# Stability and Stabilization of Linear Uncertain Systems — A Lyapunov Exponents Approach*

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## Abstract

Feedback stabilization of linear uncertain systems is frequently analyzed using quadratic Lyapunov functions that are common to all values in the uncertainty set. In this paper we use the alternative classical concept of Lyapunov exponents to analyze output feedback stabilization for systems with time varying uncertainties, and stabilization of linear equations via time varying linear output feedback with a priori bounded range of the gain matrices. It turns out that the eigenvalues and eigenspaces of periodic matrix functions in the systems semigroup play the same role for these problems as do the eigenvalues and eigenspaces of a constant matrix $A$ in the stability analysis of $\dot{x} = Ax$. Using this reduction to the periodic case, we can characterize the precise (exponential) stability and stabilization regions. A comparison of these results with those from other concepts of stabilization of linear uncertain systems shows differences for feedbacks with bounded range, and also differences for unbounded output feedback, i.e. there are systems whose stabilizability cannot be detected by quadratic Lyapunov functions, but can be analyzed using Lyapunov exponents. A numerical procedure, based on the solution of certain optimal control problems, is presented for the computation of the Lyapunov exponents and the precise stability region.



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# 1. Introduction

The analysis of uncertain systems has been one of the focal points in linear systems theory for the last decade. Of main interest in this area were, and are stability properties and performance design. Several approaches have been developed for the study of these problems, most notably $H^\infty$ theory (see e.g. Francis [Fr]) for frequency domain considerations, concepts based on the gap metric (see e.g. Zames and El-Sakkary [ZS] or Georgiou and Smith [GS]), Kharitonov type results on the stability of sets of polynomials (see e.g. Kharitonov [Kh]) as a transfer function approach, quadratic Lyapunov function criteria (see e.g. Rotea and Khargonekar [RK]), and the stability radius concept (see e.g. Hinrichsen and Pritchard [HP$^a$]) as state space approaches. Various connections between these approaches are discussed e.g. in Doyle et al. [DGKF], Rotea and Khargonekar [RK] and Townley and Ryan [TR]. In particular, it turns out that for a large class of uncertainties quadratic Lyapunov function criteria and considerations based on the complex stability radius are equivalent, compare the associated Riccati equations in Peterson [Pe] and Hinrichsen and Pritchard [HP$^c$], or Proposition 3 in Townley and Ryan [TR].
It is well known that exponential stability of the linear, time invariant differential equation $\dot{x} = Ax$ is equivalent to the existence of a quadratic (time invariant) Lyapunov function and to the condition that all eigenvalues of $A$ are in the left half plane of $\mathbb{C}$. For the stability of time varying equations $\dot{x} = A(t)x$ in $\mathbb{R}^d$, as they show up for uncertain systems with time varying uncertainty, it is not sufficient that for each $A(t)$, $t \geq 0$ there exists a (quadratic, time invariant) Lyapunov function, cp. in the context of robust stability e.g. Hinrichsen and Pritchard [HP$^d$]. Therefore the Lyapunov function approach to stability of uncertain systems, and the equivalent concepts, are based on the existence of a common quadratic Lyapunov function for all (time varying) uncertainties. A priori it is not clear, whether this idea leads to a description of the precise stability region.
Likewise, if for all $t \geq 0$ the eigenvalues of $A(t)$ are in the left half plane, then $\dot{x} = A(t)x$ need not be (exponentially) stable, see e.g. [Ha]. To cope with this fact, and in order to analyze the stability of small (nonlinear) perturbations of the equation $\dot{x} = A(t)x$, Lyapunov [Ly] introduced the concept of Lyapunov exponents and of Lyapunov regular matrix functions $A(t)$, $t \in \mathbb{R}$, for which there exists a decomposition of the state space $\mathbb{R}^d$ into linear subspaces, in which the exponential growth behavior of the solutions of $\dot{x} = A(t)x$ is exactly the Lyapunov exponent for $t \to \pm\infty$. The Lyapunov regular matrix functions include all matrix functions for which there exists a Lyapunov transformation, which maps $A(t)$ to a constant matrix $A$. Hence they include in particular all periodic functions; their Lyapunov exponents are logarithms of the characteristic (or Floquet) exponents, see Gantmacher [Ga] or Hahn [Ha]. While the use of Lyapunov exponents for the description of the precise (exponential) stability and instability region of uncertain linear systems seems intriguing, one has to overcome the difficulty that for a given uncertainty range $V$ of $d \times d$ matrices there usually exist functions with $A(t) \in V$ that are not Lyapunov regular.
In this paper we will show that the stability problem for linear systems with time varying uncertainties can be reduced to the study of certain periodic, linear differential equations to yield precise stability regions (which can be different from those obtained by the quadratic Lyapunov function approach). In particular, it turns out that the eigenvalues and eigenspaces of a class of fundamental matrices, associated with periodic equations,

govern the (exponential) stability, instability and stabilizability of linear, uncertain systems.

Throughout this paper we will use the following basic set up, although many results still hold for more general sets of uncertainties and feedbacks: Consider the linear system

$$\dot{x} = (A + v(t))x + B\hat{u}, \; y = Cx \tag{1.1}$$

where

$A \in gl(d, \mathbb{R})$, the real $d \times d$ matrices,
$B$ and $C$ are real matrices of dimension $d \times k$, and $l \times d$ respectively.

The uncertainties are denoted by $v$, and we assume: $V \subset gl(d, \mathbb{R})$ is a linear subspace, $V_1 \subset V$ a compact convex subset with $0 \in int\, V_1$, $cl(int\, V_1) = V_1$, where $int$ and $cl$ denote the interior and the closure with respect to the topology of $V$. Denote

$$\rho V_1 =: V_\rho \subset gl(d, \mathbb{R}), \; \rho \geq 0$$
$$\mathcal{V}_\rho = \{v : \mathbb{R} \to V_\rho, \text{ measurable}\}.$$

The elements of $\mathcal{V}_\rho$ are the time varying uncertainties of size $\rho$. This model includes in particular norm bounded and interval type uncertainties.
As inputs $\hat{u}$ we allow time invariant output feedbacks of the form $\hat{u} = FCx$ with: $\hat{U}$ is a linear subspace of the real $k \times l$ matrices, $\hat{U}_1 \subset \hat{U}$ a compact, connected subset with $int_{\hat{U}}\, \hat{U}_1 \neq \emptyset$. Denote $B\hat{U}C = U$, $B\hat{U}_1C = U_1$ and

$$\sigma U_1 =: U_\sigma \subset gl(d, \mathbb{R}), \; \sigma \geq 0.$$

Then any output feedback gain matrix $F$ corresponds to an element in $U$. With these notations, the system (1.1) can be written as

$$\dot{x} = (A + v(t))x + ux \quad \text{in } \mathbb{R}^d, \; v \in \mathcal{V}_\rho, \; u \in U_\sigma, \tag{1.2}$$

which represents a linear system with time varying uncertainties of size $\rho$ and constant output feedback of size $\sigma$.
The main problem discussed in this paper is the following: Varying $\rho$ and/or $\sigma$, find $u \in U_\sigma$ such that (1.2) is exponentially stable for all $v \in \mathcal{V}_\rho$. In particular, for $\rho$ (or $\sigma$) fixed, find the exact $\sigma$- (or $\rho$-) region, where (1.2) is exponentially stable. Note that even in the absence of uncertainty (i.e. $\rho = 0$), but with a priori bounded feedback gain (i.e. $\sigma < \infty$), this problem does not lead to the usual algebraic criteria for output feedback stabilization.
Define the Lyapunov exponents of the system (1.2) as

$$\lambda(x, v, u) = \limsup_{t \to \infty} \frac{1}{t} \log |\varphi(t, x, v, u)|, \; x \in \mathbb{R}^d \backslash \{0\}, \; v \in \mathcal{V}_\rho, \; u \in U, \tag{1.3}$$

where $\varphi(t,x,v,u)$ denotes the solution of (1.2) with $v \in \mathcal{V}_\rho$ as uncertainty, $u \in U$ as feedback, and $x \in \mathbb{R}^d \setminus \{0\}$ as initial value, i.e. $\varphi(0,x,v,u) = x$.
For each $u \in U$ denote by

(1.4) $$\kappa(\rho,u) = \sup_{v\in\mathcal{V}_\rho} \sup_{x\neq 0} \lambda(x,v,u)$$

the maximal Lyapunov exponent of (1.2) for the feedback $u$, and by

(1.5) $$\kappa(\rho,\sigma) = \inf_{u\in U_\sigma} \kappa(\rho,u)$$

the minimal exponent of the $\kappa(\rho,u)$'s, which can be achieved by a feedback in $U_\sigma$. Our main problem then has the following obvious answer:
Given $u \in U$, then (1.2) is exponentially stable for all uncertainties $v \in \mathcal{V}_\rho$, iff $\kappa(\rho,u) < 0$. And the system (1.2) is exponentially stabilizable by a feedback of size $\sigma$ for all uncertainties of size $\rho$, iff $\kappa(\rho,\sigma) < 0$.
This paper is devoted to the analysis of the quantities (1.3)–(1.5). In Section 2 we will reduce the problem to considering only periodic linear differential equations by analyzing the eigenvalues and eigenspaces of certain fundamental matrices. This is done via control analysis of a system on the space of directions in $\mathbb{R}^d$, i.e. on the projective space $\mathbb{P}^{d-1}$ in $\mathbb{R}^d$, see [CK[a]]. In Section 3 the system (1.2) is considered with $u \equiv 0$. It is shown that stabilization, where the $v \in \mathcal{V}_\rho$ are treated as controls, and destabilization, when $\mathcal{V}_\rho$ is interpreted as uncertainty, are dual concepts. We develop criteria for the problem that a matrix is stable under all uncertainties $\mathcal{V}_\rho$ for all $\rho \geq 0$, and compare the stability radius derived from the Lyapunov exponents approach with those for time invariant uncertainties and those obtained via quadratic Lyapunov functions. In Section 4 the main problem is treated. We prove continuity and monotonicity properties of the exponential growth coefficients (1.4) and (1.5), based on the results of Sections 2 and 3. A variety of examples is presented to show the exact stabilization regions in comparison to other approaches. In particular, there are systems for which the various stabilization radii differ for bounded feedback intervals, and there are systems whose stabilizability via output feedback cannot be detected by using quadratic Lyapunov functions, but they can be analyzed precisely via Lyapunov exponents.
It is obvious from the problem formulation that in general there will be no explicit formulae for $\kappa(\rho,u)$ and $\kappa(\rho,\sigma)$, nor for the regions in which $\kappa(\rho,u)$ and $\kappa(\rho,\sigma)$ are negative. (However, in some cases asymptotic formulae for $\sigma \to \infty$ or $\rho \to \infty$ are available.) Therefore the $\kappa$'s and the corresponding stability regions have to be computed numerically. In Section 5 we present a general method for the computation of these quantities, based on numerical solution of certain (nonlinear) optimal control problems. The method is illustrated by computation of stability and stabilization radii for a two and a three dimensional system.

