QUASI-ERGODIC LIMITS FOR FINITE ABSORBING MARKOV CHAINS

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Abstract. We present formulas for quasi-ergodic limits of finite absorbing Markov chains. Since the irreducible case has been solved in 1965 by Darroch and Seneta [6], we focus on the reducible case, and our results are based on a very precise asymptotic analysis of the (exponential and polynomial) growth behaviour along admissible paths.

1. Introduction

The long-term statistical behaviour of Markov chains is determined by their ergodic stationary measures, in the sense that the time average of an observable of the process converges to the space average of the observable with respect to the ergodic stationary measure. In the context of absorbing Markov chains, the function of a stationary measure is naturally replaced by a quasi-stationary measure, and a quasi-stationary measure describes a statistical equilibrium distribution conditioned on that the Markov chain is not absorbed. The field of quasi-stationary measures has been very active recently, see the monograph Collet, Martinez and San Martin [5], as well as Champagnat and Villemonais [3, 4], and the survey Méléard and Villemonais [12], which, in particular, covers applications to ecology and population dynamics.

It is well known that, when analysing the long-term statistical behaviour of absorbed Markov chains, quasi-stationary measures do not have the same function as stationary measures for non-absorbed Markov chains, despite their natural correspondence. In many settings, the time average of an observable of an absorbed Markov chain exists (when conditioned to non-absorption of the Markov chain), but this quantity is in general not equal to the space average of the observable with respect to the quasi-stationary measure. It turns out that, when taking the space average, the quasi-stationary measure needs to be replaced by another measure, often called quasi-ergodic measure. This has been first established for irreducible finite Markov chains by Darroch and Seneta [6]. For more general irreducible Markov processes, Breyer and Roberts [2] analysed this systematically, and they showed that the quasi-ergodic measure is absolutely continuous with respect to the quasi-stationary measure; they also coined the term *quasi-ergodic limits* [2, Theorem 1] for these (conditioned) ergodic limits (cf. also Zhang, Li, and Song [16] and He, Zhang, and Zhu [11]). Such quasi-ergodic limits have recently been used

Date: May 21, 2020.

2010 *Mathematics Subject Classification.* 15B51, 37A25, 60J10.

Key words and phrases. Absorbing Markov chain, Quasi-stationary measure, Quasi-ergodic measure, Substochastic matrix.

Martin Rasmussen was supported by the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 643073.

to define and analyse so-called conditioned Lyapunov exponents that describe the dynamical behaviour of random dynamical systems in compact subsets of the phase space, see Engel, Lamb and Rasmussen [7].

The literature on quasi-ergodic measures and limits has exclusively focussed on irreducible stochastic processes so far, and in this article, we aim at contributing to an understanding of the reducible case. It turns out that the analysis of quasi-ergodic limits is much more complicated for reducible processes, and for this reason, we focus here on the simplest possible case, given by finite state absorbing Markov chains.

We consider a stochastic matrix $P \in \mathbb{R}^{(d+1)\times(d+1)}$ of the form

$$P = \begin{pmatrix} 1 & 0 \\ R & Q \end{pmatrix}, \tag{1.1}$$

where 0 is a row vector of zeros, and $R \in \mathbb{R}^{d\times 1}, Q \in \mathbb{R}^{d\times d}$ with $R, Q \neq 0$ and $d \geq 2$.

We denote by $(X_i)_{i\in\mathbb{N}_0}$ the Markov chain associated to the substochastic matrix Q starting in a probability vector $\pi \in \mathbb{R}^d$. We suppose that all states $\{1,\dots,d\}$ are transient, i.e. the probability of return to some state when starting in that state is less that 1, which is equivalent to saying that the eigenvalues of the matrix Q lie inside the unit circle (and in particular 1 is not an eigenvalue of Q). Thus, this stochastic process is absorbed almost surely with absorption time T, meaning that the absorption state 0 is reached at time T.

We are interested in the *quasi-ergodic limit*

$$\lim_{n\to\infty} \mathbb{E}_\pi\Big[\frac{1}{n+1}\sum_{i=0}^{n} f(X_i)\Big|T > n\Big], \tag{1.2}$$

where $f : \{1,\dots,d\} \to \mathbb{R}$ is a given observable. As we will show in Corollary 2.2 below, the expectation in (1.2) is determined by the average time the Markov chain visits its states, and hence, we have to determine the *quasi-ergodic measure*

$$\lim_{n\to\infty} \mathbb{E}_\pi\big[\tfrac{1}{n+1}\#\big\{m \in \{0,\dots,n\} : X_m = j\big\}\big|T > n\big] \quad \text{for } j \in \{1,\dots,d\}\,. \tag{1.3}$$

The main results, Theorem 3.14 and Theorem 3.20, provide formulas for this limit.

This paper is organised as follows. Section 2 provides useful representations of the expectation in (1.3). These are based on results from Darroch and Seneta [6]; the proofs are postponed to the Appendix, and we note that the quasi-ergodic limits for the the irreducible case follow easily from these representations. We consider the theoretical analysis of the reducible case in Section 3. In Subsection 3.1, we assume without loss of generality that the reducible matrix Q is given in Frobenius normal form. This can be achieved by permutations of the rows and columns, and the Frobenius normal form is unique up to certain permutations, see Gantmacher [9, Chapter XIII, §4]). We use admissible paths to reformulate the formulas for the quasi-ergodic limit, and in Subsection 3.2, the main results are stated and proved. Here, we crucially have to assume that in the Frobenius normal form, the submatrices in the diagonal are scalar if their Perron–Frobenius eigenvalue is smaller than the maximal Perron–Frobenius eigenvalue. Finally, we illustrate the theoretical results by means of several examples in Section 4.

Notation. Probability vectors π are row vectors, while all other vectors in $\mathbb{R}^n$ are column vectors. In all spaces $\mathbb{R}^n$ we abbreviate $\mathbb{1} = (1,\dots,1)^\top$. The set of natural numbers is denoted by $\mathbb{N}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The number of elements of a finite set A is denoted by $\#A$. For $k \in \mathbb{N}$ and $m \in \mathbb{Z}$

$$\Gamma_k(m) := \big\{(\eta_1, \dots, \eta_k) \in \mathbb{N}_0^k : \eta_1 + \dots + \eta_k = m\big\}$$

and note that $\Gamma_k(m) = \emptyset$ for $m < 0$. Products with an empty index set are defined as $\prod_{i\in\emptyset} x_i = 1$.

2. Quasi-ergodic limits in the irreducible case

In this section, we consider the substochastic matrix Q from (1.1), and we present results from Darroch and Seneta [6] for quasi-ergodic limits of the form (1.2) in the special case when Q is irreducible.

Recall that if the matrix Q is irreducible, then it is either eventually positive or cyclic. It follows from the Perron–Frobenius theorem that Q has a simple eigenvalue $\rho \in (0, 1)$, called the Perron–Frobenius eigenvalue, such that the absolute values of all other eigenvalues of Q are equal to or less than ρ. The left eigenvector to this eigenvalue, $u \in \mathbb{R}^d$ with $u^\top Q = \rho u^\top$, has only positive entries and describes a quasi-stationary measure when normalised via $\sum_{i=1}^d u_i = 1$, which we assume in the following. If Q is eventually positive, then the absolute values of all other eigenvalues are smaller that ρ.

The following proposition is our starting point for deriving formulas for quasi-ergodic limits. The proof is given in the Appendix. Denote

$$\pi_j(z) := \pi D_j(z) \quad \text{and} \quad Q_j(z) := QD_j(z) \quad \text{for all } z \in \mathbb{R}\,, \tag{2.1}$$

where $D_j(z)$ is the $d \times d$ diagonal matrix whose j-th diagonal element is z and all other diagonal elements are equal to 1.

Proposition 2.1. Consider a substochastic matrix $Q \in \mathbb{R}^{d\times d}$, and let $(X_i)_{i\in\mathbb{N}_0}$ be the associated Markov chain starting in π. Then the following statements hold.

(i) For all $j \in \{1, \dots, d\}$ and $n \in \mathbb{N}$, we have

$$\mathbb{E}_\pi\big[\tfrac{1}{n+1}\#\big\{m \in \{0, \dots, n\} : X_m = j\big\}\big|T > n\big] = \frac{\frac{\mathrm{d}}{\mathrm{d}z}\pi_j(z)Q_j^n(z)\mathbb{1}\Big|_{z=1}}{(n+1)\pi Q^n\mathbb{1}}.$$

(ii) Suppose that Q is eventually positive. Then for $j \in \{1, \dots, d\}$, we have

$$\mathbb{E}_\pi\big[\tfrac{1}{n+1}\#\big\{m \in \{0, \dots, n\} : X_m = j\big\}\big|T > n\big] = u_j v_j + O(\tfrac{1}{n}) \quad \text{as } n \to \infty\,,$$

where v is the positive right eigenvector of Q for the Perron–Frobenius eigenvalue ρ normalised by $u^\top v = 1$.

(iii) Suppose that Q is cyclic with period $h \in \mathbb{N} \setminus \{1\}$. Then Q^h is eventually positive, and for all $j \in \{1, \dots, d\}$, we have for the left and right normalised eigenvectors $u^\top$ and v of Q^h for the eigenvalue ρ^h that

$$\mathbb{E}_\pi\big[\tfrac{1}{hn+1}\#\big\{m \in \{0, \dots, hn\} : X_m = j\big\}\big|T > hn\big\} = u_j v_j + O(\tfrac{1}{hn}) \quad \text{as } n \to \infty\,.$$

The following corollary uses the above result for the average evaluation of an observable. The formula provided in (i) below will be the basis of our further analysis of the reducible case. It shows that, in particular, the probability of the average evaluation of an observable f is determined by the average number of times that X_i is in some state j. For the irreducible case, assertion (ii) below concerns a formula for the quasi-ergodic limit involving the normalised right and left eigenvectors for the Perron–Frobenius eigenvalue ρ of Q.

Corollary 2.2. *Consider a substochastic matrix $Q \in \mathbb{R}^{d\times d}$, and let $(X_i)_{i\in\mathbb{N}_0}$ be the associated Markov chain starting in π. Then the following statements hold.*

(i) *For all $n \in \mathbb{N}$, we have*

$$\mathbb{E}_\pi\left[\tfrac{1}{n+1}\sum_{i=0}^{n} f(X_i)\Big|T>n\right] = \sum_{j=1}^{d} f(j)\mathbb{E}_\pi\left[\tfrac{1}{n+1}\#\{m\in\{0,\dots,n\}: X_m=j\}\big|T>n\right]$$
$$= \sum_{j=1}^{d} f(j)\frac{\frac{\mathrm{d}}{\mathrm{d}z}\pi_j(z)Q_j^n(z)\mathbb{1}\Big|_{z=1}}{(n+1)\pi Q^n\mathbb{1}}.$$

(ii) *If Q is irreducible, then the quasi-ergodic limit is given by*

$$\lim_{n\to\infty}\mathbb{E}_\pi\Big[\frac{1}{n+1}\sum_{i=0}^{n} f(X_i)\Big|T>n\Big] = \sum_{i=1}^{d} f(i)u_i v_i,$$

where v and $u^\top$ are the right and left eigenvectors for the eigenvalue ρ of Q normalised as in Proposition 2.1 (ii).

Proof. (i) Using Proposition 2.1 (i), one computes for fixed $n \in \mathbb{N}$

$$\sum_{j=1}^{d} f(j)\frac{\frac{\mathrm{d}}{\mathrm{d}z}\pi_j(z)Q_j^n(z)\mathbb{1}\Big|_{z=1}}{(n+1)\pi Q^n\mathbb{1}}$$
$$= \sum_{j=1}^{d} f(j)\mathbb{E}_\pi\left[\tfrac{1}{n+1}\#\{m\in\{0,\dots,n\}: X_m=j\}\big|T>n\right]$$
$$= \tfrac{1}{n+1}\sum_{j=1}^{d}\mathbb{E}_\pi\left[f(j)\#\{m\in\{0,\dots,n\}: X_m=j\}\big|T>n\right]$$
$$= \mathbb{E}_\pi\left[\frac{1}{n+1}\sum_{i=0}^{n} f(X_i)\Big|T>n\right].$$

(ii) Proposition 2.1 (ii) and (iii) yield the assertion in the irreducible case (where the matrix Q is either eventually positive or cyclic). □

Corollary 2.2 (ii) for irreducible Q and $f(j) = j$ for $j \in \{1,\dots,d\}$ is classical and has been proved in Darroch and Seneta [6, p. 95]. Here, the left eigenvector $u^\top$ is the unique quasi-stationary measure, see van Doorn and Pollett [15, Theorem 2.1]. Thus, the quasi-ergodic limit is absolutely continuous with respect to the quasi-stationary measure.

3. Quasi-ergodic limits in the reducible case

While in the previous section, we obtained an quasi-ergodic limit formula for irreducible matrices Q, we concentrate now on the reducible case, and we suppose without loss of generality that the matrices Q from (1.1) are given in Frobenius normal form

$$Q = \begin{pmatrix} Q_{11} & 0 & & 0 \\ Q_{21} & Q_{22} & & 0 \\ & & \ddots & \\ Q_{k1} & Q_{k2} & & Q_{kk} \end{pmatrix} \tag{3.1}$$

with matrices $Q_{ij} \in \mathbb{R}^{d_i \times d_j}$, where $d_1, \dots, d_k \in \mathbb{N}$. We assume in addition that the diagonal matrices Q_{ii} are eventually positive. The results for the general case, where the diagonal matrices are irreducible (hence maybe periodic), are easy consequences, see Remark 3.21 below.

We note that $\sum_{i=1}^k d_i = d$, and introduce index sets

$$I_j = \left\{1 + \sum_{i=1}^{j-1} d_i, \dots, \sum_{i=1}^{j} d_i\right\} \quad \text{for all } \ j \in \{1, \dots, k\}\,,$$

corresponding to the diagonal blocks of the matrix Q.

3.1. **Preparations and admissible paths.** In this subsection, we reformulate the quasi-ergodic problem using admissible paths of indices. We denote the initial distribution by $\pi = (\pi_1, \dots, \pi_k)$ with $\pi_i \in \mathbb{R}^{d_i}$ and first obtain a version of Proposition 2.1 for the above systems in Frobenius normal form.

Proposition 3.1. Consider a matrix Q of the form (3.1), and let $(X_i)_{i\in\mathbb{N}_0}$ be the Markov chain associated to the substochastic matrix Q starting in π. Then the following statements hold.

(i) For $j \in \{1, \dots, d\}$ and $n \in \mathbb{N}$, the probability of the average number of times that X_i is in some state j is

$$\mathbb{E}_\pi\left[\tfrac{1}{n+1}\#\big\{m \in \{0, \dots, n\} : X_m = j\big\}\big|T > n\right] = \frac{\pi\left(\sum_{r=0}^n Q^r e_j e_j^\top Q^{n-r}\right)\mathbb{1}}{(n+1)\pi Q^n \mathbb{1}}\,.$$

(ii) For $\ell \in \{1, \dots, k\}$ and $n \in \mathbb{N}$, the probability of the average number of times that X_i is in some state in I_ℓ is

$$\mathbb{E}_\pi\left[\tfrac{1}{n+1}\#\big\{m \in \{0, \dots, n\} : X_m \in I_\ell\big\}\big|T > n\right] = \frac{\pi\left(\sum_{r=0}^n Q^r \sum_{j\in I_\ell} e_j e_j^\top Q^{n-r}\right)\mathbb{1}}{(n+1)\pi Q^n \mathbb{1}}\,.$$

Proof. (i) Using (2.1), we compute

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}z}\pi_j(z)Q_j^n(z)\mathbb{1} = \frac{\mathrm{d}}{\mathrm{d}z}\left(\pi D_j(z)\left(QD_j(z)\right)^n \mathbb{1}\right)\\ =&\pi e_j e_j^\top \left(QD_j(z)\right)^n \mathbb{1} + \pi D_j(z)\left(\sum_{r=1}^n \left(QD_j(z)\right)^{r-1} Q e_j e_j^\top \left(QD_j(z)\right)^{n-r}\right)\mathbb{1}\,.\end{aligned}$$

This implies that

$$\frac{\mathrm{d}}{\mathrm{d}z}\pi_j(z)Q_j^n(z)\mathbb{1}\Big|_{z=1} = \pi e_j e_j^\top Q^n \mathbb{1} + \pi\left(\sum_{r=1}^n Q^r e_j e_j^\top Q^{n-r}\right)\mathbb{1}\,.$$

Now the assertion follows from Proposition 2.1 (i).

