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EXTRINSIC SYMMETRIC SUBSPACES

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Abstract

An extrinsic symmetric space is a submanifold $M \subset V = \mathbb{R}^n$ which is kept invariant by the reflection s_x along every normal space N_xM. An extrinsic symmetric subspace is a connected component M' of the intersection $M \cap V'$ for some subspace $V' \subset V$ which is s_x-invariant for any $x \in M'$. We give an algebraic charactrization of all such subspaces V'.

1. Introduction

It is well known that totally geodesic subspaces of a symmetric space M correspond one-to-one to Lie subtriples of the corresponding Lie triple (which is the tangent space of M with the curvature tensor as algebraic structure). In the present note we study the same question for an important subclass of symmetric spaces, those which allow a nice embedding into euclidean space $V = \mathbb{R}^n$. These are the so called extrinsic symmetric spaces or symmetric R-spaces. More precisely, an extrinsic symmetric space is a submanifold $M \subset V$ such that for any point $x \in M$, the reflection s_x along the normal space $N = N_xM$ keeps M invariant. An extrinsic symmetric subspace $M' \subset M$ will be a connected component M' of the intersection $M \cap V'$ with a subspace $V' \subset V$ which is invariant under s_x for all $x \in M'$; in particular, $M' \subset M$ is totally geodesic. We may assume that $M' \subset V'$ is full. Our main result Theorem 2 characterizes these subspaces V' as follows. By a result of Ferus [5, 6], after splitting off an affine subspace, V is itself a Lie triple (a tangent space of another symmetric space), and our result is:

A connected component M' of $M \cap V'$ which is full in V' is an extrinsic symmetric subspace if and only if $V' \subset V$ is a Lie subtriple.

The main idea for the proof is given by an alternative approach [3] to Ferus' theorem where the Lie structure is computed in terms of submanifold geometry. At the end we briefly discuss two questions.

1. Which Lie subtriples V' actually do intersect a given extrinsic symmetric space M?

2. Suppose that $V' \subset V$ is a subspace preserved by s_x for all $x \in M \cap V'$. Suppose further that $M \cap V'$ spans V', but no single connected component of $M \cap V'$ is full in V' (e.g. $M \cap V$ could be discrete). Is $V' \subset V$ still a Lie subtriple? The answer to this question seems to be open.

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2. Extrinsic symmetric spaces and subspaces

Let $M \subset V$ be a closed submanifold (not necessarily connected) of some euclidean vector space $V = \mathbb{R}^n$. For simplicity of notation[1] we assume that M is contained in the unit sphere $\mathbb{S} = \mathbb{S}^{n-1} \subset V$. Let O_n denote the orthogonal group on $\mathbb{R}^n$,

$$O_n = \{A \in \mathbb{R}^{n \times n} : A^t A = I\} \tag{1}$$

where $\mathbb{R}^{n \times n}$ is the space of real $(n \times n)$-matrices, A^t the transposed of the matrix A and I the unit matrix. For any $x \in M$ let $s_x \in O_n$ be the reflection along the normal space $N = N_x M$, that is $s_x = I$ on N and $s_x = -I$ on $T = T_x M$. The submanifold M is called *extrinsic symmetric* if

$$s_x(M) = M \ \forall x \in M\,. \tag{2}$$

Then s_x is called the *(extrinsic) symmetry* at x and the subgroup $K \subset O_n$ generated by all s_x, $x \in M$ is the *symmetry group* of M. It acts transitively on every connected component $M^o \subset M$ since any two $y, z \in M^o$ can be connected by a geodesic $\gamma : [0, 1] \to M$, and $z = s_x y$ where $x = \gamma(\frac{1}{2})$ is the midpoint between z and y.