## 2. Lyapunov Exponents and Periodic Systems

In this section we consider a family of linear differential equations

$$\dot{x} = A(t)x \quad \text{in } \mathbb{R}^d \tag{2.1}$$

with $A \in \mathcal{A} = \{A : \mathbb{R} \to \Omega \subset gl(d, \mathbb{R}), \text{ measurable }\}$, where $\Omega$ is compact and convex. One can interpret (2.1) as a bilinear control system with bounded control range $\Omega$. Denote by $\varphi(t, x, A)$ the solutions of (2.1) with initial value $x \in \mathbb{R}^d$, i.e. $\varphi(0, x, A) = x$. The Lyapunov exponents of (2.1) are given by

$$\lambda(x, A) = \limsup_{t \to \infty} \frac{1}{t} \log |\varphi(t, x, A)|, \; x \neq 0, \tag{2.2}$$

where $|\cdot|$ is any norm in $\mathbb{R}^d$. Lyapunov [Ly] showed that for each $A \in \mathcal{A}$ there are at most $d$ different values $\lambda(x, A)$. If $A(t) \equiv A$, then these numbers are the real parts of the eigenvalues of $A$, say $\lambda_1 < \ldots < \lambda_k$ with corresponding (generalized) eigenspaces $E_1, \ldots, E_k$. Now for $i = 1 \ldots k$ we have $\lambda(x, A) = \lambda_i$ iff $x \in \bigoplus_{j=1}^{i} E_j \backslash \bigoplus_{j=1}^{i-1} E_j$. If $A(t)$ is periodic with period $T > 0$, denote by $\Phi_A$ a fundamental matrix of $\dot{x} = A(t)x$ at time $T$, and by $\mu_1, \ldots, \mu_k$ its (unique) set of characteristic (or Floquet) numbers. Then $\frac{1}{T} \log |\mu_1|, \ldots, \frac{1}{T} \log |\mu_k|$ are the Lyapunov exponents of the equation and again they are realized from $x \neq 0$ iff $x$ is in the corresponding (generalized) eigenspace of $\Phi_A$ (cp. e.g. Hahn [Ha]). The exponential stability behavior of $\dot{x} = A(t)x$ is governed by its Lyapunov exponents, and for the family (2.1) we are interested in the following growth rates:

$$\begin{cases} \kappa = \sup\limits_{A \in \mathcal{A}} \sup\limits_{x \neq 0} \lambda(x, A); \\ \kappa^* = \inf\limits_{A \in \mathcal{A}} \inf\limits_{x \neq 0} \lambda(x, A). \end{cases} \tag{2.3}$$

Clearly $\kappa < 0$ iff all equations (2.1) are exponentially stable, and $\kappa^* < 0$ iff there exists $A \in \mathcal{A}$ and $x \in \mathbb{R}^d$ such that $\varphi(t, x, A) \to 0$ exponentially as $t \to \infty$.
The analysis of Lyapunov exponents for (2.1) is greatly simplified by the following simple observation: $\lambda(x, A) = \lambda(\alpha \cdot x, A)$ for all $\alpha \in \mathbb{R}$, $\alpha \neq 0$. Therefore it suffices to consider the Lyapunov exponents in each direction in $\mathbb{R}^d$, e.g. by looking only at points with Euclidian norm 1, and furthermore identifying $x$ and $-x$. This leads to the projective space $\mathbb{P}^{d-1}$ in $\mathbb{R}^d$, which we will denote by $\mathbb{P}$, since the dimension will always be clear from the context. On the space $\mathbb{P}$ of directions the family (2.1) induces the equations

$$\dot{s} = h(s, A) = (A(t) - s^{\mathrm{T}} A(t) s Id)s, \; s \in \mathbb{P} \tag{2.4}$$

where $^{\mathrm{T}}$ denotes transpose and $Id$ the $d \times d$ identity matrix. (Just project (2.1) onto $\mathbb{P}$ and compute the projected vectorfields.) Again the family (2.4) can be considered as a control system, now on the manifold $\mathbb{P}$. A straightforward application of the chain rule yields for the Lyapunov exponents in terms of the system on $\mathbb{P}$:

$$(2.5)\qquad \lambda(x_0, A) = \limsup_{t\to\infty} \frac{1}{t} \int_0^t s^T(\tau, s_0, A) A(\tau) s(\tau, s_0, A) d\tau,$$

where $s(\tau, s_0, A)$ denotes the solution of (2.3) with initial value $s_0 = \pi_{\mathbb{P}} x_0$. Here $\pi_{\mathbb{P}}$ is the projection of $\mathbb{R}^d$ onto the projective space $\mathbb{P}$.

In order to avoid notational complications, we will assume the following local accessibility condition for the control system (2.4):

$$\text{(H)}\qquad \dim \mathcal{LA}\{h(\cdot, A);\ A \in \Omega\}(s) = d - 1 \text{ for all } s \in \mathbb{P}^{d-1}.$$

Here $\mathcal{LA}\{h(\cdot, A);\ A \in \Omega\}$ denotes the Lie algebra generated by the vectorfields $h(\cdot, A)$, $A \in \Omega$, and (H) means that at every point $s \in \mathbb{P}$ the subspace of the tangent space $T_s\mathbb{P}$ obtained by evaluating all vectorfields in $\mathcal{L}$ at $s$ coincides with $T_s\mathbb{P}$.

This condition is well-known in nonlinear control theory and guarantees that for all $t > 0$ the set of points reachable from a given point in $\mathbb{P}$ up to time $t$ has nonvoid interior (cp. e.g. [NvS], [Is]).

To restrict the analysis of Lyapunov exponents to periodic equations, we introduce a suitable class of matrix functions in $\mathcal{A}$, namely the piecewise constant periodic ones and their fundamental matrices:

Define the positive semigroup of (2.1) by

$$\mathcal{S}^+ = \{e^{t_n A_n} \cdot \ldots \cdot e^{t_1 A_1};\ A_i \in \Omega,\ t_i \geq 0,\ i = 1 \ldots n \in \mathbb{N}\}$$

and the associated group as

$$\mathcal{G} = \{e^{t_n A_n} \cdot \ldots \cdot e^{t_1 A_1};\ A_i \in \Omega,\ t_i \in \mathbb{R},\ i = 1 \ldots n \in \mathbb{N}\}.$$

Define furthermore for $t \geq 0$ $\mathcal{S}^+_{\leq t}$ as the subset of $\mathcal{S}^+$ with $\sum t_i \leq t$, and similary the negative semigroup $\mathcal{S}^-$ using $t_i \leq 0$ and $\mathcal{S}^-_{\leq t}$ via $\sum |t_i| \leq t$. Let $\mathcal{S}^+_t$ be the set of elements in $\mathcal{S}^+$ with $\sum t_i = t$.

Note that $\mathcal{S}^+ = \bigcup_{t \geq 0} \mathcal{S}^+_{\leq t}$ and for each $t > 0$ an element $g \in \mathcal{S}^+_{\leq t}$ corresponds to a piecewise constant, periodic matrix function $A_g$ via:

Let $g = e^{t_n A_n} \cdot \ldots \cdot e^{t_1 A_1}$ with $T = \sum_{i=1}^n t_i$, $T \leq t$, define

$A_g(\tau) = A_i$ for $\tau \in [t_1 + \ldots + t_{i-1}, t_1 + \ldots + t_i)$ with $t_0 = 0$, and continue $T$-periodically.

Vice versa, each piecewise constant, $T$-periodic function $A(\tau)$ corresponds to an element $g_A$ in $\mathcal{S}_{\leq T}$.

Each $g \in \mathcal{G}$ is an invertible matrix and $\mathcal{G}$ acts on $\mathbb{R}^d$ via $g(x) = g \cdot x$ for $x \in \mathbb{R}^d$. One can also define an action of $\mathcal{G}$ on $\mathbb{P}$ as $\psi : \mathcal{G} \times \mathbb{P} \to \mathbb{P}$, $\psi(g, s) = \pi_{\mathbb{P}}(g \cdot s)$. Since $g$ is linear, we have for $x \in \mathbb{R}^d$ that $\pi_{\mathbb{P}}(g \cdot \alpha x) = \pi_{\mathbb{P}}(g \cdot x)$ for $\alpha \neq 0$, i.e. $\mathcal{G}$ respects directions in $\mathbb{R}^d$. Thus $\psi$ is well-defined.

Using this set up, we have that $\mathcal{G}$ is a Lie subgroup of $Gl(d, \mathbb{R})$, the invertible $d \times d$ matrices, and $\mathcal{G}$ acts transitively on $\mathbb{P}$. Furthermore, $int\ \mathcal{S}^+_{\leq t} \neq \emptyset$ for all $t > 0$, and $cl(int\ \mathcal{S}^+_{\leq t}) = cl(\mathcal{S}^+_{\leq t})$, here *int* and *cl* are taken w.r.t. $\mathcal{G}$, compare e.g. [SJ]. Furthermore, if we equip $\mathcal{A} \subset L^\infty(\mathbb{R}, \mathbb{R}^d) = (L^1(\mathbb{R}, \mathbb{R}^d))^*$ with the weak * topology, then the piecewise constant, periodic elements of the form $A_g$ are dense in $\mathcal{A}$.

Our candidates for periodic functions are the $A_g$, $g \in int\ \mathcal{S}$. The eigenvalues and the eigenspaces of these $g$'s come into the picture, when we consider the controllability properties of the system (2.4) on $\mathbb{P}$. Recall the definition of control sets from [CK^b]:

**2.1. Definition:** A set $D \subset \mathbb{P}$ is called a control set of (2.4), if for all $s \in D$ one has $cl\ \mathcal{O}^+(s) \supset D$, and $D$ is maximal with this property. Here $\mathcal{O}^+(s_0) = \{y \in \mathbb{P};$ there is $t \geq 0$ and $A \in \mathcal{A}$ with $s(t, s_0, A) = y\}$.

Note that $cl\ \mathcal{O}^+(s) = cl(\mathcal{S}s)$ for all $s \in \mathbb{P}$.

**2.2. Remark:** There exist exactly one invariant control set $C$ and one open control set $C^-$ for (2.4) on $\mathbb{P}$. $C$ has nonvoid interior and $cl(int\ C) = C$.

The following theorem shows that eigenspaces of elements $g \in int\ \mathcal{S}$ in the interior of the systems semigroup determine the control sets $C$ and $C^-$ on $\mathbb{P}$:

**2.3. Theorem:** All points $s \in C^- \cup int\ C$ are eigenvectors for a real eigenvalue of some $g \in int\ \mathcal{S}_{\leq t}$ for some $t > 0$. Furthermore, for all $t > 0$, all $g \in int\ \mathcal{S}_{\leq t}$ we have $\pi_{\mathbb{P}} \oplus E(\lambda_{\min}(g)) \subset C^-$ and $\pi_{\mathbb{P}} \oplus E(\lambda_{\max}(g)) \subset int\ C$. Here the direct sum is taken over all eigenvalues $\lambda_{\min}(g)$ of minimal modulus and all eigenvalues $\lambda_{\max}(g)$ of maximal modulus, respectively, and $E(\lambda(g))$ is the corresponding (generalized) eigenspace; $\pi_{\mathbb{P}}$ is again the projection of $\mathbb{R}^d \backslash \{0\}$ onto $\mathbb{P}$.

This theorem is a partial restatement of Theorem 3.10 in [CK^b]. We have thus identified the eigenspaces of elements in $int\ \mathcal{S}$ as those directions for which the system (2.4) on the projective space is controllable. The corresponding eigenvalues are related to the exponential growth rates $\kappa$ and $\kappa^*$ in the following way:

**2.4. Theorem:**

(i) $\kappa = \sup\{\frac{1}{t} \log|\lambda_{\max}(g)|;\ t > 0,\ g \in \mathcal{S}_t \cap int\ \mathcal{S}\}$
$\quad = \sup\{\frac{1}{t} \log|\lambda(g)|;\ t > 0,\ g \in \mathcal{S}_t \cap int\ \mathcal{S},\ \pi_{\mathbb{P}} E(\lambda(g)) \subset int\ C\}$

(ii) $\kappa^* = \inf\{\frac{1}{t} \log|\lambda_{\min}(g)|;\ t > 0,\ g \in \mathcal{S}_t \cap int\ \mathcal{S}\}$
$\quad = \inf\{\frac{1}{t} \log|\lambda(g)|;\ t > 0,\ g \in \mathcal{S}_t \cap int\ \mathcal{S},\ \pi_{\mathbb{P}} E(\lambda(g)) \subset C^-\}$

**Proof:** By Theorem 4.1 in [CK^f] we have

$$\kappa = \limsup_{t \to \infty} \frac{1}{t} \sup_{g \in \mathcal{S}_t} \log|\lambda_{\max}(g)|.$$

Hence for $\varepsilon > 0$ there exist $t > 0$ and $g \in \mathcal{S}_t$ such that $\frac{1}{t} \log|\lambda_{\max}(g)| > \kappa - \varepsilon$. But $cl\ \mathcal{S} = cl\ int\ \mathcal{S}$, and therefore there exists $g' \in \mathcal{S}_t \cap int\ \mathcal{S}$ with $\frac{1}{t} \log|\lambda_{\max}(g')| > \kappa - 2\varepsilon$, because the eigenvalues of a matrix depend continuously on the matrix coefficients. This implies $\kappa \le \sup\{\frac{1}{t}\log|\lambda_{\max}(g)|;\ t>0,\ g\in \mathcal{S}_t\cap int\mathcal{S}\}$. The reverse inequality is clear from the definitions. The second characterization of $\kappa$ follows from Theorem 2.3. Again, by Theorem 4.1 in [CK[f]] we know

$$\kappa^* = \liminf_{t\to\infty} \frac{1}{t} \inf_{g\in \mathcal{S}_t} \log|\lambda_{\min}(g)|$$

and (ii) is obtained in the same way as (i), using again Theorem 2.3 for the second characterization of $\kappa^*$.

