possible compensation. Although both methods are splendidly combinable, most notably in the family of pseudo-monotone operators, we cannot overcome the compactness defect caused by the slow medium so that monotonicity alone must fill the gap.

Finally, let us point out that realising the necessity for a compensation of compactness is not obvious (at least to the author) from the outset. Rather, our requirements are crafted retrospectively to fit theorem §10.1 below.

§8.2. Requirements on the superposition operator

We intend to employ the well-established techniques of monotone operators in reflexive Banach spaces due to decent weak compactness theorems being are available there together with monotone operators's favourable weak continuity properties on solution sets. Eventually, we will formalise in the next chapter that monotone operators can be considered as a family of operators that behaves essentially like a single pseudo-monotone operator with respect to $\varepsilon \longrightarrow 0$. In contrast, this is generally not true if one starts with pseudo-monotone operators in the first place.

As mentioned in chapter I, not much literature seems to be available on highly heterogeneous media. All instances known to the auther work with linearity or some compensating technique that works without compactness arguments. At the time of this writing, S. Reichelt work presented in [Rei15] seems to be the only succesful non-linear technique around. There, Lipschitz continuous perturbations of the Laplacian $\varepsilon^2\Delta$ in $L^2(\Omega)$ are considered. The author suspects that Hilbert space techniques should be dispensable for Reichelt's methods but the need of the perturbations to be Lipschitz seems vital for Gronwall's lemma to be applicable, a key tool in the steps 3 and 4 of the proof of [Rei15, Thm 2.1.6].

Classically, monotonicity framework are inhabited in frameworks of reflexive L^p-spaces and their duals. It must be said though, that this not more general than [Rei15] but rather a different configuration. So, let us fix $p \in (1, \infty)$ and and $p' := p/(p-1)$. It must be mentioned that the class of monotone operators with compact perturbations excluded is not 'too large', the p-Laplacian

$$\Delta_p(u) := -\nabla \cdot \left[|\nabla u|^{p-2}\nabla u\right] \qquad \text{mapping} \quad W^{1,p}(\Omega) \longrightarrow \left(W^{1,p}(\Omega)\right)'$$

being the standard prototype. In particular, pseudo-monotone operators like the p-Laplacian perturbed by lower order terms are only encompassed in the family of monotone operators if they are monotone themselves, a quite severe restriction: the loss of the Rellich–Kondrachov embedding inside the slow domain Ω_ε^2 is not easily compensated for.

Requirements on the function inducing the superposition operator

Let us specify the coefficient functions $\vec{a}_\varepsilon$ and b_ε of (8.1) by assuming that two functions

$$\begin{cases} \mathfrak{a} : \Omega \times \mathcal{Y} \times \mathbb{R} \times \mathbb{R}^d & \longrightarrow \mathbb{R}^d \\ \mathfrak{b} : \Omega \times \mathcal{Y} \times \mathbb{R} & \longrightarrow \mathbb{R} \end{cases} \tag{8.2}$$

are given that induce $\vec{a}_\varepsilon$ for $(\varepsilon, x, x/\varepsilon, r, s,) \in \mathbb{J} \times \Omega \times \mathcal{Y} \times \mathbb{R} \times \mathbb{R}^d$ by

$$\vec{a}_\varepsilon(x, r, s) = \mathfrak{a}(x, x/\varepsilon, r, s) \tag{8.3a}$$

and b_ε for $(\varepsilon, x, x/\varepsilon, r, s,) \in \mathbb{J} \times \Omega \times \mathcal{Y} \times \mathbb{R} \times \mathbb{R}^d$ by

$$b_\varepsilon(x, r) = \mathfrak{b}(x, x/\varepsilon, r). \tag{8.3b}$$

As customary, the functions $\mathfrak{a}$ and $\mathfrak{b}$ are to induce Nemytskii operators, mapping $(u_0, \ldots, u_d) \in L^p(\Omega)^{1+d}$ to $L^{p'}(\Omega)^d$ for $\varepsilon > 0$ via

$$(u_0, \vec{u}) \longmapsto \mathcal{N}_{\mathfrak{a}}(u_0, \vec{u}) := \mathfrak{a}(x, x/\varepsilon, u_0, \vec{u}), \tag{8.4}$$

writing $\vec{u} := (u_1, \dots, u_d)$ and $\mathcal{N}_\mathfrak{a}$ for the Nemytskii operator induced by $\mathfrak{a}$. Analogously, one defines $\mathcal{N}_\mathfrak{b}$ by

$$u_0 \longmapsto \mathcal{N}_\mathfrak{b}(u_0) := \mathfrak{b}(x, x/\varepsilon, u_0). \tag{8.5}$$

In order to ensure sensible properties for the operators at hand, we need to impose several conditions on the functions $\mathfrak{a}$ and $\mathfrak{b}$. Fortunately, the requirements are standard besides the two-scale admissibility matter already discussed. We want the superposition operators to yield measurable functions that possess the desired integrability. In addition, we need coercivity and monotonicity conditions, too.

N.1) (Modified) Caratheodory maps. For all $(r, s) \in \mathbb{R} \times \mathbb{R}^d$ the maps

$$\begin{cases} \mathfrak{a}(\,\cdot\,,\,\cdot\,, r, s) : \Omega \times \mathcal{Y} \longrightarrow \mathbb{R}^d & (x, y) \longmapsto \mathfrak{a}(x, y, r, s) \\ \mathfrak{b}(\,\cdot\,,\,\cdot\,, r) : \Omega \times \mathcal{Y} \longrightarrow \mathbb{R} & (x, y) \longmapsto \mathfrak{b}(x, y, r) \end{cases} \tag{8.6a}$$

are $\mathcal{L}(\Omega)\otimes\mathcal{L}(\mathcal{Y})-\mathcal{B}(\mathbb{R}^d)$-measurable and $\mathcal{L}(\Omega)\otimes\mathcal{L}(\mathcal{Y})-\mathcal{B}(\mathbb{R})$-measurable, respectively. Secondly, one imposes continuity conditions: for λ^d-almost all $y \in \mathcal{Y}$ the maps

$$\begin{cases} \mathfrak{a}(\,\cdot\,, y,\,\cdot\,,\,\cdot\,) : \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}^d & (x, r, s) \longmapsto \mathfrak{a}(x, y, r, s) \\ \mathfrak{b}(\,\cdot\,, y,\,\cdot\,) : \mathbb{R} \longrightarrow \mathbb{R} & (x, r) \longmapsto \mathfrak{b}(x, y, r) \end{cases} \tag{8.6b}$$

are continuous in the respective Euclidean topologies.

N.2) Growth conditions. We ask for a constant $C \geq 0$ such that for λ^{2d}-almost all $(x, y) \in \Omega\times\mathcal{Y}$ and all $(r, s) \in \mathbb{R} \times \mathbb{R}^d$

$$|\mathfrak{a}(x, y, r, s)| + |\mathfrak{b}(x, y, r)| \leq C\left(1 + |r|^{p/p'} + |s|^{p/p'}\right) \tag{8.7}$$

holds.

N.3) Coercivity condition. There exist constants $C_1 > 0, C_2 \geq 0$ and an non-negative, continuous function $h_\mathfrak{a} \in L^1(\Omega \times \mathcal{Y})$ such that

$$\mathfrak{a}(x, y, r, s) \cdot s + \mathfrak{b}(x, y, r) \cdot r \geq C_1|s|^p + C_2|p|^p - h_\mathfrak{a}(x, y) \tag{8.8}$$

holds for λ^{2d}-almost all $(x, y) \in \Omega \times \mathcal{Y}$ and all $(r, s) \in \mathbb{R} \times \mathbb{R}^d$.

N.4) Monotonicity condition. $\mathfrak{a}$ and $\mathfrak{b}$ induce superposition operators that are monotone on their domain of definition in $W^{1,p}(\Omega)$. To this end we require that

$$[\mathfrak{a}(x, y, r, s_1) - \mathfrak{a}(x, y, r, s_2)] \cdot [s_1 - s_2] \geq 0, \tag{8.9a}$$

$$[\mathfrak{b}(x, y, r_1) - \mathfrak{b}(x, y, r_2)] \cdot [r_1 - r_2] \geq 0. \tag{8.9b}$$

holds for λ^{2d}-almost all $(x, y) \in \Omega \times \mathcal{Y}$ and all $(r_1, r_2, r, s_1, s_2) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$.

§8.3. Requirements on the source and boundary terms

The sequence of source terms $(f_\varepsilon)_{\varepsilon\in\mathbb{J}} \subset L^{p'}(\Omega)$ is required to be strongly two-scale convergent:

$$\exists f_0 \in L^{p'}(\Omega \times \mathcal{Y}) : \quad \left\|f_0 - \mathcal{T}_\varepsilon^*(f_\varepsilon)\right\|_{L^{p'}(\mathbb{R}^d\times\mathcal{Y})} \overset{\varepsilon\to 0}{\longrightarrow} 0 \tag{8.10}$$

and again, for modelling reasons it is desirable that f_0 does not depend on the specific choice of $\varepsilon \in \mathbb{J}$. Concerning boundary values, our intentions are quite modest: we require that there exists a function $\widetilde{g} \in W^{1,p}(\Omega)$ such that $g_{|\partial\Omega} \in W^{1-1/p,p}(\partial\Omega)$ is its resulting trace function.

§9. Statement and treatment of the individual problems

The foregoing conditions allow us to give a reasonable formulation of (8.1). Note that here $\varepsilon \in \mathbb{J}$ is fixed as we do not yet consider limiting behaviour for $\varepsilon \longrightarrow 0$. Therefore, we are going to use the label of *ε-dependent (boundary value) problems*. For what follows set $\mathscr{V} := W_0^{1,p}(\Omega)$.

§9.1 Problem (ε-dependent problem)**.** For $\varepsilon \in \mathbb{J}$ find $u_\varepsilon \in \mathscr{V}$ such that for all $\varphi \in \mathscr{V}$ we have

$$(\varepsilon\mathrm{P}) \qquad \begin{cases} \displaystyle\int_\Omega \mathfrak{a}\left(x, \frac{x}{\varepsilon}, \widetilde{g}+u_\varepsilon, \nabla_\varepsilon \widetilde{g}+\nabla_\varepsilon u_\varepsilon\right)\cdot\nabla_\varepsilon\varphi \quad + \quad \mathfrak{b}\left(x, x/\varepsilon, \widetilde{g}+u_\varepsilon\right)\varphi\,dx \\ \qquad\qquad =: \langle A_\varepsilon(u_\varepsilon), \varphi\rangle_{\mathscr{V}'\times\mathscr{V}} \quad = \quad \langle \ell_\varepsilon, \varphi\rangle_{\mathscr{V}'\times\mathscr{V}} := \displaystyle\int_\Omega f_\varepsilon\varphi\,dx. \end{cases}$$

Lemma §9.1

For every fixed $\varepsilon > 0$ the operator $A_\varepsilon : \mathscr{V} \longrightarrow \mathscr{V}'$ is monotone and coercive and $\ell_\varepsilon \in \mathscr{V}'$. Consequently, (εP) is equivalent to finding $u_\varepsilon \in \mathscr{V}$ such that $A_\varepsilon(u_\varepsilon) = \ell_\varepsilon$ in $\mathscr{V}'$.

Proof. A_ε and ℓ_ε are well-defined and their claimed properties are due to the conditions imposed on $\mathfrak{a}$ and $\mathfrak{b}$ beforehand. As aforementioned, the verification is considered standard and can be found in [Rou13, Ch. 2], for instance. Finally, same applies to equivalence of operator equation and the corresponding boundary value problem. ■

Theorem §9.1: **Solvability**

For every $\varepsilon > 0$ the operator problem corresponding to Problem §9.1 is solvable. In addition, if (8.9) is strengthened to yield equality if and only if $s_1 = s_2$ and $r_1 = r_2$ then there is a unique solution. Independently of uniqueness, the following a priori estimate can be established:

$$\exists C > 0 : \forall \varepsilon > 0 : \quad \|u_\varepsilon\|_{L^p(\Omega)} + \|\nabla_\varepsilon u_\varepsilon\|_{L^p(\Omega;\mathbb{R}^d)} \leq C \tag{9.1}$$

Proof. By lemma §9.1 we may resort to the operator equation which involves a coercive and monotone operator. Both Brezis's theorem §A.2.1 or the Browder–Minty theorem yield the solvability of the operator equation. Uniqueness of solutions is due to strict monotonicity which is elementary to verify. Finally, (9.1) is deduced by standard calculations employing the coercivity constants from (8.8). ■

The estimate (9.1) is of paramount importance and the compactness theorems in section §7.4 were tailor-made to be in line with it, inferring the next result.

Corollary §9.1: **Presumptive limit functions**

To a given sequence $\varepsilon \in \mathbb{J}$ there is a subsequence of solutions $(u_{\varepsilon'})_{\varepsilon' \in \mathbb{J}'}$ of Problem §9.1 with $\mathbb{J}' \subset \mathbb{J}$ such that for Ξ given in definition §7.1 we have

$$\exists (w_0, w_1) \in \Xi(\Omega) : (\mathcal{T}^*_{\varepsilon'}(u_{\varepsilon'}), \mathcal{T}^*_{\varepsilon'}(\nabla_{\varepsilon'} u_{\varepsilon'})) \xrightarrow{\varepsilon' \to 0} (w_0, w_1) \quad \text{in } L^p(\mathbb{R}^d \times \mathcal{Y}; \mathbb{R}^{1+d}). \tag{9.2}$$

As customary, we will denote the subsequence by $(u_\varepsilon)_{\varepsilon \in \mathbb{J}}$, again.

Proof. The estimate (9.1) stems from the coercivity constants given in (8.8) and so, it is applicable to a sequence $(u_\varepsilon)_{\varepsilon \in \mathbb{J}}$, too. Applying theorem §7.1 and theorem §7.2 to $(u_\varepsilon)_{\varepsilon \in \mathbb{J}}$ yields the result. ■

Remark §9.1: **Sequences and nets**

One may notice that we distinguish statements holding for every $\varepsilon > 0$ from those that are tied to $\varepsilon \in \mathbb{J}$. The rule of thumb is that whenever sequences are under consideration we switch to the latter. In fact, one could work with $(u_\varepsilon)_{\varepsilon > 0}$ instead of $(u_\varepsilon)_{\varepsilon \in \mathbb{J}}$ but this would require to handle nets and subnets. As far as the author is aware of, all considerable compactness theorems only yield convergent subnets of a given net, and obtaining a convergent subsequence from a given net is an extremely restrictive and involved affair involving the notion of *supersequential compactness*, c.f. [Sch97, p. 468]. In addition, the easiness of handling subsequences is most starkly contrasted by the notion of subnets. As discussed in [Sch97, Ch. 7], there are at least three highly sensible but non-equivalent notions for the latter, making 'subnet' the author's favourite candidate for the most delusive notion in analysis. Moreover, whereas monotone operators are indeed fit to handle nets and subnets, pseudo-monotone operators fail to do so as they are defined in terms of sequences; and our limiting procedure is actually pseudo-monotonic in nature. Indeed, there are generalisations of pseudo-monotone operators handling nets, c.f. [DK00], but we refrain from their use for their corresponding limiting procedures seem to be quite more involved to the author, at least at present.

§10. Establishing of a limit problem

Section §9 provides us with a two-scale convergent sequence as specified in (9.2). It is our aim to show that $(w_0, w_1) \equiv (u_0, u_1, v) \in \Xi$ solve a limit problem on their own part which is related to Problem §9.1 in 'meaningful way'. Following G. Allaire's treatment of linear problems in [All92, Sec. 4], it is straightforward to come up with a candidate for a limit problem. Alas, making this correspondence rigorous is not as easy.

§10.1. The presumtive limit problem

A close reexamination of theorem §2.1 and theorem §2.2 allows to use corollary §9.1 to guess a potential limit problem whose weak form is as follows.

§10.1 Problem. Find $w = (w_0, w_1) = (w_0, \widetilde{\nabla} w_0) \in \Xi$ such that for all $\phi = (\phi_0, \widetilde{\nabla}\phi_0) \in \Xi$ we have

$$(10.1)\qquad \begin{cases} \iint\limits_{\Omega\times\mathcal{Y}} \mathfrak{a}\left(x, y, \widetilde{g} + w_0, \nabla\widetilde{g} + \widetilde{\nabla} w_0\right)\cdot \widetilde{\nabla}\phi_0 \quad + \quad \mathfrak{b}\left(x, y, \widetilde{g} + w_0\right)\phi_0 \, dxdy \\ \qquad =: \langle A_0(w), \phi\rangle_{\Xi'\times\Xi} \quad = \quad \langle \ell_0, \phi\rangle_{\Xi'\times\Xi} := \iint\limits_{\Omega\times\mathcal{Y}} f_0\phi_0 \, dxdy, \end{cases}$$

with $\widetilde{\nabla}$ representing the two-scale limits of gradient sequences of the form

$$\widetilde{\nabla}\phi_0(x,y) = \chi_1(y)\left[\nabla_x\varphi_0 + \nabla_y\varphi_1\right] + \nabla_y\psi$$

for $\phi = (\phi_0, \phi_1) = (\phi_0, \widetilde{\nabla}\phi_0) = (\varphi_0, \varphi_1, \psi) \in \Xi$, c.f. definition §7.1.

§10.2. Stock-taking and bargaining to make fit for periodic unfolding

Our wish is link the ε-dependent Problem §9.1 to Problem §10.1 sensibly. Thanks to (8.10) we have $f_\varepsilon \xrightarrow{2s} f_0$ in $L^{p'}(\mathbb{R}^d \times \mathcal{Y})$ and therefore $\ell_\varepsilon \longrightarrow \ell_0$ is established. As a minimal objective, $A_\varepsilon(u_\varepsilon) \xrightarrow{2w} A_0(w)$ in $\left(\mathcal{T}_\varepsilon^*\left(W^{1,p}(\Omega)\right)\right)'$ needs to be shown. Let us reformulate this in handier terms.

By proposition §7.1 $\mathcal{T}_\varepsilon^*\left(W^{1,p}(\Omega)\right) \hookrightarrow L^p(\mathbb{R}^d \times \mathcal{Y}; \mathbb{R}^{d+1})$ is clear, but also

$$(10.2)\qquad \left(\mathcal{T}_\varepsilon^*\left(W^{1,p}(\Omega)\right)\right)' \hookrightarrow \left(L^p(\mathbb{R}^d\times\mathcal{Y};\mathbb{R}^{d+1})\right)' \cong L^{p'}(\mathbb{R}^d\times\mathcal{Y};\mathbb{R}^{1+d})$$

is meaningful due to extension by zero or simply the Hahn–Banach theorem.

Furthermore, the conditions imposed in N.1) and N.2) allow us to recast $A_\varepsilon(u_\varepsilon)$ as an element of $L^{p'}(\Omega;\mathbb{R}^{1+d})$ via

$$(10.3)\qquad x \longmapsto \zeta_\varepsilon := \zeta_\varepsilon(x) := \left[\mathfrak{b}\left(x, \frac{x}{\varepsilon}, \widetilde{g}(x) + u_\varepsilon(x)\right), \mathfrak{a}\left(x, \frac{x}{\varepsilon}, \widetilde{g}(x) + u_\varepsilon(x), \nabla_\varepsilon\widetilde{g}(x) + \nabla_\varepsilon u(x)\right)\right].$$

However, one must not forget about $-\nabla_\varepsilon$ or its adjoint which are not incorporated into ζ_ε but transferred to the space of functions that test ζ_ε.

The purpose of subduing $-\nabla_\varepsilon$ for the moment is that unfolding $\mathcal{T}_\varepsilon^*(\zeta_\varepsilon) \in \mathcal{T}_\varepsilon^*(L^{p'}(\Omega;\mathbb{R}^{1+d}))$ works nicely, whereas $\mathcal{T}_\varepsilon^*(W^{-1,p}(\Omega))$ is not well-defined in general, even though there is a two-scale convergence notion for distributions. Thus, passing $-\nabla_\varepsilon$ to the testing functions is a very worthwhile trade for us since $\mathcal{T}_\varepsilon^*(\nabla_\varepsilon u)$ with $u \in W^{1,p}(\Omega)$ is perfectly suitable for periodic unfolding.

Now, our main objective is to establish

$$(10.4)\qquad \begin{cases} \zeta = \left(\mathfrak{b}\left(x, y, \widetilde{g} + w_0\right), \mathfrak{a}\left(x, y, \widetilde{g} + w_0, \nabla\widetilde{g} + \widetilde{\nabla}w_0\right)\right) \quad \text{and} \\ \qquad \zeta_\varepsilon \xrightarrow{2w} \zeta \qquad \text{in } L^{p'}(\mathbb{R}^d\times\mathcal{Y})^{1+d} \end{cases}$$

for some $w = (w_0, \widetilde{\nabla}w_0) \in \Xi(\Omega)$, namely the task of characterising the weak two-scale limit ζ tested against unfolded Sobolev functions with two-scale gradients. We will approach this task with weak compactness methods and a passage to limit technique borrowed from the theory of (pseudo-)monotone operators.