(ii) This follows from (i) and

$$\begin{aligned}&\mathbb{E}_\pi\left[\tfrac{1}{n+1}\#\big\{m \in \{0, \dots, n\} : X_m \in I_\ell\big\}\big|T > n\right]\\ &= \sum_{j\in I_\ell}\mathbb{E}_\pi\left[\tfrac{1}{n+1}\#\big\{m \in \{0, \dots, n\} : X_m = j\big\}\big|T > n\right],\end{aligned}$$

which finishes the proof of this proposition. □

We now aim at understanding the terms in Proposition 3.1 better and first note that for all $n \in \mathbb{N}_0$, we get

$$Q^n = \begin{pmatrix} Q_{11}^{(n)} & 0 & & 0 \\ Q_{21}^{(n)} & Q_{22}^{(n)} & & 0 \\ & & \ddots & \\ Q_{k1}^{(n)} & Q_{k2}^{(n)} & & Q_{kk}^{(n)} \end{pmatrix},$$

where for $n \geq 1$

$$Q_{ij}^{(n)} := \sum_{\substack{s_1,\dots,s_{n-1}=1,\dots,k \\ i=s_0\geq s_1\geq s_2\geq\cdots\geq s_{n-1}\geq s_n=j}} Q_{s_0 s_1} Q_{s_1 s_2}\cdots Q_{s_{n-1}s_n}\,, \tag{3.2}$$

and for $n = 0$

$$Q_{ij}^{(0)} := \begin{cases} \mathrm{Id} & : \quad i = j\,, \\ 0 & : \quad i \neq j\,. \end{cases}$$

This follows by induction, since for $i \geq \ell$, the entries $Q_{i\ell}^{(n+1)}$ of $Q^{n+1} = Q^n Q$ are given by

$$\begin{aligned} Q_{i\ell}^{(n+1)} &= \sum_{j=\ell}^{k} \sum_{\substack{s_1,\dots,s_{n-1}=1,\dots,k \\ i\geq s_1\geq s_2\geq\cdots\geq s_{n-1}\geq j}} Q_{is_1} Q_{s_1 s_2}\cdots Q_{s_{n-1}j} Q_{j\ell} \\ &= \sum_{\substack{s_1,\dots,s_n=1,\dots,k \\ i\geq s_1\geq s_2\geq\cdots\geq s_n\geq \ell}} Q_{is_1} Q_{s_1 s_2}\cdots Q_{s_{n-1}s_n} Q_{s_n\ell}\,. \end{aligned}$$

We first consider the numerator in the formula from Proposition 3.1 (ii), which can be re-written as

$$\begin{aligned} &\pi\left(\sum_{r=0}^{n} Q^r \sum_{j\in I_\ell} e_j e_j^\top Q^{n-r}\right)\mathbb{1} \\ &= \pi\sum_{r=0}^{n} \begin{pmatrix} Q_{11}^{(r)} & 0 & & 0 \\ Q_{21}^{(r)} & Q_{22}^{(r)} & & 0 \\ & & \ddots & \\ Q_{k1}^{(r)} & Q_{k2}^{(r)} & & Q_{kk}^{(r)} \end{pmatrix} \begin{pmatrix} 0 & & & 0 \\ & \ddots & & \\ Q_{\ell 1}^{(n-r)} & & Q_{\ell\ell}^{(n-r)} & 0 \\ 0 & & & 0 \end{pmatrix}\mathbb{1} \\ &= \pi\sum_{r=0}^{n} \begin{pmatrix} 0 & & & & 0 \\ & \ddots & & & \\ Q_{\ell\ell}^{(r)} Q_{\ell 1}^{(n-r)} & & Q_{\ell\ell}^{(r)} Q_{\ell\ell}^{(n-r)} & 0 & 0 \\ & & & \ddots & \\ Q_{k\ell}^{(r)} Q_{\ell 1}^{(n-r)} & & Q_{k\ell}^{(r)} Q_{\ell\ell}^{(n-r)} & & 0 \end{pmatrix}\mathbb{1} \\ &= (\pi_\ell,\dots,\pi_k)\sum_{r=0}^{n} \begin{pmatrix} Q_{\ell\ell}^{(r)} Q_{\ell 1}^{(n-r)} & & Q_{\ell\ell}^{(r)} Q_{\ell\ell}^{(n-r)} \\ & \ddots & \\ Q_{k\ell}^{(r)} Q_{\ell 1}^{(n-r)} & & Q_{k\ell}^{(r)} Q_{\ell\ell}^{(n-r)} \end{pmatrix}\mathbb{1} \end{aligned}$$

$$= \sum_{r=0}^{n} \sum_{i=\ell}^{k} \pi_i Q_{i\ell}^{(r)} \sum_{j=1}^{\ell} Q_{\ell j}^{(n-r)} \mathbb{1} = \sum_{i=\ell}^{k} \sum_{j=1}^{\ell} \pi_i \sum_{r=0}^{n} Q_{i\ell}^{(r)} Q_{\ell j}^{(n-r)} \mathbb{1}. \tag{3.3}$$

We now aim at re-writing this product of certain sub-matrices of the matrix Q in a different way involving so-called admissible paths of indices.

Definition 3.2 (Admissible paths)**.**

(i) An *admissible path* θ *of length* $\kappa = \kappa(\theta)$ is given by a finite and strictly decreasing sequence $\theta = (\theta_1, \theta_2, \dots, \theta_\kappa)$ such that $\theta_u \in \{1, \dots, k\}$ and $Q_{\theta_u \theta_{u+1}} \neq 0$ for all $u \in \{1, \dots, \kappa - 1\}$.

(ii) The *set of admissible paths* is denoted by $\mathcal{P}$, and we denote the set of admissible paths that go from i to j by

$$\mathcal{P}_{ij} := \left\{ (\theta_1, \theta_2, \dots, \theta_\kappa) \in \mathcal{P} : \theta_1 = i \text{ and } \theta_\kappa = j \right\},$$

and define the set of admissible paths through $\ell \in \{1, \dots, k\}$ as

$$\mathcal{P}^{(\ell)} := \left\{ (\theta_1, \dots, \theta_\kappa) \in \mathcal{P} : \text{ there exists a } u \in \{1, \dots, \kappa\} \text{ with } \theta_u = \ell \right\}.$$

We note that every finite sequence of natural numbers s_i occurring in non-zero products in sums of the form $Q_{i\ell}^{(r)}$ and $Q_{\ell j}^{(n-r)}$, as defined in (3.2), must follow an admissible path. More precisely, concentrating on $Q_{i\ell}^{(n)}$, for some $(s_0, \dots, s_n)$ such that $0 \neq Q_{s_0 s_1} Q_{s_1 s_2} \cdots Q_{s_{n-1} s_n}$, there exist a $\theta = (i, \theta_2, \dots, \theta_{\kappa-1}, \ell) \in \mathcal{P}_{i\ell}$ and exponents $\eta_1, \dots, \eta_\kappa \in \mathbb{N}_0$ such that $\sum_{u=1}^{\kappa} \eta_u = n + 1 - \kappa$ and

$$Q_{s_0 s_1} Q_{s_1 s_2} \cdots Q_{s_{n-1} s_n} = Q_{\theta_1 \theta_1}^{\eta_1} Q_{\theta_1 \theta_2} Q_{\theta_2 \theta_2}^{\eta_2} Q_{\theta_2 \theta_3} \cdots Q_{\theta_{\kappa-1} \theta_\kappa} Q_{\theta_\kappa \theta_\kappa}^{\eta_\kappa}. \tag{3.4}$$

Every matrix which is subdiagonal in Q occurs at most once, and for this reason, most entries in this large matrix product are diagonal blocks $Q_{\theta_u \theta_u}$ that are ordered with respect to u and thus appear as powers of these matrices.

The number of elements in

$$\Gamma_\kappa(m) = \left\{ (\eta_1, \dots, \eta_\kappa) \in \mathbb{N}_0^\kappa : \eta_1 + \dots + \eta_\kappa = m \right\} \quad \text{for all } \kappa \in \{1, \dots, k\} \text{ and } m \in \mathbb{N}$$

is given by

$$\#\Gamma_\kappa(m) = \binom{\kappa + m - 1}{m} \tag{3.5}$$

(this is modelled by drawing $\kappa - 1$ out of $m + 1$ balls from an urn with replacement and without order). For $\theta = (\theta_1, \theta_2, \dots, \theta_\kappa) \in \mathcal{P}$ and $m \in \mathbb{N}$, let

$$Q(\theta, m) := \sum_{\eta \in \Gamma_\kappa(m+1-\kappa)} Q_{\theta_1 \theta_1}^{\eta_1} Q_{\theta_1 \theta_2} Q_{\theta_2 \theta_2}^{\eta_2} Q_{\theta_2 \theta_3} \cdots Q_{\theta_{\kappa-1} \theta_\kappa} Q_{\theta_\kappa \theta_\kappa}^{\eta_\kappa} \tag{3.6}$$

and

$$Q(\theta, 0) := \begin{cases} \mathrm{Id} & : \quad \theta \in \mathcal{P}_{ij}, \text{ where } i = j, \\ 0 & : \quad \theta \in \mathcal{P}_{ij}, \text{ where } i \neq j. \end{cases}$$

We use the following restrictions of $\theta = (\theta_1, \dots, \theta_\kappa) \in \mathcal{P}^{(\ell)}$,

$$\underline{\theta^\ell} := (\theta_1, \dots, \theta_u = \ell) \in \mathcal{P}_{\theta_1, \ell} \quad \text{and} \quad \overline{\theta^\ell} := (\theta_u = \ell, \dots, \theta_\kappa) \in \mathcal{P}_{\ell, \theta_\kappa},$$

and we write $\underline{\kappa} := \underline{\kappa}(\theta, \ell) := u$ and $\overline{\kappa} := \overline{\kappa}(\theta, \ell) := \kappa - u + 1$ for the length of $\underline{\theta^\ell}$ and $\overline{\theta^\ell}$, respectively. Hence, $\underline{\kappa} + \overline{\kappa} = \kappa + 1$.

We obtain the following corollary to Proposition 3.1.

Corollary 3.3. *Consider a matrix Q of the form (3.1), let $(X_i)_{i\in\mathbb{N}_0}$ be the Markov chain associated to the substochastic matrix Q starting in π, and let $\ell \in \{1,\dots,k\}$. Then the following two statements hold.*

(i) *We have*

$$\mathbb{E}_\pi\left[\tfrac{1}{n+1}\#\big\{m\in\{0,\dots,n\} : X_m \in I_\ell\big\}\Big| T>n\right] = \frac{\sum_{\theta\in\mathcal{P}^{(\ell)}}\pi_{\theta_1}\sum_{r=0}^n Q(\underline{\theta^\ell},r)Q(\overline{\theta^\ell},n-r)\mathbb{1}}{(n+1)\sum_{\theta\in\mathcal{P}}\pi_{\theta_1}Q(\theta,n)\mathbb{1}}\,.$$

(ii) *For $s\in I_\ell$ we have with $t(s) := s-\sum_{i=0}^{\ell-1} d_i$ and $e_{t(s)}\in\mathbb{R}^{d_\ell}$,*

$$\mathbb{E}_\pi\big[\tfrac{1}{n+1}\#\big\{m\in\{0,\dots,n\} : X_m = s\big\}\big| T>n\big] = \frac{\sum_{\theta\in\mathcal{P}^{(\ell)}}\pi_{\theta_1}\sum_{r=0}^n Q(\underline{\theta^\ell},r)e_{t(s)}e_{t(s)}^\top Q(\overline{\theta^\ell},n-r)\mathbb{1}}{(n+1)\sum_{\theta\in\mathcal{P}}\pi_{\theta_1}Q(\theta,n)\mathbb{1}}.$$

Proof. (i) First consider the denominator in Proposition 3.1 (ii). With (3.2), we get

$$\begin{aligned}(n+1)\pi Q^n\mathbb{1} &= (n+1)\sum_{i=1}^k \pi_i\sum_{j=1}^i Q_{ij}^{(n)}\mathbb{1}\\ &= (n+1)\sum_{i=1}^k\sum_{j=1}^i \pi_i \sum_{i=s_0\ge s_1\ge\cdots\ge s_{n-1}\ge s_n=j} Q_{is_1}Q_{s_1s_2}\cdots Q_{s_{n-1}j}\mathbb{1}\\ &= \sum_{i=1}^k\sum_{j=1}^i (n+1)\pi_i\sum_{\theta\in\mathcal{P}_{ij}} Q(\theta,n)\mathbb{1}\\ &= (n+1)\sum_{\theta\in\mathcal{P}}\pi_{\theta_1}Q(\theta,n)\mathbb{1}\,.\end{aligned}$$

Turning to the numerator we can write

$$\sum_{r=0}^n Q_{i\ell}^{(r)}Q_{\ell j}^{(n-r)} = \sum_{r=0}^n\left(\sum_{\theta\in\mathcal{P}_{i\ell}}Q(\theta,r)\right)\left(\sum_{\theta\in\mathcal{P}_{\ell j}}Q(\theta,n-r)\right). \tag{3.7}$$

Every admissible path $\theta\in\mathcal{P}^{(\ell)}\cap\mathcal{P}_{ij}$ corresponds to two admissible paths $\underline{\theta^\ell}\in\mathcal{P}_{i\ell}$ and $\overline{\theta^\ell}\in\mathcal{P}_{\ell j}$. Hence the numerator re-written in the form (3.3) is given by

$$\begin{aligned}&\sum_{i=\ell}^k\sum_{j=1}^\ell \pi_i\sum_{r=0}^n\left(\sum_{\theta\in\mathcal{P}_{i\ell}}Q(\theta,r)\right)\left(\sum_{\theta\in\mathcal{P}_{\ell j}}Q(\theta,n-r)\right)\mathbb{1}\\ &= \sum_{\theta\in\mathcal{P}^{(\ell)}}\pi_{\theta_1}\sum_{r=0}^n Q(\underline{\theta^\ell},r)Q(\overline{\theta^\ell},n-r)\mathbb{1}\,.\end{aligned}$$

(ii) Using Proposition 3.1 (i) and an appropriate modification of formula (3.3), one proves this analogously. □

3.2. **Formulas for quasi-ergodic limits.** In this subsection, we determine formulas for quasi-ergodic limits for matrices Q of the form (3.1).

Recall that we assume that the diagonal matrices Q_{ii} are eventually positive and that the maximal eigenvalue of Q_{ii} (the Perron–Frobenius eigenvalue) is denoted by ρ_i. For $\theta = (\theta_1, \dots, \theta_\kappa) \in \{1, \dots, k\}^\kappa$, we define $\rho(\theta) := \max\{\rho_{\theta_1}, \dots, \rho_{\theta_\kappa}\}$,

$$H^+(\theta) := \big\{u \in \{1, \dots, \kappa\} : \rho_{\theta_u} = \rho(\theta)\big\} \quad \text{and}$$
$$H^-(\theta) := \big\{u \in \{1, \dots, \kappa\} : \rho_{\theta_u} < \rho(\theta)\big\},$$

and we denote the number of elements in these sets by $h^+(\theta) := \#H^+(\theta)$ and $h^-(\theta) := \#H^-(\theta)$. Note that $h^+(\theta) + h^-(\theta) = \kappa = \kappa(\theta)$. In addition, we define $\rho_{\max} := \max\{\rho_1, \dots, \rho_k\}$ and $h^+_{\max} := \max\{h^+(\theta) : \theta \in \mathcal{P} \text{ and } \rho(\theta) = \rho_{\max}\}$.

Remark 3.4. In the terminology of Friedland and Schneider [8, p. 190], if $\rho_i = \rho_{\max}$, then Q_{ii} determines a singular vertex of the graph associated with Q, and the singular distance from i to j is given by $h^+_{\max} - 1$.

We aim at quasi-ergodic limits by taking the limit $n \to \infty$ in Corollary 3.3. In the following, we will derive a few results that help to ignore parts negligible when taking this limit. For this purpose, we say that a real sequence $(a_n)_{n\in\mathbb{N}}$ is *asymptotically equivalent* to another real sequence $(b_n)_{n\in\mathbb{N}}$ in the limit $n \to \infty$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$.

Proposition 3.5. Let $\theta = (\theta_1, \dots, \theta_\kappa) \in \{1, \dots, k\}^\kappa$. Consider the sequence

$$\xi_n := \sum_{\eta \in \Gamma_\kappa(n+\kappa-1)} \rho_{\theta_1}^{\eta_1} \cdots \rho_{\theta_\kappa}^{\eta_\kappa} \quad \text{for all } n \in \mathbb{N}.$$

Then the sequence

$$\rho(\theta)^{n+1-\kappa} \frac{n^{h^+(\theta)-1}}{(h^+(\theta)-1)!} \prod_{u \in H^-(\theta)} \frac{1}{1 - \frac{\rho_{\theta_u}}{\rho(\theta)}}$$

is asymptotically equivalent to $(\xi_n)_{n\in\mathbb{N}}$ for $n \to \infty$.

Proof. For $\ell, m \in \mathbb{N}$ and $\zeta_1, \dots, \zeta_\ell > 0$, we introduce the auxiliary function

$$\Xi_\ell^m(\zeta_1, \dots, \zeta_\ell) := \sum_{\eta \in \Gamma_\ell(m)} \zeta_1^{\eta_1} \cdots \zeta_\ell^{\eta_\ell} = \sum_{\eta_1=0}^{m} \sum_{\eta_2=0}^{m-\eta_1} \cdots \sum_{\eta_\ell=0}^{m-\eta_1-\dots-\eta_{\ell-1}} \zeta_1^{\eta_1} \cdots \zeta_\ell^{\eta_\ell},$$

and for $\zeta_\ell \neq 1$, we can write

$$\begin{aligned} \Xi_\ell^m(\zeta_1, \dots, \zeta_\ell) &= \sum_{\eta_1=0}^{m} \sum_{\eta_2=0}^{m-\eta_1} \cdots \sum_{\eta_{\ell-1}=0}^{m-\eta_1-\dots-\eta_{\ell-2}} \zeta_1^{\eta_1} \cdots \zeta_{\ell-1}^{\eta_{\ell-1}} \frac{1 - \zeta_\ell^{m+1-\eta_1-\dots-\eta_{\ell-1}}}{1-\zeta_\ell} \\ &= \frac{1}{1-\zeta_\ell} \Xi_{\ell-1}^m(\zeta_1, \dots, \zeta_{\ell-1}) - \frac{\zeta_\ell^{m+1}}{1-\zeta_\ell} \Xi_{\ell-1}^m\left(\frac{\zeta_1}{\zeta_\ell}, \dots, \frac{\zeta_{\ell-1}}{\zeta_\ell}\right). \end{aligned} \tag{3.8}$$

We note that the function Ξ_ℓ^m is symmetric in the sense that $\Xi_\ell^m(\zeta_1, \dots, \zeta_\ell) = \Xi_\ell^m(\zeta_{s(1)}, \dots, \zeta_{s(\ell)})$ for all permutations s of $1, \dots, \ell$. For this reason, the above reformulation of Ξ_ℓ^m into two terms of the form $\Xi_{\ell-1}^m$ can be made as long as not all ζ_i, $i \in \{1, \dots, \ell\}$, are equal to 1.