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The exponential growth rates $\kappa$ and $\kappa^*$ play a crucial role in the stability analysis of uncertain linear systems via Lyapunov exponents. So far we have shown that they are determined by the eigenvalues and eigenspaces of piecewise constant, periodic equations. In fact, it will be shown in [CK[g]], that the complete stability behavior of the family (2.1) can be reduced to the spectral analysis of $int\ \mathcal{S}$.
In dimension $d = 2$, the supremal (and the infimal) Lyapunov exponent can actually be realized by a periodic solution (corresponding to a not necessarily piecewise constant control), as shown by the following result from [CK[g]]:

**2.5. Theorem:** Let $d = 2$ in system (2.1). Then there are $T > 0$ and a $T$-periodic element $A \in \mathcal{A}$ such that the corresponding fundamental solution $g_A$ of (2.1) satisfies

$$\kappa = \frac{1}{T}\log|\lambda_{\max}(g_A)|.$$

Furthermore, the Lyapunov exponent $\kappa$ is either attained in an equilibrium or in a periodic trajectory on $\mathbb{P}^1$ such that in the parametrization by the angle $\psi$

$$\psi(T) - \pi = \psi(0) \text{ or } \psi(T) + \pi = \psi(0).$$

Finally, we note a result that shows from which initial values the growth rates in (2.3) can be realized. This observation will be useful for the discussion of stabilization in the next section.

**2.6. Proposition:**
(i) For all $\varepsilon > 0$ and all $x \in \mathbb{R}^d\backslash\{0\}$ there exists $A \in \mathcal{A}$ with $\lambda(x, A) > \kappa - \varepsilon$.
(ii) For all $\varepsilon > 0$ and all $x \in \mathbb{R}^d\backslash\{0\}$ with $\pi_{\mathbb{P}} x \in C^-$ there exists $A^* \in \mathcal{A}$ with $\lambda(x, A^*) < \kappa^* + \varepsilon$.
The proof of this proposition uses Theorem 2.4 and the control structure of (2.4) on the compact state space $\mathbb{P}$, see [CK[f]], Theorem 5.1.

## 3. Stabilization and Destabilization

In this section we analyze the stability of the linear system

$$\dot{x} = Ax + v(t)x \quad \text{in } \mathbb{R}^d, \tag{3.1}$$

where $v \in \mathcal{V}_\rho$ as defined in Section 1. We will use the notations and the results from Section 2 with $\Omega = A + V_\rho$.
In equation (3.1) the term $v(t)x$ can be interpreted in various ways, e.g.
(a) $v \in \mathcal{V}_\rho$ describes uncertainties of size $\rho$,
(b) $\mathcal{V}_\rho$ is a set of time varying, bounded controls.
For (a) we will consider the following problem: Given a stable matrix $A$ and the family of uncertainties $\{\mathcal{V}_\rho,\ \rho \geq 0\}$, find the largest $\rho$ such that (3.1) is exponentially stable for all $v \in \mathcal{V}_\rho$.
On the other hand, if $A$ is unstable, we are interested in finding the smallest $\rho$ such that there are $x \in \mathbb{R}^d \backslash \{0\}$ and $v \in \mathcal{V}_\rho$ with $\varphi(t, x, v) \to 0$ exponentially as $t \to \infty$. (The problem of constructing (non continuous) feedbacks to achieve this will be addressed elsewhere.) At the end of this section we will briefly discuss stability properties of (3.1), where $v(t)$ is a stochastic process.
We continue to assume condition (H) from Section 2.

First, we will analyze (3.1) as an uncertain system, and we are interested in the following function (compare (2.3))

$$\kappa(\rho) = \sup_{v \in \mathcal{V}_\rho} \sup_{x \neq 0} \lambda(x, A + v). \tag{3.2}$$

**3.1. Theorem:** Let $A \in gl(d, \mathbb{R})$, and $\mathcal{V}_\rho$ as in Section 1. Then the function $\kappa(\rho)$ is continuous and increasing for $\rho \in [0, \infty)$.

**Proof:** Monotonicity is obvious from the definitions. Lower semicontinuity (here equivalent to continuity from the left) can be seen as follows: By Theorem 2.4(i), for every $\rho > 0$ and every $\varepsilon > 0$ there is $g \in \mathcal{S}_t^\rho \cap int\ \mathcal{S}^\rho$ (here $\mathcal{S}^\rho$ denotes the respective system semigroup) with

$$\kappa(\rho) \leq \frac{1}{t} \log |\lambda_{\max}(g)| + \varepsilon.$$

Since $int\ \mathcal{S}$ is path-connected and eigenvalues depend continuously on the entries of the matrix, there is for $\rho - \rho'$ small enough an element $g \in \mathcal{S}_t^{\rho'} \cap int\ \mathcal{S}^{\rho'}$ with

$$\frac{1}{t} \log |\lambda_{\max}(g')| \geq \frac{1}{t} \log |\lambda_{\max}(g)| - \varepsilon,$$

hence

$$\kappa(\rho') \geq \kappa(\rho) - 2\varepsilon.$$

Upper semicontinuity is a consequence of the following observation: By the result of Section 2, $\kappa(\rho)$ is the supremum over eigenvalues of periodic matrix functions. Hence $\kappa(\rho)$ coincides with the supremal Bohl exponent describing uniform exponential growth behavior (cp. also Theorem 5 in [CK[c]]). Upper semicontinuity of maximal Bohl exponents is a classical result (cp. [DK], Theorem III.4.6).

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**3.2. Remark:** In general, Lyapunov exponents do not depend continuously on the resp. matrix (cp. e.g. Hahn [Ha] or for a recent contribution [LY]). Hence the result above on continuity of the supremal Lyapunov exponents appears remarkable.
The function $\kappa(\rho)$, $\rho \geq 0$, will allow us to describe the following stability radius in a more explicit way.

**3.3. Definition:** For a matrix $A \in gl(d, \mathbb{R})$ and a family $\{\mathcal{V}_\rho;\ \rho \geq 0\}$ of uncertainties denote by

$$r(A) = \inf\{\rho;\ \text{there is } v \in \mathcal{V}_\rho \text{ such that } \dot{x} = [A + v(t)]x \text{ is not exponentially stable}\}$$

the stability radius of $A$.

**3.4. Proposition:** For a matrix $A \in gl(d, \mathbb{R})$ and a family $\{\mathcal{V}_\rho;\ \rho \geq 0\}$ of uncertainties the stability radius $r(A)$ can be characterized by Lyapunov exponents in the following way:

$$r(A) = \begin{cases} \min\{\rho;\ \kappa(\rho) = 0\} & \text{if this set is not empty} \\ 0 & \text{if } \kappa(0) > 0, \text{ i.e. if } A \text{ is unstable} \\ \infty & \text{otherwise.} \end{cases}$$

**Proof:** Clear from Theorem 3.1.

∎

**3.5. Remark:** Our definition of the time-varying stability radius is based on exponential stability, i.e. Lyapunov exponents. An alternative definition in Hinrichsen et al. [HIP] uses Bohl exponents, i.e. uniform exponential stability. As observed in the proof of Theorem 3.1, the supremal Bohl and Lyapunov exponents are equal. Hence the the two concepts agree (for constant $A$).

As a first estimate for the stability radius we obtain:

**3.6. Proposition:**
(i) $r(A) > 0$ iff $A$ is stable.
(ii) $r(A) = \infty$ iff $A$ is stable and there exists a transformation matrix $T \in Gl(d, \mathbb{R})$ such that $TVT^{-1}$ consists only of skew symmetric matrices.

**Proof:**

(i) If $A$ is stable, then $\kappa(0) = \max\{\text{Re } \mu;\ \mu \in \text{spec}(A)\} < 0$, where spec$(A)$ denotes the set of eigenvalues of $A$. Hence $r(A) > 0$ by continuity of $\kappa(\rho)$. Vice versa, if $r(A) > 0$ then there exists $\rho > 0$ with $\kappa(\rho) < 0$. But $A \in A + \mathcal{V}_\rho$ for all $\rho > 0$.

(ii) If $A$ is stable and there exists a transformation matrix $T$ such that $TVT^{-1}$ consists only of skew symmetric matrices, then for all $v \in \mathcal{V}_\rho$ and all $x \neq 0$

$$\lambda(x, A + v) = \lambda(x, A) < 0.$$

Thus $r(A) = \infty$. For the converse note that $r(A) = \infty$ trivially implies stability of $A$. If there is no transformation $T$ such that $TVT^{-1}$ consists only of skew symmetric matrices then $tr(A + v) = tr\ A + tr\ v$ is not constant and can be made positive by appropriate choice of $v \in V$, i.e. by choosing $\rho$ large enough.
On the other hand, Theorem 4.5(ii)(c) in [CK[f]] shows that the system semigroup $\mathcal{G}$ is compact if $\kappa^* = \kappa = 0$. By stability of $A$ this cannot hold, hence $\mathcal{G}$ is not compact. Thus by assertion (ii)(b) in the same theorem, there exists $\rho > 0$ such that

$$\kappa(\rho) = \frac{1}{d} \max_{v \in \mathcal{V}_\rho} tr(A + v) > 0$$

where the strict inequality follows by the considerations above. Hence $r(A) < \infty$.

∎

Next, we compare the radius $r(A)$ with other stability radii available in the literature:

**3.7. Definition:** The stability radius with respect to time constant perturbations in $\mathcal{V}_\rho$ is defined by

$$r_{\mathbb{R}}(A) = \inf\{\rho \geq 0;\ \sup_{v \in \mathcal{V}_\rho} \max_{\mu \in \text{spec}(A+v)} \text{Re } \mu \geq 0\},$$

and the radius given by quadratic Lyapunov functions is

$$\begin{aligned} r_{Lf}(A) = \sup\{\rho \geq 0;\ & \text{there exist a positive definite matrix } P \in Gl(d, \mathbb{R}) \text{ and} \\ & \alpha > 0 \text{ such that for all } (x, v) \in \mathbb{R}^d \times \mathcal{V}_\rho \text{ we have} \\ & x^{\mathrm{T}}(P(A + v) + (A + v)^{\mathrm{T}} P)x \leq -\alpha |x|^2\}. \end{aligned}$$

For the definition of $r_{\mathbb{R}}(A)$ see e.g. Hinrichsen and Pritchard [HP[c]], and $r_{Lf}(A)$ is implicit in much of the quadratic stability literature, see e.g. Rotea and Khargonekar [RK]. Recall that for a large class of uncertainties $r_{Lf}$ coincides with the complex stability radius of Hinrichsen and Pritchard, see e.g. [TR] and [CK[e]].

**3.8. Proposition:** Let $A \in gl(d, \mathbb{R})$ be stable. Then

(i) $0 < r_{Lf}(A) \leq r(A) \leq r_{\mathbb{R}}(A)$.

(ii) $r_{\mathbb{R}}(A) = \infty$ if $r(A) = \infty$ iff $r_{Lf}(A) = \infty$.

(iii) All inequalities in (i) can be strict.

**Proof:**

(i) follows directly from the definitions. For (iii) compare Example 3.9 below.

(ii) Both 'if' are clear from (i). Now suppose that $r(A) = \infty$. Then by Proposition 3.6(ii) there exists $T \in Gl(d, \mathbb{R})$ such that $TVT^{-1}$ consists of skew symmetric matrices. Thus we may assume that $V$ consists of skew symmetric matrices. Since $A$ is stable there is a positive definite $P \in Gl(d, \mathbb{R})$ with $x^T(PA + AP)x \leq -\alpha|x|^2$ for all $x \in \mathbb{R}^d$, some $\alpha > 0$. Then $x^T[P(A+v) + (A+v)^T P]x = x^T(PA + A^T P)x + x^T(Pv + v^T P)x = x^T(PA + A^T P)x \leq -\alpha|x|^2$.

∎

The following example shows that the stability radii from Definitions 3.3 and 3.7 can disagree for simple, 2-dimensional systems.

**3.9. Example:** The linear oscillator with uncertain restoring force.
Consider the linear oscillator $\ddot{y} + 2b\dot{y} + (1 + v(t))y = 0$, or with $x = (x_1, x_2)^T = (y, \dot{y})^T$

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & -2b \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ -v(t) & 0 \end{pmatrix} x,$$

where $V = \left\{ \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix} ;\ \alpha \in \mathbb{R} \right\}$ and $V_\rho = \left\{ \begin{pmatrix} 0 & 0 \\ \beta & 0 \end{pmatrix} ;\ \beta \in [-\rho, \rho] \right\}$ for $\rho \geq 0$.

