A cocycle model for the equivariant Chern character and differential equivariant K-theory

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Abstract

For a finite group G, a G-vector bundle is the equivariant analogue of an ordinary vector bundle. By applying the usual Grothendieck group construction to the abelian monoid of isomorphism classes of G-vector bundles with direct sum, one arrives at an equivariant version of the K-theory functor, which was already studied by Atiyah and Segal. With the correct setup, there is also a theory of characteristic classes, and an equivariant Chern character homomorphism $\mathrm{ch}\colon K_G^* \to H_G^*$, which, just like the ordinary Chern character, is a rational isomorphism. Additionally, one has a Chern–Weil homomorphism, leading to a differential refinement of the equivariant characteristic classes.

We construct models of the classifying spaces of even and odd equivariant K-theory that are infinite-dimensional Banach manifolds. These are given by restricted versions of the usual Grassmannian and the unitary group of an infinite-dimensional Hilbert space. We show that they carry natural odd and even Chern forms that can be adapted to give (delocalized) equivariant differential forms that refine the universal equivariant Chern character.

Using this refinement, we construct a model of differential equivariant K-theory based on smooth classifying spaces, together with natural addition and inversion operations given by geometric operations directly on these spaces. We then show that the abelian group structure on $\hat{K}_G^*$ is induced by these operations. The regularity and explicitness of these maps allows us to work completely on the level of classifying spaces, and we do not require a compactness assumption on our manifolds that is present in many other descriptions of differential refinements. We therefore define the theory on the full category of smooth G-manifolds.

One of the key features of K-theory is that one can, at least in the compact case, find vector bundles as geometric representatives for any class. This also remains true in the differential refinement, where one has to consider vector bundles with the additional datum of a connection. We investigate the possibility of such a cycle description in the equivariant setting and find that a key role is played by an equivariant version of the Venice Lemma by J. Simons. We show that our model is the unique differential equivariant extension that admits both an odd and an even degree differential cycle map.

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Contents

CHAPTER I

Introduction

1. Differential K-theory

For a given cohomology theory E restricted to the category of smooth manifolds, a differential refinement $\hat{E}$ provides a theory which makes use of additional geometric information. In the case of topological K-theory, if a cycle is given by a vector bundle, then a lift to a class in $\hat{K}$ would be defined by the additional data of a connection. This connection will refine the Chern character of this bundle, in general only well-defined as a cohomology class, to a differential form. There is a set of axioms analogous to the Eilenberg–Steenrod axioms for cohomology that characterizes such extensions, given by Bunke and Schick [BS10, Definition 1.1]. For any smooth manifold M, we have a diagram of abelian groups

$$\begin{tikzcd} \hat{K}^*(M) \arrow[r, "I"] \arrow[d, "R"] & K^*(M) \arrow[d, "\mathrm{ch}"] \\ \Omega^*_{d=0}(M) \arrow[r, "\mathrm{Rham}"] & H^*(M), \end{tikzcd} \tag{1}$$

where I and R are certain functors that must come with any definition of $\hat{K}$, ch is the topological Chern character, and at the bottom, we have the de Rham map. Although this is not a cartesian diagram, the slogan still is that we combine K-theory and forms in a (homotopy theoretic) fiber product

$$\text{“Differential } K\text{-theory} = K\text{-theory} \times_{\text{de Rham}} \text{Forms”.}$$

A construction of such functors (for any generalized cohomology theory E) was given by Hopkins and Singer [HS05, Definition 4.34], and from the modern viewpoint they can be described quite efficiently in a very general setting via sheaves of spectra [BNV16].

In order to understand and compute these abstractly defined refinements, it is however important to have concrete models. Differential K-theory is an especially prominent example of this, since it appears in mathematical as well as in physics discussions, often in the form of a geometric model. In the case of K-theory, the differential version is $\mathbb{Z}_2$-graded and the even and odd part were developed independently. On the category of compact manifolds, a variety of descriptions are available. Simons and Sullivan [SS10, §3] show that even differential K-theory is defined by structured vector bundles, i.e. vector bundles with connection, equipped with a suitable equivalence relation. This picture was completed by Tradler, Wilson and Zeinalian [TWZ13, Theorem 5.7] by giving a geometric description of odd differential K-theory via operator theory. Here, classes are represented by maps into the stable unitary group $\mathrm{U} = \bigcup \mathrm{U}(n)$, where the addition is induced by a block sum operation. Later, via the Caloron correspondence, an interpretation of their model via Ω-bundles was developed in [Hek+15, Theorem 3.17].

More recently, another approach has been implemented in [TWZ16, Theorem 4.25]. The authors discuss the question of representability of the $\hat{K}$-functor. As any cohomology theory, topological K-theory is represented by homotopy classes of maps into the corresponding spectrum, i.e.

$$K^0(M) \cong [M, B\mathrm{U} \times \mathbb{Z}], \qquad K^1(M) \cong [M, \mathrm{U}].$$

For compact manifolds, this agrees with the usual description as the Grothendieck group of the monoid of complex isomorphism classes of vector bundles. For non-compact manifolds, we can take this as a definition (the vector bundle definition would not yield a cohomology theory). Since only the homotopy type of these spaces is relevant in this description, one can find good models for $B\mathrm{U} \times \mathbb{Z}$ and U, which carry the additional information needed to define a differential K-theory class from a map into them. In the end, the authors describe even and odd differential K-theory via smooth maps into explicit classifying spaces, equipped with differential forms that represent the universal Chern character. These universal forms are defined on approximations of their spaces via compact smooth manifolds (the usual finite-dimensional Grassmannians and unitary groups). Therefore, this method relies heavily on the fact that a compact smooth manifold will always map to a finite stage in the filtration. The problem with working directly on the spaces $B\mathrm{U} \times \mathbb{Z}$ and U is of course their infinite-dimensional nature. As colimits of finite-dimensional smooth manifolds, they are Fréchet manifolds, and as such, it is harder to, for example, talk about differential forms on them. In this thesis, we generalize this classifying space based approach for the equivariant setting, i.e. we have a finite group G acting on our manifold and ask for a theory that enriches the equivariant K-theory functor of Atiyah–Segal [Seg68, §2].

2. Cocycles for the equivariant Chern character

The first task at hand is to ask the right question. Going back to Diagram 1, the main thing that one needs to come up with is an equivariant generalization of the Chern character map. There are many constructions available. Most prominent is maybe the Borel–Chern character, which applies to a G-vector bundle E the Borel construction and then takes the ordinary Chern character of the vector bundle $EG \times_G E \to EG \times_G M$, ending up with an element in the Borel equivariant cohomology of M. However, one of the most important properties of the non-equivariant Chern character is that it is a rational isomorphism. Alas, this property is not shared by the Borel–Chern character, which is rationally surjective, but not injective. Therefore, defining a differential refinement using the Borel–Chern character would miss important geometric information. This problem is repaired in the delocalized equivariant cohomology of Baum, Brylinski and MacPherson

[BBM85], which, in addition to the Borel cohomology of the G-manifold (corresponding to the fixed point set for the identity element), also takes into account the topology of the fixed point sets for all the other elements $g \in G$. We define

$$H^0_G(M) = \left(\bigoplus_{g\in G} \prod_{k\in\mathbb{N}} H^{2k}(M^g;\mathbb{C}) \right)^G \quad \text{and}$$

$$H^1_G(M) = \left(\bigoplus_{g\in G} \prod_{k\in\mathbb{N}} H^{2k+1}(M^g;\mathbb{C}) \right)^G . \tag{2}$$

The action is induced by the space level action where $h \in G$ sends $x \in M^g$ to $hx \in M^{hgh^{-1}}$. This makes it possible to have a rationally injective delocalized equivariant Chern character $\mathrm{ch}\colon K^*_G \to H^*_G$. There is a de Rham model for this theory that fits in the bottom left corner of Diagram 1, and we can ask for a theory $\hat{K}^*_G$ that fits in the upper left corner.

We want to work with manifold models of classifying spaces. Our new ingredient is the use of operator theory to perform certain norm completions and slightly enlarge the spaces used in [TWZ16] described above, in order to improve their regularity. This results in the well-behaved Banach manifolds $\mathrm{Gr}_{\mathrm{res}}$ and U^1, which we then equip with natural differential forms in the classical sense. These constructions are closely related to the identification of $B\mathrm{U} \times \mathbb{Z}$ with the space of Fredholm operators via a generalized index map, as developed by Atiyah and Jänich. The idea that leads to this operator-theoretic approach can be described as follows: While K-theory is the study of stable vector bundles, it can also be interpreted as studying Hilbert space bundles with a reduction of the structure group to the stable general linear group $\mathrm{GL} \subset \mathrm{GL}(\mathscr{H})$, sitting in the (contractible) full general linear group of $\mathscr{H}$. By Palais' tame approximation theorem [Pal65, Theorem B] this group is homotopy equivalent via its natural inclusion to the group of operators which have a determinant, denoted by GL^1. Therefore we might as well study the space $B\mathrm{GL}^1$. There happens to be a model of the universal smooth principal GL^1-fiber bundle, which has appeared in the study of loop groups [PS88, Sec. 7.5] and also in applications in physics in the form of fermionic second quantization (for a mathematical treatment see [Wur01, Sec. V.2]). This bundle carries a connection, which gives rise to a universal Chern character differential form via the usual Chern–Weil formula. The degree 2-part of this form is known in the physics literature as the Schwinger cocycle, where the discussion usually focuses on line bundles. We prove that we can get representatives also for the higher dimensional parts of the Chern character (as observed by Freed in [Fre88, Theorem 3.9]), and along the way, we review some constructions in the world of restricted unitary groups, Grassmannians and Stiefel manifolds, which we could not find a good reference for. Thus, while the authors in [TWZ16] ultimately work with Chen spaces as models for the spaces $B\mathrm{U} \times \mathbb{Z}$ and U, our Banach manifolds allow us to do certain calculations directly in the universal example, without considering test manifolds. One immediate advantage of this approach is that our model will need no compactness assumption on the manifolds considered. Indeed, without considering any G-action, the program that we will line out now has been carried out in the paper [Sch19] by the author of this thesis.

One of the main ingredients in the construction of differential extensions is a cocycle refinement of the Chern character, or in our case, the equivariant Chern character. In the non-equivariant setting, Chern–Weil theory gives a way to produce differential form representatives from a given connection. Usually, this only works on finite-dimensional manifolds, and as such is not useful to construct universal cocycles directly on the classifying spaces. However, there is an infinite-dimensional Stiefel bundle $\mathrm{St}_{\mathrm{res}} \to \mathrm{Gr}_{\mathrm{res}}$, directly adapted to this situation, that makes Chern–Weil theory work in the infinite-dimensional case. This was used in loc.cit. in order to construct such a universal cocycle. In the odd case, it was shown in [Get93, Definition 1.1] that one can take traces of the Maurer–Cartan form on the stable unitary group as the odd Chern form, although this result might possibly be much older. On finite-dimensional G-manifolds, there is a version of Chern–Weil theory that is compatible with the action. Given an invariant connection, it produces a differential form representing the even equivariant Chern character, just as before. In the odd case, we still have the Maurer–Cartan form on U^1. One could therefore hope that one can use similar ideas in order to get representatives for the (delocalized) equivariant Chern character using similar techniques. Indeed, we spend the first half of this thesis on setting up the correct universal situation in order to carry out this program. The key idea is to replace the generic infinite-dimensional complex separable Hilbert space $\mathscr{H}$ underlying all the operator-theoretic constructions by the G-universe $\mathscr{H} \otimes L^2(G)$, with the action induced from the regular representation. This representation contains all the irreducible representations infinitely often and serves as a universal example. Our efforts in the end allow us to define the Chern character of a smooth map into the classifying space as simply the pullback of these delocalized forms. [1]

Theorem 2.1. *There is a curvature two form $\Omega \in \Omega^2(\mathrm{Gr}_{\mathrm{res}}; L^1)$ with values in the trace class operators L^1 that gives rise to differential forms representing the universal Chern character*

$$\mathrm{ch}_G = \bigoplus_{g \in G} \mathrm{tr}\left(g \exp\left(\frac{i}{2\pi}\Omega_g\right)\right).$$

Similarly, the trace class-valued Maurer–Cartan form $\omega \in \Omega^1(\mathrm{U}^1; L^1)$ gives rise to representatives for the odd universal Chern character

$$\mathrm{ch}_G = \bigoplus_{g \in G} \sum_{k \geq 1} \left(\frac{i}{2\pi}\right)^k \frac{(-1)^{k-1}(k-1)!}{(2k-1)!} \mathrm{tr}\left(g\,(\omega_g)^{2k-1}\right).$$

[1]Note that the statement of the theorem here is a little dubious for the degree 0 part of the Chern character, because of the infinite dimensionality. For a more precise formulation, we refer to Theorem 9.4.

Here, Ω_g and ω_g denote the restriction of the corresponding forms to the submanifold $M^g \subset M$. Now, in order to construct $\hat{K}_G^*$ as a quotient of the set of equivariant smooth maps

$$\mathrm{Map}^G_{\mathrm{Smooth}}(M, \mathrm{Gr}_{\mathrm{res}}) \qquad \text{and} \qquad \mathrm{Map}^G_{\mathrm{Smooth}}(M, \mathrm{U}^1),$$

we also need to implement an abelian group structure, induced by maps $\mathrm{Gr}_{\mathrm{res}} \times \mathrm{Gr}_{\mathrm{res}} \to \mathrm{Gr}_{\mathrm{res}}$ and $\mathrm{U}^1 \times \mathrm{U}^1 \to \mathrm{U}^1$. Topologically, there are many ways to give such maps, and they all differ only up to homotopy. When we want to give a differential refinement, a little more care is needed: Differential forms are only preserved by homotopic maps up to exact forms, and therefore, we have to make more careful choices. It is important that the maps we define are compatible with the additional differential structure. For example, ideally, we would like the Chern character to be a homomorphism for the addition, already on differential form level. We show that it is indeed possible to construct such an explicit operation, called the block sum $\boxplus$, on both $\mathrm{Gr}_{\mathrm{res}}$ and U^1. This operation is geometric in the sense that it on the nose corresponds to the direct sum of vector bundles on K-theory cycles. In the end, we present a geometric K-theory spectrum consisting of Banach manifolds instead of just spaces. There are smooth maps

$$h_{\mathrm{odd}} \colon \Omega\mathrm{Gr}_{\mathrm{res}} \to \mathrm{U}^1$$
$$h_{\mathrm{even}} \colon \Omega\mathrm{U}^1 \to \mathrm{Gr}_{\mathrm{res}},$$

which are equivariant homotopy equivalences. These maps are given by holonomy in the universal bundle in the even case, and a multiplication operator map already defined by Pressley and Segal [PS88, Sec. 6.3]. We then have the following theorem.

Theorem 2.2. *For any $n \in \mathbb{Z}$, let $g_{2n} = g_{\mathrm{even}} \colon \mathrm{Gr}_{\mathrm{res}} \to \Omega\mathrm{U}^1$ and $g_{2n+1} = g_{\mathrm{odd}} \colon \mathrm{U}^1 \to \Omega\mathrm{Gr}_{\mathrm{res}}$ be G-homotopy inverses to the G-homotopy equivalences h_{odd} and h_{even}. Then, the sequence of pointed G-spaces and pointed G-maps $(E_n, h_n)_{n\in\mathbb{Z}}$ given by*

$$E_{2n} = \mathrm{Gr}_{\mathrm{res}} \quad \text{and} \quad E_{2n+1} = \mathrm{U}^1$$
$$g_{2n} = g_{\mathrm{even}} \quad \text{and} \quad g_{2n+1} = g_{\mathrm{odd}}$$

defines a (naive) G-Ω-spectrum that represents equivariant K-theory. Furthermore, addition in equivariant K-theory is implemented by the block sum operation on both $\mathrm{Gr}_{\mathrm{res}}$ and U^1.

3. Differential equivariant K-theory

Given a classifying map $f \colon M \to \mathrm{Gr}_{\mathrm{res}}$ or $f \colon M \to \mathrm{U}^1$, the setup so far allows us to extract an equivariant Chern form $\mathrm{Ch}_G(f) = f^*\mathrm{ch}_G$, and an equivariant K-theory class $I(f) = [f]$. Of course, the set of all such smooth maps is way too big to be useful, and the question is, which equivalence relation we want to impose. We need it to be strictly finer than homotopy, in order to assure that both maps Ch_G and I are well-defined. Recall that the Chern form $\mathrm{Ch}([f])$ of a homotopy class is only well-defined up to exact forms, which is not good enough. Without any group actions, the solution is to consider so called Chern–Simons homotopies, which were defined in [TWZ16, Definition 3.4.], using the idea underlying already the equivalence relation on the structured vector bundles of [SS10]. Consider a smooth homotopy f_t between the maps $f_0, f_1 \colon M \to \mathrm{Gr}_{\mathrm{res}}$ or $f_0, f_1 \colon M \to \mathrm{U}^1$. Then, we can pull back the universal Chern character via the homotopy and integrate out the fiber I. The result is the Chern–Simons form

$$\mathrm{CS}(f_t) = \int_I \mathrm{Ch}(f_t),$$

which satisfies the fundamental equality

$$\mathrm{dCS}(f_t) = \mathrm{Ch}(f_1) - \mathrm{Ch}(f_0).$$

A Chern–Simons homotopy is now a homotopy that has an exact Chern–Simons form. Two maps are CS-homotopic if they can be connected by a Chern–Simons homotopy. If we take a CS-equivalence class $[f]_{\mathrm{CS}}$, then the Chern form of the equivalence class is well-defined. We find equivariant analogues the Chern–Simons form and define equivariant CS_G-equivalence. This discussion leads to the following definition.

Definition 3.1. Define the set-valued contravariant functors on (possibly non-compact) smooth manifolds

$$\begin{aligned} \hat{K}^0_G(M) &= \mathrm{Map}^G_{\mathrm{Smooth}}(M, \mathrm{Gr}_{\mathrm{res}}) \times \Omega^1_G(M)/\sim \quad \text{and} \\ \hat{K}^1_G(M) &= \mathrm{Map}^G_{\mathrm{Smooth}}(M, \mathrm{U}^1) \times \Omega^0_G(M)/\sim, \end{aligned}$$

assigning to M the set of equivalence classes of tupels (f, ω) of a classifying map together with a delocalized differential form. Note that the grading on the differential forms here is a $\mathbb{Z}_2$-grading, as in the definition of delocalized cohomology (Equation 2). The equivalence relation is induced by two rules: First, we identify

$$(f_1, \omega_1) \sim (f_0, \omega_0)$$

if there is a smooth G-homotopy f_t from f_0 to f_1, such that

$$\mathrm{CS}_G(f_t) = \omega_1 - \omega_0 + \text{exact}.$$

Secondly, we identify $(f, \omega) \sim (f \boxplus 1, \omega)$ for any tupel (f, ω), where 1 is the constant map to the basepoint.

We then go on to prove that the set $\hat{K}^*_G$ admits an abelian group structure, induced by the aforementioned block sum operation $\boxplus$, and the usual addition of differential forms. Inverting an element in the group corresponds to taking the operator adjoint in the odd case, and flipping the polarization on the underlying polarized Hilbert space in the even case, while simultaneously reversing the sign of the differential form. We then prove the following theorem.

Theorem 3.2. *On the category of possibly non-compact smooth G-manifolds, the abelian group-valued functors $\hat{K}^0_G$ and $\hat{K}^1_G$, together with the integration, curvature and action maps*

$$I([(f,\omega)]) = [f], \quad R([(f,\omega)]) = \mathrm{Ch}_G(f) + \mathrm{d}_G\omega, \quad a(\omega) = [(1,\omega)]$$

define a differential extension of equivariant K-theory.

We also produce a differential cycle map, which assigns a differential equivariant K-theory class to a G-vector bundle with invariant connection. This works as follows: By the equivariant Narasimhan–Ramanan Theorem [Sch80, §3], invariant connections correspond to equivariant classifying maps into some Grassmannian, up to connection preserving homotopies. From this, we can produce a map into the restricted Grassmannian $\mathrm{Gr}_{\mathrm{res}}$ and therefore obtain a $\hat{K}^0_G$-class. There is also an odd version of this, which is almost tautological in our model, since the natural geometric cycles in the odd case are just maps into the stable unitary group. The existence of such differential cycle maps is one of the crucial differences to the abstract spectrum-based construction given by Hopkins and Singer.

The result in the theorem is still slightly unsatisfactory for the following two reasons. First, it is not clear to us if the second step in the equivalence relation for $\hat{K}^*_G$ is actually needed. If one considers compact G-manifolds, we have the geometric interpretation of K-theory via G-vector bundles. On the classifying space level this translates into the following statement: There are dense submanifolds which are just the colimits of finite-dimensional Grassmannians and unitary groups. The inclusion of these submanifolds can be shown to be an equivariant homotopy equivalence. If M is compact, we can therefore assume that, up to homotopy, any map into these spaces has its image contained in some finite step in this filtration. It is also not hard to show that block sum with 1 on such a finite step is equivariantly homotopic to the identity, where the homotopy is additionally compatible with the universal Chern forms. Therefore, for any $f\colon M \to \mathrm{Gr}_{\mathrm{res}}$ or $f\colon M \to \mathrm{U}^1$, there is a CS_G-homotopy that shows

$$f \boxplus 1 \sim_{\mathrm{CS}_G} f,$$

rendering the stabilization step in the equivalence relation obsolete. This gives the slightly stronger Theorem 13.12, but just in the compact case. The non-compact case remains open.

Secondly, we would like to take the classifying map approach seriously and remove the additional differential form ω from our cycles. Incidentally, we can indeed define the $\hat{L}^*_G(M)$ groups, which are just the subgroups in $\hat{K}^*_G(M)$ of elements which admit a representative with differential form part 0. The big question is, whether this is actually already the full group $K^*_G(M)$. Classically, it was one of the achievements of Simons and Sullivan [SS10] to show that this is true in the non-equivariant case. The key lemma for this is the so called Venice Lemma, which is a statement about the surjectivity of the Chern–Simons map. Equivariantly, we can reduce our problem of removing the differential form to an equivariant version of this. We formulate the following conjecture.

Conjecture 3.3. *(Equivariant Venice Lemma) Let G be a finite group and M be a smooth G-manifold. Furthermore, let*

$$\omega \in \Omega^0_G(M) \qquad \text{or} \qquad \omega \in \Omega^1_G(M)$$

be a delocalized differential form in even or odd degree. Then, ω is up to exact forms the Chern–Simons form of a homotopy $f\colon M \times I \to \mathrm{Gr}_{\mathrm{res}}$ or $f\colon M \times I \to \mathrm{U}^1$. Additionally, f can be chosen to restrict to the constant map to the basepoint at time 0.

Unfortunately, we did not succeed in proving this statement in general. Since the rest of the setup goes through, if the conjecture were true, one would indeed get a description of $\hat{K}^*_G$ via smooth classifying maps.

There have been attempts by other authors at defining a model for differential equivariant K-theory. In the non-equivariant case, there is a strong uniqueness theorem [BS10, Theorem 1.6 and Theorem 1.7] that automatically identifies different models for differential K-theory. At the moment, no such thing is known in the equivariant case. We investigate the question of uniqueness in the presence of a differential cycle map.

Recall that equivariant K-theory admits a unique topological cycle map "cycl_G" in the sense that there are assignments that take a G-vector bundle (which we can always equip with an invariant connection) in the even, or a smooth map to the stable unitary group in the odd case, and give an equivariant K-theory class. These assignments of course only depend on the isomorphism type of the bundle, or the G-homotopy type of the map respectively. A differential lift $\widehat{\mathrm{cycl}}_G$ of the cycle map is a compatible such assignment (in the sense of Definition13.2) that makes use of the additional information which is usually lost when passing to homotopy or isomorphism classes. It therefore produces even or odd differential equivariant K-theory classes from the input data of a G-vector bundle with invariant connection, or a G-map to the stable unitary group, respectively. Differential cycle maps are invaluable for producing classes in $\hat{K}_G$, and one often tries to exhibit an unknown class explicitly as the image of some particular geometric cycle. We prove the following uniqueness theorem.

Theorem 3.4. *Let $(\hat{M}^*_G, I', a', R')$ be a differential extension of equivariant K-theory on the category of compact G-manifolds that admits a differential lift $\widehat{\mathrm{cycl}}'_G$ of the even/odd cycle map. Then, there is an isomorphism of the even/odd part of the differential extensions Φ to our theory $\hat{K}^*_G$, defined via*

$$x = \widehat{\mathrm{cycl}}_G(E, \nabla) + a(\omega) \mapsto \widehat{\mathrm{cycl}}'_G(E, \nabla) + a'(\omega).$$

In particular, our model is the unique one that supports differential lifts of both the even and the odd cycle map.

We furthermore discuss two other prominent models in the literature. The paper [Ort09] by Ortiz uses spaces of Fredholm operators as manifold models for the classifying spaces, but relies on abstract choices for all the additional structure. The main reason for this is that there is no easy way to write down a cocycle representative for the Chern character in these spaces. In particular, the abstract choices Ortiz makes are not compatible with the de Rham representatives for the Chern character, coming from an actual vector bundle with connection, and there is no cycle map. Since both ours and his model are based on classifying spaces, one would hope that one can produce a map between these spaces that induces a morphism of differential extensions, but there are some technical problems that prevent this. In the end, we use an alternative description of Ortiz' model that admits a cycle map that is compatible with the maps I and a, but not with the Chern character. We conclude that the even degree groups of Ortiz are isomorphic to ours, but not by a transformation of differential extensions which respects the Chern character.

On the other hand, Bunke and Schick define differential orbifold K-theory in [BS13, Definition 2.19], using geometric families. This approach involves more analysis and less homotopy theory. Their paper focusses mostly on the case of a compact G-manifold, allowing any compact Lie group G to act with finite stabilizers. They discuss a procedure that produces from a G-vector bundle with invariant connection something that they call a geometric family. Since this procedure is compatible with the Chern character, it gives an even cycle map, and we can identify the even part of their model with ours by the above theorem. Unfortunately, we do not know how to produce an odd geometric family from a G-map $f \in \mathscr{C}^\infty(M, U)$, and therefore, we do not have a comparison map in the odd case.

Overview. This thesis is organized as follows. In Chapter II, Section 4 and 5 review the construction of the restricted Grassmannian and the unitary group of operators that have a determinant, which will give the even respectively odd model for differential K-theory. The universal Chern class in the odd case is induced by the Maurer–Cartan form of U^1. In the even case, we review the construction of a certain universal bundle over $\mathrm{Gr}_{\mathrm{res}}$, the curvature of which gives rise to invariant representatives of the Chern character via Chern–Weil theory. In Section 6, we equip these spaces with an H-space structure. The key difference to the purely homotopy-theoretical approach is that we have to choose these structures in such a way that they are compatible with the Chern and Chern–Simons forms. For example, even though it induces addition in K-theory, operator multiplication on the unitary group will not work as an addition in $\hat{K}^1$, since it will not make the Chern character map into a monoid morphism on the level of differential forms. The content of this chapter contains no group actions, and as such was already discussed in the non-equivariant model given in [Sch19, Section 2-4].

Chapter III is dedicated to setting up for the equivariant case. Section 7 reviews a classical decomposition theorem for equivariant K-theory, that already appeared in [AS89, Theorem 2]. This splitting tells us exactly how we have to set up a cohomology theory that is a good target for an equivariant Chern character map. In Section 8 we proof that the classifying spaces $\mathrm{Gr}_{\mathrm{res}}$ and U^1 from Chapter II can be equipped with G-actions that make them into equivariant classifying spaces. We then prove the crucial Proposition 8.6, which describes the fixed point components of these spaces as simple products of the spaces itself, indexed by irreducible representations. Using the equivariant Whitehead theorem, we can leverage this result in order to show that many spaces that are homotopy equivalent are also G-homotopy equivalent in this setting. Section 9 then develops the needed cocycle representatives for the universal Chern character. We also define an equivariant version of the Chern–Simons form, and set up the corresponding CS_G-homotopy equivalence relation on maps into classifying spaces. At the end of the chapter, Section 10 then reviews geometric versions of the usual periodicity maps in the K-theory spectrum. The even to odd part is given by the holonomy map in the universal fibration, while the odd to even part is a certain multiplication operator map considered already by Pressley and Segal in their study of loop group representations [PS88, Sec. 6.3]. All of this easily lifts to the equivariant setting. It is interesting, though not a key fact for us, that this map can be used to implement the inverse of the Bott periodicity map as a smooth homomorphism of infinite-dimensional Lie groups. We also prove that the geometric spaces we use combine to a G-Ω-spectrum representing equivariant K-theory, where the addition is implemented by our block sum.

In Chapter IV, we finally define our model of differential equivariant K-theory. Section 11 puts together the results of the previous sections in order to prove that the previously discussed block sum and inversion operations equip the G-Chern–Simons equivalence classes of maps into the classifying spaces with an abelian group structure. This is achieved by finding explicit homotopies directly on our classifying spaces, which need to have vanishing Chern–Simons forms. The discussion here is simplified considerably by the simple cohomological structure of the relevant spaces and the availability of a de Rham theorem for the Banach manifolds in question[2]. Having built the abelian group structure on $\hat{K}^0_G$ and $\hat{K}^1_G$, what is left to do in Section 12 is to give the remaining structure maps for a differential extension and check the corresponding axioms. Here, the periodicity maps constructed in Section 10 play a key role. We can immediately compute the resulting groups for the special cases of a trivial or a free group action. We also see that the equivalence relation can be simplified in the compact case, where we do not need a stabilization step. This is done in Section 13. In Section 14, we discuss the need of the additional differential form in the cycles for $\hat{K}_G$ and give a comparison map between both versions. We also see that the Venice Lemma is implied by the surjectivity of the differential cycle map. In the end, Section 15 proves the Venice Lemma in the special case of an abelian group action and only in the lowest degree. We discuss the problems that arise in attempts to generalize the induction step of the non-equivariant proof.

[2]Since not all Banach manifolds admit a smooth partition of unity, this is not immediately obvious.

Chapter V deals with some applications of our model. In Section 16 we prove that our differential extension is the unique one that supports both an even and an odd differential cycle map. We also discuss comparison maps to the model given by Ortiz [Ort09], using spaces of Fredholm operators, and the geometric families model given by Bunke and Schick [BS13]. Additionally, if we have no group action, our model is isomorphic to the [TWZ16]-model of differential K-theory. This is the unique model that supports the additional structure of an S^1-integration map.

Section 17 then discusses some examples of differential K-theory classes. In Section 18, we give some open problems and ideas for further research.

CHAPTER II

Smooth infinite-dimensional classifying spaces

4. Universal representatives for the Chern character

Central to this work are the constructions of explicit smooth models for the classifying spaces of even and odd K-theory. Recall that the complex K-theory spectrum is two-periodic and consists of the spaces $B\mathrm{U} \times \mathbb{Z}$ in the even degrees and U in the odd degrees, where U is the stable unitary group, i.e. the colimit along the inclusions $\mathrm{U}(n) \hookrightarrow \mathrm{U}(n+1)$. In order to build a differential extension of K-theory, we define smooth models for both of these spaces which carry natural differential forms that represent the universal Chern character.

For the odd case, recall that on $\mathrm{U}(n)$, we have the Maurer–Cartan form ω_n. It is well-known that the real cohomology of $\mathrm{U}(n)$ is generated by the cohomology classes represented by the invariant differential forms

$$\left(\frac{i}{2\pi}\right)^k \frac{(-1)^{k-1}(k-1)!}{(2k-1)!} \mathrm{tr}\left(\omega_n^{2k-1}\right) \in \Omega^{2k-1}(\mathrm{U}(n)). \tag{3}$$

The normalizations we have chosen here are in order to make this agree with the transgression of the Chern character in the universal fibration (see Section 10). We can stabilize using the usual inclusion $\mathrm{U}(n) \hookrightarrow \mathrm{U}(n+1)$, but when one passes to the limit, one has to deal with the intricacies of infinite-dimensional manifolds. Our preferred way to do this is to work in the setting of Banach manifolds. The problem is that the Lie algebra of the stable unitary group U is supposed to consist of skew-adjoint operators of arbitrary finite rank. Since this is not a closed subspace of the bounded operators, there are some complications if we want to consider U as a smooth manifold. A simple fix is to instead go one step further and complete with respect to the trace norm

$$||X||_{L^1} = \mathrm{tr}|X| = \mathrm{tr}\sqrt{X^*X}.$$

This leads to the ideal L^1 of trace-class operators, and further to the Banach-Lie group U^1, which we now define.

Definition 4.1. Let $\mathscr{H}$ be a complex separable infinite-dimensional Hilbert space. Then U^1 is the subgroup of the unitaries of $\mathscr{H}$ given by

$$\mathrm{U}^1 = \left\{ P \in \mathrm{U}(\mathscr{H}) \mid P - 1 \in L^1 \right\},$$

with topology induced by the inclusion

$$\begin{aligned} \mathrm{U}^1 &\hookrightarrow L^1 \\ P &\mapsto P - 1. \end{aligned}$$

Palais [Pal65, Theorem B] showed that the inclusion of the stable unitary group $\mathrm{U} \hookrightarrow \mathrm{U}^1$ is a homotopy equivalence, but U^1 has better regularity, as it is actually a Banach-Lie group, locally modelled on the Banach space L^1. It is well-known that its cohomology is generated entirely by traces of odd powers of the Maurer–Cartan form ω, analogous to formula (3). It is therefore sensible to make the following definition.

Definition 4.2. The universal odd Chern character form $\mathrm{ch}_{\mathrm{odd}} \in \Omega^{\mathrm{odd}}(\mathrm{U}^1)$ is

$$\mathrm{ch}_{\mathrm{odd}} = \sum_{k\geq 1} \mathrm{ch}_{2k-1} = \sum_{k \geq 1} \left(\frac{i}{2\pi} \right)^k \frac{(-1)^{k-1}(k-1)!}{(2k-1)!} \mathrm{tr}\left(\omega^{2k-1} \right).$$

In order to find a good model for the even case, we recall the construction of universal connections. We will first review the situation for the finite-dimensional Grassmannians, and then spend the next chapter to generalize to the infinite-dimensional setting. As one would hope, these universal connections will yield well suited differential form representatives for the universal Chern character on our Grassmannian model of $B\mathrm{U} \times \mathbb{Z}$.

The Stiefel bundle over the Grassmannian manifold is given by

$$\mathrm{St}_{k,N} = \mathrm{U}(N)/I_k \times \mathrm{U}(N-k) \to \mathrm{U}(N)/\mathrm{U}(k) \times \mathrm{U}(N-k) = \mathrm{Gr}_{k,N}.$$

There is a map $S\colon \mathrm{St}_{k,N} \to M_{N\times k}$ which assigns to an element on the left a matrix $A \in M_{N\times k}$ which satisfies $A^*A = I_k$. The entries of A are just given by the first k columns of a representative of our left coset. Denote by S^* the map S followed by taking the adjoint matrix, and denote by $\mathrm{d}S$ the differential of S, which is an $M_{N\times k}$-valued differential form. Then, there is a Lie algebra-valued 1-form given by $S^*\mathrm{d}S$, and one can show that it takes values in the skew adjoint matrices and furthermore that it defines a connection for the given principal bundle. Narasimhan and Ramanan [NR61, Theorem 1] observed that the family of connections given by this construction for varying k and N have a universal property, meaning that every smooth principal bundle for a unitary group with a given connection comes from pulling back such a bundle and its respective connection by a smooth classifying map.

By Chern–Weil theory, one can define representatives for the Chern character by choosing a connection and considering traces of powers of its curvature. The curvature of $\omega = S^*\mathrm{d}S$ can be calculated explicitly as follows. An element in the tangent space at $\begin{pmatrix} I_k \\ 0 \end{pmatrix}$ of $\mathrm{St}_{k,N}$

is given by an $n \times n$ skew-hermitian block matrix $\begin{pmatrix} P & -Q^* \\ Q & 0 \end{pmatrix}$ where P is a skew hermitian $k \times k$ matrix and Q is an arbitrary $(N \times (N-k))$-matrix. The horizontal subspace is given by the kernel of ω, which corresponds to matrices which have $P = 0$. Recall that the curvature according to [KN96, Theorem 5.2] is defined to be the covariant derivative of the connection, so we have $\Omega = \mathrm{d}\omega \circ h$, where h is the horizontal projection. We calculate

$$\begin{aligned}
\Omega\left(\begin{pmatrix} P_1 & -Q_1^* \\ Q_1 & 0 \end{pmatrix}, \begin{pmatrix} P_2 & -Q_2^* \\ Q_2 & 0 \end{pmatrix}\right) &= \mathrm{d}\omega\left(\begin{pmatrix} 0 & -Q_1^* \\ Q_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -Q_2^* \\ Q_2 & 0 \end{pmatrix}\right) \\
&= -\omega\left[\begin{pmatrix} 0 & -Q_1^* \\ Q_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -Q_2^* \\ Q_2 & 0 \end{pmatrix}\right] \\
&= Q_1^* Q_2 - Q_2^* Q_1. \qquad (4)
\end{aligned}$$

Invariance under the transitive left $\mathrm{U}(N)$-action allows us to extend this form to any point in $\mathrm{St}_{k,N}$. The usual Chern–Weil theory then gives explicit differential forms on the Grassmannian after we take traces.

As in the odd case, these invariant forms stabilize under the inclusions $\mathrm{Gr}_{k,N} \hookrightarrow \mathrm{Gr}_{k,N+1}$, but again, when we want to work with a universal space, problems arise. The direct limit of the Grassmannians is not a Banach manifold, and so one needs more delicate tools to talk about connections and even differential forms on them. There is no obvious construction of a universal invariant connection for U-bundles in the stable case, and some of the problems that arise are discussed in [Fre88, Proposition 2.3]. However, there still exists an analogue to the finite-dimensional construction in the category of Banach manifolds, which we will review in the next section.

5. The restricted Stiefel manifold and the restricted Grassmannian

In the infinite-dimensional setting, for a Hilbert space $\mathscr{H}$, both the unitary group $\mathrm{U}(\mathscr{H})$ and the general linear group $\mathrm{GL}(\mathscr{H})$ are contractible by Kuiper's theorem [Kui65, Theorem 3]. Therefore, one usually restricts to appropriate subgroups in order to generate non-trivial topology. Assume that our Hilbert space $\mathscr{H}$ (complex, separable, infinite-dimensional) comes with a $\mathbb{Z}$-graded orthonormal basis $\{e_i\}_{i\in\mathbb{Z}}$, thereby defining a grading (also sometimes called polarization) into two infinite-dimensional, complementary subspaces

$$\mathscr{H} \cong \mathscr{H}_+ \oplus \mathscr{H}_- = \operatorname{span}\{e_i \mid i \geq 0\} \oplus \operatorname{span}\{e_i \mid i < 0\}.$$

The grading can also be seen as given by the involution $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

We define the Banach algebra of bounded operators

$$\mathfrak{gl}_{\mathrm{res}} = \left\{ \begin{pmatrix} X_{++} & X_{-+} \\ X_{+-} & X_{--} \end{pmatrix} \in \mathfrak{gl}(\mathscr{H}_+ \oplus \mathscr{H}_-) \mid X_{-+}, X_{+-} \in L^2 \right\}$$

with norm

$$\left\| \begin{pmatrix} X_{++} & X_{-+} \\ X_{+-} & X_{--} \end{pmatrix} \right\| = ||X_{++}|| + ||X_{--}|| + ||X_{-+}||_{L^2} + ||X_{+-}||_{L^2}.$$

Recall that L^2 denotes the ideal of Hilbert–Schmidt operators, i.e. operators that meet the summability condition $\operatorname{tr} X^*X < \infty$. One could equivalently define $\mathfrak{gl}_{\mathrm{res}}$ to be the subalgebra of bounded operators that commute with ε up to a Hilbert–Schmidt operator. The group of units in this Banach algebra is the restricted general linear group $\mathrm{GL}_{\mathrm{res}}$ of [PS88, Sec. 6.2]. It is easy to see that for $\begin{pmatrix} X_{++} & X_{-+} \\ X_{+-} & X_{--} \end{pmatrix} \in \mathrm{GL}_{\mathrm{res}}$, the operators X_{++} and X_{--} have to be Fredholm operators, since they are invertible modulo compact operators. Then, one can show that the projection

$$\begin{aligned} \psi \colon \mathrm{GL}_{\mathrm{res}} &\to \mathrm{Fred} \\ \begin{pmatrix} X_{++} & X_{-+} \\ X_{+-} & X_{--} \end{pmatrix} &\mapsto X_{++} \end{aligned} \tag{5}$$

is a homotopy equivalence [Wur06, Corollary 3.1]. There is also a restricted unitary group, given by the intersection

$$\mathrm{U}_{\mathrm{res}} = \mathrm{GL}_{\mathrm{res}} \cap \mathrm{U}(\mathscr{H}).$$

We will now consider the associated Grassmanian in this situation. Denote by $\pi_\pm$ the orthogonal projection on the subspaces $\mathscr{H}_\pm$.