We assume without loss of generality that $\theta_\kappa = \rho(\theta)$. Then

$$\begin{aligned}
\xi_n &= \sum_{\eta\in\Gamma_\kappa(n+1-\kappa)} \rho_{\theta_1}^{\eta_1}\cdots\rho_{\theta_\kappa}^{\eta_\kappa} \\
&= \sum_{\eta_1=0}^{n+1-\kappa}\sum_{\eta_2=0}^{n+1-\kappa-\eta_1}\cdots\sum_{\eta_{\kappa-1}=0}^{n+1-\kappa-\eta_1-\cdots-\eta_{\kappa-2}} \rho_{\theta_1}^{\eta_1}\cdots\rho_{\theta_{\kappa-1}}^{\eta_{\kappa-1}}\rho(\theta)^{n+1-\kappa-\eta_1-\cdots-\eta_{\kappa-1}} \\
&= \rho(\theta)^{n+1-\kappa}\sum_{\eta_1=0}^{n+1-\kappa}\sum_{\eta_2=0}^{n+1-\kappa-\eta_1}\cdots\sum_{\eta_{\kappa-1}=0}^{n+1-\kappa-\eta_1-\cdots-\eta_{\kappa-2}} \left(\frac{\rho_{\theta_1}}{\rho(\theta)}\right)^{\eta_1}\cdots\left(\frac{\rho_{\theta_{\kappa-1}}}{\rho(\theta)}\right)^{\eta_{\kappa-1}}.
\end{aligned}$$

Thus we have

$$\rho(\theta)\xi_n = \rho(\theta)^{n+2-\kappa}\Xi_{\kappa-1}^{n+1-\kappa}\left(\frac{\rho_{\theta_1}}{\rho(\theta)},\dots,\frac{\rho_{\theta_{\kappa-1}}}{\rho(\theta)}\right) \quad \text{for all } n\in\mathbb{N}.$$

If $H^-(\theta)=\emptyset$, then

$$\rho(\theta)\xi_n = \rho(\theta)^{n+2-\kappa}\Xi_{\kappa-1}^{n+1-\kappa}(1,\dots,1).$$

Otherwise, we may assume that $\rho_{\theta_{\kappa-1}} < \rho(\theta)$, and formula (3.8) yields

$$\begin{aligned}
\rho(\theta)\xi_n = \rho(\theta)^{n+2-\kappa}&\frac{1}{1-\frac{\rho_{\theta_{\kappa-1}}}{\rho(\theta)}}\Xi_{\kappa-2}^{n+1-\kappa}\left(\frac{\rho_{\theta_1}}{\rho(\theta)},\dots,\frac{\rho_{\theta_{\kappa-2}}}{\rho(\theta)}\right) \\
&-\rho_{\theta_{\kappa-1}}^{n+2-\kappa}\frac{1}{1-\frac{\rho_{\theta_{\kappa-1}}}{\rho(\theta)}}\Xi_{\kappa-2}^{n+1-\kappa}\left(\frac{\rho_{\theta_1}}{\rho_{\theta_{\kappa-1}}},\dots,\frac{\rho_{\theta_{\kappa-2}}}{\rho_{\theta_{\kappa-1}}}\right).
\end{aligned}$$

Using the properties of the function Ξ_ℓ^m, we can iteratively re-write $\rho(\theta)\xi_n$ into at most $2^{\kappa-1}$ terms of the form

$$\rho_{\theta_{\gamma(i)}}^{n+2-\kappa}K(i)\Xi_{\beta(i)}^{n+1-\kappa}(1,\dots,1), \quad \text{where } i\in\{1,\dots,2^{\kappa-1}\}, \tag{3.9}$$

with $\gamma(i)\in\{1,\dots,\kappa\}$, $K(i)\in\mathbb{R}$, and $\beta(i)\in\{0,\dots,\kappa-1\}$. In all variations of this (non-unique) iterative procedure, one has a unique term of the form

$$\rho(\theta)^{n+2-\kappa}\Xi_{h^+(\theta)-1}^{n+1-\kappa}(1,\dots,1)\prod_{u\in H^-(\theta)}\frac{1}{1-\frac{\rho_{\theta_u}}{\rho(\theta)}}, \tag{3.10}$$

and we show that this sequence is asymptotically equivalent to $(\rho(\theta)\xi_n)_{n\in\mathbb{N}}$.

Firstly, we note that $\frac{m^\kappa}{\kappa!}$ is asymptotically equivalent to $\Xi_\kappa^m(1,\dots,1)$ for $m\to\infty$. This follows from the fact that one can show that $\Xi_\kappa^m(1,\dots,1) = \#\Gamma_{\kappa+1}(m)$, and we use (3.5).

In addition, on the way to get to terms of the form (3.9), the intermediate terms are of the form

$$\rho_{\theta_\gamma}^{n+2-\kappa}K\Xi_\beta^{n+1-\kappa}\left(\frac{\rho_{\theta_{s(1)}}}{\rho_{\theta_\gamma}},\dots,\frac{\rho_{\theta_{s(\beta)}}}{\rho_{\theta_\gamma}}\right), \tag{3.11}$$

where $\gamma\in\{1,\dots,\kappa\}$, $K\in\mathbb{R}$, $\beta\in\{0,\dots,\kappa-1\}$ and $s:\{1,\dots,\beta\}\to\{1,\dots,\kappa-1\}$ is injective. This implies that if $\rho_{\theta_{\gamma(i)}} = \rho(\theta)$ in a final sequence (3.9), then, in the formulation of (3.11), we have $\rho_{s(j)} = \rho(\theta)$ for all $j\in\{1,\dots,\beta\}$. It can be seen that in all such terms that do not coincide with (3.10), we have $\beta < h^+(\theta)-1$. Hence, the term (3.10) yields an asymptotically equivalent sequence, which finishes the proof. □

In order to analyse both the denominator and numerator from Corollary 3.3, we need the following notation and elementary statements for the diagonal blocks.

Lemma 3.6 (Notation and statements for the diagonal blocks of Q)**.** *For $i \in \{1,\dots,k\}$, the normed right eigenvector of the Perron–Frobenius eigenvalue ρ_i of Q_{ii} is denoted by v_i. Since for any $i \in \{1,\dots,k\}$, the matrix Q_{ii} is eventually positive, the absolute value of all other eigenvalues is less than some constant $\rho_i^- \in (0,\rho_i)$, and we denote by V_i^- the sum of the corresponding generalised eigenspaces, so that we have the decomposition $\mathbb{R}^{d_i} = \operatorname{span}(v_i) \oplus V_i^-$. In the trivial case $d_i = 1$, we have $v_i = 1$ and $V_i^- = \{0\}$.*

(i) *We define*
$$K_1 := \max\left\{1, \max\left\{\|Q_{ij}\| : i > j\right\}\right\}.$$
(ii) *Choose γ with*
$$\max\left\{\tfrac{\rho_i^-}{\rho_i} : i \in \{1,\dots,k\}\right\} < \gamma < 1\,.$$
Then there exists a constant $K_2 \geq 1$ such that for every $i \in \{1,\dots,k\}$ and $x \in V_i^-$, we have
$$\|Q_{ii}^n x\| \leq K_2 \rho_i^n \gamma^n \|x\| \quad \text{for all } n \in \mathbb{N}\,.$$
(iii) *There exists a constant $K_3 \geq 1$ such that for all $i \in \{1,\dots,k\}$ and sequences $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^d$ with*
$$x_n = z_n + w_n \quad \text{with } z_n \in \operatorname{span}(v_i) \text{ and } w_n \in V_i^-\,,$$
the following holds: if for some $\zeta \in (0,1)$ and $K \geq 1$, one has $\|x_n\| \leq K\zeta^n$ for all $n \in \mathbb{N}$, then
$$\|z_n\| \leq KK_3\zeta^n \quad \text{and} \quad \|w_n\| \leq KK_3\zeta^n \quad \text{for all } n \in \mathbb{N}\,.$$
(iv) *Consider an admissible path $\theta \in \mathcal{P}$. We define $\alpha_\theta := \alpha_{\theta_\kappa}\cdots\alpha_{\theta_1}$, where the real numbers α_{θ_u} for all $u \in \{1,\dots,\kappa\}$ are defined by*
$$\mathbb{1} = \alpha_{\theta_\kappa} v_{\theta_\kappa} + w_{\theta_\kappa}$$
with $w_{\theta_\kappa} \in V_{\theta_\kappa}^-$, and inductively for $u \in \{\kappa-1,\kappa-2,\dots,1\}$ by
$$Q_{\theta_u\theta_{u+1}} v_{\theta_{u+1}} = \alpha_{\theta_u} v_{\theta_u} + w_{\theta_u}$$
with $w_{\theta_u} \in V_{\theta_u}^-$. If all submatrices $Q_{\theta_u\theta_{u+1}}$ are scalar, one has, writing $q_{\theta_u\theta_{u+1}} := Q_{\theta_u\theta_{u+1}}$, that the constants α_{θ_u} are given by $\alpha_{\theta_\kappa} = 1$ and $\alpha_{\theta_u} = q_{\theta_u\theta_{u+1}}$ for $u \in \{\kappa-1,\kappa-2,\dots,1\}$.

(v) *There exists a $K_4 \geq 1$ such that for all $i \in \{1,\dots,k\}$, we have*
$$\|Q_{ii}^n\| \leq K_4 \rho_i^n \quad \text{for all } n \in \mathbb{N}\,.$$

Proof. (i) and (iv) concern notation and do not need to be proved. For the proof of (ii) note that for every matrix Q_{ii}, Seneta [14, Theorem 1.2] implies that
$$Q_{ii}^n = \rho_i^n v_i u_i^\top + \mathcal{O}\left((\rho_i^-)^n\right)\,,$$
where $u_i^\top$ is the positive left eigenvector of Q_{ii} for ρ_i with $u_i^\top v_i = 1$. Then it follows for $x \in V_i^-$ that $v_i u_i^\top x = 0$, since otherwise $Q_{ii}^n x$ would grow with ρ_i^n. This implies assertion (ii). Assertion (v) is clear, since the eigenspace to the maximal real eigenvalue ρ_i of Q_{ii} is one-dimensional (we assumed that the matrix Q_{ii} is eventually positive). For assertion (iii), the observation below used for the spaces $X = \operatorname{span}(v_i) \oplus V_i^-$ yields a constant $K_i' \geq 1$ for every $i \in \{1,\dots,k\}$, the maximum

of which we denote by $K_3 \geq 1$.
Observation. Consider in a finite-dimensional space $X = Z \oplus W$ a sequence $x_n = z_n + w_n$ with $z_n \in Z, w_n \in W$, and $\|x_n\| \leq K\zeta^n$ for some $\zeta \in (0,1)$ and $K \geq 1$. Then there exists a constant $K' \geq 1$ such that $\|z_n\| \leq K'K\zeta^n$ and $\|w_n\| \leq K'K\zeta^n$ for all $n \in \mathbb{N}$.
Proof of the observation. In fact, for a norm such that $\|x\|' = \|z\|' + \|w\|'$ for $x \in X$ with $z \in Z, w \in W$, one has $\|z_n\|' \leq \|z_n\|' + \|w_n\|' = \|x_n\|' \leq K\gamma^n$, analogously for w_n. This result remains true for every norm $\|\cdot\|$, since all norms on finite-dimensional spaces are equivalent. In fact, $c^{-1}\|x\|' \leq \|x\| \leq c\|x\|'$ for some constant $c > 0$, hence $\|z_n\| \leq c\|z_n\|' \leq c\|x_n\|' \leq c^2\|x_n\| \leq K'K\zeta^n$ with $K' := c^2$. □

In the following proposition, we aim at understanding the asymptotic growth of sequences of the form $\pi_{\theta_1} Q(\theta, n)\mathbb{1}$ which occur in the denominator in Corollary 3.3.

Proposition 3.7. Consider a matrix Q of the form (3.1) and an admissible path $\theta = (\theta_1, \dots, \theta_\kappa) \in \mathcal{P}$ and suppose that for all $u \in \{1, \dots, \kappa\}$ with $\rho_{\theta_u} < \rho(\theta)$, the diagonal term $Q_{\theta_u\theta_u}$ is scalar.

(i) If $\pi_{\theta_1} v_{\theta_1} \neq 0$ and $\alpha_\theta \neq 0$, then the sequence $\pi_{\theta_1} Q(\theta, n)\mathbb{1}$ is asymptotically equivalent in the limit $n \to \infty$ to

$$\pi_{\theta_1} v_{\theta_1} \alpha_\theta \sum_{\eta \in \Gamma_\kappa(n+1-\kappa)} \rho_{\theta_\kappa}^{\eta_\kappa} \cdots \rho_{\theta_1}^{\eta_1}\,, \tag{3.12}$$

and hence, due to Proposition 3.5, also to the sequence

$$\alpha_\theta \pi_{\theta_1} v_{\theta_1} \rho(\theta)^{n+1-\kappa} \frac{n^{h^+(\theta)-1}}{(h^+(\theta)-1)!} \prod_{u \in H^-(\theta)} \frac{1}{1 - \frac{\rho_{\theta_u}}{\rho(\theta)}}\,. \tag{3.13}$$

(ii) If $\pi_{\theta_1} v_{\theta_1} = 0$ or $\alpha_\theta = 0$, then the exponential growth of $\pi_{\theta_1} Q(\theta, n)\mathbb{1}$ in the limit $n \to \infty$ is equal to or less than $\rho(\theta)^{n+1-\kappa} n^{h^+(\theta)-2}$.

Proof. Due to (3.6), we have

$$\pi_{\theta_1} Q(\theta, n)\mathbb{1} = \pi_{\theta_1} \sum_{\eta \in \Gamma_\kappa(n+1-\kappa)} Q_{\theta_1\theta_1}^{\eta_1} Q_{\theta_1\theta_2} Q_{\theta_2\theta_2}^{\eta_2} Q_{\theta_2\theta_3} \cdots Q_{\theta_{\kappa-1}\theta_\kappa} Q_{\theta_\kappa\theta_\kappa}^{\eta_\kappa} \mathbb{1}\,. \tag{3.14}$$

Since we assume that for $\rho_i < \rho(\theta)$, the diagonal term Q_{ii} is scalar, it follows that for $u \in H^-(\theta)$, the decomposition $Q_{\theta_u\theta_{u+1}} v_{\theta_{u+1}} = \alpha_{\theta_u} v_{\theta_u} + w_{\theta_u}$ from Lemma 3.6 (iv) is scalar, and hence using $v_{\theta_u} = 1$ and $w_{\theta_u} = 0$, it is of the form

$$Q_{\theta_u\theta_{u+1}} v_{\theta_{u+1}} = \alpha_{\theta_u}\,. \tag{3.15}$$

Consider the first iterative step from Lemma 3.6 (iv)

$$\mathbb{1} = \alpha_{\theta_\kappa} v_{\theta_\kappa} + w_{\theta_\kappa} \tag{3.16}$$

with $w_{\theta_\kappa} \in V_{\theta_\kappa}^-$. Using that ρ_{θ_κ} is an eigenvalue of $Q_{\theta_\kappa\theta_\kappa}$ with eigenvector v_{θ_κ}, we get

$$Q_{\theta_\kappa\theta_\kappa}^{\eta_\kappa} \mathbb{1} = \alpha_{\theta_\kappa} \rho_{\theta_\kappa}^{\eta_\kappa} v_{\theta_\kappa} + Q_{\theta_\kappa\theta_\kappa}^{\eta_\kappa} w_{\theta_\kappa}. \tag{3.17}$$