Remark §10.1: Merely weak limits?

(10.4) is a sequential continuity property of solutions in a weak topology and one may wonder whether this yield is rather poor. Nevertheless, it follows the tracks of [All92, Sec. 4] closely where weak convergence was established first, with stronger estimates following afterwards by requiring additional means. In our manner of speaking, [All92, Sec. 4] works with linear operators that are uniformly monotone such that our result can be regarded as a natural extension to L^p-spaces and in the same spirit, stronger results can be derived eventually. So, (10.4) must be understood as a result that establishes a framework. This thesis is not about being exhaustive with respect to stronger results.

§10.3. A limiting result

In what follows we will write $X := L^p(\mathbb{R}^d \times \mathcal{Y})^{1+d}$ and make use of the fact that incidence product $\langle\,\cdot\,,\,\cdot\,\rangle = \langle\,\cdot\,,\,\cdot\,\rangle_{X' \times X}$ coincides with $\langle\,\cdot\,,\,\cdot\,\rangle_{\widetilde{X}' \times \widetilde{X}}$ for $\widetilde{X} = \mathcal{T}_\varepsilon^*(\mathcal{V})$ or other closed subspaces. However, this places additional importance on keeping track of the particular test functions in use.

Theorem §10.1: Limit characterisation

Provided suitable recovery sequences do exist, (10.4) holds, and consequently, Problem §10.1 is solvable and its solution is the two-scale limit of a subsequence of solutions of Problem §9.1. Consequently, Problem §10.1 can be considered as the limit problem of Problem §9.1 under the initial assumption.

Proof. For convenience's sake, let us rephrase the statements available so far. By corollary §9.1 we have

$$w_\varepsilon := (\mathcal{T}_\varepsilon^*(u_\varepsilon), \mathcal{T}_\varepsilon^*(\nabla_\varepsilon u_\varepsilon)) \rightharpoonup w = \left(w_0 . \widetilde{\nabla} w_0\right) \in \Xi \qquad \text{in } X \tag{10.5}$$

Clearly, $A_\varepsilon : \mathcal{V} \longrightarrow \mathcal{V}$ is a bounded operator and so the boundedness $(u_\varepsilon, \nabla_\varepsilon u_\varepsilon)_{\varepsilon \in \mathbb{J}} \subset L^p(\Omega)^{1+d}$ and $(w_\varepsilon)_{\varepsilon \in \mathbb{J}} \subset X$ carries over to $(\zeta_\varepsilon)_{\varepsilon \in \mathbb{J}}$ from (10.3) being bounded in X'. Theorem §7.1 yields that there exists some $\zeta_0 \in L^{p'}(\Omega \times \mathcal{Y})^{d+1}$ such that $\mathcal{T}_\varepsilon^*(\zeta_\varepsilon) \rightharpoonup \zeta_0$ in X holds. So, we need to show that $A_0(w) = \zeta_0$. To this end we rewrite A_0 and A_ε with $\varepsilon > 0$ as being induced by a single pseudo-monotone operator, namely $\mathcal{B} : U \times \mathcal{T}_\varepsilon^*(\mathcal{V}) \longrightarrow \mathcal{V}'$ given by

$$\mathcal{B}(\mathcal{T}_\varepsilon, \mathcal{T}_\varepsilon^*(u), \mathcal{T}_\varepsilon^*(\nabla_\varepsilon u) := -\nabla_\varepsilon \cdot \widetilde{\mathfrak{a}}\left(\mathcal{T}_\varepsilon, y, \mathcal{T}_\varepsilon^*(u_\varepsilon), \mathcal{T}_\varepsilon^*(\nabla_\varepsilon u)\right) + \widetilde{\mathfrak{b}}\left(\mathcal{T}_\varepsilon, y, u \circ \mathcal{T}_\varepsilon\right), \tag{10.6}$$

with $U = \{(\mathcal{T}_\varepsilon)_{\varepsilon \in \mathbb{J}} \cup \{id_\Omega\}$. The pseudo-monotonicity of $\mathcal{B}$ is due to $\|\mathcal{T}_\varepsilon - id_\Omega\|_{L^\infty(\mathbb{R}^d \times \mathcal{Y}; \mathbb{R}^d)} \longrightarrow 0$ and can be checked by standard calculations which rely on $\mathcal{B}$ being continuous in its first argument and a splitting argument similar to handling lower-order terms in the presence of compact embeddings. For a blueprint see [Rou13, Lem. 2.32]. In fact, these calculations will be revisited in section §15 in greater generality, skipping this aspect's details for the moment. Given $\mathcal{B}$'s pseudo-monotonicity, showing (10.4) requires us to establish

$$\limsup_{\varepsilon \to 0} \langle \mathcal{B}(\mathcal{T}_\varepsilon, w_\varepsilon), w_\varepsilon - w \rangle_{X' \times X} \leq 0. \tag{10.7}$$

Having assured the solvability of Problem §9.1 in theorem §9.1 we obtain

$$\langle \mathcal{B}(\mathcal{T}_\varepsilon, w_\varepsilon), \mathcal{T}_\varepsilon^*(\varphi) - w_\varepsilon\rangle = \iint_{\Omega\times\mathcal{Y}} \mathcal{T}_\varepsilon^*(f_\varepsilon)\mathcal{T}_\varepsilon^*(\varphi - u_\varepsilon)\,dx = \langle \mathcal{T}_\varepsilon^*(\ell_\varepsilon), \mathcal{T}_\varepsilon^*(\varphi - u_\varepsilon)\rangle, \tag{10.8}$$

for $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{X'\times X}$ and all $\varphi \in W_0^{1,p}(\Omega)$. In order to derive (10.7) from (10.8), we need to impose a highly non-trivial assumption, namely the *existence of a recovery sequence* $(r_\varepsilon)_{\varepsilon\in\mathbb{J}} \subset \mathscr{V}$ with $(\mathcal{T}_\varepsilon^*(r_\varepsilon), \mathcal{T}_\varepsilon^*(\nabla_\varepsilon r_\varepsilon)) \longrightarrow w$ in X. A discussion on their existence is given below. Expanding the left-hand side of (10.7), inserting a recovery sequence and using Hölder's inequality one can derive

$$\begin{aligned}
\limsup_{\varepsilon\to0} \langle \mathcal{B}(\mathcal{T}_\varepsilon, w_\varepsilon), w_\varepsilon - w\rangle_{X'\times X} &= \limsup_{\varepsilon\to0} \left\langle \mathcal{B}(\mathcal{T}_\varepsilon, w_\varepsilon), w_\varepsilon + \mathcal{T}_\varepsilon^*(r_\varepsilon - r_\varepsilon) - w\right\rangle_{X'\times X}\\
&= \limsup_{\varepsilon\to0} \left[\left\langle \mathcal{B}(\mathcal{T}_\varepsilon, w_\varepsilon), w_\varepsilon - \mathcal{T}_\varepsilon^*(r_\varepsilon)\right\rangle_{X'\times X} + \left\langle \mathcal{B}(\mathcal{T}_\varepsilon, w_\varepsilon), \mathcal{T}_\varepsilon^*(r_\varepsilon) - w\right\rangle_{X'\times X}\right]\\
&\leq \lim_{\varepsilon\to0} \left[\left\langle \mathcal{T}_\varepsilon^*(\ell_\varepsilon), w_\varepsilon - \mathcal{T}_\varepsilon^*(r_\varepsilon)\right\rangle_{X'\times X} + \|\mathcal{B}(\mathcal{T}_\varepsilon, w_\varepsilon)\|_{X'}\, \|\mathcal{T}_\varepsilon^*(r_\varepsilon) - w\|_X\right] = 0,
\end{aligned}$$

so that (10.7) does hold. We point out that two strong two-scale convergence statements are employed here simultaneously: strong two-scale convergence of the right hand side and strong two-scale convergence of the recovery sequence.
Finally, since $\mathcal{B}$ is pseudo-monotone and (10.7) is valid, the first line of what follows can be inferred for arbitrary $v \in \Xi$ by the standard implication of pseudo-monotone operators, see (2.1) for instance:

$$\left\{\begin{aligned}
\langle \mathcal{B}(id_\Omega, w), w - v\rangle_{\Xi'\times\Xi} &\leq \liminf_{\varepsilon\to0} \langle \mathcal{B}\left(\mathcal{T}_\varepsilon, w_\varepsilon\right), w_\varepsilon - v\rangle\\
&= \liminf_{\varepsilon\to0} \langle \mathcal{B}\left(\mathcal{T}_\varepsilon, w_\varepsilon\right), w_\varepsilon - \mathcal{T}_\varepsilon^*(r_\varepsilon - r_\varepsilon) - v\rangle\\
&= \langle \ell_0, w - v\rangle.
\end{aligned}\right. \tag{10.9}$$

The fact that (10.9) is valid for all $v \in \Xi$ allows to employ the standard monotonicity trick which is also known as type (M) transition: first, assume that $\mathcal{B}(id_\Omega, w) \neq \ell_0$ in Ξ'. Since the dual pairing $\langle\cdot,\cdot\rangle_{X'\times X}$ restricted to $\Xi'\times\Xi'$ is separating, i.e. if $v' \neq w'$ in Ξ' then $\exists z \in \Xi : \langle v' - w', z\rangle \neq 0$, there exists some $v_0 \in \Xi$ such that $\langle \mathcal{B}(id_\Omega, w) - \ell_0, w - v_0\rangle < 0$. But choosing $v_1 := 2w - v_0$ yields

$$0 \overset{(10.9)}{\geq} \langle \mathcal{B}(id_\Omega, w) - \ell_0, w - v_1\rangle = -\langle \mathcal{B}(id_\Omega, w) - \ell_0, w - v_0\rangle > 0 \tag{10.10}$$

such that $\langle \mathcal{B}(id_\Omega, w) - \ell_0, w - v_0\rangle = 0$ for all $v \in \Xi$ implying $\mathcal{B}(id_\Omega, w) = \ell_0$ in Ξ' which is the desired result. ■

§10.4. A short discussion on recovery sequences

As we have seen, the availability of suitable recovery sequences are necessary for our method. More specifically, the following statement would be vital to show:

There exists a recovery sequence $(r_\varepsilon)_{\varepsilon\in\mathbb{J}}$ to every $w \in \Xi(\Omega)$, in formal terms:

$$\forall w = (w_0, w_1) \in \Xi(\Omega) : \exists (r_\varepsilon)_{\varepsilon\in\mathbb{J}} \subset \mathscr{V} : \quad \left(\mathcal{T}_\varepsilon^*(r_\varepsilon), \mathcal{T}_\varepsilon^*(\nabla_\varepsilon r_\varepsilon)\right) \overset{2s}{\longrightarrow} w \quad \text{in } X. \tag{10.11}$$

As far as the author is aware of, (10.11) has precursors for media without interfaces of standard and slow media. The standard media case was given in [MT06, Prop. 2.9 & Prop. 2.10]. An extension to highly heterogeneous media was given by H. Hanke in [Han09] for $p = 2$, using the classical two-scale calculus's device of an expansion in ε, i.e. the decomposition into good and bad portions. Since both proofs are variational in nature, the author suspects that the case $p \in (1, \infty)$ should be possible to establish as well using more refined Γ-convergence techniques for functionals of the form

$$\Phi_\varepsilon(r) := \frac{1}{p} \int_\Omega |\nabla_\varepsilon r - A_\varepsilon|^p + |r - a_\varepsilon|^p \, dx, \tag{10.12}$$

with $(a_\varepsilon) \subset L^p(\Omega)$, $(A_\varepsilon) \subset L^p(\Omega)^d$ given for $x \overset{\text{a.e.}}{\in} \Omega$ by

$$a_\varepsilon(x) := \int_{\varepsilon(Y+[x/\varepsilon])\cap\Omega} w_0\left(z, \frac{x}{\varepsilon}\right) dz, \tag{10.13a}$$

$$A_\varepsilon(x) := \int_{\varepsilon(Y+[x/\varepsilon])\cap\Omega} \chi_1\left(\frac{x}{\varepsilon}\right) w_1\left(z, \frac{x}{\varepsilon}\right) dz \tag{10.13b}$$

for $w = (w_0, w_1) \in \Xi(\Omega)$. This way, one may hope to circumvent actual Hilbert space techniques. Unfortunately, the decomposition of the given functionals' minimiser into good and bad portions poses technical difficulties too steep for the author to overcome.

§10.5. Epilogue: hints for a more abstract framework

Theorem §10.1 stems from a limiting technique that relies on two key ingredients, recovery sequences and a suitable concept of pseudo-monotonicity for families of operators. In this respect,let us point out that theorem §10.1's proof and the proof of Brezis's theorem, for instance given in [Rou13, Thm. 2.6], coincide in their handling of the passage to limit. In other words, our homogenisation problem coincides with Galerkin's method in terms of passing to the limits $\varepsilon \to 0$ and $\dim(V_k) \longrightarrow \infty$ for finite-dimensional approximation spaces V_k. Furthermore, Galerkin's method also uses a unifying operator $\mathcal{B}$ which organises underlying operators, namely the finite-dimensional restrictions, whose solvability must be ensured in the first place, too. Moreover, $\mathscr{V} = W^{1,p}(\Omega)$ possesses a Schauder base which yields the finite-dimensional approximation spaces V_k and the latter 'converge' to $\mathscr{V}$ and have a coinciding incidence product. A similar set-up is encountered for the spaces $\mathcal{T}_\varepsilon^*(\mathscr{V})$ in X, with Ξ being the space containing the corresponding two-scale limits. So, both homogenisation and Galerkin's method appear to be akin to each other, both relying on a separability condition that allows to obtain recovery sequences and not only recovery nets.

On the other hand, the finite-dimensional spaces $(V_k)_{k\in\mathbb{N}} \subset \mathscr{V}$ form a direct system ordered by inclusion and so, it is not far-fetched but rather natural and even well-known, c.f. [CLM79, Prop. 1.20], that $\mathscr{V} = \lim\operatorname{ind} V_k$ holds in $\mathbf{Ban}_1$. As we have seen in section §4.3, similar constructions hold for auxiliary domains and their corresponding L^p-spaces but although $(\mathcal{T}_\varepsilon^*(\mathscr{V}))_{\varepsilon\in\mathscr{V}}$ is a sequence of closed subspaces in X it does not seem to yield an analogue that is both simple and useful, even though Ξ may be anticipated to be a limit space in some sense.[1] In the absence

[1] $\mathcal{T}_{\varepsilon_1}^*(\mathscr{V})$ and $\mathcal{T}_{\varepsilon_1}^*(\mathscr{V})$ are not contained in each other if $\varepsilon_1 \neq \varepsilon_2$. In categorical terms, $\mathcal{T}_\varepsilon^*$ is no natural transformation from $L^p(\Omega; \mathscr{B})$ to $L^p(\Omega_\varepsilon^+; \mathscr{B})$. Again, category theory is no silver bullet but a decent organiser.

of convenient comparison maps between the spaces of $(\mathcal{T}_\varepsilon^*(\mathscr{V}))_{\varepsilon\in\mathscr{V}}$ no straightforward limit or co-limit construction is available to summarise the state of affairs concisely.

Summing up, one might anticipate that surjectivity results like Brezis's theorem and our theorem §10.1 have a common root but are not fully akin, suggesting an investigation that we carry out in the next chapter. Our main reason is twofold: first, a slightly more abstract machinery is more tolerant to the inconveniences of parabolic problems, in particular the considerably more extensive apparatus of function spaces and related convergence concepts. Secondly, the actual mechanics of the limiting procedure involved are easier to isolate this way; this is our principal motivation.

III. The parabolic regime

§11. Preparations and prerequisites

Analogous to the previous chapter II, we will turn to carry out the periodic homogenisation of a quasi-linear parabolic problems with strong heterogeneities of the form

$$(11.1) \qquad \partial_t u_\varepsilon - \nabla_\varepsilon \cdot \beta_\varepsilon(t, u_\varepsilon) = f_\varepsilon \quad \text{in} \quad \Omega_T \qquad \text{with} \qquad u_\varepsilon = u_{0,\varepsilon} \quad \text{in} \quad \{0\} \times \Omega$$

which is assumed to have homogeneous boundary values for simplicity's sake. A key aspect will be whether and how one can make us of results and related techniques developed for the stationary case. As we have seen, all machinery eventually contributes to organising arguments that vindicate passages to the limit $\varepsilon \longrightarrow 0$. In order to keep this presentation free from unnecessary technical burdens, we will restrict ourselves to monotone problems, again, although involving non-monotone lower order terms on Ω^1_ε is conceivable along standard arguments of compact embeddings, we refer to T. Roubíček's [Rou13] once more.

So, our main task is to develop the additional machinery necessary to handle parabolic evolution problems and their periodic homogenisation. We start by gathering some standard tools for parabolic initial–boundary value problems together with the corresponding notation. Our presentation is guided by [DiB93; Lad68b; Rou13].

§11.1. Fundamentals for parabolic problems

Domains

We assume $\Omega \subset \mathbb{R}^d$ to be a non-void domain of $C^{0,1}$-regularity equipped with an admissible periodic decomposition. such that a parabolic analogue of theorem §7.2 to be available.

Time interval

We fix some $T > 0$ and obtain the fundamental objects $I := (0, T)$ and $\Omega_T = I \times \Omega$. More generally, we write $S_t := (0, t) \times S$ for any given set S. In addition, $(I, \mathcal{L}(I), \lambda)$ denotes the measure space resulting from restricting $(\mathbb{R}^1, \mathcal{L}(\mathbb{R}), \lambda^1)$ to I.

Lebesgue–Bochner spaces

Here, we gather the necessary elementary results on integration of Banach space valued functions. Thorough presentations on this matter can be found in [DJ77; Din74; Edw65; Lan69]. Let us fix some notation. For a given Banach space $\mathscr{B}$ we write

$$(11.2) \qquad L^0(0, T; \mathscr{B}) := L^0(I;) \leftharpoondown := \{u : (0, T) \longrightarrow \mathscr{B} : u \text{ is Bochner measurable}\}$$

for the (vector) space of Bochner measurable functions. Here, Bochner measurability is defined via sequences of simple functions $(s_n : (0,T) \longrightarrow \mathscr{B})_{n\in\mathbb{N}}$, i.e.

$$\begin{cases} \forall n \in \mathbb{N} : \exists \ell_n \in \mathbb{N}, L_1, \ldots, L_{\ell_n} \in \mathcal{L}(I), b_1, , \ldots, b_{\ell_n} \in \mathscr{B} : \\ s_n(t) = \sum_{k=1}^{\ell_n} b_k \chi_{L_k}(t) \quad \wedge \quad (1 \le i \ne j \le \ell_n \Longrightarrow L_i \cap L_j = \emptyset) \end{cases} \tag{11.3}$$

which converge pointwise to u almost everywhere in I. Furthermore, we will employ Lebesgue–Bochner spaces given for for $p \in [1,\infty)$ by

$$L^p(0,T;\mathscr{B}) := \left\{ u \in L^0(0,T;\mathscr{B}) : \|u\|^p_{L^p(0,T;\mathscr{B})} := \int_0^T \|u(\tau)\|^p_{\mathscr{B}}\, d\tau < \infty \right\} \tag{11.4}$$

and for $p = \infty$ with obvious modifications. $L^p(0,T;\mathscr{B})$ is a Banach space for $p \in [1,\infty]$ if $\mathscr{B}$ is one. In other words, we have a functor

$$\begin{cases} L^p(0,T;\,\cdot\,) : \mathbf{Ban}_\infty & \longrightarrow \quad \mathbf{Ban}_\infty \\ \mathscr{B} & \longmapsto \quad L^p(0,T;\mathscr{B}) \\ \mathrm{Hom}_{\mathbf{Ban}_\infty}(\mathscr{B}_1,\mathscr{B}_2) \ni A & \longmapsto \quad A_* \in \mathrm{Hom}_{\mathbf{Ban}_\infty}(L^p(0,T;\mathscr{B}_1), L^p(0,T;\mathscr{B}_2)) \end{cases} \tag{11.5}$$

with A_* being the *pushforward* of A given by $A_*(u) := A \circ u$. In addition, this functor respects $\mathbf{Ban}_1$, too, preserving non-expansive morhpisms.