For each damping $b \in \mathbb{R}$ we denote by $\kappa(\rho, b)$ the maximal Lyapunov exponents, as defined in (3.2), and by $r(b)$ the stability radius as in Definition 3.3, i.e. the zero level curve of $\kappa(\rho, b)$. The computation of $\kappa(\rho, b)$ was described in [CK[f]], Section 6, and $r(b)$ can be seen in Figure 2b of [CK[f]].

It is easy to see from the computation of eigenvalues that

$$r_{\mathbb{R}}(b) = \begin{cases} 1 & \text{for } b > 0 \\ 0 & \text{for } b \leq 0. \end{cases}$$

Furthermore, the stability radius that is achievable via common quadratic Lyapunov functions, can be computed explicitly, see Example 4.3 in [CK[e]]. One obtains

$$r_{Lf}(b) = \begin{cases} 1 & \text{for } u \geq \frac{1}{\sqrt{2}} \\ 2b\sqrt{1-b^2} & \text{for } 0 \leq u \leq \frac{1}{\sqrt{2}} \\ 0 & \text{for } u \leq 0. \end{cases}$$

All three radii are shown in Figure 1, depending on the damping $b$ for $b \geq 0$.

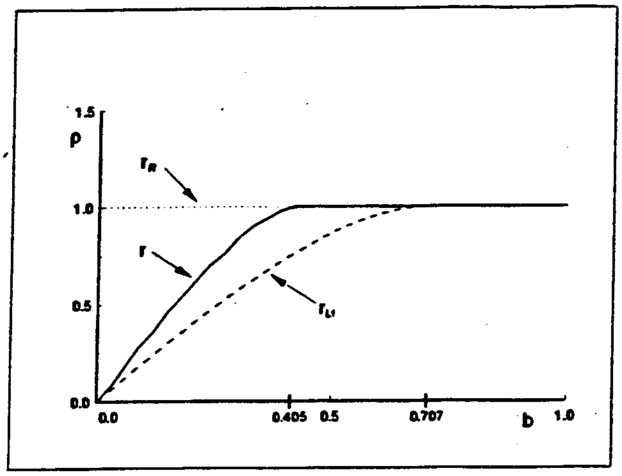


*Figure 1:* The stability radii for the linear oscillator

Next, we will turn our attention to interpretation (b) of $\mathcal{V}_\rho$ as a set of time varying, a priori bounded controls. We define

$$\kappa^*(\rho) = \inf_{v \in \mathcal{V}_\rho} \inf_{x \neq 0} \lambda(x, A + v). \tag{3.3}$$

The analysis of $\kappa^*(\rho)$ follows exactly the same lines as the one of $\kappa(\rho)$ above, by the following duality result: Denote by

$$\dot{y} = -(Ay + v(t)y) \quad \text{in } \mathbb{R}^d \tag{3.4}$$

the time reversed system of (3.1) with the same family $\{\mathcal{V}_\rho;\ \rho \geq 0\}$. Let $\kappa_-(\rho)$ and $\kappa_-^*(\rho)$ be the growth rates of (3.4) according to (3.2) and (3.3).

**3.10. Theorem:** Consider the system (3.1) and its time reversed system (3.4) with the notations above. Then for all $\rho \geq 0$

$$-\kappa^*(\rho) = \kappa_-(\rho) \text{ and } -\kappa(\rho) = \kappa_-^*(\rho). \tag{3.5}$$

This theorem is an easy consequence of Theorem 4.1 in [CK$^f$]. In particular, continuity and monotonicity of $\kappa^*(\rho)$ follow from Theorem 3.1. We can use $\kappa^*(\rho)$ in order to analyze the following instability radius.

**3.11. Definition:** For a matrix $A \in gl(d, \mathbb{R})$ and a family $\{\mathcal{V}_\rho;\ \rho \geq 0\}$ of controls, denote by

$$r^*(A) = \inf\{\rho \geq 0;\ \text{there exist } x \in \mathbb{R}^d \backslash \{0\} \text{ and } v \in \mathcal{V}_\rho \text{ such that } \varphi(t, x, v) \to 0 \text{ exponentially as } t \to \infty\}$$

the instability radius of $A$ w.r.t. time varying controls in $\{\mathcal{V}_\rho;\ \rho \geq 0\}$.
A characterization analogous to Proposition 3.4 can be given as follows.

**3.12. Proposition:** The instability radius satisfies

$$r^*(A) = \begin{cases} \max\{\rho;\ \kappa^*(\rho) = 0\} & \text{if this set is nonempty} \\ 0 & \text{if } \kappa^*(0) < 0 \\ \infty & \text{otherwise.} \end{cases}$$

One sees easily that $\rho > r^*(A)$ means that for all $0 \neq x \in \mathbb{R}^d$ with $\pi_{\mathbf{P}} x \in C^-$ there is $v \in \mathcal{V}_\rho$ such that $\varphi(t, x, v) \to 0$ exponentially as $t \to \infty$. Also the results 3.6 and 3.8 have obvious analogues for this concept of stabilization, where for the corresponding definition of $r^*_{Lf}(A)$ and the pertinent statements in Proposition 3.8 we have to assume that $A$ is totally unstable, i.e. all eigenvalues are in the right halfplane of $\mathbb{C}$.

**3.13. Example:** Stabilization of the linear oscillator with controlled restoring force. We continue Example 3.9, but now for $b \leq 0$, i.e. for the unstable case. We denote by $\kappa^*(\rho, b)$ the minimal Lyapunov exponent, as defined in (3.3), depending on $b \leq 0$. The computation of $\kappa^*(\rho, b)$ was described in [CK$^f$], Section 6, and the corresponding instability radius $r^*(b)$ is the zero level curve from Figure 4b in [CK$^f$]. Again computing eigenvalues shows that $r^*_{\mathbb{R}}(b) = 1$ for $b < 0$, and $r^*_{\mathbb{R}}(b) = 0$ for $b = 0$. Here $r^*_{\mathbb{R}}$ denotes the largest $\rho$ such that all matrices in $A + V_\rho$ have only eigenvalues with nonnegative real parts.
The radii $r^*(b)$ and $r_{\mathbb{R}}(b)$ are shown in Figure 2. The duality between $\kappa(\rho, b)$ and $\kappa^*(\rho, b)$ shows up by comparing Figures 1 and 2 as the fact that $r(b) = r^*(-b)$ for $b \geq 0$.

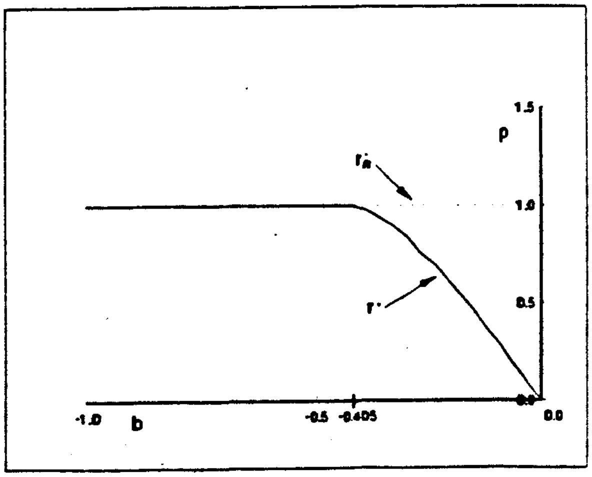


*Figure 2:* The instability radii for the linear oscillator

**3.14. Remark:** The uniform stabilization problem for all initial values $x \neq 0$ leads to the definition of the following quantity:

$$\bar{\kappa}(\rho) = \inf_{v \in \mathcal{V}_\rho} \sup_{x \neq 0} \lambda(x, A + v),$$

which was analyzed in [CK[f]]. Thus $\bar{\kappa}(\rho)$ describes the smallest Lyapunov exponent that can be realized from all $x \neq 0$. It follows from [CK[f]] that

$$\begin{aligned} \bar{\kappa}(\rho) &= \inf_{v \in \mathcal{V}_\rho} \sup_{s \in C} \lambda(x, A + v) \\ &= \inf_{g \in int\ S} \lambda_{\max}(g) = \inf_{g \in int\ S} \{\lambda_{\max}(g);\ \pi_{\mathbf{P}} E(\lambda_{\max}(g)) \subset int\ C\}. \end{aligned}$$

Furthermore, $\bar{\kappa}(\rho)$ is left continuous in $\rho$, which can be seen in the same way as in the first part of the proof of Theorem 3.1. Let us define

$$\bar{r}(A) = \begin{cases} \max\{\rho;\ \bar{\kappa}(\rho) = 0\} & \text{if this set is nonempty} \\ 0 & \text{if } \bar{\kappa}(0) < 0, \text{ i.e. if } A \text{ is stable} \\ \infty\ , & \text{otherwise.} \end{cases}$$

The uniform instability radius $\bar{r}(A)$ can be interpreted as the precise stabilization region of a linear system with linear, time varying output feedback: Consider the system $\dot{x} = Ax + B\hat{u}$, $y = Cx$ with output feedbacks $\hat{u} = F(t)Cx$. Then we can write $\dot{x} = Ax + v(t)x$, where $v(t) \in V_\rho = B\hat{V}_\rho C$, and $\hat{V}_\rho$ is the range of the gain matrices $F(t)$. In this set up we have $\rho > \bar{r}(A)$ iff the linear system is stabilizable with linear, time varying output feedbacks of size $\rho$.

**3.15. Remark:** The case of stochastic uncertainties has been discussed by Willems and Willems [WW] and, in the context of Lyapunov exponents, by the authors in [CK[c]]. Loosely speaking it turns out that there always exists a stationary process, which leads to the extremal Lyapunov exponents $\kappa(\rho)$, $\kappa^*(\rho)$, and $\bar{\kappa}(\rho)$. This result merely reflects the fact that periodic trajectories can be considered as stationary stochastic processes. For less degenerate processes, such as e.g. stationary solutions of nondegenerate diffusion processes, the situation is more complicated: The resulting almost sure Lyapunov exponent is always contained in the interval of deterministic Lyapunov exponents, which can be realized from points in the maximal control set $C$. On the other hand the $p$-th moment Lyapunov exponents $\lambda(p, \rho)$ converges towards $\kappa(\rho)$ as $p \to \infty$ for each nondegenerate diffusion process as uncertainty. Therefore, stability of all moments for this kind of stochastic uncertainty boils down to the analysis of $\kappa(\rho)$ and $r(\rho)$. For these results, and for connections to the theory of large deviations, see [AK] and [CK[c]]. Stabilization by noise via a high gain approach is discussed e.g. in [ACW].

## 4. Stabilization of Uncertain Systems

We return to the set up (1.2) of Section 1 and discuss the stabilization of linear systems with time varying uncertainties under linear, constant output feedback. Throughout this section we assume for the projected system

$$\dim \mathcal{LA}\{h(s, A+v+u),\ v \in V_\rho\}(s) = d-1 \text{ for all } s \in \mathbb{P},\text{ all } u \in U \text{ and } \rho > 0$$

(cp. condition (H) in Section 2).
In equation (1.2)

$$\dot{x} = (A + v(t))x + ux,\ v \in \mathcal{V}_\rho,\ u \in U_\sigma$$

the constant output feedback $u$ can be considered as a parameter varying in $U \subset gl(d, \mathbb{R})$. In particular, once a $u \in U$ is chosen, all the results obtained in Section 3 hold for the system $\dot{x} = (A+u)x + v(t)x$, where in Section 3 the matrix $A$ is replaced by $A + u$. Therefore we will study here the functions $\kappa(\rho, u)$ and $\kappa(\rho, \sigma)$ (see Equations (1.4) and (1.5)) in their dependence on $u$ and $\sigma$.

**4.1. Theorem:** With the above notations and assumptions we have:
(i) $\kappa : [0, \infty) \times U \to \mathbb{R}$, $(\rho, u) \mapsto \kappa(\rho, u)$ is continuous, and increasing in $\rho$.
(ii) $\kappa : [0, \infty) \times [0, \infty) \to \mathbb{R}$, $(\rho, \sigma) \mapsto \kappa(\rho, \sigma)$ is continuous, and increasing in $\rho$, decreasing in $\sigma$.

