A NEW APPROACH TO THE STABILITY ANALYSIS OF PARAMETER IDENTIFICATION PROBLEMS

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Abstract. The estimation of the friction coefficient from observation of the state is studied as output least squares problem. Stability with respect to the observation and the admissible parameter set is investigated using sufficient conditions for optimality.

Keywords. Stability in inverse problems, Tychonoff regularization, boundary value problem.

INTRODUCTION

This paper is concerned with parameter identification in a two point boundary value problem. We attempt to estimate infinite dimensional parameters in an output least squares (OLS-) formulation and we study in particular continuous dependence of the - not necessarily unique - optimal solutions of this nonlinear minimization problem on the observation and the constraints. Our approach is based on perturbation theory of infinite dimensional optimization problems and - partially - on a regularization of the problem, similar to Tychonoff-regularization.

The OLS-problem can be formulated as follows

$$\text{Minimize } \frac{1}{2}\,|\Phi(c) - z|^2 \quad \text{over} \quad c \in U_{ad} \qquad \text{(OLS)}$$

where z is interpreted as the actual observation, U_{ad} is the set of admissible parameters and $\Phi(c)$ is the observation generated by the parameter c in the (supposedly known) mathematical model (e.g. a differential equation).

In general, it will not be known if z is in the set

$$\mathcal{V} = \{u(c) : c \in U_{ad}\}$$

of attainable observations.

The results in this paper are related to G. Chavent's [5] notion of 'Output Least Squares Identifiability (OLSI)': The parameter c is called OLSI, if there exists a neighborhood $\hat{\mathcal{V}}$ of $\mathcal{V}$ such that for every $z \in \hat{\mathcal{V}}$ there exists a unique solution of (OLS) depending continuously on z. Observe that uniqueness of a solution of (OSL) requires uniqueness of the projection z_{pr} of z on $\mathcal{V}$ as well as uniqueness of the inverse of Φ.

Since the properties required in OLSI are difficult to verify, a recent contribution by C. Kravaris and J. H. Seinfeld [8] using Tychonoff regularization is interesting. Here (OLS) is changed to

$$\text{Minimize } \frac{1}{2}\,|\Phi(c) - z|^2 + \beta|c|^2_{cp}$$

where $0 < \beta << 1$, and $|\cdot|_{cp}$ is the norm of a space compactly embedded into the space of parameters. The assumption is made that z is uniquely attainable. Then solutions of the regularized problem depend continuously on z, while its solution can be related to the solution of (OLS).

The present paper drops the assumption that the solution of (OLS) is unique and that $z \in \mathcal{V}$ and concentrates on the question whether local solutions depend continuously on the observation z and the constraints defining U_{ad}. We call this property Output-Least Squares Stability (OLS-stability), and we also investigate the advantages of a regularization similar (but not equal) to the one above.

The applicability of our approach, which is based on a recent result by W. Alt [1] for perturbations of optimization problems will be demonstrated by analyzing a class of two point boundary value problems. Complete proofs of the results given here appear in [6].

PROBLEM FORMULATION AND PRELIMINARIES

We consider the two-point boundary value problem

$$-(a(x)u_x(x))_x + c(x)u(x) = f(x),\ x \in (0,1) \qquad (2.1)$$

with boundary conditions

$$R_i u = 0, \quad i = 1,2,\ldots \qquad (2.2)$$

where $f \in H^0 := H^0(0,1) = L^2(0,1)$, $a \in C^1(0,1)$, $a(x) \geq \underline{a} > 0$, and

$$R_i u = \alpha_{i1}u(o) + \alpha_{i2}u_x(o) + \alpha_{i3}u(1) + \alpha_{i4}u_x(1),$$
$$\alpha_{ij} \in \mathbb{R}.$$

The unknown parameter c is allowed to vary in the set

$$U_{ad} = \{c \in H^o: c(x) \geq \alpha \text{ a.e.}, |c| \leq \gamma\} \quad (2.3)$$

where $\alpha < \gamma$.

The Output-Least Squares (OLS-) problem consists in minimizing

$$\frac{1}{2}|u(c) - z|^2$$

where $u(c)$ solves (2.1), (2.2), $c \in U_{ad}$, and $z \in H^o$ is a given observation.

Here and throughout the inner product and norm in H^o are denoted by $(\cdot,\cdot)$ and $|\cdot|$.

Let $A(c)$ be the differential operator associated with (2.1), (2.2), i.e.

$$D(A): = \{\phi \in H^o: \phi \in H^2 \text{ and } R_i\phi = 0,\ i=1,2\},$$
$$A\phi \quad = -(a\phi_x)_x + c\phi.$$

Throughout we make the following assumptions on the coefficients appearing in (2.1), (2.2).