For $p \in (1,\infty)$ set $p' := p/(p-1)$ and for $p = 1$ use $1' := \infty$. There is a canonical $\mathbf{Ban}_1$-isomorphism

$$\left(L^p(0,T;\mathscr{B})\right)' \quad \cong_{\mathbf{Ban}_1} L^{p'}(0,T;\mathscr{B}'), \tag{11.6}$$

provided $\mathscr{B}$ has the Radon–Nikodym property, which is fulfilled if $\mathscr{B}$ is reflexive or has a separable pre-dual space. [DJ77] is very extensive on the Radon–Nikodym property and related questions. Besides, $\mathbf{Ban}_1$-isomorphisms mean that there is a isometric, linear homeomorphism between the corresponding normed spaces. As James' space shows [Jam51], a Banach space can be $\mathbf{Ban}_1$-isomorphic to its bidual without being reflexive. Thus, it must be emphasised that this $\mathbf{Ban}_1$-isomorphism can be chosen to be canonical which is vital to get the representation of the dual product in terms of

$$\langle v, u\rangle_{(L^p(0,T;\mathscr{B}))' \times L^p(0,T;\mathscr{B})} = \int_0^t \langle v(\tau), u(\tau)\rangle_{\mathscr{B}'\times\mathscr{B}}\, d\tau \tag{11.7}$$

for $u \in L^p(0,T;\mathscr{B}), v \in (L^p(0,T;\mathscr{B}))'$ is valid. As a most important consequence, $L^p(0,T;\,\cdot\,)$ preserves reflexive spaces and separable spaces for $p \in (1,\infty)$ and Hilbert spaces for $p = 2$.

Finally, the importance of separability of $\mathscr{B}$ stems from $L^0(I;\mathscr{B})$ being 'effectively void' otherwise: as [Mei08, Sec. 2] points out, $(x,y) \longmapsto \chi_{\{x<y\}}$ is not Bochner-measurable as a function $(0,1) \longrightarrow L^\infty(0,1)$. A short explanation is that a countable family of simple functions is insufficient for approximating a given function u almost everywhere unless u is *essentially separably valued*: the image of $u_{|\widetilde{I}}$ is separable in $\mathscr{B}$ and $\lambda(I\setminus\widetilde{I}) = 0$. In practical terms, it is hardly verifiable

whether or not a given function is essentially separably valued. As a remedy, one restricts to separable Banach spaces $\mathscr{B}$ such that all functions $u : I \longrightarrow \mathscr{B}$ are essentially separably valued.

Distributional derivatives of vector–valued functions

Every $u \in L^p(I;\mathscr{B})$ can be identified canonically with a vector–valued distribution thanks to $\phi \longmapsto u(\phi) := \int_I u(t)\phi(t)\,dt \in \mathscr{B}$ for $\phi \in \mathscr{D}(I)$. Consequently, speaking of distributional derivatives $\partial_t u$ makes sense thanks to $(\partial_t u)(\phi) := u(\phi')$.[1]

Sobolev–Bochner spaces

We speak of weak differentiability if the distributional derivative $\partial_t u$ of a function $u \in L^p(I;\mathscr{B})$ with $p, q \in [1,\infty]$ is sufficiently regular such that it can be represented by a function $v \in L^q(0,T;\mathscr{B})$, calling v the weak derivative and writing $v = \partial_\tau u$, again. As customary, $W^{1,p}(I;\mathscr{B})$ denotes the Sobolev–Bochner space which is the subspace of $L^p(I;\mathscr{B})$ containing functions whose weak derivatives are again in $L^p(I;\mathscr{B})$; equipped with the norm

$$\|u\|_{W^{1,p}(I;\mathscr{B})} = \|u\|_{L^p(I;\mathscr{B})} + \|\partial_t u\|_{L^p(I;\mathscr{B})} \tag{11.8}$$

it is a Banach space. Solutions of parabolic standard problems are usually not elements of $W^{1,p}(\Omega_T)$. Rather, they inhabit spaces of varying regularity of the following kind: for $p, q \in [1,\infty]$ and given Banach spaces $\mathscr{B}_1, \mathscr{B}_2$ with $\mathscr{B}_1 \hookrightarrow \mathscr{B}_2$ we set

$$W^{1,p,q}(I;\mathscr{B}_1,\mathscr{B}_2) := L^p(I;\mathscr{B}_1) \cap W^{1,q}(I;\mathscr{B}_2) \tag{11.9a}$$

$$V_0^{q,p}(\Omega_T) := L^p\left(I; W_0^{1,p}(\Omega)\right) \cap L^\infty\left(I; L^q(\Omega)\right). \tag{11.9b}$$

Again, properties of $\mathscr{B}_1$ and $\mathscr{B}_2$ like reflexivity or separability carry over for $p, q \in (1,\infty)$. The most relevant case for us will be $q = p'$ with $\mathscr{B}_2 = \mathscr{B}_1'$ and $\mathscr{B}_1 = W_0^{1,p}(\Omega)$.

A Gelfand triplet is given by a Banach space V and a Hilbert space H with a dense embedding $\iota : V \hookrightarrow H$. Due to density, the adjoint $\iota' : H' \hookrightarrow V'$ is well-defined and many authors write (V, H, V') for a Gelfand triplet, referring to the sequence $V \hookrightarrow H \cong H' \hookrightarrow V'$.

A Gelfand triplet (V, H, ι) allows to establish continuity of the linear embedding

$$W^{1,p,p'}(I;V,V') \hookrightarrow C^0(\bar{I};H) \qquad u \longmapsto u. \tag{11.10}$$

Suppressing ι in most instances, this yields the integration by parts formula

$$\int_{t_1}^{t_2} \langle u', v\rangle_{V'\times V}\,d\tau = \langle u(t_2), v(t_2)\rangle_{H'\times H} - \langle u(t_1), v(t_1)\rangle_{H'\times H} - \int_{t_1}^{t_2} \langle v', u\rangle_{V'\times V}\,d\tau \tag{11.11}$$

for $u, v \in W^{1,p,p'}(I;V,V')$ and $t_1, t_2 \in \bar{I}$ which is of vital importance in order to incorporate initial values given in H.

Yet another word on compactness

So far, we have swept compactness of embeddings of the spaces involved under the rug. In most instances, one encounters Gelfand triplets (V, H, ι) whose embedding map ι is not merely

[1] The topologies on $\mathscr{D}'(I;\mathscr{B})$ are defined analogously to the topologies of periodic distributions of definition §4.2.

continuous but even compact. As a prototype, consider $V = W^{1,p}(\Omega) \cap L^2(\Omega) \overset{\iota}{\hookrightarrow} H = L^2(\Omega)$ for $p \in (1,\infty)$ large enough such that the Rellich–Kondrachov theorem renders ι compact, i.e. $2 < p^*$ with p^* being the Sobolev exponent of p.

As we have mentioned already and seen in the elliptic case, the Rellich–Kondrachov embedding is not applicable on Ω_ε^2 since uniform bounds on the gradients are not available there. It will hardly be a surprise that time-dependency does not heal this defect. More explicitly, in the absence of stationary compact embeddings one cannot hope to obtain time-dependent compact embeddings. In fact, progressing from the former to the latter is a major step in the theory with the famous *Aubin–Lions–Simon lemma* being the most prominent prototype. However, any such result – at least as far as the author is aware of – requires compactly embedded spaces in the elliptic regime; a prerequisite that we cannot fulfil from the very outset! [Ama00] is a rather new and very extensive study on this issue.

For what it is worth, the absence of compactness foils any straightforward attempt to carry out a periodic homogenisation procedure of a Stefan problem in highly heterogeneous media which includes phase changes in the slow domain. The standard media case has been covered by A. Visintin in [Vis07a], making crucial use of the Aubin–Lions–Simon lemma but a transfer without compactness methods seems out of the author's scope, though it was the very initial motivation of our work.

To cut a long story short, compactness theorems are unavailable and this defect must be overcome by methods that are sufficiently tolerant. Again, monotonicity is fit for this endeavour making it our method of choice. Nevertheless, compact embeddings are available on $\Omega_{1,\varepsilon}$ so that augmentations of the methods presented here are conceivable. In particular, such embeddings open the possibility of incorporating phase changes in $\Omega_{1,\varepsilon}$ following Visintin's work. Since our work focuses on presenting a generalised framework that isolates the actual mechanisms of passage to the limit $\varepsilon \longrightarrow 0$, we refrain from elaborating on the aforementioned task, leaving them to subsequent research.

§11.2. Periodic unfolding in a time-dependent framework

An elementary question is whether and how $\mathcal{T}_\varepsilon^* : L^p(\Omega) \longrightarrow L^p(\Omega_\varepsilon^+ \times \mathcal{Y})$ carries over to parabolic spaces like $L^p(\Omega_T)$ and $L^p(I; W^{1,q}(\Omega))$. First, let $\mathscr{B}$ be a function space on which periodic unfolding has been defined, i.e $\mathscr{B} = L^p(\Omega)$ or $\mathscr{B} = W_0^{1,p}(\Omega)$. For $u \in L^p(I;\mathscr{B})$ we extend periodic unfolding in a canonical fashion via

$$\left\{ \begin{array}{rcl} \mathcal{T}_\varepsilon^* : L^q(I;\mathscr{B}) & \longrightarrow & L^q(I;\mathcal{T}_\varepsilon^*(\mathscr{B})) \\ (t \mapsto u(t)) & \longmapsto & \big(t \mapsto \mathcal{T}_\varepsilon^*(u(t))\big) \end{array} \right. \tag{11.12}$$

to obtain induced parabolic periodic unfolding for $q \in [1,\infty]$ and $\mathscr{B} = L^p(\Omega)$ or $\mathscr{B} = W_0^{1,p}(\Omega)$ with $p \in [1,\infty)$. This construction is very straightforward but exploits two aspects crucially;

a) $\mathcal{T}_\varepsilon^*$ is an isometry that respects both Borel sets and Lebesgue zero sets. Therefore, both integrability and measurability properties of L^p-spaces remain unaltered.

b) Our version of periodic unfolding exclusively deals with spatial oscillations such that no extension for time-dependent oscillations of periodic unfolding to all of Ω_T is necessary.

The latter aspect does contain quite an assumption which rules out considerable applications with time-oscillating coefficients. On the other hand, it allows lifting periodic unfolding from the elliptic realm to the parabolic regime with minor effort.

Last but not least, recall the key observation of remark §7.2 realising that time-dependent analogues of theorem §7.1 and theorem §7.2 hold without any restrictions provided temporal L^1-regularity is available. We will exploit this fact tacitly from now on, referring to theorem §7.1 and theorem §7.2 in a somewhat sloppy way.

§12. Families of initial–boundary value problems with oscillating coefficients

We will formulate and treat a family of initial–boundary value problems indexed by fixed $\varepsilon \in \mathbb{J}$ first. First, we will only state the problem in an ad hoc fashion, specifying suitable conditions afterwards. To keep the notation as slim as possible, we fix the following notation.

a) The functions' arguments (t, x) and (t, x, y) are mostly neglected.

b) Recall $\mathcal{T}_\varepsilon^*(ab) = \mathcal{T}_\varepsilon^*(a)\mathcal{T}_\varepsilon^*(b)$ for all functions a, b whose product is in L^1, and this multiplicativity extends to scalar products, too.

c) $\beta_\varepsilon(u_\varepsilon, \nabla_\varepsilon u_\varepsilon) := \beta(t, x, x/\varepsilon, u_\varepsilon, \nabla_\varepsilon u_\varepsilon)$ for almost all $(t, x) \in \Omega_T$. Later, we will impose growth conditions on β ensuring sufficient regularity for its periodically unfolded counterpart to make sense, which reads reads: for almost all $(t, x, y) \in \Omega_T \times \mathcal{Y}$:

$$\mathcal{T}_\varepsilon^*(\beta_\varepsilon(u_\varepsilon, \nabla_\varepsilon u_\varepsilon)) = \beta\left(t, \mathcal{T}_\varepsilon(x, y), y, \mathcal{T}_\varepsilon^*(u_\varepsilon), \mathcal{T}_\varepsilon^*(\nabla_\varepsilon u_\varepsilon)\right). \tag{12.1}$$

d) We set $\mathcal{V} := W_0^{1,p}(\Omega) \cap L^2(\Omega)$ and $\mathcal{H} := L^2(\Omega)$ such that $\mathcal{V} \hookrightarrow \mathcal{H}$ forms a Gelfand triplet.

§12.1 Problem (The ε-dependent parabolic IVP)**.** For a fixed $\varepsilon > 0$ let $f_\varepsilon \in L^{p'}(\Omega_T)$ and $u_{0,\varepsilon} \in \mathcal{H}$ be given. The ε-dependent (heterogeneous) parabolic initial–boundary value problem reads: find $u_\varepsilon \in W^{1,p,p'}(I; \mathcal{V}, \mathcal{V}')$ such that for all $\varphi \in W^{1,p,p'}(I; \mathcal{V}, \mathcal{V}')$ the following variational equality holds:

$$(\varepsilon\text{PP}) \quad \left\{ \begin{aligned} &\int_\Omega u_\varepsilon(T)\varphi(T)\,dx + \iint_{\Omega_T} -u_\varepsilon\varphi'\,dxdt + \iint_{\Omega_T} \beta_\varepsilon(u_\varepsilon, \nabla_\varepsilon u_\varepsilon) \cdot \nabla_\varepsilon\varphi\,dxdt \\ &\qquad = \iint_{\Omega_T} f_\varepsilon\varphi\,dxdt + \int_\Omega u_{0,\varepsilon}\varphi(0)\,dx. \end{aligned} \right.$$

Naturally, one can rewrite (εPP) via periodic unfolding. Yet, quite an inconvenience comes into play namely $L^{p'}(I; \mathcal{V}') \nsubseteq L^1_{\text{loc}}(\Omega_T)$ such that φ' from (εPP) is not regular enough to be unfolded.

As a remedy, one can write

$$(\varepsilon\text{PP'})\qquad \begin{cases} \displaystyle\iint\limits_{\mathbb{R}^d\times\mathcal{Y}} \mathcal{T}_\varepsilon^*(u_\varepsilon(T)\varphi(T))\,dxdy + \iiint\limits_{\mathbb{R}^d_T\times\mathcal{Y}} -\mathcal{T}_\varepsilon^*\,(u_\varepsilon\varphi')\;dydxdt \\ \displaystyle\quad + \iiint\limits_{\mathbb{R}^d_T\times\mathcal{Y}} \mathcal{T}_\varepsilon^*\,(\beta_\varepsilon(u_\varepsilon)\cdot\nabla_\varepsilon\varphi)\;dydxdt = \iiint\limits_{\mathbb{R}^d_T\times\mathcal{Y}} \mathcal{T}_\varepsilon^*\,(f_\varepsilon\varphi)\;dxdt \\ \displaystyle\qquad\qquad + \iint\limits_{\mathbb{R}^d\times\mathcal{Y}} \mathcal{T}_\varepsilon^*\left(u_{0,\varepsilon}\varphi(0)\right)\;dx \end{cases}$$

for all $\varphi \in W^{1,p,p'}(I;\mathscr{V},\mathscr{H}')$. Since the latter function space is not really the natural habitat for an existence theory of problem §12.1 we prefer (εPP) for this endeavour and save (εPP') for the limiting procedure.

§12.1. Requirements on the heat flux

Our treatment of the elliptic ε-dependent problems (εP) provides the blueprint for handling (εPP). Explicitly, we aim at imposing conditions on on the heat flux function β such that a monotone and coercive operator is at hand. Later, the 'right hand side data' made up by the initial values $u_{0,\varepsilon}$ and the source functions f_ε will be required to fulfil suitable restricitons, too. Let us start by assuming that there is a heat flux function

$$\beta : I\times\Omega\times\mathcal{Y}\times\mathbb{R}\times\mathbb{R}^d \longrightarrow \mathbb{R}^d \qquad (t,x,y,r,\vec{s}) \longmapsto \beta(t,x,y,r,\vec{s}).$$

which fulfils the following conditions.

H.1) β is a Caratheodory function with respect to its $(r,\vec{s})$-arguments.

$$(12.2a)\qquad \begin{cases} \forall (r,\vec{s}) \in \mathbb{R}\times\mathbb{R}^d : \quad (t,x,y) \longmapsto \beta(t,x,y,r,\vec{s}) \quad \text{is measurable and} \\ \forall (t,y) \overset{\text{a.e.}}{\in} I\times\mathcal{Y} : \quad (x,r,\vec{s}) \longmapsto \beta(t,x,y,r,\vec{s}) \quad \text{is continuous.} \end{cases}$$

H.2) β suffices $L^p - L^{p'}$-growth conditions.

$$(12.2b)\qquad \begin{cases} \exists C_{\beta,1} > 0 : \forall (t,x,y,r,\vec{s}) \in I\times\Omega\times\mathcal{Y}\times\mathbb{R}\times\mathbb{R}^d : \\ \quad |\beta(t,x,y,r,\vec{s})|^{p'} \le C_{\beta,1}\,(1 + |r|^p + |\vec{s}|^p) \end{cases}$$

H.3) β is L^p-coercive in s.

$$(12.2c)\qquad \begin{cases} \exists C_{\beta,2} > 0, C_{\beta,3} \in \mathbb{R} : \\ \forall (t,x,y,r,\vec{s}) \in I\times\Omega\times\mathcal{Y}\times\mathbb{R}\times\mathbb{R}^d : \beta(t,x,y,r,\vec{s})\cdot\vec{s} \ge C_{\beta,2}|\vec{s}|^p + C_{\beta,3} \end{cases},$$

H.4) β is monotone in its $\vec{s}$-argument.

$$(12.2d)\qquad \begin{cases} \forall (t,x,y) \overset{\text{a.e.}}{\in} I\times\Omega\times\mathcal{Y}, (r_1,\vec{s}_1), (r_2,\vec{s}_2) \in \mathbb{R}\times\mathbb{R}^d : \\ (\beta(t,x,y,r_1,\vec{s}_1) - \beta(t,x,y,r_1,\vec{s}_1))\cdot(\vec{s}_1 - \vec{s}_2) \ge 0 \end{cases}$$

The modified Caratheodory condition (12.2a) is necessary for measurability of the resulting functions and admissibility in the two-scale sense. The remaining conditions (12.2b), (12.2c) and (12.2d) are more or less standard conditions from the theory of monotone operators, see [Rou13, Sec. 8.6] for a thorough discussion. In contrast to the elliptic case, we suppress monotone lower order terms. These can be included of course but since we want to focus on the limiting procedure itself, we decided to neglect them. Anyhow, monotone lower order terms can be included without any substantial modifications.

Finally, let us point out that (12.2d) effectively rules out r-dependence of β. We do keep this argument refraining from disposing of it prematurely since monotonicity only serves as a limiting technique. One may wonder whether future research will yield additional means to enlarge this family of operators, for instance, by incorporating Reichelt's method from [Rei15].

Remark §12.1: **On heat fluxes and energy conservation**

The idea of heat fluxes stems from the law of *energy conservation*, an invaluable concept of physics which is well-known to be an emanation of symmetry laws by E. Noether's theorem, c.f. [Kos10]. Observe that *Fourier's law* tacitly inspires our heat flux function since we (aim to) arrive at parabolic problems. We follow this approach recognising its shortcomings such as not being in full accord with the finite propagation speed of light. We point out that generalised Fourier laws like *Cattaneo's law* are available but the resulting problems are no longer parabolic but hyperbolic in nature![a] It is therefore not too surprising that Fourier's law is much better received in mathematics than its alternatives. The latter are predominantly debated about in physicists' literature, for instance in [LJC08, Sec. 7.1].

[a]Our limiting procedure relies on monotonicity methods whose applications to hyperbolic problems is rather scarce, see [Kan97] for an exception.

§12.2. Conditions on initial data and heat sources

Here, we impose strong two-scale convergence conditions on the sequences $(u_{0,\varepsilon})_{\varepsilon\in\mathbb{J}} \subset L^2(\Omega \times \mathcal{Y})$ and $(f_\varepsilon)_{\varepsilon\in\mathbb{J}} \subset L^{p'}(\Omega_T \times \mathcal{Y})$, namely

$$\exists u_{0,0} \in L^2(\Omega \times \mathcal{Y}) : \lim_{\varepsilon\to 0} \|\mathcal{T}_\varepsilon^*(u_{0,\varepsilon}) - u_{0,0}\|_{L^2(\mathbb{R}^d\times\mathcal{Y})} = 0, \tag{12.3}$$

$$\exists f_0 \in L^{p'}(\Omega_T \times \mathcal{Y}) : \lim_{\varepsilon\to 0} \|\mathcal{T}_\varepsilon^*(f_\varepsilon) - f_0\|_{L^{p'}(I\times\mathbb{R}^d\times\mathcal{Y})} = 0. \tag{12.4}$$

Again, the soundness of the model at hand might suggest the desirability of the limits $u_{0,0}$ and f_0 being independent of the specific sequence $\varepsilon \in \mathbb{J}$ in use.

§12.3. Solving (εPP) as an equation of monotone operators

First, we need to ensure the solvability of (εPP) which we do by casting it into an abstract operator inclusion that is treatable by Browder's theorem of pseudo-monotone perturbations of maximal monotone mappings: see appendix §A.2 for further reference and [Zei90, Thm 32.A] or [Le11] for full expositions. Of course, Browder's theorem is much stronger than we need but its abstract simplicity make it our method of choice.