**Proof:** The monotonicity statements follow directly from the definitions. Continuity can be proven following essentially the same lines as in the proof for Theorem 3.1.

**4.2. Corollary:** The zero level sets of the functions $\kappa$

$$\Gamma(U) = \{(\rho, u) \in [0, \infty) \times U;\ \kappa(\rho, u) = 0\}$$
$$\Gamma = \{(\rho, \sigma) \in [0, \infty) \times [0, \infty);\ \kappa(\rho, \sigma) = 0\}$$

are closed and connected, maybe empty.
As in Section 3 this will allow us to obtain a characterization of the precise stabilization radius defined as follows.

**4.3. Definition:** For a matrix $A \in gl(d, \mathbb{R})$, uncertainties $\{\mathcal{V}_\rho,\ \rho \geq 0\}$ and output feedbacks $\{U_\sigma,\ \sigma \geq 0\}$ define the stabilization radius for constant $u \in U$ by

$$r(u) := \inf\{\rho \geq 0;\text{ there is } v \in \mathcal{V}_\rho \text{ such that (1.2) is not exponentially stable}\}$$

and for feedbacks $u$ of size at most $\sigma$

$$r(\sigma) := \inf\{\rho \geq 0;\text{ for every } u \in U_\sigma \text{ there is } v \in \mathcal{V}_\rho \text{ such that (1.2) is not exponentially stable}\}.$$

If $A + u + v(\cdot)$ is exponentially stable for all $v \in \mathcal{V}_\rho$, $\rho \geq 0$, then we set $r(u) = \infty$, similarly for $r(\sigma)$.

**4.4. Corollary:** The stabilization radii defined above are characterized by

$$r(u) = \begin{cases} \min\{\rho \geq 0;\ \kappa(\rho, u) = 0\} & \text{if } \Gamma(u) \cap \{\binom{\rho}{u};\ \rho \geq 0\} \neq \emptyset \\ 0 & \text{if } \kappa(0, u) > 0, \text{ i.e. if } A + u \text{ is unstable} \\ \infty & \text{otherwise} \end{cases}$$

$$r(\sigma) = \begin{cases} \min\{\rho \geq 0;\ \kappa(\rho, \sigma) = 0\} & \text{if } \Gamma \cap \{\binom{\rho}{u};\ \rho \geq 0\} \neq \emptyset \\ 0 & \text{if } \kappa(0, \sigma) > 0 \\ \infty & \text{otherwise.} \end{cases}$$

**Proof:** The expressions above with min replaced by inf are clear from the definitions. Since the functions $\kappa$ are continuous, the assertions follow.

∎

**4.5. Remark:** One can also take an alternative route to defining stabilization radii: Let

$$q(\rho) := \max\{\sigma \geq 0; \text{ there exists } u \in U_\sigma \text{ such that for every } v \in \mathcal{V}_\rho \text{ (1.2) is not exponentially stable}\}.$$

Then by continuity and monotonicity of $\kappa(\rho, \sigma)$ from Theorem 4.1 we have: $\rho < r(\sigma)$ iff $\sigma > q(\rho)$, compare Figure 3. Since $\kappa(\rho, u)$ need not be increasing in $u$, a similar definition to describe $r(u)$ is not possible. Hence Definition 4.3 is more natural, because $\kappa(\rho, u)$ and $\kappa(\rho, \sigma)$ are increasing in $\rho$.

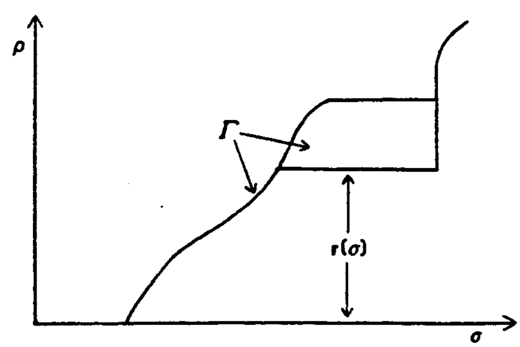


*Figure 3:* The zero level set $\Gamma$ and the stabilization radius $r(\sigma)$

The continuity and monotonicity properties of $\kappa$ imply also certain continuity and monotonicity results for the stabilization radii:

**4.6. Corollary:** Denote by $\text{dom}_u := \{u \in U;\ r(u) < \infty\}$ and by $\text{dom}_\sigma := \{\sigma \geq 0;\ r(\sigma) < \infty\}$ the effective domains of $r(u)$, and of $r(\sigma)$ respectively. Then

(i) $r(\sigma)$ is right continuous with left hand limits, and increasing in $\text{dom}_\sigma$, in particular $r(\sigma)$ is lower semi continuous;

(ii) $r(u)$ is semi continuous in $\text{dom}_u$, i.e. if $u_0 \in \text{dom}_u$ is a point of discontinuity, then there is a sequence $\{u_n,\ n \in \mathbb{N}\}$ in $\text{dom}_u$, such that $u_n \to u_0$ and $r(u_n) \to r(u_0)$ as $n \to \infty$.

**Proof:** Both parts follow from Theorem 4.1 and Corollary 4.2 via standard analysis arguments.

∎

The domains of the stabilization radii can be determined in the following way.

**4.7. Proposition:**

(i) $r(\sigma) = \sup_{u \in U_\sigma} r(u)$

(ii) $\sigma \in \mathrm{dom}_\sigma$ iff $u \in \mathrm{dom}_u$ for all $u \in U_\sigma$

(iii) $r(u) > 0$ iff $A + u$ is stable

(iv) $r(u) = \infty$ iff $A + u$ is stable and there exists a transformation matrix $T \in Gl(d, \mathbb{R})$ such that $TVT^{-1}$ consists only of skew symmetric matrices.

**Proof:** (i) and (ii) follow directly from the definitions.

(iii) If $A + u$ is stable, then $\kappa(0, u) = \max\{\mathrm{Re}\,\mu;\ \mu \in \mathrm{spec}(A + u)\} < 0$, where $\mathrm{spec}(A + u)$ denotes the set of eigenvalues of $A + u$. Hence $r(u) > 0$ by continuity of $\kappa(\cdot, u)$. Vice versa, if $r(u) > 0$, then there exists $\rho > 0$ with $\kappa(\rho, u) < 0$. But $A + u \in A + u + V_\rho$ for all $\rho > 0$.

(iv) follows directly from Proposition 3.6(ii) applied to $A + u$ instead of $A$.

∎

**4.8. Remark:** Note that the complex stability radius depends continuously on structured variations in the systems matrix $A$, while the real stability radius (for time invariant uncertainties) is lower semicontinuous (see [HP$^b$], [HP$^c$], [HP$^d$]).

The following remarks and examples comment on the continuity and monotonicity properties of the stabilization radii.

**4.9. Example:** The radii $r(u)$ and $r(\sigma)$ need not be continuous.
Consider again the linear oscillator, but this time with uncertain damping and with controlled restoring force, i.e. $\ddot{y} + 2(b + v(t))\dot{y} + uy = 0$, or

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & -2b \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ 0 & -2v(t) \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ -u & 0 \end{pmatrix} x$$

with $v(t) \in [-\rho, \rho]$, $\rho \geq 0$ and $u \in \mathbb{R}$. Note that the Lie algebra condition is satisfied for this system, except for $u = 0$. The stabilization radius $r(u)$ has the following form

$$r(u) = \begin{cases} 0 & \text{for } u \leq 0 \\ b & \text{for } u > 0 \end{cases}$$

by Example 4.2 in [CK$^e$].

Set $U_\sigma := \left\{ \begin{pmatrix} 0 & 0 \\ -\alpha & 0 \end{pmatrix};\ \alpha \in [-\sigma, \sigma] \right\}$, $\sigma \geq 0$. Then we obtain for $r(\sigma)$

$$r(\sigma) = \begin{cases} 0 & \text{for } \sigma = 0 \\ b & \text{for } \sigma > 0, \end{cases}$$

and hence neither $r(u)$ nor $r(\sigma)$ are continuous, compare also Example 4.14, below.

**4.10. Example:** The radius $r(\sigma)$ need not be strictly increasing.
We return to Example 3.9, i.e. the linear oscillator with uncertain restoring force, where we use the damping $b$ now as the output feedback $u$. A description in terms of (1.1) is e.g.

$$\dot{x} = \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -v(t) & 0 \end{pmatrix}\right] x + \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \hat{u}, \; y = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} x,$$

where with $F = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}$ the resulting closed loop system is

$$\dot{x} = \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -v(t) & 0 \end{pmatrix}\right] x + \begin{pmatrix} 0 & 0 \\ 0 & -2u_4 \end{pmatrix} x.$$

Choose $U_\sigma = \left\{\begin{pmatrix} 0 & 0 \\ 0 & -2\alpha \end{pmatrix};\ \alpha \in [0,\sigma]\right\}$, then $r(\sigma)$ is the function $r(b)$ (for $b = \sigma$) from Figure 1, and $r(\sigma) = r(\sigma_0) = 1$ for $\sigma \geq \sigma_0$, with $\sigma_0 \sim 0.405$.

**4.11. Example:** The radius $r(u)$ need not be increasing, and may be positive only on bounded intervals.
Consider the system

$$\begin{aligned} \dot{x} &= \left[\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -v(t) & 0 \end{pmatrix}\right] x + \begin{pmatrix} 0 & 0 \\ 0 & -2u \end{pmatrix} x \\ &=: \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} x + A(v,u)x, \end{aligned}$$

which is the equation from Example 4.10 with an added term $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} x$.
Denote by $\lambda(x,v,u)$ the Lyapunov exponents of Example 4.10 (for a definition see (1.3)) and by $\lambda(x,v,u;\alpha)$ the exponents of the system above. Since $A_\alpha = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$ commutes with all matrices, we obtain from Formula (2.5) with $s_\alpha(\tau) := s_\alpha(\tau, s_0, A_\alpha + A(v,u))$:

$$\begin{aligned} \lambda(x,v,u;\alpha) &= \limsup_{t\to\infty} \frac{1}{t} \int_0^T s_\alpha^{\mathrm{T}}(\tau)(A_\alpha + A(v,u))s_\alpha(\tau)d\tau \\ &= \limsup_{t\to\infty} \frac{1}{t} \int_0^T s_\alpha^{\mathrm{T}}(\tau)A_\alpha s_\alpha(\tau)d\tau + \limsup_{t\to\infty} \frac{1}{t} \int_0^T s_\alpha^{\mathrm{T}}(\tau)A(v,u)s_\alpha(\tau)d\tau \\ &= \alpha + \lambda(x,v,u). \end{aligned}$$

Hence $\kappa(\rho,u;\alpha) = \alpha + \kappa(\rho,u)$ and

$$r(u;\alpha) = \min\{\rho;\ \kappa(\rho,u) \geq -\alpha\}.$$

Therefore, for $\alpha \in (0,1)$, the level curves of $\kappa(\rho,u)$ for the level $-\alpha$ are the stabilization radii of the system $\dot{x} = (A_\alpha + A(v,u))x$. Figure 4 shows the stabilization radii $r(u;\frac{1}{4})$ for $\alpha = \frac{1}{4}$ and $r(\sigma;\frac{1}{4})$ (with $U_\sigma = [0,\sigma]$), as obtained from Figure 2b in [CK$^{\mathrm{f}}$]. Note that for the system without uncertainty, i.e. for $\rho = 0$, we have $\kappa(0,u;\frac{1}{4}) = 0$ iff $u = \frac{1}{4}$ or $u = \frac{34}{16} = 2.125$.

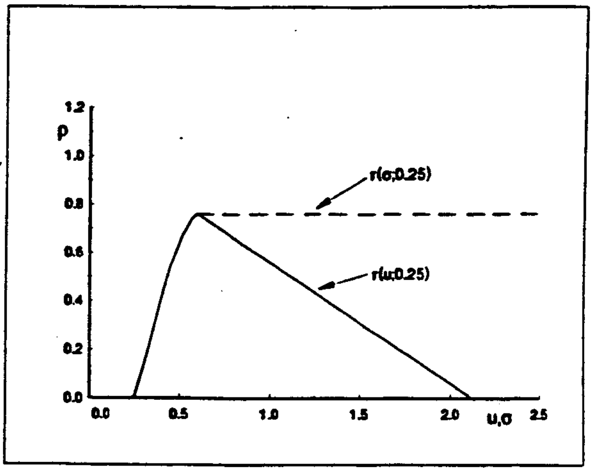


*Figure 4:* The stabilization radii $r(u;\frac{1}{4})$ and $r(\sigma;\frac{1}{4})$

The same radius is obtained by using the explicit solution of an optimal control problem as in [Go], compare also [CK$^f$], Section 6.