(H1) There exists a constant $k > o$ such that

$$(A(c)\phi,\phi) \geq k|\phi|_{H^1}$$

for all $\phi \in D(A)$ and all $c \in H^o$ with $c(x) \geq \alpha$ a.e..

(H2) The boundary conditions in (2.2) are such that $A(c)$ is selfadjoint.

The hypotheses (H1), (H2) ensure in particular, that for every ε-neighborhood U of U_{ad} there are constants $k_1 > 0$, $k_2 > 0$ such that for all $\phi \in D(A)$ and $c \in U$

$$k_1|\phi|_{H^{-i+2}} \leq |A(c)\phi|_{\tilde{H}^{-i}} \leq k_2|\phi|_{H^{-i+2}},\ i = 0,2,$$

where $\tilde{H}^{-i}$ is the completion of $C^\infty(0,1)$ with respect to the norm (cp. [11])

$$|u|_{-i} = \sup\{\frac{|(u,v)|}{|v|_{H^i}}: v \in C^\infty(0,1),$$
$$R_j v = o,\ j = 1,2\},\quad u \in C^\infty$$

(observe that $\tilde{H}^o = H^o$).

Furthermore, it follows that the mapping $c \to u(c)$: $U \subset H^o \to H^2$ is twice continuously Fréchet-differentiable with first derivative $\eta(h): = u_c(c)h$ and second derivative $\xi(h,k): = u_{cc}(c)(h,k)$ given as solutions of

$$A(c)\eta = -hu(c) \text{ and } A(c)\xi = -ku_c(c)(h)-hu_c(c)(k), \quad (2.4)$$

respectively.

OUTPUT LEAST SQUARES STABILITY

In this section, we analyze continuous dependence of the (not necessarily unique) solutions of the OLS-Problem on the observation z, the (upper) norm bound γ, and the (lower) pointwise bound α, which is considered as a constant function in $C = C(0,1)$. This continuity property is called Output-Least-Square (OLS-) Stability and it will be analyzed using perturbation theory of infinite dimensional optimization problems.

Consider the following parametrized family of OLS-problems:

$$\text{Minimize } f(c,z) = \frac{1}{2}|u(c) - z|^2 \qquad (OLS)^w$$
$$\text{for } c \in U_{ad}: = \{c \in H^o: c(x) \geq \alpha \text{ a.e.}, |c| \leq \gamma\}.$$

where the perturbation parameter $w = (z,\alpha,\gamma)$ is an element of $W: = H^o \times C \times \mathbb{R}$ (observe that U_{ad} depends on w).

These problems are considered as perturbations of the reference problem $(OLS)^{w^o} = (OLS)$ where $w^o = (z^o,\alpha^o,\gamma^o)$.

The following definition specifies the property that we are interested in. It is a streamlined version of the notion introduced in [6]. By $B(x,r)$ we denote an open ball with center x and radius r.

Definition: The unknown parameter c is called Output-Least-Square (OLS-) stable in U_{ad} at the local solution c^o of $(OLS)^{w^o}$ if there exist a neighborhood V of w^o and constants $d > 0$, $r > 0$ such that for all $w \in V$ there is a local solution $c_w^o \in B(c_{w^o}^o,r)$ of $(OLS)^w$ and every local solution $c_w^o \in B(c_{w^o}^o,r)$ satisfies

$$|c_w^o - c^o| \leq d||w - w^o||_W^{1/2}.$$

Remark. Before we analyze OLS-stability we note that the boundedness assumption in U_{ad} itself implies some weak continuous dependence of the solution c_w^o of $(OLS)^w$ on w. For let $w^n = (z^n, \alpha^n,\gamma^n) \to w^o = (z^o,\alpha^o,\gamma^o) \in W$. Then a subsequence of corresponding solutions $c_{w^n}^o$ converges weakly to some solution $\tilde{c}$ of $(OLS)^{w^o}$. However OLS-stability of a parameter requires more. Weak continuous dependence is replaced by strong continuous dependence and if c is OLS-stable at the local solution c^o then the perturbed problems must have local solutions in a neighborhood of c^o.

We can obtain OLS-stability either by admitting only a finite dimensional parameter space or by exploiting the consequence of the norm bound defining the set U_{ad}. Alternatively, one can also use a regularization approach, as will be shown in the next section.

First we consider the OLS-problem over the set $U^N: = U_{ad} \cap H^N$, where H^N is a finite dimensional subspace of L^∞. We prepare the following lemma, where $\xi = \xi(h,h)$ and $\eta = \eta(h)$.