Lemma §12.1: Equivalent operator equation

For every $\varepsilon > 0$, *Problem* §12.1 can be cast into the form of an abstract operator equation given as follows. Set $X := L^p(I;\mathscr{V})$, and let $b_\varepsilon \in X'$ be given. Find $u_\varepsilon \in X$ such that the inclusion

$$A_\varepsilon(u_\varepsilon) + B_\varepsilon(u_\varepsilon) = b_\varepsilon \tag{12.5}$$

is fulfilled for two operators $A_\varepsilon, B_\varepsilon$ which we specify as

$$A_\varepsilon : X \longrightarrow X' \quad \text{is maximally monotone and } \operatorname{dom}(A_\varepsilon) \subset X \tag{12.6}$$

$$B_\varepsilon : X \longrightarrow X' \quad \text{is pseudo-monotone, bounded and demi-continuous} \tag{12.7}$$

together with the following coercivity condition

$$\begin{cases} \exists v_0 \in X, R \geq \|v_0\|_X : \quad \forall u \in \operatorname{dom}(A_\varepsilon) \cap B_R(0) \neq \emptyset : \\ \qquad \langle A_\varepsilon(u) + B_\varepsilon(u) - f_\varepsilon, u - v_0 \rangle_{X' \times X} > 0. \end{cases} \tag{12.8}$$

Proof. Set $\operatorname{dom}(A_\varepsilon) := \{u \in W^{1,p,p'}(I;\mathscr{V},\mathscr{V}') \subset X : \|u(0)-u_{0,\varepsilon}\|_{\mathscr{H}} = 0\}$ and define A_ε via the weak derivative $A_\varepsilon(u) := \partial_t u$ such that A_ε maps $\operatorname{dom}(A_\varepsilon)$ into X'. To check monotonicity, let $u, v \in \operatorname{dom}(A_\varepsilon)$ and exploit integration by parts and the equality of the initial values to obtain

$$\langle A_\varepsilon(u) - A_\varepsilon(v), u - v \rangle_{X' \times X} = \int_0^T \langle \partial_t(u-v), u-v \rangle_{\mathscr{V}' \times \mathscr{V}} \, dt = \frac{1}{2}\|u(T) - v(T)\|^2_{\mathscr{H}} \geq 0. \tag{12.9}$$

To establish maximality, use the linearity of A_ε and the monotonicity of its adjoint A'_ε to establish that A_ε is closed which infers maximality. For a similar proof see [Rou13, p.289]. B_ε can be defined on all of X via $\langle B_\varepsilon(u), v\rangle_{X' \times X} := \iint_{\Omega_T} \beta(t,x,x/\varepsilon,u,\nabla_\varepsilon u) \cdot \nabla_\varepsilon v \, dxdt$. The remaining claims on B_ε and coercivity are standard due to the conditions imposed on β, we refer to [Rou13] for a full display. Finally, due to the density of $\mathscr{V} \hookrightarrow \mathscr{H}$ the right hand side $\langle b_\varepsilon, v\rangle_{Y' \times Y} := \iint_{\Omega_T} f_\varepsilon v \, dxdt$ establishes the equivalence of (12.5) to (εPP). ■

Theorem §12.1: Solvability via Browder's theorem

Problem §12.1 is solvable for every $\varepsilon > 0$. Moreover, the solution is unique and the following a priori estimates can be established:

$$\exists C > 0 : \forall \varepsilon > 0 : \qquad \|u_\varepsilon\|_{L^\infty(I;L^p(\Omega))} + \|\nabla_\varepsilon u_\varepsilon\|_{L^p(\Omega_T;\mathbb{R}^d)} + \left\|u'_\varepsilon\right\|_{L^{p'}(I;\mathscr{V}')} \leq C. \tag{12.10}$$

Proof. (12.5) is solvable by Browder's theorem §A.2.2 and thus problem §12.1 is solvable, too. Uniqueness can be inferred from (εPP) by considering two solutions u_1, u_2 with

$$\partial_t (u_1 - u_2) - \nabla_\varepsilon \cdot [\beta(u_1) - \beta(u_2)] = 0 \qquad \text{in } L^{p'}(I;\mathscr{V}) \tag{12.11}$$

which can be tested with $\varphi = u_1 - u_2 \in L^p(I;\mathscr{V})$ such that integration by parts and monotonicity yield

$$\forall t \overset{\text{a.e.}}{\in} I: \qquad \|u_1(t) - u_2(t)\|^2_{L^2(\Omega)} \leq 0 \tag{12.12}$$

which infers equality in $W^{1,p,p'}(I;\mathscr{V},\mathscr{V}')$ due to the Du Bois-Reymond lemma. Finally, deriving (12.10) is standard as well: choose $\varphi = u_\varepsilon$ in (εPP) with $T = t \overset{\text{a.e.}}{\in} I$ to get

$$\begin{cases} C_{\beta,3} + \frac{1}{2}\|u_\varepsilon(t)\|^2_{L^2(\Omega)} + C_{\beta,2}\|\nabla_\varepsilon u_\varepsilon\|^p_{L^p(\Omega_t)} \\ \quad \leq \dfrac{1}{2}\|u_\varepsilon(t)\|^2_{L^2(\Omega)} + \displaystyle\iint_{\Omega_t} \beta(u_\varepsilon,\nabla_\varepsilon u_\varepsilon)\cdot\nabla_\varepsilon u_\varepsilon\,dxdt \\ \quad = \dfrac{1}{2}\|u_{0,\varepsilon}\|^2_{L^2(\Omega)} + \displaystyle\iint_{\Omega_t} f_\varepsilon u_\varepsilon\,dxdt \leq \dfrac{1}{2}\|u_{0,\varepsilon}\|^2_{L^2(\Omega)} + \|f_\varepsilon\|_{L^{p'}(\Omega_T)}\|u_\varepsilon\|_{L^p(\Omega_T)} \end{cases} \tag{12.13}$$

by using the coercivity estimate (12.2c) for the first inequality and Hölder's inequality for the latter. Arguing with contradiction in (12.13) yields the claimed a priori results, implicitly using a Poincaré-inequality which is valid for fixed $\varepsilon > 0$. ■

§13. Establishing the parabolic limiting problem – a plan of action

We turn to the periodic homogenisation of (εPP) by retrieving weak sequential compactness results. Note that u_ε' is not sufficiently regular to be unfolded, foiling its ability to have a reasonable two-scale limit. This defect is no major obstacle but will require some additional discussion to be compensated for.

§13.1. Preparations: weak compactness results

Analogous to corollary §9.1 in chapter II, (12.10) allows to invoke theorem §7.2's time analogue.

Corollary §13.1: Weak sequential two-scale compactness

There exists a subsequence of $\varepsilon \in \mathbb{J}$ again denoted by $\varepsilon \in \mathbb{J}$ such that for $\varepsilon \to 0$ the following weak convergence statements holds. First, there are $(w_0, \widetilde{\nabla} w_0) \in L^p(I;\Xi(\Omega))$ such that

$$\begin{cases} \mathcal{T}_\varepsilon^*(u_\varepsilon) \overset{2w}{\rightharpoonup} w_0 & \text{in } L^p(I\times\mathbb{R}^d\times\mathcal{Y}) \\ \mathcal{T}_\varepsilon^*(\nabla_\varepsilon u_\varepsilon) \overset{2w}{\rightharpoonup} \widetilde{\nabla} w_0 & \text{in } L^p(I\times\mathbb{R}^d\times\mathcal{Y})^d \end{cases} \tag{13.1}$$

with $w_0(x,y) = u_0(x)+v_0(x,y)$ and $\widetilde{w}_0(x,y) = \chi_1(y)\left[\nabla_x u_0(x) + \nabla_y u_1(x,y)\right]+\nabla_y v_0(x,y)$ for some $u_0 \in L^p(I;\mathscr{V})$, $u_1 \in L^p(\Omega_T;W^{1,p}(\mathcal{Y}_1)/\mathbb{R})$ and $v_0 \in L^p(\Omega_T;W^{1,p}_0(\mathcal{Y}_2))$. In addition, there exists a function $z_0 \in L^p(I;\mathscr{V}')$ such that

$$u_\varepsilon' \rightharpoonup z_0 \quad \text{in } L^{p'}(I;\mathscr{V}') \tag{13.2}$$

is valid. Moreover, there exists a function $\xi_0^{\circledast} \in L^{p'}(\Omega_T \times \mathcal{Y})^d$ such that

$$\mathcal{T}_\varepsilon^*\left(\beta(t,x,x/\varepsilon,u_\varepsilon,\nabla_\varepsilon u_\varepsilon)\right) \xrightarrow{2w} \xi_0^{\circledast} \quad \text{in } L^{p'}(I \times \mathbb{R}^d \times \mathcal{Y})^d. \tag{13.3}$$

Warning: besides suppressing the arguments of u_ε, (13.3) tacitly uses the isometry described in (10.3) which allows identifying $\mathcal{T}_\varepsilon^*(\beta_\varepsilon)$ – which is tested by functions from $L^p\left(I;\mathcal{T}_\varepsilon^*(\mathcal{V})\right)$ – with an element in $L^{p'}(I \times \mathbb{R}^d \times \mathcal{Y})^d$. There, weak compactness results are easily available. Finally, one must return to the actual set-up of testing with the right functions. Observe that this identification processes depends on the $\mathcal{T}_\varepsilon^*$ being an isometry such that its image is weakly closed.

Proof. (13.1) is entirely due to the time-dependent modification of theorem §7.2 mentioned in remark §7.2. (13.2) is a consequence of $\mathcal{V}$ being separable and reflexive such that $L^{p'}(I;\mathcal{V}')$ inherits these properties for $p \in (1,\infty)$. Combining the Banach–Alaoglu and Eberlein–Šmulian theorems yields the claim. Finally, (13.3) is due to weakly (two-scale) converging sequences being bounded and β being a bounded operator such that the foregoing arguments can be repeated. ■

For future reference we will need the following lemma.

Lemma §13.1: Parabolic energy estimate

Let $u_\varepsilon \in W^{1,p,p'}(I,\mathcal{V},\mathcal{V}')$ be a a weak solution of (εPP), then we have

$$\frac{1}{2}\|u_\varepsilon(T)\|_{\mathcal{H}}^2 - \frac{1}{2}\|u_{0,\varepsilon}\|_{\mathcal{H}}^2 + \iint\limits_{\Omega_T} \beta_\varepsilon(u_\varepsilon,\nabla_\varepsilon u_\varepsilon)\cdot\nabla_\varepsilon u_\varepsilon \, dtdx = \iint\limits_{\Omega_T} f_\varepsilon u_\varepsilon \, dtdx, \tag{13.4}$$

which can be periodically unfolded and rewritten as

$$\left\{\begin{aligned} &\iiint\limits_{I\times\mathbb{R}^d\times\mathcal{Y}} \beta_\varepsilon\left(\mathcal{T}_\varepsilon(x,y),y,\mathcal{T}_\varepsilon^*(u_\varepsilon),\mathcal{T}_\varepsilon^*(\nabla_\varepsilon u_\varepsilon)\right)\cdot\mathcal{T}_\varepsilon^*(\nabla_\varepsilon u_\varepsilon)\, dydxdt \\ &\qquad = \iiint\limits_{I\times\mathbb{R}^d\times\mathcal{Y}} \mathcal{T}_\varepsilon^*(f_\varepsilon u_\varepsilon)\, dydxdt \\ &\qquad\qquad + \frac{1}{2}\left\|\mathcal{T}_\varepsilon^*(u_{0,\varepsilon})\right\|_{L^2(\mathbb{R}^d\times\mathcal{Y})}^2 - \frac{1}{2}\|\mathcal{T}_\varepsilon^*(u_\varepsilon(T))\|_{L^2(\mathbb{R}^d\times\mathcal{Y})}^2. \end{aligned}\right. \tag{13.5}$$

Proof. Take $\varphi = u_\varepsilon$ in (εPP) and integrate by parts. Since sufficient spatial regularity is available, periodic unfolding is admissible and a straightforward calculation yields (13.5). ■

§13.2. Stock-taking

So far, we have obtained the following:

a) Solvability of (εPP) for all $\varepsilon \in \mathbb{J}$ invoking acceptable restrictions on β.

b) Existence of a sequence $\varepsilon \in \mathbb{J}$ together with suitable weak (two-scale) limits under suitable conditions on the initial values and the source functions.

Of course, there are also gaps in our treatment.

a) We are unable to identify a weak two-scale limit of $(u_\varepsilon')_{\varepsilon\in\mathbb{J}}$ due to lacking spatial regularity.

b) At present we are unable to characterise $\xi_0^{\circledast}$. Similarly to the elliptic case one would expect

$$(\heartsuit) \qquad \xi_0^{\circledast}(t,x,y) = \beta\left(t,x,y,w_0(t,x,y),\widetilde{\nabla}w_0(t,x,y)\right).$$

for $\left(w_0,\widetilde{\nabla}w_0\right) \in L^p(I;\Xi(\Omega))$ from corollary §13.1.

So, we are in a situation similar to chapter II where ($\heartsuit$)'s analogue was treated via monotonicity methods, the latter's sole purpose being a correct passage to the limit $\varepsilon \longrightarrow 0$.

§14. Limiting procedures – Part I: spaces

As we have seen in chapter II, the presence of non-linearities and insufficient a priori estimates make the passage to the limit a major issue in both homogenisation and existence proofs – and in non-linear analysis, in general. Actually, we already invoked the idea that a suitable family of (pseudo-)monotone operators is not only available for existence proofs – most prominently carried out by Galerkin's method or Rothe's method – but also for periodic homogenisation. In the latter case, the limiting machinery is less formalised, therefore most proofs are usually carried out in ad-hoc fashion, just like our theorem §10.1.

We intend to formalise matters in order to carry out periodic homogenisation proofs analogously to existence proofs. Whereas the machinery for existence proofs is quite mature and rather well-established nowadays, an adaptation to periodic homogenisation will require more refined constructions. Basically, there are two aspects to attend to, namely function spaces of suitable limits and families of operators which operate with the corresponding function spaces in a well-defined manner, admitting properties which allow for a sensible passage to the limit.

Both questions will be addressed in separate sections, starting with the space issue which is to place the headstone for the operators' discussion.

A word on spaces of limits

In order to focus on the actual mechanics of the problem under consideration we will abandon the topic of periodic homogenisation altogether for a moment and favour to work with a more abstract description. Our plan is to revisit existence proofs very briefly for the sake of comparison. Afterwards, we will find a description of a space containing sensible limits which is constructed by a modified Cauchy completion procedure.

So, let X be a Banach space and $(X_n)_{n\in\mathbb{N}}$ a family of closed subspaces of X which do not necessarily form an ascending chain, i.e. $X_n \hookrightarrow X_{n+1}$ is not available in general. The following questions are important to us.

a) Are there non-trivial strongly converging sequences of the form $(x_n)_{n\in\mathbb{N}} \subset X$ with $x_n \in X_n$ holding for all $n \in \mathbb{N}$?

b) If so, can the corresponding limits be considered as a vector space?

c) If the strong limits form a vector space, is it a closed subspace of X or can it be considered as some other Banach space, at least?

d) Does the same hold for weakly convergent sequences, weak limits and the resulting space of weak limits?

e) Do the spaces of strong and weak limits coincide in a meaningful way?

Lamentably, the importance of these questions is not obvious, in fact, they stem from carefully revising ‘standard proofs’ of existence theorem seeking adaptations to periodic homogenisation. This process of anticipation is strongly guided by finite-dimensional approximation which we recall in brief.

§14.1. Example: finite-dimensional approximation

In order to avoid technical subtleties, we will restrict ourselves to Banach spaces X with a (countable) Schauder base. Our applications will work in a given Banach space X; for instance, $X = L^p(\Omega; \mathscr{B})^k$ will be sufficient to keep in mind as a prototype, with Ω being a domain, $\mathscr{B}$ a reflexive Banach space with Schauder basis (thus being separable, too) and $k \in \mathbb{N}$.

Again, we only work with the directed set $\mathbb{N}$, avoiding nets that are no sequences. Indeed, this matter can be subtle. For instance, every Banach space $\mathscr{B}$ is weakly sequentially complete, and weakly quasi-complete but not weakly complete unless $\mathscr{B}$ coincides with its algebraic bidual, a condition which is fulfilled for normed spaces only if $\dim(\mathscr{B}) < \infty$, c.f. [Köt83, § 20.9.(2)]. In addition, all sequences in a given set form a set themselves. In contrast, the totality of all nets in a given set is no set in general – a key aspect advocating the use of filters and (bounded) Cauchy filters for completion precedures, c.f. [Köt79] and [Low89] in particular.

We fix some concepts. X is a Banach space with a Schauder basis, i.e. there exists a linearly independent sequence $\exists (e_k)_{k\in\mathbb{N}} \subset X$ whose linear span is dense in X. If X is a Hilbert space one usually prefers to work with a orthonormalised Schauder basis. One writes X_k for the span of $\{e_1, \ldots, e_k\}$ so that an ascending sequence of finite-dimensional spaces with $\forall k \in \mathbb{N} : X_k \hookrightarrow X_{k+1}$ is at hand.

Thanks to this ascending chain of subspaces the first two questions can be answered positively by considering eventually constant sequences of the form $(x_0, \ldots, x_k, x, x, \ldots)$ for arbitrary $x_i \in X_i$ for $i = 1, \ldots, k$ and some fixed $x \in X_{k+1} \subset X_{k+2} \subset \ldots$. Clearly, such sequences are Cauchy sequences in X and their limits form a vector space. Since the limits are elements in X the space of limits is a subspace of X, namely the union of all X_k. Referring to the weak topology in the fourth question, the same procedure yields weak Cauchy sequences and spaces of weak limits.

Finally, one can address the third and the fifth question by considering the strong and weak closure of the spaces of limits of eventually constant functions. This yields closed vector spaces and these coincide since weak and strong closures coincide on convex sets due to Mazur's lemma.

To sum up, one can answer the above questions positively in the instance of Galerkin's method with a Schauder base. The task which we now turn to is a modification of the foregoing construction. Starting with the time-independent configuration, we will have a null sequence $\varepsilon = \varepsilon_n \longrightarrow 0$ together with closed subspaces $X_n = \mathcal{T}^*_{\varepsilon_n}(\mathscr{V})$ at hand, which are no ascending chain but yet subspaces of $X = L^p(\mathbb{R}^d \times \mathcal{Y})^{1+d}$. Also, one has the very important property that $(\mathcal{T}_{\varepsilon_n}(v))_{n\in\mathbb{N}}$ is a Cauchy sequence in X for fixed $u \in \mathscr{V}$.

A word on category theory and completion

Galerkin's method stems from the fact that $X = \lim\operatorname{ind} X_k$ holds in $\mathbf{Ban_1}$ and more generally from the fact that every Banach space is the co-limit of its finite-dimensional subspaces which form a directed set, too, see [CLM79; Cas10], for instance. In hands on terms, co-limits in $\mathbf{Ban_1}$ are constructed by considering the span of co-limit in **Set**, i.e. span $(\cup_{k\in\mathbb{N}} X_k)$, and to take its closure. In categorical terms, this construction can be deemed very easy.

Now, the unfolded spaces X_n form no ascending chain so that no directed set is easily accessible. As a result, the foregoing procedure cannot be initiated or mimed easily. Despite the author's considerable affection of category theory there seems to be no useful tool easily available for our task. More explicitly, the concept of Cauchy completion is very well-known in a heavily generalised form in category theory but it seems to be entirely useless to our purpose as it excels at describing consequences of completeness and its properties but fails to give decent construction guidance, a state in which sceptics surely rejoice in schadenfreude.

§14.2. An abstract description of the limiting procedure

This subsection sketches a more abstract description of the limiting procedure at hand. Due to its vital importance for the elliptic case, we will discuss the existence of recovery sequences, too, leaving conclusive steps open, though. For convenience's sake, appendix §A.4 gathers the notions related to uniform structures necessary for Cauchy completions.

Idempotence of a sequential completion procedure of quasi-complete, uniform spaces

Recall X to be a Banach space and $(X_k)_{k\in\mathbb{N}}$ a sequence of closed subspaces of X. First, writing $\prod$ for the Cartesian product, let us set $Y_0 := \prod_{k\in\mathbb{N}} X$ and $Z_0 := \prod_{k\in\mathbb{N}} X_k$, considered as algebraic vector spaces without any topology for the moment. Observe that the inclusions $\iota_k : X_k \hookrightarrow X$ are linear maps of vector spaces for all $k \in \mathbb{N}$ and thus, $\iota : Z_0 \hookrightarrow Y_0$ via the product map $\iota = (\iota_k)_{k\in\mathbb{N}}$ makes sense.