In the next step, we decompose

$$Q_{\theta_{\kappa-1}\theta_\kappa} v_{\theta_\kappa} = \alpha_{\theta_{\kappa-1}} v_{\theta_{\kappa-1}} + w_{\theta_{\kappa-1}} \tag{3.18}$$

with $\alpha_{\theta_{\kappa-1}} \in \mathbb{R}$ and $w_{\theta_{\kappa-1}} \in V^-_{\theta_{\kappa-1}}$. Hence, Lemma 3.6 (ii) implies

$$\left\| Q^{\eta_{\kappa-1}}_{\theta_{\kappa-1}\theta_{\kappa-1}} w_{\theta_{\kappa-1}} \right\| \leq K_2 \rho^{\eta_{\kappa-1}}_{\theta_{\kappa-1}} \gamma^{\eta_{\kappa-1}} \|w_{\theta_{\kappa-1}}\| , \tag{3.19}$$

and we decompose

$$Q_{\theta_{\kappa-1}\theta_\kappa} Q^{\eta_\kappa}_{\theta_\kappa\theta_\kappa} w_{\theta_\kappa} = \beta^{(\eta_\kappa)}_{\theta_{\kappa-1}} v_{\theta_{\kappa-1}} + w^{(\eta_\kappa)}_{\theta_{\kappa-1}} \tag{3.20}$$

with $\beta^{(\eta_\kappa)}_{\theta_{\kappa-1}} \in \mathbb{R}$ and $w^{(\eta_\kappa)}_{\theta_{\kappa-1}} \in V^-_{\theta_{\kappa-1}}$. Due to Lemma 3.6 (i),(ii), the left hand side of (3.20) satisfies

$$\left\| Q_{\theta_{\kappa-1}\theta_\kappa} Q^{\eta_\kappa}_{\theta_\kappa\theta_\kappa} w_{\theta_\kappa} \right\| \leq K_1 \left\| Q^{\eta_\kappa}_{\theta_\kappa\theta_\kappa} w_{\theta_\kappa} \right\| \leq K_1 K_2 \rho^{\eta_\kappa}_{\theta_\kappa} \gamma^{\eta_\kappa} \left\| w_{\theta_\kappa} \right\| .$$

This implies for the right hand side of (3.20) by Lemma 3.6 (iii) that

$$\left\| w^{(\eta_\kappa)}_{\theta_{\kappa-1}} \right\| \leq K_1 K_2 K_3 \rho^{\eta_\kappa}_{\theta_\kappa} \gamma^{\eta_\kappa} \left\| w_{\theta_\kappa} \right\| \tag{3.21}$$

and

$$\left| \beta^{(\eta_\kappa)}_{\theta_{\kappa-1}} \right| = \left\| \beta^{(\eta_\kappa)}_{\theta_{\kappa-1}} v_{\theta_{\kappa-1}} \right\| \leq K_1 K_2 K_3 \rho^{\eta_\kappa}_{\theta_\kappa} \gamma^{\eta_\kappa} \left\| w_{\theta_\kappa} \right\| . \tag{3.22}$$

Together this yields

$$\begin{aligned}
& Q^{\eta_{\kappa-1}}_{\theta_{\kappa-1}\theta_{\kappa-1}} Q_{\theta_{\kappa-1}\theta_\kappa} Q^{\eta_\kappa}_{\theta_\kappa\theta_\kappa} \mathbb{1} \\
\overset{(3.17)}{=} \quad & Q^{\eta_{\kappa-1}}_{\theta_{\kappa-1}\theta_{\kappa-1}} Q_{\theta_{\kappa-1}\theta_\kappa} \left(\alpha_{\theta_\kappa} \rho^{\eta_\kappa}_{\theta_\kappa} v_{\theta_\kappa} + Q^{\eta_\kappa}_{\theta_\kappa\theta_\kappa} w_{\theta_\kappa} \right) \\
= \quad & \alpha_{\theta_\kappa} \rho^{\eta_\kappa}_{\theta_\kappa} Q^{\eta_{\kappa-1}}_{\theta_{\kappa-1}\theta_{\kappa-1}} Q_{\theta_{\kappa-1}\theta_\kappa} v_{\theta_\kappa} + Q^{\eta_{\kappa-1}}_{\theta_{\kappa-1}\theta_{\kappa-1}} Q_{\theta_{\kappa-1}\theta_\kappa} Q^{\eta_\kappa}_{\theta_\kappa\theta_\kappa} w_{\theta_\kappa} \\
\overset{(3.18),(3.20)}{=} \quad & \alpha_{\theta_\kappa} \rho^{\eta_\kappa}_{\theta_\kappa} Q^{\eta_{\kappa-1}}_{\theta_{\kappa-1}\theta_{\kappa-1}} \left(\alpha_{\theta_{\kappa-1}} v_{\theta_{\kappa-1}} + w_{\theta_{\kappa-1}} \right) \\
& + Q^{\eta_{\kappa-1}}_{\theta_{\kappa-1}\theta_{\kappa-1}} \left(\beta^{(\eta_\kappa)}_{\theta_{\kappa-1}} v_{\theta_{\kappa-1}} + w^{(\eta_\kappa)}_{\theta_{\kappa-1}} \right) \\
= \quad & \alpha_{\theta_\kappa} \alpha_{\theta_{\kappa-1}} \rho^{\eta_\kappa}_{\theta_\kappa} Q^{\eta_{\kappa-1}}_{\theta_{\kappa-1}\theta_{\kappa-1}} v_{\theta_{\kappa-1}} + \alpha_{\theta_\kappa} \rho^{\eta_\kappa}_{\theta_\kappa} Q^{\eta_{\kappa-1}}_{\theta_{\kappa-1}\theta_{\kappa-1}} w_{\theta_{\kappa-1}} \\
& + Q^{\eta_{\kappa-1}}_{\theta_{\kappa-1}\theta_{\kappa-1}} \beta^{(\eta_\kappa)}_{\theta_{\kappa-1}} v_{\theta_{\kappa-1}} + Q^{\eta_{\kappa-1}}_{\theta_{\kappa-1}\theta_{\kappa-1}} w^{(\eta_\kappa)}_{\theta_{\kappa-1}} \\
= \quad & \alpha_{\theta_\kappa} \alpha_{\theta_{\kappa-1}} \rho^{\eta_\kappa}_{\theta_\kappa} \rho^{\eta_{\kappa-1}}_{\theta_{\kappa-1}} v_{\theta_{\kappa-1}} + \alpha_{\theta_\kappa} \rho^{\eta_\kappa}_{\theta_\kappa} Q^{\eta_{\kappa-1}}_{\theta_{\kappa-1}\theta_{\kappa-1}} w_{\theta_{\kappa-1}} \\
& + \beta^{(\eta_\kappa)}_{\theta_{\kappa-1}} \rho^{\eta_{\kappa-1}}_{\theta_{\kappa-1}} v_{\theta_{\kappa-1}} + Q^{\eta_{\kappa-1}}_{\theta_{\kappa-1}\theta_{\kappa-1}} w^{(\eta_\kappa)}_{\theta_{\kappa-1}} .
\end{aligned}$$

The last three summands satisfy the estimates

$$\begin{aligned}
\left\| \alpha_{\theta_\kappa} \rho^{\eta_\kappa}_{\theta_\kappa} Q^{\eta_{\kappa-1}}_{\theta_{\kappa-1}\theta_{\kappa-1}} w_{\theta_{\kappa-1}} \right\| \quad & \leq \quad \left| \alpha_{\theta_\kappa} \right| \rho^{\eta_\kappa}_{\theta_\kappa} \left\| Q^{\eta_{\kappa-1}}_{\theta_{\kappa-1}\theta_{\kappa-1}} w_{\theta_{\kappa-1}} \right\| \\
& \overset{(3.19)}{\leq} \quad K_2 \left| \alpha_{\theta_\kappa} \right| \rho^{\eta_\kappa}_{\theta_\kappa} \rho^{\eta_{\kappa-1}}_{\theta_{\kappa-1}} \gamma^{\eta_{\kappa-1}} \|w_{\theta_{\kappa-1}}\|
\end{aligned}$$

and

$$\left\| \beta^{(\eta_\kappa)}_{\theta_{\kappa-1}} \rho^{\eta_{\kappa-1}}_{\theta_{\kappa-1}} v_{\theta_{\kappa-1}} \right\| \leq \left| \beta^{(\eta_\kappa)}_{\theta_{\kappa-1}} \right| \rho^{\eta_{\kappa-1}}_{\theta_{\kappa-1}} \|v_{\theta_{\kappa-1}}\| \overset{(3.22)}{\leq} K_1 K_2 K_3 \rho^{\eta_\kappa}_{\theta_\kappa} \rho^{\eta_{\kappa-1}}_{\theta_{\kappa-1}} \gamma^{\eta_\kappa} \left\| w_{\theta_\kappa} \right\|$$

and

$$\begin{aligned}
\left\| Q^{\eta_{\kappa-1}}_{\theta_{\kappa-1}\theta_{\kappa-1}} w^{(\eta_\kappa)}_{\theta_{\kappa-1}} \right\| \quad & \leq \quad \left\| Q^{\eta_{\kappa-1}}_{\theta_{\kappa-1}\theta_{\kappa-1}} \right\| \left\| w^{(\eta_\kappa)}_{\theta_{\kappa-1}} \right\| \overset{\text{Lemma 3.6 (v)}}{\leq} K_4 \rho^{\eta_{\kappa-1}}_{\theta_{\kappa-1}} \left\| w^{(\eta_\kappa)}_{\theta_{\kappa-1}} \right\| \\
& \overset{(3.21)}{\leq} \quad K_1 K_2 K_3 K_4 \rho^{\eta_\kappa}_{\theta_\kappa} \rho^{\eta_{\kappa-1}}_{\theta_{\kappa-1}} \gamma^{\eta_\kappa} \left\| w_{\theta_\kappa} \right\| .
\end{aligned}$$

If $\kappa \notin H^+(\theta)$, then by (3.15), it follows that $w_{\theta_\kappa} = 0$, and hence, of the last three summands above, only

$$\alpha_{\theta_\kappa} \rho^{\eta_\kappa}_{\theta_\kappa} Q^{\eta_{\kappa-1}}_{\theta_{\kappa-1}\theta_{\kappa-1}} w_{\theta_{\kappa-1}}$$

can be different from 0. If $\kappa-1 \notin H^+(\theta)$, then by (3.15), it follows that $w_{\theta_{\kappa-1}} = 0$, and hence, this summand vanishes. Together with the estimates derived above, it follows that each of the additional three summands vanishes, if both $\kappa, \kappa-1 \notin H^+(\theta)$, and the norm of each of the additional summands can be estimated by a constant multiplied with

$$\begin{aligned} \rho_{\theta_\kappa}^{\eta_\kappa}\rho_{\theta_{\kappa-1}}^{\eta_{\kappa-1}}\gamma^{\eta_\kappa} \qquad & \text{if } \kappa \in H^+(\theta) \text{ and } \kappa-1 \notin H^+(\theta)\,, \\ \rho_{\theta_\kappa}^{\eta_\kappa}\rho_{\theta_{\kappa-1}}^{\eta_{\kappa-1}}\gamma^{\eta_{\kappa-1}} \qquad & \text{if } \kappa \notin H^+(\theta) \text{ and } \kappa-1 \in H^+(\theta)\,, \\ \rho_{\theta_\kappa}^{\eta_\kappa}\rho_{\theta_{\kappa-1}}^{\eta_{\kappa-1}}\gamma^{\eta_\kappa} \text{ or } \rho_{\theta_\kappa}^{\eta_\kappa}\rho_{\theta_{\kappa-1}}^{\eta_{\kappa-1}}\gamma^{\eta_{\kappa-1}} \qquad & \text{if } \kappa \in H^+(\theta) \text{ and } \kappa-1 \in H^+(\theta)\,. \end{aligned}$$

After κ decomposition steps, we arrive at the following result: any term in the sum in (3.14) (i.e. for a fixed $\eta \in \Gamma_\kappa(n+1-\kappa)$) is equal to the sum of

$$\pi_{\theta_1}\alpha_\theta\rho_{\theta_\kappa}^{\eta_\kappa}\cdots\rho_{\theta_1}^{\eta_1}v_{\theta_1} = \alpha_\theta\pi_{\theta_1}v_{\theta_1}\rho_{\theta_\kappa}^{\eta_\kappa}\cdots\rho_{\theta_1}^{\eta_1} \tag{3.23}$$

and up to $2^{h^+(\theta)}-1$ summands that contain in addition to the factor $\rho_{\theta_\kappa}^{\eta_\kappa}\cdots\rho_{\theta_1}^{\eta_1}$ some factor γ^{η_u} with $u \in H^+(\theta)$, hence, $\gamma\rho_{\theta_u} < \rho(\theta)$. It follows from Proposition 3.5 that these additional terms have in the limit $n \to \infty$ exponential growth equal to or less than $\rho(\theta)^{n+1-\kappa}n^{h^+(\theta)-2}$. Hence for $\pi_{\theta_1}\alpha_\theta v_{\theta_1} \neq 0$, the sequences in (3.23) and (3.14) are asymptotically equivalent, and assertion (i) holds. If $\pi_{\theta_1}\alpha_\theta v_{\theta_1} = 0$ the terms in (3.23) vanish and the estimate for the exponential growth of the other summands implies that (ii) holds. □

Remark 3.8. If for an admissible path $\theta \in \mathcal{P}$, all submatrices $Q_{\theta_u\theta_{u+1}}$ are scalar, the assumption $\alpha_\theta = \alpha_{\theta_\kappa}\cdots\alpha_{\theta_1} \neq 0$ in Proposition 3.7 holds. In the general case, it is generically satisfied: for matrices Q of the form (3.1), recall that v_i denotes the Perron–Frobenius eigenvector of the diagonal block Q_{ii} for $i \in \{1,\dots,k\}$. Then the set of matrices Q such that the decomposition $\mathbb{1} = \alpha_k v_k + w_k$ with $\alpha_k \in \mathbb{R}$ and $w_k \in V_k^-$ satisfies $\alpha_k \neq 0$, is open and dense. Similarly, for all $i, j \in \{1,\dots,k\}$ with $i > j$, the set of matrices such that the decomposition

$$Q_{ij}v_j = \alpha_{ij}v_i + w_{ij} \text{ with } \alpha_{ij} \in \mathbb{R} \text{ and } w_{ij} \in V_i^-,$$

satisfies $\alpha_{ij} \neq 0$, is open and dense. This implies (cf. Lemma 3.6 (iv)) that the set of matrices Q such that for every admissible path $\theta \in \mathcal{P}$, all numbers $\alpha_{\theta_\kappa},\dots,\alpha_{\theta_1}$ are nonzero, is open and dense.

The following two examples further illustrate the assumptions of Proposition 3.7.

Example 3.9. We demonstrate now that the assumption $\alpha_\theta = \alpha_{\theta_\kappa}\cdots\alpha_{\theta_1} \neq 0$ in Proposition 3.7 (i) is not satisfied in general. Consider the matrix

$$Q = \begin{pmatrix} Q_{11} & 0 \\ Q_{21} & Q_{22} \end{pmatrix}, \quad \text{where } Q_{11}, Q_{21}, Q_{22} \in \mathbb{R}^{n\times n}.$$

Suppose that Q_{11} and Q_{22} are eventually positive with Perron–Frobenius eigenvalues ρ_1 and ρ_2 with normalised positive right eigenvectors v_1 and v_2, respectively. Furthermore, let $V_2^- \subset \mathbb{R}^n$ be the subspace spanned by the eigenvectors of Q_{22} corresponding to the eigenvalues with smaller magnitude than ρ_2, as defined in Lemma 3.6. Suppose that Q_{21} is a matrix and $Q_{21}v_1 \in V_2^-$. For $\theta = (\theta_1, \theta_2) = (2, 1)$ and $\kappa = 2$, one has $v_{\theta_\kappa} = v_1$ and $v_{\theta_{\kappa-1}} = v_2$. Then it follows that $\alpha_{\theta_{\kappa-1}} = \alpha_2 = 0$, since due to (3.18), we have

$$Q_{21}v_1 = 0\cdot v_2 + w_2\,, \quad \text{with } w_2 \in V_2^-\,.$$

We now take a closer look at the assumption requiring certain diagonal terms to be scalar.