Definition 5.1. The restricted Grassmannian $\mathrm{Gr}_{\mathrm{res}}$ is the set of all closed subspaces $W \subset \mathscr{H}$ such that the orthogonal projection $\pi_+ \colon W \to \mathscr{H}_+$ is a Fredholm operator and $\pi_- \colon W \to \mathscr{H}_-$ is a Hilbert–Schmidt operator.

Loosely speaking, we only consider subspaces here which are comparable in size with $\mathscr{H}_+$, in the sense of a perturbation by a Hilbert–Schmidt operator. As in the finite-dimensional case, there are many equivalent descriptions of the Grassmannian.

Proposition 5.2. *A point in* $\mathrm{Gr}_{\mathrm{res}}$ *can be thought of as*

(i) A subspace $W \subset \mathscr{H}$ *such that* $\pi_+|_W \in \mathrm{Fred}$ *and* $\pi_-|_W \in L^2$.
(ii) A self-adjoint projection operator π *on* $\mathscr{H}$ *that satisfies* $\pi - \pi_+ \in L^2$.
(iii) A self-adjoint involution F *on* $\mathscr{H}$ *that satisfies* $F - \varepsilon \in L^2$.
(iv) An equivalence class $[X] \in \mathrm{U}_{\mathrm{res}}/\mathrm{U}(\mathscr{H}_+) \times \mathrm{U}(\mathscr{H}_-)$.

(v) An equivalence class $[X] \in \mathrm{GL}_{\mathrm{res}}/\mathrm{P}$*, where* P *is the subgroup*

$$P = \left\{ \begin{pmatrix} X_{++} & X_{-+} \\ X_{+-} & X_{--} \end{pmatrix} \in \mathrm{GL}_{\mathrm{res}} \mid X_{+-} = 0 \right\}.$$

PROOF. For (iv) and (v), we check that both $\mathrm{U}_{\mathrm{res}}$ and $\mathrm{GL}_{\mathrm{res}}$ act transitively on $\mathrm{Gr}_{\mathrm{res}}$, with the respective stabilizer at $\mathscr{H}_+$ (see for example [Wur01, Proposition III.5]).

In order to prove the equivalence of (i) and (ii), first note that the conditions given obviously imply that the subspace W in (i) and the image $\mathrm{im}(\pi)$ in (ii) both have infinite dimension and infinite codimension. Such a subspace can always be written as $W = A(\mathscr{H}_+)$, for some unitary $A \in \mathrm{U}(\mathscr{H})$. The associated projection operator is then $\pi_W = A\pi_+A^*$. We now have the following equivalence of conditions:

$$\begin{aligned} \pi_W - \pi_+ \in L^2 &\Leftrightarrow A\pi_+A^* - \pi_+ \in L^2 \\ &\Leftrightarrow [A, \pi_+]A^* \in L^2 \\ &\Leftrightarrow [A, \pi_+] \in L^2 \\ &\Leftrightarrow [A, \varepsilon] \in L^2 \\ &\Leftrightarrow A \in \mathrm{U}_{\mathrm{res}}. \end{aligned}$$

Lastly, we need to prove that (ii) is equivalent to (iii). From a projection π, we construct the corresponding involution $F = 2\pi - 1_{\mathscr{H}}$, which restricts to $+1$ on the image $\mathrm{im}(\pi)$, and which restricts to -1 on $\mathrm{im}(\pi)^\perp$. Of course, F is self-adjoint if and only if π was self-adjoint, and we can also go back from F to π. In order to see the equivalence of the summability conditions, we calculate:

$$\begin{aligned} F - \varepsilon &= 2\pi - 1_{\mathscr{H}_+} - \varepsilon \\ &= 2\pi - \pi_+ - \pi_- - \pi_+ + \pi_- \\ &= 2(\pi - \pi_+). \end{aligned}$$

Therefore, $F - \varepsilon \in L^2$ if and only if $\pi - \pi_+ \in L^2$, and we are done. $\square$

It is often convenient to have multiple descriptions of $\mathrm{Gr}_{\mathrm{res}}$. Note that, for example by using (iv), we can endow $\mathrm{Gr}_{\mathrm{res}}$ with the structure of a Hilbert manifold modelled on

$$T_1\mathrm{Gr}_{\mathrm{res}} \cong \mathfrak{u}_{\mathrm{res}}/\mathfrak{u}(\mathscr{H}_+) \times \mathfrak{u}(\mathscr{H}_-) \cong L^2(\mathscr{H}_+, \mathscr{H}_-).$$

Since both $\mathrm{U}(\mathscr{H})$ and $\mathrm{GL}(\mathscr{H})$ are contractible, it is easy to deduce that also the stabilizer groups that appear in the homogenous space structures (iv) and (v) in the Proposition are contractible. Since the projection maps $\mathrm{U}_{\mathrm{res}} \to \mathrm{Gr}_{\mathrm{res}}$ and $\mathrm{GL}_{\mathrm{res}} \to \mathrm{Gr}_{\mathrm{res}}$ define locally trivial principal bundles, these projections are therefore actually homotopy equivalences, in sharp contrast to the finite-dimensional case (for details, see [Wur06, Lemma 2.1]). Note that it follows from this that the inclusion $\mathrm{GL}_{\mathrm{res}} \hookrightarrow \mathrm{U}_{\mathrm{res}}$ is also an equivalence. Using the homotopy equivalence (5), we conclude that the restricted Grassmannian has infinitely many diffeomorphic path-components, indexed by $\mathbb{Z}$, which can be recovered from a given subspace W by its virtual dimension

$$\mathrm{virt.dim}(W) = \dim(\ker(\pi_+ \colon W \to \mathscr{H}_+)) - \dim(\mathrm{coker}(\pi_+ \colon W \to \mathscr{H}_+)).$$

If $W = X(\mathscr{H}_+)$ for $X \in \mathrm{U}_{\mathrm{res}}$, then $\mathrm{virt.dim}(W) = \mathrm{ind}(X_{++})$. As in the finite-dimensional case, there is a corresponding Stiefel manifold. We want to restrict the kind of possible basis that we allow here, in order to get the right structure group for our universal bundle.

Definition 5.3. Let

$$w = \begin{pmatrix} w_+ \\ w_- \end{pmatrix} \in B(\mathscr{H}_+, \mathscr{H})$$

be a bounded operator, which is a continuous isomorphism onto its image. If w satifies the two summability conditions

(i) $w_+ - 1_{\mathscr{H}_+} \in L^1$ and
(ii) $w_- \in L^2$,

then, in particular, $\mathrm{im}(w) \in \mathrm{Gr}^0_{\mathrm{res}}$. In this case, we call w an *admissible base* for the subspace $W = \mathrm{im}(w)$.

Definition 5.4. The *restricted Stiefel manifold* is the set of all admissible bases for all subspaces in the identity component $W \in \mathrm{Gr}^0_{\mathrm{res}}$. We endow this set with the topology and smooth structure coming from the inclusion as an open subset into the Banach space $L^1 \times L^2$.

Proposition 5.5. *The restricted Stiefel manifold is contractible.*

PROOF. Consider an orthonormal basis $\{e_i\}_{i\in\mathbb{Z}}$ for $\mathscr{H}$, such that $\mathscr{H}_+$ is spanned by the e_i for $i \geq 0$. Define, for any $N \in \mathbb{Z}$, the subspaces

$$\mathscr{H}_N = \mathrm{span}\{e_i \mid i \geq N\}.$$

Since $\mathscr{H}_+ = \mathscr{H}_0$ in this notation, for $N \geq 0$, we can define submanifolds $\mathrm{St}^N_{\mathrm{res}} \subset \mathrm{St}_{\mathrm{res}}$ of embeddings $w \colon \mathscr{H}_+ \hookrightarrow \mathscr{H}$, which have the following properties

(i) The restriction of w to $\mathscr{H}_N$ is the inclusion $\mathscr{H}_N \hookrightarrow \mathscr{H}$.
(ii) The image of w is contained in $\mathscr{H}_{-N}$.

Observe that there is an inclusion $\mathrm{St}^N_{\mathrm{res}} \hookrightarrow \mathrm{St}^{N+1}_{\mathrm{res}}$. By Palais' tame approximation theorem [Pal65, Theorem A], $\mathrm{St}_{\mathrm{res}}$ is homotopy equivalent to the inductive limit of the subspaces $\mathrm{St}^N_{\mathrm{res}}$.

Furthermore, $\mathrm{St}^N_{\mathrm{res}}$ is diffeomorphic to the usual Stiefel manifold $\mathrm{St}_{N,2N}$, by restricting an arbitrary embedding w to the subspace spanned by the e_i for $0 \leq i \leq N-1$. The limit of the stabilization procedure given by the inclusions above is then just the total space for the usual universal $\mathrm{GL}(\infty)$-bundle, where $\mathrm{GL}(\infty)$ is the stable general linear group. As such, this space is contractible, and the claim follows. $\square$

We have set up a situation very similar to the finite-dimensional one, where one has a principal $\mathrm{U}(k)$-bundle $\mathrm{St}_{k,N} \to \mathrm{Gr}_{k,N}$. It turns out that the correct structure group in our case is the group of invertible operators which have a determinant, given by

$$\mathrm{GL}^1 = \left\{P \in \mathrm{GL}(\mathscr{H}_+) \mid P - 1 \in L^1\right\}$$

This group clearly acts on $\mathrm{St}_{\mathrm{res}}$ on the right via change of basis $(w, Q) \mapsto wQ$. With this action, we have the following proposition.

Proposition 5.6. *The map*

$$\begin{aligned} q\colon \mathrm{St}_{\mathrm{res}} &\to \mathrm{Gr}^0_{\mathrm{res}} \\ w &\mapsto w(\mathscr{H}_+) \end{aligned}$$

defines a smooth principal GL^1*-bundle over the path-component of the basepoint* $\mathscr{H}_+$ *in the restricted Grassmannian.*

Proof. The action is smooth since it is just multiplication of operators, and it is also clear that it is free. For fiberwise transitivity, we need to check that two admissible bases for the same subspace are related by right multiplication with elements in GL^1. Let w, w' be two admissible bases for W. Then $w' = wQ$, where $Q = w^{-1}w' \in \mathrm{GL}(\mathscr{H}_+)$ and we need to show that $Q \in \mathrm{GL}^1$. We calculate:

$$1 \equiv w'_+ = w_+Q \equiv Q \mod L^1.$$

The only thing left to show is local triviality. As in the finite-dimensional case, there exist graph coordinates for the restricted Grassmannian (cf. [PS88, Ch. 7]). At $W \in \mathrm{Gr}_{\mathrm{res}}$, those are given by the map

$$\begin{aligned} L^2(W, W^\perp) &\to U \subset \mathrm{Gr}_{\mathrm{res}} \\ T &\mapsto \Gamma_T = \{v + Tv \mid v \in W\}, \end{aligned}$$

which is a diffeomorphism onto its image U. In order to construct the needed local section, choose $X \in \mathrm{U}_{\mathrm{res}}$ such that $W = X(\mathscr{H}_+)$. Then, we define a local section s by setting $s(T) = X\big|_{\mathscr{H}_+} + TX\big|_{\mathscr{H}_+} \in \mathrm{St}_{\mathrm{res}}$. $\square$

Remark 5.7. It would be convenient if one could reduce the structure group of this bundle to the unitary group U^1. Interestingly, this is actually not possible, since it would determine a homogeneous connection which would ultimately imply that the bundle is trivial. This is discussed after Proposition 3.15 in [Fre88].

Corollary 5.8. *The smooth fiber bundle of Banach manifolds*

$$\mathrm{GL}^1 \to \mathrm{St}_{\mathrm{res}} \to \mathrm{Gr}^0_{\mathrm{res}}$$

is a model for the universal GL^1*-fibration.*

We will now construct a connection form for this principal bundle that is supposed to represent the limit of the finite dimensional connections on the bundles $\mathrm{St}_{k,N} \to \mathrm{Gr}_{k,N}$. It will in particular generate representatives for the Chern character which are compatible with the finite-dimensional versions. Consider the coordinate map

$$\begin{aligned} w\colon \mathrm{St}_{\mathrm{res}} &\to L^1 \times L^2 \\ \begin{pmatrix} w_+ \\ w_- \end{pmatrix} &\mapsto \begin{pmatrix} w_+ - 1 \\ w_- \end{pmatrix}, \end{aligned}$$

and consider its differential $\mathrm{d}w$ as an operator-valued differential form on $\mathrm{St}_{\mathrm{res}}$. Furthermore, we can associate to $w \in \mathrm{St}_{\mathrm{res}}$ the projection operator $\pi_W \in \mathrm{Gr}^0_{\mathrm{res}}$ onto $W = w(\mathscr{H}_+)$. Since $\pi_W \in \mathfrak{gl}_{\mathrm{res}}$ as an operator, this gives another operator-valued differential form $\mathrm{d}\pi_W$ on $\mathrm{St}_{\mathrm{res}}$.

Proposition 5.9. *The assignment $\Theta = w^{-1}\pi_W \mathrm{d}w$ defines a principal connection on* GL^1*-bundle* $\mathrm{St}_{\mathrm{res}} \to \mathrm{Gr}^0_{\mathrm{res}}$. *The curvature of Θ is given by the expression*

$$\Omega = \mathrm{d}\Theta + \frac{1}{2}[\Theta, \Theta] = w^{-1}\pi_W \mathrm{d}\pi_W \mathrm{d}\pi_W w.$$

PROOF. We first check that Θ is L^1-valued. We can write $\Theta = w^{-1}\pi_W(\pi_+ + \pi_-)\mathrm{d}w$, and since $\pi_+\mathrm{d}w$ is trace class, it remains to show that the second summand is also trace class. From Proposition 5.2 we know that $\pi_W \in \mathrm{Gr}_{\mathrm{res}}$ is equivalent to $\pi_+ - \pi_W \in L^2$. Therefore, using that $\pi_-\mathrm{d}w = \mathrm{d}(\pi_- w) \in L^2$, we have

$$\begin{aligned} w^{-1}\pi_W\pi_-\mathrm{d}w &= w^{-1}(\pi_+ + (\pi_W - \pi_+))\pi_-\mathrm{d}w \\ &= w^{-1}\underbrace{(\pi_W - \pi_+)}_{\in L^2}\underbrace{\pi_-\mathrm{d}w}_{\in L^2} \in L^1. \end{aligned}$$

We now check the defining properties of a connection form. On the fundamental vector fields for $X \in L^1$ of the form $\widetilde{X}_w = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} w\exp(tX)$, we clearly have $\Theta(\widetilde{X}) = X$. On the other hand, we have

$$(R_Q^*\Theta)_w = (wQ)^{-1}\pi_W(\mathrm{d}w)Q = \mathrm{Ad}_{Q^{-1}}\Theta_w,$$

finishing the proof that Θ is a connection form. For the calculation of the curvature, we will need the identities

$$\begin{aligned} \mathrm{d}w &= \mathrm{d}(\pi_W w) = \mathrm{d}\pi_W w + \pi_W \mathrm{d}w \\ \mathrm{d}\pi_W &= \mathrm{d}(ww^{-1}\pi_W) = \mathrm{d}ww^{-1}\pi_W + w\mathrm{d}(w^{-1}\pi_W). \end{aligned}$$

From the second identity, it follows that

$$\mathrm{d}(w^{-1}\pi_W) = w^{-1}\pi_W\mathrm{d}\pi_W - w^{-1}\pi_W\mathrm{d}ww^{-1}\pi_W.$$

We now calculate

$$\begin{aligned}
\mathrm{d}\Theta &= \mathrm{d}(w^{-1}\pi_W \mathrm{d}w) \\
&= \mathrm{d}(w^{-1}\pi_W)\mathrm{d}w \\
&= (w^{-1}\pi_W \mathrm{d}\pi_W - w^{-1}\pi_W \mathrm{d}w w^{-1}\pi_W)\mathrm{d}w \\
&= w^{-1}\pi_W \mathrm{d}\pi_W(\mathrm{d}\pi_W w + \pi_W \mathrm{d}w) - w^{-1}\pi_W \mathrm{d}w w^{-1}\pi_W \mathrm{d}w \\
&= w^{-1}\pi_W \mathrm{d}\pi_W \mathrm{d}\pi_W w - \frac{1}{2}[\Theta, \Theta],
\end{aligned}$$

since $\pi_W \mathrm{d}\pi_W \pi_W = 0$. This finishes the proof. □

Since the curvature form is trace class-valued, the usual arguments from Chern–Weil theory go through and give representatives for the Chern character of the universal GL^1 bundle over $\mathrm{Gr}^0_{\mathrm{res}}$ (cf. [Fre88, Theorem 1.13]). One difference to the bundles over the finite-dimensional Grassmannians is that our form Θ does not have left-invariance properties for the action of a unitary group. In fact, there is no left action on $\mathrm{St}_{\mathrm{res}}$, since the summability conditions that we required for admissible bases in Definition 5.4 are not preserved under left multiplication by unitary matrices - not even if we restrict to $\mathrm{U}_{\mathrm{res}}$. However, we still have that after taking traces, the forms $\operatorname{tr}\Omega^k$ make sense as left-invariant differential forms on $\mathrm{Gr}^0_{\mathrm{res}}$, which invariantly extend to the other diffeomorphic components of $\mathrm{Gr}_{\mathrm{res}}$. We make the following definition.

Definition 5.10. The universal even Chern character form $\mathrm{ch}_{\mathrm{even}} \in \Omega^{\mathrm{even}}(\mathrm{Gr}_{\mathrm{res}})$ is

$$\mathrm{ch}_{\mathrm{even}} = \sum_{k\geq 0} \mathrm{ch}_{2k} = \mathrm{ch}_0 + \sum_{k\geq 1} \left(\frac{i}{2\pi}\right)^k \frac{1}{k!} \operatorname{tr}\left(\Omega^k\right),$$

where $\Omega = \pi_W \mathrm{d}\pi_W \mathrm{d}\pi_W$ is a trace class operator-valued form. Here, $\mathrm{ch}_0 \colon \mathrm{Gr}_{\mathrm{res}} \to \mathbb{Z}$ is the map that assigns to W its virtual dimension.

The positive degree forms are actually invariant: Since the action of $\mathrm{U}_{\mathrm{res}}$ is by conjugation of both π_W and $\mathrm{d}\pi_W$ by a unitary, it leaves the trace invariant. Thus, it is useful to explicitly work out what happens at the tangent space of $\mathscr{H}_+$. Recall that

$$T_{\pi_+}\mathrm{Gr}_{\mathrm{res}} \cong \mathfrak{u}_{\mathrm{res}}/\mathfrak{u}(\mathscr{H}_+) \times \mathfrak{u}(\mathscr{H}_-) \cong \left\{ \begin{pmatrix} 0 & -c^* \\ c & 0 \end{pmatrix} \mid c \in L^2(\mathscr{H}_+, \mathscr{H}_-) \right\}. \tag{6}$$

Set $w_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathrm{St}_{\mathrm{res}}$. For $w = Xw_0 \in \mathrm{St}_{\mathrm{res}}$, we have that $\pi_W = \pi_{X(\mathscr{H}_+)} = X\pi_+ X^*$ and therefore $(\mathrm{d}\pi_W)_{\pi_+} = [-, \pi_+]$, where the bracket indicates the commutator. Therefore, evaluation of $\Omega = \pi_W \mathrm{d}\pi_W \mathrm{d}\pi_W$ at the point π_+ yields

$$\begin{aligned}
&\Omega_{\pi_+}\left(\begin{pmatrix} 0 & -c_1^* \\ c_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -c_2^* \\ c_2 & 0 \end{pmatrix}\right) \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\Bigg(\left[\begin{pmatrix} 0 & -c_1^* \\ c_1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]\left[\begin{pmatrix} 0 & -c_2^* \\ c_2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] - \\
&\qquad \left[\begin{pmatrix} 0 & -c_2^* \\ c_2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]\left[\begin{pmatrix} 0 & -c_1^* \\ c_1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]\Bigg) \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\left(\begin{pmatrix} 0 & c_1^* \\ c_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & c_2^* \\ c_2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & c_2^* \\ c_2 & 0 \end{pmatrix}\begin{pmatrix} 0 & c_1^* \\ c_1 & 0 \end{pmatrix}\right) \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} c_1^*c_2 - c_2^*c_1 & 0 \\ 0 & c_1c_2^* - c_2c_1^* \end{pmatrix} = \begin{pmatrix} c_1^*c_2 - c_2^*c_1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad (7)
\end{aligned}$$

and we recover the familiar formula from the finite-dimensional case (4).

There are natural smooth inclusions of the finite-dimensional Grassmannians into the restricted Grassmannian, given as follows: Pick a $\mathbb{Z}$-graded orthonormal basis $\{e_i\}$ for $\mathscr{H}$, where $\mathscr{H}_+ \cong \operatorname{span}\{e_i \mid i \geq 0\}$. Considering for $N \in \mathbb{Z}$ the subspaces

$$\mathscr{H}_N = \operatorname{span}\{e_i \mid i \geq N\},$$

one sees that the subsets

$$\mathrm{Gr}_{\mathrm{res},N} = \{W \in \mathrm{Gr}_{\mathrm{res}} \mid \mathscr{H}_N \subset W \subset \mathscr{H}_{-N}\}$$

are isomorphic to the full finite-dimensional Grassmannians $\mathrm{Gr}(\mathbb{C}^{2N}) = \coprod_{k\leq 2N} \mathrm{Gr}_{k,2N}$ by mapping W to $W/\mathscr{H}_N \subset \mathscr{H}_{-N}/\mathscr{H}_N \cong \mathbb{C}^{2N}$. The inclusion of $\mathrm{Gr}_{\mathrm{res},N}$ into $\mathrm{Gr}_{\mathrm{res},N+1}$ corresponds to sending $V \in \mathrm{Gr}(\mathbb{C}^{2N})$ to $\{0\} \oplus V \oplus \mathbb{C} \in \mathrm{Gr}(\mathbb{C}^{2(N+1)})$. The union of these finite-dimensional Grassmannians, denoted by $\mathrm{Gr}_{\mathrm{res},\infty}$, is dense in $\mathrm{Gr}_{\mathrm{res}}$, and the intersection $\mathrm{Gr}_{\mathrm{res},N} \cap \mathrm{Gr}^k_{\mathrm{res}}$ is diffeomorphic to $\mathrm{Gr}_{N+k,2N}$ (cf. [Wur01, Proposition III.5]). All in all, we have inclusion maps

$$\begin{aligned}
i\colon \mathrm{Gr}_{k,2N} = \mathrm{Gr}_{N+(k-N),2N} &\to \mathrm{Gr}_{\mathrm{res},\infty} \subset \mathrm{Gr}_{\mathrm{res}} \\
W &\mapsto W \oplus \mathscr{H}_N,
\end{aligned}$$

which are easily seen to be compatible with the chosen Chern character differential forms in the following sense:

Proposition 5.11. *Under the natural inclusion $i\colon \mathrm{Gr}_{k,2N} \hookrightarrow \mathrm{Gr}_{\mathrm{res}}$, the universal Chern character form $\mathrm{ch}_{\mathrm{even}}$ pulls back to the corresponding forms on the finite-dimensional Grassmannian, which are given by the Chern–Weil forms of the universal connection (see Equation 4).*

Proof. On the level of projections, with the above mentioned identification of $\mathbb{C}^{2N}$ with a subset of $\mathscr{H}$, we see that π_W gets mapped by i to $\pi_W + \pi_N$, where π_N is the projection to $\mathscr{H}_N$. This map is equivariant for the conjugation action by $\mathrm{U}(2N)$, where $\mathrm{U}(2N)$ acts on $\mathrm{Gr}_{\mathrm{res}}$ by extension with the identity. Since our forms are invariant under this action, and also the action is transitive on $\mathrm{Gr}_{k,2N}$, it is enough to check the statement at the basepoint. But here, it is easy to see that the curvature form from Equation 7 restricts to the finite-dimensional one from Equation 4. □

Remark 5.12. We can use this calculation to cook up a cycle map (see Definition 13.2): Given a connected manifold M and a class in $\hat{K}^0(M)$ represented by a formal difference $[V, \nabla_V] - [W, \nabla_W]$ of smooth hermitian vector bundles with compatible connections of dimension k and k', we can use the Narasimhan–Ramanan theorem to get classifying maps $f_V\colon M \to \mathrm{Gr_{k,2N}}$, $f_W\colon M \to \mathrm{Gr_{k',2N}}$. Employing our above defined inclusions, we may as well assume that the target of these maps is actually $\mathrm{Gr}_{\mathrm{res}}$. Then, using the flip and block sum map defined in Section 6, we get a smooth map to the restricted Grassmannian, given by $f_V \boxplus \mathrm{flip}(f_W)$, which is supposed to represent the differential K-theory class in our model. Note that $\mathrm{ch}_0(f_V \boxplus \mathrm{flip}(f_W)) = (k-N) - (l-N) = k - l = \mathrm{ch}_0(V) - \mathrm{ch}_0(W)$, which justifies our definition of the degree zero part ch_0 of the Chern character. We will further discuss cycle maps in Section 13.

6. Chern–Simons forms, the block sum and the inversion operation

We begin this chapter by discussing the transgressions of the Chern character in the path-loop fibration. The resulting Chern–Simons forms have first appeared in [CS74, Sec. 3] and they were one of the key ideas that led to the development of differential cohomology theories.

Let us consider the universal situation of the smooth path-loop fibration over U^1 and $\mathrm{Gr}_{\mathrm{res}}$. There are some subtleties when one wants to consider path and loop spaces as smooth manifolds, but all we need is to have well-defined pullbacks to finite-dimensional manifolds. This situation can be made precise by Chen's notion of diffeological spaces [Che77, Definition 1.2.1]. However, the identities that we need are provable via topological arguments, so this viewpoint is not too important for the present thesis, and one might as well interpret the next paragraph as an informal motivation for the second part of Definition 6.2.

Let us fix the topology on mapping spaces. It will be enough for our purposes to consider compact source manifolds, where all of the sensible topologies coincide. Let M be a compact manifold and N be smooth Banach manifold. We first equip the set of r-times differentiable functions $\mathscr{C}^r(M, N)$ with a topology. Assume that $f\colon M \to N$ is an r-times differentiable map. Let (φ, U) and (ψ, V) be charts on M and N. Furthermore, assume that $K \subset U$ is a compact set such that $f(K) \subset V$, and let $0 < \varepsilon \leq \infty$. Then, one can define the sets

$$N^r(f; (\varphi, U), (\psi, V), K, \varepsilon),$$

given by the set of smooth maps $g\colon M \to N$ such that $g(K) \subset V$ and

$$||D^k(\psi f \varphi^{-1})(x) - D^k(\psi g \varphi^{-1})(x)|| < \varepsilon$$

for all $x \in \varphi(K)$ and $0 \leq k \leq r$. This means that the local representatives of f and g, together with their first k derivatives are within ε at each point of K. The (weak) Whitney topology on $\mathscr{C}^r(M,N)$ is generated by these sets. A neighborhood of f is thus any set containing the intersection of a finite number of sets of this type. For details, we refer to [Hir76, Chapter 2.1].

Definition 6.1. Let M and N be smooth Banach manifolds. Then, we denote by

$$\mathrm{Map}(M,N)$$

the set of smooth $\mathscr{C}^\infty$ maps from M to N. If M is compact, we equip $\mathrm{Map}(M,N)$ with the Whitney topology, which is the union of the topologies induced by the inclusion maps

$$\mathrm{Map}(M,N) \hookrightarrow \mathscr{C}^r(M,N).$$

Furthermore, if N has a basepoint n_0, we denote by $PN \subset \mathrm{Map}(I,N)$ and $\Omega N \subset \mathrm{Map}(S^1,N)$ the space of smooth paths, and the space of smooth loops, based at n_0 at time 0.

By pulling back along the evaluation maps $P\mathrm{Gr}_{\mathrm{res}} \times I \to \mathrm{Gr}_{\mathrm{res}}$ and $P\mathrm{U}^1 \times I \to \mathrm{U}^1$ and then fiber integrating, we can define the universal Chern–Simons forms

$$\begin{aligned} \mathrm{cs}_{\mathrm{odd}} &= \int_I \mathrm{ev}_t^*(\mathrm{ch}_{\mathrm{even}}) \in \Omega^{\mathrm{odd}}(P\mathrm{U}^1) \quad \text{and} \\ \mathrm{cs}_{\mathrm{even}} &= \int_I \mathrm{ev}_t^*(\mathrm{ch}_{\mathrm{odd}}) \in \Omega^{\mathrm{even}}(P\mathrm{Gr}_{\mathrm{res}}). \end{aligned}$$

The base points we use here are the identity $1 \in \mathrm{U}^1$ and the space $\mathscr{H}_+ \in \mathrm{Gr}_{\mathrm{res}}$. These forms famously fit into the equation

$$\mathrm{dcs} = \mathrm{ev}_1^*\mathrm{ch} - \mathrm{ev}_0^*\mathrm{ch},$$

which can be seen by an application of Stokes' theorem. When we pull back the Chern–Simons forms to the based loop space in order to get a form cs_Ω, this identity shows that cs_Ω is a transgression of ch in the path-loop fibration. Using our universal representatives, we can now associate certain differential forms to a map into U^1 or $\mathrm{Gr}_{\mathrm{res}}$.

Definition 6.2. Let M be a smooth manifold. We define the maps

$$\begin{aligned} \mathrm{Ch}\colon \mathrm{Map}(M,\mathrm{U}^1) &\to \Omega^{\mathrm{odd}}_{\mathrm{cl}}(M) \\ \mathrm{Ch}\colon \mathrm{Map}(M,\mathrm{Gr}_{\mathrm{res}}) &\to \Omega^{\mathrm{even}}_{\mathrm{cl}}(M), \end{aligned}$$

given by pullback of the universal Chern forms (Definition 4.2 and Definition 5.10). Furthermore, we define the maps

$$\begin{aligned} \mathrm{CS}\colon \mathrm{Map}(M \times I, \mathrm{U}^1) &\to \Omega^{\mathrm{even}}(M) \\ \mathrm{CS}\colon \mathrm{Map}(M \times I, \mathrm{Gr}_{\mathrm{res}}) &\to \Omega^{\mathrm{odd}}(M), \end{aligned}$$

given by "pullback of the universal Chern–Simons forms" via smooth homotopies, i.e. $\mathrm{CS}(H_t) = \int_I H_t^* \mathrm{ch}$.

We define a refined notion of homotopy by using these forms, following [TWZ16, Definition 3.4]. It is designed to retain more information in an equivalence class than just the isomorphism type of the corresponding bundle. One important feature is that we will have a well-defined map that assigns to a CS-homotopy equivalence class (see Definition 6.3) of maps the pullback of its universal Chern form, which is only possible up to exact forms for a homotopy class.

Definition 6.3. Let $f, g\colon M \to \mathcal{U}$ for $\mathcal{U} \in \{\mathrm{Gr}_{\mathrm{res}}, \mathrm{U}^1\}$ be smooth maps. We say that f and g are Chern–Simons homotopic or CS-homotopic if there is a smooth homotopy H_t connecting them such that the resulting Chern–Simons form given by integrating the universal Chern character

$$\mathrm{CS}_{\mathrm{odd/even}}(H) = \int_I H_t^*(\mathrm{ch}_{\mathrm{even/odd}}) \in \Omega^{\mathrm{odd/even}}(M)$$

is exact.

We will also define the block sum operation, which works in general for operators on an infinite-dimensional Hilbert space $\mathscr{H}$. It will be used to implement addition in differential K-theory. In order to be explicit, we choose a specific isomorphism $\rho : \mathscr{H} \to \mathscr{H} \oplus \mathscr{H}$. When a polarization on $\mathscr{H}$ is given, our isomorphism is designed to respect the grading.

Definition 6.4. Let $\rho\colon \mathscr{H} \to \mathscr{H} \oplus \mathscr{H}$ be the isometric isomorphism

$$\begin{aligned} \rho : \quad e_{2k} &\mapsto (e_k, 0) \\ e_{2k+1} &\mapsto (0, e_k), \end{aligned}$$

given on an orthonormal basis $\{e_i\}$ indexed by $\mathbb{N}$ or $\mathbb{Z}$. We define the corresponding block sum map

$$\begin{aligned} \boxplus_\rho\colon \mathfrak{gl}(\mathscr{H}) \times \mathfrak{gl}(\mathscr{H}) &\to \mathfrak{gl}(\mathscr{H}) \\ (A, B) &\mapsto \rho^*(A \oplus B)\rho. \end{aligned}$$

Note that various subgroups of operators, which we consider, are preserved by this construction, most importantly $\mathrm{U}_{\mathrm{res}}$ and U^1. This also induces a well-defined operation on $\mathrm{Gr}_{\mathrm{res}}$, where it corresponds to a direct sum of subspaces: If $W = X(\mathscr{H}_+)$ and $V = Y(\mathscr{H}_+)$ for $X, Y \in \mathrm{U}_{\mathrm{res}}$, then

$$W \boxplus_\rho V = (X \boxplus_\rho Y)(\mathscr{H}_+) = \rho^*(X \oplus Y)\rho(\mathscr{H}_+) = \rho^*(V \oplus W),$$

where in the last expression we interpret $V \oplus W$ as a subspace of $\mathscr{H} \oplus \mathscr{H}$.

Via pointwise application, we can now make sense of the block sum of two maps f, g from a manifold into the bounded linear operators $\mathfrak{gl}(\mathscr{H})$. We write

$$f \boxplus_\rho g = \rho^*(f \oplus g)\rho.$$

Ultimately, one wants this block sum operation on maps to not depend on the chosen unitary isomorphism ρ up to the right equivalence relation. This is easily seen to be true for homotopy classes of maps by using path-connectedness of the unitary group U. The following technical lemma will show the corresponding statement for the more restricted class of CS-homotopies. Note that for maps into $\mathrm{U}_{\mathrm{res}}$, we also have a notion of CS-equivalence, this time with respect to the universal Chern character that one gets from pulling back $\mathrm{ch}_{\mathrm{even}}$ via the projection $\mathrm{U}_{\mathrm{res}} \to \mathrm{Gr}_{\mathrm{res}}$.

Lemma 6.5. *Let $f\colon M \to \mathrm{U}^1$, $g\colon M \to \mathrm{U}_{\mathrm{res}}$ and $h\colon M \to \mathrm{Gr}_{\mathrm{res}}$ be smooth maps and consider $A \in \mathrm{U}(\mathscr{H})$ and $B \in \mathrm{U}(\mathscr{H}_+) \times \mathrm{U}(\mathscr{H}_-) \subset \mathrm{U}_{\mathrm{res}}$. Then the pairs of maps*

(i) $f\colon M \to \mathrm{U}^1$ *and* $AfA^*\colon M \to \mathrm{U}^1$
(ii) $g\colon M \to \mathrm{U}_{\mathrm{res}}$ *and* $BgB^*\colon M \to \mathrm{U}_{\mathrm{res}}$
(iii) $h\colon M \to \mathrm{Gr}_{\mathrm{res}}$ *and* $Bh\colon M \to \mathrm{Gr}_{\mathrm{res}}$

are CS-*homotopic.*

Proof. For the first case, choose a smooth path A_t from $A_0 = 1$ to $A_1 = A$. Then there is a smooth universal homotopy

$$\begin{aligned} H_t\colon \mathrm{U}^1 \times I &\to \mathrm{U}^1 \\ (X, t) &\mapsto A_t X A_t^*, \end{aligned}$$

which yields a homotopy as stated for any $f\colon X \to \mathrm{U}^1$ by composition. We need to show that its CS-form is exact. We have

$$\mathrm{dCS}_{2k}(H_t) = \mathrm{d}\int_I H_t^*\mathrm{ch}_{2k+1} = H_1^*\mathrm{ch}_{2k+1} - H_0^*\mathrm{ch}_{2k+1} =$$
$$-\left(\frac{1}{2\pi i}\right)^{k+1} \frac{(k)!}{(2k+1)!}(\mathrm{tr}(AX^*\mathrm{d}XA^*)^{2k+1} - \mathrm{tr}(X^*\mathrm{d}X)^{2k+1}) = 0.$$

Since the positive even cohomology of U^1 vanishes, this implies that the Chern–Simons forms for $k > 0$ are exact. For $k = 0$, we make a direct calculation. Note that the differential of H_t splits according to the decomposition of the tangent space of $\mathrm{U}^1 \times I$ into a direct sum of a space part with a time derivative. Our notation for the space derivative is $\mathrm{d}H_t$, while we denote the time derivative by $\dot{H}_t$. We have

$$\begin{aligned} \dot{H}_t &= \dot{A}_t X A_t^* - A_t X A_t^* \dot{A}_t A_t^* \\ \mathrm{d}H_t &= A_t \mathrm{d}X A_t^*. \end{aligned}$$

Recall that ω denotes the Maurer–Cartan form on U^1. We need to calculate

$$\mathrm{CS}_0(H_t) = \int_I H_t^* \mathrm{ch}_1 = \left(\frac{i}{2\pi}\right) \int_I \iota_{\partial_t}(H_t^*(\mathrm{tr}(\omega))).$$

The integrand yields

$$\begin{aligned}\iota_{\partial_t}(H_t^*(\mathrm{tr}(\omega))) &= \mathrm{tr}(A_t X^* A_t^* (\dot{A}_t X A_t^* - A_t X A_t^* \dot{A}_t A_t^*)) \\ &= \mathrm{tr}(X^* A_t^* \dot{A}_t X - \dot{A}_t A_t^*) = 0\end{aligned}$$

for all t and therefore, CS_0 vanishes.

For the second case, we choose again a smooth path from 1 to B in order to define a homotopy H_t starting at $H_0 = C_B$ and ending at $H_1 = \mathrm{id}_{\mathrm{U}_{\mathrm{res}}}$, where C_B denotes conjugation by B. By the vanishing of $H^{\mathrm{odd}}(\mathrm{Gr}_{\mathrm{res}})$, it is enough to show that the CS-form is closed, i.e. that H_0 and H_1 have the same Chern form. We argue as follows: The projection $\mathrm{U}_{\mathrm{res}} \to \mathrm{Gr}_{\mathrm{res}}$ sends a matrix X to the projection $X\pi_+ X^*$. Conjugating X by B yields

$$BXB^*\pi_+ BX^*B^* = BX\pi_+X^*B^*,$$

and therefore, using the invariance of ch, the conjugation map $C_B \colon \mathrm{U}_{\mathrm{res}} \to \mathrm{U}_{\mathrm{res}}$ pulls back the universal Chern form to itself. The third case follows by the same argument, using the invariance of ch one more time.