Using the setup in Section 1 and the stabilization radii from Definition 4.3 we have the following considerations for the feedback stabilization of uncertain linear systems: Given the differential equation $\dot{x} = Ax$ as a model of a physical system, the coefficients $a_{ij}$ of the matrix $A$ can be

- fixed (structural) parameters, e.g. the 0's and 1's in a companion form matrix,
- uncertain parameters, due to modelling or measurement problems, where usually bounds or statistical properties are known,
- controllable parameters in the sense that they can be varied by inputs in a certain range, usually through feedback of the observable output.

For the design of such a system it is desirable to know exactly the influence of the uncertain parameters with given bounds (or statistical properties), and of the achievable output feedback (with given range) on the systems behavior. For the case of time varying (or stochastic) uncertainties of size $\rho \geq 0$ and of time invariant, linear output feedbacks of size $\sigma \geq 0$ the stabilization radius $r(\sigma)$ provides exactly this information: The system (1.2) is (exponentially) stabilizable via constant output feedback of size $\sigma$ for all uncertainties of size $\rho$ iff $r(\sigma) > \rho$. Furthermore, $r(u)$ provides the additional information, for which $u \in U_\sigma$ the stability radius will be maximal. In the light of Example 4.11 this is important, because we can have $r(\sigma) \equiv \text{const}$ for $\sigma \geq \sigma_0$, but $r(u)$ is decreasing for $u \in U_\sigma$, $\sigma \geq \sigma_0$, and may be even zero for $u \in U_\sigma$, $\sigma \geq \sigma_1$. These effects, and the fact that gain matrices usually have to satisfy various restrictions in the real world, suggest to take bounded feedbacks into consideration. (Note that by Example 4.18 below also the stabilization radius $r_{Lf}(u)$ based on quadratic Lyapunov functions need not be monotone in $u$.)

The rest of this section is devoted to some simple properties of the radii $r(u)$ and $r(\sigma)$, and to the comparison with other concepts of stabilization of uncertain linear systems. In Section 5 we will present a method for the computation of our stabilization radii.

**4.12. Remark:** (positivity of $r(u)$ and $r(\sigma)$)
$r(0) =: r_0 > 0$ iff $A$ is stable, by Proposition 3.6(i). Hence $r(\sigma) \geq r_0$ for all $\sigma \geq 0$, and there exists $\sigma > 0$ such that $r(u) > 0$ for all $u \in U_\sigma$, by Theorem 4.1(i). If $A$ is unstable, then $r(0) = 0$, and the following cases can occur:

a) $r(\sigma) = 0$ for all $\sigma \geq 0$, see Example 4.13,
b) $r(\sigma) > 0$ for all $\sigma > 0$, see Example 4.9,
c) $r(\sigma) > 0$ for all $\sigma > \sigma_0 > 0$, see Example 4.11,
d) $r(\sigma) = \infty$ for all $\sigma > \sigma_0 > 0$, see Example 4.14.

**4.13. Example:** $r(\sigma) = 0$ for all $\sigma \geq 0$
Consider again Example 4.11, now for $\alpha \geq 1$. Then $r(u) = r(\sigma) = 0$ for all $u \geq 0$ and all $\sigma \geq 0$. Note that for the system without uncertainty, i.e. for $\rho = 0$, we have $\min_{u \geq 0} \kappa(0, u; \alpha) = \alpha - 1$, which is nonnegative for $\alpha \geq 1$.

**4.14. Example:** $r(\sigma) = \infty$ for all $\sigma$ large enough
Recall first of all that by Proposition 3.6 we have $r(A) = \infty$ iff $A$ is stable and there exists a transformation matrix $T \in Gl(d, \mathbb{R})$ such that $TVT^{-1}$ consists only of skew symmetric matrices. Consider the following system

$$\dot{x} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x + v(t) \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} x + u \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x,$$

with $V_\rho = [-\rho, \rho]$, $\rho \geq 0$, and $U_\sigma = [-\sigma, \sigma]$ for $\sigma \geq 0$. Then we have by the criterion above and by simple eigenvalue computations:

$$r(u) = \begin{cases} 0 & \text{for } u \geq -1 \\ \infty & \text{for } u < -1, \end{cases}$$

and hence

$$r(\sigma) = \begin{cases} 0 & \text{for } \sigma \in [0,1] \\ \infty & \text{for } \sigma > 1. \end{cases}$$

This example shows at the same time that $r(u)$ and $r(\sigma)$ need not be continuous (compare also Example 4.9), and here the Lie algebra assumption is satisfied for all $s \in \mathbb{P}$, all $u \in U$, and all $\rho > 0$.
Stabilization radii for the system (1.2) can also be defined using the concepts from Definition 3.7:

**4.15. Definition:** The stability radius with respect to time invariant perturbations in $V_\rho$ is defined for each $u \in U$ by

$$r_{\mathbb{R}}(u) = \inf\{\rho \geq 0;\ \sup_{v \in V_\rho} \max_{\mu \in \mathrm{spec}(A+v+u)} \mathrm{Re}\, \mu \geq 0\},$$

and the radius achievable by quadratic Lyapunov functions is for $u \in U$ denoted by $r_{Lf}(u)$, where in Definition 3.7 we again replace $A + v$ by $A + v + u$.
As before, we set for the corresponding stabilization radii

$$r_{\mathbb{R}}(\sigma) = \sup_{u \in U_\sigma} r_{\mathbb{R}}(u) \text{ and } r_{Lf}(\sigma) = \sup_{u \in U_\sigma} r_{Lf}(u).$$

Compare for these definitions Townley and Ryan [TR], and Rotea and Khargonekar [RK], where also an open loop concept of quadratic stabilizability is presented.
With these definitions we can compare the various stabilization radii for a priori bounded feedbacks $u \in U_\sigma$, and for the case $u \in U$.

**4.16. Remark:** (positivity and finiteness of $r_{\mathbb{R}}$ and $r_{Lf}$)
By Proposition 3.8 we have for a stable matrix $A$ and for all $u \in U$: $0 < r_{Lf}(u) \leq r(u) \leq r_{\mathbb{R}}(u)$ and $r_{\mathbb{R}}(u) = \infty$ if $r(u) = \infty$ iff $r_{Lf}(u) = \infty$. Hence the corresponding results also hold for $r_{\mathbb{R}}(\sigma)$ and $r_{Lf}(\sigma)$.

**4.17. Remark:** (differences between the stabilization radii for bounded $U_\sigma$)
Write Example 3.9 in the following form:

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x + v(t) \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} x + u \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} x,$$

with $v(t) \in [-\rho, \rho]$, and $u \in U_\sigma = [0, \sigma]$. Then we have

$$r_{Lf}(\sigma) < r(\sigma) < r_{\mathbb{R}}(\sigma) \text{ for } 0 < \sigma < \sigma_0 \text{ with } \sigma_0 \sim 0.405,$$

$$r_{Lf}(\sigma) < r(\sigma) = r_{\mathbb{R}}(\sigma) \text{ for } \sigma_0 \leq \sigma < \frac{1}{\sqrt{2}},$$

$$r_{Lf}(\sigma) = r(\sigma) = r_{\mathbb{R}}(\sigma) \text{ for } \sigma \geq \frac{1}{\sqrt{2}}.$$

Note that in all cases the radii as a function of $\sigma$ are the same as those for $u$, if $u = \sigma$. Therefore the stabilization criterion based on quadratic Lyapunov functions is too conservative for $\sigma < \frac{1}{\sqrt{2}}$, and the one based on constant uncertainties is too optimistic for $\sigma < 0.405$, if we are interested in time varying uncertainties. For $\sigma$ large enough, the three criteria agree, compare also Remark 4.20.

**4.18. Example:** (non monotonicity of $r_{Lf}(u)$ and $r_{\mathbb{R}}(u)$)
In [CK$^e$], Example 4.4, we presented the following example:

$$\dot{x} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} x + v(t)x + u \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} x,$$

with $V = gl(2, \mathbb{R})$, $V_\rho = \{v \in V,\ |v| \leq \rho\}$, $\rho \geq 0$, where $|\cdot|$ denotes the Euclidian norm in $gl(2, \mathbb{R})$, and $u \geq 0$. For this system we have

$$r_{Lf}(u) = \begin{cases} s_2(A(u)) & \text{for } u \in [0, u_0] \\ \frac{2\sqrt{u}}{1+u} & \text{for } u \geq u_0, \end{cases}$$

where $s_2(A(u))$ denotes the second singular value of $A(u) = \begin{pmatrix} -1 & 1 \\ -u & -1 \end{pmatrix}$, and $u_0$ is the unique zero of $u^3 + u^2 + 3u - 1 = 0$, i.e. $u_0 \sim 0.296$. $r_{Lf}(u)$ has a unique maximum at $u = 1$ with $r_{Lf}(1) = 1$, and decreases for $u > 1$ with $\lim_{u \to \infty} r_{Lf}(u) = 0$. This shows that $r_{Lf}(u)$ need not be monotone.
Now consider again Example 4.11:

$$(4.1) \qquad \dot{x} = \left[ \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -v(t) & 0 \end{pmatrix} \right] x + \begin{pmatrix} 0 & 0 \\ 0 & -2u \end{pmatrix} x,$$

where $v(t) \in [-\rho, \rho]$, $u \in U = [0, \sigma]$. This system can be interpreted as an output feedback system with structured uncertainties as demonstrated in Example 4.10. We compute the stabilization radii $r_{\mathbb{R}}(u)$ and $r_{Lf}(u)$ and compare them with $r(u)$ from Example 4.10. The radius $r_{\mathbb{R}}(u)$ is computed directly via eigenvalues of the system matrices, which yields

$$r_{\mathbb{R}}(u) = \begin{cases} 0 & \text{for } 0 \le u \le \frac{1}{4}, \ \frac{17}{8} \le u \\ \frac{1}{2}(\frac{17}{8} - u) & \text{for } \frac{1}{4} < u < \frac{17}{8}. \end{cases}$$

The radius $r_{Lf}(u)$ can be computed by solving a family of parametrized Riccati-equations (see [HKL]), or by finding a positive definite matrix $P$ that satisfies the requirements of Definition 3.5, see Example 4.3 in [CK$^e$]. It turns out that $r_{Lf}(u)$ has a unique maximum at $(0.77, 0.67)$, and decreases along the line $\frac{1}{2}(\frac{17}{8} - u)$ as $u \to \frac{17}{8}$. The corresponding graphs are shown in Figure 5.

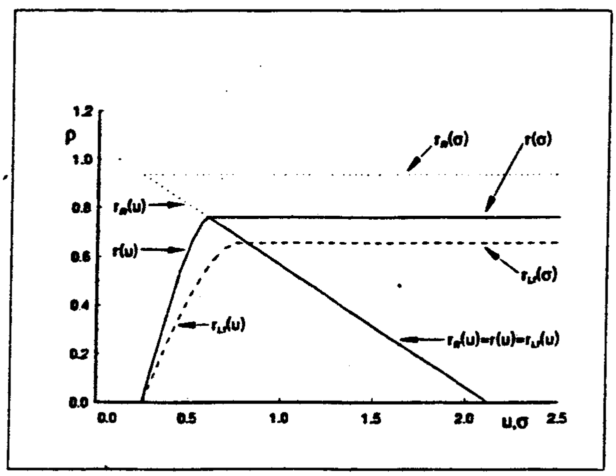


*Figure 5:* The stability radii $r_{\mathbb{R}}(u)$ and $r_{Lf}(u)$ for system (4.1)

**4.19. Example:** (bound invariant Lyapunov functions and $r(\sigma) = \infty$ for $\sigma < \infty$)
In [Ho], Hollot introduced the concept of bound invariant Lyapunov functions to analyze the stabilization of uncertain linear systems. This means that the positive definite matrix $P$ in Definition 3.7 is independent of $\rho \ge 0$. Here we present an example that shows that bound invariant Lyapunov functions can exist, even if $r(\sigma) < \infty$ for all $\sigma \ge 0$.