Lemma 3.1: Let $c \in U_{ad}^N$. If $u(c) > o$ on $[0,1]$, then for all $h \in H^N$ and η,ξ defined by (2.4), the following inequality holds:

$$|\eta|^2+(u(c)-z,\xi) \geq |\eta|(k_2^{-1}\min_x u(c)(x)|h|_{\tilde{H}^{-2}} - 2k_1^{-1}|u(c)-z|\ |h|_{L^\infty}).$$

Proof: (See [6])

Theorem 3.1: Let κ be chosen such that $|h|_{L^\infty} \leq \kappa|h|_{\tilde{H}^{-2}}$ for all $h \in H^N$. If for a local solution c^o of (OLS) in U^N_{ad}

(i) $u(c^o) > 0$ or $u(c^o) < 0$ on $[0,1]$ and

(ii) $|u(c^o) - z| < \frac{1}{2} k_1 k_2^{-1}\kappa^{-1} \min_x |u(c^o)(x)|$

holds, then c is OLS-stable in U^N_{ad} at the local solution c^o of $(OLS)^{w^o}$.

Proof: Using a result by W. Alt [1] we have to check regularity of the constraint set and second order sufficient optimality conditions. Since regularity follows by straightforward arguments [6], we only deal with the latter property. Second order sufficient optimality conditions for infinite-dimensional problems with constraints require that the second derivative of the Lagrangian F is strictly positive [10]. We obtain for (OLS) using the chain rule and the preceeding lemma

$$\begin{aligned} F_{cc}(c^o)(h,h) &= |\eta|^2 + (u(c^o)-z,\xi) - \lambda_2|h|^2 \\ &\geq |h|_{L^\infty}|\eta|(k_2^{-1}\kappa^{-1}\min_x|u(c^o)(x)| \\ &\quad - 2k_1^{-1}|u(c^o)-z|) \\ &\geq \tilde{\delta}|h|^2 \end{aligned} \tag{3.1}$$

for some $\tilde{\delta} > 0$ which is independent of h (here $\lambda_2 \leq 0$ is the Lagrange multiplier corresponding to the norm constraint). Then the assertion follows.

Instead of restricting the admissible parameters c to a finite dimensional subspace as above, we can consider a different fit-to-data criterion in order to establish OLS-stability. In [6, Theorem 5.3] the output-least-square criterion with respect to H^1-norm is minimized. The practical relevance of this criterion is given by interpolation of point observations. Then conditions which ensure that the multiplier λ_2 (cp. 3.1) does not vanish imply continuous dependence on (z,α,γ) by similar arguments as above. Instead of discussing this result in detail we turn in the next section to the regularization approach.

OUTPUT LEAST SQUARES STABILITY BY REGULARIZATION

In the last section we imposed further assumptions on the problem data in order to establish OLS-stability. An alternative is to add a regularization term to the fit-to-data criterion (see e.g. [8]). Then this regularized problem is analyzed with respect to continuous dependence on the observation z and the constraint defining the set of admissible parameters. For $\beta > 0$ we define the identification problem by regularized output least squares minimization as:

$$\text{minimize } \tfrac{1}{2}|u(c) - z|^2 + \beta|c|^2 \quad \text{(ROLS)}$$

over $c \in \tilde{U}_{ad}$;

here $\tilde{U}_{ad} = \{c \in H^o\colon c(x) \geq \alpha \text{ a.e.}\}$. Of course, α is chosen so that (H1) holds. It is simple to see that for each $\beta > 0$ there exists a solution c^β of (ROLS) in $\tilde{U}_{ad}$. At times we shall write $(\text{ROLS})_\beta$ to specify the value of the regularization parameter in (ROLS).

Having omitted the norm constraint in $\tilde{U}_{ad}$ we instead make the following assumption:

(H3) There exists a solution c^o of

$$\text{minimize } \tfrac{1}{2}|u(c) - z|^2 \text{ over } c \in \tilde{U}_{ad}.$$

We assume that (H1) - (H3) hold throughout this section and we again refer to the optimization problem in (H3) as (OLS) and define $\mathscr{V}\colon = \{u(c)\colon c \in \tilde{U}_{ad}\}$. The set of perturbation parameters $w = (z,\alpha)$ is now given by

$$\tilde{W} = H^o \times C,$$

and, when appropriate, we denote dependence on w by $(\text{OLS})^w$. First we study the relations between (OLS) and (ROLS).

Proposition 4.1: For $\beta_n \to 0^+$ let c^{β_n} be a sequence of (global) solutions for $(\text{ROLS})_{\beta_n}$ in $\tilde{U}_{ad}$. Then there exists a convergent subsequence of c^{β_n} and every such subsequence converges to a minimum norm solution of (OLS).