The independence of the block sum up to CS-equivalence is now easily deduced, since

$$\rho'^* \rho (f \boxplus_\rho g) \rho^* \rho' = \rho'^* \rho \rho^* (f \oplus g) \rho \rho^* \rho' = f \boxplus_{\rho'} g$$

and therefore the two block sums defined by ρ and ρ' just differ by a conjugation with the unitary matrix $\rho'\rho^*$ on $\mathscr{H}$, which in the polarized case respects the grading. □

Remark 6.6. We have shown that conjugation by a fixed matrix of of a certain form does not change the Chern–Simons equivalence class. In particular, for any other unitary isomorphism $\rho' \colon \mathscr{H} \to \mathscr{H} \oplus \mathscr{H}$ (respecting the grading in the polarized case), we have that $f \boxplus_\rho g$ and $f \boxplus_{\rho'} g$ are CS-homotopic. By the preceding Lemma, it is therefore safe to suppress ρ in our notation. For two elements in the restricted unitary group, which are by definition 2 by 2 block operators, we write

$$f \boxplus g = \begin{pmatrix} f_{++} & 0 & f_{-+} & 0 \\ 0 & g_{++} & 0 & g_{-+} \\ f_{+-} & 0 & f_{--} & 0 \\ 0 & g_{+-} & 0 & g_{--} \end{pmatrix}. \tag{8}$$

Proposition 6.7. *Let $f, g, h \colon M \to \mathcal{U}$ for $\mathcal{U} \in \{\mathrm{Gr}_{\mathrm{res}}, \mathrm{U}^1\}$ be smooth maps. Then, the operation induced by block sum is commutative and associative up to CS-homotopy, i.e. we have*

$$f \boxplus g \sim_{\mathrm{CS}} g \boxplus f \quad \text{and} \quad f \boxplus (g \boxplus h) \sim_{\mathrm{CS}} (f \boxplus g) \boxplus h.$$

PROOF. This is just a consequence of Lemma 6.5, since the difference in each case is just a permutation of the basis. For commutativity in the case of U^1, one sees that for $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{U}(\mathscr{H} \oplus \mathscr{H})$, one has

$$g \boxplus f = \rho^* U \rho (f \boxplus g) \rho^* U \rho.$$

For the even case, acting by the same matrix $\rho^* U \rho$ on $\mathscr{H}_+$ and $\mathscr{H}_-$ separately does the job. For associativity, one has that

$$\begin{aligned} (f \boxplus g) \boxplus h &= \rho^*(\rho^* \times \mathrm{id}) \begin{pmatrix} f & & \\ & g & \\ & & h \end{pmatrix} (\rho \times \mathrm{id})\rho \\ &= \rho^*(\rho^* \times \mathrm{id})(\mathrm{id} \times \rho)\rho(f \boxplus (g \boxplus h))\rho^*(\mathrm{id} \times \rho^*)(\rho \times \mathrm{id})\rho \end{aligned}$$

in the U^1 case. Acting by the same matrix on $\mathscr{H}_+$ and $\mathscr{H}_-$ separately show the $\mathrm{Gr}_{\mathrm{res}}$ case. $\square$

We will now discuss the involution on $\mathrm{U}_{\mathrm{res}}$ that will implement inversion in differential K-theory. In an orthonormal $\mathbb{Z}$-basis $\{e_i\}$ adapted to the polarization, let $\mathscr{H}_+$ be spanned by the e_i for $i \geq 0$, and $\mathscr{H}_-$ be spanned by the e_i for $i < 0$. Let furthermore $U \in \mathrm{U}(\mathscr{H})$ be the unitary transformation that reverses the polarization by sending e_i to e_{-i-1} for any i. In the basis $\{e_i\}$, it is given by the matrix $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Definition 6.8. We define the polarization flip map

$$\begin{aligned} \mathrm{flip}\colon \mathrm{U}_{\mathrm{res}} &\to \mathrm{U}_{\mathrm{res}} \\ X &\mapsto UXU, \end{aligned}$$

which conjugates every element with the matrix $U \in \mathrm{U}(\mathscr{H})$. On the space of smooth maps from a manifold M to $\mathrm{U}_{\mathrm{res}}$, this induces the operation

$$f \mapsto \mathrm{flip}(f) = \mathrm{flip} \circ f = UfU.$$

Explicitly, for any point $m \in M$, we have

$$\mathrm{flip}(f)(m) = \begin{pmatrix} f_{--}(m) & f_{+-}(m) \\ f_{-+}(m) & f_{++}(m) \end{pmatrix}, \quad \text{where} \quad f(m) = \begin{pmatrix} f_{++}(m) & f_{-+}(m) \\ f_{+-}(m) & f_{--}(m) \end{pmatrix}.$$

Note that there is an induced flip map on the restricted Grassmannian, which corresponds to taking the orthogonal complement of a subspace and then changing the polarization. We have

$$W = X(\mathscr{H}_+) \mapsto \mathrm{flip}(X)(\mathscr{H}_+) = UXU(\mathscr{H}_+) = UX(\mathscr{H}_-) = U(W^\perp) = \mathrm{flip}(W),$$

which also extends to maps $M \to \mathrm{Gr}_{\mathrm{res}}$ via composition. One furthermore sees that pullback by flip preserves left invariance of forms: If $L_Y^*\eta = \eta$, we have that

$$L_Y^*(\mathrm{flip}^*\eta) = (\mathrm{flip} \circ L_Y)^*(L^*_{\mathrm{flip}(Y^{-1})}\eta) = \mathrm{flip}^*\eta,$$

since flip is a group homomorphism on $\mathrm{U}_{\mathrm{res}}$. The following proposition shows compatibility of the inversion and addition operations on the classifying spaces with the Chern and Chern–Simons forms.

Proposition 6.9. *Consider smooth maps $f, g\colon M \to \mathcal{U}$ and smooth homotopies $H_t, G_t\colon M \times I \to \mathcal{U}$ for $\mathcal{U} \in \{\mathrm{Gr}_{\mathrm{res}}, \mathrm{U}^1\}$. Then:*

(i) The maps Ch *and* CS *are monoid morphisms, i.e.* $\mathrm{Ch}(f \boxplus g) = \mathrm{Ch}(f) + \mathrm{Ch}(g)$ *and* $\mathrm{CS}(H_t \boxplus G_t) = \mathrm{CS}(H_t) + \mathrm{CS}(G_t)$.

(ii) $\mathrm{CS}(H_t * G_t) = \mathrm{CS}(H_t) + \mathrm{CS}(G_t)$, *where $*$ denotes composition of homotopies.*

(iii) $\mathrm{Ch}_{\mathrm{even}}(\mathrm{flip}(f)) = -\mathrm{Ch}_{\mathrm{even}}(f)$ *and* $\mathrm{Ch}_{\mathrm{odd}}(f^*) = -\mathrm{Ch}_{\mathrm{odd}}(f)$.

(iv) $\mathrm{CS}_{\mathrm{odd}}(\mathrm{flip}(H_t)) = -\mathrm{CS}_{\mathrm{odd}}(H_t)$ *and* $\mathrm{CS}_{\mathrm{even}}(H_t^*) = -\mathrm{CS}_{\mathrm{even}}(H_t)$.

Proof. The monoid morphism property follows directly from the additivity of the trace under block sum and linearity of the integral, and the additivity under composition of homotopies follows from additivity of the integral under partition of the interval.

We check the third identity directly on $\mathrm{Gr}_{\mathrm{res}}$ and U^1. For $\mathrm{Gr}_{\mathrm{res}}$, we need to compute the pullback of the curvature $\mathrm{flip}^*\Omega$, and it suffices to do this in the tangent space at $\mathscr{H}_+$ by left invariance. Take $X = \begin{pmatrix} 0 & X_{-+} \\ X_{+-} & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & Y_{-+} \\ Y_{+-} & 0 \end{pmatrix} \in T_{\mathscr{H}_+}\mathrm{Gr}_{\mathrm{res}}$. We have

$$(\mathrm{flip}^*\Omega)_{\mathscr{H}_+}(X, Y) = \Omega_{\mathscr{H}_+}\left(\begin{pmatrix} 0 & X_{+-} \\ X_{-+} & 0 \end{pmatrix}, \begin{pmatrix} 0 & Y_{+-} \\ Y_{-+} & 0 \end{pmatrix}\right) = Y_{+-}X_{-+} - X_{+-}Y_{-+}$$

Therefore, we see that $(\mathrm{flip}^*(\mathrm{tr}\,\Omega^k))_{\mathscr{H}_+}(X^1, \dots, X^{2k})$ equals

$$\begin{aligned} &\frac{1}{2^k}\sum_{\sigma \in S_{2k}} \mathrm{sgn}(\sigma)\mathrm{tr}\,\Omega(\mathrm{flip}X^{\sigma(1)}, \mathrm{flip}X^{\sigma(2)}) \cdots \Omega(\mathrm{flip}X^{\sigma(2k-1)}, \mathrm{flip}X^{\sigma(2k)}) \\ = &\frac{1}{2^k}\sum_{\sigma \in S_{2k}} \mathrm{sgn}(\sigma)\mathrm{tr}\,(X_{+-}^{\sigma(2)}X_{-+}^{\sigma(1)} - X_{+-}^{\sigma(1)}X_{-+}^{\sigma(2)}) \cdots (X_{+-}^{\sigma(2k)}X_{-+}^{\sigma(2k-1)} - X_{+-}^{\sigma(2k-1)}X_{-+}^{\sigma(2k)}). \end{aligned}$$

This is just a big sum of products of $2k$-operators with many redundant terms. One sees that it is equal to

$$\begin{aligned} &\sum_{\sigma \in S_{2k}} \mathrm{sgn}(\sigma)\mathrm{tr}\,X_{+-}^{\sigma(2)}X_{-+}^{\sigma(1)}X_{+-}^{\sigma(4)}X_{-+}^{\sigma(3)} \cdots X_{+-}^{\sigma(2k)}X_{-+}^{\sigma(2k-1)} \\ = &\sum_{\sigma \in S_{2k}} \mathrm{sgn}(\sigma)\mathrm{tr}\,X_{-+}^{\sigma(1)}X_{+-}^{\sigma(4)}X_{-+}^{\sigma(3)} \cdots X_{+-}^{\sigma(2k)}X_{-+}^{\sigma(2k-1)}X_{+-}^{\sigma(2)} \\ = -&\sum_{\sigma \in S_{2k}} \mathrm{sgn}(\sigma)\mathrm{tr}\,X_{-+}^{\sigma(2)}X_{+-}^{\sigma(1)}X_{-+}^{\sigma(4)}X_{+-}^{\sigma(3)} \cdots X_{-+}^{\sigma(2k)}X_{+-}^{\sigma(2k-1)} \\ = -&(\mathrm{tr}\,\Omega^k)_{\mathscr{H}_+}(X^1, \dots, X^{2k}). \end{aligned}$$

The first equality is cyclic invariance of the trace, the second one follows after applying a cyclic permutation with $2k$-elements. Finally, the last equality comes from going through the same calculation without applying the flip map. This proves (iii) for the even case.

For the odd case, note that from $0 = \mathrm{d}(ff^*) = \mathrm{d}ff^* + f\mathrm{d}f^*$ it follows that $\mathrm{tr}(f\mathrm{d}f^*)^{2k-1} = -\mathrm{tr}(\mathrm{d}ff^*)^{2k-1} = -\mathrm{tr}(f^*\mathrm{d}f)^{2k-1}$. Therefore pulling back $\mathrm{ch}_{\mathrm{odd}}$ via the adjoint-operator map $\mathrm{U}^1 \to \mathrm{U}^1$ gives the desired minus sign. Part (iv) of the Proposition now easily follows from part (iii) by definition of the Chern–Simons form. $\square$

CHAPTER III

Equivariant smooth classifying spaces

7. A decomposition theorem for complex equivariant K-theory

The idea of differential cohomology is to glue together differential form information and cohomology classes via the Chern character map. In order to discuss an equivariant version of this process, we will therefore need to have an equivariant version of the Chern character map. The goal of this section is to develop the necessary theory for the delocalized equivariant Chern character.

Let G be a finite group. One well-known way to go from a non-equivariant cohomology theory to the equivariant world is by applying the Borel construction. Any cohomology theory E^* gives rise to an equivariant version E^*_G via

$$E^*_G(M) = E^*(EG \times_G M).$$

Therefore, we can immediately obtain an equivariant version of the Chern character

$$\mathrm{ch}_{\mathrm{Bor},G}\colon K^*_{\mathrm{Bor},G}(M) \cong K^*(EG \times_G M) \to H^{\mathrm{even/odd}}(EG \times_G M; \mathbb{R}) \cong H^*_{\mathrm{Bor},G}(M; \mathbb{R}),$$

which by the usual theory is an isomorphism if we tensor with the reals on the left. The problem is that one would like equivariant K-theory to be represented by geometric objects at least on compact G-manifolds M. From this perspective, the interesting generalization of K-theory to the equivariant setting is not $K^*_{\mathrm{Bor},G}$, but rather the Atiyah-Segal $K^*_G(M)$ defined in [Seg68, §2], which we will now review.

Definition 7.1. A G-vector bundle on a G-manifold M is a vector bundle $\pi\colon E \to M$ together with a G-space structure on E such that

(i) π is an equivariant map,
(ii) if $g \in G$ and $m \in M$, then $g\colon \pi^{-1}(m) \to \pi^{-1}(gm)$ is a linear map.

We say that two G-vector bundles are isomorphic if there exists a vector bundle isomorphism that is also a G-map. It is also clear that the usual constructions like direct sum and tensor product of vector bundles apply directly to G-bundles. We can now employ the usual Grothendieck construction in order to define equivariant K-theory.

Definition 7.2. Equivariant K-theory $K_G(M)$ of a compact G-manifold M is defined to be the Grothendieck group of the abelian monoid given by isomorphism classes of G-vector bundles over M with direct sum.

An element in $K_G(M)$ is therefore given by a formal difference of G-vector bundles $[V] - [W]$. As in the non-equivariant case, the assignment $M \mapsto K_G(M)$ satisfies certain properties that we require from an equivariant cohomology theory, i.e. G-homotopy equivalence, excision and also additivity under disjoint union. It is therefore reasonable to set $K_G^0(M) = K_G(M)$ and define $K_G^{-n}(M)$ by the suspension isomorphism. Denote by $\widetilde{K}_G^*(M)$ the reduced version of equivariant K-theory, i.e. the kernel of the map

$$K_G^*(M) \to K_G^*(*)$$

given by the inclusion of a basepoint. Then, we have

$$\begin{aligned} K_G^{-1}(M) = \widetilde{K}_G^{-1}(M_+) \cong \widetilde{K}_G^0(S^1 \wedge M_+) &\cong \widetilde{K}_G^0((S^1 \times M)/(\{1\} \times M)) \\ &\cong K_G^0(S^1 \times M, \{1\} \times M), \end{aligned}$$

where we equip S^1 always with the trivial G-action, and M_+ denotes the disjoint union of M with a basepoint. From the pair sequence in equivariant K-theory, we now get

$$\cdots \to K_G^0(S^1 \times M, \{1\} \times M) \to K_G^0(S^1 \times M) \to K_G^0(\{1\} \times M) \to \cdots .$$

Since the first map in this sequence is injective, one can describe $K_G^{-1}(M)$ via G-vector bundles on $S^1 \times M$, which have the additional property to be trivial when restricted to $\{1\} \times M$. In other words, $K_G^{-1}(M)$ is exactly the kernel of the map

$$K_G^0(S^1 \times M) \to K_G^0(M) \tag{9}$$

given by the pullback with the inclusion $\{1\} \times M \to S^1 \times M$. There is an equivariant version of Bott periodicity that takes care of all K_G^n for $n \in \mathbb{Z}$.

Proposition 7.3. *The group $K_G^k(M)$ is naturally isomorphic to $K_G^{k-2}(M)$, where the map is given by multiplication by the Bott element in $K_G^{-2}(\mathrm{pt})$.*

PROOF. See [Seg68, Proposition 3.5]. □

It turns out that this is just a special case of a much more general statement, which is, in Segal's words, the most important theorem in equivariant K-theory: The equivariant Thom isomorphism theorem. If M is non-compact, we denote by M^+ the one-point compactification. If M is already compact, we set $M^+ = M_+$. Define the compactly supported theory

$$(K_G^*)_{\mathrm{cpt}}(M) = \widetilde{K}_G^*(M^+).$$

We then have the following theorem.

Proposition 7.4. *For a G-vector bundle $E \to M$ on a compact G-manifold M, the Thom homomorphism*

$$K_G^*(M) \to (K_G^*)_{\mathrm{cpt}}(E)$$

with target the compactly supported equivariant K-theory of E, is an isomorphism.

PROOF. See [Seg68, Proposition 3.3]. □

Example 7.5. It follows directly from the definition that the equivariant K-theory of a point is the representation ring $R(G)$, concentrated in degree 0.

Example 7.6. Take a complex representation V of G. Then, we can examine the representation sphere $S^V = V_+ \cong V \cup \{\infty\}$, where the point at infinity is fixed by the action. Proposition 7.4 now gives that

$$\widetilde{K}^*_G(S^V) \cong (K^*_G)_{\mathrm{cpt}}(V) \cong K^*_G(*) \cong R(G),$$

where the first equality is by definition.

We close our short digression on equivariant K-theory by computing $K^*_G(M)$ for the two extreme cases of G-actions: The trivial action, and free actions. For the latter we have the following result.

Lemma 7.7. *If G acts freely on M, the projection $\pi\colon M \to M/G$ induces an isomorphism*

$$\pi^*\colon K^*_G(M) \to K^*(M/G).$$

PROOF. If G acts freely on M and if $E \to M$ is a G-vector bundle, the quotient E/G is an ordinary vector bundle over M/G. We claim that the map induced by the assignment $E \mapsto E/G$ is an inverse to π^*. We have a commutative diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d] & M \times_{M/G} E/G \arrow[r, equal] \arrow[d] & \pi^*(E/G) \\ M \arrow[r, equal] & M & \end{tikzcd},$$

where the map on the top is induced by the vector bundle projection $E \to M$ and the projection $E \to E/G$. It is easy to see that this is an isomorphism of G-vector bundles.

For the other direction, if we are given a vector bundle F on M/G, the projection onto the second factor induces an isomorphism $\pi^* F = M \times_{M/G} F \to F$. The odd case follows from the same argument applied to $M \times S^1$. □

On the other hand, if G acts trivially on M, then each fiber E_m is a G-representation. If we are given a representation V, regarded as a trivial G-vector bundle over M, we can consider the ordinary vector bundle given by $\mathrm{Hom}_G(V, E)$. As an example, the trivial representation is assigned to the vector bundle E^G, i.e. the fixed point set of E. It turns out that this way of probing our G-vector bundle with representations gives all the equivariant information, if we consider all possible irreducible representations $V \in \mathrm{Irr}(G)$.

Lemma 7.8. *If G acts trivially on M, the natural map*

$$\mu\colon K^*(M) \otimes R(G) \to K^*_G(M)$$

is an isomorphism.

PROOF. We will construct an inverse to μ. Because G acts trivially on M, it acts on each fiber of a G-vector bundle $E \to M$, and there is an operation of averaging over G, varying continuously. Therefore, for any representation V, the functor

$$E \mapsto \mathrm{Hom}_G(V, E) = \mathrm{Hom}(V, E)^G$$

induces a homomorphism of abelian groups $K^0_G(M) \to K^0(M)$. We claim that the map

$$\begin{aligned} \phi\colon K^0_G(M) &\to K^0(M) \otimes R(G) \\ [E] &\mapsto \sum_{V \in \mathrm{Irr}(G)} [\mathrm{Hom}_G(V, E)] \otimes V \end{aligned}$$

is an inverse to μ. If E is a G-vector bundle over M, then it can be decomposed canonically into isotypical components

$$E \cong \bigoplus_{V \in \mathrm{Irr}(G)} \mathrm{Hom}_G(V, E) \otimes V,$$

which proves that $\mu \circ \phi = \mathrm{id}$.

On the other hand, let V and W be irreducible representations. We consider the element $\phi \circ \mu(E \otimes W)$, where G acts trivially on E. Clearly,

$$\mathrm{Hom}_G(V, E \otimes W) \cong \mathrm{Hom}_G(V, W) \otimes E.$$

By Schur's Lemma, the last bundle is either E or 0, which proves that also $\phi \circ \mu = \mathrm{id}$. The odd case follows by applying the same reasoning to the space $M \times S^1$. □

Example 7.9. For a subgroup $H \subset G$, the homogeneous spaces G/H are often called equivariant points, since they appear as the building blocks for G-CW complexes. One can show that the category of G-vector bundle on a homogeneous space G/H is equivalent to the category H-representations, i.e. every G-vector bundle E over G/H is of the form

$$E = G \times_H V$$

for some H-representation V. It follows that $K^0_G(G/H) = R(H)$. Along the same lines, G-vector bundles on $G/H \times S^1$ with the trivial action on S^1 are equivalent to H-vector bundles on S^1 by restriction. Therefore,

$$K^0_G(G/H \times S^1) \cong K^0_H(S^1) \cong K^0(S^1) \otimes R(H) \cong R(H).$$

It is now clear that we have

$$K^{-1}(G/H) = \ker\left(K^0_G(G/H \times S^1) \to K^0_G(G/H)\right) = 0.$$

We will now compare the geometric version of equivariant K-theory, defined via G-vector bundles, with the naive version that comes from applying the Borel construction to ordinary K-theory. Keep in mind that the overarching goal is the construction of a good equivariant version of the Chern character. Recall that the projection map $\pi_2\colon EG \times M \to M$ induces a map

$$\begin{tikzcd} K_G^*(M) \arrow[r, "\pi_2^*"] \arrow[rrr, bend right, "\alpha"'] & K_G^*(EG \times M) \arrow[r, "\cong"] & K^*(EG \times_G M) \arrow[r, "="] & K^*_{Bor,G}(M), \end{tikzcd}$$

and by composing with $\mathrm{ch}_{\mathrm{Bor},G}$, we immediately get a Chern character map valued in Borel cohomology. The problem with this approach is that the map α is far from being an isomorphism. In fact, the Atiyah–Segal completion theorem [AS69a, Proposition 4.2.] tells us that α precisely induces an isomorphism after we apply the I_G-adic completion at the dimension ideal I_G of $K_G^*(M)$. Thus, working with the Borel–Chern character in the main diagram of differential cohomology would mean ignoring important information.

The solution is to enrich the target of the Chern character map to a better suited equivariant cohomology theory. The easiest version of such a theory is the delocalized equivariant cohomology defined in [BBM85][BC88, §1]. We review the construction. Recall that a class function on G is a function $f\colon G \to \mathbb{C}$ that is constant on each conjugacy class. We have the following well-known fact from the representation theory of finite groups.

Lemma 7.10. *The irreducible characters span the space of complex-valued class functions on G, i.e. the character homomorphism*

$$\chi\colon R(G) \to \mathbb{C}[G]^G$$
$$[V] - [W] \mapsto \mathrm{tr}_V - \mathrm{tr}_W$$

is an isomorphism after tensoring with $\mathbb{C}$.

Proof. See [Ser77, Section 2, Theorem 6]. □

The following splitting result according to [AS89, Theorem 2] is ultimately just a consequence of the simple linear algebra result in the previous lemma. Recall that the action of G permutes fixed point sets in the following way: If $x \in M^g$, and we act with a group element $h \in G$, then one has

$$(hgh^{-1})(hx) = hx.$$

Therefore, left multiplication by h sends M^g to $M^{hgh^{-1}}$. One can define the so called Brylinski space

$$\widehat{M} = \{(x,g) \in M \times G \mid gx = x\} \cong \coprod_{g \in G} M^g$$

which admits a G-action given by

$$h(x,g) = (hx, hgh^{-1}).$$

The K-theory of $\widehat{M}$ is the direct sum

$$K^*(\widehat{M}) \cong \bigoplus_{g \in G} K^*(M^g),$$

and there is an action induced by the space level action just described. We have the following theorem.

Theorem 7.11. *Let G be a finite group, and M be a G-manifold. Then, there is a natural map*

$$\Phi\colon K_G^*(M) \to \left(\bigoplus_{g\in G} K^*(M^g)\otimes\mathbb{C}\right)^G, \tag{10}$$

where M^g is the fixed point set of $g\in G$. Furthermore, when tensored with the complex numbers on the left, this map becomes an isomorphism.

Proof. We first describe the map appearing in the theorem. Note that there is a pullback map, induced by inclusion of fixed point sets

$$K_G^*(M) \xrightarrow{i^*} K_G^*(\widehat{M})$$

We want to restrict further to the fixed point set M^g, but M^g is not a G-space. Instead, denote by C the cyclic subgroup of G generated by g. We can restrict to the subgroup $C\subset G$ and then pull back to the fixed point set. Since C acts trivially on $M^g = M^C$, we have an isomorphism

$$K_C^*(M^g) \xrightarrow{\phi} K^*(M^g)\otimes R(C)$$

by Lemma 7.8. In order to get from a representation V to a complex number, we now just apply the character homomorphism and evaluate the resulting class function at $g\in C$:

$$R(C) \xrightarrow{\chi} \mathbb{C}[C]^C \xrightarrow{\mathrm{ev}_g} \mathbb{C}.$$

All in all, we have the following long composition:

$$\begin{tikzcd}
K_G^*(M) \arrow[rr, "i_g^*"] & & K_C^*(M^g) \arrow[dll, "\phi"] \\
K^*(M^g)\otimes R(C) \arrow[r, "\mathrm{id}\otimes\chi"] & K^*(M^g)\otimes\mathbb{C}[C]^C \arrow[r, "\mathrm{id}\otimes\mathrm{ev}_g"] & K^*(M^g)\otimes\mathbb{C}.
\end{tikzcd}$$

Let us describe this map explicitly on a cycle coming from a G-vector bundle $E\to M$. We can restrict to a C-vector bundle $E_g\to M^g$ over the fixed point set for $g\in G$. On this bundle, g acts fiberwise with finite order, and therefore, the bundle splits canonically into a direct sum of subbundles $E_{g,\lambda}\to M^g$ for each eigenvalue λ of the g-action. The map Φ is then given as

$$[E]\mapsto \sum_{g\in G}\sum_{\lambda\in\mathrm{Eig}(g)} \lambda[E_{g,\lambda}].$$

We now check that the resulting class is indeed invariant under pullbacks by the global G-action on $\widehat{M}$, as claimed in the statement of the theorem. The reason for this is that we started with a G-equivariant bundle E: Consider for $h\in G$ the pullback bundle $h^*E_{hgh^{-1},\lambda}$.

Then, we have a bundle isomorphism

$$\begin{array}{ccc} h^*E_{hgh^{-1},\lambda} & \longrightarrow & E_{g,\lambda} \\ \downarrow & & \downarrow \\ M^g & \xrightarrow{\mathrm{id}} & M^g, \end{array}$$

where the top arrow is given by $(m, v) \mapsto h^{-1}v$. Since $K_G^{-1}(M)$ is equal to the kernel of the restriction map $K_G^0(M \times S^1) \to K_G^0(M)$ (see Equation 9), we can establish well-definedness in degree -1 by the same construction applied to $S^1 \times M$.

We sketch the proof that $\Phi \otimes \mathrm{id}_{\mathbb{C}}$ is an isomorphism. First, one checks that both sides of Equation 10 define $\mathbb{Z}_2$-graded equivariant cohomology theories that satisfy the equivariant Eilenberg–Steenrod axioms. Recall that in the non-equivariant case, a natural transformation $T^*\colon E^* \to F^*$ of generalized cohomology theories is an isomorphism if and only if it induces an isomorphism on the point, i.e. if

$$T^n(*)\colon E^n(*) \to F^n(*)$$

is an isomorphism for all n. In the equivariant case, there is a similar theorem, which says that the isomorphism property has to be checked on equivariant points, meaning on homogeneous spaces G/H for all subgroups $H \subset G$. In our case, on the left hand side, we have that $K_G^0(G/H) \cong R(H)$ and $K_G^1(G/H) \cong 0$ (see Example 7.9). For the right hand side, we can first rewrite as a sum over conjugacy classes

$$\left(\bigoplus_g K^*((G/H)^g) \otimes \mathbb{C}\right)^G \cong \bigoplus_{[g]} \left(K^*((G/H)^g) \otimes \mathbb{C}\right)^{Z_g}, \tag{11}$$

where Z_g is the centralizer of g in G. Note that this is an isomorphism, since the information for the other representatives $hgh^{-1} \in [g]$ can be reconstructed by invariance. Now, since the fibers of the projection from $EZ_g \times_{Z_g} (G/H)^g \to (G/H)^g/Z_g$ are rationally acyclic (they are classifying spaces of finite groups), we can make our action free, and the invariants agree with the K-theory of the quotient:

$$\begin{aligned} \left(K^*((G/H)^g) \otimes \mathbb{C}\right)^{Z_g} &\cong \left(K^*(EZ_g \times (G/H)^g) \otimes \mathbb{C}\right)^{Z_g} \\ &\cong K^*(EZ_g \times_{Z_g} (G/H)^g) \otimes \mathbb{C} \\ &\cong K^*((G/H)^g/Z_g) \otimes \mathbb{C}. \end{aligned}$$

Now, $(G/H)^g/Z_g$ is a discrete space, and we need to count how many points it has. Note that there is a map

$$\begin{aligned} f\colon \mathrm{Conj}(H) &\to \mathrm{Conj}(G) \\ \langle g \rangle_H &\mapsto \langle g \rangle_G, \end{aligned}$$

which assigns to an H-conjugacy class the corresponding conjugacy class in G. A class $\langle h \rangle_H$ is in $f^{-1}(\langle g \rangle_G)$ if there is a $k \in G$ such that $khk^{-1} = g$. In order to get the number of preimages, we need to count the $k's$ up to elements in H, and up to elements centralizing g. Therefore,

$$(K^*((G/H)^g) \otimes \mathbb{C})^{Z_g} = \mathbb{C}^{|f^{-1}(\langle g\rangle_G)|}.$$

Summing over these terms for all conjugacy classes in G then yields a copy of $\mathbb{C}$ for each conjugacy class in H. Recall that the ring of class functions $\mathbb{C}[H]^H$ consists of functions $H \to \mathbb{C}$ which are constant on conjugacy classes. This ring is therefore also generated by $|\mathrm{Conj}(H)|$ elements. We end up with the following commutative diagram

$$\begin{tikzcd}
K^*(G/H) \otimes \mathbb{C} \arrow[r, "\Phi\otimes \mathrm{id}_{\mathbb{C}}"] \arrow[d, "\cong"] & \left(\bigoplus_g K^*((G/H)^g) \otimes \mathbb{C}\right)^G \arrow[d, "\cong"] \\
R(H) \otimes \mathbb{C} \arrow[r, "\chi"] & \mathbb{C}[H]^H,
\end{tikzcd}$$

and so, we showed the Φ in this case reduces to the character homomorphism χ from representation theory. By Lemma 7.10, χ becomes an isomorphism when tensored with $\mathbb{C}$, and the claim follows. $\square$

Remark 7.12. There is another way to interpret the result of this theorem. For a G-manifold M, since $K_G^*(M)$ is a $K_G^*(*) = R(G)$-module, we have that $K_G^*(M) \otimes \mathbb{C}$ is a module over the ring of class functions

$$\mathbb{C}[G]^G \cong \mathbb{C}^{|\mathrm{Conj}(G)|}.$$

For a conjugacy class $\langle g\rangle \in \mathrm{Conj}(G)$, there is a corresponding prime ideal in this ring, given by the class functions that vanish at $\langle g\rangle$. It is clear that any $\mathbb{C}[G]^G$-module must split into a direct sum of the localized $\mathbb{C}[G]^G_{\langle g\rangle} \cong \mathbb{C}$-modules. We recall the localization theorem by Atiyah and Segal (see [AS68, Theorem 1.1]). Denote by

$$M^{\langle g\rangle} = \bigcup_{h\in\langle g\rangle} M^h$$

the fixed point set for a conjugacy class. Then, its content is that the restriction

$$K_G^*(M) \to K_G^*(M^{\langle g\rangle})$$

is an isomorphism after localizing at the prime ideal given by $\langle g\rangle$. Therefore, we get a splitting

$$\begin{aligned}
K_G^*(M) \otimes \mathbb{C} &\cong \bigoplus_{\langle g\rangle} K_G^*(M)_{\langle g\rangle} \otimes \mathbb{C} \\
&\cong \bigoplus_{\langle g\rangle} K_G^*(M^{\langle g\rangle})_{\langle g\rangle} \otimes \mathbb{C}.
\end{aligned}$$

Now, the result can be rephrased by saying that the $\langle g\rangle$-part is isomorphic to $(K^*(M^g)\otimes\mathbb{C})^{Z_g}$, i.e. that the inclusion

$$\coprod_{h\in\langle g\rangle} M^h \hookrightarrow \bigcup_{h\in\langle g\rangle} M^h = M^{\langle g\rangle}$$

from the disjoint union into the ambient union induces an isomorphism. While entirely obvious when the fixed point components are disjoint, it is somewhat surprising that this holds in general.

The previous theorem exposes the problem with the Borel–Chern character. One can see now that there is a factorization

$$\begin{tikzcd}
K^*_G(M) \arrow[r, "\mathrm{ch}_{\mathrm{Bor}}"] \arrow[d, "\Phi"] & H^*_{\mathrm{Bor},G}(M;\mathbb{C}) \\
\left(\bigoplus_{g\in G} K^*(M^g)\otimes\mathbb{C}\right)^G \arrow[d] & \\
K^*(M^e)\otimes\mathbb{C} \arrow[uur, bend right] &
\end{tikzcd},$$

where the lower vertical map is the projection onto the "untwisted sector" for the identity element $e\in G$. Therefore, all the contributions that come from the other fixed point sets of the G-action are completely ignored. For this reason, one says that the Borel–Chern character is localized at the identity element. But the decomposition theorem also gives a recipe to fix the situation. We just have to enlarge the target cohomology theory in order to account for all the additional fixed point sets.

Definition 7.13. [BC88, §1] The $\mathbb{Z}_2$-graded (delocalized) equivariant cohomology of a G-manifold M is

$$H^0_G(M) = \left(\bigoplus_{g\in G}\prod_{k\in\mathbb{N}} H^{2k}(M^g;\mathbb{C})\right)^G$$

$$H^1_G(M) = \left(\bigoplus_{g\in G}\prod_{k\in\mathbb{N}} H^{2k+1}(M^g;\mathbb{C})\right)^G.$$

Functoriality in delocalized equivariant cohomology is realized by pullback of ordinary cohomology classes via the induced maps on fixed point sets.

Definition 7.14. [BC88, §1] The (delocalized) equivariant Chern character is given by the composition

$$\begin{tikzcd} K_G^*(M) \arrow[r, "\Phi"] \arrow[rr, bend right, "\mathrm{ch}_G"'] & \left(\bigoplus_{g\in G} K^*(M^g)\otimes\mathbb{C}\right)^G \arrow[r, "\bigoplus_g \mathrm{ch}"] & H_G^*(M) \end{tikzcd}$$

where ch is just the ordinary Chern character tensored with the identity on $\mathbb{C}$.

Proposition 7.15. *The equivariant Chern character is a well-defined homomorphism of rings. Furthermore, after tensoring with $\mathbb{C}$, it is an isomorphism.*

Proof. This is almost a triviality, since all the work was in establishing the corresponding properties for Φ in Theorem 7.11. In order to check well-definedness, consider an element

$$\bigoplus_g [E_g] \in \left(\bigoplus_g K^*(M^g)\otimes\mathbb{C}\right)^G.$$

Then, for $h \in G$, this means that $[h^*E_g] = [E_{hgh^{-1}}]$. Therefore, since the Chern character is natural under pullback, we have

$$h^*\mathrm{ch}([E_g]) = \mathrm{ch}([h^*E_g]) = \mathrm{ch}([E_{hgh^{-1}}]),$$

which is what we had to show. The homomorphism properties follow from the corresponding properties of the non-equivariant Chern character. The equivariant Chern character is an isomorphism after tensoring with $\mathbb{C}$, since the same is true for the ordinary Chern character. $\square$

Example 7.16. Let us compute the cohomology and Chern character of the representation sphere S^V for the non-trivial one dimensional $\mathbb{Z}_2$-representation. Denote by $\tau \in \mathbb{Z}_2$ the generator. The fixed point sets are

$$(S^V)^e \cong S^2 \qquad \text{and} \qquad (S^V)^\tau \cong S^0.$$

Therefore, the delocalized cohomology is

$$\begin{aligned} H^*_{\mathbb{Z}_2}(S^V) &\cong H^*((S^V)^e;\mathbb{C})^{\mathbb{Z}_2} \oplus H^*((S^V)^\tau;\mathbb{C}) \\ &\cong H^*(S^2/\mathbb{Z}_2;\mathbb{C}) \oplus H^*(S^0;\mathbb{C}) \\ &\cong H^*(S^2;\mathbb{C}) \oplus H^*(S^0;\mathbb{C}). \end{aligned}$$

With this result in mind, we turn to the equivariant K-theory. Since the space in question is a representation sphere of a complex representation, we see via Example 7.6 that

$$K_{\mathbb{Z}_2}(S^V) \cong R(\mathbb{Z}_2) \oplus R(\mathbb{Z}_2) \cong (\mathbb{Z}[X]/(X^2-1))^2,$$

concentrated in even degree. We can give explicit representatives for the bundles that generate this ring. We get two representatives by equipping the trivial line bundle with the two possible $\mathbb{Z}_2$-representations (call these bundles $\mathbb{C}_+$ and $\mathbb{C}_-$). The other representatives are induced by the Hopf bundle $H_\pm$ with a trivial and a non-trivial action. Write the Hopf bundle $H_\pm$ as

$$H_\pm = \{([x_0 : x_1], (\lambda x_0, \lambda x_1)) \mid \lambda \in \mathbb{C}\},$$

where $[x_0 : x_1]$ are understood as homogeneous coordinates for $\mathbb{C}P^1 \cong S^2$. The action on $H_\pm$ by the generator $\tau \in \mathbb{Z}_2$ is given by

$$\tau\left([x_0 : x_1], (\lambda x_0, \lambda x_1)\right) = \left([\pm x_0 : x_1], (\pm\lambda x_0, \lambda x_1)\right). \tag{12}$$

We compute the Chern character in these four cases.

The target complex cohomology of our space consists of four copies of $\mathbb{C}$, all in even degree. We will denote elements in it by four-tupels

$$(a_1, a_2, a_3, a_4)^T \in H^0(S^2) \oplus H^2(S^2) \oplus H^0(S \in S_0) \oplus H^1(N \in S_0),$$

where N and S denote the north and south pole in $S^0 \subset S^2$, respectively. The first two entries will come from the ch_e-part of the equivariant Chern character, while the latter two come from ch_τ. After choosing suitable generators of the respective cohomology groups, the results are given in the table below. The Chern character part for the fixed point set of the identity element is always just the ordinary non-equivariant Chern character and therefore independent of the group action. This explains the first two entries in the respective vector. For the third and fourth component, we have to consider the τ-eigenvalue decomposition of the fiber over both points of $S^0 = \{N, S\}$. Since it is just a line bundle, the entry is either going to be $+1$, if τ acts by the identity, or -1, if it acts by flipping the sign.

$\mathbb{C}_+$	$\mathbb{C}_-$	H_+	H_-
$\begin{pmatrix}1\\0\\1\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\-1\\-1\end{pmatrix}$	$\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\\1\\-1\end{pmatrix}$

8. Equivariant classifying spaces

We have seen in the last chapter that there are geometric models $\mathrm{Gr}_{\mathrm{res}}$ and U^1 for the classifying spaces of even and odd K-theory, together with universal cocycle representatives for the universal Chern character. We are now going to generalize this to the equivariant setting that was introduced in the last section.

In the construction of $\mathrm{Gr}_{\mathrm{res}}$ and U^1, we used a generic infinite-dimensional separable Hilbert space $\mathscr{H}$, in the even case together with a $\mathbb{Z}_2$-grading. In the non-equivariant setting, this played the role of a sufficiently big ambient space, containing all possible finite-dimensional vector spaces. In the equivariant setting, we therefore have to replace $\mathscr{H}$

with a representation big enough to contain all possible finite-dimensional representations. Such a G-representation can easily be constructed by just taking the tensor product with the regular representation $L^2(G)$, given by the vector space of functions $f\colon G \to \mathbb{C}$ with action $(gf)(x) = f(g^{-1}x)$. We define the polarized Hilbert space

$$\begin{aligned}\mathscr{H}_G = \mathscr{H} \otimes L^2(G) &\cong (\mathscr{H}_G)_+ \oplus (\mathscr{H}_G)_- \\ &\cong \mathscr{H}_+ \otimes L^2(G) \oplus \mathscr{H}_- \otimes L^2(G),\end{aligned}$$

with the linear polarization preserving G-action on the second factor. This induces an action on the space of bounded operators, given by conjugation:

$$\begin{aligned} G \times \mathfrak{gl}(\mathscr{H}_G) &\to \mathfrak{gl}(\mathscr{H}_G) \\ (g, A) &\mapsto gAg^{-1}. \end{aligned}$$

Observe that subspaces like $\mathrm{GL}_{\mathrm{res}}(\mathscr{H}_G)$, $\mathrm{U}_{\mathrm{res}}(\mathscr{H}_G)$, $\mathrm{U}^1(\mathscr{H}_G)$ and $\mathrm{Fred}(\mathscr{H}_G)$ are preserved by this action, and furthermore, all of these actions are smooth. Furthermore, we also have smooth G-actions on $\mathrm{Gr}_{\mathrm{res}}(\mathscr{H}_G)$ and $\mathrm{St}_{\mathrm{res}}(\mathscr{H}_G)$, given by $W \mapsto gW \in \mathrm{Gr}_{\mathrm{res}}(\mathscr{H}_G)$ and $w \mapsto gwg^{-1} \in \mathrm{St}_{\mathrm{res}}(\mathscr{H}_G)$.