Example 3.10. In Proposition 3.7, the assumption that the diagonal term $Q_{\theta_u\theta_u}$ is scalar for all $u \in \{1, \dots, \kappa\}$ with $\rho_{\theta_u} < \rho(\theta)$, is necessary and cannot be omitted in general. Let $d = 3$ and consider a matrix of the form

$$Q = \begin{pmatrix} \rho_1 & 0 \\ Q_{21} & Q_{22} \end{pmatrix},$$

where $Q_{22} \in \mathbb{R}^{2\times 2}$ is eventually positive with the simple eigenvalues $\rho_2 > \rho_2^- > 0$ and $\rho_1 > \rho_2$. Here $\rho_1 = \rho(\theta)$ and $Q_{11} = (\rho_1)$, and we assume that $Q_{21} = (q_{21}, q_{31})^\top$ has positive entries, and hence, the path $\theta = (\theta_1, \theta_2) = (2, 1)$ with $\kappa = 2$ is admissible. Then $H^+(\theta) = \{1\}$, $H^-(\theta) = \{2\}$ and

$$\theta_\kappa = \theta_2 = 1 \quad \text{and} \quad \theta_{\kappa-1} = \theta_1 = 2\,.$$

The eigenvalue of Q_{11} is $\rho_1 = \rho(\theta)$ with normed eigenvector $1 \in \mathbb{R}$. The decomposition (3.16) reads as $\mathbb{1} = \alpha_{\theta_\kappa} v_{\theta_\kappa} + w_{\theta_\kappa}$ in $\mathbb{R}$ with $\theta_\kappa = \theta_2 = 1$, and we get $\alpha_{\theta_\kappa} = \alpha_1 = \rho_1$, $v_{\theta_\kappa} = v_1 = 1$ and $w_{\theta_\kappa} = w_1 = 0$. The matrix Q_{22} has normalised eigenvectors v_2 for ρ_2 and v_2^- for ρ_2^-. Thus the subspace V_2^- is spanned by the eigenvector v_2^-, and hence, the decomposition (3.18) in $\mathbb{R}^2$ has the form

$$Q_{21}1 = \begin{pmatrix} q_{21} \\ q_{31} \end{pmatrix} = \alpha_2 v_2 + w_2 = \alpha_2 v_2 + c v_2^- \quad \text{with } c v_2^- \in V_2^- \text{ for some } c \in \mathbb{R}.$$

We further assume that $w_2 \neq 0$, and hence $c \neq 0$. The decomposition (3.20) is trivial, since $w_{\theta_\kappa} = w_1 = 0$, showing that $\beta^{(\eta_\kappa)}_{\theta_{\kappa-1}} = 0$ and $w^{(\eta_\kappa)}_{\theta_{\kappa-1}} = 0$. Together, this yields the formula

$$Q_{22}^{\eta_2} Q_{21} Q_{11}^{\eta_1} 1 = \rho_1 \alpha_2 \rho_1^{\eta_2} \rho_2^{\eta_1} v_2 + \rho_1 \rho_1^{\eta_2} Q_{22}^{\eta_1} w_2 = \rho_1 \alpha_2 \rho_1^{\eta_2} \rho_2^{\eta_1} v_2 + \rho_1 \rho_1^{\eta_2} c \left(\rho_2^-\right)^{\eta_1} v_2^-\,.$$

Now we sum the right hand side over all $\eta = (\eta_1, \eta_2) \in \mathbb{N}_0^2$ with $\eta_1 + \eta_2 = n + 1 - \kappa = n - 1$. This gives for the first summand of the right hand side

$$\begin{aligned}
\sum_{\eta \in \Gamma_2(n-1)} \rho(\theta) \alpha_2 \rho^{\eta_2} \rho_2^{\eta_1} v_2 &= \rho(\theta) \alpha_2 \sum_{\eta_1=0}^{n-1} \rho(\theta)^{n-1-\eta_1} \rho_2^{\eta_1} v_2 \\
&= \rho(\theta) \alpha_2 \rho(\theta)^{n-1} \sum_{\eta_1=0}^{n-1} \left(\frac{\rho_2}{\rho(\theta)}\right)^{\eta_1} v_2 = \alpha_2 \rho(\theta)^n \frac{1 - \left(\frac{\rho_2}{\rho(\theta)}\right)^n}{1 - \frac{\rho_2}{\rho(\theta)}} v_2\,.
\end{aligned}$$

Similarly, the second summand yields

$$\sum_{\eta \in \Gamma_2(n-1)} \rho(\theta)^{\eta_2+1} c \left(\rho_2^-\right)^{\eta_1} v_2^- = c\rho(\theta)^n \sum_{\eta_1=0}^{n-1} \left(\frac{\rho_2^-}{\rho(\theta)}\right)^{\eta_1} v_2^- = c\rho(\theta)^n \frac{1 - \left(\frac{\rho_2^-}{\rho(\theta)}\right)^n}{1 - \frac{\rho_2^-}{\rho(\theta)}} v_2^-\,.$$

Thus, we get with $\pi_1 = \pi_{\theta_2}$ and $(\pi_{2,1}, \pi_{2,2}) = \pi_2 = \pi_{\theta_1}$ that

$$
\begin{aligned}
(n+1)\pi_{\theta_1} Q(\theta, n)\mathbb{1} &= (n+1)(\pi_{2,1}, \pi_{2,2}) \sum_{\eta \in \Gamma_2(n-1)} Q_{22}^{\eta_2} Q_{21} Q_{11}^{\eta_1} 1 \\
&= (n+1)(\pi_{2,1}, \pi_{2,2}) v_2 \alpha_2 \rho(\theta)^n \frac{1 - \left(\frac{\rho_2}{\rho(\theta)}\right)^n}{1 - \frac{\rho_2^-}{\rho(\theta)}} \\
&\quad + (n+1)(\pi_{2,1}, \pi_{2,2}) v_2^- c \rho(\theta)^n \frac{1 - \left(\frac{\rho_2^-}{\rho(\theta)}\right)^n}{1 - \frac{\rho_2^-}{\rho(\theta)}} .
\end{aligned}
$$

Since $\left(\frac{\rho_2^-}{\rho(\theta)}\right)^n \to 0$ for $n \to \infty$, one concludes that for $(\pi_{2,1}, \pi_{2,2}) v_2^- \neq 0$, the first summand is asymptotically equivalent to (3.13), and the second summand is not asymptotically equivalent to 0, so the assertion of Proposition 3.7 does not hold in this case.

Remark 3.11. Proposition 3.7 sharpens Theorem 9.4 in the survey Schneider [13] which asserts that for any matrix of the form (3.1), the submatrix $Q_{ij}^{(n)}$ has exponential growth rate $s(i,j)^n n^{d(i,j)}$, where $s(i,j)$ is the maximum of ρ_k which lie on an admissible path from i to j and $d(i,j)+1$ is the number of k with $\rho_k = s(i,j)$. In our terminology, $d(i,j) + 1 = h^+(\theta)$, hence this theorem implies that the exponential growth rate is given by $\rho_{\max}^n n^{\max_\theta h^+(\theta) - 1}$, where the maximum is taken over all admissible paths θ from i to j.

Proposition 3.7 can be immediately applied to the summands in the denominators of Corollary 3.3. In the following, we show that also the asymptotic behaviour of the terms in the numerator of Corollary 3.3 can be understood via Proposition 3.7, but for this purpose, we need to replace the matrix Q by the following matrix. For $\ell \in \{1, \dots, k\}$, consider

$$
\hat{Q}^{(\ell)} := \begin{pmatrix}
Q_{11} & & & & & 0 \\
\vdots & \ddots & & & & \\
Q_{\ell 1} & \dots & Q_{\ell\ell} & & & \\
& & \mathrm{Id}_{d_\ell} & Q_{\ell\ell} & & \\
& & & \vdots & \ddots & \\
0 & & & Q_{k\ell} & \dots & Q_{kk}
\end{pmatrix} .
$$

We note that any sub-matrix Q_{ij} with $i > \ell$ and $j < \ell$ does not appear in this matrix. The matrix $\hat{Q}^{(\ell)}$ is also of the form (3.1) with $k+1$ blocks, one more block than the matrix Q. Denote the elements of $\Gamma_{\kappa+1}(n+1-\kappa)$ by $\hat{\eta} = (\eta_1, \dots, \eta_\ell, \hat{\eta}_\ell, \eta_{\ell+1}, \dots, \eta_\kappa)$, and define for $\theta = (\theta_1, \dots, \theta_\kappa) \in \mathcal{P}^{(\ell)}$,

$$
\hat{Q}^{(\ell)}(\theta, n) = \sum_{\hat{\eta} \in \Gamma_{\kappa+1}(n+1-\kappa)} Q_{\theta_1\theta_1}^{\eta_1} Q_{\theta_1\theta_2} \cdots Q_{\theta_{u-1}\ell} Q_{\ell\ell}^{\eta_\ell} Q_{\ell\ell}^{\hat{\eta}_\ell} Q_{\ell\theta_{u+1}} \cdots Q_{\theta_{\kappa-1}\theta_\kappa} Q_{\theta_\kappa\theta_\kappa}^{\eta_\kappa} .
$$

We now aim at understanding the asymptotic growth of sequences of the form $\pi_{\theta_1} \hat{Q}^{(\ell)}(\theta, n)\mathbb{1} = \pi_{\theta_1} \sum_{r=0}^n Q(\underline{\theta^\ell}, r) Q(\overline{\theta^\ell}, n-r)\mathbb{1}$, which occur in the numerators in Corollary 3.3. The main idea is to apply Proposition 3.7, where $Q(\theta, n)$ is replaced by $\hat{Q}^{(\ell)}(\theta, n)$.

Proposition 3.12. Consider a matrix Q of the form (3.1), a number $\ell \in \{1,\dots,k\}$ and an admissible path $\theta = (\theta_1,\dots,\theta_\kappa) \in \mathcal{P}^{(\ell)}$. Furthermore, suppose that for all $u \in \{1,\dots,\kappa\}$ with $\rho_{\theta_u} < \rho(\theta)$, the diagonal term $Q_{\theta_u\theta_u}$ is scalar. Then for every $\theta \in \mathcal{P}^{(\ell)}$, we have

$$\hat{Q}^{(\ell)}(\theta,n) = \sum_{r=0}^{n} Q(\underline{\theta^\ell}, r) Q(\overline{\theta^\ell}, n-r)\,, \tag{3.24}$$

and the following two statements hold.

(i) If $\alpha_\theta \pi_{\theta_1} v_{\theta_1} \neq 0$, then for the sequence $\pi_{\theta_1}\hat{Q}^{(\ell)}(\theta,n)\mathbb{1}$, an asymptotically equivalent sequence in the limit $n\to\infty$ is given by

$$\alpha_\theta \pi_{\theta_1} v_{\theta_1} \rho(\theta)^{n+1-\kappa} \frac{n^{h^+(\theta)}}{h^+(\theta)!} \prod_{u\in H^-(\theta)} \frac{1}{1-\frac{\rho_{\theta_u}}{\rho(\theta)}} \quad \text{if } \rho_\ell = \rho(\theta) \tag{3.25}$$

and

$$\alpha_\theta \pi_{\theta_1} v_{\theta_1} \rho(\theta)^{n+1-\kappa} \frac{n^{h^+(\theta)-1}}{(h^+(\theta)-1)!} \frac{1}{1-\frac{\rho_\ell}{\rho(\theta)}} \prod_{u\in H^-(\theta)} \frac{1}{1-\frac{\rho_{\theta_u}}{\rho(\theta)}} \quad \text{if } \rho_\ell < \rho(\theta)\,. \tag{3.26}$$

(ii) If $\alpha_\theta \pi_{\theta_1} v_{\theta_1} = 0$, then the exponential growth of the sequence $\pi_{\theta_1}\hat{Q}^{(\ell)}(\theta,n)\mathbb{1}$ is for $\rho_\ell = \rho(\theta)$ equal to or less than $\rho(\theta)^{n+1-\kappa} n^{h^+(\theta)-1}$ and for $\rho_\ell < \rho(\theta)$ equal to or less than $\rho(\theta)^{n+1-\kappa} n^{h^+(\theta)-2}$.

Proof. In a first step, we show that (3.24) holds for every $\theta \in \mathcal{P}^{(\ell)}$, i.e.

$$\begin{aligned} &\sum_{\hat\eta \in \Gamma_{\kappa+1}(n+1-\kappa)} Q^{\eta_1}_{\theta_1\theta_1} Q_{\theta_1\theta_2} \cdots Q_{\theta_{u-1}\ell} Q^{\eta_\ell}_{\ell\ell} Q^{\hat\eta_\ell}_{\ell\ell} Q_{\ell\theta_{u+1}} \cdots Q_{\theta_{\kappa-1}\theta_\kappa} Q^{\eta_\kappa}_{\theta_\kappa\theta_\kappa} \\ &= \sum_{r=0}^{n} \sum_{\substack{\eta\in\Gamma_{\underline\kappa}(r+1-\underline\kappa)\,,\\ \zeta\in\Gamma_{\overline\kappa}(n-r+1-\overline\kappa)}} Q^{\eta_1}_{\theta_1\theta_1} \cdots Q^{\eta_{\underline\kappa}}_{\ell\ell} Q^{\zeta_1}_{\ell\ell} \cdots Q^{\zeta_{\overline\kappa}}_{\theta_\kappa\theta_\kappa}\,. \end{aligned}$$

We order the summands on the left hand side by putting together the summands with equal sum of the first $\underline\kappa$ exponents, say $\psi_1 + \cdots + \psi_{\underline\kappa} = r+1-\underline\kappa$ for some $r \in \{\underline\kappa - 1, \dots, n+1-\overline\kappa\}$ where $\overline\kappa = \kappa + 1 - \underline\kappa$. Hence, the sum of the last $\overline\kappa$ exponents equals $n+1-\kappa-(r+1-\underline\kappa) = n-r+1-\overline\kappa$. Then the left hand side equals

$$\sum_{r=\underline\kappa-1}^{n+1-\overline\kappa} \sum_{\substack{\eta\in\Gamma_{\underline\kappa}(r+1-\underline\kappa)\,,\\ \zeta\in\Gamma_{\overline\kappa}(n-r+1-\overline\kappa)}} Q^{\eta_1}_{\theta_1\theta_1} \cdots Q^{\eta_{\underline\kappa}}_{\ell\ell} Q^{\zeta_1}_{\ell\ell} \cdots Q^{\zeta_{\overline\kappa}}_{\theta_\kappa\theta_\kappa}\,.$$

Since by definition $\Gamma_{\underline\kappa}(r+1-\underline\kappa) = \emptyset$ for $r+1-\underline\kappa < 0$ and $\Gamma_{\overline\kappa}(n-r+1-\overline\kappa) = \emptyset$ for $n-r+1-\underline\kappa < 0$, equality (3.24) follows.

We apply Proposition 3.7 to the matrix $\hat{Q}^{(\ell)}$ instead of the matrix Q. For this extended matrix, we consider the admissible path of length $\kappa+1$

$$\hat\theta^{(\ell)} := (\theta_1+1,\dots,\theta_{u-1}+1, \ell+1, \ell, \theta_{u+1},\dots,\theta_\kappa), \tag{3.27}$$

where $\theta_u = \ell$. The matrix $\hat{Q}^{(\ell)}(\theta,n)$ corresponds to the admissible sequence $\hat\theta^{(\ell)}$ for $\hat{Q}^{(\ell)}$, more precisely, one sees that

$$\hat{Q}^{(\ell)}(\theta,n) = (\hat{Q}^{(\ell)})(\hat\theta^{(\ell)}, n+1), \tag{3.28}$$

since $\Gamma_{\kappa+1}(n+1-\kappa) = \Gamma_{\kappa+1}((n+1)+1-(\kappa+1))$, where the right hand side of (3.28) is defined as (3.6) with Q replaced by $\hat{Q}^{(\ell)}$.

Proposition 3.7 yields the following two statements.

(i) If $\pi_{\theta_1} v_{\theta_1} \neq 0$ and $\alpha_\theta \neq 0$, then the sequence $\pi_{\theta_1}\hat{Q}^{(\ell)}(\theta,n)\mathbb{1}$ is asymptotically equivalent in the limit $n\to\infty$ to

$$\pi_{\theta_1} v_{\theta_1} \alpha_\theta \sum_{\eta\in\Gamma_{\kappa+1}(n+1-\kappa)} \rho_{\theta_\kappa}^{\eta_\kappa+1}\cdots\rho_{\theta_{u+1}}^{\eta_{u+2}}\rho_\ell^{\eta_{u+1}}\rho_\ell^{\eta_u}\rho_{\theta_{u-1}}^{\eta_{u-1}}\cdots\rho_{\theta_1}^{\eta_1},$$

where α_θ is equal to the corresponding quantity for the matrices Q and $\hat{Q}^{(\ell)}$, since for $\hat{Q}^{(\ell)}$ the block in row $\ell+1$ and column ℓ is the identity matrix. If $\rho_\ell = \rho(\theta)$, then $h^+(\hat{\theta}^{(\ell)}) = h^+(\theta)+1$, and if $\rho_\ell < q(\theta)$, then $h^+(\hat{\theta}^{(\ell)}) = h^+(\theta)$. Hence, Proposition 3.7 shows that the sequence above is asymptotically equivalent to

$$\alpha_\theta \pi_{\theta_1} v_{\theta_1}\rho(\theta)^{n+1-\kappa}\frac{(n+1)^{h^+(\theta)}}{h^+(\theta)!}\prod_{u\in H^-(\theta)}\frac{1}{1-\frac{\rho_{\theta_u}}{\rho(\theta)}} \qquad \text{if } \rho_\ell = \rho(\theta)$$

and

$$\alpha_\theta \pi_{\theta_1} v_{\theta_1}\rho(\theta)^{n+1-\kappa}\frac{(n+1)^{h^+(\theta)-1}}{(h^+(\theta)-1)!}\frac{1}{1-\frac{\rho_\ell}{\rho(\theta)}}\prod_{u\in H^-(\theta)}\frac{1}{1-\frac{\rho_{\theta_u}}{\rho(\theta)}} \qquad \text{if } \rho_\ell < \rho(\theta).$$

This proves assertion (i), since $(n+1)^{h^+(\theta)}$ is asymptotically equivalent to $n^{h^+(\theta)}$.

(ii) If $\pi_{\theta_1} v_{\theta_1} = 0$ or $\alpha_\theta = 0$, then the exponential growth of the sequence $\pi_{\theta_1}\hat{Q}^{(\ell)}(\theta,n)\mathbb{1}$ is for $\rho_\ell = \rho(\theta)$ equal to or less than $\rho(\theta)^{n+1-\kappa}n^{h^+(\theta)-1}$ and for $\rho_\ell < \rho(\theta)$ equal to or less than $\rho(\theta)^{n+1-\kappa}n^{h^+(\theta)-2}$.

This finishes the proof of the proposition. □

So far, we have fixed a particular admissible path $\theta\in\mathcal{P}$, and in both Proposition 3.7 and Proposition 3.12, we made certain assumptions relating to this particular θ. In the following assumption for our main results, we consider all relevant admissible paths. Recall that $\rho_{\max} = \max\{\rho_1,\dots,\rho_k\}$ and $h^+_{\max} := \max\{h^+(\theta) : \theta\in\mathcal{P} \text{ and } \rho(\theta)=\rho_{\max}\}$, and define the set of maximal admissible paths $\mathcal{P}_{\max}$ by

$$\mathcal{P}_{\max} := \left\{\theta\in\mathcal{P} : h^+(\theta) = h^+_{\max} \text{ and } \rho(\theta) = \rho_{\max}\right\},$$

and let $\mathcal{P}^{(\ell)}_{\max} := \mathcal{P}^{(\ell)}\cap\mathcal{P}_{\max}$.