EXAMPLE: ORTHOGONAL GROUP. Let $M := O_p \subset V = \mathbb{R}^{p \times p}$ be the orthogonal group (1). This is extrinsic symmetric (with two connected components): the symmetry at $x \in M$ is $s_x(v) = xv^t x$ for all $x \in M$ and $v \in V$. Clearly, $\det s_x(v) = \det v$, hence s_x preserves the two connected components of M.

REMARK. In this example, the symmetry group does not act transitively on $M = O_p$ since the connected components of M are preserved. However, the full isometry group of all orthogonal maps of V preserving M does act transitively since it contains the left (or right) translations with all elements of O_p.

A subset $M' \subset M$ is called an *extrinsic symmetric subspace* if M' is a connected component of $M \cap V'$ for some linear subspace $V' \subset V$ with

$$s_x(V') = V' \text{ for all } x \in M'\,. \tag{3}$$

Given an extrinsic symmetric subspace $M' \subset M$, there might be several subspaces $V' \subset V$ with (3) such that M' is a connected component of $M \cap V'$; we will always choose V' to be the smallest one (the intersection of all such subspaces), or equivalently, V' is just the linear span of M' or M' is full in V'.

Every s_x, $x \in M'$, preserves V' and its orthogonal complement V'', thus it decomposes these spaces into its (± 1)-eigenspaces which are the intersections with T and N,

$$V' = T' \oplus N'\,,\ V'' = T'' \oplus N'' \tag{4}$$

where $T' = T \cap V'$, $N' = N \cap V'$ etc. Let $\pi_T : V \to T$ and $\pi_N : V \to N$ be the orthogonal projection onto T and N. Then $\pi_T(V') = T'$ by (4). Hence $\pi_T|_{M'} : M' \to T'$ is a diffeomorphism near x. Thus M' is a submanifold of both M and V', and the tangent and normal

[1] It turns out that indecomposable extrinsic symmetric spaces (other than straight lines) lie in euclidean spheres, cf. [1].

spaces of $M' \subset V'$ at x are T' and N'.

Let α, α' denote the second fundamental forms of $M \subset V$ and $M' \subset V'$, respectively. E.g. $\alpha(v, w) = \pi_N(\partial_v w) = (\partial_v w)^N$ for all $v, w \in T$, where ∂_v is the directional derivative, $\partial_v w = \frac{d}{dt}\big|_{t=0} w(x + tv)$. Then

$$\alpha'(v, w) = (\partial_v w)^{N'} = (\partial_v w)^N = \alpha(v, w) \tag{5}$$

for all $v, w \in T'$ since $\partial_v w \in V'$ and $(\partial_v w)^N \in V' \cap N = N'$. As a consequence we obtain

Lemma 1. *Every connected component M' of $M \cap V'$ is totally geodesic in M and extrinsic symmetric in V'.*

Proof. $M' \subset M$ is totally geodesic by (5). Further, the group

$$K' = \{k \in K : k(V') = V'\}$$

contains the symmetries s_x, $x \in M'$, and any s_x preserves both M and V' and thus $M \cap V'$, and its connected component through x which is M'. Hence $M' \subset V'$ is extrinsic symmetric by (4). □

Example: Grassmannians. Let $V = \mathbb{R}^{p\times p}$ and $M = O_p$ as in the previous example. Let $V' = S_p = \{x \in V : x^t = x\}$. Then $O_p \cap S_p$ is the set of involutions in O_p ("reflections") since for each $x \in O_p$, that is $x^t = x^{-1}$, the condition $x^t = x$ is the same as $x^{-1} = x$. Orthogonal reflections are in 1:1 correspondence to their fixed spaces, thus $O_p \cap S_p$ can be considered as the union of all Grassmannians $G_k = G_k(\mathbb{R}^p)$ with $k \in \{0, \dots, p\}$. These are the connected component of $M \cap V'$. Hence each Grassmannian G_k is an extrinsic symmetric subspace of one of the components of M. The map $x \mapsto -x$ on $\mathbb{R}^{p\times p}$ is an isometry of O_p which interchanges G_k and G_{p-k} while G_k and G_l for $l \neq k, p-k$ are non-isometric.