We have seen earlier that even degree K-theory is classified by the space of Fredholm operators. The original proof by Atiyah can be generalized to the equivariant setting, and one has the following result:

Theorem 8.1. *The G-space $\mathrm{Fred}(\mathscr{H}_G)$ classifies equivariant K-theory in degree 0, i.e. for any compact manifold M, we have*

$$K^0_G(M) \cong [M, \mathrm{Fred}(\mathscr{H}_G)]_G.$$

The group structure on the right is induced by the block sum (Definition 6.4).

Proof. See the main theorem in [Mat71]. The second statement follows from the concrete form of Atiyah's index isomorphism, where a block sum of Fredholm operators corresponds exactly to the sum of G-vector bundles of its kernel and cokernel. □

Our goal is to show that we can again, as in the non-equivariant case, replace Fredholm operators by a more structured space of operators. Recall the concept of a homotopy equivalence in the equivariant setting.

Definition 8.2. A G-homotopy equivalence between two G-spaces X and Y is an equivariant map $f\colon X \to Y$ which has an inverse $g\colon Y \to X$ such that $f \circ g \sim \mathrm{id}_Y$ and $g \circ f \sim \mathrm{id}_X$ with homotopies through equivariant maps.

If two G-spaces X, Y are G-homotopy equivalent, we have that the sets of G-homotopy classes of maps

$$[M, X]_G \cong [M, Y]_G$$

agree for any G-space M. There is a convenient way to check if a map is a G-homotopy equivalence.

Theorem 8.3. *(Equivariant Whitehead Theorem) Let $f\colon X \to Y$ be an equivariant map between two G-CW-complexes. Then, f is a G-homotopy equivalence if and only if for all subgroups H, the induced maps on fixed point sets $f^H\colon X^H \to Y^H$ are (weak) homotopy equivalences in the usual sense.*

PROOF. See for example [Wan80, Theorem 3.4]. □

By using the equivariant Whitehead theorem, it is possible to understand the equivariant situation by analyzing carefully the fixed point sets of the relevant spaces. Recall that we essentially passed to the equivariant setting by looking at operator spaces on $L^2(G) \otimes \mathscr{H}$ instead of just $\mathscr{H}$. It turns out that the fixed point sets of such operator spaces are often of a quite simple form.

Lemma 8.4. *Let $\mathscr{H}$ be an infinite-dimensional complex separable Hilbert space and G be a finite group. Fix a complete representing set of irreducible G-representations $V \in \mathrm{Irr}(G)$. There is an equivariant decomposition into isotypical components*

$$L^2(G) \otimes \mathscr{H} \cong \bigoplus_{V \in \mathrm{Irr}(G)} V \otimes \mathscr{H}.$$

Furthermore, consider an operator $A \in \mathfrak{gl}(L^2(G) \otimes \mathscr{H})^G$ in the fixed point set for the conjugation action. Then, with respect to the above decomposition, the operator is block-diagonal, and each component is of the form

$$A\big|_{V\otimes\mathscr{H}} = \mathrm{id}_V \otimes B_V$$

for some operator $B_V \in \mathfrak{gl}(\mathscr{H})$.

PROOF. The decomposition for $\mathscr{H}_G$ just follows from the regular representation $L^2(G)$ being decomposable. The dimensions of the isotypical components do not matter, since we tensor with an infinite-dimensional space anyway.

Now, an operator A fixed by the G-action has the property that

$$gAg^{-1} = A, \quad \text{for all } g \in G,$$

in other words, it has to commute with the G-action on $\mathscr{H}_G$. The decomposition into isotypicals gives orthogonal projections

$$\pi_V\colon \mathscr{H}_G \to V \otimes \mathscr{H}$$

for each component. Consider the restricted operator $A_{V,W} = \pi_W \circ A\big|_{V\otimes\mathscr{H}}$. We claim that this operator is either identically 0, if V is not equal to W, or of the form $\mathrm{id}_V \otimes B_V$ for some operator B_V, if $V = W$.

In order to prove this, choose an orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ of $\mathscr{H}$. There are orthogonal projections

$$\eta_i \colon W \otimes \mathscr{H} \to W \otimes \langle e_i \rangle,$$

and we consider the operator $\eta_i \circ A_{V,W}\big|_{V\otimes\langle e_j\rangle}$ for some natural numbers i, j. By Schur's Lemma, this restricted map is either identically 0, or a scalar multiple of the identity in the case that W is equal to V. It follows that $A_{V,W}$ is zero, unless $V = W$, in which case $A_{V,V}$ is of the form $\mathrm{id}_V \otimes B_V$, as claimed. □

Definition 8.5. Let $\mathscr{H}$ be a complex separable infinite-dimensional Hilbert space. Let φ be a condition that can be imposed on an operator in $\mathfrak{gl}(\mathscr{H})$, for example invertibility, Fredholmness, etc. Then φ defines a subspace $X_\varphi \subset \mathfrak{gl}(\mathscr{H})$. We say that φ is *finitely additive*, if the following is true: If an operator is block diagonal with respect to a decomposition into finitely many infinite-dimensional subspaces

$$\mathscr{H} = \mathscr{H}_1 \oplus \cdots \oplus \mathscr{H}_k,$$

i.e. we have

$$A = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_k \end{pmatrix},$$

then A satisfies φ if and only if all of the A_i satisfy φ.

Proposition 8.6. *Let $X_\varphi(\mathscr{H}_G)$ be a subspace of $\mathfrak{gl}(\mathscr{H}_G)$, given by a finitely additive condition φ. Then, for any subgroup $H \subset G$, the fixed point sets X_φ^H split block diagonally with respect to irreducible H-representations, i.e. for a complete representing set of irreducible representations, we have*

$$X_\varphi^H(\mathscr{H}_G) \cong \prod_{V\in\mathrm{Irr}(H)} X_\varphi(\mathscr{H}).$$

Proof. Consider first the case of $H = G$. In this case, the statement is implied by Lemma 8.4, since the relevant property φ of the operators is preserved under the restriction.

For the case of a proper subgroup $H \subset G$, let V be an irreducible G-representation. Then, we can always restrict this representation to H. As an H-representation, V now decomposes into a direct sum of isotypical components

$$V \cong \bigoplus_{W\in\mathrm{Irr}(H)} W^{n_{V,W}}.$$

If we fix one of the W in the list of irreducible H-representations, because of Frobenius reciprocity, we can always find an irreducible G-representation V such that W is a subrepresentation of the H-restriction of V. This asserts that for any fixed W, at least one $n_{V,W}$ is non-zero, and we conclude that we have an isomorphism of Hilbert spaces with

H-action

$$\bigoplus_{V \in \mathrm{Irr}(G)} W^{n_{V,W}} \otimes \mathscr{H} \cong W \otimes \mathscr{H},$$

using that $\mathscr{H}$ has a countable basis. We now have the following chain of isomorphisms of Hilbert spaces with H-action:

$$\begin{aligned}\mathscr{H}_G \cong L^2(G) \otimes \mathscr{H} &\cong \bigoplus_{V \in \mathrm{Irr}(G)} V \otimes \mathscr{H} \cong \bigoplus_{V \in \mathrm{Irr}(G)} \left(\bigoplus_{W \in \mathrm{Irr}(H)} W^{n_{V,W}} \right) \otimes \mathscr{H} \\ &\cong \bigoplus_{W \in \mathrm{Irr}(H)} \bigoplus_{V \in \mathrm{Irr}(G)} W^{n_{V,W}} \otimes \mathscr{H} \cong \bigoplus_{W \in \mathrm{Irr}(H)} W \otimes \mathscr{H} \cong \mathscr{H}_H.\end{aligned}$$

This gives rise to an equivariant diffeomorphism of spaces with H-action $X_\varphi(\mathscr{H}_G) \cong X_\varphi(\mathscr{H}_H)$, and we have successfully reduced to the case $H = G$, which was handled in the beginning of the proof. □

Corollary 8.7. *We have a splitting as in the above Proposition in particular for the fixed point sets of the spaces* $\mathrm{Fred}(\mathscr{H}_G)$, $\mathrm{U}_{\mathrm{res}}(\mathscr{H}_G)$ *and* $\mathrm{U}^1(\mathscr{H}_G)$. *Furthermore, we also have the splitting for* $\mathrm{Gr}_{\mathrm{res}}(\mathscr{H}_G)$, $\mathrm{St}_{\mathrm{res}}(\mathscr{H}_G)$, *and the loop and path spaces of all the above spaces.*

Proof. For the first three spaces, it is enough to observe that the defining condition is finitely additive (Definition 8.5). The restricted Grassmannian, when seen as a space of self-adjoint projection operators, is also an operator space defined by a finitely additive condition (see Proposition 5.2). After identifying $\mathscr{H}_+ \otimes L^2(G)$ and $\mathscr{H} \otimes L^2(G)$, the same is true for the restricted Stiefel manifold.

As for the path and loop spaces, we note that if Y is a space with trivial G-action, we the following splitting of smooth mapping spaces:

$$\begin{aligned}\mathrm{Map}(Y, X_\varphi)^H \cong \mathrm{Map}(Y, X_\varphi^H) &\cong \mathrm{Map}(Y, \prod_{V \in \mathrm{Irr}(H)} X_\varphi(\mathscr{H})) \\ &\cong \prod_{V \in \mathrm{Irr}(H)} \mathrm{Map}(Y, X_\varphi(\mathscr{H})).\end{aligned}$$

Since G acts trivially on both S^1 and the interval I when we form the smooth loop and path spaces, it follows that the fixed point sets split into a product in these cases as well. □

In the remainder of this section, we will use the previous results in order to show that $\mathrm{Gr}_{\mathrm{res}}(\mathscr{H}_G)$ and $\mathrm{U}^1(\mathscr{H}_G)$ are classifying spaces for equivariant K-theory. Recall the projection map

$$\begin{aligned}\psi \colon \mathrm{U}_{\mathrm{res}} &\to \mathrm{Fred} \\ \begin{pmatrix} X_{++} & X_{-+} \\ X_{+-} & X_{--} \end{pmatrix} &\mapsto X_{++}\end{aligned} \tag{13}$$

from Equation 5.

Theorem 8.8. *There is a zig-zag of G-homotopy equivalences*

$$\mathrm{Gr}_{\mathrm{res}} \xleftarrow{\pi} \mathrm{U}_{\mathrm{res}} \xrightarrow{\psi} \mathrm{Fred},$$

where π is the projection and ψ is the map from Equation 13.

Proof. Equivariance of ψ and π is obvious. We first check that ψ induces homotopy equivalences on fixed point sets. Let $H \subset G$ be a subgroup. Given a complete representing set of irreducible H-representations, we deduce from Corollary 8.7 the commutative diagram

$$\begin{tikzcd}
\mathrm{U}_{\mathrm{res}}^H \arrow[r, "\psi"] \arrow[d, "\cong"] & \mathrm{Fred}^H \arrow[d, "\cong"] \\
\prod\limits_{V \in \mathrm{Irr}(H)} \mathrm{U}_{\mathrm{res}} \arrow[r, "\prod \psi"] & \prod\limits_{V \in \mathrm{Irr}(H)} \mathrm{Fred}.
\end{tikzcd}$$

Since we know that each ψ in the bottom row is a homotopy equivalence from the non-equivariant case, it follows that the map on the top also is one. The claim follows by employing Theorem 8.3. Again by Corollary 8.7, we get another commutative diagram

$$\begin{tikzcd}
\mathrm{U}_{\mathrm{res}}^H \arrow[r, "\pi"] \arrow[d, "\cong"] & \mathrm{Gr}_{\mathrm{res}}^H \arrow[d, "\cong"] \\
\prod\limits_{V \in \mathrm{Irr}(H)} \mathrm{U}_{\mathrm{res}} \arrow[r, "\prod \pi"] & \prod\limits_{V \in \mathrm{Irr}(H)} \mathrm{Gr}_{\mathrm{res}}.
\end{tikzcd}$$

Since we know that the bottom map is a homotopy equivalence from the non-equivariant case, the claim follows. ☐

Working towards the odd case, recall that over $\mathrm{Gr}_{\mathrm{res}}^0$, there exists a universal smooth GL^1-bundle $\mathrm{St}_{\mathrm{res}} \to \mathrm{Gr}_{\mathrm{res}}^0$ with connection Θ (cf. Corollary 5.8 and Proposition 5.9). Consider the loop space $\Omega\mathrm{Gr}_{\mathrm{res}}$, based at $(\mathscr{H}_G)_+ \in \mathrm{Gr}_{\mathrm{res}}^0$. Parallel transport via the connection Θ gives rise to the holonomy map, which assigns to such a loop the fiber coordinate of the endpoint of the horizontal lift of this loop, starting at $w_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathrm{St}_{\mathrm{res}}$. We can now make the following definition:

Definition 8.9. The odd periodicity map h_{odd} is the composition of the holonomy map $\Omega\mathrm{Gr}_{\mathrm{res}} \to \mathrm{GL}^1$ with the homotopy equivalence given by polar decomposition, which maps $T \mapsto T|T|^{-1} \in \mathrm{U}^1$.

Theorem 8.10. *The map h_{odd} is a G-homotopy equivalence $\Omega\mathrm{Gr}_{\mathrm{res}} \to \mathrm{U}^1$.*

Proof. Since h_{odd} implements holonomy in the fibration $\mathrm{U} \to E\mathrm{U} \to B\mathrm{U}$, it is clear that it is a non-equivariant homotopy equivalence. To see that it is also G-equivariant, let $\tilde{f}\colon S^1 \to \mathrm{St}_{\mathrm{res}}$ be a horizontal lift of the loop $f\colon S^1 \to \mathrm{Gr}_{\mathrm{res}}^0$. This means that $\Theta(f'(t)) = 0$

for all $t \in S^1$. Then $g\tilde{f}$ is still horizontal, since, if we write $f(t) = w \in \mathrm{St}_{\mathrm{res}}$, and $W = \pi(w) \in \mathrm{Gr}_{\mathrm{res}}$ for the corresponding subspace in $\mathrm{Gr}_{\mathrm{res}}$, we have

$$\begin{aligned}\Theta((g\tilde{f})'(t)) = \Theta(g\tilde{f}'(t)) &= gw^{-1}g^{-1}g\pi_W g^{-1}\mathrm{d}(gwg^{-1}) \\ &= gw^{-1}\pi_W \mathrm{d}wg^{-1} \\ &= g\Theta(\tilde{f}'(t))g^{-1} = 0.\end{aligned}$$

Therefore, $g\tilde{f}$ is a horizontal lift of gf. Its endpoint is just the conjugation of the endpoint of $\tilde{f}$. Furthermore, the homotopy equivalence given by polar decomposition is also equivariant with respect to conjugation with G. Therefore, h_{odd} is an equivariant map.

We additionally have to check that it induces a homotopy equivalence on all fixed point sets. Note that in the splitting of the fixed point set

$$\Omega\mathrm{Gr}_{\mathrm{res}}^H \cong \prod_{V\in\mathrm{Irr}(H)} \Omega\mathrm{Gr}_{\mathrm{res}},$$

the holonomy map restricts to a product of holonomy maps on each of the blocks, which we know to be a homotopy equivalence. Therefore, we have a G-homotopy equivalence $\Omega\mathrm{Gr}_{\mathrm{res}} \to \mathrm{U}^1$. □

The following Corollary is now an immediate consequence of the previous two theorems.

Corollary 8.11. *We have isomorphisms of abelian groups*

$$K_G^0(M) \cong [M, \mathrm{Gr}_{\mathrm{res}}]_G, \qquad K_G^{-1}(M) \cong [M, \mathrm{U}^1]_G,$$

where the addition is pulled back from Atiyah's spaces of Fredholm operators via the above homotopy equivalences.

Remark 8.12. We will improve on this in the next chapter by giving a much more concrete geometric implementation of the group structure on $K_G^0(M)$ and $K_G^{-1}(M)$ directly on our models.

9. Cocycle representatives in the equivariant case

We have proven that the functors $K_G^0(-)$ and $K_G^1(-)$ are representable. As such, by the Yoneda lemma, a natural transformation like the Chern character corresponds to an element in the cohomology of the classifying space $\mathrm{Gr}_{\mathrm{res}}$ and U^1. We will now describe this element more explicitly. Keep in mind that, in the even case, each fixed point set is of the form

$$\mathrm{Gr}_{\mathrm{res}}^g \cong \prod_{V\in\mathrm{Irr}(\langle g\rangle)} \mathrm{Gr}_{\mathrm{res}}.$$

Then, the g-component of the universal even Chern character is a certain class in

$$\prod_{k\in\mathbb{Z}} H^{2k}(\mathrm{Gr}^g_{\mathrm{res}};\mathbb{C}) \cong \prod_{k\in\mathbb{Z}} H^{2k}\left(\prod_{V\in\mathrm{Irr}(\langle g\rangle)} \mathrm{Gr}_{\mathrm{res}};\mathbb{C}\right).$$

In the same way, the universal odd Chern form is a certain form in

$$\prod_{k\in\mathbb{Z}} H^{2k-1}((\mathrm{U}^1)^g;\mathbb{C}) \cong \prod_{k\in\mathbb{Z}} H^{2k-1}\left(\prod_{V\in\mathrm{Irr}(\langle g\rangle)} \mathrm{U}^1;\mathbb{C}\right).$$

It turns out that we can represent the full equivariant universal Chern character with differential forms that are constructed from the corresponding forms in the non-equivariant case. The groups that these forms live in are the following.

Definition 9.1. The groups of even and odd delocalized equivariant differential forms on a G-manifold M are

$$\Omega^0_G(M) = \left(\bigoplus_{g\in G}\prod_{k\in\mathbb{Z}} \Omega^{2k}(M^g;\mathbb{C})\right)^G \quad \text{and}$$
$$\Omega^1_G(M) = \left(\bigoplus_{g\in G}\prod_{k\in\mathbb{Z}} \Omega^{2k+1}(M^g;\mathbb{C})\right)^G.$$

Note that the factor-wise exterior differential is compatible with the translation action: If we denote by L_h the translation by $h\in G$, we have

$$L_h^*\left(\bigoplus_{g\in G} \mathrm{d}\omega_g\right) = \bigoplus_{g\in G} \mathrm{d}\left(L_h^*\omega_g\right).$$

Therefore, we can define the delocalized exterior differentials

$$\mathrm{d}^0_G = \bigoplus_{g\in G}\mathrm{d}\colon \Omega^0_G(M)\to\Omega^1_G(M) \quad \text{and}$$
$$\mathrm{d}^1_G = \bigoplus_{g\in G}\mathrm{d}\colon \Omega^1_G(M)\to\Omega^0_G(M).$$

Proposition 9.2. *The groups of delocalized differential forms give a de Rham model for delocalized cohomology (Definition 7.13). By this, we mean that the natural maps*

$$\varphi^0\colon \ker(\mathrm{d}^0_G)/\mathrm{im}(\mathrm{d}^1_G)\to H^0_G(M) \quad \text{and}$$
$$\varphi^1\colon \ker(\mathrm{d}^1_G)/\mathrm{im}(\mathrm{d}^0_G)\to H^1_G(M).$$

given by $\varphi^i\left(\left[\bigoplus_{g\in G}\omega_g\right]\right) = \bigoplus_{g\in G}[\omega_g]$ *are isomorphisms.*

Proof. It is obvious that φ^i is a group homomorphism. In order to show injectivity, assume that $\varphi^i([\omega]) = 0$ for $\omega\in\Omega^i_G(M)$. This means that $[\omega_g] = 0 \in H^*(M^g;\mathbb{C})$, and therefore, for any $g\in G$, there is an η_g such that $\omega_g = \mathrm{d}\eta_g$. Denote the collection of these forms by $\eta = (\eta_g)$. Averaging over the group action, we see that

$$\begin{aligned}\omega = \bigoplus_{g\in G} \omega_g &= \bigoplus_{g\in G} \left(\frac{1}{|G|} \bigoplus_{h\in G} L_h^* \omega_g \right) \\ &= \bigoplus_{g\in G} \frac{1}{|G|} \bigoplus_{h\in G} L_h^* (\mathrm{d}\eta_g) \\ &= \bigoplus_{g\in G} \mathrm{d} \frac{1}{|G|} \bigoplus_{h\in G} L_h^* \eta_g \\ &= \mathrm{d}_G^{i-1} \left(\frac{1}{|G|} \bigoplus_{h\in G} L_h^* \eta \right),\end{aligned}$$

and so $\omega \in \mathrm{im}(\mathrm{d}^{i-1})$.

For surjectivity, assume that $c \in H^i(M)$. Then, for each $g \in G$, $c_g \in H^*(M^g; \mathbb{C})$ is represented by a closed differential form ω_g. Pick such a form for each g, and denote the collection of these forms by $\omega = (\omega_g)$. Averaging over the group action, we define the form

$$\eta = \frac{1}{|G|} \bigoplus_{h\in G} L_h^* \omega.$$

This is a delocalized differential form which gets mapped to c by φ^i. $\square$

Remark 9.3. Note that the first part of the proof shows that there is no difference between d_G-exactness and exactness of all the components of a delocalized form. Therefore, there is no ambiguity when talking about exactness of delocalized differential forms.

Theorem 9.4. *Let*

$$\mathrm{ch}_{\mathrm{even}} = \mathrm{ch}_0 + \sum_{k\geq 1} \left(\frac{i}{2\pi} \right)^k \frac{1}{k!} \mathrm{tr}\left(\Omega^k \right) \in \Omega^{\mathrm{even}}(\mathrm{Gr}_{\mathrm{res}}) \quad \text{and}$$

$$\mathrm{ch}_{\mathrm{odd}} = \sum_{k\geq 1} \left(\frac{i}{2\pi} \right)^k \frac{(-1)^{k-1}(k-1)!}{(2k-1)!} \mathrm{tr}\left(\omega^{2k-1} \right) \in \Omega^{\mathrm{odd}}(\mathrm{U}^1)$$

be the even and odd universal non-equivariant Chern characters from Definition 5.10 and Definition 4.2. Let i_g be either the inclusion of $\mathrm{Gr}_{\mathrm{res}}^g$ into $\mathrm{Gr}_{\mathrm{res}}$, or the inclusion of $(\mathrm{U}^1)^g$ into U^1. Denote by $C = \langle g \rangle$ the cyclic group generated by $g \in G$. Let furthermore

$$\pi_W : \mathrm{Gr}_{\mathrm{res}}^g \cong \prod_{V\in \mathrm{Irr}(C)} \mathrm{Gr}_{\mathrm{res}} \to \mathrm{Gr}_{\mathrm{res}}$$

be the projection onto the W-factor for some $W \in \mathrm{Irr}(C)$. Then, the collection of forms

$$
\begin{aligned}
(\mathrm{ch}_G)_0 &= \bigoplus_{g\in G} \sum_{V\in \mathrm{Irr}(C)} \mathrm{tr}_V(g)\pi_V^*\mathrm{ch}_0 \in \left(\bigoplus_{g\in G} \Omega^0(\mathrm{Gr}_{\mathrm{res}}^g;\mathbb{C})\right)^G \\
(\mathrm{ch}_G)_{2k} &= \left(\frac{i}{2\pi}\right)^k \frac{1}{k!} \bigoplus_{g\in G} \mathrm{tr}\left(g\,(i_g^*\Omega)^k\right) \in \left(\bigoplus_{g\in G} \Omega^{2k}(\mathrm{Gr}_{\mathrm{res}}^g;\mathbb{C})\right)^G \\
(\mathrm{ch}_G)_{2k-1} &= \left(\frac{i}{2\pi}\right)^k \frac{(-1)^{k-1}(k-1)!}{(2k-1)!} \bigoplus_{g\in G} \mathrm{tr}\left(g\,(i_g^*\omega)^{2k-1}\right) \in \left(\bigoplus_{g\in G} \Omega^{2k-1}((\mathrm{U}^1)^g;\mathbb{C})\right)^G
\end{aligned}
$$

for $k \in \mathbb{N}$ represents the universal equivariant Chern character.

Remark 9.5. In order to make sense of the term in the trace, recall that Ω can be interpreted as a 2-form with values in the adjoint bundle

$$\mathrm{ad}(\mathrm{St}_{\mathrm{res}}) = \mathrm{St}_{\mathrm{res}} \times_{\mathrm{Ad}} L^1(L^2(G) \otimes \mathscr{H}_+),$$

which carries the left-multiplication G-action induced by the action on $L^2(G)$. In the odd case, ω is just the Maurer–Cartan form with values in the Lie algebra of trace-class operators $L^1((\mathscr{H}_G)_+)$, and again, we make sense of the G-action using the $L^2(G)$-factor.

Proof. We first show that this is a well-defined assignment, i.e. that the $(\mathrm{ch}_G)_k$ are G-invariant in both cases. In the even case G acts by unitary transformations in $\mathrm{U}_+ \times \mathrm{U}_-$ on the left on $\mathrm{Gr}_{\mathrm{res}}$. If L_h denotes left-multiplication by $h \in G$, for $k > 0$, this has the following effect on the g-component of ch_{2k}^G:

$$
\begin{aligned}
L_h^*\left(\mathrm{tr}\; g\,(i_g^*\Omega)^k)\right) &= \mathrm{tr}\; g\,(i_g \circ L_h)^*\Omega^k \qquad (14)\\
&= \mathrm{tr}\; g\,(L_h \circ i_{h^{-1}gh})^*\Omega^k \\
&= \mathrm{tr}\; g\,(i_{h^{-1}gh})^* L_h^*\Omega^k \\
&= \mathrm{tr}\; g\, h(i_{h^{-1}gh})^*\Omega^k h^{-1} \\
&= \mathrm{tr}\; h^{-1}gh(i_{h^{-1}gh}^*\Omega^k).
\end{aligned}
$$

The result is therefore exactly the $h^{-1}gh$-component, which is what we needed to show. For the ch_0-component, note that $C = \langle g\rangle$ has the same representation theory as the conjugated group $h^{-1}Ch$. We therefore also have that

$$
\begin{aligned}
L_h^* \sum_{V\in\mathrm{Irr}(C)} \mathrm{tr}_V(g)\pi_V^*\mathrm{ch}_0 &= \sum_{V\in\mathrm{Irr}(C)} \mathrm{tr}_V(g) L_h^*\pi_V^*\mathrm{ch}_0 \\
&= \sum_{V\in\mathrm{Irr}(h^{-1}Ch)} \mathrm{tr}_V(h^{-1}gh)\pi_V^*\mathrm{ch}_0
\end{aligned}
$$

is equal to the $h^{-1}gh$-component. Here, we used that the action is by elements in $\mathrm{U}_+ \times \mathrm{U}_-$: If $W = X((\mathscr{H}_G)_+)$ for some $X \in \mathrm{U}_{\mathrm{res}}$, then $hW = hX((\mathscr{H}_G)_+)$, where

$$hX = \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix} \begin{pmatrix} X_{++} & X_{-+} \\ X_{+-} & X_{--} \end{pmatrix} = \begin{pmatrix} hX_{++} & hX_{-+} \\ hX_{+-} & hX_{--} \end{pmatrix}.$$

Since hX_{++} and X_{++} have the same Fredholm index, W and hW have the same virtual dimension and therefore lie in the same path-component of $\mathrm{Gr}_{\mathrm{res}}$, which implies $L_h^*\mathrm{ch}_0 = \mathrm{ch}_0$. For the odd case, we now just repeat the calculation from Equation 14, replacing Ω by ω. Therefore, the collection of forms ch_G defines an element in the de Rham cohomology version of delocalized cohomology of the classifying spaces. We now show that it has a universal property.

Consider the even case and let M be some compact G-manifold. Furthermore, denote again by $C = \langle g \rangle$ the cyclic group generated by g. The top row of the following diagram describes by definition the g-component ch_g of the equivariant Chern character (cf. Definition 7.14):

$$\begin{tikzcd}
K_G^0(M) \arrow[r, "i_g^*"] & K_C^0(M^C) \arrow[r, "\phi"] & K^0(M^C) \otimes R(C) \arrow[r, "\mathrm{ch}\otimes\mathrm{tr}(g)"] & H^*(M^C) \\
{[M, \mathrm{Gr}_{\mathrm{res}}]_G} \arrow[u, "\cong"] \arrow[r] & {[M^C, \mathrm{Gr}_{\mathrm{res}}^C]_C} \arrow[u, "\cong"] \arrow[r] & \prod_{V \in \mathrm{Irr}(C)} [M^C, \mathrm{Gr}_{\mathrm{res}}] \arrow[u] \arrow[r] & H^*(M^C). \arrow[u, "="]
\end{tikzcd}$$

Here, ϕ is the isomorphism that appeared in Lemma 7.8, which exists since the C-action on M^C is obviously trivial. Tracing a G-homotopy class $[f]$ through the lower horizontal part of the diagram yields

$$[f] \longmapsto [f \circ i_g] \longmapsto \sum_{V \in \mathrm{Irr}(C)} [\pi_V \circ f \circ i_g] \longmapsto \sum_{V \in \mathrm{Irr}(C)} [\mathrm{tr}_V(g)(\pi_V \circ f \circ i_g)^* \mathrm{ch}_{\mathrm{even}}].$$

Denote by $f^g = f \circ i_g$ the restriction of f to the fixed point set M^g. Then, for $k > 0$, we can simplify the $2k$-part of the form on the right further:

$$\begin{aligned}
&\left(\frac{i}{2\pi}\right)^k \frac{1}{k!} \sum_{V \in \mathrm{Irr}(C)} [\mathrm{tr}_V(g)(\pi_V \circ f^g)^* \mathrm{tr}\,\Omega^k] \\
&= \left(\frac{i}{2\pi}\right)^k \frac{1}{k!} (f^g)^* \left(\sum_{V \in \mathrm{Irr}(C)} [\mathrm{tr}_V(g)\ \pi_V^* \mathrm{tr}\,\Omega^k] \right) \\
&= \left(\frac{i}{2\pi}\right)^k \frac{1}{k!} (f^g)^* \left(\sum_{V \in \mathrm{Irr}(C)} [\mathrm{tr}\, g\, \pi_V^* \Omega^k] \right) \\
&= \left(\frac{i}{2\pi}\right)^k \frac{1}{k!} (f^g)^* \mathrm{tr}\left(g\,(i_g^*\Omega)^k\right).
\end{aligned}$$

The second equality uses that g acts by a multiple of the identity on each factor $V \otimes \mathscr{H}_+$, while the third equality uses that the trace converts block sum of matrices into sums. This shows the claim for ch_{2k}, $k > 0$, and a minor variation of the argument also shows the case ch_0.

For the odd degree part of the statement, we copy the above argument, replacing K^0 by K^1 and $\mathrm{Gr}_{\mathrm{res}}$ by U^1, respectively. In the same way, we arrive at the calculation

$$\begin{aligned}
&\sum_{V \in \mathrm{Irr}(C)} [\mathrm{tr}_V(g)(\pi_V \circ f^g)^* \mathrm{ch}_{2\mathrm{k}-1}] \\
&= \left(\frac{i}{2\pi}\right)^k \frac{(-1)^{k-1}(k-1)!}{(2k-1)!} \sum_{V \in \mathrm{Irr}(C)} [\mathrm{tr}_V(g)(\pi_V \circ f^g)^* \mathrm{tr}\, \omega^{2k-1}] \\
&= \left(\frac{i}{2\pi}\right)^k \frac{(-1)^{k-1}(k-1)!}{(2k-1)!} (f^g)^* \mathrm{tr}\left(g\, (i_g^* \omega)^{2k-1}\right),
\end{aligned}$$

thereby finishing the proof. ☐

Remark 9.6. Note that in particular, we have

$$(\mathrm{ch}_G)_{\mathrm{even}} \in \Omega^0_G(\mathrm{Gr}_{\mathrm{res}}) \qquad \text{and} \qquad (\mathrm{ch}_G)_{\mathrm{odd}} \in \Omega^1_G(\mathrm{U}^1).$$

Remark 9.7. We have complexified our differential forms, as well as our version of delocalized cohomology, as is standard in most treatments of this gadget. The reason for this is that the equivariant Chern character is in general complex-valued, since it is built from the complex characters of a group. There is, however, the following canonical real structure: Complexified equivariant K-theory $K_G(M) \otimes \mathbb{C}$ carries a conjugation action from the $\mathbb{C}$-factor, which induces a real structure on H_G as well, since Ch_G is an isomorphism. This structure can be described more explicitly. Note that the fixed point sets for g and g^{-1} always agree. From the description in Theorem 9.4, we furthermore always have

$$\begin{aligned}
[\mathrm{ch}_{g^{-1}}] &= \sum_{V \in \mathrm{Irr}(G)} \mathrm{tr}_V(g^{-1})(\pi_V)^*[\mathrm{ch}] \\
&= \sum_{V \in \mathrm{Irr}(G)} \overline{\mathrm{tr}_V(g)}(\pi_V)^*[\mathrm{ch}] = \overline{[\mathrm{ch}_g]},
\end{aligned} \tag{15}$$

since the non-equivariant Chern character is real. Therefore, the induced real structure on $H^*_G(M)$ is given by the assignment $c \mapsto \overline{c}$, where $(\overline{c})_g = \overline{c_{g^{-1}}}$. The fixed point set $(H_G)^*_{\mathbb{R}}(M)$ is the real version of delocalized cohomology. Similarly, we can define the real version of delocalized differential forms, which give a cocycle model for $(H_G)^*_{\mathbb{R}}(M)$.

The relevance of this is that the equivariant Chern–Weil forms of an invariant unitary connection are real delocalized forms for this real structure ([Ort09, Page 5]). Indeed, for the universal cocycles we have chosen, the calculation in Equation 15 is true on cocycle level, which means that we can get rid of all the equivalence brackets.

Having constructed an equivariant Chern form, we can also make sense of an equivariant Chern–Simons form. Imitating the $\mathbb{Z}_2$-graded notation for the delocalized cohomology, we make the following definition.

Definition 9.8. The equivariant Chern–Simons form of a smooth G-homotopy $H\colon M\times I\to \mathrm{Gr}_{\mathrm{res}}$ or $H\colon M\times I\to \mathrm{U}^1$ is the form

$$\mathrm{CS}_G(H)=\bigoplus_{g\in G}\int_I (H_t^g)^*\mathrm{ch}_g,$$

which is easily seen to be an element in $\Omega^0_G(M)$ or $\Omega^1_G(M)$ respectively. We say that two G-maps into $\mathrm{Gr}_{\mathrm{res}}$ or U^1 are CS_G-homotopic, if they are G-homotopic through a homotopy with exact CS_G-form.

Remark 9.9. Note that in [Ort09, Page 15], Ortiz asks for universal cocycle representatives for ch_G and cs_G that can be constructed from the Chern–Weil method. Our forms in principle answer this question: The Chern forms are defined via a curvature form Ω, and the universal CS_G-forms are their transgressions in the path-loop fibrations, i.e. for example $H\colon P\mathrm{Gr}_{\mathrm{res}}\times I\to \mathrm{Gr}_{\mathrm{res}}$ would be just the evaluation map of paths, which is a G-map. The caveat is that one has to employ a bit more machinery in order to properly interpret CS_G as a differential form on the loop space $\Omega\mathrm{Gr}_{\mathrm{res}}$. We already discussed this in the beginning of Section 6.

The flip map from Definition 6.8, as well as the block sum from Definition 6.4 still make sense: The new Hilbert space we use is $\mathscr{H}_G=L^2(G)\otimes\mathscr{H}$, and we can just do everything on the second factor. For example, the new splitting map $\rho_G\colon \mathscr{H}_G\to\mathscr{H}_G\oplus\mathscr{H}_G$ used for the block sum is just $\mathrm{id}_{L^2(G)}\otimes\rho$.

Lemma 9.10. *The block sum is an equivariant map* $\mathrm{Gr}_{\mathrm{res}}\times\mathrm{Gr}_{\mathrm{res}}\to\mathrm{Gr}_{\mathrm{res}}$ *and* $\mathrm{U}^1\times\mathrm{U}^1\to \mathrm{U}^1$*, where the left hand side carries the diagonal G-action. Furthermore, the maps*

$$\mathrm{flip}\colon \mathrm{Gr}_{\mathrm{res}}\to\mathrm{Gr}_{\mathrm{res}}\qquad\text{and}\qquad *\colon \mathrm{U}^1\to\mathrm{U}^1$$

are G-equivariant.

Proof. For the block sum, note that for $V=X((\mathscr{H}_G)_+), W=Y((\mathscr{H}_G)_+)\in\mathrm{Gr}_{\mathrm{res}}$ and $g\in G$, we have

$$\begin{aligned}gV\boxplus gW&=(gXg^*\boxplus gYg^*)((h_G)_+)\\&=\rho_G^*(gXg^*\oplus gYg^*)\rho_G((\mathscr{H}_G)_+)\\&=g\rho_G^*(X\oplus Y)\rho_G g^*((\mathscr{H}_G)_+)\\&=g\rho_G^*(X\oplus Y)\rho_G((\mathscr{H}_G)_+)\\&=g(V\boxplus W).\end{aligned}$$

The flip map is also equivariant, since the action

$$\begin{aligned} gW = gX((\mathscr{H}_G)_+) &= \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} X_{++} & X_{-+} \\ X_{+-} & X_{--} \end{pmatrix} \begin{pmatrix} g^* & 0 \\ 0 & g^* \end{pmatrix} (\mathscr{H}_G)_+ \\ &= \begin{pmatrix} gX_{++}g^* & gX_{-+}g^* \\ gX_{+-}g^* & gX_{--}g^* \end{pmatrix} (\mathscr{H}_G)_+ \end{aligned}$$

is block-wise and therefore commutes with switching around the blocks. Finally, we have $(gAg^*)^* = gA^*g^*$, showing that "$*$" commutes with the G-action. $\square$

Thanks to equivariance, the forms also retain their good compatibility properties.

Proposition 9.11. *The block sum is commutative and associative up to* CS_G*-homotopy. Furthermore, the equivariant Chern character and Chern–Simons form induce monoid morphisms, i.e.*

$$\begin{aligned} \mathrm{Ch}_G(f \boxplus g) &= \mathrm{Ch}_G(f) + \mathrm{Ch}_G(g) \\ \mathrm{CS}_G(H_t \boxplus G_t) &= \mathrm{CS}_G(H_t) + \mathrm{CS}_G(G_t). \end{aligned}$$

Additionally, CS_G *is compatible with concatenation of homotopies*

$$\mathrm{CS}_G(H_t * G_t) = \mathrm{CS}_G(H_t) + \mathrm{CS}_G(G_t).$$

Lastly, both maps respect the relevant inversion operations

$$\begin{aligned} (\mathrm{Ch}_G)_{\mathrm{even}}(\mathrm{flip}(f)) = -(\mathrm{Ch}_G)_{\mathrm{even}}(f) \qquad &\textit{and} \qquad (\mathrm{Ch}_G)_{\mathrm{odd}}(f^*) = -(\mathrm{Ch}_G)_{\mathrm{odd}}(f) \\ (\mathrm{CS}_G)_{\mathrm{odd}}(\mathrm{flip}(H_t)) = -(\mathrm{CS}_G)_{\mathrm{odd}}(H_t) \qquad &\textit{and} \qquad (\mathrm{CS}_G)_{\mathrm{even}}(H_t^*) = -(\mathrm{CS}_G)_{\mathrm{even}}(H_t). \end{aligned}$$

Proof. All of these are just fixed point set-wise applications of the results of Proposition 6.7 and Proposition 6.9. Note that the h-component of the equivariant Chern character of a block sum $(\mathrm{Ch}_h)(f \boxplus g)$ is the pullback of a universal form ch_h on $\mathrm{Gr}_{\mathrm{res}}^h$ or $(\mathrm{U}^1)^h$ by the induced map on fixed point sets $(f \boxplus g)^h$. But clearly

$$(f \boxplus g)^h = (f^h \boxplus g^h),$$

and therefore

$$\mathrm{Ch}_h(f \boxplus g) = ((f \boxplus g)^h)^* \mathrm{ch}_h = (f^h \boxplus g^h)^* \mathrm{ch}_h = (f^h)^* \mathrm{ch}_h + (g^h)^* \mathrm{ch}_h,$$

where the last equality follows from additivity of the trace under block sum similar to the non-equivariant case. This now easily implies additivity also for the Chern–Simons form. The additivity under concatenation of G-homotopies is due to the additivity of the integral under partitioning the integration interval.