Next, we will introduce two gauges on X to have notions of Cauchy sequences available, namely the gauge given by the norm of X, and $D_{\sigma(X,X')}$, the gauge stemming from the weak topology $\sigma(X, X')$ on X. For instance, the pseudo-metric $p : X\times X \longrightarrow [0,\infty)$ fulfils $p \in D_{\sigma(X,X')}$ if there exists $f \in X'$ such that $p(x, y) = |f(x - y)|$.

Definition §14.1: **Spaces of Cauchy sequences**

Having two gauges, and thus two uniform structures at hand, the following definitions are unambiguous.

(14.1a)	$Y_{1,s}$	the *space of strong Cauchy sequences in* X,
(14.1b)	$Y_{1,\sigma(X,X')}$	the *space of weak Cauchy sequences in* X,
(14.1c)	$Z_{1,s}$	the *space of strong Cauchy* X_k*-sequences in* X, and
(14.1d)	$Z_{1,\sigma(X,X')}$	the *space of weak Cauchy* X_k*-sequences in* X.

The following inclusions hold by construction: $Z_{1,s}, Z_{1,\sigma(X,X')} \hookrightarrow Z_0$, $X_{1,s}, X_{1,\sigma(X,X')} \hookrightarrow X_0$, $Z_{1,s} \hookrightarrow Y_{1,s}$, and $Z_{1,\sigma(X,X')} \hookrightarrow Y_{1,\sigma(X,X')}$. Furthermore, $Y_{1,s} \hookrightarrow Y_{1,\sigma(X,X')}$ and $Z_{1,s} \hookrightarrow Z_{1,\sigma(X,X')}$ hold, as well.

Let us note that the question whether the linear vector spaces $Z_{1,s}$ and $Z_{1,\sigma(X,X')}$ are non-trivial at all, i.e. whether non-zero elements exist in these spaces, is open. It is straightforward to come up with pathological examples, for instance, sett $X_k = \operatorname{span}\{e_k\}$ for some orthonormal basis $\{e_k : k \in \mathbb{N}\}$ in $X = L^2(0, 1)$. Thus, considerable restrictions will be necessary in order to arrive at meaningful structures.

The next step is to introduce appropriate uniform structures on the spaces of Cauchy sequences themselves. The idea is, to place the focus on the limits of the sequences at hand. So, one endows $Y_{1,s}$ with a pseudo-metric induced by the norm of X, namely

$$d_s\big((x_k)_{k\in\mathbb{N}}, (y_k)_{k\in\mathbb{N}}\big) := \lim_{k\to 0} \|x_k - y_k\|_X, \tag{14.2a}$$

which is actually a pseudo-norm. Analogously, $D_{\sigma(X,X')}$ induces a gauge on $Y_{1,\sigma(X,X')}$, more specifically, for $p \in D_{\sigma(X,X')}$ set

$$p'\big((x_k)_{k\in\mathbb{N}}, (y_k)_{k\in\mathbb{N}}\big) := \lim_{k\to 0} p\,(x_k, y_k) \tag{14.2b}$$

and denote the resulting gauge by D_{Y_1}. Tacitly, we equip $Z_{1,s}$ and $Z_{1,\sigma(X,X')}$ with the respective relative uniform structure. Similarly, we impose the corresponding relative topology on the spaces $Z_{1,s}$ and $Z_{1,\sigma(X,X')}$ which stems from the uniform topology of $Y_{1,s}$ and $Y_{1,\sigma(X,X')}$. Of course, this means that one uses the aforementioned pseudo-norms restricted to $Z_{1,s}$ and $Z_{1,\sigma(X,X')}$, respectively.

A major step is to turn the spaces of limits from (14.1) into Hausdorff spaces. In this respect recall that (sequentially) complete uniform spaces are not necessarily closed, unless they are Hausdorff. Cutting a long story short, the next step is to obtain a Hausdorff modification of the topological and uniform structures at hand.

Since linear spaces are at hand, it is sufficient, to factor out the closed subspace given by the closure of $\{0\}$ with respect to the topology induced by d_s and the uniform topology of D_{Y_1}:

$$Y_{2,s} := Y_{1,s}/\left(\overline{\{0\}}^{d_s}\right) \quad \text{and} \quad Y_{2,\sigma(X,X')} := Y_{1,\sigma(X,X')}/\left(\overline{\{0\}}^{D_{Y_1}}\right) \tag{14.3a}$$

$$Z_{2,s} := Z_{1,s}/\left(\overline{\{0\}}^{d_s}\right) \quad \text{and} \quad Z_{2,\sigma(X,X')} := Z_{1,\sigma(X,X')}/\left(\overline{\{0\}}^{D_{Y_1}}\right). \tag{14.3b}$$

After these preparatory steps, let us note that $Y_{1,s}$ and $Y_{1,\sigma}$ are sequentially complete just as $Y_{2,s}$ and $Y_{2,\sigma}$, c.f. [Kel55; Köt83]. However, for $Z_{1,s}$, $Z_{1,\sigma}$, $Z_{2,s}$ and $Z_{2,\sigma}$ the situation is not classical since the resulting space of limits does not contain almost all of its respective sequence's elements.

Assuming that sequential completeness can be shown, though, let us recall the following fact. (Quasi-)completion procedures are idempotent in the sense that X and $Y_{2,s}$ are linearly homeomorphic and even $\mathbf{Ban_1}$-isomorphic. More specifically, $Y_{2,s}$ is the sequential completion of the Banach space X and there is isometric isomorphism, namely $I : Y_{2,s} \longrightarrow X$, given by the equivalence class of constant sequences; its inverse is obvious.

Concerning $Y_{2,\sigma(X,X')}$, the situation is not as clear, since the latter is neither the completion, nor the quasi-completion but the sequential completion of of $(X, \sigma(X, X'))$, the latter being a quasi-complete Hausdorff vector space itself. Since quasi-completeness implies sequential completeness, it is natural that the very same map I can be used again to obtain a linear and continuous isomorphism between $Y_{2,\sigma(X,X')}$ and $(X, \sigma(X, X'))$, making $Y_{2,\sigma(X,X')}$ quasi-complete spaces, as well. Note that at this point one my infer $Z_{2,s}$ and $Z_{2,\sigma(X,X')}$ to be identifiable with sequentially closed subspaces of X if they are sequentially complete.

Let us close with pointing out that at this point the existence of recovery sequences is not touched upon. Eventually, one would like to show that $Z_{2,s} 0 Z_{2,\sigma}$. Even if sequential completeness can be shown, this is a highly non-trivial claim since the non-ascending structure of the spaces X_n involved requires very refined criteria for existence since a too general statement cannot be expected, even for Banach spaces.

§15. Limiting procedures – Part II: suitable families of operators

Let us now consider the following set-up: we are provided with a reflexive Banach space X which has a Schauder basis together with a sequence of subspaces $(X_k)_{k\in\mathbb{N}}$ such that the limiting procedure of the foregoing section is non-trivial, i.e. the sequences of the form $x_k \in X_k$ for all $k \in \mathbb{N}$ yield non-trivial spaces of weak and strong limits and the latter spaces coincide, forming the space of limits Ξ. Moreover and most importantly, we assume that appropriate recovery sequences exist, which means that any weak limit can be retrieved as a strong limit. Also, we ask for the incidence products to coincide, i.e. $\langle v', v\rangle_{X'\times X} = \langle v', v\rangle_{X_k'\times X_k}$ for $v' \in X_k', v \in X_k$. Note in this context, that $\iota_k^* : X' \hookrightarrow X_k'$ holds due to the adjoint map of the injection map $\iota_k : X_k \hookrightarrow X$, whereas X_k' can also be identified with $\{x' \in X' : X_k \not\subseteq \ker(x')\}$ via a topological linear homeomorphism by using the Hahn–Banach theorem.

Turning to families of mappings, we assume that a familiy of operators $(B_k)_{k\in\mathbb{N}_0}$ is given such that

$$\forall k \in \mathbb{N}: \quad B_k : X_k \longrightarrow X_k' \tag{15.1a}$$

$$B_0 : \Xi \longrightarrow \Xi' \tag{15.1b}$$

which fulfils the following compatibility condition:

$$[\forall k \in \mathbb{N} : v_k \in X_k \ \wedge\ \exists v_0 \in \Xi : v_k \longrightarrow v_0 \quad \text{in } X] \Longrightarrow [B_k(v_k) \longrightarrow B_0(v_0) \quad \text{in } X'] . \tag{15.1c}$$

Again, a prototypical instance is given by Galerkin's method on $X = \Xi$ with a pseudo-monotone operator $B : X \longrightarrow X'$ and the B_k being the restrictions of B to the finite-dimensional spaces X_k. Taking values in X_k' is to be understood in the sense, that one would use only test functions from X_k itself, a procedure which requires growth conditions to be sensible.

Returning to $(B_k)_{k\in\mathbb{N}_0}$, one is interested in the archetypical question of which conditions to impose on the family described in (15.1), such that it is possible to secure a passage to the limit? Of course, we can only hope to give highly restricted frameworks for such an endeavour. To be more specific, we want to work with families of monotone or pseudomonotone operators; therefore, we aim to establish

$$\left.\begin{array}{r} X_k \ni x_k \rightharpoonup x_0 \in \Xi \text{ in } X \\ B_k(x_k) \rightharpoonup \xi_0 \in \Xi' \text{ in } X' \\ \limsup_{k\to\infty}\langle B_k(x_k), x_k\rangle_{X'\times X} \leq \langle \xi_0, x_0\rangle_{X'\times X} \end{array}\right\} \Longrightarrow B_0(x_0) = \xi_0 \text{ in } X', \tag{M$_{\text{lim}}$}$$

which is a straightforward modification of the so-called *M-transition* or *type (M)* property from [Zei90]. A second notion is given by *pseudomonotone operators* which are closed under addition, a fact which does not necessarily hold for operators which allow M-transitions. Therefore, we aim to generalise pseudomonotonicity in a natural manner: the following implication is again a straightforward generalisation of classical pseudomonotonicity:

$$\left\{\begin{array}{l} \left(X_k \ni x_k \rightharpoonup x_0 \in \Xi \text{ in } X \quad \wedge \quad \limsup\limits_{k\to\infty}\langle B_k(x_k), x_k - x_0\rangle_{X'\times X} \leq 0\right) \\ \quad \Longrightarrow \left(\forall y_k \longrightarrow y_0 \in \Xi : \ \langle B_0(x_0), x_0 - y_0\rangle_{X'\times X} \leq \liminf\limits_{k\to\infty}\langle B_k(x_k), x_k - y_k\rangle_{X'\times X}\right). \end{array}\right. \tag{PM$_{\text{lim}}$}$$

Several questions are in order: first, is ($M_{\lim}$) actually sufficient for a passage to the limit? Secondly, does ($PM_{\lim}$) imply ($M_{\lim}$)? And finally, what are reasonable conditions to impose in (15.1a) and (15.1b) to actually get ($PM_{\lim}$) or at least ($M_{\lim}$)?

Rather trivially, the first question can be answered positively since ($M_{\lim}$) is tailor-made for the purpose of characterising limits. The second question is answered by a

Lemma §15.1

Let the family of operators in (15.1) be given, then ($PM_{\lim}$) implies ($M_{\lim}$) provided recovery sequences are available.

Proof. Let $x_k \rightharpoonup x_0 \in \Xi$ and $B_k(x_k) \rightharpoonup \xi_0 \in \Xi'$. Then, $\limsup_{k\to\infty}\langle B_k(x_k), x_k\rangle_{X'\times X} \leq \langle \xi_0, x_0\rangle_{X'\times X}$ implies $\limsup_{k\to\infty}\langle B_k(x_k), x_k - x_0\rangle_{X'\times X} \leq 0$ and thus ($PM_{\lim}$) yields $\langle B_0(x_0), x_0 - y_0\rangle \leq \liminf_{k\to\infty}\langle B_k(x_k), x_k - y_k\rangle$ for every strongly convergent sequence $X_k \ni y_k \longrightarrow y_0 \in \Xi$. Now, one deduces $\liminf_{k\to\infty}\langle B_k(x_k), x_k - y_k\rangle = \liminf_{k\to\infty}\left[\langle B_k(x_k), x_k\rangle - \langle B_k(x_k), y_k\rangle\right] = \langle \xi_0, x_0 - y_0\rangle$. Since a recovery sequence is available for every $y_0 \in \Xi$, the foregoing step implies $\langle B_0(x_0), x_0 - y_0\rangle \leq \langle \xi_0, x_0 - y_0\rangle$ for every $y_0 \in \Xi$ which yields $B_0(x_0) = \xi_0$ via an obvious contradiction. ■

Theorem §15.1

Let the family in (15.1) consist of continuous, monotone operators, then ($PM_{\lim}$) holds provided recovery sequences are available.

Proof. Our proof is an adaption of the proof of [Rou13, Lemma 2.32]. First, let $X_k \ni v_k \longrightarrow x_0 \in \Xi$, $\delta \in [0,1]$ be a fixed and set $u_k := (1-\delta)v_k + \delta y_k = v_k + \delta(y_k - v_k) \in X_k$. Since every operator $B_k : X_k \longrightarrow X_k'$ is monotone, we have $0 \leq \langle B_k(x_k) - B_k(u_k), x_k - u_k\rangle$ for all $k \in \mathbb{N}$. Rewriting the latter expression leads to

$$\delta\langle B_k(x_k), v_k - y_k\rangle \geq -\langle B_k(x_k), x_k - v_k\rangle + \langle B_k(u_k), x_k - v_k\rangle + \delta\langle B_k(u_k), v_k - y_k\rangle. \tag{15.2}$$

Now, we want to estimate the right hand side suitably from below. By assumption, ($PM_{\lim}$) assumes $0 \leq -\limsup_{k\to\infty}\langle B_k(x_k), x_k - x_0\rangle = \liminf_{k\to\infty} -\langle B_k(x_k), x_k - x_0\rangle$ to hold. Also, $v_k \longrightarrow x_0$ such that $\lim_{k\to\infty}\langle B_k(x_k), x_0 - v_k\rangle = 0$ is valid, so we have

$$\liminf_{k\to\infty} -\langle B_k(x_k), x_k - v_k\rangle = -\limsup_{k\to\infty}\langle B_k(x_k), x_k - x_0\rangle + \lim_{k\to\infty}\langle B_k(x_k), x_0 - v_k\rangle \geq 0. \tag{15.3}$$

Furthermore, we have $\lim_{k\to\infty}\langle B_k(u_k), x_k - v_k\rangle = 0$ since $u_k \longrightarrow u_0 := (1-\delta)x_0 + \delta y_0$ implies $B_k(u_k) \longrightarrow B_0(u_0)$ due to (15.1c) and $x_k - v_k \rightharpoonup x_0 - x_0 = 0$ holds. Consequently, (15.2) yields

$$\begin{aligned}\delta \liminf_{k\to\infty}&\langle B_k(x_k), v_k - y_k\rangle \\ &\geq \liminf_{k\to\infty}\left[-\langle B_k(x_k), x_k - v_k\rangle + \langle B_k(u_k), x_k - v_k\rangle + \delta\langle B_k(u_k), v_k - y_k\rangle\right] \\ &\quad \geq \liminf_{k\to\infty} \delta\langle B_k(u_k), v_k - y_k\rangle = \lim_{k\to\infty} \delta\langle B_k(u_k), v_k - y_k\rangle = \delta\langle B_0(u_0), x_0 - y_0\rangle\end{aligned} \tag{15.4}$$

for all $\delta \in (0,1]$. Dividing by δ we obtain for $\delta \longrightarrow 0$

$$\liminf_{k\to\infty}\langle B_k(x_k), v_k - y_k\rangle \geq \lim_{\delta\to 0}\langle B_0(u_0), x_0 - y_0\rangle = \langle B_0(x_0), x_0 - y_0\rangle \tag{15.5}$$

since $\lim_{\delta\to 0}\|B_0(u_0)-B_0(x_0)\|_{X'}=0$ holds due to the continuity of B_0 and $\lim_{\delta\to 0}\|u_0-x\|_X=0$.

Finally, all of the preceding argumentation relies on the availability of recovery sequences for all $x_0, y_0 \in \Xi$. Assuming to have sufficiently many recovery sequences at hand, we may conclude the statement to be proven. ■

Remark §15.1

The key argument of the forgoing proof is (to discuss) the asymptotic negligibility of the first two right hand side terms in (15.2). Futre work may conceive of more general set-ups than ours where the continuity condition (15.1c) holds, aiming to compensate for the loss of the Rellich–Kondrachov embedding. As a first step, one may try to incorporate lower order terms on the fast medium may by adapting the model proof of [Rou13, Lemma 2.32]. Also note, that one may try to weaken the continuity restrictions from theorem §15.1 to work with hemicontinuity instead; we do not elaborate on this, though.

§16. Continuation of the homogenisation process

Let us connect (εPP) to our set-up first. Writing $\mathscr{V}=W_0^{1,p}(\Omega)$, we set $X := L^p(I\times\mathbb{R}^d\times\mathcal{Y})^{1+d}$,

$$\mathcal{T}_\varepsilon^*\left(L^p(I;\mathscr{V})\right) := \left\{(w_0,\vec{w}_1)\in X : \exists v_0\in L^p(I;\mathscr{V}) : w_0=\mathcal{T}_\varepsilon^*(v_0)\ \wedge\ \vec{w}_1=\mathcal{T}_\varepsilon^*(\nabla_\varepsilon v_0)\right\} \tag{16.1}$$

and $X_\varepsilon := \mathcal{T}_\varepsilon^*\left(L^p(I;\mathscr{V})\right)$. Naturally, the space of (parabolic) two-scale limits, stems from the space Ξ given in (7.21) for the elliptic set-up.

Proposition §16.1: **Parabolification of Ξ**

Assuming the existence of recovery sequences, the space of limits of all weakly or strongly two-scale convergent sequences of the form $(u_\varepsilon)_{\varepsilon\in\mathbb{J}}\subset X$ with $u_\varepsilon\in X_\varepsilon\subset X$ for every $\varepsilon\in\mathbb{J}$ is is induced by Ξ by applying the functor $L^p(I,\,\cdot\,)$ to obtain the parabolic analogue $L^p(I;\Xi)$.

Proof. Having recovery sequences at hand, the statement stems from the functor $L^p(I,\,\cdot\,)$ operating solely on the temporal variable whereas unfolding is restricted to spatial coordinates. Thus, the two operations can be considered to commute. ■

Returning to the homogenisation of (εPP), we gather the convergence statements available from (12.3), (12.4) and corollary §13.1. There exists a subsequence $\varepsilon\in\mathbb{J}$ such that

$$\begin{cases} \mathcal{T}_\varepsilon^*(f_\varepsilon) \longrightarrow f_0 & \text{in } L^{p'}(I\times\mathbb{R}^d\times\mathcal{Y}),\\ \mathcal{T}_\varepsilon^*(u_{0,\varepsilon}) \longrightarrow u_{0,0} & \text{in } L^2(\mathbb{R}^d\times\mathcal{Y}),\\ \mathcal{T}_\varepsilon^*(u_\varepsilon) \rightharpoonup w_0 & \text{in } L^p(I\times\mathbb{R}^d\times\mathcal{Y}),\\ \mathcal{T}_\varepsilon(\nabla_\varepsilon u_\varepsilon) \rightharpoonup \widetilde{\nabla} w_0 & \text{in } L^p(I\times\mathbb{R}^d\times\mathcal{Y})^d,\\ \mathcal{T}_\varepsilon^*(B_\varepsilon(u_\varepsilon)) \rightharpoonup \xi_0^{\circledast} & \text{in } L^{p'}(I\times\mathbb{R}^d\times\mathcal{Y})^d. \end{cases} \tag{16.2}$$

As mentioned before, the natural minimal objective is characterise the limit $\xi_0^{\circledast}$ as

$$(\heartsuit) \qquad \xi_0^{\circledast}(t,x,y) = B_0(w_0, \widetilde{\nabla} w_0)(t,x,y) := \beta\left(t,x,y,u_0,\widetilde{\nabla} u_0\right) \qquad \text{in } L^{p'}(I;\Xi')$$

which is essentially a weak continuity property along a sequence of solutions and would imply correctness of the homogenisation process, at least in a distributional sense of $\mathscr{D}'(\Omega_T \times \mathcal{Y})$. Of course, such a characterisation is the very purpose of ($\mathrm{M}_{\mathrm{lim}}$). To this end, recall (13.5)

$$\begin{cases} \langle B_\varepsilon(u_\varepsilon), u_\varepsilon\rangle &= \displaystyle\iiint_{I\times\mathbb{R}^d\times\mathcal{Y}} \beta_\varepsilon\left(\mathcal{T}_\varepsilon(x,y), y, \mathcal{T}_\varepsilon^*(u_\varepsilon), \mathcal{T}_\varepsilon^*(\nabla_\varepsilon u_\varepsilon)\right)\cdot \mathcal{T}_\varepsilon^*(\nabla_\varepsilon u_\varepsilon)\, dtdydx \\ &= \displaystyle\iiint_{I\times\mathbb{R}^d\times\mathcal{Y}} \mathcal{T}_\varepsilon^*(f_\varepsilon u_\varepsilon)\, dtdx + \frac{1}{2}\left\|\mathcal{T}_\varepsilon^*(u_{0,\varepsilon})\right\|^2_{L^2(\Omega\times\mathcal{Y})} - \frac{1}{2}\|\mathcal{T}_\varepsilon^*(u_\varepsilon(T))\|^2_{L^2(\Omega\times\mathcal{Y})} \end{cases}$$

for which we need to show $\limsup_{\varepsilon\to 0}\langle B_\varepsilon(u_\varepsilon), u_\varepsilon\rangle \le \langle \xi_0^{\circledast}, u_0\rangle$. At this point, a workaround is necessary to deal with lacking regularity since

$$\mathcal{T}_\varepsilon^*(\partial_t u_\varepsilon) \xrightarrow{2w} \partial_t w_0 \qquad \text{in } L^1_{\mathrm{loc}}(\Omega_T\times\mathcal{Y})$$

is meaningless. Quasi-linear set-up generally yield only very weak spatial regularity of time derivatives, see [Lad68b] and [DiB93]. Consequently, u_0' is not available with sufficient spatial regularity to be fit for periodic unfolding. Therefore, we will work with Steklov averages and show $w_0 \in C^0\left(\bar{I}; L^2(\Omega\times\mathcal{Y})\right)$ so that $\langle \xi_0^{\circledast}, w_0\rangle$ is unambiguous.