Assumption 3.13. Consider a matrix Q of the Frobenius normal form (3.1), and let $(X_i)_{i\in\mathbb{N}_0}$ be the Markov chain associated to the substochastic matrix Q starting in π. We assume that

(i) for all $i\in\{0,\dots,k\}$ with $\rho_i < \rho_{\max}$, the diagonal term Q_{ii} is scalar, and

(ii) there exists a maximal admissible path $\theta = (\theta_1,\dots,\theta_\kappa)\in\mathcal{P}_{\max}$ such that the constant $\alpha_\theta\neq 0$ and the initial distribution π satisfies $\pi_{\theta_1}v_{\theta_1}\neq 0$, where α_θ and the Perron–Frobenius eigenvector v_{θ_1} of $Q_{\theta_1\theta_1}$ are defined as in Lemma 3.6.

By combining Proposition 3.7 and Proposition 3.12, we arrive at the following formulas for the quasi-ergodic limits.

Theorem 3.14 (Quasi-ergodic limits for finite absorbing Markov chains)**.** *Suppose that Assumption 3.13 holds, and let $\ell \in \{1, \dots, k\}$. Then the following statements hold:*

(i) *If $\rho_\ell < \rho_{\max}$ or $\mathcal{P}^{(\ell)}_{\max} = \emptyset$, then*

$$\lim_{n\to\infty} \mathbb{E}_\pi\left[\tfrac{1}{n+1}\#\{m \in \{0,\dots,n\} : X_m \in I_\ell\}\middle| T > n\right] = 0\,.$$

(ii) *If $\rho_\ell = \rho_{\max}$ and $\mathcal{P}^{(\ell)}_{\max} \neq \emptyset$, then*

$$\begin{aligned}&\lim_{n\to\infty} \mathbb{E}_\pi\left[\tfrac{1}{n+1}\#\{m \in \{0,\dots,n\} : X_m \in I_\ell\}\middle| T > n\right]\\ &= \frac{\sum_{\theta\in\mathcal{P}^{(\ell)}_{\max}} \alpha_\theta \pi_{\theta_1} v_{\theta_1} \prod_{u\in H^-(\theta)} \frac{1}{\rho_{\max}-\rho_{\theta_u}}}{h^+_{\max}\sum_{\theta\in\mathcal{P}_{\max}} \alpha_\theta \pi_{\theta_1} v_{\theta_1} \prod_{u\in H^-(\theta)} \frac{1}{\rho_{\max}-\rho_{\theta_u}}}\,.\end{aligned}$$

Proof. We first assume $\rho_\ell = \rho_{\max}$ and show that in this case, we have

$$\begin{aligned}&\lim_{n\to\infty} \mathbb{E}_\pi\left[\tfrac{1}{n+1}\#\{m \in \{0,\dots,n\} : X_m \in I_\ell\}\middle| T > n\right]\\ &= \lim_{n\to\infty} \frac{\sum_{\theta\in\mathcal{P}^{(\ell)}} \pi_{\theta_1} \hat{Q}^{(\ell)}(\theta,n)\mathbb{1}}{\sum_{\theta\in\mathcal{P}}(n+1)\pi_{\theta_1} Q(\theta,n)\mathbb{1}}\\ &= \lim_{n\to\infty} \frac{\sum_{\theta\in\mathcal{P}^{(\ell)}_{\max}} \pi_{\theta_1} \hat{Q}^{(\ell)}(\theta,n)\mathbb{1}}{\sum_{\theta\in\mathcal{P}_{\max}}(n+1)\pi_{\theta_1} Q(\theta,n)\mathbb{1}}\,. \qquad (3.29)\end{aligned}$$

The first equality follows from Corollary 3.3 (i) and Proposition 3.12 applied to the numerator in Corollary 3.3 (i).

For the second equality, we identify summands in both denominator and numerator that dominate for $n \to \infty$. By Assumption 3.13 (ii) there exists a maximal admissible path $\theta \in \mathcal{P}_{\max}$ with $\alpha_\theta \pi_{\theta_1} v_{\theta_1} \neq 0$, hence Proposition 3.7 shows that for the sequence $\pi_{\theta_1} Q(\theta,n)\mathbb{1}$ an asymptotically equivalent sequence for $n \to \infty$ is given by (3.13). Thus $(n+1)\pi_{\theta_1} Q(\theta,n)$ has exponential growth rate equal to $\rho(\theta)^n n^{h^+(\theta)} = \rho^n_{\max} n^{h^+_{\max}}$. For any $\theta \in \mathcal{P}\setminus\mathcal{P}_{\max}$, one has $h^+(\theta) < h^+_{\max}$, and the summand $(n+1)\pi_{\theta_1} Q(\theta,n)\mathbb{1}$ grows at most with the smaller exponential growth rate $\rho(\theta)^n n^{h^+(\theta)}$, again by Proposition 3.7. This justifies replacing $\mathcal{P}$ by $\mathcal{P}_{\max}$ in the denominator. For any $\theta \in \mathcal{P}^{(\ell)}$ in the numerator, Proposition 3.12 shows that the summand $\pi_{\theta_1} \hat{Q}^{(\ell)}(\theta,n)\mathbb{1}$ grows at most with exponential growth rate $\rho(\theta)^n n^{h^+(\theta)} = \rho^n_{\max} n^{h^+(\theta)}$, see (3.25) (the exponential growth can be smaller than that by Proposition 3.12 (ii), when $\alpha_\theta \pi_{\theta_1} v_{\theta_1} = 0$). This justifies replacing $\mathcal{P}^{(\ell)}$ by $\mathcal{P}^{(\ell)}_{\max}$ in the numerator using again $h^+(\theta) < h^+_{\max}$ for $\theta \in \mathcal{P}^{(\ell)} \setminus \mathcal{P}^{(\ell)}_{\max}$ (note that for $\mathcal{P}^{(\ell)}_{\max} = \varnothing$ the limits for $n \to \infty$ equal 0).

Now for the denominator in (3.29), Proposition 3.7 (i) yields the following: Let $\theta \in \mathcal{P}_{\max}$ with $\alpha_\theta \pi_{\theta_1} v_{\theta_1} \neq 0$ (recall that the existence is clear due to Assumption 3.13 (ii)). Then for the corresponding summand, given by $(n+1)\pi_{\theta_1} Q(\theta,n)\mathbb{1}$, an asymptotically equivalent sequence is

$$\Psi_n(\theta) = \alpha_\theta \pi_{\theta_1} v_{\theta_1} \rho_{\max}^{n+1-\kappa(\theta)} \frac{n^{h^+_{\max}}}{(h^+_{\max}-1)!} \prod_{u\in H^-(\theta)} \frac{1}{1-\frac{\rho_{\theta_u}}{\rho_{\max}}}\,.$$

Note that for $\theta \in \mathcal{P}_{\max}$ with $\alpha_\theta \pi_{\theta_1} v_{\theta_1} = 0$, the corresponding summand given by $(n+1)\pi_{\theta_1} Q(\theta,n)\mathbb{1}$ has weaker exponential growth for $n \to \infty$, equal to or less than $\rho^n_{\max} n^{h^+_{\max}-1}$. Since $\Psi_n(\theta) = 0$ for those θ, this implies that an asymptotically equivalent term for the denominator is given by $\sum_{\theta\in\mathcal{P}_{\max}}\Psi_n(\theta)$ with exponential growth $\rho_{\max}^n n^{h^+_{\max}}$.

For the numerator, suppose first that there exists $\theta \in \mathcal{P}^{(\ell)}_{\max}$ with $\alpha_\theta\pi_{\theta_1}v_{\theta_1} \neq 0$. Then by Proposition 3.12, for the corresponding summand $\pi_{\theta_1}\hat{Q}^{(\ell)}(\theta,n)\mathbb{1}$, an asymptotically equivalent sequence is

$$\Xi_n(\theta) := \alpha_\theta\pi_{\theta_1}v_{\theta_1}\rho_{\max}^{n+1-\kappa(\theta)}\frac{n^{h^+_{\max}}}{h^+_{\max}!}\prod_{u\in H^-(\theta)}\frac{1}{1-\frac{\rho_{\theta_u}}{\rho_{\max}}}\,.$$

By Proposition 3.12 (ii), for $\theta\in\mathcal{P}^{(\ell)}_{\max}$ with $\alpha_\theta\pi_{\theta_1}v_{\theta_1}=0$ the corresponding summand given by $\pi_{\theta_1}\hat{Q}^{(\ell)}(\theta,n)\mathbb{1}$ has weaker exponential growth for $n\to\infty$, equal to or less than $\rho_{\max}^n n^{h^+_{\max}-1}$. Since $\Xi_n(\theta)=0$ for those θ, this implies that an asymptotically equivalent term for the numerator is given by $\sum_{\theta\in\mathcal{P}^{(\ell)}_{\max}}\Xi_n(\theta)$. Hence, the formula given in (ii) holds in this case. Otherwise, $\mathcal{P}^{(\ell)}_{\max}=\emptyset$ or for all $\theta\in\mathcal{P}^{(\ell)}_{\max}$ we have $\alpha_\theta\pi_{\theta_1}v_{\theta_1}=0$. Then the limits for $n\to\infty$ are equal to 0, hence assertion (ii) and also the second statement in (i) follow.

It remains to show (i) under the assumption $\rho_\ell<\rho_{\max}$. In this case, exactly like above, an asymptotically equivalent term for the denominator is given by $\sum_{\theta\in\mathcal{P}_{\max}}\Psi_n(\theta)$. Consider a summand in the numerator, so let $\theta\in\mathcal{P}^{(\ell)}$. Then Proposition 3.12 shows that the exponential growth for $n\to\infty$ of the numerator is bounded above by either $\rho(\theta)^n n^{h^+(\theta)}$ if $\rho(\theta)<\rho_{\max}$, or by $\rho_{\max}^n n^{h^+_{\max}-1}$ if $\rho(\theta)=\rho_{\max}$. In both cases, the exponential growth is weaker than for the denominator, determined by $\sum_{\theta\in\mathcal{P}_{\max}}\Psi_n(\theta)$ with exponential growth $\rho_{\max}^n n^{h^+_{\max}}$. This finishes the proof of the theorem. □

Remark 3.15. We note that due to Remark 3.8, the condition $\alpha_\theta\neq 0$ in Assumption 3.13 (ii) is generically satisfied, and the condition $\pi_{\theta_1}v_{\theta_1}\neq 0$ in this assumption is not restrictive. In fact, suppose that for a given initial distribution π, there is no $\theta\in\mathcal{P}_{\max}$ with $\pi_{\theta_1}v_{\theta_1}\neq 0$. Then define

$$\begin{aligned} h^{+,\pi}_{\max} &:= \max\left\{h^+(\theta) : \theta\in\mathcal{P} \text{ and } \rho(\theta)=\rho_{\max}, \pi_{\theta_1}v_{\theta_1}\neq 0\right\},\\ \mathcal{P}^\pi_{\max} &:= \left\{\theta\in\mathcal{P} : \rho(\theta)=\rho_{\max} \text{ and } h^+(\theta)=h^{+,\pi}_{\max}\right\}. \end{aligned}$$

Then Theorem 3.14 remains valid with $\mathcal{P}_{\max}$ and $\mathcal{P}^{(\ell)}$ replaced by $\mathcal{P}^\pi_{\max}$ and $\mathcal{P}^{(\ell),\pi}_{\max} := \mathcal{P}^{(\ell)}\cap\mathcal{P}^\pi_{\max}$, respectively.

The formula for the quasi-ergodic limit in the above theorem can be simplified in certain special cases.

Corollary 3.16. *Assume that in the setting of Theorem 3.14, there is only one maximal path, i.e.* $\mathcal{P}_{\max}=\{\theta=(\theta_1,\dots,\theta_\kappa)\}$. *If* $\ell=\theta_u$ *with* $\rho_\ell=\rho_{\max}$ *for some* $u\in\{1,\dots,\kappa\}$, *then*

$$\lim_{n\to\infty}\mathbb{E}_\pi\left[\tfrac{1}{n+1}\#\left\{m\in\{0,\dots,n\} : X_m\in I_\ell\right\}\middle| T>n\right]=\frac{1}{h^+_{\max}}\,,$$

and this limit vanishes whenever $\rho_\ell<\rho_{\max}$.

Proof. The assertion is an immediate consequence of Theorem 3.14. □

In the scalar case, one obtains the following formulas which are directly given in terms of the matrix Q.

Corollary 3.17. *Consider a matrix Q of the Frobenius normal form* (3.1)*, and let $(X_i)_{i\in\mathbb{N}_0}$ be the Markov chain associated to the substochastic matrix Q starting in π. We assume that all submatrices $q_{ij} := Q_{ij}$ are scalar and the initial distribution π satisfies $\pi_{\theta_1} \neq 0$ for some maximal admissible path $\theta = (\theta_1, \dots, \theta_\kappa) \in \mathcal{P}_{\max}$. Then for all $\ell \in \{1, \dots, k\}$, the following holds.*

(i) *If $\rho_\ell < \rho_{\max}$ or $\mathcal{P}^{(\ell)}_{\max} = \emptyset$, then*

$$\lim_{n\to\infty} \mathbb{E}_\pi \left[\tfrac{1}{n+1}\#\big\{m \in \{0,\dots,n\} : X_m \in I_\ell\big\}\big|T > n\right] = 0.$$

(ii) *If $\rho_\ell = \rho_{\max}$ and $\mathcal{P}^{(\ell)}_{\max} \neq \emptyset$, then*

$$\lim_{n\to\infty} \mathbb{E}_\pi \left[\tfrac{1}{n+1}\#\big\{m \in \{0,\dots,n\} : X_m \in I_\ell\big\}\big|T > n\right]$$
$$= \frac{1}{h^+_{\max}} \frac{\sum_{\theta\in\mathcal{P}^{(\ell)}_{\max}} \pi_{\theta_1} q_{\theta_1\theta_2}\cdots q_{\theta_{\kappa-1}\theta_\kappa} \prod_{u\in H^-(\theta)} \frac{1}{\rho_{\max}-\rho_{\theta_u}}}{\sum_{\theta\in\mathcal{P}_{\max}} \pi_{\theta_1} q_{\theta_1\theta_2}\cdots q_{\theta_{\kappa-1}\theta_\kappa} \prod_{u\in H^-(\theta)} \frac{1}{\rho_{\max}-\rho_{\theta_u}}}\,.$$

Proof. In the scalar case considered here, one finds, for an admissible path $\theta = (\theta_1, \dots, \theta_\kappa)$, that the constants α_{θ_u} defined in Lemma 3.6 (iv) are given by $\alpha_{\theta_\kappa} = 1$ and $\alpha_{\theta_u} = q_{\theta_u\theta_{u+1}}$ for $u \in \{\kappa-1, \kappa-2, \dots, 1\}$. Hence, we get

$$\alpha_\theta \pi_{\theta_1} v_{\theta_1} = \pi_{\theta_1} q_{\theta_1\theta_2}\cdots q_{\theta_{\kappa-1}\theta_\kappa} \neq 0\,,$$

and thus, the assertion follows from Theorem 3.14. □

Remark 3.18. In the scalar case, suppose that all maximal admissible paths begin in the same element, i.e. $\theta_1 = \theta'_1$ for all $\theta, \theta' \in \mathcal{P}_{\max}$. Then the quasi-ergodic limits are independent of the initial distribution π. In fact, here all scalars π_{θ_1} and $\pi_{\theta'_1}$ coincide and Corollary 3.17 yields for $\rho_\ell = \rho_{\max}$ and $\mathcal{P}^{(\ell)}_{\max} \neq \emptyset$

$$\lim_{n\to\infty} \mathbb{E}_\pi \big[\tfrac{1}{n+1}\#\big\{m \in \{0,\dots,n\} : X_m \in I_\ell\big\}\big|T > n\big]$$
$$= \frac{1}{h^+_{\max}} \frac{\sum_{\theta\in\mathcal{P}^{(\ell)}_{\max}} q_{\theta_1\theta_2}\cdots q_{\theta_{\kappa-1}\theta_\kappa} \prod_{u\in H^-(\theta)} \frac{1}{\rho_{\max}-\rho_{\theta_u}}}{\sum_{\theta\in\mathcal{P}_{\max}} q_{\theta_1\theta_2}\cdots q_{\theta_{\kappa-1}\theta_\kappa} \prod_{u\in H^-(\theta)} \frac{1}{\rho_{\max}-\rho_{\theta_u}}}\,.$$

Next we discuss the behaviour within the blocks. In order to deal with the numerator in Corollary 3.3 (ii), we consider for $\ell \in \{1, \dots, k\}$ instead of the matrix $\hat{Q}^{(\ell)}$ the following matrix for any $t \in \{1, \dots, d_\ell\}$ and $e_t \in \mathbb{R}^{d_\ell}$,

$$\hat{Q}^{(\ell,t)} := \begin{pmatrix} Q_{11} & & & & & 0 \\ \vdots & \ddots & & & & \\ Q_{\ell 1} & \dots & Q_{\ell\ell} & & & \\ & & e_t e_t^\top & Q_{\ell\ell} & & \\ & & & \vdots & \ddots & \\ 0 & & & Q_{k\ell} & \dots & Q_{kk} \end{pmatrix}.$$

This matrix is also of the form (3.1) with one more block, i.e. $k+1$ blocks. Denote the elements of $\Gamma_{\kappa+1}(n+1-\kappa)$ by $\hat{\eta} = (\eta_1, \dots, \eta_\ell, \hat{\eta}_\ell, \dots, \eta_\kappa)$ and define,

$$\hat{Q}^{(\ell,t)}(\theta, n) = \sum_{\hat{\eta}\in\Gamma_{\kappa+1}(n+1-\kappa)} Q^{\eta_1}_{\theta_1\theta_1} Q_{\theta_1\theta_2}\cdots Q_{\theta_{u-1}\ell} Q^{\eta_\ell}_{\ell\ell} e_t e_t^\top Q^{\hat{\eta}_\ell}_{\ell\ell} Q_{\ell\theta_{u+1}}\cdots Q_{\theta_{\kappa-1}\theta_\kappa} Q^{\eta_\kappa}_{\theta_\kappa\theta_\kappa}.$$

The following proposition applies to the terms in the numerator of Corollary 3.3 (ii). It is convenient to change the notation slightly by writing $u^{(i)}$ and $v^{(i)}$ instead of

u_i and v_i for the left and right eigenvectors, respectively, of the Perron–Frobenius eigenvalue of Q_{ii}.