Finally, the inversion operations both respect the decomposition of a fixed point set into products, and as such, the formulas also follow from the non-equivariant case. $\square$

10. A geometric spectrum representing equivariant K-theory

The goal of this section is to establish explicit equivariant homotopy equivalences $\Omega\mathrm{Gr}_{\mathrm{res}} \to \mathrm{U}^1$ and $\Omega\mathrm{U}^1 \to \mathrm{Gr}_{\mathrm{res}}$ which are compatible with the equivariant versions of the Chern and Chern–Simons forms. In the first case, this will just be the holonomy map, which already appeared in the last chapter, while the second case is handled by a certain operator map constructed by Pressley and Segal.

The basic strategy will be to compute transgressions of the universal Chern forms in the path-loop fibration. A good collection of some basic properties of the transgression map is given in [BS08, Appendix A]. Let us begin with the even case. It is well-known that the transgression of $[\mathrm{ch}_{\mathrm{even}}] \in H^{\mathrm{even}}(B\mathrm{U};\mathbb{R})$ in the universal fibration $\mathrm{U} \to E\mathrm{U} \to B\mathrm{U}$ is the class $[\mathrm{ch}_{\mathrm{odd}}] \in H^{\mathrm{odd}}(\mathrm{U};\mathbb{R})$. We can actually recover this fact from the universal bundle which we constructed in Section 5 by a direct calculation.

Lemma 10.1. *The transgression map $T\colon H^k(\mathrm{Gr}^0_{\mathrm{res}};\mathbb{R}) \to H^{k-1}(\mathrm{GL}^1;\mathbb{R}) \cong H^{k-1}(\mathrm{U}^1;\mathbb{R})$ in the universal GL^1-fibration $\mathrm{St}_{\mathrm{res}} \to \mathrm{Gr}^0_{\mathrm{res}}$ maps the even Chern character to the odd one, i.e. $T([\mathrm{ch}_{2k}]) = [\mathrm{ch}_{2k-1}]$.*

PROOF. We use the connection Θ as constructed in Proposition 5.9. Then the Chern character is given by an invariant polynomial, evaluated at the curvature Ω. For this situation, Chern and Simons [CS74, Sec. 3] gave a formula for the transgression form. Define the time dependent form

$$\varphi_t = t\Omega + \frac{t^2-t}{2}[\Theta,\Theta].$$

Then, one defines the Chern–Simons transgression form

$$\eta_{2k-1} = \left(\frac{i}{2\pi}\right)^k \frac{1}{k!}\int_0^1 k\mathrm{tr}(\Theta \wedge \varphi_t^{k-1})\mathrm{d}t. \tag{16}$$

By definition of transgression, we will be done if we can show that η_{2k-1} satisfies the two identities $\mathrm{d}\eta_{2k-1} = \pi^*\mathrm{ch}_{2k}$ and $i^*\eta_{2k-1} = \mathrm{ch}_{2k-1}$. We will check this in detail, since this calculation is often skipped in the literature, and we need to check that the normalizations we picked for the even and odd Chern character are compatible.

We first check the second identity. Note that the pullback of the curvature Ω to the fiber is 0. On the other hand, the pullback of the connection Θ to the fiber is the Maurer–Cartan form ω. Furthermore, we have the identity $[\omega,\omega] = 2\omega\wedge\omega$. Calculating the pullback with

i yields

$$
\begin{aligned}
i^*\eta_{2k-1} &= \left(\frac{i}{2\pi}\right)^k \frac{1}{k!} k \operatorname{tr}\omega^{2k-1} \int_0^1 (t^2-t)^{k-1} \\
&= \left(\frac{i}{2\pi}\right)^k \frac{1}{k!} k \operatorname{tr}\omega^{2k-1} (-1)^{k-1} \frac{((k-1)!)^2}{(2k-1)!} \\
&= \left(\frac{i}{2\pi}\right)^k \frac{(-1)^{k-1}(k-1)!}{(2k-1)!} \operatorname{tr}\omega^{2k-1} \\
&= \mathrm{ch}_{2k-1},
\end{aligned}
$$

and we see that this agrees with Definition 4.2. In order to show the other identity, define the function $f(t) = \operatorname{tr}(\varphi_t \wedge \cdots \wedge \varphi_t)$. Now, the fundamental theorem of calculus yields

$$
\operatorname{tr}\Omega^k = f(1) - f(0) = \int_0^1 f'(t)\mathrm{d}t.
$$

It will therefore follow that $\mathrm{d}\eta_{2k-1} = \pi^*\mathrm{ch}_{2k}$, if we can show (compare with Equation 16) that

$$
\frac{1}{k} f'(t) = \mathrm{d}\operatorname{tr}(\Theta \wedge \varphi_t^{k-1}). \tag{17}
$$

The left hand side is equal to

$$
\operatorname{tr}\dot{\varphi}_t \wedge \varphi_t^{k-1} = \operatorname{tr}((\Omega + (2t-1)\Theta\wedge\Theta) \wedge \varphi_t^{k-1}). \tag{18}
$$

For the right hand side, note that φ_t formally fulfills the Bianchi identity of a curvature:

$$
\begin{aligned}
\mathrm{d}\varphi_t &= t\mathrm{d}\Omega + \frac{t^2-t}{2}\mathrm{d}[\Theta,\Theta] \\
&= t[\Omega,\Theta] + (t^2-t)[\Omega,\Theta] \\
&= [\Omega + (t-1)\Omega, t\Theta] \\
&= [t\Omega, t\Theta] \\
&= [\varphi_t, t\Theta].
\end{aligned}
$$

Here, we used the Jacobi identity to conclude that $[[\Theta,\Theta],\Theta] = 0$ in the second and in the last equality. Therefore, continuing our calculation yields

$$
\begin{aligned}
\mathrm{d}\operatorname{tr}(\Theta\wedge\varphi_t^{k-1}) &= \operatorname{tr}(\mathrm{d}\Theta\wedge\varphi_t^{k-1}) - (k-1)\operatorname{tr}(\Theta\wedge\mathrm{d}\varphi_t\wedge\varphi_t^{k-2}) \\
&= \operatorname{tr}((\Omega - \Theta\wedge\Theta)\wedge\varphi_t^{k-1}) - (k-1)\operatorname{tr}(\Theta\wedge[\varphi_t, t\Theta]\wedge\varphi_t^{k-2}) \\
&= \frac{1}{k}f'(t) - 2t\operatorname{tr}\Theta\wedge\Theta\wedge\varphi_t^{k-1} + (k-1)\operatorname{tr}(\Theta\wedge[t\Theta,\varphi_t]\wedge\varphi_t^{k-2}) \\
&= \frac{1}{k}f'(t) - \operatorname{tr}[t\Theta,\Theta]\wedge\varphi_t^{k-1} + (k-1)\operatorname{tr}(\Theta\wedge[t\Theta,\varphi_t]\wedge\varphi_t^{k-2}).
\end{aligned}
$$

In the third step, we used Equation 18. Now, the sum of the second and third term in the last line vanishes by the invariance of the trace under the adjoint action (as a symmetric polynomial). We have therefore shown Equation 17. We conclude that ch_{2k-1} represents a transgression of ch_{2k}. □

From this basic result we can deduce the needed compatibility between the equivariant versions of the Chern and Chern–Simons forms. Recall that we have already defined an odd periodicity map in Definition 8.9, which was just the composition $h_{\mathrm{odd}} \colon \Omega\mathrm{Gr}^0_{\mathrm{res}} \to \mathrm{GL}^1 \to \mathrm{U}^1$ of the holonomy map and the retraction $\mathrm{GL}^1 \to \mathrm{U}^1$ given by $X \mapsto X|X|^{-1}$.

Proposition 10.2. *Let $H \colon M \times I \to \mathrm{Gr}^0_{\mathrm{res}}$ be a smooth G-homotopy, starting and ending with $H_0 = H_1 = \mathrm{const}_{(\mathscr{H}_G)_+}$ with adjoint map $\hat{H} \colon M \to \Omega\mathrm{Gr}^0_{\mathrm{res}}$. Then, on the level of delocalized differential forms, we have the congruence modulo exact forms*

$$(\mathrm{CS}_G)_{\mathrm{odd}}(H) = (h_{\mathrm{odd}} \circ \hat{H})^*(\mathrm{ch}_G)_{\mathrm{odd}} + \mathrm{exact} \in \Omega^1_G(M).$$

Proof. The strategy is to analyze the universal case by comparing trangression along the two different fibrations (in the non-equivariant sense)

$$\begin{tikzcd}
\mathrm{GL}^1 \arrow[r] & \mathrm{St}_{\mathrm{res}} \arrow[r] & \mathrm{Gr}_{\mathrm{res}} \\
\Omega\mathrm{Gr}_{\mathrm{res}} \arrow[r] \arrow[u, "\mathrm{hol}_\Omega"] & P\mathrm{Gr}_{\mathrm{res}} \arrow[r] \arrow[u, "\mathrm{hol}"] & \mathrm{Gr}_{\mathrm{res}}, \arrow[u, equal]
\end{tikzcd}$$

which are connected by the map of fibrations given by holonomy. Recall that all paths are based at the standard basepoint $(\mathscr{H}_G)_+ \in \mathrm{Gr}_{\mathrm{res}}$. For the Bredon cohomology version of transgression, we need to restrict these fibrations to the fixed point sets for the action of a cyclic subgroup $C = \langle g \rangle \subset G$. From Corollary 8.7, we know that these fixed point sets have the form of a product of a copy of the respective space for each irreducible C-representation, while all the maps in the above diagram reduce to the block-wise product of the same map on the V-component for an irreducible representation V. More concretely, restricting to g-fixed point sets, the following is a diagram of fibrations:

$$\begin{tikzcd}
\prod \mathrm{GL}^1 \arrow[r] & \prod \mathrm{St}_{\mathrm{res}} \arrow[r] & \prod(\mathrm{Gr}_{\mathrm{res}}) \\
\prod \Omega(\mathrm{Gr}_{\mathrm{res}}) \arrow[r] \arrow[u, "\prod \mathrm{hol}_\Omega"] & \prod P(\mathrm{Gr}_{\mathrm{res}}) \arrow[r] \arrow[u, "\prod \mathrm{hol}"] & \prod(\mathrm{Gr}_{\mathrm{res}}). \arrow[u, equal]
\end{tikzcd}$$

Note that every horizontal map here is actually a product of many copies of the corresponding map above. In the base space, we have the differential form

$$\mathrm{ch}_g \in \Omega^{\mathrm{even}}\left(\prod_{\mathrm{Irr}(C)} \mathrm{Gr}_{\mathrm{res}}\right) = \Omega^{\mathrm{even}}\left(\mathrm{Gr}^g_{\mathrm{res}}\right).$$

By definition, the transgression of the cohomology class $[\mathrm{ch}_g]$ in the path-loop fibration is the g-component of the universal Chern–Simons form, $[\mathrm{CS}_g] \in H^{\mathrm{odd}}(\Omega(\mathrm{Gr}_{\mathrm{res}}))$ (see Definition 9.8). On the other hand, we can transgress the same form in the upper fibration:

Pulling back along the projection $\prod p_V$ yields

$$\begin{aligned}(\prod p_V)^*\mathrm{ch}_g &= (\prod p_V)^* \sum_{V\in\mathrm{Irr}(C)} [\mathrm{tr}_V(g)\ \pi_V^*\mathrm{ch}_{\mathrm{even}}] \\ &= \sum_{V\in\mathrm{Irr}(C)} [\mathrm{tr}_V(g)\ (\pi_V\circ p_V)^*\mathrm{ch}_{\mathrm{even}}] \in H^{\mathrm{even}}(\prod \mathrm{St}_{\mathrm{res}}).\end{aligned}$$

Now, taking η_{2k-1} as in Lemma 10.1, we can construct an integrating form

$$\sum_{V\in\mathrm{Irr}(C)} \mathrm{tr}_V(g)\,\eta_{2k-1} \in \Omega^{2k-1}(\mathrm{St}^g_{\mathrm{res}}).$$

Pulling this back to the fiber and further to $\prod \mathrm{U}^1 \subset \prod \mathrm{GL}^1$ gives

$$\sum_{V\in\mathrm{Irr}(C)} \mathrm{tr}_V(g)\,\mathrm{ch}_{2k-1}.$$

When summed over all k, this is exactly ch_g, the g-component of the odd equivariant Chern character. That means that transgression in the upper fibration maps the g-component of the even universal Chern character $[\mathrm{ch}_{g,\mathrm{even}}]$ in cohomology to the respective odd component. Now since transgression commutes with maps of fibrations, if we denote transgression in the upper and lower fibration by T_u and T_l respectively, we see that

$$[\mathrm{CS}_g] = T_l[\mathrm{ch}_{g,\mathrm{even}}] = (\prod \mathrm{hol}_\Omega)^*(T_u[\mathrm{ch}_{g,\mathrm{even}}])) = (\prod \mathrm{hol}_\Omega)^*[\mathrm{ch}_{g,\mathrm{odd}}] = h^*_{\mathrm{odd}}[\mathrm{ch}_{g,\mathrm{odd}}]$$

as cohomology classes on $\Omega(\mathrm{Gr}^0_{\mathrm{res}})^g$. If we are now given any smooth G-homotopy $H\colon M\times I\to \mathrm{Gr}_{\mathrm{res}}$, we can pull back the above equation via the adjoint map $\hat{H}$ and get

$$\begin{aligned}(h^g_{\mathrm{odd}}\circ \hat{H}^g)^*[\mathrm{ch}_{g,\mathrm{odd}}] &= (\hat{H}^g)^*[\mathrm{CS}_g] \\ &= (\hat{H}^g)^*\int_I (\mathrm{ev}^g_t)^*[\mathrm{ch}_{g,\mathrm{even}}] \\ &= \int_I (\hat{H}^g\times \mathrm{id}_I)^*(\mathrm{ev}^g_t)^*[\mathrm{ch}_{g,\mathrm{even}}] \\ &= \int_I (H^g_t)^*[\mathrm{ch}_{g,\mathrm{even}}] \\ &= [\mathrm{CS}_g(H)].\end{aligned}$$

Since the domain now is a finite-dimensional manifold, we can smoothly approximate the maps $h^g_{\mathrm{odd}}\circ \hat{H}^g$ up to homotopy and see that the claimed equality is true on the level of differential forms, up to exact forms. □

Remark 10.3. This argument (and also the one in the odd case given below) avoids the discussion of $h_{\mathrm{odd}}\colon \Omega\mathrm{Gr}^0_{\mathrm{res}}\to \mathrm{U}^1$ as a smooth map between infinite-dimensional manifolds. It would be interesting if one could compute directly the derivative of h_{odd} and pull back the equivariant Chern character as a differential form.

The construction of the map for the even case has appeared in [CM00, Appendix 2], based on the ideas of [PS88, Sec. 6.3]. We will adapt it to the equivariant setting. We choose a concrete model for the generic polarized Hilbert space that has been used before. Let $\mathscr{H}^\infty = L^2(S^1, \mathscr{H}) = L^2(S^1)\hat{\otimes}\mathscr{H}$ be the space of L^2-functions on the circle to the infinite-dimensional separable complex Hilbert space $\mathscr{H}$ with basis $\{e_i\}_{i\geq 0}$. There is a natural $\mathbb{Z}_2$-grading given by the positive and negative exponent part of the Fourier decomposition, respectively. This means that we have

$$\mathscr{H}^\infty_+ = \left\{ f \in \mathscr{H}^\infty \mid f = \sum_{k\geq 0} f_k z^k, f_k \in \mathscr{H} \right\} \quad \text{and}$$

$$\mathscr{H}^\infty_- = \left\{ f \in \mathscr{H}^\infty \mid f = \sum_{k< 0} f_k z^k, f_k \in \mathscr{H} \right\},$$

where $z = \exp(i\theta)$. We are now ready to define our periodicity map for the even case.

Definition 10.4. The multiplication operator map

$$h_{\mathrm{even}} \colon \Omega \mathrm{U}^1(\mathscr{H}) \to \mathrm{U}(\mathscr{H}^\infty)$$
$$\gamma \mapsto M_\gamma$$

maps a loop γ to the operator defined by the rule $(M_\gamma f)(\theta) = \gamma(\theta) f(\theta)$.

Lemma 10.5. *The map h_{even} has its image contained in the restricted unitary group. Furthermore, as a map to $\mathrm{U}_{\mathrm{res}}$, it is a homotopy equivalence.*

Proof. We will rely on the corresponding statements for the finite-dimensional version of this map, which were proved by Pressley and Segal. First, we have that the analogously defined map $h^n_{\mathrm{even}} \colon \Omega\mathrm{U}(n) \to \mathrm{U}(\mathscr{H}^{(n)})$ has its image contained in $\mathrm{U}_{\mathrm{res}}(\mathscr{H}^{(n)}) \subset \mathrm{U}_{\mathrm{res}}(\mathscr{H}^\infty)$, where $\mathscr{H}^{(n)}$ is the finite-dimensional version of our space $\mathscr{H}^\infty$, i.e. $\mathscr{H}^{(n)} = L^2(S^1, \mathbb{C}^n)$, and the inclusion $\mathbb{C}^n \hookrightarrow \mathscr{H}$ is via the first n basis vectors of a chosen orthonormal basis for $\mathscr{H}$. The fact that the image is in the restricted unitary group is a consequence of the decay condition on the Fourier coefficients of the loop γ, using the boundedness of its first derivative (for more details, see [PS88, Proposition 6.3.1]). Furthermore, this multiplication operator map on a finite step is $(2n-2)$-connected, as is also shown by Pressley and Segal (see [PS88, Proposition 8.8.1]).

Now, note that we can define a stabilized version of this map on the loop space of the stable unitary group $\Omega\mathrm{U}$. This is still continuous by the same argument that is used for the finite-dimensional case: The norm of the multiplication operator M_γ is bounded by the supremum of $||\gamma(z)||$ for $z \in S^1$, and therefore, loops which are close to γ in the Whitney topology must give multiplication operators with similar norm. Since the L^1-norm is stronger than the operator norm, we immediately conclude that the version on $\Omega\mathrm{U}^1$ is continuous. The restriction of h_{even} to $\Omega\mathrm{U}(n)$ has its image contained in $\mathrm{U}_{\mathrm{res}}$ by the previous paragraph. The union of these loop spaces is the loop space $\Omega\mathrm{U}$ of the stable unitary group, and, applying the same map, this still has image contained in $\mathrm{U}_{\mathrm{res}}$, since S^1 is compact and we are always in a finite step of the colimit. But $\Omega\mathrm{U}$ is a dense subspace of $\Omega\mathrm{U}^1$, and therefore the first claim follows, since $\mathrm{U}_{\mathrm{res}}$ is a complete Riemannian manifold.

Since the connectivity of these maps increases, we can use a similar argument for h_{even} being a homotopy equivalence. It is enough to show that it induces an isomorphism on all homotopy groups. We have the commutative diagram

$$\begin{tikzcd} \Omega\mathrm{U}(n) \arrow[r, "h^n_{\mathrm{even}}"] \arrow[d, "i"] & \mathrm{U}_{\mathrm{res}}(\mathscr{H}^{(n)}) \arrow[d, "j"] \\ \Omega\mathrm{U}^1(\mathscr{H}) \arrow[r, "h_{\mathrm{even}}"] & \mathrm{U}_{\mathrm{res}}(\mathscr{H}^{\infty}), \end{tikzcd}$$

where both horizontal maps come from the inclusion of $\mathbb{C}^n$ into $\mathscr{H}$ and then filling up with the identity matrix. The maps h^n_{even} and i are $(2n-2)$-connected, while the map j is a homotopy equivalence and therefore h_{even} is also $(2n-2)$-connected, for any n. Finally, since $\mathrm{U}_{\mathrm{res}}$ and U^1 are metrizable Banach manifolds, by [Pal66, Theorem 5], they have the homotopy type of CW-complexes. Therefore, a weak homotopy equivalence $\Omega\mathrm{U}^1 \to \mathrm{U}_{\mathrm{res}}$ is already a homotopy equivalence. □

The map h_{even} realizes the inverse of the Bott periodicity map as a homomorphism of infinite-dimensional Lie groups. We can append the projection $\mathrm{U}_{\mathrm{res}} \to \mathrm{Gr}_{\mathrm{res}}$ in order to get our desired periodicity map for our Grassmannian model, and we will also denote this map by h_{even}. Put a G-action on $\mathscr{H}$ and $\mathscr{H}^{\infty}$ as usual by tensoring with $L^2(G)$. Our next goal is to show that h_{even} has good properties with respect to the actions.

Proposition 10.6. *The map h_{even} is equivariant with respect to the G-action on* $\mathrm{Gr}_{\mathrm{res}}$ *and* $\Omega\mathrm{U}^1$. *Furthermore, it is a G-homotopy equivalence.*

Proof. For the first claim, we recall that the projection $\mathrm{U}_{\mathrm{res}} \to \mathrm{Gr}_{\mathrm{res}}$ is equivariant, and even a G-homotopy equivalence (see Theorem 8.8). It is thus sufficient to show that the multiplication operator map from Definition 10.4 is equivariant. For $g \in G$, $\gamma \in \Omega\mathrm{U}^1$, $z \in S^1$, we have

$$(M_{g.\gamma}(f))(z) = (M_{g\gamma g^{-1}}(f))(z) = g\gamma(z)g^{-1}f(z).$$

Recall that $f \in L^2(G) \otimes L^2(S^1)\hat{\otimes}\mathscr{H}$, and therefore $f(z) \in L^2(G) \otimes \mathscr{H}$, where G acts on the left factor. On the other hand, if we act on the target manifold, we have

$$((g.M_{\gamma})(f))(z) = ((gM_{\gamma}g^{-1})(f))(z) = g\gamma(z)g^{-1}f(z),$$

which proves equivariance.

For the claim that h_{even} is a G-homotopy equivalence, we use the splitting over irreducible representations (Corollary 8.7). We have the commutative diagram

$$\begin{tikzcd} (\Omega \mathrm{U}^1)^H \arrow[r, "h_{\text{even}}"] \arrow[d, "\cong"] & \mathrm{U}_{\text{res}}^H \arrow[d, "\cong"] \\ \prod_{V \in \mathrm{Irr}(H)} \Omega \mathrm{U}^1 \arrow[r, "\prod h_{\text{even}}"] & \prod_{V \in \mathrm{Irr}(H)} \mathrm{U}_{\text{res}}, \end{tikzcd}$$

and all the maps in the product in the bottom row are homotopy equivalences. Therefore, the map on the top is one as well, finishing the proof. $\square$

Proposition 10.7. *Let $H\colon M \times I \to \mathrm{U}^1$ be a smooth G-homotopy, starting and ending with $H_0 = H_1 = \mathrm{const}_{\mathrm{id}}$ with adjoint map $\hat{H}\colon M \to \Omega\mathrm{U}^1$. Let $h_{\text{even}}\colon \Omega\mathrm{U}^1 \to \mathrm{U}_{\text{res}} \to \mathrm{Gr}_{\text{res}}$ be the assignment of the corresponding multiplication operator, composed with the G-homotopy equivalence given by the projection. Then, on the level of delocalized differential forms, we have the congruence modulo exact forms*

$$(\mathrm{CS}_G)_{\text{even}}(H) = (h_{\text{even}} \circ \hat{H})^*(\mathrm{ch}_G)_{\text{even}} + \text{exact} \in \Omega^0_G(M).$$

Proof. In the proof of Proposition 10.2 we have seen that the even equivariant Chern character transgresses to the odd equivariant Chern character as cohomology classes in the path-loop fibration over $\mathrm{Gr}^0_{\text{res}}$. Now, since the Chern character is compatible with Bott periodicity on the level of cohomology theories, if we apply transgression twice, we get $T^2([\mathrm{ch}_G])) = [\mathrm{ch}_G]$ after identifying $\Omega^2\mathrm{Gr}^0_{\text{res}}$ with Gr_{res} explicitly via $h_{\text{even}} \circ (\Omega h_{\text{odd}})$. This allows us to make the calculation

$$(h_{\text{even}} \circ \Omega h_{\text{odd}})^*[\mathrm{ch}_G] = T(T([\mathrm{ch}_G)) = T(h^*_{\text{odd}}[\mathrm{ch}_G]) = (\Omega h_{\text{odd}})^* T([\mathrm{ch}_G]),$$

where the first equality is Bott periodicity, the second one is from (the proof of) Proposition 10.2, and the last one follows from the naturality of transgression in the diagram of fibrations (which restricts to a diagram of fibrations of fixed point sets)

$$\begin{tikzcd} \Omega\mathrm{U}^1 \arrow[r] & P\mathrm{U}^1 \arrow[r] & \mathrm{U}^1 \\ \Omega^2\mathrm{Gr}^0_{\text{res}} \arrow[r] \arrow[u, "\Omega \mathrm{h}_{\text{odd}}"] & P\Omega\mathrm{Gr}^0_{\text{res}} \arrow[r] \arrow[u, "Ph_{\text{odd}}"] & \Omega\mathrm{Gr}^0_{\text{res}}. \arrow[u, "h_{\text{odd}}"] \end{tikzcd}$$

Since Ωh_{odd} is a G-homotopy equivalence, we get that $T[\mathrm{ch}_G] = h^*_{\text{even}}[\mathrm{ch}_G]$ for the even equivariant Chern character. Pulling back by H and using the same argument as in the even case now gives the claim. $\square$

We finish the chapter with a geometric version of the equivariant K-theory spectrum, including spaces, structure maps, and an explicit construction of the addition. The following lemmas will be needed in the proof of Theorem 10.11.

Lemma 10.8. *Let* $\mathrm{Gr}_{\mathrm{res},\infty}$ *be the equivariant stable Grassmannian, i.e. the colimit of the sequence*

$$\begin{aligned}\mathrm{Gr}((\mathbb{C}^0 \oplus \mathbb{C}^0) \otimes L^2(G)) \to \mathrm{Gr}((\mathbb{C}^1 \oplus \mathbb{C}^1) \otimes L^2(G))\\ \to \mathrm{Gr}((\mathbb{C}^2 \oplus \mathbb{C}^2) \otimes L^2(G)) \to \dots,\end{aligned}$$

where the maps are $V \mapsto \{0\} \oplus V \oplus L^2(G)$*, i.e. we stabilize along the regular representation of* G*. Via the inclusions* $\mathbb{C}^n \oplus \mathbb{C}^n \hookrightarrow \mathscr{H}$ *for each* n*, this is a subspace of* $\mathrm{Gr}_{\mathrm{res}}$*. Furthermore, let* U *be the colimit of the sequence*

$$\mathrm{U}(\mathbb{C}^0 \otimes L^2(G)) \to \mathrm{U}(\mathbb{C}^1 \otimes L^2(G)) \to \dots,$$

where the maps are $A \mapsto A \oplus 1$*. Via the inclusions* $\mathbb{C}^n \to \mathscr{H}_+$*, this is a subspace of* U^1*. Then, the inclusions*

$$i\colon \mathrm{Gr}_{\mathrm{res},\infty} \hookrightarrow \mathrm{Gr}_{\mathrm{res}}$$
$$i\colon \mathrm{U} \hookrightarrow \mathrm{U}^1$$

are G*-homotopy equivalences.*

Proof. We check the fixed point sets and use the equivariant Whitehead theorem. Recall that taking fixed points commutes with filtered colimits in the case of finite groups and closed inclusions (see for example [Mal14, Proposition 1.2 (3)]). Furthermore, we have that

$$L^2(G) \cong \bigoplus_{V \in \mathrm{Irr}(G)} V^{\dim(V)}.$$

Combining these two facts in the odd case yields

$$\begin{aligned}\mathrm{U}^G &= \Big(\operatorname*{colim}_n \mathrm{U}(\mathbb{C}^n \otimes L^2(G))\Big)^G\\ &= \operatorname*{colim}_n \big(\mathrm{U}(\mathbb{C}^n \otimes L^2(G))^G\big)\\ &= \operatorname*{colim}_n \prod_{V\in\mathrm{Irr}(G)} \mathrm{U}(\mathbb{C}^{n\dim(V)}) = \prod_{V\in\mathrm{Irr}(G)} \mathrm{U}.\end{aligned}$$

The inclusion into U^1 therefore induces a homotopy equivalence, since this is true on each factor (see [Pal65, Theorem B]). We need to show the corresponding statement also for the fixed point sets for the non-trivial subgroups $H \subset G$. Recall that (see Proposition 8.6), as H-Hilbert spaces,

$$\begin{aligned} L^2(G) &\cong \bigoplus_{V\in\mathrm{Irr}(G)} V^{\dim(V)} \\ &\cong \bigoplus_{V\in\mathrm{Irr}(G)} \bigoplus_{W\in\mathrm{Irr}(H)} W^{\dim(V)\, n_{V,W}} \\ &\cong \bigoplus_{W\in\mathrm{Irr}(H)} W^{\sum_{V\in\mathrm{Irr}(G)} \dim(V)\, n_{V,W}} \\ &= \bigoplus_{W\in\mathrm{Irr}(H)} W^{k_W}. \end{aligned}$$

Therefore, similarly to the above calculation,

$$\begin{aligned} \mathrm{U}^H &= \operatorname*{colim}_n \left(\mathrm{U}(\mathbb{C}^n \otimes L^2(G))^H\right) \\ &= \operatorname*{colim}_n \prod_{W\in\mathrm{Irr}(H)} \mathrm{U}(\mathbb{C}^{nk_W}) = \prod_{W\in\mathrm{Irr}(H)} \mathrm{U}. \end{aligned}$$

By the equivariant Whitehead theorem, the odd case of the statement follows.

For the even case, we use a similar argument. We have

$$\begin{aligned} \mathrm{Gr}^H_{\mathrm{res},\infty} &= \operatorname*{colim}_n \left(\mathrm{Gr}((\mathbb{C}^n \oplus \mathbb{C}^n) \otimes L^2(G))^H\right) \\ &= \operatorname*{colim}_n \prod_{W\in\mathrm{Irr}(H)} \mathrm{Gr}(\mathbb{C}^{nk_W} \oplus \mathbb{C}^{nk_W}) = \prod_{W\in\mathrm{Irr}(H)} \mathrm{Gr}_{\mathrm{res},\infty}, \end{aligned}$$

and again, the inclusion into $\mathrm{Gr}_{\mathrm{res}}$ induces a homotopy equivalence on fixed point sets by the non-equivariant case (see [Wur01, Proposition III.5]). □

Lemma 10.9. *The two different H-space structures on $\Omega\mathrm{U}^1$ and $\Omega\mathrm{U}_{\mathrm{res}}$ given by composition of loops and pointwise multiplication of loops agree, i.e. there is a G-homotopy from the map $(\gamma,\eta) \mapsto \gamma * \eta$ to the map $(\gamma,\eta) \mapsto \gamma \cdot \eta$.*

Proof. This is an application of the usual Eckmann–Hilton argument. Consider the loops $\gamma *_{1-t} 1$ and $1 *_t \eta$, where 1 denotes the constant loop at the identity. The index is supposed to indicate the time parameter at which we are done traversing the first loop. In the case of the first loop, we traverse γ in the interval $[0,t]$, and then stay at the point with the constant loop for the rest of the time $[t,1]$. Varying the parameter t from $1/2$ to 1 gives a homotopy of paths from $\gamma *_{1/2} 1$ to γ.

We now take the pointwise product of these families of loops. This gives a family of loops starting at $\gamma * \eta$ and ending at $\gamma \cdot \eta$. Since the G-action on the loop spaces is by pointwise conjugation, it is clear that this is a G-homotopy. □

Lemma 10.10. *The two different H-space structures on $\mathrm{U}_{\mathrm{res}}$ and U^1 given by block sum and multiplication of operators are equivalent, i.e. there is a G-homotopy from the map $(A,B) \mapsto A \boxplus B$ to the map $(A,B) \mapsto AB$.*

PROOF. We discuss the $\mathrm{U}_{\mathrm{res}}$ case first. On the direct sum

$$\mathscr{H}_G \oplus \mathscr{H}_G \cong (\mathscr{H}_G)_+ \oplus (\mathscr{H}_G)_+ \oplus (\mathscr{H}_G)_- \oplus (\mathscr{H}_G)_-, \tag{19}$$

we let C_t denote the grading preserving rotation

$$C_t = \begin{pmatrix} \cos(t) & \sin(t) & 0 & 0 \\ -\sin(t) & \cos(t) & 0 & 0 \\ 0 & 0 & \cos(t) & \sin(t) \\ 0 & 0 & -\sin(t) & \cos(t) \end{pmatrix}.$$

We have the homotopy

$$\begin{aligned} \mathrm{U}_{\mathrm{res}} \times \mathrm{U}_{\mathrm{res}} \times I &\to \mathrm{U}_{\mathrm{res}} \\ (A, B, t) &\mapsto (A \boxplus 1) C_t (1 \boxplus B) C_t^*, \end{aligned}$$

which, when we run it from $t = 0$ to $t = \pi/2$, shows that $A \boxplus B \sim AB \boxplus 1$. It is easy to see that this is again a homotopy through G-maps, since

$$\begin{aligned} (gAg^{-1} \boxplus 1) C_t (1 \boxplus gBg^{-1}) C_t^* &= g(A \boxplus 1) g^{-1} C_t g (1 \boxplus B) g^{-1} C_t^* \\ &= g(A \boxplus 1) C_t (1 \boxplus B) C_t^* g^{-1}. \end{aligned}$$

The fact that allows us to make these modifications is that the G-action is diagonal with respect to the decomposition in Equation 19. We are therefore left to show that the map $f\colon A \mapsto A \boxplus 1$ is G-homotopic to the identity. It is enough to show this over every path-component of $\mathrm{U}_{\mathrm{res}}$, so we can restrict to $\mathrm{U}^0_{\mathrm{res}}$. The claim will follow from the following two facts: First, recall that the projection $\pi\colon \mathrm{U}_{\mathrm{res}} \to \mathrm{Gr}_{\mathrm{res}}$ is a G-homotopy equivalence, which also respects taking block sums of operators, thus commutes with f.

Secondly, we can write the G-CW-complex $\mathrm{Gr}^0_{\mathrm{res}}$ as the colimit over finite-dimensional, compact Grassmannians as in Lemma 10.8. Then, as a subspace $\mathrm{Gr}_{\mathrm{res},\infty} \subset \mathrm{Gr}^0_{\mathrm{res}}$, we have the subspaces of $\mathscr{H} \otimes L^2(G)$, which only differ from $(\mathscr{H}_G)_+$ by a finite-dimensional subspace. We now argue as follows: Consider the G-homotopy class of f in $[\mathrm{U}^0_{\mathrm{res}}, \mathrm{U}^0_{\mathrm{res}}]_G$. Let $i\colon \mathrm{Gr}_{\mathrm{res},\infty} \hookrightarrow \mathrm{Gr}_{\mathrm{res}}$ be the inclusion and denote by π^{-1} and i^{-1} arbitrary G-homotopy inverses for π and i. We then have a chain of isomorphism by pre- and postcomposition

$$\begin{array}{ccccc} [\mathrm{U}^0_{\mathrm{res}}, \mathrm{U}^0_{\mathrm{res}}]_G & \xrightarrow{\cong} & \left[\mathrm{Gr}^0_{\mathrm{res}}, \mathrm{Gr}^0_{\mathrm{res}}\right]_G & \xrightarrow{\cong} & [\mathrm{Gr}_{\mathrm{res},\infty}, \mathrm{Gr}_{\mathrm{res},\infty}]_G \\ \Cup & & \Cup & & \Cup \\ [\mathrm{f}] & \longmapsto & [\pi \circ f \circ \pi^{-1}] & \longmapsto & [i^{-1} \circ \pi \circ f \circ \pi^{-1} \circ i]\,. \end{array}$$

Since f commutes with π, we have

$$[i^{-1} \circ \pi \circ f \circ \pi^{-1} \circ i] = [i^{-1} \circ f \circ i].$$

Now $f \circ i$ is the restriction of the block sum to the direct limit $\mathrm{Gr}_{\mathrm{res},\infty}$. Recall the notation

$$\mathscr{H}_n = \{e_i \mid i \geq n\}$$

for $\{e_i\}$ a $\mathbb{Z}$-indexed basis of $\mathscr{H}$. Note that the compact set $\mathrm{Gr}_{n|G|} \subset \mathrm{Gr}_{\mathrm{res},\infty}$ consists of spaces of the form $\{0\} \oplus W \oplus \mathscr{H}_n \otimes L^2(G)$, i.e. finite-dimensional spaces $W \subset \mathbb{C}^n \otimes \mathbb{C}^n \otimes L^2(G)$ with infinitely many copies of $L^2(G)$ added to them. Restricted to such a subspace the block sum f does the following: First, the subspace W gets shifted into even coordinates under the inclusion into even coordinates $\mathbb{C}^n \hookrightarrow \mathbb{C}^{2n}$. Then, we take the direct sum with the subspace spanned by all odd coordinates in the second copy of $\mathbb{C}^n$, tensored with $L^2(G)$. The resulting subspace lives in $\mathrm{Gr}_{2n|G|}$. This map differs from the inclusion $\mathrm{Gr}_{n|G|} \hookrightarrow \mathrm{Gr}_{2n|G|}$ just by a permutation of coordinates, which can be achieved by multiplying with a unitary matrix which respects the polarization. But this implies that the map is G-homotopic to the inclusion, by a rotation as in Lemma 6.5 (the Lemma even shows that this is a Chern–Simons homotopy. It follows that $i^{-1} \circ f \circ i$ is G-homotopic to the identity, and therefore, f also is.

The argument for the case of U^1 is similar. Here, we use that the inclusion of the stable unitary group $\mathrm{U} \hookrightarrow \mathrm{U}^1$ is a G-homotopy equivalence (Lemma 10.8), and the block sum with 1 is G-homotopic to the identity map on U. ☐

Theorem 10.11. *For any $n \in \mathbb{Z}$, let $g_{2n} = g_{\mathrm{even}} \colon \mathrm{Gr}_{\mathrm{res}} \to \Omega\mathrm{U}^1$ and $g_{2n+1} = g_{\mathrm{odd}} \colon \mathrm{U}^1 \to \Omega\mathrm{Gr}_{\mathrm{res}}$ be G-homotopy inverses to the G-homotopy equivalences h_{odd} and h_{even}. Then, the sequence of pointed G-spaces and pointed G-maps $(E_n, h_n)_{n\in\mathbb{Z}}$ given by*

$$E_{2n} = \mathrm{Gr}_{\mathrm{res}} \quad \text{and} \quad E_{2n+1} = \mathrm{U}^1$$
$$g_{2n} = g_{\mathrm{even}} \quad \text{and} \quad g_{2n+1} = g_{\mathrm{odd}}$$

defines a (naive) G-Ω-spectrum that represents equivariant K-theory. Furthermore, addition in K-theory is implemented by the block sum operation on both $\mathrm{Gr}_{\mathrm{res}}$ and U^1.

Proof. Since we already showed that the structure maps are homotopy equivalences, the first part of the theorem follows. For the second part, recall that the addition in the cohomology theory associated to a spectrum is induced by loop composition, identifying the space in the spectrum with a loop space via the structure maps. We have to prove that the block sum is homotopic to composition of loops, i.e. that the squares

$$\begin{tikzcd} \Omega\mathrm{U}^1 \times \Omega\mathrm{U}^1 \arrow[r, "*"] & \Omega\mathrm{U}^1 \arrow[d, "h_{\mathrm{even}}"] \\ \mathrm{Gr}_{\mathrm{res}} \times \mathrm{Gr}_{\mathrm{res}} \arrow[u, "g_{\mathrm{even}} \times g_{\mathrm{even}}"] \arrow[r, "\boxplus"] & \mathrm{Gr}_{\mathrm{res}} \end{tikzcd} \qquad \begin{tikzcd} \Omega\mathrm{Gr}_{\mathrm{res}} \times \Omega\mathrm{Gr}_{\mathrm{res}} \arrow[r, "*"] & \Omega\mathrm{Gr}_{\mathrm{res}} \arrow[d, "h_{\mathrm{odd}}"] \\ \mathrm{U}^1 \times \mathrm{U}^1 \arrow[u, "g_{\mathrm{odd}} \times g_{\mathrm{odd}}"] \arrow[r, "\boxplus"] & \mathrm{U}^1 \end{tikzcd}$$

commute up to an equivariant homotopy, where the star denotes loop composition.