3. Lie triples and submanifold geometry

A connected extrinsic symmetric space $M \subset \mathbb{S} \subset V$ is extrinsic homogeneous, $M = Kx$ for some $x \in \mathbb{S}$. In other words, it is an orbit of a representation. By a theorem of D. Ferus [3, 6], both the representation and the point x are very special. The vector space V carries the structure of an *orthogonal Lie triple* $\mathfrak{p}$ (cf. [7]) and x satisfies

$$(\mathrm{ad}_x)^3 = -\,\mathrm{ad}_x\,. \tag{6}$$

More precisely, $V = \mathfrak{p}$ is a linear subspace of a Lie algebra $\mathfrak{g}$ with an involution σ with (± 1)-eigenspace decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \tag{7}$$

hence $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. In other words, $V = \mathfrak{p}$ is the tangent space of another symmetric space $P = G/K$, and the Lie triple structure on $V = \mathfrak{p} \subset \mathfrak{g}$ is $R(u, v)w := -[[u, v], w]$. The Lie bracket can be chosen such that $M = \mathrm{Ad}(K)x$ where $x \in \mathfrak{p}$ satisfies (6).

Recall from [3] that the Lie structure on $\mathfrak{p}$ can be derived from the submanifold geometry of $M \subset \mathfrak{p}$ as follows. Consider the decomposition $\mathfrak{k} = \mathfrak{k}_+ \oplus \mathfrak{k}_-$ where $\mathfrak{k}_+$ is the Lie algebra of the stabilizer group of $x \in M$ and $\mathfrak{k}_-$ denotes the space of infinitesimal transvections at x (the Killing fields A with $\nabla A = 0$ at x) which can be identified with the tangent space $T = T_x M$.

Then the infinitesimal transvection S_v corresponding to any $v \in T$ is essentially[2] the second fundamental form $\alpha : S(T) \to N$ of $M \subset V$:

$$S_v : \begin{cases} T \;\to\; N \;:\; w \;\mapsto\; (\partial_v w)^N = \alpha(v,w)\,, \\ N \;\to\; T \;:\; \xi \;\mapsto\; (\partial_v \xi)^T = -A_\xi v\,. \end{cases} \tag{8}$$

Moreover, the Lie brackets on $\mathfrak{p}$ are also given in terms of α: for all $v, w \in T$ and $\xi, \eta \in N$ we have by [3]:

$$\begin{array}{rcccl} [v,w] & = & [S_v, S_w] & \in & \mathfrak{k}_+\,, \\ [v,\xi] & = & S_{A_\xi v} & \in & \mathfrak{k}_-\,, \\ [\xi,\eta] & = & -[A_\xi, A_\eta] & \in & \mathfrak{k}_+\,. \end{array} \tag{9}$$

On the other hand, when $M = \mathrm{Ad}(K)x \subset \mathfrak{p}$ and $(\mathrm{ad}_x)^3 = -\,\mathrm{ad}_x$, the extrinsic symmetry s_x can be expressed by the Lie structure of $\mathfrak{p} \subset \mathfrak{g}$ as follows:

$$s_x = e^{\pi\,\mathrm{ad}_x} \tag{10}$$

since ad_x is a complex structure on $\mathfrak{k}_- \oplus T$ interchanging these two subspaces while it vanishes on $\mathfrak{k}_+ \oplus N$.