The fulcrum of our analysis will be a suitable limit system of (εPP). Assume the existence of recovery sequences, let us pick test functions $\varphi_\varepsilon \in W_0^{1,p}(\Omega_T)$ which are strongly two-scale convergent to a given $\varphi_0 = (\varphi_1,\varphi_2,\varphi_3) \in W_0^{1,p}(I;\Xi)$ in the sense that $\varphi_\varepsilon' \xrightarrow{2s} \varphi_0'$ holds in $L^p(I\times\mathbb{R}^d\times\mathcal{Y})$, too. Inserting such test functions in (εPP) and passing to the limit $\varepsilon\longrightarrow 0$ yields the interim limit

$$(16.3) \qquad \left\{ \iiint_{\Omega_T\times\mathcal{Y}} -w_0\varphi_0' + \xi_0^{\circledast}\cdot \underbrace{\left[\chi_1\left(\nabla_x\varphi_1 + \nabla_y\varphi_2\right) + \nabla_y\varphi_3\right]}_{=:\, \widetilde{\nabla}\varphi_0} - f_0\varphi_0\, dydxdt = 0. \right.$$

Unfortunately, neither (16.2) nor (16.3) yield $w_0 \in C^0\left(\bar{I}; L^2(\Omega\times\mathcal{Y})\right)$ directly. Therefore, we cannot give meaning to initial values like $w_0(t=0) = u_{0,0}$ at the moment. Nevertheless, restricting to I is unproblematic, as mirrored by (16.3). Next, we shall retrieve a little more temporal regularity of w_0 by employing Steklov averages, and eventually, (12.3) and (12.4) will yield ($\heartsuit$).

§16.1. Application of Steklov averages to homogenisation

Let us return to the matter of lacking temporal regularity of u_0 in (16.3). The following result is a two-scale adaptation of a classical result [Lad68b, Ch. III, Lemma 4.1].

Proposition §16.2: (Hölder) continuity in time

Let (16.3) and (16.2) hold for w_0 from $\in L^\infty(I;L^2(\Omega\times\mathcal{Y}))\cap L^p(I;\Xi)$. Then, $t\longmapsto w_0(t)\in L^2(I\times\mathbb{R}^d\times\mathcal{Y})$ is uniformly continuous and can be considered unambiguously to fulfil

$$w_0\in C^0\left(\bar{I};L^2(\Omega\times\mathcal{Y})\right). \tag{16.4}$$

In addition, $\lim_{h\to 0}1/h\|w_0(t+h)-w_0(t)\|^2_{L^2((0,T-h_0)\times\Omega\times\mathcal{Y})}=0$ holds for every $h_0\in(0,T)$.

Proof. We intend to use the Arzelà–Ascoli theorem for vector–valued functions, see [Sch97, Thm. 18.35] for instance, to show that the family of Steklov averages $((w_0)_h)_{0<h\ll h_0}\subset C^0(\bar{I};L^2(\Omega\times\mathcal{Y}))$ converges to w_0, carrying over uniform continuity, as well. Our key arguments rely on favourable properties of the Steklov averages and the integrability of f_0 and $\xi_0^{\circledast}$.
Let $\varphi\in W_0^{1,p}(I;\Xi)$, $0<h<h_0$ and insert $\varphi_0=\varphi_{\bar{h}}$ as a test function in (16.3). Transferring the average via lemma §A.3.5 and using partial integration, we get

$$\int_0^{T-h_0}\iint_{\Omega\times\mathcal{Y}}(w_0)'_h\,\varphi+\left(\xi_0^{\circledast}\right)_h\cdot\widetilde{\nabla}\varphi-(f_0)_h\,\varphi\,dydxdt=0. \tag{16.5}$$

Now, let a compact intervall $J:=[t_1,t_2]\subset\subset[0,T)$ be given. Assume $h_0>0$ to be sufficiently small such that $J\subset[0,T-h_0)$ holds. Also, let $\zeta_\delta\in\mathcal{D}(I)$ be the mollification of χ_J. Note that for sufficiently small $\delta>0$ the support of ζ_δ is contained in $(-h_0,T-h_0)$, using tacitly an extension by zero of w_0 outside of I for the moment.
Now, as a first test function in (16.5) choose $\varphi=\zeta_\delta(w_0)_h$ and let $\delta\to 0$ to get

$$\frac{1}{2}\|(w_0)_h\|^2_{L^2(\Omega\times\mathcal{Y}}\Big|_{t_1}^{t_2}=R_1:=\int_J\iint_{\Omega\times\mathcal{Y}}-\left(\xi_0^{\circledast}\right)_h\cdot\widetilde{\nabla}\,(w_0)_h+(f_0)_h\,(w_0)_h\,dydxdt. \tag{16.6}$$

Observe that $R_1=R_1(t_1,t_2,h)$ fulfils $\lim_{t_1\to t_2}R_1=0$ uniformly for $h\in(0,h_0)$. Consequently, one can argue that for a given $h_0>0$ the family $((w_0)_h)_{0<h\ll h_0}$ is uniformly equicontinuous in $C^0\left([0,T-h_0];L^2(\Omega\times\mathcal{Y})\right)$. Similarly, one can use backward Steklov averages to construct a family $((w_0)_{\bar{h}})_{0<h\ll h_0}$ which is uniformly equicontinuous in $C^0([h_0,T];L^2(\Omega\times\mathcal{Y}))$. In both cases, w_0 is the pointwise limit of both families a.e.
As a second step, we need to show that the family of Steklov averages is pointwise relatively compact. To this end, subtract (16.5) for $0<h_1,h_2\ll T-t_2$, choose $\varphi=\left((w_0)_{h_1}-(w_0)_{h_2}\right)\zeta_\delta\in W_0^{1,p}(I;\Xi)$ and let $\delta\to 0$ to get

$$\begin{cases}\frac{1}{2}\left\|(w_0)_{h_1}-(w_0)_{h_2}\right\|^2_{L^2(\Omega\times\mathcal{Y})}\Big|_{t_1}^{t_2}\\ \qquad=R_2:=\int_J\iint_{\Omega\times\mathcal{Y}}-\left[\left(\xi_0^{\circledast}\right)_{h_1}-\left(\xi_0^{\circledast}\right)_{h_2}\right]\cdot\widetilde{\nabla}\left[(w_0)_{h_1}-(w_0)_{h_2}\right]\\ \qquad\qquad+\left[(f_0)_{h_1}-(f_0)_{h_2}\right]\left[(w_0)_{h_1}-(w_0)_{h_2}\right]\,dydxdt.\end{cases} \tag{16.7}$$

Again, observe that independently of $t_1,t_2\in I$, R_2 involves uniformly integrable integrands. Moreover, this holds uniformly for $h_1,h_2\to 0$, too, a consequence of(16.2). By lemma §A.3.2

sufficiently regular Steklov averages converge strongly to functions which are already sufficiently regular themselves. Consequently, we have $\lim_{t_1\to t_2} R_2 = 0$ uniformly in h_1 and h_2. As a result we get $\lim_{h_1,h_2\to 0}\lim_{t_1\to t_2} R_2 = 0 = \lim_{t_1\to t_2}\lim_{h_1,h_2\to 0} R_2$ for every $t_2 \in I$. Arguing similarly for backwards Steklov averages, makes the Arzelà–Ascoli theorem applicable for both averages. Which yields a uniformly continuous function $\widetilde{u} \in C^0\left(\overline{I}; L^2(\Omega\times\mathcal{Y})\right)$ as the limit of the Steklov averages. Moreover, $\widetilde{u}$ coincides with w_0 a.e. in I, in other words, $\widetilde{u} \in [w_0] = w_0 \in L^p(I;\Xi)$, i.e. w_0 is in the same equivalence class like $\widetilde{u}$.
To obtain the claimed Hölder continuity, return to (16.5) and insert $\varphi = (\tau_h - Id)w_0$ with $\tau_h(w_0)(t) := w_0(t+h)$ being defined a.e. in $[0, T-h_0]$ for every $0 < h < h_0$. Using $\partial_t(w_0)_h(t) = 1/h\,[w_0(t+h) - w_0(t)]$ everywhere in the foregoing interval one arrives at $1/h\int_J\int_{\Omega\times\mathcal{Y}} |w_0(t+h) - w_0(t)|^2\,d(x,y)dt$ on the left hand side for every $[t_1,t_2] = J \subset [0, T-h_0]$. Transferring the differences appropriately allows to exploit the integrability of the right hand side functions. Explicitly, the right hand side reads

$$
(16.8)\qquad R_3 := \begin{cases} \displaystyle\int_{t_1}^{t_1+h}\iint_{\Omega\times\mathcal{Y}} (-\xi_0^{\circledast}(t))_h\cdot\widetilde{\nabla}w_0(t) + (f_0(t))_h w_0(t)\,dydxdt \\[2ex] \displaystyle\qquad + \int_{t_2}^{t_2+h}\iint_{\Omega\times\mathcal{Y}} (-\xi_0^{\circledast}(t))_h\cdot\widetilde{\nabla}w_0(t) + (f_0(t))_h w_0(t)\,dydxdt \\[2ex] \displaystyle\qquad + \int_{t_1+h}^{t_2}\iint_{\Omega\times\mathcal{Y}} \left[(-\xi_0^{\circledast}(t))_h - (-\xi_0^{\circledast}(t-h))_h\right]\cdot\widetilde{\nabla}w_0(t) \\[2ex] \displaystyle\qquad\qquad + \left[(f_0(t))_h - (f_0(t-h))_h\right] w_0(t)\,dydxdt. \end{cases}
$$

Similarly as above, the regularity of the integrands and the regularity of Steklov averaging ensure the applicability of Lebesgue's dominated convergence theorem and as a result $\lim_{h\to 0} R_3 = 0$. Thanks to backward Steklov averaging, this limiting can be carried out for any interval $J \subset \overline{I}$ and consequently, the claim is shown. ■

Corollary §16.1: **Initial values for the interim limit**

Assuming the existence of recovery sequences, one may extend (16.3) such that the limit function w_0 $L^p(I;\Xi)$ satisfies

$$
(16.9)\qquad \begin{cases} \displaystyle\iiint_{I\times\mathbb{R}^d\times\mathcal{Y}} -w_0\varphi_0' + \xi_0^{\circledast}\cdot\widetilde{\nabla}\varphi_0 - f_0\varphi_0\,dydxdt \\[2ex] \displaystyle\qquad = \iint_{\Omega\times\mathcal{Y}} u_{0,0}\varphi_0(0) - w_0(T)\varphi_0(T)\,dydx \end{cases}
$$

for all $\varphi_0 \in W_0^{1,\min(2,p)}([0,T]\times\Omega\times\mathcal{Y})$. In particular, the initial value $u_{0,0}$ is attained by the limit function w_0 from (16.2) in the sense of

$$
(16.10)\qquad \|w_0(0) - u_{0,0}\|_{L^2(\Omega\times\mathcal{Y})} = 0.
$$

Proof. Assuming (16.10), (16.9) follows by inserting strongly two-scale convergent test functions into (εPP), i.e. using the existence of recovery sequences, together with passing to the limit $\varepsilon \longrightarrow 0$, applying partial integration and making use of the increased temporal regularity of w_0.
(16.10) is a direct consequence of $u_{0,0} \longleftarrow \mathcal{T}_\varepsilon^*(u_\varepsilon)(0) \longrightarrow w_0(0) \in L^2(\Omega \times \mathcal{Y})$ in $L^2(\mathbb{R}^d \times \mathcal{Y})$. ■

We now turn to inquiring (♥) by considering (13.5), again. This time, not only (16.2) is available but also additional temporal regularity of w_0.

Theorem §16.1: Characterisation of the limit problem

Given the limits from (16.2) which satisfy (16.4) and (16.9) with (16.10), we have

$$\limsup_{\varepsilon \to 0} \langle \mathcal{T}_\varepsilon^*(B_\varepsilon(u_\varepsilon)), \mathcal{T}_\varepsilon^*(u_\varepsilon)\rangle_{X_\varepsilon' \times X_\varepsilon} \leq \langle \xi_0^{\circledast}, w_0 \rangle_{\Xi' \times \Xi} \tag{16.11}$$

Assuming the existence of recovery sequences, (♥) holds and one can identify the limit problem of (εPP) for $\varepsilon \longrightarrow 0$. More specifically, without loss of generality the sequence of solutions $(u_\epsilon) \subset L^p(I; W_0^{1,p}(\Omega))$ of (εPP) are two-scale convergent to a limit function, in more detail

$$\begin{cases} \left(w_0, \widetilde{\nabla} w_0\right) \in L^p(I;\Xi) \ \wedge \ w_0 \in C^0(\bar{I}; L^2(\Omega \times \mathcal{Y})) \\ u_\varepsilon \xrightarrow{2w} w_0 \text{ in } L^p(I \times \mathbb{R}^d \times \mathcal{Y}) \quad \& \quad \nabla_\varepsilon u_\varepsilon \xrightarrow{2w} \widetilde{\nabla} w_0 \text{ in } L^p(I \times \mathbb{R}^d \times \mathcal{Y}; \mathbb{R}^d) \end{cases} \tag{16.12}$$

holds. Moreover, w_0 fulfils (16.10) and solves the following limit problem of (εPP):

$$\begin{cases} \displaystyle\iiint_{I \times \Omega \times \mathcal{Y}} -w_0 \varphi_0' + \beta\left(t, x, y, w_0, \widetilde{\nabla} w_0\right) \cdot \widetilde{\nabla} \varphi_0 - f_0 \varphi_0 \, dy dx dt \\ \displaystyle\qquad = \iint_{\Omega \times \mathcal{Y}} u_{0,0} \varphi_0(0) - w_0(T) \varphi_0(T) \, dy dx. \end{cases} \tag{16.13}$$

for every $\varphi_0 \in L^p(I;\Xi) \cap W^{1,p'}(I;\Xi')$.

Remark §16.1

Implicitly, we assume $\Xi \hookrightarrow L^2(\Omega \times \mathcal{Y})$ throughout, which essentially represents the case $p \geq 2$. For $p \in (1,2)$ minor modifications are necessary, most notably, one has ask that the elements $\left(w_0, \widetilde{\nabla} w_0\right) \in L^p(I;\Xi) \in \Xi$ also fulfil $w_0 \in L^2(I \times \Omega \times \mathcal{Y})$.

Proof. Recall that the unfolded form of $\langle B_\varepsilon(u_\varepsilon), u_\varepsilon\rangle$ from (13.5) fulfils the following equality:

$$\begin{cases} \langle \mathcal{T}_\varepsilon^*(B_\varepsilon(u_\varepsilon)), \mathcal{T}_\varepsilon^*(u_\varepsilon)\rangle_{X_\varepsilon'\times X_\varepsilon} \\ \qquad = \iiint\limits_{I\times\mathbb{R}^d\times\mathcal{Y}} \beta_\varepsilon\left(\mathcal{T}_\varepsilon(x,y), y, \mathcal{T}_\varepsilon^*(u_\varepsilon), \mathcal{T}_\varepsilon^*(\nabla_\varepsilon u_\varepsilon)\right)\cdot \mathcal{T}_\varepsilon^*(\nabla_\varepsilon u_\varepsilon)\, dtdydx \\ \qquad = \iiint\limits_{I\times\mathbb{R}^d\times\mathcal{Y}} \mathcal{T}_\varepsilon^*(f_\varepsilon u_\varepsilon)\, dtdydx \\ \qquad\qquad + \frac{1}{2}\left\|\mathcal{T}_\varepsilon^*(u_{0,\varepsilon})\right\|^2_{L^2(\mathbb{R}^d\times\mathcal{Y})} - \frac{1}{2}\|\mathcal{T}_\varepsilon^*(u_\varepsilon(T))\|^2_{L^2(\mathbb{R}^d\times\mathcal{Y})} =: R_\varepsilon. \end{cases}$$

First, let us remark that all contributions of R_ε are uniformly bounded for all $\varepsilon > 0$ and thus also for $\varepsilon \to 0$. Considering $\limsup_{\varepsilon\to 0} R_\varepsilon$, observe that by (16.2) $\lim_{\varepsilon\to 0}\left\|\mathcal{T}_\varepsilon^*(u_{0,\varepsilon})\right\|^2_{L^2(\mathbb{R}^d\times\mathcal{Y})} = \|u_{0,0}\|^2_{L^2(\Omega\times\mathcal{Y})}$ holds, together with $\lim_{\varepsilon\to 0}\iiint_{I\times\mathbb{R}^d\times\mathcal{Y}} \mathcal{T}_\varepsilon^*(f_\varepsilon u_\varepsilon)\, dtdxdy = \iiint_{I\times\Omega\times\mathcal{Y}} f_0 w_0\, dtdxdy$. Since norms of Hilbert spaces are sequentially weakly lower semi-continuous, $\mathcal{T}_\varepsilon^*(u_\varepsilon)(T) \rightharpoonup w_0(T)$ in $L^2(\Omega\times\mathcal{Y})$ implies $\liminf_{\varepsilon\to 0}\|\mathcal{T}_\varepsilon^*(u_\varepsilon)(T)\|_{L^2(\mathbb{R}^d\times\mathcal{Y})} \geq \|w_0(T)\|_{L^2(\Omega\times\mathcal{Y})}$. Thus, we can estimate $\limsup_{\varepsilon\to 0} R_\varepsilon$ from above:

$$(16.14)\quad \begin{cases} \limsup\limits_{\varepsilon\to 0} R_\varepsilon \leq \iiint\limits_{I\times\Omega\times\mathcal{Y}} f_0 w_0\, dtdxdy + \frac{1}{2}\|u_{0,0}\|^2_{L^2(\Omega\times\mathcal{Y})} \\ \qquad\qquad - \frac{1}{2}\liminf\limits_{\varepsilon\to 0}\|\mathcal{T}_\varepsilon^*(u_\varepsilon)(T)\|^2_{L^2(\mathbb{R}^d\times\mathcal{Y})} \\ \qquad \leq \iiint\limits_{I\times\Omega\times\mathcal{Y}} f_0 w_0\, dtdx + \frac{1}{2}\|u_{0,0}\|^2_{L^2(\Omega\times\mathcal{Y})} - \frac{1}{2}\|w_0(T)\|^2_{L^2(\Omega\times\mathcal{Y})} =: Z. \end{cases}$$

Now, consider (16.9) and insert a suitable test function: we choose $\varphi_{\overline{h}} = ((\widetilde{w}_0)_h)_{\overline{h}}$ which stems from averaging $\widetilde{w}_0$, the extension of $t \mapsto w_0(t) \in L^2(\Omega\times\mathcal{Y})$ outside of $[0,T]$ by $\widetilde{u}_0(t > T) \equiv w_0(T)$ and $\widetilde{u}_0(t < 0) \equiv u_{0,0}$. This test function is admissible since w_0 is spatially sufficiently regular in $(0,T)$ and temporally in $[0,T]$ for the resulting integral to be meaningful. With this test function at hand, we claim that the following identity can be established:

$$(16.15)\quad \begin{cases} Z = \lim\limits_{h\to 0}\iiint\limits_{I\times\Omega\times\mathcal{Y}} w_0\varphi_{\overline{h}}' + f_0\varphi_{\overline{h}}\, dydxdt + \frac{1}{2}\iint\limits_{\Omega\times\mathcal{Y}} u_{0,0}\varphi_{\overline{h}}(0) - w_0(T)\varphi_{\overline{h}}(T)\, dydx \\ \quad = \lim\limits_{h\to 0}\iiint\limits_{I\times\Omega\times\mathcal{Y}} \xi_0^{\circledast}\cdot\widetilde{\nabla}\varphi_{\overline{h}}\, dydxdt \\ \quad = \iiint\limits_{I\times\Omega\times\mathcal{Y}} \xi_0^{\circledast}\cdot\widetilde{\nabla} w_0\, dydxdt = \langle \xi_0^{\circledast}, w_0\rangle_{L^{p'}(I;\Xi')\times L^p(I;\Xi)} \end{cases}$$