Proposition 3.19. Consider a matrix Q of the form (3.1), numbers $\ell \in \{1, \dots, k\}$ and $t \in \{1, \dots, d_\ell\}$, and an admissible path $\theta = (\theta_1, \dots, \theta_\kappa) \in \mathcal{P}^{(\ell)}$. Furthermore, suppose that $\rho_\ell = \rho(\theta)$ and that for all $u \in \{1, \dots, \kappa\}$ with $\rho_{\theta_u} < \rho(\theta)$, the diagonal term $Q_{\theta_u \theta_u}$ is scalar. Then

$$\hat{Q}^{(\ell,t)}(\theta, n) = \sum_{r=0}^{n} Q(\underline{\theta^\ell}, r) e_t e_t^\top Q(\overline{\theta^\ell}, n-r), \tag{3.30}$$

and the following two statements hold:

(i) If $\alpha_\theta \pi_{\theta_1} v^{(\theta_1)} \neq 0$, then for the sequence $\pi_{\theta_1} \hat{Q}^{(\ell,t)}(\theta, n)\mathbb{1}$, an asymptotically equivalent sequence in the limit $n \to \infty$ is given by

$$u_t^{(\ell)} v_t^{(\ell)} \alpha_\theta \pi_{\theta_1} v^{(\theta_1)} \rho(\theta)^{n-\kappa} \frac{n^{h^+(\theta)}}{h^+(\theta)!} \prod_{u \in H^-(\theta)} \frac{1}{1 - \frac{\rho_{\theta_u}}{\rho(\theta)}},$$

where $u^{(\ell)\top} = (u_1^{(\ell)}, \dots, u_{d_\ell}^{(\ell)})$ and $v^{(\ell)} = (v_1^{(\ell)}, \dots, v_{d_\ell}^{(\ell)})^\top$ are the positive left and right eigenvector of $Q_{\ell\ell}$ for ρ_ℓ, respectively, normalised in the sense of $\sum_{i=1}^{d_\ell} v_i^{(\ell)} = 1$ and $u^{(\ell)\top} v^{(\ell)} = 1$.

(ii) If $\alpha_\theta \pi_{\theta_1} v^{(\theta_1)} = 0$, then the exponential growth of the sequence $\pi_{\theta_1} \hat{Q}^{(\ell,t)}(\theta, n)\mathbb{1}$ is equal to or less than $\rho(\theta)^{n+1-\kappa} n^{h^+(\theta)-1}$.

Proof. We proceed as in Proposition 3.12, where we have used Proposition 3.7 with the matrix $\hat{Q}^{(\ell)}$ instead of the matrix Q. Here we can argue analogously using the matrix $\hat{Q}^{(\ell,t)}$ instead of Q.

In a first step we verify equality (3.30), i.e.

$$\begin{aligned} &\sum_{\hat{\eta} \in \Gamma_{\kappa+1}(n+1-\kappa)} Q_{\theta_1\theta_1}^{\eta_1} Q_{\theta_1\theta_2} \cdots Q_{\theta_{u-1}\ell} Q_{\ell\ell}^{\eta_\ell} e_t e_t^\top Q_{\ell\ell}^{\hat{\eta}_\ell} Q_{\ell\theta_{u+1}} \cdots Q_{\theta_{\kappa-1}\theta_\kappa} Q_{\theta_\kappa\theta_\kappa}^{\eta_\kappa} \\ &= \sum_{r=0}^{n} \sum_{\substack{\eta \in \Gamma_{\underline{\kappa}}(r+1-\underline{\kappa}), \\ \zeta \in \Gamma_{\overline{\kappa}}(n-r+1-\overline{\kappa})}} Q_{\theta_1\theta_1}^{\eta_1} \cdots Q_{\ell\ell}^{\eta_{\underline{\kappa}}} e_t e_t^\top Q_{\ell\ell}^{\zeta_1} \cdots Q_{\theta_\kappa\theta_\kappa}^{\zeta_{\overline{\kappa}}}. \end{aligned}$$

This follows as in the proof of equality (3.24) in Proposition 3.12.

Now, we will apply Proposition 3.7 to the matrix $\hat{Q}^{(\ell,t)}(\theta, n)$ using the admissible sequence $\hat{\theta}^{(\ell)}$ defined in (3.27). The only difference to the proof of Proposition 3.12 occurs when we determine the factor α_θ from Proposition 3.7. In the earlier proof, the block in row $\ell+1$ and column ℓ of the matrix $\hat{Q}^{(\ell)}$ is the identity matrix, hence this did not lead to a change in this factor in Proposition 3.12. Here, however, the identity matrix is replaced by $e_t e_t^\top$, and we will show that we get the additional factor $\tilde{\alpha} = u_t^{(\ell)} v_t^{(\ell)}$.

To determine this additional factor $\tilde{\alpha}$, we have to uniquely decompose

$$e_t e_t^\top v^{(\ell)} = \tilde{\alpha} v^{(\ell)} + \tilde{w},$$

where $\tilde{\alpha} \in \mathbb{R}$ and $\tilde{w} \in V_\ell^-$ as in Lemma 3.6 (iv). We show now that with $\tilde{\alpha} = u_t^{(\ell)} v_t^{(\ell)}$, we have

$$\tilde{w} = e_t e_t^\top v^{(\ell)} - u_t^{(\ell)} v_t^{(\ell)} v^{(\ell)} \in V_\ell^-.$$

To see this, we determine the exponential growth of $Q_{\ell\ell}^n \tilde{w}$ in the limit $n \to \infty$. Using that $e_t e_t^\top v^{(\ell)} = (0, \dots, 0, v_t^{(\ell)}, 0, \dots, 0)^\top$, we have

$$Q_{\ell\ell}^n \tilde{w} = Q_{\ell\ell}^n (0, \dots, 0, v_t^{(\ell)}, 0, \dots, 0)^\top - u_t^{(\ell)} v_t^{(\ell)} \underbrace{Q_{\ell\ell}^n v^{(\ell)}}_{=\rho_\ell^n v^{(\ell)}} .$$

We multiply with the left eigenvector $u^{(\ell)\top}$ and get

$$\begin{aligned} u^{(\ell)\top} Q_{\ell\ell}^n \tilde{w} &= u^{(\ell)\top} Q_{\ell\ell}^n (0, \dots, 0, v_t^{(\ell)}, 0, \dots, 0)^\top - u_t^{(\ell)} v_t^{(\ell)} \rho_\ell^n u^{(\ell)\top} v^{(\ell)} \\ &= \rho_\ell^n u^{(\ell)\top} (0, \dots, 0, v_t^{(\ell)}, 0, \dots, 0)^\top - u_t^{(\ell)} v_t^{(\ell)} \rho_\ell^n = 0 . \end{aligned}$$

This shows that $Q_{\ell\ell}^n \tilde{w}$ is in the orthogonal complement of the one-dimensional space $\operatorname{span}(u^{(\ell)})$. Since $\operatorname{span}(v^{(\ell)})$ is not orthogonal to $\operatorname{span}(u^{(\ell)})$, it follows that $\tilde{w} \in V_\ell^-$, because otherwise, the component relating to $\operatorname{span}(v^{(\ell)})$ would become dominant. □

This observation can be used to determine the quasi-ergodic behaviour within the blocks.

Theorem 3.20. *Assume the setting of Theorem 3.14, and let $\ell \in \{1, \dots, k\}$ such that $\rho_\ell = \rho_{\max}$ and $\mathcal{P}_{\max}^{(\ell)} \neq \emptyset$. Then for all $s \in I_\ell$, we have*

$$\begin{aligned} &\lim_{n\to\infty} \mathbb{E}_\pi \left[\tfrac{1}{n+1} \#\big\{ m \in \{0, \dots, n\} : X_m = s \big\} \big| T > n \right] \\ &= u_{t(s)}^{(\ell)} v_{t(s)}^{(\ell)} \frac{1}{h_{\max}^+} \frac{\sum_{\theta \in \mathcal{P}_{\max}^{(\ell)}} \alpha_\theta \pi_{\theta_1} v_{\theta_1} \prod_{u \in H^-(\theta)} \frac{1}{\rho_{\max} - \rho_{\theta_u}}}{\sum_{\theta \in \mathcal{P}_{\max}} \alpha_\theta \pi_{\theta_1} v_{\theta_1} \prod_{u \in H^-(\theta)} \frac{1}{\rho_{\max} - \rho_{\theta_u}}} , \end{aligned}$$

where $t(s) := s - \sum_{i=1}^{\ell-1} d_i$. Here $u^{(\ell)\top} = (u_1^{(\ell)}, \dots, u_{d_\ell}^{(\ell)})$ and $v^{(\ell)} = (v_1^{(\ell)}, \dots, v_{d_\ell}^{(\ell)})^\top$ are the positive left and right eigenvector of $Q_{\ell\ell}$ for ρ_ℓ, respectively, normalised in the sense of $\sum_{i=1}^{d_\ell} v_i^{(\ell)} = 1$ and $u^{(\ell)\top} v^{(\ell)} = 1$.

Proof. The proof is the same as the proof of Theorem 3.14, with the difference that instead of Proposition 3.12, we need to use Proposition 3.19 here. □

Finally, we consider the case that some of the blocks Q_{ii} are not eventually positive.

Remark 3.21 (The irreducible case). Consider a matrix Q in Frobenius normal form (3.1) with diagonal matrices Q_{ii} which are irreducible and possibly periodic. Then there exists an $N \in \mathbb{N}$ such that the diagonal blocks of Q^N are eventually positive. Denote by $\tilde{X}$ the Markov chain induced by Q^N and the corresponding stopping time by $\tilde{T}$. We get for any $j \in \{1, \dots, k\}$,

$$
\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}_\pi \left[\tfrac{1}{n+1} \#\big\{m \in \{0,\dots,n\} : X_m = j\big\} \big| T \geq n\big\}\right] \\
&= \lim_{n\to\infty} \sum_{i=0}^{N-1} \mathbb{E}_\pi \left[\tfrac{1}{n+1} \#\big\{m \in N\mathbb{Z} + i : m \leq n \text{ and } X_m = j\big\} \big| T \geq n\big\}\right] \\
&= \lim_{n\to\infty} \sum_{i=0}^{N-1} \mathbb{E}_{\pi Q^i} \left[\tfrac{1}{n+1} \#\big\{m \in N\mathbb{Z} + i : m \leq n \text{ and } \tilde{X}_{\frac{m-i}{N}} = j\big\} \big| T \geq n\big\}\right] \\
&= \lim_{n\to\infty} \sum_{i=0}^{N-1} \mathbb{E}_{\pi Q^i} \left[\tfrac{1}{Nn+1} \#\big\{m \in \{0,\dots,n\} : \tilde{X}_m = j\big\} \big| T \geq Nn\big\}\right] \\
&= \frac{1}{N} \sum_{i=0}^{N-1} \lim_{n\to\infty} \mathbb{E}_{\pi Q^i} \left[\tfrac{1}{n+1} \#\big\{m \in \{0,\dots,n\} : \tilde{X}_m = j\big\} \big| \tilde{T} \geq n\big\}\right] .
\end{aligned}
$$

Hence corresponding formulas for the case of not eventually positive matrices follow from the formulas for the eventually positive case.

4. Examples

We present a number of examples illustrating the previous results and we start with the following simple example from Benaïm, Cloez, Panloup [1, Example 3.5], slightly modified in order to get a lower diagonal matrix Q.

Example 4.1. Consider the transition matrix given by

$$
Q = \begin{pmatrix} \rho_1 & 0 \\ 1-\rho_2 & \rho_2 \end{pmatrix}, \qquad \text{where } 0 < \rho_1 < \rho_2 < 1 .
$$

This example is reducible with scalar blocks and $k = 2$. The maximal admissible paths are $\theta = (2)$ with $\kappa(\theta) = 1$ and $\theta' = (2, 1)$ with $\kappa(\theta') = 2$, hence $\mathcal{P}_{\max} = \{(2), (2, 1)\} = \mathcal{P}^{(2)}_{\max}$ and $\mathcal{P}^{(1)}_{\max} = (2, 1)$. Furthermore, $H^+(\theta) = \{2\} = H^+(\theta')$ and $H^-(\theta') = \{1\}$ and hence $h^+_{\max} = 1$. Applying Corollary 3.17 to this example, one finds that for $\pi_2 > 0$, the quasi-ergodic limit for $\ell = 1$ is given by

$$
\lim_{n\to\infty} \mathbb{E}_\pi \left[\tfrac{1}{n+1} \#\big\{m \in \{0,\dots,n\} : X_m = 1\big\} \big| T > n\right] = 0 ,
$$

since $\rho_1 < \rho_{\max}$, and for $\ell = 2$, we have

$$
\lim_{n\to\infty} \mathbb{E}_\pi \left[\tfrac{1}{n+1} \#\big\{m \in \{0,\dots,n\} : X_m = 2\big\} \big| T > n\right] = 1 ,
$$

since these two probabilities sum up to 1. The classification of the quasi-stationary distributions in van Doorn and Pollett [15], in particular [15, Theorem 4.3], shows that for this example there is a unique quasi-stationary distribution given by the normalised left eigenvector $u^\top = \big(\frac{1-\rho_2}{1-\rho_1}, \frac{\rho_2-\rho_1}{1-\rho_1}\big)$ to the eigenvalue ρ_2 of the matrix Q.

Next we consider a class of three-dimensional matrices.

Example 4.2. Let $\rho \in (0, 1)$, and consider

$$
Q = \begin{pmatrix} \rho & 0 & 0 \\ q_{21} & \rho & 0 \\ q_{31} & q_{32} & \rho \end{pmatrix} .
$$

In this scalar case, one finds that under the condition $q_{21}, q_{32} \neq 0$, one has

$$\mathcal{P}_{\max} = \{\theta = (3,2,1)\} \quad \text{with} \quad h^+_{\max} = h^+(\theta) = 3\,.$$

Corollary 3.16 yields that for $\ell \in \{1,2,3\}$, the quasi-ergodic limits are independent of the initial distribution π, and we have

$$\lim_{n\to\infty} \mathbb{E}_\pi\left[\frac{1}{n+1}\#\{m \in \{0,\dots,n\} : X_m = \ell\}\,|T \geq n\}\right] = \frac{1}{h^+_{\max}} = \frac{1}{3}.$$

The situation is different if instead we assume that $q_{21}, q_{31} \neq 0$ and $q_{32} = 0$. Then

$$\mathcal{P}_{\max} = \{\theta = (3,1), \theta' = (2,1)\} = \mathcal{P}^{(1)}_{\max}\,,$$
$$h^+_{\max} = h^+(\theta) = h^+(\theta') = 2 \text{ and } \kappa(\theta) = \kappa(\theta') = 2\,.$$

Here $\mathcal{P}^{(2)}_{\max} = \{\theta'\}$ and $\mathcal{P}^{(3)}_{\max} = \{\theta\}$. Corollary 3.17 yields that for $\pi_1 < 1$,

$$\lim_{n\to\infty} \mathbb{E}_\pi\left[\tfrac{1}{n+1}\#\big\{m \in \{0,\dots,n\} : X_m = 1\big\}\big|T > n\right] = \frac{1}{2}\,,$$
$$\lim_{n\to\infty} \mathbb{E}_\pi\left[\tfrac{1}{n+1}\#\big\{m \in \{0,\dots,n\} : X_m = 2\big\}\big|T > n\right] = \frac{1}{2}\frac{\pi_2 q_{21}}{\pi_2 q_{21} + \pi_3 q_{31}}\,,$$
$$\lim_{n\to\infty} \mathbb{E}_\pi\left[\tfrac{1}{n+1}\#\big\{m \in \{0,\dots,n\} : X_m = 3\big\}\big|T > n\right] = \frac{1}{2}\frac{\pi_3 q_{31}}{\pi_2 q_{21} + \pi_3 q_{31}}\,.$$

Thus, the quasi-ergodic measure is

$$\left(\frac{1}{2}, \frac{1}{2}\frac{\pi_2 q_{21}}{\pi_2 q_{21} + \pi_3 q_{31}}, \frac{1}{2}\frac{\pi_3 q_{31}}{\pi_2 q_{21} + \pi_3 q_{31}}\right).$$

This case illustrates in particular that the lower diagonal entries Q_{ij} for $i > j$ are relevant for the quasi-ergodic limits, which also may depend on the initial distribution π.