4. Extrinsic symmetric subspaces

Theorem 2. *Let $M = \mathrm{Ad}(K)x \subset \mathfrak{p}$ with $(\mathrm{ad}_x)^3 = -\,\mathrm{ad}_x$ be an extrinsic symmetric space and $\mathfrak{p}'$ a linear subspace of $\mathfrak{p}$ intersecting M. Let M' be a connected component of $M \cap \mathfrak{p}'$ and suppose that $\mathfrak{p}'$ is the linear span of M'. Then $M' \subset \mathfrak{p}'$ is an extrinsic symmetric subspace if and only if $\mathfrak{p}'$ is a Lie subtriple.*

Proof. Let $\mathfrak{p}' \subset \mathfrak{p}$ be a Lie subtriple intersecting M. Let M' be a connected component of $M \cap \mathfrak{p}'$ and $x \in M'$. We have to show that the symmetry s_x preserves $\mathfrak{p}'$. Since $s_x = e^{\pi\,\mathrm{ad}_x}$ by (10), it has a natural extension to an automorphism of the full Lie algebra $\mathfrak{g}$. Now $x \in \mathfrak{p}'$ lies in the Lie subalgebra $\mathfrak{g}' = \mathfrak{p}' + [\mathfrak{p}', \mathfrak{p}'] \subset \mathfrak{g}$. Thus s_x preserves both $\mathfrak{g}'$ and $\mathfrak{p}$ and hence its intersection $\mathfrak{g}' \cap \mathfrak{p} = \mathfrak{p}'$ is preserved. Therefore $M' \subset \mathfrak{p}'$ is an extrinsic symmetric subpace.

Vice versa, let $M' \subset \mathfrak{p}'$ be an extrinsic symmetric subspace. Choose $x \in M'$. Let $T' = T_x M'$ and $N' = \mathfrak{p}' \ominus T'$ be the tangent and normal spaces of $M' \subset \mathfrak{p}'$. We want to show that $\mathfrak{p}'$ is a Lie subtriple. We know already that $M' \subset M$ is totally geodesic (see Lemma 1), thus the second fundamental form α' of $M' \subset \mathfrak{p}'$ satisfies $\alpha' = \alpha|_{S(T')}$. Hence by (9), the restriction of the Lie bracket of $\mathfrak{p}$ to $\mathfrak{p}' = T' \oplus N'$ takes values in $\mathfrak{k}'$. Thus $[\mathfrak{p}', \mathfrak{p}'] \subset \mathfrak{k}'$ and $\mathfrak{p}' \subset \mathfrak{p}$ is a Lie subtriple. □

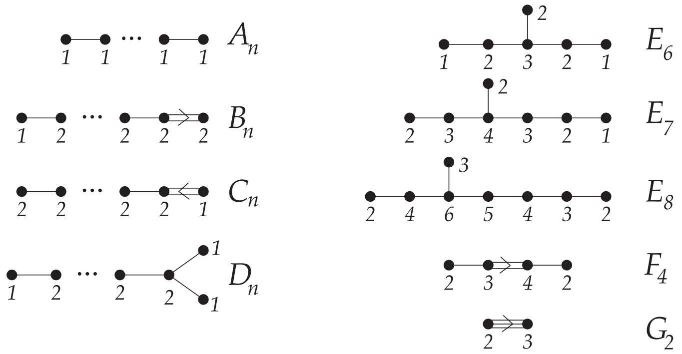
5. Lie subtriples $\mathfrak{p}' \subset \mathfrak{p}$ intersecting $M \subset \mathfrak{p}$

It remains to determine those Lie subtriples $\mathfrak{p}'$ which have non-empty intersection with M. This can be seen from M and the Dynkin diagrams of $\mathfrak{p}$ and $\mathfrak{p}'$.

Let $x \in \mathfrak{p}$ be an extrinsic symmetric vector, that is x satisfies (6) or in other words, $i, 0, -i$ are the eigenvalues of ad_x. We choose a maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$ containing x and a simple root system $\alpha_1, \ldots, \alpha_r$ with $\alpha_i(x) \geq 0$ for $i = 1, \ldots, r$. Let α be any positive root.

[2] Note that $(\xi \mapsto A_\xi) : N \to S(T)$ is the adjoint of $\alpha : S(T) \to N$.