We proceed as follows: the second equality is clear from (16.9); the third equality is due to lemma §A.3.2 and proposition §A.3.1 which ensure $\lim_{h\to 0}\|\widetilde{\nabla}(w_0 - \varphi_{\overline{h}})\|_{L^{p'}(I\times\Omega\times\mathcal{Y};\mathbb{R}^d)} = 0$. The final equality is correct by definition, thus only the first equality is open. Similarly to the preceding proofs, we transfer the backwards Steklov average to get

$$\left\{\begin{aligned} &\iiint_{I\times\Omega\times\mathcal{Y}} (w_0)_h\,(\widetilde{w}_0)'_h + (f_0)_h\,(\widetilde{w}_0)_h\;dydxdt \\ &\qquad + \iint_{\Omega\times\mathcal{Y}} u_{0,0}\varphi_{\overline{h}}(0) - (w_0)_h\,(T)\,(\widetilde{w}_0)_h\,(T)\,dxdy. \end{aligned}\right. \tag{16.16}$$

For the first term on the left hand side we defer

$$\left\{\begin{aligned} &\iiint_{I\times\Omega\times\mathcal{Y}} (w_0)_h\,(\widetilde{w}_0)'_h\,dydxdt = \iiint_{I\times\Omega\times\mathcal{Y}} (\widetilde{w}_0)_h\,(\widetilde{w}_0)'_h\,dydxdt \\ &\qquad = \frac{1}{2}\,\|(\widetilde{u}_0)_h(t)\|^2_{L^2(\Omega\times\mathcal{Y})}\Big|_{t=0}^{t=T} \overset{h\to 0}{\longrightarrow} \frac{1}{2}\,\|w_0(t)\|^2_{L^2(\Omega\times\mathcal{Y})}\Big|_{t=0}^{t=T} \end{aligned}\right. \tag{16.17}$$

The remaining terms and the convergence statement for $h \longrightarrow 0$ are clear from (16.10) and the strong convergence properties of Steklov averaging for sufficiently regular as (16.4) holds. To establish (♥), we use ($\mathrm{M}_{\mathrm{lim}}$) via ($\mathrm{PM}_{\mathrm{lim}}$). Thanks to theorem §15.1, the family of operators

$$\langle B_\varepsilon(u_\varepsilon,\nabla_\varepsilon u_\varepsilon),\varphi\rangle_{L^p(I;X_\varepsilon)\times L^{p'}(I;X'_\varepsilon)} := \iiint_{I\times\mathbb{R}^d\times\mathcal{Y}} \beta\left(t,\mathcal{T}_\varepsilon(x,y),y,\mathcal{T}_\varepsilon^*(u_\varepsilon),\mathcal{T}_\varepsilon^*(\nabla_\varepsilon u_\varepsilon)\right)\cdot\nabla_\varepsilon\varphi\,dtdxdy$$

$$\langle B_0(w_0,\widetilde{\nabla} w_0),\varphi\rangle_{L^p(I;\Xi)\times L^{p'}(I;\Xi')} := \iiint_{I\times\Omega\times\mathcal{Y}} \beta\left(t,x,y,w_0,\widetilde{\nabla} w_0\right)\cdot\widetilde{\nabla}\varphi\,dtdxdy$$

is a subject for the implication ($\mathrm{PM}_{\mathrm{lim}}$): first, β was assumed to fulfil the respective monotonicity condition in (12.2d). Secondly, to verify (15.1c) let $u_\varepsilon \overset{2s}{\longrightarrow} w_0$ in $L^p(I\times\mathbb{R}^d\times\mathcal{Y})$ with $\nabla_\varepsilon u_\varepsilon \overset{2s}{\longrightarrow} \widetilde{\nabla} w_0$ in $L^p(I\times\mathbb{R}^d\times\mathcal{Y};\mathbb{R}^d)$ be given. Recall that $\lim_{\varepsilon\to 0}\|\mathcal{T}_\varepsilon(x,y) - id_\Omega\|_{L^\infty(\mathbb{R}^d\times\mathcal{Y})} = 0$ holds and since β induces a continuous superposition operator thanks to (12.2a) and (12.2b), we get

$$\lim_{\varepsilon\to 0}\left\|B_\varepsilon\left(u_\varepsilon,\nabla_\varepsilon u_\varepsilon\right) - B_0\left(w_0,\widetilde{\nabla} w_0\right)\right\|_{L^{p'}(I\times\mathbb{R}^d\times\mathcal{Y};\mathbb{R}^d)} = 0. \tag{16.18}$$

Finally, as ($\mathrm{PM}_{\mathrm{lim}}$) is an admissible implication, (16.11) is precisely the criterion named in ($\mathrm{PM}_{\mathrm{lim}}$) to yield (♥). To obtain the limit system, one inserts (♥) into (16.9) to get (16.13) ■

§17. Epilogue

This work concludes with a short summary and a prospect on related problems and questions. In short, an elliptic and a parabolic family of quasi-linear problems were subject to periodic homogenisation. Our main difficulty was to compensate for a loss of compactness due to degenerated a priori estimates caused by the slow medium. In technical terms, our method relied on carrying over a limiting technique for existence proofs – monotone operators – to our problems. Unfortunately, the vital step of ensuring the existence of recovery sequences could not established but it could be shown that if it holds, one can adapt monotonicity accordingly, allowing to characterise weak limits in both elliptic and parabolic set-ups.

Leaving the recovery sequences' issue aside, there are several natural extensions of weak limit characterisations which aim at improving and quantifying convergence. In this respect, it must be said that for merely monotone operators stronger results cannot expected, even for existence proofs. However, stronger concepts analogous to families of uniformly monotone operators are conceivable and may yield stronger modes of convergence along the lines of [All92, Prop. 4.6] where smoothness of the corrector functions was assumed to establish strong two-scale convergence to the solution of the limit problem. We refrained from elaborating on this matter since unlike to the linear case, such smoothness assumptions are either highly non-trivial or quite non-credible in the quasi-linear regime. So, regularity issues play an important role and such considerations are presently too immature to be incorporated in the aforementioned machinery. However, for future research [Lad68a; Bec16; Lad68b; DiB93] form sound points of departure.

Moreover, even in the original domain of existence proofs, quantification of convergence is a precarious issue. Nevertheless, the analogy of periodic homogenisation to finite-dimensional approximation leads the author to hazard the guess that these two share a common root; more concretely, restriction operators to finite-dimensional subspaces and periodic unfolding are alike and consequently, finite element error estimates and error estimates for homogenised problem are 'two leafs on the same tree'.

Furthermore, let us mention the recent works of A. Koutsoukou-Argyraki with U. Kohlenbach – we refer to [KK15; KK16; Kou17b; Kou17a] – where error estimates are derived from (classical) existence proofs invoking accretive operators. Perhaps it is possible to transfer such techniques to homogenisation, provided the latter and existence proofs are sufficiently linked. As a result, such an endeavour would yield error estimates from mere limiting proofs.

Finally, the families of operators fit for limiting procedures should be sought to be enlarged in favour of a larger framework. Indeed, S. Reichelt's thesis [Rei15] can be considered as complementary to our presentation, but both works have abandoned compactness methods altogether and turned to a *compensation by properness*: given strongly convergent right hand side functions, one wants to infer convergence of solutions. Indeed, this concept has been cast in great generality by W. V. Petryshyn into the notions of *A-properness* and *pseudo-A-properness* in [Pet75] and [Pet93]. Adding to the possibility of a unifying frame, there is an alternative formulation of pseudo-monotonicity in optimisation which links pseudo-convex potentials to pseudo-monotone gradient functions in Euclidean spaces, we refer to [HSW12]. We did not enquire this topic any further but the relation to Γ-convergence is quite obvious, in particular since pseudo-monotonicity of gradient operators imply weak lower sequential semi-continuity of the potential. So, exploring a common origin of homogenisation and existence proofs is strongly encouraged by the present state of affairs but is left to future research.

A. Appendix

§A.1. Our essentials of category theory

We refer to [Sem71, Ch. III] and [Cas10] for analysis-oriented introductions, to [Bra17] for a state-of-the-art introduction in German, and to [AHS04] as an extensive presentation of category theory. More classical literature like [Mac71; KS06] is available, too. Also, [nLa19] is a very serious and valuable resource and a significant hub of related communities. Moreover, [Hel89; Hel06] are very extensive analysis-centred works; however, by analysis we mean functional analysis, ignoring such highly sophisticated matters like *secondary calculus on diffieties*, see [Vin01] for further reference.

§A.1.1. Categories

A category $\mathscr{C}$ is pair of sets or even classes

$$\mathscr{C} = \Big(Ob(\mathscr{C}), \big(\mathrm{Mor}_{\mathscr{C}(C_1,C_2)}\big)_{C_1,C_2 \in Ob(\mathscr{C})}\Big) \tag{1.1}$$

The elements $C \in Ob(\mathscr{C})$ are the objects of $\mathscr{C}$ and for $C_1, C_2 \in Ob(\mathscr{C})$ $\mathrm{Mor}_{\mathscr{C}}(C_1, C_2)$ denotes the $\mathscr{C}$-morphisms from C_1 to C_2. It is required that $\mathrm{Mor}_{\mathscr{C}}(C_1, C_1)$ contains an identity element. Moreover, morphisms can be composed, i.e. $\forall f_1 \in \mathrm{Mor}_{\mathscr{C}}(C_1, C_2), f_2 \in \mathrm{Mor}_{\mathscr{C}}(C_2, C_3)$ there is some composition operation $\circ_{\mathscr{C}}$ such that $f_2 \circ_{\mathscr{C}} f_1 \in \mathrm{Mor}_{\mathscr{C}}(C_1, C_3)$ holds an this composition is associative and respects identity maps, making the latter unique. $\mathscr{C}$ is locally small if all morphism classes $\mathrm{Mor}_{\mathscr{C}}(C_1; C_2)$ are sets for all $C_1, C_2 \in \mathscr{C}$. $\mathscr{C}$ is a small category if it is locally small and its class of objects is a set.

Some examples of categories

A most central category is the category of sets **Set** whose objects are sets with the morphisms being all set maps between the corresponding sets. Secondly, **Top** is the category of all topological spaces and continuous maps as morphisms; **Vect** is the category of all linear vector spaces with linear maps as morphisms. Thirdly, **Ban**, $\mathbf{Ban}_\infty$ and $\mathbf{Ban}_1$ are the (quite natural) categories whose objects are Banach spaces together with linear maps, continuous linear maps and non-expansive linear maps as morphisms, respectively. In likewise fashion one can formalise almost everything. To give you an impression the following form categories: uniform spaces (**Unif**), pseudo-metric spaces (**PMet**, $\mathbf{PMet}_1$) metric spaces (**Met**, $\mathbf{Met}_1$), topological vector spaces (**Tvs**), locally convex spaces (**Lcs**), finite-dimensional vector spaces (**FinVect**), Hilbert spaces (**Hilb**), smooth manifolds (**Diff**) partially ordered sets (**PoSet**), measurable spaces (**Meas**), measure spaces (**Measure**) and even the category of all categories (**CAT**) and the category of all small categories (**cat**). None of the aforementioned categories is small, but all except **CAT** are locally small. We underline four aspects:

a) Not all categories are defined equally by all authors so one must take care. In particular, this holds for the morphisms underlying a category. Roughly put, in most instances morphisms are 'even more important' than the corresponding objects.

b) A lot can be said and is being discussed about related and additional structures such as Abelian categories, Grothendieck topologies, sheaves and topoi. We do not elaborate on these aspects.

c) Again, almost everything can be formalised as a category but at the end of the day, the important question is whether or not the category under consideration has favourable properties for the applications one has in mind. Our focus will be on limits and co-limits; related concepts are completeness and co-completeness of a category.

d) More generally and going back to A. Grothendieck, is the insight that it is better to have a well-behaved collection of bad objects than a collection of good objects that misbehaves. Obviously Sobolev spaces are a positive example which supersede more classical function spaces like $C^{k,\alpha}$. As we have seen in chapter I, **Diff** is no favourable properties though its objects are even smooth.

For further information consider the given literature.

Isomorphisms in a category

Next, a $\mathscr{C}$-morphism $f : A \longrightarrow B$ is a $\mathscr{C}$-isomorphism if there is a $\mathscr{C}$-morphism $g : B \longrightarrow A$ such that $g \circ_{\mathscr{C}} f = id_A$ and $f \circ_{\mathscr{C}} g = id_B$ in $\mathscr{C}$. In fact, this concept is very natural in conventional functional analysis where a linear isomorphism, a **Vect**-isomorphism, is not necessarily continuous, at least not as long as the (full) axiom of choice holds. Moreover, it is the very essence of the open mapping theorem that for a $\mathbf{Ban}_\infty$-morphism to be an $\mathbf{Ban}_\infty$-isomorphism, i.e. it has a continuous and linear inverse, it is sufficient to be a **Set**-isomorphism, i.e. a bijection.

§A.1.2. Functors and natural transformations

A (covariant) functor is a map between categories $F : \mathscr{C}_1 \longrightarrow \mathscr{C}_2$ that sends objects from $\mathscr{C}_1$ to $\mathscr{C}_2$ together with their morphisms. It is required that a functor respects identity elements and composition of morphisms. Thus, $F(id_{C_1}) = id_{F(C_1)}$ for all $C_1 \in \mathscr{C}_1$ and $F(g \circ f) = F(g) \circ F(f)$ for all respective $\mathscr{C}_1$-morphisms. In practical terms, defining a functor is 'more difficult' than just defining a map since one must specify what to do with the underlying morphisms. A contravariant functor is defined in the same manner but it inverts the succession of maps, i.e. $F(g \circ f) = F(f) \circ F(g)$ for all respective $\mathscr{C}_1$-morphisms. Both contravariant and covariant functors are fundamentally important.

Two examples of functors

A prominent example is the *forgetful functor* which maps a category of structures into a greater category with weakened structure conditions. For instance, the sequence

$$\mathbf{Ban}_1 \longrightarrow \mathbf{Ban}_\infty \longrightarrow \mathbf{Ban} \longrightarrow \mathbf{Vect} \longrightarrow \mathbf{Set}$$

can be considered in this way since forgetful functors leave the morphisms untouched.

A most important instance are function spaces, for example the functor $\mathscr{B} \longmapsto L^1((0,1),\mathscr{B})$ given in chapter III. This concept is strongly advocated in [Cas10].

Another important concept are completions: let $\mathbf{NormVect}_{\mathbb{R}}$ be the category of normed vector spaces over $\mathbb{R}$ with continuous linear maps as morphisms. Then, a functor can be defined by sending $X \in \mathbf{NormVect}$ to its Cauchy completion $\overline{X}$. Since continuous linear maps are uniformly continuous their extension to the completion is unambiguous.

Natural transformations

At the very heart of category theory are natural transformations. Given two functors $F, G : \mathscr{C}_1 \longrightarrow \mathscr{C}_2$ a natural transformation from F to G is a family of maps denoted $\eta : F \longrightarrow G$ such that for every $C \in \mathscr{C}_1$ there is a $\mathscr{C}_2$-morphism $\eta_C : F(C) \longrightarrow G(C)$ making the diagram

$$\begin{array}{ccc} F(C_1) & \xrightarrow{\eta_{C_1}} & G(C_1) \\ {\scriptstyle F(f)}\big\downarrow & & \big\downarrow{\scriptstyle G(f)} \\ F(C_2) & \xrightarrow{\eta_{C_2}} & G(C_2) \end{array} \tag{1.2}$$

commutative for all $C_1, C_2 \in Ob(\mathscr{C}_{\frown 1), f \in \mathrm{Hom}_{\mathscr{C}_1}(C_1,C_2)}$.

As an example consider a continuous linear map between Banach spaces $A : \mathscr{B}_1 \longrightarrow \mathscr{B}_2$. Then, $\mathscr{B} \longmapsto L^1((0,1);\mathscr{B}_1)$ is mapped to $\mathscr{B} \longmapsto L^1((0,1);\mathscr{B}_2)$ by prolongation, also known as pushforward along A.

A most crucial fact to us is that periodic unfolding admits no natural transformation, at least none that the author managed to find. This state of affairs is extremely inconvenient and a single major source of challenging issues.

§A.1.3. Co-limits and limits

Among the purposes of providing additional categories are the notions of co-limits and limits in a given category. To start with co-limits, let $J = (J, \lhd)$ be a directed set and let $(X_j)_{j\in J}$ be a family of $\mathscr{C}$ objects together with a collection of $\mathscr{C}$-morphisms $\xi_{i,j} : X_i \longrightarrow X_j$ for $i, j \in J$ with $i \lhd j$ (the case of absent morphisms is explicitly allowed). Such a family $((X_j)_{j\in J}, (\xi_{i,j})_{i,j\in J})$ is called an inductive system or inductive familiy (in $\mathscr{C}$). A co-cone or target to this family of objects and morphisms is an object $Y \in \mathscr{C}$ with $\mathscr{C}$-morphisms $\xi_j : X_j \longrightarrow Y$ such that $\xi_i = \xi_j \circ \xi_{i,j}$ for all $i, j \in J$ with $i \lhd j$. The cone is universal if any other cone is factored by Y, i.e. if there exists another cone Z with cone maps $\zeta_j : X_j \longrightarrow Z$ then there exists a unique $\mathscr{C}$-morphism $\alpha : Y \longrightarrow Z$ such that $\zeta_j = \alpha \circ \xi_j$ for all $j \in J$. A co-limit to a given family of objects and morphisms indexed by a directed set is a universal co-cone. Co-limits are also known as *direct limits* or *inductive limits* and one writes $\lim\operatorname{ind} X_j = Y$ or $\operatorname{colim} X_j = Y$.[1]

Likewise, limits are given by *dualisiation*, the reversing of arrows: for $i, j \in J$ with $i \lhd j$ there may be maps $\xi_{i,j} : X_i \longleftarrow X_j$ for $X_J \in \mathscr{C}$ for $j \in J$ such that a $((X_j)_{j\in J}, (\xi_{i,j})_{i,j\in J})$ is a projective system or projective familiy (in $\mathscr{C}$). The corresponding cone or source is an $\mathscr{C}$-object Y with $\mathscr{C}$-morphisms $\xi_j : X_j \longleftarrow Y$ for all $j \in J$ such that $\xi_i = \xi_{i,j} \circ \xi_j$ for all $i, j \in J$ with $i \lhd j$. Again, the limit is defined as a universal cone to a given family of objects and morphisms. It is also known as *inverse limit* or *projective limit* and we write $\lim X_j = Y$ or $\lim\operatorname{proj} X_j = Y$; however, we

[1] Originally, co-limits generalised inductive limits but we use both notions interchangeably.

prefer the latter notation to keep projective limit constructions apart from conventional limits of analysis. In fact, the latter are conceivable as inverse limits by the help of filters in **Set**.

A family of objects can have different limits or co-limits with respect to different underlying categories. Also, our co-limits and limits are *small* in the sense that J is a set and not a proper class. More generally, category theory is impressed by set theoretic concepts which touch upon the antinomy of sets and classes. Fortunately, the overwhelming majority of problems of analysis circumvent the related problems altogether – there are exceptions, though, for instance, the totality of all nets in a given set. For numerical mathematics the situation is quite unclear since floating-point arithmetic effectively abandons mathematically well-behaved objects like the fields $\mathbb{R}$ and $\mathbb{C}$, entering mathematical realms that are not too well-understood.

Construction and examples

Most often, the idea is to construct or co-limits limits in **Set** first, endowing them with additional structure afterwards. This approach is based on the fact that many categories – and all of our categories – can be considered as subcategories of **Set** via forgetful functors. This does not apply to every mathematical subject, leading to *higher category theory*, a field which we avoid.

Concerning examples, one can consider all finite-dimensional sub-spaces of a given Banach space as a family of Banach spaces directed by inclusion. The corresponding co-limit in $\mathbf{Ban_1}$ is the original Banach space, c.f. [CLM79]: every finite-dimensional approximation procedure rests on this result.