This result shows that for small β, solutions of $(\text{ROLS})_\beta$ are close to solutions of (OLS). However, also a partial converse holds, indicating that in some sense every minimal norm solution of (OLS) can be recovered from $(\text{ROLS})_\beta$. More precisely, we have the following result:

Proposition 4.2: Decompose the set C of all minimum norm solutions of (OLS) into its connected components M in the weak H^o-topology. Then for every M and every sequence $\beta_n \to 0^+$ there exist local solutions c^{β_n} of $(\text{ROLS})_{\beta_n}$ in $\tilde{U}_{ad}$ converging in the H^o-norm to an element of M.

The next definition specifies the property we are interested in:

Definition: The unknown parameter c in (OLS) is called Ouput-Least-Square stable by Regularization (ROLS-stable) in $\tilde{U}_{ad}$ at $w^o \in \tilde{W}$ for $\beta \in J \subset (0,\infty)$, if for every global solution $c^\beta_{w^o}$ of $(\text{ROLS})_\beta$ with $\beta \in J$ there exist a neighborhood V of w^o and constants $d > 0$, $r > 0$, such that for all $w \in V$ there exists a local solution $c^\beta_w \in B(c^\beta_{w^o},r)$ of $(\text{ROLS})^w_\beta$ and every local solution $c^\beta_w \in B(c^\beta_{w^o},r)$ satisfies

$$|c^\beta_w - c^\beta_{w^o}| \leq d\,||w - w^o||^{1/2}_{\tilde{W}}.$$

Loosely speaking, ROLS-stability means OLS-stability of all global solutions of the regularized problems.

Results in the spirit of section 2, but under weaker assumptions, can be proved for ROLS-stability [6]. Here we turn to some results which are specific for the regularization approach. Note that $|c^\beta|$ converges to $|c^o|$ from below, as $\beta \to 0^+$, where c^o is a minimum norm solution of $(OLS)^{w^o}$ and c^β is any solution of $(ROLS)_\beta^{w^o}$.

Theorem 4.1: Choose $\bar{\beta}$ such that for a minimal norm solution c^o of $(OLS)^{w^o}$

$$k_1^2 - 2|c^o|^2 + 2 \sup|c^{\bar{\beta}}|^2 > 0$$

and define $\underline{\beta} \geq 0$ by

$$\underline{\beta} := \text{dist}\,(z^o \mathscr{V})^2 \,(k_1^2 - 2|c^o|^2 + 2 \sup|c^{\bar{\beta}}|)^{-1};$$

here the supremum is taken over all solutions $c^{\bar{\beta}}$ of $(ROLS)_{\bar{\beta}}^{w^o}$ in $\tilde{U}_{ad}$.

If $\underline{\beta} < \bar{\beta}$ then the parameter c is ROLS-stable at w^o in $\tilde{U}_{ad}$ for all $\beta \in (\underline{\beta}, \bar{\beta})$.

In particular, if $z^o \in \mathscr{V}$, then c is (ROLS)-stable in $\tilde{U}_{ad}$ at $w^o \in \hat{W}$ for $\beta \in (0, \bar{\beta})$.

Proof: (see [6])

The condition $\underline{\beta} < \bar{\beta}$ of Theorem 4.1 constitutes a certain relationship between the convergence rate of c^β to c^o, the bound k_1 and $\text{dist}(z^o, \mathscr{V})$: fast convergence rates of $|c^\beta|$, large bounds k_1 and small distances $\text{dist}(z^o \mathscr{V})$ are favorable.

If the condition $\underline{\beta} < \bar{\beta}$ is violated, a natural idea is to try to get a better observation z^o, i.e. to lower $\text{dist}(z^o, \tilde{\ })$. Next we show that this is a reasonable strategy which - at least theoretically - leads to success.

Theorem 4.2: Let $z_n^o \to z_o^o$ in H^o, with $z_o^o \in \mathscr{V}$, and assume that solutions c_n^o of $(OLS)_n^{w^o}$, with $w_n^o = (z_n^o, \alpha^o)$, exist with $\sup\{|c_n^o| : n = 0,1,2,\ldots\} < \infty$.

Then there exists $\tilde{\beta} > 0$ with the following property: For all $\beta^* \in (0, \tilde{\beta})$ there exists a natural number $N(\beta^*)$ and a neighborhood $J(\beta^*)$ of β^* such that for all $n \geq N(\beta^*)$ and all $\beta \in J(\beta^*)$ the parameter c is ROLS-stable in $\tilde{U}_{ad}$ at $w_n^o = (z_n^o, \alpha^o)$ for $\beta \in J(\beta^*)$.

Proof: (see [6])

Although we have only treated continuous dependence of the inverse problem on the coefficient c in this paper, we expect that our methods are suited to study output least squares stability of other parematers in (2.1), (2.2) as well as in more complex equations.

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