On the corresponding root space $\mathfrak{g}_\alpha \subset \mathfrak{g} \otimes \mathbb{C}$ we have $\mathrm{ad}_x = i\alpha(x) \cdot \mathrm{id}$. Thus $\alpha(x) \in \{0, \pm 1\}$. In particular this holds for the highest root, $\alpha = \delta = \sum_i n_i \alpha_i$, hence $\delta(x) = \sum_i n_i \alpha_i(x) = 1$. Since all $n_i \geq 1$, the element x must be a *dual root* $x = \xi_j$ for some $j \in \{1, \dots, r\}$, that is $\alpha_j(x) = 1$ for some j with $n_j = 1$ and $\alpha_i(x) = 0$ for all $i \neq j$. Below we display the Dynkin diagrams of the simple root systems[3] with the numbers n_j attached to α_j [7, p. 477]. The extrinsic symmetric elements x are precisely the dual vectors to simple roots α_j with $n_j = 1$.



When we have a Lie subtriple $\mathfrak{p}' \subset \mathfrak{p}$, we may choose maximal abelian subalgebras $\mathfrak{a}'$, $\mathfrak{a}$ of $\mathfrak{p}'$, $\mathfrak{p}$ with $\mathfrak{a}' \subset \mathfrak{a}$. Since M is an $\mathrm{Ad}(K)$-orbit, it intersects $\mathfrak{a}$ at some point x in a closed Weyl chamber $\bar{C} \subset \mathfrak{a}$, and x is a dual root of weight 1 for the simple root system corresponding to the Weyl chamber C. Thus:

Theorem 3. *Let $M = \mathrm{Ad}(K)x$ with $x \in \mathfrak{a}$. Then $M \cap \mathfrak{p}'$ is non-empty if and only if $x \in \mathfrak{a}'$ up to transformations of the Weyl group W_P of $P = G/K$ (with $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$), more precisely, if (up to Weyl transformations) x is a dual root of weight one with respect to a simple root system of $\mathfrak{p}'$.*

An obvious necessary condition is that $\mathfrak{p}'$ contains dual roots of weight one at all. In particular we see:

Corollary 4. *If $\mathfrak{p}' \subset \mathfrak{p}$ is a Lie subtriple of the same rank as $\mathfrak{p}$, then $M \cap \mathfrak{p}'$ is nonempty and its connected components are extrinisic symmetric subspaces.*

EXAMPLES. 1. Let $\mathfrak{p} = \mathbb{R}^{p \times p}$ and $M_\pm \subset \mathfrak{p}$ be the connected components of O_p (with $M_+ = SO_p$). Further, let $\mathfrak{p}' = S_p \subset \mathfrak{p}$ be the space of symmetric $(p \times p)$-matrices. This is the example of the real Grassmannians (see end of section 2). Then $\mathfrak{p}'$ is of type AI [7, p. 532] with Dynkin diagram A_{p-1}. The maximal abelian subalgebra of $\mathfrak{p}$ is the space of diagonal matrices $\mathfrak{a}$. Since $\mathfrak{a} \subset \mathfrak{p}'$, the Lie triples $\mathfrak{p}'$ and $\mathfrak{p}$ have the same rank and hence $\mathfrak{p}'$ intersects $M_\pm$. The positive dimensional connected components of $M_\pm \cap \mathfrak{p}'$ are the real Grassmannians $G_k(\mathbb{R}^p)$, $k = 1, \dots, p-1$, which correspond to the $p-1$ dual roots with weight one in the table above.