Consider the following situation in terms of equation (1.1)

$$A=\begin{pmatrix}-1 & 0\\ 1 & -1\end{pmatrix},\ V=\left\{\begin{pmatrix}v_1 & 0\\ v_2 & v_2\end{pmatrix};\ v_1,v_2\in\mathbb{R}\right\},\ B=\begin{pmatrix}1\\0\end{pmatrix},\ C=Id,$$

with reads in the notation of (1.2)

$$\dot{x}=\left[\begin{pmatrix}-1 & 0\\ 1 & -1\end{pmatrix}+\begin{pmatrix}v_1(t) & 0\\ v_2(t) & v_2(t)\end{pmatrix}\right]x+\begin{pmatrix}1\\0\end{pmatrix}Fx.$$

This system does not satisfy the conditions mentioned in Example 4.14. However, the system is stabilizable via a bound invariant Lyapunov function, if there is a positive definite matrix $P$ such that

(i) $$(0\ \ 1)(AP+PA^{\mathrm{T}})\begin{pmatrix}0\\1\end{pmatrix}<0, \text{ and}$$

(ii) $$(1\ \ 1)P\begin{pmatrix}0\\1\end{pmatrix}=0,$$

compare Lemma 4.4 in Zhou and Khargonekar [ZK]. One computes easily that $P=\begin{pmatrix}2 & -1\\ -1 & 1\end{pmatrix}$ satisfies these conditions. This means in particular that $\lim_{\sigma\to\infty} r(\sigma)\geq \lim_{\sigma\to\infty} r_{Lf}(\sigma)=\infty$.

**4.20. Remark:** (stabilization radii for $\sigma=\infty$)
If no a priori bounds are given, one can analyze the behavior of the stabilization radii as $\sigma\to\infty$. A class of systems, for which $\lim_{\sigma\to\infty} r_{\mathbb{R}}(\sigma)=\lim_{\sigma\to\infty} r_{Lf}(\sigma)$, and hence also $\lim_{\sigma\to\infty} r(\sigma)=\lim_{\sigma\to\infty} r_{\mathbb{R}}(\sigma)$, was characterized by Townley and Ryan [TR]. In particular, they prove that all 2-dimensional stabilizable systems with "structured uncertainties", see [HP$^c$], are of this type. Remark 4.17 shows that also for these systems $r_{Lf}(\sigma)<r(\sigma)<r_{\mathbb{R}}(\sigma)$ is possible for $\sigma<\infty$. On the other hand, Example 4.21 shows that there are systems with structured, time varying uncertainties and with output feedback, for which $\infty>\max_u r_{\mathbb{R}}(u)>\max_u r_{Lf}(u)>0$, and hence $\lim_{\sigma\to\infty} r_{\mathbb{R}}(\sigma)>\lim_{\sigma\to\infty} r(\sigma)>\lim_{\sigma\to\infty} r_{Lf}(\sigma)$. Thus for these systems the precise stabilization regions (w.r.t. time varying uncertainties) for unbounded output feedback cannot be determined using eigenvalues of the matrices in $A+U+V_\rho$, nor by using quadratic Lyapunov functions.

**4.21. Example:** Consider again the system (4.1) in Example 4.18 as an output feedback system. Its stability radii depending on $\sigma$ are

$$r_{\mathbb{R}}(\sigma)=\frac{15}{16}\text{ for }\sigma>\frac{1}{4}$$
$$r_{Lf}(\sigma)\approx 0.67\text{ for }\sigma>0.77$$
$$r(\sigma)\approx 0.77\text{ for }\sigma>0.59,$$

compare Figure 5. Hence we have

$$r_{\mathbb{R}}(\sigma) > r(\sigma) > r_{Lf}(\sigma) \text{ for all } \sigma > \frac{1}{4}.$$

**Going back to the interpretation of (4.1) as an output feedback system as in Example 4.10, we have for the form $\dot{x} = Ax + v(t)x + B\hat{u}$: $A = \begin{pmatrix} \frac{1}{4} & 1 \\ -1 & \frac{1}{4} \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$. Hence the pair $(A, B)$ is controllable in this case. Furthermore, shifting $A$ to $\tilde{A} = \begin{pmatrix} \frac{1}{4} & 1 \\ -1 & \frac{1}{4} - \beta \end{pmatrix}$ with $\frac{1}{4} < \beta < \frac{15}{8}$, results in a system $\dot{x} = \tilde{A}x + v(t)x + B\hat{u}$, whose matrix $\tilde{A}$ is stable. However, (for this system) the various stabilization radii are different for all $\sigma > 0$ and also in the limit as $\sigma \to \infty$, compare the results of Townley and Ryan [TR] for unbounded state feedback. Note that in this example the values of the radii in a bounded $u$-interval determine the possibilities of stabilization, i.e. stabilization via bounded output feedback for linear systems with time varying uncertainties is not a high gain problem, but should be formulated realistically with bounds on the size of the uncertainty and on the size of the feedback gain.**

## 5. Numerical Computation of the Stabilization Radius

In this section a method for the numerical computation of the stabilization radius is presented. It is based on the computation of the extremal Lyapunov exponents $\kappa(\rho, u)$ using numerical optimal control algorithms. Then the stability radius $r(u)$ is extracted as the zero level curve; finally taking infima over feedbacks $u \in U_\sigma$, one obtains the domains of feedbacks where stabilization is possible. Two examples in dimension two and three, respectively, are analyzed in detail.

Consider the general uncertain system with bounded output feedback introduced in Section 1

$$\dot{x} = (A + v(t))x + ux, \; v \in V_\rho, \; u \in U_\sigma. \tag{1.2}$$

The Lyapunov exponents are

$$\lambda(x, v, u) = \limsup_{t \to \infty} \frac{1}{t} \log |\varphi(t, x, u)| \tag{1.3}$$

and

$$\kappa(\rho, u) = \sup_{v \in V_\rho} \sup_{x \neq 0} \lambda(x, v, u). \tag{1.4}$$

As described in the preceeding sections, $\kappa(\rho, u)$ can be determined as the optimal value of the following control problem (considering $v$ as an open loop control) in the space $\mathbb{P} = \mathbb{P}^{d-1}$ of directions:

$$\text{(5.1)}\qquad \sup_{v\in\mathcal{V}_\rho}\sup_{s_0\in\mathbb{P}}\limsup_{t\to\infty}\frac{1}{t}\int_0^t q(s,A+u+v)d\tau$$

where $s(\tau),\ \tau\geq 0$, is the solution of the differential equation

$$\text{(5.2)}\qquad s(0)=s_0,\ \dot{s}(t)=h(s(t),A+u+v(t)),\ t\geq 0;$$

here the function $q$ and the vectorfield $h$ on the sphere (and hence on projective space $\mathbb{P}$) are given by

$$\text{(5.3)}\qquad \begin{aligned} q(s,A+u+v)&=s^{\mathrm{T}}(A+u+v)s\\ h(s,A+u+v)&=[A+u+v-s^{\mathrm{T}}(A+u+v)s\cdot Id]s. \end{aligned}$$

As shown in Proposition 2.6 the optimal value in (5.1) can approximately be obtained from every $s_0\in\mathbb{P}$, in particular, if $s_0\in int\ C$, then the corresponding trajectory remains in $int\ C$. Hence we may take an arbitrary, fixed initial point $s_0\in\mathbb{P}$, preferably in $int\ C$. Furthermore arguments from ergodic theory (cp. [CK$^{\mathrm{d}}$]) imply that there are $s_{\mathrm{opt}}\in C$ and $v_{\mathrm{opt}}\in\mathcal{V}_\rho$ such that

$$\sup_{v\in\mathcal{V}_\rho}\sup_{s_0\in\mathbb{P}}\limsup_{t\to\infty}\frac{1}{t}\int_0^t q(s,A+u+v)d\tau$$

is attained in $(s_{\mathrm{opt}},v_{\mathrm{opt}})$ and the lim sup is actually a limit. Hence we can approximate the optimal value of (5.1) by the value of the following (nonlinear) optimal control problem in $\mathbb{P}$ with $T$ "large":

$$\text{(5.4)}\qquad \begin{aligned} &\sup_{v\in\mathcal{V}_\rho}\frac{1}{T}\int_0^T q(s(\tau),A+u+v(\tau))d\tau\\ &s(0)=s_0,\ \dot{s}(t)=h(s(t),A+u+v(t)),\ t\geq 0. \end{aligned}$$

This problem can — in principle — be solved by numerical optimal control algorithms. However, using this formulation, in both examples below we could obtain convergence of the employed algorithms only in very special cases, where the (local) optimal solutions converge for $T\to\infty$ to an equilibrium. This problem is due to a very small range of convergence caused by the nonlinear nature of the optimal control problem and the averaging over time in the performance criterion. Hence, in addition to (5.4) we use another numerically more feasible formulation. It is based on the approximation by periodic solutions contained in Theorems 2.3, 2.4:
We know (see Theorem 2.4) that

$$\kappa = \sup\{\frac{1}{t}\log|\lambda(g)|;\ t \geq 0,\ g \in int\ \mathcal{S}_{\leq t},\ \pi_{\mathbb{P}} E(\lambda(g)) \subset int\ C\}.$$

Every $T$-periodic trajectory in *int* $C$ corresponds to an eigenspace of a real eigenvalue $\lambda$ of some element in *cl* $\mathcal{S}$, and $\frac{1}{T}\log|\lambda|$ is the corresponding Lyapunov exponent.
This leads us to the following optimal control problem in *int* $C \subset \mathbb{P}^{d-1}$:

$$\text{(5.5)}\qquad \sup_{v\in\mathcal{V}_\rho} \frac{1}{T}\int_0^T q(s(\tau), A+u+v(\tau))d\tau$$
$$s(0) = s(T) \in int\ C,\ \dot{s}(t) = h(s(t), A+u+v(t)),\ t \geq 0.$$

The optimal control problems (5.4) and (5.5) differ only with respect to the boundary conditions. In (5.4) the initial point is fixed and the final point is free, while in (5.5) we impose the periodicity requirement that the initial and final point coincide.
For fixed $T > 0$, standard results in optimal control theory imply the existence of an optimal solution.
We note that in general we cannot guarantee that maximation also over $T > 0$ leads to an optimal solution of (5.4). However, in the examples below we will also maximize over $T$ and accept the obtained optimal value as an estimate for the optimal solution of (5.4). (Note that for $d = 2$ this is justified by Theorem 2.5).
For the numerical solution of the optimal control problems (5.4) and (5.5) we used two software packages, MUSCOD and PARFIT, developed by H.G. Bock, Augsburg, and his coworkers (see e.g. [BC] and [BP]). MUSCOD uses a direct approach, transforming the infinite-dimensional optimal control problem into a finite dimensional optimization problem by discretizing the control $v$. PARFIT is based on an indirect approach: the optimal control problem is solved via necessary optimality conditions resulting from Pontryagin's maximum principle; this leads to a boundary value problem, which is solved via multiple shooting. This method does not require a priori information about the structure of the optimal control whereas, using the direct method, the discretization for $v$ must be chosen adequately. However, the indirect method needs very good starting values for the trajectories (and the adjoint variables). These starting values can be generated e.g. by MUSCOD.
In the two examples below we follow this two-step procedure. We remark, that in general the indirect approach requires a considerable amount of analytic computations, since the adjoint equations as well as the switching functions and their time derivatives have to be determined analytically.
Our first example takes up the linear oscillator from Example 3.9. There we referred to [CK$^f$] for determination of the stability radius, presented in Figure 1. Here we use this result, which is based on considerations valid only for this very special two-dimensional system, in order to validate our general approach based on numerical optimal control.

**5.1. Example:** Consider again the linear oscillator with uncertain restoring force

$$\ddot{y} + 2b\dot{y} + (1 + v(t))y = 0, \ v(t) \in V_\rho := [-\rho, \rho]. \tag{5.6}$$

For the computation of the maximal Lyapunov exponent $\kappa(\rho, b)$ we have to solve the following optimal control problem on $\mathbb{P}^1$, written in polar coordinates with $\varphi \in [0, \pi)$:

$$\sup_{v \in \mathcal{V}_\rho} -\frac{1}{T}\int_0^T \sin\varphi[v(t)\cos\varphi + 2b\sin\varphi]dt \tag{5.7}$$
$$\dot{\varphi} = -(1 + v\cos^2\varphi + 2b\sin\varphi\cos\varphi)$$

with $T > 0$ fixed or free and suitable boundary conditions according to (5.4) and (5.5), respectively.