Since the projection $\mathrm{U}_{\mathrm{res}} \to \mathrm{Gr}_{\mathrm{res}}$ is a G-homotopy equivalence of H-spaces for the block sum, we can replace $\mathrm{Gr}_{\mathrm{res}}$ by $\mathrm{U}_{\mathrm{res}}$ in both diagrams. For the left one, recall that h_{even} assigns to a loop the corresponding multiplication operator in $\mathrm{U}_{\mathrm{res}}$ and then projects to $\mathrm{Gr}_{\mathrm{res}}$. It is clear that h_{even} respects the alternative H-space structures on target and domain: the pointwise multiplication of two loops maps to the product of their operators in $\mathrm{U}_{\mathrm{res}}$. On the other hand, in the right square, one easily verifies that the holonomy map h_{odd} takes composition of loops to products of operators.

But by Lemma 10.9 and 10.10, loop composition can be replaced by pointwise multiplication, which in turn can be replaced by block sum, proving the desired commutativity of the diagrams up to homotopy. $\square$

CHAPTER IV

Differential equivariant K-theory

11. The differential equivariant K-theory groups

Let us recall the axiomatic definition of differential extensions according to [BS10, Definition 1.1], adapted to the special case of K-theory. Let $V = K^*(*) \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}[u, u^{-1}]$ be the coefficients of complex K-theory with the Bott element of degree 2, and let $\Omega^*(M; V) = \mathscr{C}^\infty(M, \Lambda^* T^* M \otimes_{\mathbb{R}} V)$ denote the V-valued differential forms on M, where the degree is induced by the sum of the degrees as a differential form and as an element of V. There is also a version of cohomology with coefficients in the graded vector space V, which we will denote by $H^*(M; V)$. In the following, we will consider all $\mathbb{Z}$-graded vector spaces in sight as $\mathbb{Z}_2$-graded by restricting to even or odd degrees. Denote by $\mathrm{ch}\colon K^*(M) \to H^*(M; V)$ the topological Chern character.

Definition 11.1. A differential extension of K-theory is a contravariant functor from the category of smooth manifolds to $\mathbb{Z}_2$-graded abelian groups, together with natural transformations

(i) $R\colon \hat{K}^*(M) \to \Omega^*_{d=0}(M; V)$, called the curvature,
(ii) $I\colon \hat{K}^*(M) \to K^*(M)$, called the underlying class,
(iii) $a\colon \Omega^{*-1}(M; V)/\mathrm{im}(d) \to \hat{K}^*(M)$, called the action of forms,

such that

(i) the diagram
$$\begin{array}{ccc} \hat{K}^*(M) & \xrightarrow{\quad I \quad} & K^*(M) \\ \Big\downarrow{\scriptstyle R} & & \Big\downarrow{\scriptstyle \mathrm{ch}} \\ \Omega^*_{d=0}(M; V) & \xrightarrow{\mathrm{Rham}} & H^*(M; V) \end{array} \tag{20}$$
commutes,
(ii) $R \circ a = d$, so the action map is a lift of the exterior derivative, and
(iii) we have the exact sequence
$$K^{*-1}(M) \xrightarrow{\mathrm{Ch}} \Omega^{*-1}(M; V)/\mathrm{im}(d) \xrightarrow{a} \hat{K}^*(M) \xrightarrow{I} K^*(M) \longrightarrow 0.$$

A description of a functor $\hat{K}$, based on the classifying spaces $\mathrm{Gr}_{\mathrm{res}}$ and U^1, that fits in the above definition has been worked out in [Sch19, Section 6+7]. In the following, we will develop the generalization for equivariant K-theory for any finite group G.

In order to generalize Definition 11.1 to the equivariant setting, one has to decide which functors to put into Diagram 20. We decided earlier that the correct equivariant replacement for K-theory is the equivariant K-theory of Atiyah and Segal. One possible way to fill up the diagram would be to use Borel equivariant cohomology in the bottom right. The obvious Chern character map $\mathrm{ch}\colon K_G \to H_{\mathrm{Bor},G}$ then is the Borel–Chern character, which maps a G-vector bundle over the compact manifold M to the induced vector bundle over $EG \times_G M$ and then takes its ordinary Chern character. As discussed earlier, the problem with this approach is that the Borel–Chern character ignores the data coming from the fixed point sets of the non-trivial elements in G. Consequently, refining K_G to a differential extension using the Borel Chern character would also lose this information. This was already observed in [SV10], where Bredon cohomology was used in order to repair this defect. While Bredon cohomology can be defined for compact Lie groups, if one is only interested in finite groups, the delocalized cohomology from Definition 7.13 is equivalent and a little easier to handle.

Having decided on the right hand side of Diagram 20, we still need an equivariant replacement for differential forms. But here, the obvious thing works: as in Definition 9.1, an equivariant differential form should just be a collection of forms on the fixed point sets M^g, for every group element g. This equivariant de Rham model was already successfully employed in [Ort09, Sec. 2.2]. Summarizing, we can make the following definition.

Definition 11.2. A differential extension of G-equivariant K-theory is a contravariant functor from the category of smooth G-manifolds to $\mathbb{Z}_2$-graded abelian groups, together with natural transformations

(i) $R\colon \hat{K}^*_G(M) \to \Omega^*_{G,\mathrm{d}_G=0}(M)$, called the curvature,
(ii) $I\colon \hat{K}^*_G(M) \to K^*_G(M)$, called the underlying class,
(iii) $a\colon \Omega^{*-1}_G(M)/\mathrm{im}(\mathrm{d}_G) \to \hat{K}^*_G(M)$, called the action of forms,

such that

(i) the diagram
$$\begin{array}{ccc} \hat{K}^*_G(M) & \xrightarrow{\quad I \quad} & K^*_G(M) \\ \big\downarrow R & & \big\downarrow \mathrm{ch}_G \\ \Omega^*_{G,\mathrm{d}_G=0}(M) & \xrightarrow{\mathrm{Rham}} & H^*_G(M). \end{array}$$
commutes,
(ii) $R \circ a = d_G$, so the action map is a lift of the exterior derivative, and
(iii) we have the exact sequence
$$K^{*-1}_G(M) \xrightarrow{\mathrm{Ch_G}} \Omega^{*-1}_G(M)/\mathrm{im}(\mathrm{d}_G) \xrightarrow{a} \hat{K}^*_G(M) \xrightarrow{I} K^*_G(M) \longrightarrow 0.$$

We will now give a concrete implementation of such a differential equivariant refinement via our smooth classifying spaces. The goal of this section is to define the underlying group-valued functors $\hat{K}^*_G$. We will first prove that the set of smooth CS_G-homotopy classes into our classifying spaces carries an abelian group structure. We will use the letter L to reserve $\hat{K}$ for another definition.

Definition 11.3. Let M be a smooth G-manifold and

$$\mathscr{H}_G \cong (\mathscr{H}_G)_+ \oplus (\mathscr{H}_G)_- \cong L^2(G) \otimes (\mathscr{H}_+ \oplus \mathscr{H}_-)$$

be a complex separable graded G-Hilbert space with both $\mathscr{H}_+$ and $\mathscr{H}_-$ infinite-dimensional. Define the set-valued contravariant functors on smooth manifolds

$$\hat{L}^0_G(M) = \mathrm{Map}^G_{Smooth}(M, \mathrm{Gr}_{\mathrm{res}}(\mathscr{H} \otimes L^2(G)))/\sim$$
$$\hat{L}^1_G(M) = \mathrm{Map}^G_{Smooth}(M, \mathrm{U}^1(\mathscr{H}_+ \otimes L^2(G)))/\sim .$$

The equivalence relation is generated by the following two relations:

(i) Chern–Simons homotopy equivalence (see Definition 9.8).
(ii) Stabilization: for any map f, we identity f with $f \boxplus \mathrm{const}_*$, where $*$ is the basepoint, i.e. the subspace $\mathscr{H}_+ \otimes L^2(G) \in \mathrm{Gr}_{\mathrm{res}}$ or the identity in U^1.

Pullback in $\hat{L}^*_G$ is given by precomposition with smooth functions.

Remark 11.4. One easily checks that the pullback is well-defined: Let $g\colon M \to N$ be a smooth equivariant map, and let furthermore f_0 be CS_G-homotopic to f_1 via a homotopy f_t. Then $f_0 \circ g$ is also CS_G-homotopic to $f_1 \circ g$, since $\mathrm{CS}_G(f_t \circ g) = g^*\mathrm{CS}_G(f_t)$ and the pullback of exact forms is exact. On the other hand, the pullback of $f \boxplus 1$ is just $(f \circ g) \boxplus 1$, and so stabilization is also respected.

Lemma 11.5. *The operation $\boxplus$ induces an abelian group structure on $\hat{L}^0_G(M)$ and $\hat{L}^1_G(M)$. The neutral elements are given by the equivalence class of the constant map to the basepoint, and inversion is given by $f \mapsto f^*$ and $f \mapsto \mathrm{flip}(f)$ in the odd/even case respectively.*

Proof. We need to check well-definedness. If $f_0 \sim_{\mathrm{CS}} f_1$ and $g_0 \sim_{\mathrm{CS}} g_1$, then we need to show that $f_0 \boxplus g_0 \sim_{\mathrm{CS}} f_1 \boxplus g_1$. This is achieved by the homotopy $f_t \boxplus g_t$, which is again a CS_G-homotopy by Proposition 9.11. Furthermore, the matrices

$$(f \boxplus 1) \boxplus (g \boxplus 1) \qquad \text{and} \qquad (f \boxplus g) \boxplus 1 = (f \boxplus g) \boxplus (1 \boxplus 1)$$

are CS-equivalent, and so stabilization is also fine. Commutativity and associativity are also proven in the same Proposition. That block summing with const_* is the identity is built into the definition of our equivalence relation (the stabilization step). It therefore remains to show that inversion is given by the proposed operations.

Start with the even case and consider the rotation matrix in the 2-3-plane, given by

$$C_t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(t) & -\sin(t) & 0 \\ 0 & \sin(t) & \cos(t) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We define the universal homotopy $H\colon \mathrm{U}_{\mathrm{res}} \times [0,\pi/2] \to \mathrm{U}_{\mathrm{res}}$, which at time t is

$$\begin{aligned} H_t(X) &= \rho^* C_t^*(X \oplus \mathrm{flip}(X)) C_t \rho \\ &= \begin{pmatrix} X_{++} & \sin(t)X_{-+} & \cos(t)X_{-+} & 0 \\ \sin(t)X_{+-} & X_{--} & 0 & \cos(t)X_{+-} \\ \cos(t)X_{+-} & 0 & X_{--} & -\sin(t)X_{+-} \\ 0 & \cos(t)X_{-+} & -\sin(t)X_{-+} & X_{++} \end{pmatrix}. \end{aligned}$$

Obviously, $H_t(gXg^{-1}) = gH_t(X)g^{-1}$. Furthermore, this is a unitary matrix whose $(+-)$- and $(-+)$-components are L^2-operators, and so it really lies in $\mathrm{U}_{\mathrm{res}}$. The map $H_{\frac{\pi}{2}}$ has values always in the subgroup $\mathrm{U}_+ \times \mathrm{U}_- \subset \mathrm{U}_{\mathrm{res}}$, while H_0 is just $X \mapsto X \boxplus \mathrm{flip}(X)$. This homotopy furthermore induces a well-defined homotopy on the quotient $\mathrm{Gr}_{\mathrm{res}}$, which can be seen as follows: If we consider another representative $XV = X \begin{pmatrix} v_+ & 0 \\ 0 & v_- \end{pmatrix}$ for unitary matrices $v_\pm$, it is easy to see that the operator $V \oplus \mathrm{flip}(V)$ commutes with C_t. Therefore, we have

$$\begin{aligned} \rho^* C_t^*(XV \oplus \mathrm{flip}(XV)) C_t \rho &= \rho^* C_t^*(X \oplus \mathrm{flip}(X))(V \oplus \mathrm{flip}(V)) C_t \rho \\ &= \rho^* C_t^*(X \oplus \mathrm{flip}(X))\, C_t \rho \rho^*(V \oplus \mathrm{flip}(V))\rho \end{aligned}$$

and since $\rho^*(V \oplus \mathrm{flip}(V))\rho \in \mathrm{U}_+ \times \mathrm{U}_-$, the homotopy is well-defined as a map $\mathrm{Gr}_{\mathrm{res}} \times [0,\pi/2] \to \mathrm{Gr}_{\mathrm{res}}$. As it goes from $X \boxplus \mathrm{flip}(X)$ to the constant map to the basepoint, we are reduced to showing that H is an equivariant CS-homotopy.

It is enough to show that the Chern–Simons form $\mathrm{CS}_G(H)$ is d_G-closed, since all the fixed point sets $\mathrm{Gr}_{\mathrm{res}}^g$ have no odd dimensional cohomology, and therefore $H^1_G(\mathrm{Gr}_{\mathrm{res}})$ vanishes. We calculate

$$\begin{aligned} \mathrm{d}_G\left(\bigoplus_g \int_I (H_t^g)^* \mathrm{ch}_g\right) &= \bigoplus_g \mathrm{d}\int_I (H_t^g)^* \mathrm{ch}_g = \bigoplus_g (H_1^g)^* \mathrm{ch}_g - (H_0^g)^* \mathrm{ch}_g \\ &= \bigoplus_g (X \boxplus \mathrm{flip}X)^* \mathrm{ch}_g - \mathrm{const}^*_{(\mathscr{H}_G)_+} \mathrm{ch}_g = \bigoplus_g (X \boxplus \mathrm{flip}X)^* \mathrm{ch}_g. \end{aligned}$$

By Proposition 9.11, the block sum is additive for the Chern character. Furthermore, by the same Proposition, pulling back by the flip map yields a minus sign, finishing the proof that the Chern–Simons form is closed and therefore exact.

For the odd case, we use the homotopy from Lemma [TWZ13, Lemma 3.7] and adapt their calculations to the universal equivariant case. Define

$$\begin{aligned} H_t\colon \mathrm{U}^1 \times I &\to \mathrm{U}^1 \\ (A,t) &\mapsto (A \boxplus 1) C_t (1 \boxplus A^*) C_t^*, \end{aligned}$$

where $C_t = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$ is again a rotation matrix. This is a homotopy through G-maps from $H_0(A) = A \boxplus A^*$ to $H_{\frac{\pi}{2}}(A) = 1$. Since the $H^0_G(\mathrm{U}^1)$ does not vanish, we cannot argue as in the even case. Instead, we will show by hand that the Chern–Simons form is not only exact, but actually vanishes in this case. Consider the g-component of the Chern–Simons form for $g \in G$, in the decomposition $\mathrm{CS}_G(H) = \bigoplus_g \mathrm{CS}_g(H)$. Furthermore, fix a $k \geq 0$. Then, according to Theorem 9.4, we have

$$(\mathrm{CS}_g)_{2k}(H) = \int_I (H_t^g)^* \mathrm{ch}_{2k+1}^g = \left(\frac{i}{2\pi}\right)^{k+1} \frac{(-1)^k k!}{(2k+1)!} \int_I (H_t^g)^* \mathrm{tr}\left(g\,(i_g^*\omega)^{2k+1}\right), \tag{21}$$

where $i_g^*\omega$ is just the Maurer–Cartan form of U^1, restricted to a fixed point set. In order to compute the integral, we calculate:

$$\begin{aligned} H_t^* &= C_t(1 \boxplus A) C_t^* (A^* \boxplus 1) \\ \mathrm{d}H_t &= (\mathrm{d}A \boxplus 0) C_t (1 \boxplus A^*) C_t^* + (A \boxplus 1) C_t (0 \boxplus \mathrm{d}A^*) C_t^* \\ \dot{H}_t &= (A \boxplus 1) \dot{C}_t (1 \boxplus A^*) C_t^* - (A \boxplus 1) C_t (1 \boxplus A^*) J C_t^*, \end{aligned}$$

again using the notation $J = C_t^* \dot{C}_t = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. This yields

$$\begin{aligned} H_t^* \dot{H}_t &= C_t (1 \boxplus A) J (1 \boxplus A) C_t^* - J = C_t \begin{pmatrix} 0 & A^* \\ -A & 0 \end{pmatrix} C_t^* - J \\ H_t^* \mathrm{d}H_t &= C_t(1 \boxplus A) C_t^* (A^* \boxplus 1) \left\{ (\mathrm{d}A \boxplus 0) C_t (1 \boxplus A^*) C_t^* + (A \boxplus 1) C_t (0 \boxplus \mathrm{d}A^*) C_t^* \right\} \\ &= C_t (1 \boxplus A) \left\{ C_t^* (A^* \boxplus 1)(\mathrm{d}A \boxplus 0) C_t + (0 \boxplus \mathrm{d}A^*)(1 \boxplus A) \right\} (1 \boxplus A^*) C_t^* \end{aligned}$$

and therefore

$$\begin{aligned} (H_t^* \mathrm{d}H_t)^{2k} &= C_t (1 \boxplus A) \ \left\{ C_t^* (A^* \boxplus 1)(\mathrm{d}A \boxplus 0) C_t + (0 \boxplus \mathrm{d}A^*)(1 \boxplus A) \right\}^{2k} \ (1 \boxplus A^*) C_t^* \\ &= C_t (1 \boxplus A) \ \left\{ C_t^* (A^* \mathrm{d}A \boxplus 0) C_t - (0 \boxplus A^* \mathrm{d}A) \right\}^{2k} \ (1 \boxplus A^*) C_t^* \\ &= C_t (1 \boxplus A)(A^* \mathrm{d}A)^{2k} \left\{ \begin{pmatrix} \cos(t)^2 & \cos(t)\sin(t) \\ \cos(t)\sin(t) & \sin(t)^2 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}^{2k} (1 \boxplus A^*) C_t^* \\ &= C_t (1 \boxplus A)(A^* \mathrm{d}A)^{2k} \left\{ \begin{pmatrix} \cos(t)^2 & \cos(t)\sin(t) \\ \cos(t)\sin(t) & -\cos(t)^2 \end{pmatrix} \right\}^{2k} (1 \boxplus A^*) C_t^* \\ &= C_t (1 \boxplus A)(A^* \mathrm{d}A)^{2k} \begin{pmatrix} \cos(t)^{2k} & 0 \\ 0 & \cos(t)^{2k} \end{pmatrix} (1 \boxplus A^*) C_t^*. \end{aligned}$$

Recall that we are trying to compute the integral in Equation 21. For this, the $(2k+1)$-form under the integral is contracted by the time vector ∂t on $\mathrm{U}^1 \times I$, and then evaluated with

$2k$-vectors from the tangent space of U^1. Therefore, we calculate

$$\iota_{\partial_t}((H_t^g)^*\mathrm{tr}\left(g\,(i_g^*\omega)^{2k+1}\right)) = \mathrm{tr}\left(g\,H_t^*\dot{H}_t(H_t^*\mathrm{d}H_t)^{2k}\right)$$

$$= \mathrm{tr}\left(gC_t\begin{pmatrix}0 & 1\\ -A & 0\end{pmatrix}(A^*\mathrm{d}A)^{2k}\begin{pmatrix}\cos(t)^{2k} & 0\\ 0 & \cos(t)^{2k}A^*\end{pmatrix}C_t^*\right)$$

$$-\mathrm{tr}\left(gJC_t\begin{pmatrix}1 & 0\\ 0 & A\end{pmatrix}(A^*\mathrm{d}A)^{2k}\begin{pmatrix}\cos(t)^{2k} & 0\\ 0 & \cos(t)^{2k}A^*\end{pmatrix}C_t^*\right).$$

Using cyclic invariance of the trace, the fact that the G-action commutes with C_t, and the identity $C_t^*JC_t = J$, we arrive at

$$\mathrm{tr}\left(\begin{pmatrix}0 & g(A^*\mathrm{d}A)^{2k}\cos(t)^{2k}A^*\\ -gA(A^*\mathrm{d}A)^{2k}\cos(t)^{2k} & 0\end{pmatrix}\right)$$

$$-\mathrm{tr}\left(\begin{pmatrix}0 & gA(A^*\mathrm{d}A)^{2k}\cos(t)^{2k}A^*\\ -g(A^*\mathrm{d}A)^{2k}\cos(t)^{2k} & 0\end{pmatrix}\right),$$

which clearly vanishes. Therefore, since the integrand in Equation 21 vanishes identically, H_t is indeed a CS-homotopy. □

An alternative definition. It is technically convenient to include an additional differential form in the cycles for a differential extension (see for example [TWZ13, Appendix A], and also [FL10]). Indeed it follows easily from the previous lemma that we can work with a more traditional model, separating the homotopy information contained in the classifying map from the differential form input.

Definition 11.6. Let M be a smooth G-manifold and

$$\mathscr{H}_G \cong (\mathscr{H}_G)_+ \oplus (\mathscr{H}_G)_- \cong L^2(G) \otimes (\mathscr{H}_+ \oplus \mathscr{H}_-)$$

be a complex separable graded G-Hilbert space with both $\mathscr{H}_+$ and $\mathscr{H}_-$ infinite-dimensional. We define the set-valued contravariant functors on smooth manifolds

$$\hat{K}_G^0(M) = \mathrm{Map}^G_{\mathit{Smooth}}(M, \mathrm{Gr}_{\mathrm{res}}(\mathscr{H}\otimes L^2(G))) \times \Omega^1_G(M)/\sim \quad \text{and}$$
$$\hat{K}_G^1(M) = \mathrm{Map}^G_{\mathit{Smooth}}(M, \mathrm{Gr}_{\mathrm{res}}(\mathscr{H}\otimes L^2(G))) \times \Omega^0_G(M)/\sim .$$

The equivalence relation is again generated by two relations:

(i) Chern–Simons homotopy equivalence: We identify

$$(f_1,\omega_1) \sim (f_0,\omega_0)$$

if there is a smooth G-homotopy f_t from f_0 to f_1, such that

$$\mathrm{CS}_G(f_t) = \omega_1 - \omega_0 + \text{exact}.$$

(ii) Stabilization: We identify $(f,\omega) \sim (f \boxplus 1, \omega)$ for any tupel (f,ω).

Pullback by maps is given by pulling back the classifying map in the first component and the differential form in the second component separately. This is well-defined by the argument from Remark 11.4.

Corollary 11.7. *The sets* $\hat{K}^0_G(M)$ *and* $\hat{K}^1_G(M)$ *carry an abelian group structure given by*

$$(f,\omega)+(g,\eta)=(f\boxplus g,\omega+\eta).$$

Inversion is given by

$$(f,\omega)^{-1}=(f^*,-\omega)\qquad\text{and}\qquad(f,\omega)^{-1}=(\mathrm{flip}(f),-\omega)$$

in the odd/even case respectively, and the neutral element is given by the equivalence class of $(1,0)$*, where* 1 *is the constant map to the basepoint.*

Remark 11.8. We will discuss the need of this additional differential form, as well as the relation between the $\hat{K}_G$ and $\hat{L}_G$ groups in detail in Section 14. For now, let us remark that there is an obvious comparison map

$$\begin{aligned}\Phi\colon \hat{L}_G &\to \hat{K}_G\\ [f] &\mapsto [(f,0)]\end{aligned}$$

into the bigger cycle set, which is clearly injective. The question about surjectivity of Φ is a question about a certain surjectivity property of the map CS_G.

12. Natural transformations and exact sequences

Now that we have an abelian group $\hat{K}$ that includes differential geometric and homotopical information, it remains to define the curvature map R, the integration map I and the action map a from the definition of a differential extension and see that they have the required properties. Recall that this means that we need a commutative diagram

$$\begin{tikzcd} \hat{K}^*_G(M) \arrow[r, "I"] \arrow[d, "R"] & K^*_G(M) \arrow[d, "\mathrm{Ch}_G"] \\ \Omega^*_{G,\mathrm{d}_G=0}(M) \arrow[r, "\mathrm{Rham}"] & H^*_G(M), \end{tikzcd} \tag{22}$$

as well as an exact sequence

$$K^{*-1}_G(M)\xrightarrow{\mathrm{Ch}_G}\Omega^{*-1}_G(M)/\mathrm{im}(\mathrm{d}_G)\xrightarrow{a}\hat{K}^*_G(M)\xrightarrow{I}K^*_G(M)\longrightarrow 0,$$

and furthermore, $R\circ a=\mathrm{d}_G$ is the delocalized exterior differential.

Definition 12.1. Let $[(f,\omega)]\in\hat{K}_G(M)$. Denote by $[f]$ the G-homotopy class of the map f. Then, we define the underlying class $I(f,\omega)$ and curvature $R(f,\omega)$ by

$$I([f,\omega])=[f]\qquad\text{and}\qquad R([(f,\omega)])=\mathrm{Ch}_G(f)+\mathrm{d}_G\omega.$$

Furthermore, let $\omega \in \Omega^*_G(M)$. Then, the action map

$$a\colon \Omega^{*-1}_G(M)/\mathrm{im}(\mathrm{d}_G) \to \hat{K}^*_G(M)$$

is given by $a(\omega) = [(1, \omega)]$, where 1 is the G-homotopy class of the constant map to the basepoint.

We now have all the definitions to state and prove our Main Theorem.

Theorem 12.2. *On the category of possibly non-compact smooth G-manifolds, the abelian group-valued functors $\hat{K}^0_G$ and $\hat{K}^1_G$ from Definition 11.6, together with the maps I, R and a from Definition 12.1 define a differential extension of G-equivariant K-theory.*

PROOF. We first check that our definitions of R, I and a indeed give well-defined homomorphisms. Concerning I, if (f_1, ω_1) and (f_0, ω_0) are CS_G-equivalent, then f_1 and f_0 are in particular G-homotopic. Furthermore, I is also compatible with the block sum, since the block sum defines addition in equivariant K-theory by Theorem 10.11. For the curvature, we see that

$$\begin{aligned} R(f_1, \omega_1) - R(f_0, \omega_0) &= \mathrm{Ch}_G(f_1) - \mathrm{Ch}_G(f_0) + \mathrm{d}(\omega_1 - \omega_0) \\ &= \mathrm{dCS}_G(f_t) - \mathrm{d}(\omega_1 - \omega_0) = 0. \end{aligned}$$

The curvature is also a homomorphism, since Chern forms are additive under block sum (Proposition 9.11). Lastly, if we apply a to an exact form ω, we get $a(\omega) = (1, \omega)$. Since the constant homotopy from 1 to 1 has vanishing Chern–Simons form, we have

$$a(\omega) = (1, \omega) = (1, 0),$$

and a is well-defined. It is obviously a homomorphism.

Commutativity of Diagram 22 is checked by the calculation

$$\begin{aligned} \mathrm{Rham} \circ R(f, \omega) &= \mathrm{Rham}(\mathrm{Ch}_G(f) + \mathrm{d}\omega) \\ &= [\mathrm{Ch}_G(f)] = \mathrm{Ch}_G([f]) = (\mathrm{Ch}_G \circ I)(f, \omega). \end{aligned}$$

Furthermore, we have

$$R \circ a(\omega) = R(1, \omega) = \mathrm{ch}_G(1) + \mathrm{d}\omega = \mathrm{d}\omega.$$

Lastly, we check exactness of the above sequence. At $K^*_G(M)$, we need I to be surjective, which is obviously the case. Next, at $\hat{K}^*_G(M)$, we easily check that

$$I \circ a(\omega) = I(1, \omega) = 1$$

is trivial. If $I(f, \omega) = 1$, we know that there is a homotopy f_t starting from $f_1 = f$ and ending at the constant map to the basepoint $f_0 = 1$. Therefore,

$$(f, \omega) = (1, \omega - \mathrm{CS}(f_t)) = a(\omega - \mathrm{CS}(f_t))$$

is in the image of a.

Showing exactness at the spot $\Omega_G^{*-1}(M)/\mathrm{im}(\mathrm{d}_G)$ will require a little more effort. Consider the composition

$$a \circ \mathrm{Ch}_G(f) = (1, \mathrm{Ch}_G(f)).$$

We need to show that this is 0, which means that there is a homotopy from 1 to 1 which has $\mathrm{Ch}_G(f)$ as its Chern–Simons form, modulo d_G-exact forms. Recall that we have equivariant homotopy equivalences given bya explicit periodicity maps

$$\Omega\mathrm{Gr}_{\mathrm{res}} \to \mathrm{U}^1 \qquad \text{and} \qquad \Omega\mathrm{U}^1 \to \mathrm{Gr}_{\mathrm{res}},$$

defined in Section 10. We will denote them both by the letter h. Consider the composition $h \circ h^{-1} \circ f$, which is G-homotopic to f. Note that $h^{-1} \circ f$ is the adjoint of a map $\widehat{h^{-1} \circ f}$, defined on $M \times I$. It follows from Proposition 10.2 and 10.7 that we have

$$\mathrm{Ch}_G(f) = \mathrm{Ch}_G(h \circ (h^{-1} \circ f)) = \mathrm{CS}_G(\widehat{h^{-1} \circ f}) + \text{exact}.$$

Now $\widehat{h^{-1} \circ f}$ is a homotopy from the constant map 1 to itself, which has the required Chern form as its Chern–Simons form, as desired. Conversely, if we have a differential form ω with $a(\omega) = 0$, we have a homotopy H from the constant map 1 to itself which realizes ω as a Chern–Simons form

$$\mathrm{CS}_G(H) = \omega.$$

We have to show that ω is also realized as the Chern form of a K-theory class. Reversing the argument from above, we see that

$$\mathrm{CS}_G(H) = \mathrm{Ch}_G(\widehat{h \circ H}) + \text{exact},$$

and therefore, the equivalence class of $\widehat{h \circ H}$ in $K_G^{*-1}(M)$ maps to ω under Ch_G. $\square$

Remark 12.3. Differential equivariant K-theory is functorial for CS_G-equivalence classes of maps: if two maps $f_0, f_1 \colon M \to N$ are G-homotopic and the homotopy f_t satisfies the additional condition that for any $g \colon Y \to \mathrm{Gr}_{\mathrm{res}}$ or $g \colon Y \to \mathrm{U}^1$, the Chern–Simons form $\mathrm{CS}_G(f_t \circ g)$ is exact, then f_0 and f_1 induce the same map on $\hat{K}_G^0$ and $\hat{K}_G^1$. This feature of a descent to a quotient category of smooth manifolds is a general property of differential cohomology theories and is discussed in [TWZ16, Corollary 2.5].

13. First computations and the compact case

In the non-equivariant setting, one can define a K-theory class on a manifold M by giving a vector bundle E over M. The additional data needed to lift such a class to the differential refinement is a connection ∇ on E. Likewise, if an equivariant K-theory class in $K_G^0(M)$ is given by a G-vector bundle E on a G-manifold M, what is needed to get a $\hat{K}_G^0(M)$-class is a connection that is compatible with the group action.

Definition 13.1. Let (E, ∇) be a G-vector bundle over M with connection ∇. We say that ∇ is invariant, if for any $g \in G$, the pullback connection $g^*\nabla$ is equal to ∇.

For any given model for differential equivariant K-theory, a rule for assigning differential K-theory classes to invariant connections is usually called a cycle map. Let Vect_G^∇ be the functor that assigns to a G-manifold the commutative monoid of isomorphism classes of G-vector bundles with invariant connections, let cycl_G be the topological cycle map that assigns to V a class in $K_G^0(M)$, and recall that R and I denote the curvature functor and the underlying class functor of a differential extension $\hat{K}_G$.

Definition 13.2. A differential refinement of the topological cycle map is a natural transformation

$$\widehat{\mathrm{cycl}}_G \colon \mathrm{Vect}_G^\nabla \to \hat{K}_G^0$$

of semigroup-valued functors satisfying

(i) $R(\widehat{\mathrm{cycl}}_G(V,\nabla)) = \mathrm{Ch}_G(V,\nabla)$
(ii) $I(\widehat{\mathrm{cycl}}_G(V,\nabla)) = \mathrm{cycl}_G(V)$.

Here, $\mathrm{Ch}_G(E,\nabla)$ is the delocalized differential form that one gets from equivariant Chern–Weil theory, with g-component

$$\mathrm{Ch}_g(E,\nabla) = \mathrm{tr}\left(g \exp\left(\frac{i\Omega}{2\pi}\right)\right).$$

One of the advantages of a classifying space based approach like ours is that writing down such a cycle map is rather easy, especially since we use an actual Grassmannian in the even case.

Proposition 13.3. *There is a differential refinement $\widehat{\mathrm{cycl}}_G$ for the classifying space based functors from Theorem 12.2.*

Proof. The key result here is an equivariant version of the Narasimhan–Ramanan theorem. In fact, one can easily see that the universal connections from Section 4 turn out to be invariant, if one chooses the correct equivariant version of the Grassmannians. The crucial fact is the additional $\mathrm{U}(N)$-left-invariance of the universal connections on $\mathrm{U}(N)/\mathrm{U}(k) \times \mathrm{U}(N-k)$ that we already observed earlier. For details, we refer to [Sch80, §3].

In particular, if we have a G-vector bundle $E \to M$ with invariant connection ∇ over a G-manifold M, then there is a classifying map to the finite-dimensional (full) Grassmannian

$$f \colon M \to \mathrm{Gr}((\mathbb{C}^n \oplus \mathbb{C}^n) \otimes L^2(G)),$$

using a big enough power of the regular representation. We have already seen in Lemma 10.8 that, just as in the non-equivariant case, we have an embedding

$$i \colon \mathrm{Gr}((\mathbb{C}^n \oplus \mathbb{C}^n) \otimes L^2(G)) \hookrightarrow \mathrm{Gr}_{\mathrm{res}},$$

given by stabilizing with infinite copies of the regular representations in the second component. In order to get rid of the arbitrary number n, note that

$$[E] = [E] - [0] \in K^0_G(M),$$

where 0 is the 0-dimensional G-vector bundle over M, classified by the constant map

$$0\colon M \to \mathrm{Gr}((\mathbb{C}^n \oplus \mathbb{C}^n) \otimes L^2(G)).$$

With this motivation, we define the $\hat{K}^0_G(M)$ class of the geometric cycle (E, ∇) as

$$\widehat{\mathrm{cycl}}_G(E, \nabla) = [(i \circ f \boxplus \mathrm{flip}(i \circ 0), 0)].$$

The addition of $\mathrm{flip}(i \circ 0)$ makes our class independent of the choice of n. It is proven in [Sch80, §4] that the classifying map f is unique up to connection preserving G-homotopy. Therefore, the CS_G-homotopy class $[f]$ is independent of the choice of classifying map, and our class is well-defined. The identity $I \circ \widehat{\mathrm{cycl}}_G = \mathrm{cycl}_G$ holds by construction. We also have $R \circ \widehat{\mathrm{cycl}}_G = \mathrm{Ch}_G$, since the inclusion is compatible with the Chern character. ☐

Remark 13.4. The cycle map factors through the alternative $\hat{L}^0_G$-groups, so we have a diagram

$$\begin{tikzcd} \mathrm{Vect}^\nabla_G(M) \arrow[r] \arrow[rr, bend right, "\widehat{\mathrm{cycl}}_G"'] & L^0_G(M) \arrow[r, "\Phi"] & \hat{K}^0_G(M), \end{tikzcd}$$

where Φ is the natural transformation defined in Remark 11.8.

The relevance of the cycle map is to actually write down elements in differential equivariant K-theory. In fact, one of the best ways to make sense of what it means to compute a class $x \in \hat{K}_G$ is to give an actual G-vector bundle with connection that gets mapped to x under the cycle map.

Recall that odd K-theory also has somewhat of a geometric description, since one can define it (see Equation 9) as the kernel of the map

$$K^0_G(S^1 \times M) \to K^0_G(M)$$

given by the inclusion at $1 \in S^1$. In terms of a G-homotopy class

$$[f] \in [M, \mathrm{U}]_G \cong K^1_G(M),$$

this can be described as follows (see also [Bun13, Example 4.80]): There is a G-homotopy equivalence $\mathrm{U} \sim \Omega B\mathrm{U}$. Under the loop-suspension adjunction, f now corresponds to a map from the suspension ΣM_+ to $B\mathrm{U}$, which gives a well-defined element in $K^0_G(S^1 \times M)$. This class is in the kernel of the inclusion map, and therefore corresponds to the desired element in $K^1_G(M)$, which we call $\mathrm{cycl}_G(f)$. This assignment is well-known to be a homomorphism with respect to the product structure induced by the multiplication $\mu\colon \mathrm{U} \times \mathrm{U} \to \mathrm{U}$. If one wants to define an odd cycle map, it is now reasonable (compare [Bun13, Problem 4.81]) to ask for the following:

Definition 13.5. A differential refinement of the odd cycle map is a natural transformation

$$\widehat{\mathrm{cycl}}_G \colon \mathscr{C}^\infty(-, \mathrm{U}) \to \hat{K}^1_G$$

of semigroup-valued functors satisfying

(i) $R(\widehat{\mathrm{cycl}}_G(f) = \mathrm{Ch}_G(f)$
(ii) $I(\widehat{\mathrm{cycl}}_G(f) = \mathrm{cycl}_G(f)$.

Here, $\mathrm{Ch}_G(f) = f^*\mathrm{ch}_G$ means the odd Chern character given by traces of powers of the Maurer–Cartan form (see Section 4). It was shown by Bunke that such a cycle map can only exist when the image of the maps is contained in the circle group $\mathrm{U}(1) \subset \mathrm{U}$. The reason is that there are no 2-forms on $\mathrm{U}(k)$ for $k \geq 2$ which are primitive for the multiplication $\mathrm{U} \times \mathrm{U} \to \mathrm{U}$, which is inducing the semigroup structure on the left.

Our stance on this is that the problem is from using the wrong product structure on the left. In this thesis, instead of the multiplication map, we focus on the block sum, which is more natural when one deals with vector bundles. When we read "semigroup-valued functor" with regards to the block sum, we still have that the topological cycle map cycl_G is a homomorphism, since block sum and multiplication agree up to homotopy (see Lemma 10.10). Almost trivially, we now have the following result.

Proposition 13.6. *There is a differential refinement $\widehat{\mathrm{cycl}}_G$ of the odd cycle map for the classifying space based functors from Theorem 12.2.*

Proof. By composing with the inclusion $\mathrm{U} \hookrightarrow \mathrm{U}^1$, we can assume that we start with a map $f\colon M \to \mathrm{U}^1$. Assign to f the odd differential equivariant K-theory class

$$\widehat{\mathrm{cycl}}_G(f) = [(f, 0)].$$

By definition, this induces a natural transformation of semigroup-valued functors, and is compatible with the underlying class functor I and the curvature R. □

As in the odd case, it is immediately clear that the cycle map factors through the $\hat{L}^*_G$ groups (see Remark 13.4). Another important property of differential extensions is the so called homotopy formula. While homotopic maps of course induce the same map on cohomology, one cannot expect this from a differential extension, since differential forms are not homotopy invariant. Still, one has some control over the situation. The proof from the non-equivariant case, for example [BS09, Lemma 5.1], applies almost verbatim. We spell it out for the convenience of the reader.