2. Let $\mathfrak{p} = \mathfrak{u}_n$ with $n = 2m$ be the Lie algebra of U_n with maximal abelian subspace $\mathfrak{a} = \{i\,\mathrm{diag}(x_1, \dots, x_n) : x_i \in \mathbb{R}\}$. We identify $\mathfrak{a}$ with $\mathbb{R}^n$ and let $\{e_k : k = 1, \dots, n\}$ denote the standard orthonormal basis of $\mathbb{R}^n$. The root system is of type A_{n-1}. The fundamental roots are $\alpha_k = e_k - e_{k+1}$ with $k = 1, \dots, n-1$; all of them have weight one. The dual root vector for α_k is $\xi_k = \frac{1}{2}(\sum_{i=1}^k e_i - \sum_{j=k+1}^n e_j)$. The corresponding extrinsic symmetric space

[3]Remind that BC_n and B_n have the same simple root system.

$M_k = \mathrm{Ad}(U_n)\xi_k$ is isomorphic to the complex Grassmannian of k-planes in $\mathbb{C}^n$.

Now consider $\mathfrak{p}' = \mathfrak{so}_n \subset \mathfrak{p}$. Passing to a conjugate $\tilde{\mathfrak{p}}' = g\,\mathfrak{p}'\,g^{-1}$ for some suitable $g \in U_n$, the maximal abelian subspace of $\tilde{\mathfrak{p}}'$ becomes

$$\tilde{\mathfrak{a}}' = \{x \in \mathfrak{a} : x_{j+m} = -x_j \text{ for all } j = 1, \dots, m\}.$$

This contains $\xi_k \in \mathfrak{a}$ precisely for $k = m$, and ξ_m is a complex structure in $\mathfrak{so}_n$. Hence $M_k \cap \mathfrak{p}' = \emptyset$ for $k \neq m$, and $M_m \cap \mathfrak{p}'$ is the space SO_n/U_m of complex structures in $\mathfrak{so}_n$. This has two connected components which are conjugate in O_n and hence in U_n; these correspond to the two fundamental roots with weight 1 at the bifurcation of the Dynkin diagram D_m of SO_{2m}.

3. An extrinsic symmetric space $M \subset \mathfrak{p}$ is hermitian symmetric if and only if $\mathfrak{p}$ is a Lie algebra, $\mathfrak{p} = \mathfrak{g}$, and all other extrinsic symmetric spaces are the real forms of hermitian symmetric spaces, see [1, p. 310f] or [2]. The real forms are obtained as extrinsic symmetric subspaces from a hermitian extrinsic symmetric space $M = \mathrm{Ad}(G)x \subset \mathfrak{g}$ as follows. Let σ be an involution on $\mathfrak{g}$ with eigenspace decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and $x \in \mathfrak{p}$. Then $M' := M \cap \mathfrak{p}$ is a real form of M, and every real form arises that way [8].

E.g. let $G = U_n$. Then $M \subset \mathfrak{g}$ is the complex Grassmannian $G_p(\mathbb{C}^n)$ for $p \in \{1, \dots, n-1\}$. There are three types of real forms: real Grassmannians, quaternionic Grassmannians if both p, n are even, and the unitary group U_p if $n = 2p$. Let us consider the latter case, $M' = U_p$. The embedding of U_p into the Grassmannian $G_p(\mathbb{C}^{2p})$ is by assigning to each $A \in U_p$ its graph $E_A = \{(x, Ax) : x \in \mathbb{C}^p\} \subset \mathbb{C}^p \times \mathbb{C}^p$. The subgroup $U_p \times U_p \subset U_{2p}$ acts transitively on it since for all $(B, C) \in U_p \times U_p$,

$$\begin{aligned} \left(\begin{smallmatrix} B & \\ & C \end{smallmatrix}\right) E_A &= \{(Bx, CAx) : x \in \mathbb{C}^p\} \\ &= \{(\tilde{x}, CAB^{-1}\tilde{x}) : \tilde{x} \in \mathbb{C}^p\} \\ &= E_{CAB^{-1}}. \end{aligned}$$