In the next example, a Perron–Frobenius eigenvalue $\rho_i < \rho_{\max}$ is present, and the two maximal admissible paths start in the same element and have different lengths.

Example 4.3. Consider

$$Q = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 \\ 0 & q_{32} & \rho_3 & 0 \\ q_{41} & 0 & q_{43} & \rho \end{pmatrix}$$

with $0 < \rho_3 < \rho < 1$ and $q_{32}, q_{41}, q_{43} \neq 0$. We have

$$\mathcal{P}_{\max} = \{\theta = (4,3,2), \theta' = (4,1)\} = \mathcal{P}^{(4)}_{\max},$$
$$h^+_{\max} = h^+(\theta) = h^+(\theta') = 2 \text{ and } \kappa(\theta) = 3, \kappa(\theta') = 2,$$

and

$$\mathcal{P}^{(1)}_{\max} = \{\theta' = (4,1)\}, \mathcal{P}^{(2)}_{\max} = \mathcal{P}^{(3)}_{\max} = \{\theta = (4,3,2)\}.$$

Since both maximal paths start in the same element, Remark 3.18 yields that the quasi-ergodic limits do not depend on π provided $\pi_4 \neq 0$. Corollary 3.17 yields that in this case, the quasi-ergodic measure is given by

$$\left(\frac{1}{2}\frac{q_{41}}{q_{41} + q_{43}q_{32}\frac{1}{\rho-\rho_3}}, \frac{1}{2}\frac{q_{43}q_{32}\frac{1}{\rho-\rho_3}}{q_{41} + q_{43}q_{32}\frac{1}{\rho-\rho_3}}, 0, \frac{1}{2}\right).$$

In the next example, three maximal paths are present, they have different lengths, and start in different elements.

Example 4.4. Consider

$$Q = \begin{pmatrix} \rho_1 & 0 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 & 0 \\ q_{31} & 0 & \rho & 0 & 0 \\ 0 & 0 & q_{43} & \rho & 0 \\ 0 & q_{52} & 0 & 0 & \rho \end{pmatrix}$$

with $0 < \rho_1 < \rho < 1$ and $q_{31}, q_{43}, q_{52} \neq 0$. We have

$$\mathcal{P}_{\max} = \{(5,2),(4,3),(4,3,1)\}, h^+_{\max} = 2, \kappa(5,2) = \kappa(4,3) = 2, \kappa(4,3,1) = 3,$$

and

$$\mathcal{P}^{(1)}_{\max} = \{(4,3,1)\}, \mathcal{P}^{(2)}_{\max} = \mathcal{P}^{(5)}_{\max} = \{(5,2)\}, \mathcal{P}^{(3)}_{\max} = \mathcal{P}^{(4)}_{\max} = \{(4,3),(4,3,1)\}.$$

Suppose that $\pi_4 \neq 0$ or $\pi_5 \neq 0$. Corollary 3.17 yields that

$$\lim_{n\to\infty} \mathbb{E}_\pi \left[\tfrac{1}{n+1} \#\big\{ m \in \{0,\dots,n\} : X_m = 1 \big\} \big| T \geq n \big\} \right] = 0,$$

and for $\ell = 2, 5$, we get

$$\begin{aligned} &\lim_{n\to\infty} \mathbb{E}_\pi \left[\tfrac{1}{n+1} \#\big\{ m \in \{0,\dots,n\} : X_m = \ell \big\} \big| T \geq n \big\} \right] \\ &= \frac{1}{2} \frac{\pi_5 q_{52}}{\pi_4 q_{43} + \pi_4 q_{43} q_{31} \frac{1}{\rho - \rho_1} + \pi_5 q_{52}}, \end{aligned}$$

and for $\ell = 3, 4$, we get

$$\begin{aligned} &\lim_{n\to\infty} \mathbb{E}_\pi \left[\tfrac{1}{n+1} \#\big\{ m \in \{0,\dots,n\} : X_m = 3 \big\} \big| T \geq n \big\} \right] \\ &= \frac{1}{2} \frac{\pi_4 q_{43} + \pi_4 q_{43} q_{31} \frac{1}{\rho-\rho_1}}{\pi_4 q_{43} + \pi_4 q_{43} q_{31} \frac{1}{\rho - \rho_1} + \pi_5 q_{52}}. \end{aligned}$$

Suppose, for instance, that the diagonal terms are given by $\rho_1 = 0.5, \rho = 0.75$ and $q_{31} = q_{43} = q_{52} = 0.1$, and the initial distribution is of the form $\pi = (*, *, *, 0.5, 0.3)$. Then an evaluation of the formulas above yields the quasi-ergodic measure

$$(0, 0.15, 0.35, 0.35, 0.15).$$

Finally, we present an example with a non-scalar diagonal matrix $Q_{\ell\ell} \in \mathbb{R}^{d_\ell \times d_\ell}$, where $d_\ell > 1$.

Example 4.5. Consider the matrix

$$Q = \begin{pmatrix} Q_{11} & 0 \\ Q_{21} & \rho_2 \end{pmatrix} \in \mathbb{R}^{3\times 3},$$

with an eventually positive matrix $Q_{11} \in \mathbb{R}^{2\times 2}, 0 \neq Q_{21} \in \mathbb{R}^{1\times 2}$, and $\rho_2 > 0$. Let the eigenvalues ρ_1, ρ_1^- of Q_{11} satisfy $\rho := \rho_1 > |\rho_1^-|$ and $\rho_1 > \rho_2$. Here the index sets are $I_1 = \{1,2\}$ and $I_2 = \{3\}$, and we have

$$\mathcal{P}_{\max} = \mathcal{P}^{(1)}_{\max} = \{(1),(2,1)\}, \mathcal{P}^{(2)}_{\max} = \{(2,1)\} \text{ and } h^+_{\max} = 1.$$

Suppose that $\alpha_\theta \neq 0$ for $\theta = (1)$ or $(2,1)$ and the initial distribution $\pi = (\pi^{(1)}, \pi^{(2)})$ satisfies $\pi^{(2)} \neq 0$. Then Theorem 3.20 yields

$$\lim_{n\to\infty} \mathbb{E}_\pi \left[\tfrac{1}{n+1} \#\big\{ m \in \{0,\dots,n\} : X_m = 3 \big\} \big| T > n \right] = 0.$$

Furthermore, let $u^{(1)}$ and $v^{(1)}$ be the left and right eigenvectors, respectively, for the eigenvalue ρ of Q_{11}, normalised by $v_1^{(1)} + v_2^{(1)} = 1$ and $u^{(1)\top} v^{(1)} = 1$. Using $\mathcal{P}_{\max} = \mathcal{P}_{\max}^{(1)}$ one obtains for $s \in \{1, 2\}$ (with $t(s) = s$) that

$$\lim_{n\to\infty} \mathbb{E}_\pi \left[\tfrac{1}{n+1} \#\{m \in \{0, \dots, n\} : X_m = s\} \middle| T > n\right] = u_s^{(1)} v_s^{(1)}. \tag{4.1}$$

As a specific example, consider the eventually positive matrix

$$Q_{11} = \begin{pmatrix} 0.2 & 0.1 \\ 0.1 & 0 \end{pmatrix} \quad \text{with } (Q_{11})^2 = \begin{pmatrix} 0.05 & 0.02 \\ 0.02 & 0.01 \end{pmatrix}$$

and eigenvalues $\rho = \rho_1 = 0.1(1 + \sqrt{2})$ and $\rho_1^- = 0.1(1 - \sqrt{2})$. The left and right normalised eigenvectors of ρ are

$$u^{(1)\top} = \frac{1}{2}\big(1 + \sqrt{2}, 1\big) \quad \text{and} \quad v^{(1)} = \frac{1}{2 + \sqrt{2}}\big(1 + \sqrt{2}, 1\big)^\top.$$

For $\theta = (1)$, one finds that the constant $\alpha_\theta = \alpha_{(1)} \neq 0$, it is determined by the decomposition

$$\mathbb{1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \alpha_\theta v^{(1)} + w_\theta = \alpha_{(1)} \frac{1}{2 + \sqrt{2}} \begin{pmatrix} 1 + \sqrt{2} \\ 1 \end{pmatrix} + w_\theta\,,$$

with w_θ in the eigenspace for the eigenvalue ρ^- of Q_{11}. Thus, for an initial distribution with $\pi^{(2)} \neq 0$, one obtains from (4.1) that

$$\lim_{n\to\infty} \mathbb{E}_\pi \left[\tfrac{1}{n+1} \#\{m \in \{0, \dots, n\} : X_m = 1\} \middle| T > n\right] = \frac{3 + 2\sqrt{2}}{4 + 2\sqrt{2}}\,,$$
$$\lim_{n\to\infty} \mathbb{E}_\pi \left[\tfrac{1}{n+1} \#\{m \in \{0, \dots, n\} : X_m = 2\} \middle| T > n\right] = \frac{1}{4 + 2\sqrt{2}}\,.$$

Appendix A. Proof of Proposition 2.1

The proof of Proposition 2.1 provided in this appendix is based on methods from Darroch and Seneta [6].

The following preparations are necessary.

Lemma A.1. *For any $j \in \{1, \dots, d\}$ and $n \in \mathbb{N}_0$, we have*

$$\sum_{\ell=0}^{n} \ell\, \mathbb{P}_\pi\big[\#\{m \in \{0, \dots, n\} : X_m = j\} = \ell \text{ and } T = n + 1\big] = \frac{\mathrm{d}}{\mathrm{d}z} \pi_j(z) Q_j^n(z) R\Big|_{z=1}\,.$$

Proof. For a given finite sequence $(x_0, x_1, \dots, x_n) \in \{1, \dots, d\}^{n+1}$, the probability $\mathbb{P}_\pi[X_i = x_i \text{ for all } i \in \{0, \dots, n\} \text{ and } T = n]$ is given by

$$\pi_{x_0} q_{x_0,x_1} q_{x_1,x_2} \cdots q_{x_{n-1},x_n} r_{x_n}\,,$$

where q_{ij} denote the entries of the matrix Q, and

$$\pi Q^n R = \sum_{x_0=1}^{d} \sum_{x_1=1}^{d} \cdots \sum_{x_n=1}^{d} \pi_{x_0} q_{x_0,x_1} q_{x_1,x_2} \cdots q_{x_{n-1},x_n} r_{x_n}\,.$$

Define for $j \in \{0, \dots, d\}$

$$\gamma_{x_0,x_1,\dots,x_n}(z) := z^{\#\{m \in \{0,\dots,n\} : x_m = j\}} \pi_{x_0} q_{x_0,x_1} q_{x_1,x_2} \cdots q_{x_{n-1},x_n} r_{x_n}\,.$$

Then it follows that

$$\pi_j(z)Q_j^n(z)R = \sum_{x_0=1}^{d}\sum_{x_1=1}^{d}\cdots\sum_{x_n=1}^{d}\gamma_{x_0,x_1,\dots,x_n}(z)\,.$$

Hence, $z \mapsto \pi_j(z)Q_j^n(z)R$ is the probability generating function for $\ell \in \{0,1,\dots,n\}$ having the probability $\mathbb{P}_\pi\big[\#\big\{m \in \{0,\dots,n\} : X_m = j\big\} = \ell \text{ and } T = n+1\big]$. This means that its expectation is given by $\frac{\mathrm{d}}{\mathrm{d}z}\pi_j(z)^\top Q_j^n(z)R\big|_{z=1}$, which finishes the proof of this lemma. □

We actually do not need this lemma, but the following lemma, which can be proved analogously.

Lemma A.2. *For any $j \in \{1,\dots,d\}$ and $n \in \mathbb{N}_0$, we have*

$$\sum_{\ell=0}^{n} \ell\,\mathbb{P}_\pi\big[\#\big\{m \in \{0,\dots,n\} : X_m = j\big\} = \ell \text{ and } T > n\big] = \frac{\mathrm{d}}{\mathrm{d}z}\pi_j(z)Q_j^n(z)\mathbb{1}\Big|_{z=1}\,.$$

We now start the proof of Proposition 2.1.

Proof of Proposition 2.1. (i) Note first that $\mathbb{P}_\pi(T > n) = \pi Q^n\mathbb{1}$. We get

$$\begin{aligned}
&\mathbb{E}_\pi\big[\tfrac{1}{n+1}\#\big\{m \in \{0,\dots,n\} : X_m = j\big\}\big|T > n\big]\\
&= \tfrac{1}{n+1}\sum_{\ell=0}^{n}\ell\,\frac{\mathbb{P}_\pi\big[\#\big\{m \in \{0,\dots,n\} : X_m = j\big\} = \ell \text{ and } T > n\big]}{\mathbb{P}_\pi(T > n)}\\
&\overset{\text{Lemma } A.2}{=} \frac{\frac{\mathrm{d}}{\mathrm{d}z}\pi_j(z)Q_j^n(z)\mathbb{1}\big|_{z=1}}{(n+1)\pi Q^n\mathbb{1}}\,.
\end{aligned}$$

(ii) The product rule implies

$$\frac{\mathrm{d}}{\mathrm{d}z}Q_j^n(z) = \sum_{m=0}^{n-1} Q_j^m(z)\left(\frac{\mathrm{d}}{\mathrm{d}z}Q_j(z)\right)Q_j^{n-1-m}(z),$$

and hence

$$\frac{\mathrm{d}}{\mathrm{d}z}Q_j^n(z)\Big|_{z=1} = \sum_{m=0}^{n-1} Q^m q_j e_j^\top Q^{n-1-m},$$

where q_j is the jth column of Q, and e_j is the jth unit vector. If Q is eventually positive, Seneta [14, Theorem 1.2] implies that

$$Q^m = \rho^m v u^\top + O(m^{d-1}|\rho'|^m)\,. \tag{A.1}$$

This yields

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}z}Q_j^n(z)\Big|_{z=1} &= \sum_{m=0}^{n-1}\Big(\rho^{n-1}vu^\top q_j e_j^\top v u^\top + O(\rho^m|\rho'|^{n-1-m}(n-1-m)^{d-1})\\
&\quad + O(\rho^{n-1-m}|\rho'|^m m^{d-1}) + O(m^{d-1}|\rho'|^m)O((n-1-m)^d|\rho'|^{n-1-m}\Big) \qquad \text{(A.2)}\\
&= n\rho^n u_j v_j v u^\top + O(n^2\rho^n),
\end{aligned}$$

since $u^\top q_j = \rho u_j$ and $e_j^\top v = v_j$. Thus,

$$\frac{\frac{\mathrm{d}}{\mathrm{d}z}\pi_j(z)Q_j^n(z)\mathbb{1}\Big|_{z=1}}{(n+1)\pi Q^n\mathbb{1}} \overset{(A.1),(A.2)}{=} \frac{(0,\dots,\pi_j,\dots,0)Q^n\mathbb{1} + \pi(n\rho^n u_j v_j v u^\top + O(\rho^n))\mathbb{1}}{(n+1)\pi(\rho^n v u^\top + O(n^{d-1}|\rho_1|^n))\mathbb{1}}$$

$$= \frac{\pi(\rho^n u_j v_j v u^\top)\mathbb{1}}{\pi(\rho^n v u^\top)\mathbb{1}} + O(\tfrac{1}{n}) = u_j v_j + O(\tfrac{1}{n})\,.$$

(iii) If Q is cyclic with period h, then by Seneta [14, Theorem 1.4] the matrix Q^h is eventually positive. Hence by assertion (ii)

$$\mathbb{E}_\pi\left[\tfrac{1}{n+1}\#\big\{m\in\{0,1,\dots,n\} : X_{mh}=j\big\}\big|T>nh\right] = u_j v_j + O(\tfrac{1}{n})$$

for the right and left normalised eigenvectors v and $u^\top$ of Q for the eigenvalue ρ, which are also eigenvectors for Q^h for the eigenvalue ρ^h. Naturally, $O(\frac{1}{nh}) = O(\frac{1}{n})$ for $n\to\infty$.

One also finds for every $\ell\in\{1,\dots,h-1\}$

$$\mathbb{E}_\pi\left[\tfrac{1}{n+1}\#\big\{m\in\{0,1,\dots,n-1\} : X_{mh+\ell}=j\big\}\big|T>nh+i\right] = u_j v_j + O(\tfrac{1}{n})$$

Summing for $\ell = 0,\dots,h-1$ one finds

$$\mathbb{E}_\pi\left[\frac{1}{nh}\#\big\{\,\big\{m\in\{0,\dots,nh\} : X_m=j\big\}\big|T>nh\big\}\right]$$

$$= \mathbb{E}_\pi\left[\frac{1}{nh}\sum_{\ell=0}^{h-1}\left[\#\big\{m\in\{0,1,\dots,(n-1)\} : X_{mh+\ell}=j\big\}\big|T>nh\right]\right]$$

$$= \frac{1}{h}\sum_{\ell=0}^{h-1}\mathbb{E}_\pi\left[\frac{1}{n}\left[\#\big\{m\in\{0,1,\dots,(n-1)\} : X_{mh+\ell}=j\big\}\big|T>nh\right]\right]$$

$$= \frac{1}{h}\sum_{\ell=0}^{h-1}\big(u_j v_j + O(\tfrac{1}{n})\big) = u_j v_j + O(\tfrac{1}{n}).$$

This concludes the proof of Proposition 2.1. ☐

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