Lemma 13.7. *Let M be a smooth G-manifold. Let $x \in \hat{K}^*_G(M \times I)$, and let $i_0, i_1 \colon M \to M \times I$ be the inclusion at the endpoints. Then, we have*

$$i_1^*x - i_0^*x = a\left(\int_I R(x)\right).$$

The integral here is defined component-wise. That means that for a delocalized equivariant differential form $\omega = \bigoplus \omega_g$, we have

$$\int_I \omega = \bigoplus_g \int_I \omega_g.$$

Proof. Let $p \colon M \times I \to M$ be the projection onto the first factor. Since $x - p^*i_0^*x$ is in the kernel of I, we can write

$$x = p^*i_0^*x + a(\omega)$$

for some equivariant differential form ω. After applying the curvature map, we have

$$R(x) = p^*i_0^*R(x) + \mathrm{d}_G\omega.$$

Now, the homotopy formula follows from

$$\begin{aligned} i_1^*x - i_0^*x &= i_1^*(p^*i_0^*x + a(\omega)) - i_0^*(p^*i_0^*x + a(\omega)) \\ &= i_1^*a(\omega) - i_0^*a(\omega) \\ &= a(i_1^*\omega - i_0^*\omega) \\ &= a\left(\int_I \mathrm{d}_G\omega - \mathrm{d}_G \int_I \omega\right) \\ &= a\left(\int_I \mathrm{d}_G\omega\right) \\ &= a\left(\int_I R(x) - p^*i_0^*R(x)\right) \\ &= a\left(\int_I R(x)\right). \end{aligned}$$

Here, we used Stokes' theorem for delocalized differential forms, which easily follows from the usual version by applying it component-wise. Furthermore, in the last step, we use that the form $p^*i_0^*R(x)$ is constant in the direction of the interval. □

Remark 13.8. Note that we only used the axioms of a differential extension in the proof. Therefore, the homotopy formula is independent of the model used.

There are a few special cases in which one can easily compute the full differential equivariant K-theory groups. First, if one sets G equal to the trivial group, one would expect to get back ordinary differential K-theory. This is indeed the case.

Example 13.9. In the case of $G = \{e\}$, we recover ordinary differential K-theory as defined in [Sch19, Theorem A], up to complexification. It is proved there that, non-equivariantly, the version without a differential form, in our notation $\hat{L}^*_G$, is also a differential extension of K-theory. An analysis of the map Φ from Remark 11.8 then easily shows that it is a natural transformation of differential extensions, meaning that it commutes with all relevant structure maps. Since natural transformations of differential extensions are always isomorphisms, the groups $\hat{L}^*_G$ and $\hat{K}^*_G$ are isomorphic.

Recall that for equivariant K-theory, we have a reduction to the non-equivariant situation in a slightly more general case: If G acts on M trivially, then by Lemma 7.8,

$$K^*_G(M) \cong K^*(M) \otimes R(G).$$

Indeed, one has the same splitting in the differential extension.

Proposition 13.10. *If G acts trivially on M, we have an isomorphism*

$$\hat{K}^*_G(M) \cong \hat{K}^*(M) \otimes R(G).$$

Proof. We only consider the even case. Let (f, ω) be a cycle for $\hat{K}^0_G(M)$. Since G acts trivially on M, the image of f is contained in

$$\mathrm{Gr}^G_{\mathrm{res}} \cong \prod_{V \in \mathrm{Irr}(G)} \mathrm{Gr}_{\mathrm{res}}.$$

Therefore, f is naturally a product of maps f_V, indexed by irreducible representations V. Similarly, ω is just a collection of differential forms on M, indexed by the conjugacy classes in G. There is a natural isomorphism

$$\Omega^1_G(M) \cong \Omega^{\mathrm{odd}}(M) \otimes R(G),$$

induced by the character map. We therefore have a map

$$\begin{aligned} \hat{K}^0_G(M) &\to \hat{L}^0_{\{e\}}(M) \otimes R(G) \\ [(f, \omega)] &\mapsto \sum_{V \in \mathrm{Irr}(G)} [(f_V, \omega_V)] \otimes [V], \end{aligned}$$

which is easily seen to be an isomorphism. Finally, as seen in Example 13.9, for the trivial group, we have an isomorphism $\hat{L}_{\{e\}}(M) \cong \hat{K}_{\{e\}}(M) = \hat{K}(M)$. □

On the other hand, if G acts on M freely, we have an isomorphism

$$K^*_G(M) \cong K^*(M/G)$$

by Lemma 7.7. This suggests that the same statement is true in differential equivariant K-theory, and indeed, we have the following Proposition.

Proposition 13.11. *If G acts freely on M, we have an isomorphism*

$$\hat{K}^*_G(M) \cong \hat{K}^*(M/G).$$

PROOF. We concentrate on the even case again. Let $\pi\colon M \to M/G$ be the projection and $i\colon \mathrm{Gr}_{\mathrm{res}}(\mathscr{H}) \hookrightarrow \mathrm{Gr}_{\mathrm{res}}(L^2(G)\otimes\mathscr{H})$ the inclusion given by $W \mapsto L^2(G)\otimes W$. Then, we have a map

$$\begin{aligned}\hat{K}^0(M/G) \cong \hat{L}^0_{\{e\}}(M/G) &\to \hat{K}^0_G(M)\\ ([f,\omega)] &\mapsto [(i\circ f\circ\pi),\pi^*\omega].\end{aligned}$$

Note that the assignment in the first component, applied to homotopy classes $[f]$, induces an isomorphism in K-theory (cf. Lemma 7.7). Likewise, the group of equivariant forms on M is isomorphic to the group of forms on M/G under pullback. Therefore, this map fits into the following commutative diagram:

$$\begin{array}{ccccccccc} K^0(M/G) & \xrightarrow{\mathrm{Ch}} & \Omega^{\mathrm{odd}}(M/G)/\mathrm{im(d)} & \xrightarrow{a} & \hat{K}^0(M/G) & \xrightarrow{I} & K^0(M/G) & \longrightarrow & 0\\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow\\ K^0_G(M) & \xrightarrow{\mathrm{Ch}_G} & \Omega^1_G(M)/\mathrm{im}(\mathrm{d}_G) & \xrightarrow{a} & \hat{K}^0_G(M) & \xrightarrow{I} & K^0_G(M) & \longrightarrow & 0. \end{array}$$

By the axioms of differential extensions, both rows are exact, and therefore, the 5-Lemma implies that the middle map is an isomorphism. $\square$

In the case of a compact G-manifold M, one has a slight simplification of the equivalence relation defining differential equivariant K-theory. Recall that the relation consisted of two steps, namely CS-equivalence and stabilization. If M is compact, we can get rid of the second step.

Theorem 13.12. *On the category of compact smooth G-manifolds, the abelian group-valued functors*

$$\begin{aligned}\hat{K}^0_G(M) &= Map^G_{\mathrm{Smooth}}(M, \mathrm{Gr}_{\mathrm{res}}(\mathscr{H}\otimes L^2(G)))\times\Omega^1_G(M)/\sim \quad \text{and}\\ \hat{K}^1_G(M) &= Map^G_{\mathrm{Smooth}}(M, \mathrm{Gr}_{\mathrm{res}}(\mathscr{H}\otimes L^2(G)))\times\Omega^0_G(M)/\sim,\end{aligned}$$

with equivalence relation induced by CS_G-equivalence and addition given by block sum and addition of differential forms, define a differential extension of equivariant K-theory. Recall that this means that

$$(f_1,\omega_1)\sim(f_0,\omega_0),$$

if there is a smooth G-homotopy f_t from f_1 to f_0 such that

$$\mathrm{CS}_G(f_t) = \omega_1 - \omega_2 + \text{exact}.$$

PROOF. We just need to show that block summing with the constant map to the basepoint does not change the CS_G-equivalence class, i.e.

$$(f\boxplus 1,\omega)\sim_{\mathrm{CS}_G}(f,\omega)$$

for any map $M\to\mathrm{Gr}_{\mathrm{res}}$ or $M\to\mathrm{U}^1$, and any odd/even differential form ω. The key to this is that our classifying spaces are up to homotopy equivalence colimits of compact spaces. Recall from Lemma 10.8 that the inclusions

$$i\colon \mathrm{Gr}_{\mathrm{res},\infty} \hookrightarrow \mathrm{Gr}^0_{\mathrm{res}}$$
$$i\colon \mathrm{U} \hookrightarrow \mathrm{U}^1$$

are G-homotopy equivalences. It was shown in the proof of Lemma 10.10 that on these subspaces, block sum with the basepoint is G-homotopic to the identity. The homotopy given there has vanishing CS_G-form, since it uses only rotations in the coordinates.

We concentrate on the even case. Let (f, ω) be a cycle for even differential equivariant K-theory. Since M is compact, there is a map $g\colon M \to \mathrm{Gr}_{\mathrm{res},\infty}$ such that f is G-homotopic to $i \circ g$. Let H_t be such a G-homotopy. Then, we have an equivalence of cycles

$$\begin{aligned}(f \boxplus 1, \omega) &\sim ((i \circ g) \boxplus 1, \mathrm{CS}_G(H_t) + \omega)\\ &\sim (i \circ (g \boxplus 1), \mathrm{CS}_G(H_t) + \omega)\\ &\sim (i \circ g, \mathrm{CS}_G(H_t) + \omega)\\ &\sim (f, \omega),\end{aligned}$$

where the second equivalence uses that the pushforward with the inclusion i is a homomorphism for the block sum. $\square$

14. Comparison of the $\hat{K}$ and $\hat{L}$ groups

We now address the difference between the $\hat{K}$ and $\hat{L}$ groups. In $\hat{K}$, the cycle set consists of pairs of a classifying map f, together with a differential form ω. This separates the information about the isomorphism class of the G-bundle, given by the homotopy class $[f]$, from the differential information, given by ω. The combination of these parts is then implemented in the equivalence relation, which is a generalized form of Chern–Simons equivalence.

On the other hand, in $\hat{L}^*_G$, we truly use the spaces $\mathrm{Gr}_{\mathrm{res}}$ and U^1 as classifying spaces, where the CS_G-homotopy class of a map f is enough to give a unique $\hat{K}^*_G$ class. Given such a CS_G-class $[f]$, we can always go back to $\hat{K}^*_G$ by the map discussed in Remark 11.8:

$$\Phi\colon \hat{L}^*_G \to \hat{K}^*_G$$
$$[f] \mapsto [(f, 0)].$$

Note that this map is well-defined and injective, as the equivalence relation in $\hat{K}^*$, when restricted to cycles of the form $[(f, 0)]$, is just CS_G-homotopy. We have proven in Section 12 that one can equip $\hat{K}^*_G$ with the needed additional structure maps a, R and I in order to get a differential extension of K^*_G. We are interested whether one can do the same for $\hat{L}^*_G$.

We need to analyze the image of Φ in $\hat{L}^*_G$. The map Φ would be surjective precisely if for any given tupel (f_1, ω), we could find a representative in its equivalence class that has vanishing differential form part, i.e.

$$(f_1, \omega) \sim (f_0, 0).$$

Now by definition of the relation, this equivalence means that there is a G-homotopy f_t between f_0 and f_1 such that

$$\mathrm{CS}_G(f_t) = \omega + \text{exact}.$$

Using the decomposition

$$(f, \omega) \sim (f, 0) + (1, \omega),$$

we can make an even more precise statement. Assume that there is a G-homotopy g_t from the constant map to the basepoint $g_1 = 1$ to some map g_0 that has Chern–Simons form $\mathrm{CS}_G(g_t) = \omega + \text{exact}$. Then,

$$(f, \omega) \sim (f, 0) + (1, \omega) \sim (f, 0) + (g_0, 0) \sim (f \boxplus g_0, 0).$$

Therefore, the groups $\hat{L}_G^*$ and $\hat{K}_G^*$ are isomorphic if and only if every form is the Chern–Simons form of some null-homotopy g_t, up to exact forms.

Let us, for a moment, forget about the group actions and consider the non-equivariant case. Here, it was one of the achievements of Simons and Sullivan [SS10] to show that the geometric data of a bundle with connection, or equivalently, a classifying map, is indeed all the data needed to define a $\hat{K}$-class. One can then equip the set of such classifying maps with the correct equivalence relation and completely drop the additional differential form. The key statement that one needs to prove for this, is exactly the surjectivity statement for the Chern–Simons form discussed above. Unfortunately, we were not able to prove an equivariant version of the Venice Lemma. Since all the other steps in the proof that $\hat{L}_G^*$ is also a model for differential equivariant K-theory do in fact translate to the equivariant setting, we still find it worthwhile to formulate the needed lemma as a conjecture, and then briefly explore its consequences.

Conjecture 14.1. *(Equivariant Venice Lemma) Let G be a finite group and M be a smooth G-manifold. Furthermore, let*

$$\omega \in \Omega_G^0(M) \qquad \text{or} \qquad \omega \in \Omega_G^1(M)$$

be a delocalized differential form in even or odd degree. Then, ω is up to exact forms the Chern–Simons form of a G-homotopy $f\colon M \times I \to \mathrm{U}^1$ or $f\colon M \times I \to \mathrm{Gr}_{\mathrm{res}}$. Additionally, f can be chosen to restrict to the constant map to the basepoint at time 0.

If the conjecture is true, it follows immediately that $\hat{L}_G^*$ defines a differential extension of G-equivariant K-theory, using the structure maps for the $\hat{K}_G$ groups under the isomorphism Φ. Specifically, a CS_G-homotopy class $[f]$ gets mapped to its Chern form and its homotopy class by the curvature and underlying class map respectively:

$$R([f]) = \mathrm{Ch}_G(f) \qquad \text{and} \qquad I([f]) = [f].$$

The action map is supposed to induce an isomorphism of the kernel of I with the group

$$\Omega^{*-1}(M)/\mathrm{im}(\mathrm{d}_G)/\mathrm{im}(\mathrm{Ch}_G).$$

Since, if the conjecture is true, every form $\omega \in \Omega_G^{*-1}(M)$ is up to exact forms the Chern–Simons form of a null-homotopy g_t, we see that we might define

$$a(\omega) = [g_1].$$

With these definitions, the equivariant setting would therefore be exactly parallel to the non-equivariant extension of ordinary topological K-theory that was explored in [Sch19, Section 7]. Of course, since the considerations in that paper show that the $\hat{K}_G^*$ and $\hat{L}_G^*$ groups are in fact isomorphic when one restricts to the trivial group, it follows that the same is true in the two special cases that were already discussed in Section 13.

Proposition 14.2. *Let M be a smooth G-manifold. If the G-action on M is free or trivial, the natural homomorphism*

$$\begin{aligned} \Phi\colon \hat{L}_G^*(M) &\cong \hat{K}_G^*(M) \\ [f] &\mapsto [(f, 0)] \end{aligned}$$

is an isomorphism. Therefore, in this case, Chern–Simons homotopy classes of G-maps into $\mathrm{Gr}_{\mathrm{res}}$ and U^1 do indeed give differential equivariant K-theory.

Recall the differential lift of the cycle map that takes a G-vector bundle with connection and spits out a $\hat{K}_G^0$-class. In the non-equivariant case, this differential cycle map is surjective when extended to virtual bundles on compact manifolds, and therefore, one can always find a geometric representative for a differential K-theory class. This interplay between the geometric and homotopic data is certainly fundamental in the study of K-theory. Therefore, we think that the following Proposition makes a strong case that one should believe in the Venice Lemma.

Proposition 14.3. *Let M be a compact G-manifold. Assume that the even cycle map*

$$\widehat{\mathrm{cycl}}_G\colon \mathrm{Vect}_G^{\nabla} \to \hat{K}_G^0$$

defined in Proposition 13.3 is surjective, when we extend the domain to virtual bundles. Then, the even degree Venice Lemma is true on M.

Proof. Let ω be an odd delocalized differential form. Then, the class

$$a(\omega) = [(1, \omega)] \in \hat{K}_G^0(M)$$

is in the image of the cycle map by assumption, so

$$[(1,\omega)] = \widehat{\mathrm{cycl}}_G(E,\nabla) = [(f,0)]$$

for some map $f\colon M \to \mathrm{Gr}_{\mathrm{res}}$. By definition of the equivalence relation in $\hat{K}^0_G(M)$, there now must be a null-homotopy of f with Chern–Simons form ω. $\square$

Remark 14.4. This is actually almost an equivalence: Assume that the Venice Lemma is true and take a cycle (f,ω). By compactness of M, f is homotopic by a G-homotopy H_t to a map g with image $\mathrm{im}(g)$ contained in a finite-dimensional Grassmannian. Therefore,

$$(f,\omega) \sim (g, \omega + \mathrm{CS}_G(H_t)) \sim (g,0) + (1, \omega + \mathrm{CS}_G(H_t)).$$

The first summand is clearly in the image of the cycle map: Pull back the universal vector bundle with connection (E_u, ∇_u) via g. Then

$$[(g,0)] = \widehat{\mathrm{cycl}}_G(g^*E_u, g^*\nabla_u) - [(i\circ 0_n, 0)] = \widehat{\mathrm{cycl}}_G(g^*E_u, g^*\nabla_u) - \widehat{\mathrm{cycl}}_G(\mathbb{C}^n, \mathrm{d}).$$

On the other hand, the form $\omega + \mathrm{CS}(H_t)$ can be written by assumption up to exact forms as the Chern–Simons form of a G-homotopy $G_t\colon M\times I \to \mathrm{Gr}_{\mathrm{res}}$ with $G_0 = 1$. But then, by the equivalence rules in $\hat{K}^0_G(M)$,

$$(1, \omega + \mathrm{CS}_G(H_t)) \sim (G_1, 0).$$

The only thing we cannot show is that G_1 can be chosen to have image in a finite-dimensional Grassmannian, without changing the differential form part. Therefore, G_1 may not correspond to an actual finite-dimensional bundle with invariant connection.

15. The abelian Venice Lemma for forms of degree 1 and 2

In this section, we will prove the equivariant Venice Lemma in the special case of low dimensional forms and abelian groups. Following the original non-equivariant proof, the Venice Lemma can be deduced from a similar surjectivity statement for the Chern–Character, instead of the Chern–Simons map.

Even non-equivariantly, there is no good description of the image of the Chern–Character map on differential form level. In other words, it is not known how to distinguish forms which are Chern–Weil Chern–Character forms of vector bundles with connection. In the case of a trivial bundle, one knows by the cohomological theory that the Chern form of any connection has to be exact. In this special case of trivial bundles, it is then known that one actually has surjectivity: For any exact form ω, it is shown in [PT14, Proposition 1] how to explicitly construct a trivial bundle with connection $(E, \mathrm{d}+A)$ such that

$$\mathrm{Ch}(E, \mathrm{d}+A) - \mathrm{Ch}(E,\mathrm{d}) = \omega.$$

If we translate this statement to the equivariant setting, we get the following lemma:

Lemma 15.1. *Assume that the following is true: On any G-manifold M, every exact delocalized differential form $\mathrm{d}_G\omega \in \Omega^0_G(M)$ or $\mathrm{d}_G\omega \in \Omega^1_G(M)$ is the Chern form of an* *equivariantly null-homotopic map* $f\colon M \to \mathrm{Gr}_{\mathrm{res}}$ *or* $f\colon M \to \mathrm{U}^1$*. Additionally, if* $\mathrm{d}_G\omega = 0$ *on some open invariant set* $U \subset M$*, then, on* U*,* f *can be chosen to be the constant map to the basepoint in* $\mathrm{Gr}_{\mathrm{res}}$ *or* U^1*, respectively. Then, the equivariant Venice Lemma is true.*

Proof. Let $S^1 \subset \mathbb{C}$ be the unit circle in the complex plane. From a given form ω on M, we can construct a form $\widetilde{\omega}$ on $M \times S^1$ which restricts under the inclusion i_{-1} to $i_{-1}^*\widetilde{\omega} = \omega$, and which vanishes in an open neighborhood of $M \times \{1\} \subset M \times S^1$. By assumption, we can write $\mathrm{d}\widetilde{\omega} = \mathrm{Ch}_G(f_t)$ for some map on $M \times S^1$, which restricts to the constant map to the basepoint around some open neighborhood of $M \times \{1\}$. Now denote by $f_t\colon M \times I \to \mathrm{Gr}_{\mathrm{res}}$ or $f_t\colon M \times I \to \mathrm{U}^1$ the homotopy that one gets from restricting to the upper half circle. Using Stokes' theorem, we have

$$\begin{aligned}\mathrm{CS}_g(f_t) = \int_I (f_t^g)^*\mathrm{ch}_g = \int_I (\mathrm{d}\widetilde{\omega})_g &= i_{-1}^*\widetilde{\omega}_g - i_1^*\widetilde{\omega}_g + \mathrm{d}\int_I \omega_g \\ &= \omega_g + \text{exact}.\end{aligned}$$

□

In the even case, restated in bundle language, this means that for any even form ω, we can find a trivial G-vector bundle E with connection $\mathrm{d} + A$ such that

$$\mathrm{Ch}_G(E, \mathrm{d} + A) - \mathrm{Ch}_G(E, \mathrm{d}) = \omega.$$

The non-equivariant proof of the Venice Lemma works by explicitly constructing the needed connections, and then using an induction argument on the degree of the form. We will discuss in the following how much of this proof can be adapted to the equivariant setting, and what the challenges are that stop us from proving the full equivariant Venice Lemma.

Lemma 15.2. *If* $E = M \times \mathbb{C}^n$ *is a* G*-vector bundle with underlying trivial bundle with connection* $\nabla = \mathrm{d} + A$*, then the pullback connection* $g^*\nabla$ *satisfies*

$$g^*\nabla = \mathrm{d} + g^{-1}(g^*A)g,$$

where g^*A *is the pullback of the matrix-valued* 1*-form* A*.*

Proof. Since $E = M \times \mathbb{C}^n$ is a trivial bundle, consider the standard basis of $\mathbb{C}^n$, which gives sections

$$\begin{aligned} e_i\colon M &\to E \\ m &\mapsto (m, e_i). \end{aligned}$$

Any other section $s\colon M \to E$ can be developed in this basis as $s = \sum s_i e_i$, where the coefficients are functions $s_i\colon M \to \mathbb{C}$. By definition, we have

$$\nabla s = \sum_i \mathrm{d}s_i \otimes e_i + s_i \nabla e_i = \sum_{i,j} \mathrm{d}s_i \otimes e_i + s_i A_{ji} \otimes e_j,$$

where A is a matrix of 1-forms that determines ∇. Any such section s can be pulled back with the g-action to get a section $\widetilde{s}$, which satisfies

$$\widetilde{s}(m) = g^{-1}s(gm).$$

The pullback connection $g^*\nabla$ on such sections satisfies by definition the identity

$$((g^*\nabla)\widetilde{s})(m) = g^{-1}(\nabla s)(gm) \circ T_m g.$$

Here $T_m g$ denotes the derivative of the g-action on M at the point m. We have to calculate $(g^*\nabla)e_i$ for all i. Note that

$$\widetilde{e_i}(m) = g^{-1}e_i(gm) = g^{-1}(gm, e_i) = (m, g^{-1}e_i),$$

where the action in the last step is by the representation given on $\mathbb{C}^n$. Note that G also acts on the space of sections, where it modifies the basis vectors e_i and $\widetilde{e_i}$ in the following way:

$$ge_i = \sum_j g_{ji}e_j, \qquad g\widetilde{e_i} = \sum_j g_{ji}\widetilde{e_j},$$

where the g_{ji} are just the matrix entries of the matrix g in the representation on $\mathbb{C}^n$. We calculate:

$$\begin{aligned}
((g^*\nabla)e_i)(m) &= \sum_j ((g^*\nabla)g_{ji}\widetilde{e_j})(m) \\
&= \sum_j g_{ji}((g^*\nabla)\widetilde{e_j})(m) \\
&= \sum_j g_{ji}\, g^{-1}(\nabla e_j)(gm) \circ T_m g \\
&= \sum_{j,k} g_{ji}\, g^{-1}(A_{kj} \otimes e_k)(gm) \circ T_m g \\
&= \sum_{j,k,l} g_{ji}\, g^{-1}_{lk}(g^*A_{kj} \otimes e_l)(m) \\
&= \sum_l ((g^{-1}(g^*A)g)_{li} \otimes e_l)(m).
\end{aligned}$$

This proves the claim. $\square$

Proposition 15.3. *(Abelian Venice Lemma for forms of degree* 2*) Let G be a finite abelian group. Let $\omega \in \Omega^0_G(M)$ be an exact delocalized differential form, which only consists of differential forms of degree* 2*. Then, there is a trivial G-vector bundle $E \to M$ with invariant connection $\nabla = \mathrm{d} + A$ such that*

$$\mathrm{Ch}_G(E, \nabla) - \mathrm{Ch}_G(E, \mathrm{d}) = \omega.$$

Additionally, if ω vanishes on some invariant open subset $U \subset M$, we can arrange that $A = 0$ on U.

PROOF. Consider the trivial line bundle $E_V = M \times V$ over M, given by an irreducible representation V. Let α_V be a 1-form on M. Then $-2\pi i\alpha_V$ is a connection 1-form on V. Now the curvature of the connection

$$\nabla_V = \mathrm{d} - 2\pi i\alpha_V \tag{23}$$

is given by

$$\nabla_V^2 = -2\pi i \mathrm{d}\alpha_V$$

which results in the Chern form

$$\begin{aligned}\mathrm{Ch}(\nabla_V) &= \mathrm{Ch}_0 + \mathrm{Ch}_2 \\ &= 1 + \frac{i}{2\pi}\mathrm{tr}(\nabla_V^2) \\ &= 1 + \mathrm{d}\alpha_V. \end{aligned}\tag{24}$$

This calculation corresponds to the component Ch_e for the identity of the equivariant Chern character. Computing Ch_g in general is not much more difficult, since our bundle contains only one irreducible representation. We just have

$$\mathrm{Ch}_g(\nabla_V) = \mathrm{tr}_V(g)\mathrm{Ch}(\nabla_V) = \chi_V(g)\mathrm{Ch}(\nabla_V) \in \Omega^{\mathrm{even}}(M^g).$$

If we do this construction for all $V \in \mathrm{Irr}(G)$ and take the direct sum bundle with direct sum connection $E = \bigoplus E_V$, then the total Chern form is the sum of all Chern forms. Enumerating all irreducible representations and taking one representative g_i for each conjugacy class, we have the linear system of equations

$$\begin{aligned}\mathrm{Ch}_{g_1}(E) &= \chi_1(g_1)\mathrm{Ch}(V_1) + \chi_2(g_1)\mathrm{Ch}(V_2) + \cdots + \chi_{|G|}(g_1)\mathrm{Ch}(V_{|G|}) \\ \mathrm{Ch}_{g_2}(E) &= \chi_1(g_2)\mathrm{Ch}(V_1) + \chi_2(g_2)\mathrm{Ch}(V_2) + \cdots + \chi_{|G|}(g_2)\mathrm{Ch}(V_{|G|}) \\ &\vdots \\ \mathrm{Ch}_{g_{|G|}}(E) &= \chi_1(g_{|G|})\mathrm{Ch}(V_1) + \chi_2(g_{|G|})\mathrm{Ch}(V_2) + \cdots + \chi_{|G|}(g_{|G|})\mathrm{Ch}(V_{|G|})\end{aligned}\tag{25}$$

Since the coefficient matrix $(\chi_i(g_j))_{ij}$ is invertible by Lemma 7.10, we can solve for the left hand side. On the other hand, by Equation 24, we can make the $\mathrm{Ch}(V_i)$ be any exact form we want in degree 2.

Getting back to the claim we want to prove, recall that we are given a differential form $\omega = \mathrm{d}_G\alpha \in \Omega^0_G(M)$, which is just a collection of invariant exact forms $\mathrm{d}\alpha_g \in \Omega^{\mathrm{even}(M^g)}$. We can certainly extend all the α_g to invariant forms defined on all of M, by just extending as differential forms defined on a submanifold and then averaging over G. Using Equation 25, we can regrade by the irreducible representations. Now, choosing the bundle $E = \bigoplus E_V$ with the E_V defined as trivial bundles with connections ∇_V, we have almost achieved our goal.

We still have to check that the connection we just constructed is invariant. But all the α_V we constructed are invariant, and therefore, by Lemma 15.2 it immediately follows that ∇ is invariant. Also, if $\omega = 0$ on some invariant open subset $U \subset M$, then we can arrange that $\alpha = 0$ on the same subset. By Equation 23, it follows that $A = 0$.

□

Proposition 15.4. *(Abelian Venice Lemma for forms of degree* 1*) Let G be a finite abelian group. Let $\omega \in \Omega^1_G(M)$ be an exact delocalized differential form, which only consists of differential forms of degree* 1*. Then, there is a classifying map $f\colon M \to \mathrm{U}^1$ such that*

(i) The equivariant Chern form of f is equal to ω, i.e.

$$\mathrm{Ch}_G(f) = \omega.$$

(ii) The map f is equivariantly null-homotopic.

Additionally, if $\omega = 0$ on some invariant open subset $U \subset M$, we can choose f to be the constant map to the basepoint on U.

Proof. As in the even case, we will build a map with image contained in the fixed point set

$$(\mathrm{U}^1)^G \cong \prod_{V \in \mathrm{Irr}(G)} \mathrm{U}^1.$$

Then f splits into a product of f_V. The Chern form of each f_V is

$$\mathrm{Ch}_g(f_V) = \chi_V(g)\mathrm{Ch}(f_V).$$

Same as in the even case, the left hand side will be prescribed by the form $\omega = \bigoplus \omega_g$, where we again need to extend the ω_g to forms on all of M. We can invert the coefficient matrix to find the $\mathrm{Ch}(f_V) = \mathrm{d}g_V$, which will be some exact 1-forms. Now choose the f_V to be

$$f_V = \exp\left(\frac{2\pi}{i} g_V\right), \tag{26}$$

which has Chern form

$$\frac{i}{2\pi}\mathrm{tr}(f_V^* \mathrm{d} f_V) = \mathrm{d}g_V.$$

Since g_V was an invariant function $M \to \mathbb{C}$, f_V is an equivariant map. Then $f = \prod_V (f_V)$ will do the job.

If $\omega = 0$ on some open invariant subset, then we can choose $g_V = 0$ for all V on U. By Equation 26, we will get $f_V = 1$, the constant map to the basepoint, on U. □

Remark 15.5. Note that it was crucial for the proof that G is abelian in both the even and the odd case: We extended the invariant forms $\alpha_g \in \Omega(M^g)^G$ to invariant forms on all of M. If G is not abelian, we run into the problem that the forms ω_g we start with are only centralizer $Z(g)$-invariant forms on the fixed point set M^g. One idea would be to choose an extension to a form defined on all of M, and then average over the $Z(g)$-action. This would give a $Z(g)$-invariant form that agrees over M^g with the form we started with. With a slight modification of the above argument, taking the dimension of the irreducible connections into account, one can cook up a bundle with connection in the even case or a map into U^1 in the odd case that has the correct Chern forms. The problem is that the connection will not be invariant, and the map will not be equivariant. If one could always extend a $Z(g)$-invariant form on M^g to a G-invariant form on M, this would solve the case of low dimensional forms for all finite groups G.

Concluding the discussion of the restriction to abelian groups, let us see now what goes wrong in the induction step, when one tries to go for higher degree forms. Since we run into similar problems in the odd and even case, let us focus on the even case. Recall that the higher Chern forms arise by taking higher powers of the curvature form Ω or the Maurer–Cartan form ω. Therefore, if our target differential forms happen to be wedge products of 1-forms, we can hope to reduce to the already known cases. The non-equivariant proof exactly implements this idea: Suppose we have an exact $(2k+2)$-form $\omega = \mathrm{d}\alpha$. Assume that α is a basic form, i.e. there are global functions $f_i\colon M \to \mathbb{C}$ with

$$\alpha = f_1 \mathrm{d}f_2 \wedge \cdots \wedge \mathrm{d}f_{2k+2}. \tag{27}$$

Since it follows from the Whitney Embedding Theorem that any form is a finite linear combination of such forms, it is enough to consider this case. Now ω can be written in a slightly different way, namely

$$\omega = \frac{1}{(k+1)!}(\mathrm{d}f_1 \wedge \mathrm{d}f_2 + \cdots + \mathrm{d}f_{2k+1} \wedge \mathrm{d}f_{2k+2})^{k+1}.$$

This achieves our goal of writing ω as a product. Now, let V be a trivial line bundle over M with

$$\nabla = \mathrm{d} - 2\pi i(f_1 \mathrm{d}f_2 + \cdots + f_{2k+1}\mathrm{d}f_{2k+2})$$

so that

$$\nabla^2 = -2\pi i(\mathrm{d}f_1 \mathrm{d}f_2 + \cdots + \mathrm{d}f_{2k+1}\mathrm{d}f_{2k+2}).$$

Then $\mathrm{Ch}(\nabla)$ has the correct $(2k+2)$-Chern form, and $\mathrm{Ch}(\nabla) - \omega$ is an exact form of degree $\leq 2k$. By induction, the claim follows.

Now, if one tries to implement this strategy in the equivariant situation, one runs into the following problem. After restricting to basic forms, we need that the form

$$f_1 \mathrm{d}f_2 + \cdots + f_{2k+1}\mathrm{d}f_{2k+2}$$

is invariant – otherwise we will not get an invariant connection. One might conjecture that one can restrict to basic forms with only invariant functions f_i by using some kind of equivariant Whitney embedding theorem (which does exist), but the statement is unfortunately already wrong for linear representations (see Example 15.6 below). Therefore, at the present time, we do not have a fix for this, at least not without severely restricting the kind of actions one allows. We remark that there are some results in this direction in the case of finite reflection groups given in [Sol63, Main Theorem].

Example 15.6. (Invariant functions are not enough) Consider for $G = \mathbb{Z}_2 = \langle \tau \rangle$ the $\mathbb{Z}_2$-manifold $\mathbb{R}^2$ with antipodal action

$$\tau(x, y) = (-x, -y).$$

Then, certainly, the volume form $\mathrm{d}x \wedge \mathrm{d}y$ is invariant. But any invariant function $f \colon \mathbb{R}^2 \to \mathbb{C}$ necessarily has to have a vanishing derivative at $(0, 0)$, since

$$-\mathrm{d}f_{(0,0)}(v, w) = \mathrm{d}f_{(0,0)}(-(v, w)) = \mathrm{d}f_{(0,0)}(v, w).$$

This means that the form $\mathrm{d}x \wedge \mathrm{d}y$ cannot be written as a finite linear combination of basic forms as in Equation 27, with invariant functions f_i, as all of those would have to vanish at $(0, 0)$.

CHAPTER V

Applications

16. Comparison to other models

We would like to compare our model to other attempts at defining differential equivariant K-theory. Let $\hat{K}_G^*$ and $\hat{K}_G'^*$ be two differential extensions of equivariant K-theory. The natural notion of a map between differential extensions is the following one:

Definition 16.1. A natural transformation of smooth extensions of equivariant K-theory is a natural transformation $\Phi\colon \hat{K}_G^* \to \hat{K}_G'^*$ such that for any manifold M, Φ commutes with all the structure maps, i.e.

$$I' \circ \Phi = I$$
$$R' \circ \Phi = R$$
$$\Phi \circ a = a'.$$

Proposition 16.2. *Any natural transformation of smooth extensions is an isomorphism.*

Proof. By definition, the diagram

$$\begin{tikzcd}
K_G^{*-1}(M) \arrow[r, "\mathrm{Ch}_G"] \arrow[d, "="] & \Omega_G^{*-1}(M)/\mathrm{im}(\mathrm{d}_G) \arrow[r, "a"] \arrow[d, "="] & \hat{K}_G^*(M) \arrow[r, "I"] \arrow[d, "\Phi"] & K_G^*(M) \arrow[r] \arrow[d, "="] & 0 \arrow[d, "="] \\
K_G^{*-1}(M) \arrow[r, "\mathrm{Ch}_G"] & \Omega_G^{*-1}(M)/\mathrm{im}(\mathrm{d}_G) \arrow[r, "a'"] & \hat{K}_G'^*(M) \arrow[r, "I'"] & K_G^*(M) \arrow[r] & 0
\end{tikzcd}$$

commutes and has exact rows. By the five lemma, Φ is an isomorphism. $\square$

Because of this result, in the following, all the difficulty will lie in the construction of a natural map between two models.

Comparison via cycle maps in the compact case. Let M be a compact G-manifold and consider the even degree case. Let $x \in \widehat{K}_G^0(M)$. Since M is compact, we can assume that x is represented by a cycle (f, ω), where $\mathrm{im}(f)$ is contained in a finite-dimensional Grassmannian. Therefore, by pulling back the universal connection, we get a G-vector bundle with invariant connection (E, ∇), such that

$$x = [(f,\omega)] = [(f,0)] + [(1,\omega)] = \widehat{\mathrm{cycl}}_G(E,\nabla) + a(\omega).$$

Assume that the target also has a cycle map $\widehat{\mathrm{cycl}}'_G$ (see Definition 13.2). Then, we can define the transformation Φ from our theory to the second theory, using the cycle and action map of the new model:

$$\Phi(x) = \widehat{\mathrm{cycl}}'_G(E, \nabla) + a'(\omega).$$

Proposition 16.3. *The map Φ is a well-defined natural transformation of differential extensions. It follows that Φ is an isomorphism.*

Proof. Let $x \in \widehat{K}^0_G(M)$ and

$$x = \widehat{\mathrm{cycl}}_G(E_0, \nabla_0) + a(\omega_0) = \widehat{\mathrm{cycl}}_G(E_1, \nabla_1) + a(\omega_1)$$

be two different decompositions of x. Since $I \circ a = 0$ and $I \circ \widehat{\mathrm{cycl}}_G = \mathrm{cycl}_G$, we know that E_0 and E_1 represent the same element in $K^0_G(M)$. Therefore, a bundle $E \to M \times I$ exists, which restricts to $E_0 \oplus K$ and $E_1 \oplus K$ under the inclusions i_0, i_1 at the endpoints, where K is a G-vector bundle that is trivial as a vector bundle. For example by pulling back the universal connection, we can assume that there is an invariant connection ∇ on E, which restricts to $\nabla_0 \oplus \mathrm{d}$ and $\nabla_1 \oplus \mathrm{d}$ on the endpoints. It follows that

$$\begin{aligned}
a(\omega_0 - \omega_1) &= \widehat{\mathrm{cycl}}_G(E_1, \nabla_1) - \widehat{\mathrm{cycl}}_G(E_0, \nabla_0) \\
&= \widehat{\mathrm{cycl}}_G(E_1 \oplus K, \nabla_1 \oplus \mathrm{d}) - \widehat{\mathrm{cycl}}_G(E_0 \oplus K, \nabla_0 \oplus \mathrm{d}) \\
&= \widehat{\mathrm{cycl}}_G(i_1^* E, i_1^* \nabla) - \widehat{\mathrm{cycl}}_G(i_0^* E, i_0^* \nabla) \\
&= i_1^* \widehat{\mathrm{cycl}}_G(E, \nabla) - i_0^* \widehat{\mathrm{cycl}}_G(E, \nabla) \\
&= a\left(\int_I R(\widehat{\mathrm{cycl}}_G(E, \nabla)) \right) \\
&= a\left(\int_I \mathrm{Ch}_G(E, \nabla) \right).
\end{aligned}$$

In the second to last step, we used the homotopy formula (Lemma 13.7), while the last step follows from the definition of differential cycle maps (Definition 13.2). We have shown that

$$\int_I \mathrm{Ch}_G(E, \nabla) - (\omega_0 - \omega_1) \in \ker(a) = \mathrm{im}(\mathrm{Ch}_G) = \ker(a'),$$

by the axioms of a differential extension. Therefore, going backwards in the above chain of equalities, we have

$$\begin{aligned}
a'(\omega_0 - \omega_1) &= a'\left(\int_I \mathrm{Ch}_G(E, \nabla) \right) \\
&= a'\left(\int_I R'(\widehat{\mathrm{cycl}}'_G(E, \nabla)) \right) \\
&= i_1^* \widehat{\mathrm{cycl}}'_G(E, \nabla) - i_0^* \widehat{\mathrm{cycl}}'_G(E, \nabla) \\
&= \widehat{\mathrm{cycl}}'_G(E_1, \nabla_1) - \widehat{\mathrm{cycl}}'_G(E_0, \nabla_0),
\end{aligned}$$

which shows that Φ is well-defined.