The embedding of $M = G_p(\mathbb{C}^{2p})$ into $\mathfrak{g} = \mathfrak{u}_{2p}$ is obtained by assigning to a p-dimensional subspace $E \subset \mathbb{C}^{2p}$ the matrix r_E with eigenvalues i on E and $-i$ on $E^\perp$. This matrix is not only in $\mathfrak{u}_{2p}$ but also in U_{2p}. In particular, for the subspace $E_I = \{(x, x) : x \in \mathbb{C}^p\}$ we have $r_{E_I} = i\left(\begin{smallmatrix} & I \\ I & \end{smallmatrix}\right)$. The group U_{2p} acts by conjugation on these matrices, hence for $E' = \left(\begin{smallmatrix} B & \\ & C \end{smallmatrix}\right) E_I$ we have

$$r_{E'} = \left(\begin{smallmatrix} B & \\ & C \end{smallmatrix}\right) i \left(\begin{smallmatrix} & I \\ I & \end{smallmatrix}\right) \left(\begin{smallmatrix} B^* & \\ & C^* \end{smallmatrix}\right) = i \left(\begin{smallmatrix} & BC^* \\ CB^* & \end{smallmatrix}\right) = \left(\begin{smallmatrix} & A \\ -A^* & \end{smallmatrix}\right)$$

with $A = iBC^*$. Thus $M' \subset \mathfrak{p}' := \left\{\left(\begin{smallmatrix} & X \\ -X^* & \end{smallmatrix}\right) : X \in \mathbb{C}^{p \times p}\right\}$. Vice versa, if $\left(\begin{smallmatrix} & X \\ -X^* & \end{smallmatrix}\right) \in M \subset U_{2p}$, then $X \in U_p$, thus $M' = M \cap \mathfrak{p}'$. The subtriple $\mathfrak{p}'$ belongs to the Grassmannian $G_p(\mathbb{C}^{2p})$ and has Dynkin diagram C_p, see [7, pp. 517, 532], which has just one weight 1.

6. Open problems

In some sense, $\hat{M}' := M \cap \mathfrak{p}'$ should be considered as one single object with several connected components, like in the case of the Grassmannians. However, given $\hat{M}'$, we are not able to show that in general the smallest linear subspace $\mathfrak{p}'$ containing $\hat{M}'$ is a Lie triple. The question is easy when $\hat{M}'$ is the fixed set of a group of isometries: any isometry of $M \subset \mathfrak{p}$ extends as a linear isometry to the ambient space $\mathfrak{p}$, see [4], and $\mathfrak{p}'$ is the common fixed space of these extensions which is a Lie subtriple. In general, if M'_i are the connected components of $\hat{M}'$, then $\mathfrak{p}' = \sum \mathfrak{p}'_i$ where $\mathfrak{p}'_i$ is the linear span of M'_i, and all $\mathfrak{p}'_i$ are Lie triples acted on by the same group $K' = \{k \in K : k(\mathfrak{p}') = \mathfrak{p}'\}$ containing the symmetries s_x for all $x \in \hat{M}'$. But is $\mathfrak{p}'$ itself a Lie triple? So far, we have no information on the Lie brackets $[\mathfrak{p}'_i, \mathfrak{p}'_j]$ for $i \neq j$.

Maybe the worst case is when $\hat{M}'$ is discrete. This happens when $\mathfrak{p}'$ is abelian. In particular, when $\mathfrak{p}' = \mathfrak{a}$ is maximal abelian in $\mathfrak{p}$, then $\hat{M}' = M \cap \mathfrak{a}$ is a Weyl orbit: It is the intersection of the Ad(K)-orbit M on $\mathfrak{p}$ with the section $\mathfrak{a}$ of this polar representation. We conjecture that the converse is also true:

Conjecture 5. *Let $M \subset \mathfrak{p}$ be extrinsic symmetric and $\mathfrak{p}' \subset \mathfrak{p}$ a Lie subtriple intersecting M. Then $M' = M \cap \mathfrak{p}'$ is discrete (finite) if and only if $\mathfrak{p}'$ is abelian.*

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