Another example is the construction of the space of test functions which is a co-limit construction in **Lcs**; one can verify that the corresponding family of objects is in $\mathbf{Ban_1}$, too, and has a co-limit there, as well. However, the latter is trivial, a fact which is related to locally compact Banach spaces being necessarily finite-dimensional. For more applications, we refer to the main text. In general, an important question is whether a given category can be expected to possess 'reasonable' co-limits or limits and in fact, the corresponding notions are *co-completeness* and *completeness* of a given category. For instance, **Set** and $\mathbf{Ban_1}$ are both co-complete and complete but $\mathbf{Ban}_\infty$ is not.

Let us close with remarking that there are several more concepts from category theory which deserve more attention by the analysis community. Most interestingly, let us mention *sheaves* and *co-sheaves*, *adjunctions of functors* and the related concept of *Kan extensions*. All of these are extensively used throughout mathematics. For instance, the completion functor and the forgetful functor are adjoint in a meaningful way. However, for the present work such conceptional means we not necessary, making it our main reason of omission.

§A.2. Monotone operators

Our references on the theory of monotone operators are [Lio69; Klu79; Zei90; Le11] and [Rou13], the latter being our preferred choice. Given a reflexive Banach space X an operator $A : X \longrightarrow X'$ is monotone if for all $x, y \in X$ we have $\langle Ax - Ay, x - y\rangle_{X'\times X} \geq 0$ and stictly monotone if $\langle Ax - Ay, x - y\rangle_{X'\times X} > 0$ for $x \neq y$. Throughout the literature, it is customary to write Ax even if A is not linear. One can show that monotone operators are locally bounded and continuous as a map from V in the strong topology to V' equipped with its weak topology if they are hemi-continuous, meaning $[0, 1] \ni \tau \longmapsto \langle A(x + \tau y), z\rangle_{X'\times X}\mathbb{R}$ is continuous for all $x, y, z \in X$. An operator $A : \mathrm{dom}(A) \subset X \longrightarrow X'$ is maximally monotone if it is monotone and if its graph is maximal in the sense that there is no monotone extension of A but A itself. An operator $A : X \longrightarrow X'$ is pseudo-monotone if it is bounded and the following implication holds for weakly convergent sequences $x_k \rightharpoonup x_0$ in X:

$$\limsup_{k\to\infty} \langle Ax_k, x_k - x_0\rangle \leq 0 \Longrightarrow \left[\forall y \in X : \ \langle Ax_0, x_0 - y\rangle \leq \liminf_{k\to\infty} \langle Ax_k, x_k - y\rangle \right]. \tag{2.1}$$

We call $A : X \longrightarrow X'$ (strongly) coercive if $\lim_{\|x\|\to\infty} \langle Ax, x\rangle / \|x\| = +\infty$. The following implications hold: a hemi-continuous monotone operator is maximally monotone, every monotone operator is pseudo-monotone, every compact operator is pseudo-monotone, sums of pseudo-monotone operators are pseudo-monotone. The next result is a major theorem of the theory.

Theorem §A.2.1: **Brezis's theorem, c.f. [Rou13, Thm. 2.6]**

Every pseudo-monotone, coercive operator is surjective and strict monotonicity is sufficient to be a bijection, i.e. a **Set**-isomorphism.

Brezis's theorem generalises the *Browder–Minty theorem* which restricts to monotone operators and is strongly related to the Lax–Milgram theorem. Perhaps the pinnacle of monotone operators' existence theory is the next result.

Theorem §A.2.2: **Browder's theorem, c.f. [Zei90, Thm. 32.A]**

Let $A : X \longrightarrow X'$ be maximally monotone and $B : X \longrightarrow X'$ be pseudo-monotone such that

$$\begin{cases} \exists v_0 \in X, R \geq \|v_0\|_X : \quad \forall u \in \mathrm{dom}(A_\varepsilon) \cap B_R(0) \neq \emptyset : \\ \qquad \langle A_\varepsilon(u) + B_\varepsilon(u) - f_\varepsilon, u - v_0\rangle_{X' \times X} > 0 \end{cases} \tag{2.2}$$

holds. Then $A + B$ is a surjection.

Finally, all of the above definitions and results have multivalued analogues which we neither present nor use here. A full account, including proofs, is given in the German textbook [Klu79].

§A.3. Steklov averages

Here, we gather the material on Steklov averages required for our purposes. We omit the proofs of all classical statements, tending only relevant two-scale results. In general, the respective proofs are entirely 'classical and elementary', i.e. most people know or anticipate and use the result but only few people feel the urge to write down proofs. A classical ressource is [Lad68b, pp. 84-86, pp. 141-142] but the more recent [CDG17] is very extensive and contains proofs, too. We will give a formulation in the two-scale set-up on $\Omega \times \mathcal{Y}$ since this is what we are going to work with. Of course, one can drop $\mathcal{Y}$, Ω or $\Omega \times \mathcal{Y}$ to work with $\mathbb{R}$-valued functions, instead. Throughout, $I = (0, T)$ for some $T > 0$.

Definition §A.3.1: Steklov averages

Let $p_1, p_2 \in [1,\infty]$, $u \in L^{p_1}(I; L^{p_2}(\Omega \times \mathcal{Y}))$ and $h_0 \geq h > 0$. We call

$$u_h(t,x,y) := \fint_t^{t+h} u(\tau,x,y)\chi_{[0,T-h]}(\tau)\,d\tau := \frac{1}{h}\int_t^{t+h} u(\tau,x,y)\chi_{[0,T-h]}(\tau)\,d\tau \tag{3.1a}$$

the *(forward) Steklov average* of u. The *(backward) Steklov average* of u is given by

$$u_{\overline{h}}(t,x,y) := \fint_{t-h}^{t} u(\tau,x,y)\chi_{[0,T]}(\tau)\,d\tau. \tag{3.1b}$$

Clearly, $u \longmapsto u_h$ and $u \longmapsto u_{\overline{h}}$ are linear mappings for all $h > 0$ and regularise temporal regularity. Let us formulate several fundamental properties as lemmata.

Lemma §A.3.1: Continuity of Steklov averaging

For $u \in L^{p_1}(I; L^{p_2}(\Omega \times \mathcal{Y}))$ with $p_1, p_2 \in [1,\infty]$ we have $u_h \in C^0\left(\overline{I}; L^{p_2}(\Omega \times \mathcal{Y})\right)$ and

$$\begin{cases} \|u_h\|_{L^{p_1}(I;L^{p_2}(\Omega\times\mathcal{Y}))} & \leq \quad \|u\|_{L^{p_1}(I;L^{p_2}(\Omega\times\mathcal{Y}))} \\ \|u_h(t)\|_{L^{p_2}(\Omega\times\mathcal{Y})} & \leq \quad h^{-1/p_1}\|u(t)\|_{L^{p_2}(\Omega\times\mathcal{Y})} \quad \text{a.e. in } I. \end{cases} \tag{3.2}$$

The very same statements hold for backward averages. Thus, Steklov averages are continuous, linear endomorphism of $L^{p_1}(I; L^{p_2}(\Omega \times \mathcal{Y}))$.

Lemma §A.3.2: Approximation property of Steklov averages

For all $u \in L^{p_1}(I; L^{p_2}(\Omega \times \mathcal{Y}))$ with $p_1 \in [1,\infty)$, $p_2 \in [1,\infty]$ we have

$$\begin{cases} \lim_{h\to 0}\|u - u_h\|_{L^{p_1}(I;L^{p_2}(\Omega\times\mathcal{Y}))} & = 0 = \quad \lim_{h\to 0}\|u - u_{\overline{h}}\|_{L^{p_1}(I;L^{p_2}(\Omega\times\mathcal{Y}))} \\ \lim_{h\to 0}\|(u - u_h)(t)\|_{L^{p_2}(\Omega\times\mathcal{Y})} & = 0 = \quad \lim_{h\to 0}\|(u - u_{\overline{h}})(t)\|_{L^{p_2}(\Omega\times\mathcal{Y})} \end{cases} \tag{3.3}$$

almost everywhere in I.

Lemma §A.3.3: Temporal differentiability of Steklov averages

For all $u \in L^{p_1}(I; L^{p_2}(\Omega \times \mathcal{Y}))$ with $p_1 \in [1,\infty)$, $p_2 \in [1,\infty]$ we have $u_h \in W^{1,p_1}(0, T - h; L^{p_2}(\Omega \times \mathcal{Y}))$ and $u_{\overline{u}} \in W^{1,p_1}(h, T; L^{p_2}(\Omega \times \mathcal{Y}))$ thanks to

$$\begin{cases} \left(\partial_t(u_h)\right)(t) = 1/h\,[u(t+h) - u(t)] & \text{a.e. in } (0, T-h), \text{ and} \\ \left(\partial_t(u_{\overline{h}})\right)(t) = 1/h\,[u(t) - u(t-h)] & \text{a.e. in } (h, T). \end{cases} \tag{3.4}$$

If in addition $u \in C^0(I; L^{p_2}(\Omega \times \mathcal{Y}))$ then the foregoing statements hold pointwise.

Lemma §A.3.4: **Uniform approximation by Steklov averages**

Let $u \in B(0,T) := L^p(0,T;L^p(\Omega;W^{1,p}(\mathcal{Y}))) \cap C^0(0,T;L^2(\Omega\times\mathcal{Y}))$ then for every $h_0 > 0$

$$\lim_{h\to 0} \|u - u_h\|_{B(0,T-h_0)} = 0 \qquad \& \qquad \lim_{h\to 0} \|u - u_{\overline{h}}\|_{B(h_0,T)} = 0 \tag{3.5}$$

hold with $\|u\|_{B(0,T)} = \max_{t\in[0,T]} \|u(t)\|_{L^2(\Omega\times\mathcal{Y})} + \|u\|_{L^p(0,T;L^p(\Omega;W^{1,p}(\mathcal{Y})))}$.

Lemma §A.3.5: **Partial integration for Steklov averages**

For $u \in L^p(\Omega_T\times\mathcal{Y})$, $w \in L^{p'}(\Omega_T\times\mathcal{Y})$ we have

$$\int_h^T \iint_{\Omega\times\mathcal{Y}} u w_{\overline{h}}\, dydx\, dt = \int_0^{T-h} \iint_{\Omega\times\mathcal{Y}} u_h w\, dydx\, dt. \tag{3.6}$$

Thus, backward Steklov averaging can be considered to be the adjoint operator of forward Steklov averaging for appropriately truncated functions.

Proposition §A.3.1: **Steklov averaging commutes with spatial derivatives and periodic unfolding**

Let $u \in L^p(I;W_0^{1,p}(\Omega\times\mathcal{Y}))$ and $w \in L^p(\Omega_T))$ then we have

$$\nabla_{(x,y)}(u_h) = \left(\nabla_{(x,y)} u\right)_h \tag{3.7}$$

$$\mathcal{T}_\varepsilon^*(u_h) = \left(\mathcal{T}_\varepsilon^*(u)\right)_h \tag{3.8}$$

Proof. For the first statement let $C^1(\bar{I};C_0^1(\Omega\times\mathcal{Y})) \ni u_n \longrightarrow u$ in $L^p(I;W_0^{1,p}(\Omega\times\mathcal{Y}))$. The second statement is trivial by definition. ■

Corollary §A.3.1: **Steklov averaging for weak two-scale limits**

Let $L^p(\Omega_T) \ni u_\varepsilon \xrightarrow{2w} u_0 \in L^p(\Omega_T\times\mathcal{Y})$ in $L^p(I\times\mathbb{R}^d\times\mathcal{Y})$ then we have the following commutativity statement for $0 < h \leq h_0 \leq T$ and $\varphi \in L^{p'}(\Omega_T\times\mathcal{Y})$ for forward Steklov averaging, while similar backward counterparts hold, too.

$$(3.9)\qquad \int_0^T \iint_{\Omega\times\mathcal{Y}} u_0\varphi\, dydxdt = \lim_{\varepsilon\to0}\lim_{h\to0} \int_0^T \iint_{\mathbb{R}^d\times\mathcal{Y}} \left(\mathcal{T}_\varepsilon^*(u_\varepsilon)\right)_h \varphi\, dydxdt$$

$$(3.10)\qquad \int_{h_0}^T \iint_{\Omega\times\mathcal{Y}} u_0\varphi\, dydxdt = \lim_{h\to0}\lim_{\varepsilon\to0} \int_0^{T-h_0} \iint_{\mathbb{R}^d\times\mathcal{Y}} \left(\mathcal{T}_\varepsilon^*(u_\varepsilon)\right)_h \varphi\, dydxdt$$

$$(3.11)\qquad \int_{h_0}^T \iint_{\Omega\times\mathcal{Y}} u_0\varphi\, dydxdt = \begin{cases} \lim\limits_{\varepsilon\to0}\lim\limits_{h\to0} \displaystyle\int_{h_0}^T \iint_{\mathbb{R}^d\times\mathcal{Y}} \left(\mathcal{T}_\varepsilon^*(u_\varepsilon)\right)_h \varphi\, dydxdt \\ \lim\limits_{h\to0}\lim\limits_{\varepsilon\to0} \displaystyle\int_0^{T-h_0} \iint_{\mathbb{R}^d\times\mathcal{Y}} \left(\mathcal{T}_\varepsilon^*(u_\varepsilon)\right)_h \varphi\, dydxdt. \end{cases}$$

Proof. Since $\lim_{h\to0} \| \left(\mathcal{T}_\varepsilon^*(u_\varepsilon)\right)_h - \mathcal{T}_\varepsilon^*(u_\varepsilon)\|_{L^p(I\times\mathbb{R}^d\times\mathcal{Y})} = 0$, the first integral is tantamount to weak two-scale convergence. For the middle integral we use lemma §A.3.5 to obtain $\int_{h_0}^T \iint_{\mathbb{R}^d\times\mathcal{Y}} \mathcal{T}_\varepsilon^*(u_\varepsilon)\varphi_{\overline{h}}\, dxdydt$ with $\varphi_{\overline{h}} \in L^{p'}(\Omega_T\times\mathcal{Y})$ so that for $\varepsilon\to0$ the expression converges to $\int_{h_0}^T \iint_{I\times\mathbb{R}^d\times\mathcal{Y}} u_0\varphi_{\overline{h}}\, dxdydt$. Then, applying $h\to0$ yields the claim thanks to lemma §A.3.2. The final identity follows from combining the first two identities. ■

§A.4. Uniform structures and Cauchy completion

Here, we gather the concepts of Cauchy completion and uniform structures used in section §14.2 which are entirely classical, though. We refer to [Kel55; Bou64] as classical resources, to [Sch97] for a analysis-themed presentation and to [Low89; MP09] for specialised literature. Throughout the literature uniform spaces and completion concepts are handled in varying formulations, for instance, uniform spaces can be defined via entourages or via families of pseudo-metrics. Likewise, completion procedures may be based on Cauchy filters or equivalently, on Cauchy nets.

§A.4.1. Uniform spaces, gauges and completion

(P, d_P) is a pseudo-metric space if P is a set together with a pseudo-metric $d_P : P\times P \longrightarrow [0,\infty)$ fulfilling $\forall x,y,z\in P : (x=y) \Rightarrow (d(x,y)=0) \ \wedge\ d(x,y) \le d(x,z)+d(z,y) \ \wedge\ d(x,y)=d(y,x)$. A metric is pseudo-metric for which $\forall x,y\in P : (d_P(x,y)=0) \Longrightarrow (x=y)$ holds.

Next, we introduce uniform spaces. Large parts of the literature employ vicinities and entourages to do so, we prefer an equivalent definition invoking pseudo-metric spaces. A uniform space $(U, \mathcal{D}_U)$, also known as gauge space, is given by a set U and a gauge $\mathcal{D}_U$, also known as uniform structure. More specifically, $\mathcal{D}_U$ is family of pseudo-metrics on U, such that (U, d_U) is a pseudo-metric space for all $d_U \in \mathcal{D}_U$. Gauges and topologies are different concepts but they are naturally linked by defining convergence of nets or sequences via convergence with respect to every pseudo-metric of the gauge. Such topologies are named uniform topologies.

We use nets and gauges to define all necessary completeness notions. By a net in a set S, we mean elements $(n_\alpha)_{\alpha\in A}$ indexed by a given directed set $(A, \triangleleft)$ such that $\forall\alpha\in A : n_\alpha \in S$ holds.

Let $(U, \mathcal{D}_U)$ be a uniform space with $\mathcal{D} = (d_j)_{j \in J}$ and an index set J. A net $(u_\alpha)_{\alpha \in A}$ is a Cauchy net if $\forall j \in J, \delta > 0 : \exists \alpha_j^\delta \in A$ such that

$$\forall \alpha_0, \alpha_1 \in A : \alpha_j^\delta \lhd \alpha_0, \alpha_1 \Longrightarrow d_j(u_{\alpha_0}, u_{\alpha_1}) < \delta. \tag{4.1}$$

$(U, \mathcal{D}_U)$ is complete if every Cauchy net converges in U. A uniform space $(V, \mathcal{D}_V)$ is the (uniform) completion of U if U is uniformly homeomorphic to a dense subset of V. A uniform homeomorphism is a homeomorphism which, together with its inverse, is uniformly continuous with respect to all pseudo-metrics of all gauges.

In general, completions of pseudo-metric spaces are more convenient since a pseudo-metric space is complete as a uniform space if and only if every Cauchy sequence converges in P (a Cauchy sequence is a sequence which is a Cauchy net).

Concerning existence, every uniform space has a completion. More explicitly, to every uniform space $(U, \mathcal{D}_U)$ there exists a complete uniform space $(V, \mathcal{D}_V)$ containing U as a subset such that restricting the elements of $\mathcal{D}_V$ to U yields the elements of $\mathcal{D}_U$. In addition, if the uniform topology of $(U, \mathcal{D}_U)$ is Hausdorff, so is the uniform topology of its completion. For a full proof we refer to [Sch97, Thm 19.36].

§A.4.2. Completeness of topological vector spaces

Not every topological space is a uniform space but for topological groups and topological vector spaces the situation is quite convenient, see [Sch97, Thm 26.29]: given a topological vector space, there exists a non-unique gauge whose uniform topology coincides with the given topology. In addition, all such gauges are equivalent in the sense that their Cauchy nets coincide.

As a consequence, any topological vector space can be considered as a uniform space so that corresponding notions from uniform spaces – and completeness in particular – carry over to topological vector spaces sensibly. For instance, weak topologies also correspond to a uniform structure: let (E, τ) be a given topological vector space over $\mathbb{F} = \mathbb{R}, \mathbb{C}$. One can verify that the weak topology $\sigma(E; E')$ induced by E' is the uniform topology given by the following gauge of semi-norms

$$\mathcal{D}_{\sigma(E,E')} := \left\{ d_f(x_1, x_2) := |f(x_1 - x_2)|_{\mathbb{F}} : f \in E' \right\}. \tag{4.2}$$

Again, such gauges are not unique. For instance, excluding trivial dual spaces, restricting the gauge given in (4.2) to $f \in E'$ with $\|f\|_{E'} = 1$ yields another gauge whose uniform topology is the weak topology.

Returning to completeness, it turns out that completeness alone is not the only notion one is after. Referring to weak topologies, one rather aims at a slightly weaker notion. A topological vector space (E, τ) is termed sequentially complete if every Cauchy sequence in E converges in E and quasi-complete if every bounded and closed set $A \subset E$ is complete as a uniform space. Equivalently, every bounded Cauchy net converges in E (Cauchy nets are not necessarily bounded).

Of course, completeness implies quasi-completeness which in turn implies sequential completeness; the reverse is true if a pseudo-metric space is at hand. Quasi-completeness can be derived from completeness by the help of quasi-closed sets: a subset of a topological vector space is quasi-closed if it contains all closure points of its bounded subsets. The quasi-closure of a subset A is the intersection of all quasi-closed sets of containing A. The quasi-completion of a topological vector space E is its quasi-closure in its completion. Sequential closures can be defined analogously, outside pseudo-metric spaces, the sequential closure of a set is not necessarily sequentially closed itself, though.

For illustration, consider a Banach space $(E, \|\cdot\|_E)$ and its continuous dual $(E', \|\cdot\|_{E'})$. Both spaces are is complete with respect to the uniform structure induced by the norm. In contrast, let $\mathcal{D}_{w*}$ denote the gauge given by all pseudo-metrics $d(\ell_1, \ell_2) := \sup\{|\ell_1(x) - \ell_2(x)| : x \in E \text{ with } \|x\|_E = 1\}$, whose uniform topology is the weak-* topology $\sigma(E', E)$. It is a classical result that $(E, \mathcal{D}_{w*})$ is quasi-complete, but it is complete if and only if $\dim(E') < \infty$.

Finally, the uniform space completion of a topological vector space (E, τ) is a topological vector space, too. In addition, every pair of such completions is topologically isomorphic, if τ is a Hausdorff topology. In case of a Hausdorff topology, the quasi-completions are isomorphic, too. Thus, every topological vector space that is complete (resp. quasi-complete) is topologically isomorphic to its completion (resp. quasi-completion).

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