The homomorphism property of Φ is obvious. Furthermore, we easily check that, given $x = \widehat{\mathrm{cycl}}'_G(E, \nabla) + a'(\omega)$, we have

$$\begin{aligned} I' \circ \Phi(x) &= I'(\widehat{\mathrm{cycl}}'_G(E, \nabla) + a'(\omega)) \\ &= I'(\widehat{\mathrm{cycl}}'_G(E, \nabla)) \\ &= \mathrm{cycl}_G(E) \\ &= I(\widehat{\mathrm{cycl}}_G(E, \nabla)) \\ &= I(\widehat{\mathrm{cycl}}_G(E, \nabla) + a(\omega)) \\ &= I(x), \end{aligned}$$

using that $I \circ a = 0$. We also have

$$\begin{aligned} R' \circ \Phi(x) &= R'(\widehat{\mathrm{cycl}}'_G(E, \nabla) + a'(\omega)) \\ &= \mathrm{Ch}_G(E, \nabla) + \mathrm{d}\omega \\ &= R(\widehat{\mathrm{cycl}}_G(E, \nabla) + a(\omega)) \\ &= R(x) \end{aligned}$$

and

$$\begin{aligned} \Phi \circ a(\omega) &= \Phi(a(\omega)) \\ &= a'(\omega). \end{aligned}$$

This finishes the proof. $\square$

It is clear that the same proof applies to the odd cycle map. We therefore have the following theorem.

Theorem 16.4. *Let $(\hat{M}^*_G, I', a', R')$ be a differential extension of equivariant K-theory on the category of compact G-manifolds that admits a differential lift $\widehat{\mathrm{cycl}}_G$ of the even/odd cycle map. Then, there is an isomorphism of the even/odd part of the differential extensions Φ to our theory $\hat{K}^*_G$, defined via*

$$x = \widehat{\mathrm{cycl}}_G(E, \nabla) + a(\omega) \mapsto \widehat{\mathrm{cycl}}'_G(E, \nabla) + a'(\omega).$$

In particular, our model is the unique one that supports differential lifts of both the even and the odd cycle map.

The Ortiz model. In [Ort09, Section 2.2], the author defines a differential extension of equivariant K-theory based on the original model of Hopkins–Singer. As smooth models for the classifying spaces, he uses the Atiyah–Singer [AS69b, page 6] spaces $\mathscr{F}^k$ of Fredholm operators with Clifford action, which can also be translated to the equivariant world by the same trick used before: replace $\mathscr{H}$ with $\mathscr{H} \otimes L^2(G)$.

Definition 16.5. Let M be a G-manifold. A cycle for $\hat{K}^k_G(M)$ is a triple

$$(f,\eta,\omega) \in \mathrm{Map}^G_{\mathrm{smooth}}(M,\mathscr{F}^k_G) \times \Omega^{k-1}_G(M) \times \Omega^k_G(M)_{\mathrm{cl}}$$

consisting of a smooth map f, a closed delocalized differential form ω and another delocalized differential form η in one degree lower. A cycle has to fulfill the relation

$$\mathrm{d}_G\eta = \omega - \mathrm{Ch}_G(f).$$

Two cycles (f,η,ω) and (f',η',ω') are equivalent if $\omega = \omega'$ and there is a concordance

$$F \in \mathrm{Map}^G_{\mathrm{smooth}}(M \times I, \mathscr{F}^k_G)$$

such that

$$\eta' = \eta - \mathrm{CS}_G(F) + \mathrm{exact},$$

where $\mathrm{CS}_G(F)$ and also the Chern character above has to be interpreted with the definition of $\mathrm{Ch}_G(F)$ given by Ortiz in mind.

Compared to our model, Ortiz uses triples instead of tupels in the cycle set, but this is of course only a formal difference, since the same information is contained. A cycle (f,ω) in our model corresponds to the cycle $(f,\omega,\mathrm{Ch}_G(f)+\mathrm{d}\omega)$ in the world of triples. Secondly, there is the use of different smooth models for the classifying spaces. Using spaces of Fredholm operators as in Ortiz's model leads to the problem that there simply is no known good cocycle representative of the universal Chern character on spaces of Fredholm operators that is compatible with a suitable addition. Therefore, the cocycle is only abstractly chosen in his paper. This leads to a very inconcrete description of both the Chern character map and also the addition on $\hat{K}_G$ in Ortiz's paper, since one cannot make concrete choices and check compatibilities. To this end, our work seems to be the first complete reference that deals with all the technical issues that arise from equipping $\hat{K}_G$ with an abelian group structure, when working with the homotopical approach.

One would like to construct comparison maps between Ortiz's and our model. These should be induced by equivariant homotopy equivalences between the respective classifying spaces. The challenge is to find smooth representatives of such maps, which are compatible with the universal Chern forms. Beware that Ortiz does not choose a specific representative for the universal Chern form, but instead just abstractly chooses a cycle on $\mathscr{F}^0$, which is then transgressed via explicit homotopy equivalences to the other $\mathscr{F}^k$.

Let us discuss the situation in degree zero. Here, we have the zig-zag of G-homotopy equivalences,

$$\mathrm{Gr}_{\mathrm{res}} \leftarrow \mathrm{U}_{\mathrm{res}} \rightarrow \mathrm{Fred},$$

where the space we use is on the left, and Ortiz's model on the right. The maps are given by the projection on the left and the map $\begin{pmatrix} X_{++} & X_{-+} \\ X_{+-} & X_{--} \end{pmatrix} \mapsto X_{++}$. Both of these are smooth, but there are no known smooth inverses. Recall that the smooth principal bundle $\mathrm{U}_{\mathrm{res}} \to \mathrm{Gr}_{\mathrm{res}}$ with contractible structure group $\mathrm{U}_+ \times \mathrm{U}_-$ is topologically trivial and therefore admits a continuous section. Nevertheless, one cannot immediately find a smooth section. Such a smooth section would correspond to a smooth version of Kuiper's theorem [Kui65, Theorem 3], which is not known. Similarly, we do not know a smooth equivariant map $\mathrm{U}^1 \to \mathscr{F}^1_G$ that could be a candidate for the odd case.

But even if one does not succeed in finding such natural maps, one really only needs to have smoothness on finite-dimensional test manifolds. Therefore, we expect there to be a way to work on finite-dimensional approximations of the classifying spaces in order to make the comparison, as described in [BS13, Section 6]. Granted the existence of such a "diffeologically smooth" G-homotopy equivalence $\varphi\colon \mathrm{Gr}_{\mathrm{res}} \to \mathscr{F}^0$, which fulfills $\varphi^*\mathrm{ch}_{\mathrm{Ortiz}} = \mathrm{ch}$, it would give rise to a natural group homomorphism

$$\begin{aligned} \varphi_*\colon \hat{K}^0_G(M) &\to \hat{K}^0_{G,\mathrm{Ortiz}}(M) \\ [(f,\omega)] &\mapsto [(\varphi\circ f, \omega, \mathrm{Ch}_{G,\mathrm{Ortiz}}(f) + \mathrm{d}\omega)] \end{aligned}$$

that commutes with the structure maps I, R and a. Therefore, by Lemma 16.2, it would induce an isomorphism of differential extensions. This method of proof by approximation would only produce an abstract map, though. At this point, it would probably be more worthwhile to go directly for a proof of uniqueness of differential extension of equivariant K-theory, analogous to the theorem known in the non-equivariant case. It was already conjectured that this should be true in [BS13, Section 6].

Since we did not succeed in giving a full comparison map, we remark here that Ortiz in [Ort09, Proposition 3.4] gives a version of his theory that is generated entirely by triples (E, ∇, η) of a vector bundle with connection, together with a differential form. This gives an obvious map that assigns to a geometric cycle (E, ∇) the triple $(E, \nabla, 0)$. This is not a cycle map in the usual sense though, since the identity $R \circ \widehat{\mathrm{cycl}}_G = \mathrm{Ch}_G$ cannot be guaranteed (recall that $\mathrm{Ch}_G(E, \nabla)$ denotes the delocalized differential form that comes from the equivariant Chern–Weil method, see the discussion after Definition 13.2). Instead, the cocycle representatives that appear in Ortiz's paper are just abstractly chosen and do not have anything to do with these geometric representatives. If one could make universal choices to guarantee this compatibility, then by the results discussed in the beginning of this section, one would get an isomorphism at least of the even degree parts, in the compact case. For now, we only get that the group-valued functors $\hat{K}^0_G$ and $K^0_{G,\mathrm{Ortiz}}$ are isomorphic, where the isomorphism commutes with the integration map I and the action map a, while the notion of Chern forms differs.

The Bunke–Schick model. In [BS13, Definition 2.19], Bunke and Schick define a version of differential equivariant K-theory using as cycles what they call geometric families, together with a differential form. These objects were introduced by Bunke in [Bun02, Section II.4] in order to have a short name for the data needed to define a Bismut super-connection (see [BGV03, Proposition 10.15]). This approach, compared to our homotopical model, is more on the analytical side. The beauty of this description is that it uses the same geometric objects to describe $\hat{K}^0_G$ as well as $\hat{K}^1_G$, where the difference is just in the dimension of some vertical fiber. Unfortunately, the definition of geometric families and the equivalence relation of “paired” geometric families involves quite a lot of data and setup. Additionally, the paper uses the language of orbifolds. Therefore, it would probably not be very helpful to include the definition here.

We would still like to discuss the existence of a comparison map from our model to the Bunke–Schick model. First, we need to make the transition to the orbifold world. A compact G-manifold M can be interpreted as a presentation of the presentable orbifold $B = [M/G]$. Then, the equivariant K-theory $K^*_G(M)$ corresponds to the orbifold K-theory $K^*(B)$. In [BS13, Section 2.2.5], the authors describe a way to get from a G-vector bundle with connection to a class in their model, which gives rise to a cycle map. By the results of the beginning of this Section, their even degree theory therefore is isomorphic to ours. It is not clear to us how to produce a geometric family from a map $f\colon M \to \mathrm{U}$, and therefore, we unfortunately cannot give a map in the odd case.

The Tradler–Wilson–Zeinalian model. As stated before, in the non-equivariant case, there is a strong uniqueness property for differential cohomology theories, proved by Bunke and Schick in [BS10, Theorem 1.6]. Applied to K-theory, it says that the even part is always unique, while the odd part is unique when we add in the additional requirement of an S^1-integration map. It turns out that when we restrict to the compact case and the trivial group, our model of differential K-theory is isomorphic to the one proposed in [TWZ16, Theorem 4.25], which is exactly the unique model supporting an S^1-integration.

We start by remarking that in the case of a trivial group, we have $L^*_{\{e\}}(M) \cong K^*_{\{e\}}(M)$ by Proposition 14.2. Recall that $L^*_{\{e\}}(M)$ consists of Chern–Simons homotopy classes of classifying maps.

The [TWZ16]-model is also based on smooth classifying spaces. For the odd part, they use the stable unitary group U and define

$$\hat{K}^1_T(M) = \mathrm{Map}(M, \mathrm{U})/\mathrm{CS}{-}\mathrm{equivalence}.$$

Since U does not admit a Banach manifold structure, the authors work with universal cocycles given by the finite-dimensional differential forms (3) on the filtration defined by the inclusions of $\mathrm{U}(n)$ for $n \in \mathbb{N}$. It is immediately clear that our Chern forms $\mathrm{ch}_{\mathrm{odd}} \in \Omega^{\mathrm{odd}}(\mathrm{U}^1)$ pull back to their Chern forms under the natural inclusions

$$\mathrm{U}(n) \hookrightarrow \mathrm{U} \xhookrightarrow{i} \mathrm{U}^1.$$

The second map also preserves the block sum. Since CS-homotopies go to CS-homotopies, it induces a well-defined homomorphism $i_*\colon \hat{K}^1_{\mathrm{T}}(M) \to \hat{K}^1(M)$.

Proposition 16.6. *The natural homomorphism $i_*\colon \hat{K}^1_T \to \hat{K}^1$ preserves all the structure of a differential extension, i.e. $I \circ i_* = I_T$, $i_* \circ a_T = a$ and $R \circ i_* = R_T$. Furthermore, i_* is an isomorphism.*

PROOF. The compatibilities are easy to check and follow from i being a homotopy equivalence and pulling back ch to ch_T. It follows from Lemma 16.2 that i_* is an isomorphism. $\square$

The even part of the [TWZ16]-theory is given by maps into the space of finite rank projections on $\mathbb{C}^\infty_{-\infty} = \bigoplus_\mathbb{Z} \mathbb{C} \subset \mathscr{H}$, defined as

$$\begin{aligned}\mathrm{Proj} &= \left\{\pi \in \mathrm{End}(\mathbb{C}^\infty_{-\infty}) \mid \pi^* = \pi, \mathrm{Spec}(\pi) \subset \{0,1\}, \mathrm{rank}(\pi - \pi_{\mathbb{C}^0_{-\infty}}) < \infty)\right\} \\ &\cong \left\{V \subset \mathbb{C}^\infty_{-\infty} \mid \mathbb{C}^p_{-\infty} \subset V \subset \mathbb{C}^q_{-\infty} \text{ for some } p, q \in \mathbb{Z}\right\}.\end{aligned}$$

Their basepoint is the space $\mathbb{C}^0_{-\infty}$. Apart from a change of basis, we can identify Proj with the colimit of the finite-dimensional Grassmannians, which we denoted by $\mathrm{Gr}_{\mathrm{res},\infty}$ in Section 5, as follows: Denote by $A\colon \mathscr{H} \to \mathscr{H}$ the change of basis which maps e_i to e_{-i} for all i. Then, we have a natural map

$$\begin{aligned} i\colon \mathrm{Proj} &\to \mathrm{Gr}_{\mathrm{res},\infty} \overset{\sim}{\hookrightarrow} \mathrm{Gr}_{\mathrm{res}} \\ \pi &\mapsto A\pi A.\end{aligned}$$

This is well-defined, since $A\pi A - \pi_+ = A(\pi - \pi_{\mathbb{C}^0_{-\infty}})A$ has image contained in some $\mathscr{H}_N \subset \mathrm{im}(A\pi A) \subset \mathscr{H}_{-N}$. We check that it is a homomorphism for the block sum. We have that $\pi_1 \boxplus \pi_2 = \rho^*\pi_1 \oplus \pi_2\rho$ gets mapped to $A\rho^*(\pi_1 \oplus \pi_2)\rho A$. On the other hand, the block sum of the images is $\rho^*(A \oplus A)(\pi_1 \oplus \pi_2)(A \oplus A)\rho$. Comparing $(A \oplus A)\rho$ and ρA as operators from $\mathscr{H} \to \mathscr{H} \oplus \mathscr{H}$ (see Definition 6.4), we see that they both map basis vectors e_{2i} to $(e_{-i}, 0)$. On odd basis vectors, we have

$$(A \oplus A)\rho(e_{2i+1}) = (0, e_i) \quad \text{and} \quad \rho A(e_{2i+1}) = \rho(e_{-2(i+1)+1}) = (0, e_{i+1}).$$

Therefore, if we have $f, g\colon M \to \mathrm{Proj}$, then $i \circ (f \boxplus g)$ and $(i \circ f) \boxplus (i \circ g)$ differ only by conjugation with a fixed unitary matrix $B \in \mathrm{U}_+ \times \mathrm{U}_-$ which shifts odd basis vectors by one. By Lemma 6.5, these are therefore CS-equivalent. We conclude that i induces a homomorphism of differential K-theory groups.

Proposition 16.7. *The homomorphism $i_*\colon \hat{K}^0_T \to \hat{K}^0$ preserves all the structure of a differential extension, i.e. $I \circ i_* = I_T$, $i_* \circ a_T = a$ and $R \circ i_* = R_T$. Furthermore, i_* is an isomorphism.*

PROOF. As in the even case, we have that i is a homotopy equivalence. We need to check that $i^*\mathrm{ch} = \mathrm{ch}_T$. The path-components of Proj are given by the rank map, where by definition $\mathrm{rank}(V) = \dim(V/\mathbb{C}^{-N}_{-\infty}) - N$, if we assume that $\mathbb{C}^{-N}_{-\infty} \subset V \subset \mathbb{C}^N_{-\infty}$. This agrees with the path-component of the image, which is indexed by $\mathrm{virt.dim}(A(V))$. In order to check that the positive degree parts of ch are compatible as well, we note that for the inclusion $\mathrm{Gr}_{k,2N} \hookrightarrow \mathrm{Proj}$, the [TWZ16]-Chern character is calculated in terms of traces of powers of the differential forms $\pi \mathrm{d}\pi \mathrm{d}\pi$. Pulling back along the composition

$$\mathrm{Gr}_{k,2N} \to \mathrm{Proj} \to \mathrm{Gr}_{\mathrm{res},\infty}$$

on the other hand gives the forms $A\pi \mathrm{d}\pi \mathrm{d}\pi A$, whose powers of traces agree with the ones above. Since any map from a compact manifold factors through one of these Grassmannians, we are done. $\square$

17. Examples

Already the point is an interesting example, since it illustrates the role that is played by CS_G-homotopies.

Proposition 17.1. *We have isomorphisms*

$$\hat{K}^0_G(*) \cong K^0_G(*) \cong R(G)$$
$$\hat{K}^1_G(*) \cong \mathbb{C}[G]^G/R(G).$$

Proof. Since there are no odd forms on the point, every even cycle is of the form $(f, 0)$. Furthermore, for the same reason, every homotopy has vanishing Chern–Simons form, and the statement follows. In the odd case, start with a cycle (f, ω), which we will simplify step by step. By equivariance, any map from the point has to go to the fixed point set

$$(\mathrm{U}^1)^G \cong \prod_{V \in \mathrm{Irr}(G)} \mathrm{U}^1.$$

For a homotopy f_t, we write $f_t = (\prod f_V)_t$. The components of the CS_G-form then are

$$\mathrm{CS}_g(f_t) = \int_I (f_t^g)^*(\mathrm{ch}_g)_1 = \frac{i}{2\pi} \sum_{V \in \mathrm{Irr}(C)} \int_I \mathrm{tr}_V(g)\mathrm{tr}(f_V)_t^{-1}(\dot{f}_V)_t. \tag{28}$$

We have an equivariant splitting induced by the (Fredholm)-determinant map

$$S\mathrm{U}^1 \rtimes \mathrm{U}(1) \cong \mathrm{U}^1. \tag{29}$$

Under the isomorphism, the semi-direct group structure is given by

$$(n_1, h_1) \cdot (n_2, h_2) = (n_1 h_1 n_2 h_1^{-1}, h_1 h_2), \quad (n, h)^{-1} = (h^{-1} n^{-1} h, h^{-1}),$$

which yields that for $(f_V)_t = (n_t, h_t)$, we have

$$(f_V)_t^{-1}(\dot{f}_V)_t = h_t^{-1} n_t^{-1} \dot{n}_t h_t + h_t^{-1} \dot{h}_t,$$

where the first term is in $\mathfrak{su}(n)$ and the second one in $\mathfrak{u}(1)$. Since $\mathfrak{su}(n)$ consists of matrices with trace zero, every homotopy that leaves the second factor in (29) constant has vanishing CS_G-form. Therefore, the map f in our starting cycle (f, ω) can be simplified to have image in

$$\prod_{V \in \mathrm{Irr}(G)} \mathrm{U}(1).$$

We can furthermore always rotate it to map to the identity element in each factor, for the price of adding the CS_G-form of the homotopy to ω. Therefore, any cycle is equivalent to a cycle of the form $(1, \omega)$, and we know that $\hat{K}^1_G(*)$ is a quotient of $\Omega^0_G(*) \cong \mathbb{C}[G]^G$, the ring of class functions on G. Now by the above considerations, the only possible way to alter our cycle is by going around the circle an integral amount of times. By Equation 28, going around the circle once in the V-factor changes our cycle the following way:

$$(1, \omega) = (1, \bigoplus_g \omega_g) \sim (1, \bigoplus_g \omega_g + \mathrm{tr}_V(g)).$$

Therefore, what we get is exactly the quotient of the ring of class functions with the integral multiples of the characters, which can be identified with the representation ring $R(G)$. □

Next, we will study the circle S^1. One can deduce from the exact sequence

$$0 \to \Omega^{*-1}_G(S^1)/\mathrm{im}(d_G)/\mathrm{im}(\mathrm{Ch}_G) \xrightarrow{a} \hat{K}^*_G(S^1) \xrightarrow{I} K^*_G(S^1) \to 0$$

coming from the axioms of $\hat{K}_G$ that there are exact sequences

$$\begin{aligned} 0 \to H^1_G(S^1)/\mathrm{im}(\mathrm{Ch}_G) \to \hat{K}^0_G(S^1) \to K^0_G(S^1) \to 0 \\ \text{and} \quad 0 \to \Omega^0_G(S^1)/R(G) \to \hat{K}^1_G(S^1) \to K^1_G(S^1) \to 0. \end{aligned} \tag{30}$$

The interesting part is to understand the kernel of I. Assume that the action on S^1 is trivial. Then, we can give a concrete description both cases. The following computation already appeared in [BS13, Lemma 5.1], in a different geometric model.

Lemma 17.2. *Let $(E^\pm, \nabla^\pm)$ be a pair of G-vector bundles with compatible hermitian connection over S^1 with $(E^+)_{x_0} \cong (E^-)_{x_0}$ as G-representations at some basepoint $x_0 \in S^1$. For each $g \in G$, the bundles split into a direct sum of bundles*

$$E^\pm \cong \bigoplus_{V \in \mathrm{Irr}(\langle g \rangle)} E^\pm_V,$$

according to the irreducible representations of the cyclic group $\langle g \rangle$ (see Lemma 7.8). Then, the corresponding element in $\hat{K}^0(S^1)$ is

$$[(E^+, \nabla^+)] - [(E^-, \nabla^-)] = a\left(\bigoplus_{g \in G} \frac{1}{2\pi i} \sum_{V \in \mathrm{Irr}(\langle g \rangle)} \mathrm{tr}_V(g) \log \frac{\det \mathrm{hol}(E^+_V, \nabla^+_V)}{\det \mathrm{hol}(E^-_V, \nabla^-_V)} \mathrm{vol}\right),$$

where vol *is a representative of an integral generator of $H^1(S^1)$ and $\mathrm{hol}(E^\pm_V, \nabla^\pm_V) \in \mathrm{U}(\dim(E^\pm_V))/\text{conjugation}$ denotes the holonomy of the bundle.*

Proof. Since the action on S^1 is trivial, we can reduce to the non-equivariant case. Let $f_\pm$ be the classifying maps of the bundles with connections, which have image in some finite-dimensional Grassmannian subspace of $\mathrm{Gr}_{\mathrm{res}}$. Then the induced $\hat{K}_G$-class is given by the tupel $(f, 0) = (f^+ \boxplus \mathrm{flip}(f^-), 0)$.

The image of $f^\pm$ is contained in $\mathrm{Gr}^G_{\mathrm{res}}$, and so they split into a product of maps $\prod_{V\in\mathrm{Irr}(G)}(f^\pm)_V$. Since their representations at x_0 are isomorphic, for each V, there is a null-homotopy to a constant map of the classifying map of the virtual bundle $E^+_V - E^-_V$ (see Remark 5.12), given by a block sum of homotopies

$$(f_V)_t = (f^+_V)_t \boxplus \mathrm{flip}(f^-_V)_t \colon S^1 \to \mathrm{Gr}^0_{\mathrm{res}}.$$

By the construction of the map a, the differential form corresponding to $f_1 = f$ is the CS_G-form of a null-homotopy to $\mathrm{const}_{\mathscr{H}_+}$. Note that $f_t = \prod_{V\in\mathrm{Irr}(G)}(f_V)_t$ is a null-homotopy to a constant map to some subspace in $\mathrm{Gr}^0_{\mathrm{res}}(V \otimes \mathscr{H})$. Any such map can be connected by a CS-homotopy to $\mathrm{const}_{V\otimes\mathscr{H}_+}$. We need to calculate $\mathrm{CS}_G(f_t)$. This form is given by

$$\mathrm{CS}_G(f_t) = \bigoplus_{g\in G} \mathrm{CS}_g(f_t) = \bigoplus_{g\in G} \sum_{V\in\mathrm{Irr}(\langle g\rangle)} \mathrm{tr}_V(g) \int_I \mathrm{Ch}((f_V)_t).$$

We are therefore interested in the (non-equivariant) Chern–Simons form of $(f_V)_t$.

Since the image $(f_V)_t$ is contained in some finite-dimensional Grassmannian, we have a path of actual finite-dimensional bundles with connections. We compute

$$\begin{aligned}\mathrm{CS}((f_V)_t) = \int_I (f_V)^*_t \mathrm{ch}_2 &= \int_I (f^+_V)^*_t \mathrm{ch}_2 - (f^-_V)^*_t \mathrm{ch}_2 \\ &= \frac{i}{2\pi} \int_I \Omega_{\det((f^+_V)_t)} - \Omega_{\det((f^-_V)_t)}.\end{aligned}$$

In the last step, we used that ch_2 of a bundle is the same as the first Chern class of its determinant line bundle, i.e. the integral over $\frac{i}{2\pi}$ times its curvature. Recall that we are interested in the cohomology class of this form in the exact sequence in Equation 30. The integral over the circle gives an isomorphism $H^1_{\mathrm{de\,Rham}}(S^1) \cong \mathbb{R}$. Therefore, we can find out the multiple of an integral generator vol that we are looking at by integrating. This yields

$$\begin{aligned}\int_{S^1} \mathrm{CS}((f_V)_t) &= \frac{i}{2\pi} \int_{S^1} \int_I \Omega_{\det((f^+_V)_t)} - \Omega_{\det((f^-_V)_t)} \\ &= \frac{1}{2\pi i} \left(-\int_{D^2} \Omega_{\det((f^+_V)_t)} + \int_{D^2} \Omega_{\det((f^-_V)_t)} \right) \\ &= \frac{1}{2\pi i} \left(\log \mathrm{hol}(\det(f^+_V)) - \log \mathrm{hol}(\det(f^-_V)) \right),\end{aligned}$$

which shows that the function f_V corresponds to the logarithm of the determinant of the holonomy of its induced bundles. Putting this together for all irreducible $\langle g\rangle$-representations V gives the claim. □

Proposition 17.3. *A delocalized cohomology class on S^1 represented by a 1-form $\omega \in \Omega^1_G(S^1)$ gets mapped in the exact sequence (30) to $a([\omega]) = [f_\omega]$, where f_ω is the classifying map of a trivial bundle*

$$E = \bigoplus_{V \in \mathrm{Irr}(G)} E_V = \bigoplus_{V \in \mathrm{Irr}(G)} L_V \otimes V,$$

where each summand is a line bundle with connection tensored with a representation (as a trivial bundle with trivial connection). The connection on L_V is given by the local connection form $i\alpha_V$, where the collection $\alpha = (\alpha_{V_1}, \alpha_{V_2}, \ldots, \alpha_{|\mathrm{Conj}(G)|})$ can be constructed from $\omega = (\omega_{g_1}, \omega_{g_2}, \ldots, \omega_{g_{|\mathrm{Conj}(G)|}})$ via the rule

$$\alpha = A^{-1}\omega.$$

Here, the matrix A is given by $A = (\mathrm{dim(V_i)}\chi_i(g_j))_{ij}$. In the odd case, a function $\varphi = \bigoplus_g \varphi_g \in \Omega^0_G(S^1)$ gives rise to an element in $\hat{K}^1(S^1)$ via the exponential map in the following way: Reindex as in the even case using the irreducible representations by applying the matrix $B = (\chi_i(g_j))_{ij}$ and setting $\alpha = (\alpha_{V_1}, \ldots, \alpha_{V_{|G|}})$ equal to

$$\alpha = B^{-1}\varphi.$$

Then, we have

$$a(\varphi) = [\prod_{V \in \mathrm{Irr}(G)} \exp(\frac{2\pi}{i}\alpha_V)].$$

Proof. Let $i\alpha_V \in \Omega^1(S^1; \mathfrak{u}(1))$ be a local connection form for a trivial line bundle $L_V \to M$. Consider an irreducible G-representation V. Then, after choosing a basis for V, there is an induced connection on the trivial frame bundle

$$P_V = \mathrm{Fr}(E_V) = \mathrm{Fr}(L_V \otimes V),$$

which we denote by

$$i\beta_V = i \prod_{i=1}^{\dim(V)} \alpha_V.$$

Let

$$\begin{aligned} s\colon I &\to P \cong S^1 \times \mathrm{U}(\dim(V)) \\ t &\mapsto (z, H(t)) \end{aligned}$$

be a horizontal lift of the fundamental loop on S^1, starting at the identity, where $z = \exp(2\pi i t)$. This lift is block diagonal for the product decomposition into copies of U(1)'s, so we have $H = \prod H_i$, with $H_i = H_j$ for all i, j. If we write $\alpha_z = \alpha(z)\mathrm{d}z$, the blocks are determined by the equation

$$i\alpha_V(z)\mathrm{d}z = H_i(z)^{-1}H_i'(z)\mathrm{d}z,$$

which we can integrate over the interval and exponentiate in order to get

$$\exp\left(i\int_{S^1}\alpha_V(z)\mathrm{d}z\right) = \exp\left(\int_I H(z)^{-1}H'(z)\mathrm{d}z\right) = H_i(1).$$

In the last step, we used that for any path $h\colon I \to \mathrm{U}(1)$ starting at $h(0) = 1$, we have

$$h(s) = \exp\left(\int_0^s h_t^{-1}\dot{h}_t\mathrm{d}t\right),$$

which can be seen by noting that $k(s) = h_s^{-1}\exp(\int_0^s h_t^{-1}\dot{h}_t\mathrm{d}t)$ satisfies $k(0) = 1$ and $\dot{k}(s) = 0$ for all s. Note that $H(1) = \prod H_i(1)$ is precisely the holonomy of the connection $i\beta_V$. Since there is only one irreducible representation in the decomposition of E_V, by Lemma 17.2, we have

$$\begin{aligned}
[E_V, \mathrm{d} + i\beta_V] &= a\left(\bigoplus_{g\in G}\frac{1}{2\pi i}\mathrm{tr}_V(g)\log\det\mathrm{hol}(E_V)\,\mathrm{vol}\right)\\
&= a\left(\bigoplus_{g\in G}\frac{1}{2\pi i}\mathrm{tr}_V(g)\log\prod\det\mathrm{H_i}(1)\,\mathrm{vol}\right)\\
&= a\left(\bigoplus_{g\in G}\dim(V)\mathrm{tr}_V(g)\left(\frac{1}{2\pi}\int_{S^1}\alpha_V(z)\mathrm{d}z\right)\mathrm{vol}\right)\\
&= a\left(\bigoplus_{g\in G}\dim(V)\mathrm{tr}_V(g)\alpha_V\right)\\
&= a\left(\omega_V\right).
\end{aligned}$$

Note that in the third step we replace a form by another form cohomologous to it. Consequently, if we do this construction for all irreducible G-representations V, we can consider the bundle $E = \bigoplus E_V$ with the direct sum connection form of all the $i\beta_V$. Then, it is clear that

$$\begin{aligned}
[E, \bigoplus_V \mathrm{d} + i\beta_V] &= a\left(\sum_{V\in\mathrm{Irr}(G)}\omega_V\right)\\
&= a(\omega).
\end{aligned}$$

We are interested in the g-component of the form ω. This basically means that we have to reindex from using the $|\mathrm{Conj}(G)|$-many irreducible G-representations to the actual conjugacy classes by using the character map.

If we enumerate the irreducible G-representations V_i with characters χ_i as well as a representative g_i for each conjugacy class in G, we have the system of linear equations

$$\begin{aligned}
\omega_{g_1} &= \dim(V_1)\chi_1(g_1)\alpha_{V_1} + \dim(V_2)\chi_2(g_1)\alpha_{V_2} + \cdots + \dim(V_{|G|})\chi_{|G|}(g_1)\alpha_{V_{|G|}} \\
\omega_{g_2} &= \dim(V_1)\chi_1(g_2)\alpha_{V_1} + \dim(V_2)\chi_2(g_2)\alpha_{V_2} + \cdots + \dim(V_{|G|})\chi_{|G|}(g_2)\alpha_{V_{|G|}} \\
&\vdots \\
\omega_{g_{|G|}} &= \dim(V_1)\chi_1(g_{|G|})\alpha_{V_1} + \dim(V_2)\chi_2(g_{|G|})\alpha_{V_2} + \cdots + \dim(V_{|G|})\chi_{|G|}(g_{|G|})\alpha_{V_{|G|}}.
\end{aligned}$$

Note that the coefficient matrix $A = (\dim(V_i)\chi_i(g_j))_{i,j}$ is always invertible, since the characters form an orthonormal basis of the set of class functions on G (Lemma 7.10). Therefore, for any given $\omega = \bigoplus_g \omega_g$, there is a collection of differential forms α_{V_i}, such that

$$\omega_g = \sum_{V_i} \dim(V_i)\chi_i(g)\alpha_{V_i}.$$

Finally, as shown in the first part of the proof, the bundle $E = \bigoplus E_V$ with connection the direct sum connection, as constructed, is the image under a of the differential form $\omega \in \Omega^1_G(S^1)$.

For the odd part, consider a cycle $(f, 0) \in \hat{K}_G(S^1)$. Since the action is trivial, every cycle is equivalent to a cycle of this form. Again, $f = \prod f_V$ splits into a product over the irreducible G-representations. Assume that $(f, 0) \in \ker(I)$. That means that there are null-homotopies $(f_V)_t$ from $(f_V)_1 = f_V$ to the constant map to the basepoint for all V, which can be put together to a null-homotopy f_t from f to 1. The Chern–Simons form is

$$\omega_{f_V}(z) = \mathrm{CS}((f_V)_t)(z) = \frac{i}{2\pi}\int_I \mathrm{tr}\left((f_V)_t^*(z)(\dot{f}_V)_t(z)\right) \mathrm{d}t.$$

By the splitting in (29), we can assume that $(f_V)_t(z)$ takes values in $\mathrm{U}(1) \subset \mathrm{U}^1$. Then, for fixed z, the integral on the right is over a path that starts at $1 \in S^1$, and we use the same argument as in the even case to conclude that

$$\exp\left(\frac{2\pi}{i}\omega_{f_V}(z)\right) = \exp\left(\int_I \mathrm{tr}\left((f_V)_t^*(z)(\dot{f}_V)_t(z)\right) \mathrm{d}t\right) = f_V(z). \tag{31}$$

Now, similar to the even case, if we are given a form $\omega = \bigoplus_g \omega_g \in \Omega^0_G(S^1)$, we want to reindex to using the irreducible G-representations. This works by applying the inverse of the matrix $B = (\chi_i(g_j))_{i,j}$. It follows that by defining $f\colon S^1 \to \mathrm{U}^1$ to be the product for all V of the maps on the left hand side of (31), followed by the inclusion $\mathrm{U}(1) \hookrightarrow \mathrm{U}^1$, we have successfully recovered the function f from the given form ω. $\square$

18. Further research and open questions

Groups. Our model only allows finite groups to act on our manifolds. While this is already an interesting case, ideally, one would like to extend to a bigger class of groups. The two obvious candidates are infinite discrete groups and compact Lie groups. In both cases, there are some serious technical difficulties to overcome. For example, our definition of delocalized cohomology does not make sense for infinite groups.

In the first case, if one is willing to restrict to proper actions of discrete groups, Lück and Oliver showed in [LO01, Theorem 5.5] that there is still an equivariant Chern character. The target of this Chern character is Bredon cohomology, which is isomorphic to delocalized cohomology for finite groups. The problem that arises with our classifying space based approach is then that the representation theory for discrete groups is quite complicated. Recall that a lot of our constructions relied quite heavily on the fact that we can write the fixed point sets just as products over all irreducible representations. This will not be possible anymore in the case of an infinite discrete group.

For compact Lie groups, we still have more control over the representation theory, since we have the Peter–Weyl theorem. Therefore, it seems much more likely that one can generalize to this case. Nevertheless, one needs to be careful that the summability conditions on our spaces are preserved throughout all the constructions, since, for example, the sum of infinitely many trace class operators is not guaranteed to be trace class anymore. Another problem is that there is in general no equivariant Chern character for compact Lie groups. For example in [Hae85], Haeberly constructs an example where such an isomorphism is impossible. However, the example uses a circle action with fixed points. On the other hand, it is shown in [AR01, Corollary 5.5, Remark 5.11] that almost free actions of compact Lie groups do in fact admit appropriate equivariant Chern characters.

Product. Differential equivariant K-theory admits a product structure. On the spaces of Fredholm operators in [Ort09], this is explicitly induced by a sort of tensor product of Fredholm operators, similarly to how the block sum induces the addition. It should be possible to come up with a tensor product formula for the spaces of operators that we use in our model. The challenge here is again that the summability conditions that we put on our operators are much more rigid and harder to preserve then just Fredholmness when defining a product map. For this reason, the formulas from [Jän65] do not work on the nose and have to be tweaked. We suspect that this can be done.

Pushforward. Ortiz constructs a push forward map in his model [Ort09, Section 5]. For an equivariant fiber bundle $p\colon X \to Y$, where X is compact and the fiber has dimension n, this is a map

$$K_G^*(X) \to K_G^{*-n}(Y).$$

He then conjectures an index theorem in differential K-theory. It would be interesting to see if his map has an explicit description in our model.

Calculations. Differential K-theory is in general quite hard to compute, and there are definitely not enough examples that have been evaluated. Since the spaces $\mathrm{Gr}_{\mathrm{res}}$ and U^1 have been studied a lot and are by now quite well understood, we hope that our model can help enlarge the list of computable examples in the future.

Representations of loop groups. One of the main sources for the properties of the space $\mathrm{Gr}_{\mathrm{res}}$ is the book [PS88, Section 7] by Pressley and Segal. Here, this space is studied for a completely different reason, namely in order to understand the representation theory of loop groups. These are the infinite-dimensional Lie groups that arise from taking the loop space of a Lie group. They are in some sense the simplest infinite-dimensional Lie groups, since they often behave like compact groups. Let $K \to \mathrm{U}(n)$ be a unitary representation for a compact Lie group K. Then, the free loop group $LK = \mathscr{C}^\infty(S^1, K)$ acts unitarily on the Hilbert space $\mathscr{H} = L^2(S^1, \mathbb{C}^n)$ by

$$\begin{aligned} i\colon LK &\to \mathrm{U}(\mathscr{H}) \\ \varphi &\mapsto M_\varphi, \end{aligned}$$

where $M_\varphi(f)(t) = \varphi(t) \cdot f(t)$. This is the multiplication operator map from Lemma 10.5. Now as discussed in that lemma, the image of this map is in the restricted unitary group, and we can compose with the projection to get a map to $\mathrm{Gr}_{\mathrm{res}}$. Actually, when one takes the quotient with respect to the based constant loops $K \subset LK$, one gets an embedding of the based loop space

$$\Omega K \cong LK/K \hookrightarrow \mathrm{Gr}_{\mathrm{res}}.$$

The properties of this embedding are discussed in [SW00]. In this sense, a representation for a compact Lie group K gives rise to maps from both the free and based loop space into the restricted Grassmannian, which we might interpret as differential K-theory classes. It would be interesting to understand this connection better.

Uniqueness of equivariant extensions. In the non-equivariant case, there is a strong uniqueness property that asserts that there is up to isomorphism only one differential extension of topological K-theory that admits an S^1-integration map. Since the underlying homotopy theory for equivariant K-theory shares the basic relevant features, it was conjectured already by Bunke and Schick [BS13, Section 6] that a similar uniqueness theorem can be established in the equivariant case. It will definitely be necessary to make some assumptions about compatibility with additional structure like integration maps, since even non-equivariantly, there are infinitely many differential extensions of odd K-theory (see [BS10, Section 6]). On this note, we remark that we proved in Section 16 that there is a unique differential extension of equivariant K-theory that supports differential lifts of the cycle maps.

Extension to infinite-dimensional manifolds. Our approach with infinite-dimensional classifying manifolds in principle gives a straight-forward definition of differential equivariant K-theory for any object that supports the notion of differential form, and for which we can make sense of smooth maps into a Banach manifold. Certainly, a generalization of the definition of $\hat{K}_G$ to the category of smooth Banach manifolds, or even smooth Banach G-manifolds, seems natural. Ideally, one would like the functors K_G^* to be defined on the classifying objects itself. The main challenge for such an extension is, that it it no longer possible to smoothly approximate any continuous map by a smooth on an infinite-dimensional source manifold. This makes it harder to prove the realization results of differential forms as Chern forms that we need. In particular, in the proof of Proposition 10.2 and 10.7, we can no longer cop out and use a cohomological argument, since we cannot smoothly approximate in the end. As already mentioned in this chapter, one possible solution would be to consider the periodicity maps h_{odd} and h_{even} as smooth maps between infinite-dimensional manifolds, and to try to explicitly calculate the pullback of the equivariant Chern character to the loop space.

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