For the invariant control set $C$, two cases can be distinguished (cp. [CK[f]]):

(i) If $\rho \leq b^2 - 1$, the eigenvalues for all $v \in V_\rho$ are real and

$$C = \left[\arctan\left(-b + \sqrt{b^2 - 1 - \rho}\right), \arctan\left(-b + \sqrt{b^2 - 1 + \rho}\right)\right].$$

(ii) If $\rho > b^2 - 1$, the eigenvalues for all $v \in V_\rho$ with $|v| > b^2 - 1$ are complex and $C = \mathbb{P}^1$. Thus in this case, $C$ contains periodic solutions for certain constant $v \in V_\rho$; if additionally $-\rho \leq b^2 - 1$, then $C$ contains also equilibria.

According to Theorem 2.5, the optimal value of $\kappa$ is either attained in an equilibrium or in a proper periodic solution. Thus we consider the problem formulation (5.4), i.e. with fixed $\varphi_0 \in int\ C$. In case (ii), also formulation (5.5) with proper periodic boundary condition

$$\varphi(T) = \varphi(0) - \pi$$

and optimization also over $T$ is considered.

For the numerical solution with MUSCOD, $v(\cdot)$ has to be discretized (we chose piecewise constant control on $M = 10$ intervals) and starting values for the whole trajectory $\varphi$ and for the control function $v$ have to be given.

The numerical results are:

(a) If MUSCOD converges in the problem (5.4) (we could always achieve this in case (i) above), then $v_{opt} \equiv -\rho$ and $\kappa(\rho, b) = \lambda_{max}(-\rho)$.

(b) If MUSCOD converges in the problem (5.5), then

$$v_{opt} = \begin{cases} -\rho & \text{for } 0 \leq t \leq t_s \\ \rho & \text{for } t_s < t \leq T. \end{cases}$$

In case (b), the switching time $t_s$ cannot be approximated by increasing the number $M$ of discretization intervals, unless $t_s$ is a discretization point. There are two possibilities to overcome this difficulty:

Either assume that the optimal control has bang-bang structure as indicated above and try to determine the switching time as a parameter (instead of $v$); or solve the optimal control problem via an indirect approach, using the program PARFIT. Then the switching time is calculated automatically by searching for zeros of the switching function. Here the results of a MUSCOD run are taken as starting values. Both possibilities were tried and yield essentially identical results. Figure 6 shows $\kappa(\rho, b)$ for $\rho = 0.5$ and 1.0, and Figure 7 shows the stability radius $r(b)$. Taking infima over $b$ a corresponding stabilization radius can be determined. Both results are in good agreement with those obtained in [CK[e]].

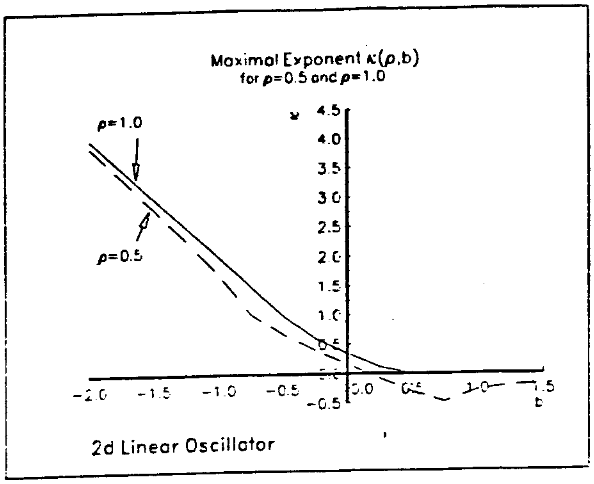


Figure 6

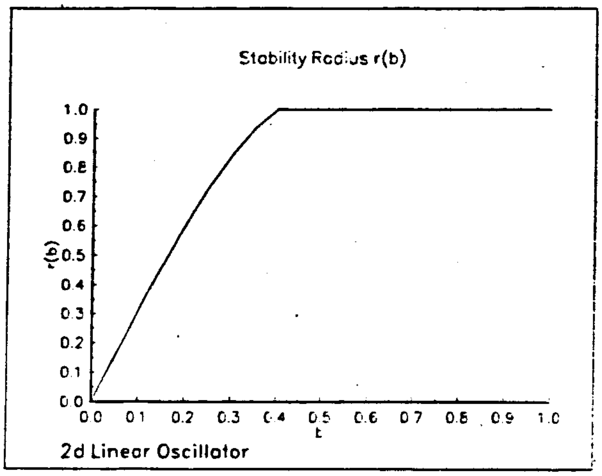


Figure 7

**5.2. Example:** We consider the following uncertain system in dimension $d = 3$:

$$\dddot{y}(t) + a\ddot{y}(t) + b\dot{y}(t) + [c + v(t)]y(t) = 0 \tag{5.8}$$

where $a, b, c \in \mathbb{R}$ and $v(t) \in [-\rho, \rho]$, $\rho > 0$.
The standard transformation (cp. e.g. Bronstein/Semendjajew [BS]) $z(t) := e^{\frac{1}{3}at}y(t)$ allows us to get rid of the second order term and yields the equation

$$\dddot{z}(t) + b_1\dot{z}(t) + b_0 z(t) = 0 \tag{5.9}$$

with $b_1 = b - \frac{1}{3}a^2$, $b_0 = c + v(t) - \frac{1}{3}ab + \frac{2}{27}a^3$.
Writing (5.9) as a first order system for $x = (z, \dot{z}, \ddot{z})^T$ we get

$$\dot{x} = Ax \tag{5.10}$$

with $A = A(a, b, c, v(t)) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -b_0 & -b_1 & 0 \end{pmatrix}$.

Since $x = e^{\frac{1}{3}at} B \begin{pmatrix} y \\ \dot{y} \\ \ddot{y} \end{pmatrix}$

with the invertible matrix $B = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3}a & 1 & 0 \\ \frac{1}{9}a^2 & \frac{2}{3}a & 1 \end{pmatrix}$,

one easily sees that the Lyapunov exponents of (5.8) and (5.9) (or (5.10)) are related by

(5.11) $$\lambda(\bar{y}_0) = \lambda(B\bar{y}_0) - \frac{1}{3}a$$

where $\bar{y}_0 = (y(0), \dot{y}(0), \ddot{y}(0))^{\mathrm{T}}$.
Projection of (5.10) onto the sphere $S^2$ gives in spherical coordinates $\theta, \varphi$ with

(5.12) $$S^2 \ni (x, y, z) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta),\ \theta \in (0, \pi),\ \varphi[-\pi, \pi)$$

$$\begin{aligned} \dot{\theta} = h_1(\theta, \varphi, v) &:= \cos\theta\sin\varphi[\sin\theta\cos\varphi + \cos\theta] \\ &+ sin^2\theta[(c + v - \frac{1}{3}ab + \frac{2}{27}a^3)\cos\varphi + (b - \frac{1}{3}a^2)\sin\varphi] \\ \dot{\varphi} = h_2(\theta, \varphi) &:= -\sin^2\varphi + \frac{\cos\theta}{\sin\theta}\cos\varphi. \end{aligned}$$

(If trajectories include a pole $(x, y, z) = (0, 0, \pm 1)$, a different parametrization is necessary.)
Observe that the projection on $\mathbb{P}^2$ identifies two points on $S^2$ given by $(\theta_1, \varphi_1)$ and $(\theta_2, \varphi_2)$, $\varphi_2 < \varphi_1$, respectively, iff

(5.13) $$\theta_2 = \pi - \theta_1,\ \varphi_2 = \varphi_1 - \pi.$$

Furthermore

$$\begin{aligned} q(\theta, \varphi, v) := \sin\theta\big[\sin\theta\sin\varphi\cos\varphi - \left(c + v - \frac{1}{3}ab + \frac{2}{27}a^3\right)\cos\theta\cos\varphi \\ \left(1 - b + \frac{1}{3}a^2\right)\cos\theta\sin\varphi\big]. \end{aligned}$$

Thus we arrive at the following optimal control problems:

(5.14) $$\begin{aligned} &\max_{v \in \mathcal{V}_\rho} \frac{1}{T}\int_0^T q(\theta(t), \varphi(t), v(t))dt \\ &\text{s.t. } \dot{\theta}(t) = h_1(\theta(t), \varphi(t), v(t)) \\ &\qquad \dot{\varphi}(t) = h_2(\theta(t), \varphi(t)), \quad t \geq 0 \end{aligned}$$

with suitable boundary conditions according to (5.4) or (5.5).

First, we will discuss the eigenvalues of $A(a,b,c,v)$ for $v \equiv$ const. With $D := \left(\frac{b_0}{2}\right)^2 + \left(\frac{b_1}{3}\right)^3$, the following cases can be distinguished for the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and their maximal real part $\lambda_{\max}$:

(i) $D > 0$: $\lambda_1 \in \mathbb{R}$, $\lambda_2 = \bar{\lambda}_3 \in \mathbb{C}\backslash\mathbb{R}$,

$$\lambda_{\max} = \begin{cases} \lambda_1 = \sqrt[3]{-\frac{b_0}{2} + \sqrt{D}} + \sqrt[3]{-\frac{b_0}{2} - \sqrt{D}} & \text{if } b_0 < 0 \\ 0 & \text{if } b_0 = 0 \\ \operatorname{Re} \lambda_{2,3} = -\frac{1}{2}\left(\sqrt[3]{-\frac{b_0}{2} + \sqrt{D}} + \sqrt[3]{-\frac{b_0}{2} - \sqrt{D}}\right) & \text{if } b_0 > 0. \end{cases}$$

(ii) $D = 0$: $\lambda_1 = \lambda_2 = \lambda_3 = 0 = \lambda_{\max}$ if $b_0 = b_1 = 0$

$\lambda_1 \in \mathbb{R}$, $\lambda_2 = \lambda_3 \in \mathbb{R}$, $\lambda_{\max} = 2\sqrt{-\frac{b_1}{3}}$ if $b_0 < 0$

$\lambda_1 \in \mathbb{R}$, $\lambda_2 = \lambda_3 \in \mathbb{R}$, $\lambda_{\max} = \sqrt{-\frac{b_1}{3}}$ if $b_0 > 0$.

(iii) $D < 0$: Here $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$; for $b_0 < 0$ determine $\lambda_{\max}$ from

$\lambda_1 = 2\sqrt{-\frac{b_1}{3}} \cos\left(\frac{1}{3}\arctan\left(\frac{2\sqrt{-D}}{b_0}\right)\right)$,

$\lambda_{2,3} = -\sqrt{-\frac{b_1}{3}}\left[\cos\left(\frac{1}{3}\arctan\left(\frac{2\sqrt{-D}}{b_0}\right)\right) \mp \sqrt{3}\sin\left(\frac{1}{3}\arctan\left(\frac{2\sqrt{-D}}{b_0}\right)\right)\right]$. For $b_0 > 0$, the signs of $\lambda_i$ are reversed, and for $b_0 = 0$ one has $\lambda_1 = 0$, $\lambda_{2,3} = \pm\sqrt{b_1}$.

If we consider $D$ as a function of $v$, then $D$ is strictly convex and attains its minimal value in $v_0 = \frac{1}{3}ab - \left(c + \frac{2}{27}a^3\right)$. Thus $D(v_0) \geq 0$ implies $D(v) > 0$ for all $v \neq v_0$ and $A(a,b,c,v)$ has complex (i.e. nonreal) eigenvalues for every $v \neq v_0$. The maximal real part occurs at the real eigenvalue for $v \leq v_0$ and at the complex eigenvalues for $v > v_0$ (clearly, $D(v_0) > 0$ if $b_1 = b - \frac{1}{3}a^2 > 0$).

On the other hand, if $D(v_0) < 0$, then for $v \in [v_1, v_2]$ with $v_{1,2} := v_0 \mp 2\sqrt{\left(-\frac{b_1}{3}\right)^3}$, there are three real eigenvalues. For $v < v_1$, there exist two complex eigenvalues, and the maximal eigenvalue is real; for $v > v_2$ there exist again two complex eigenvalues and they are maximal. If there is only one (complex conjugate pair of ) eigenvalue(s), with maximal real part, the typical trajectories of the projected system converge to the projection of the corresponding eigenspace. This projected eigenspace is an equilibrium for a real eigenvalue and a periodic orbit for a pair of complex eigenvalues. All eigenvalues corresponding to some maximal eigenvalue ly in the invariant control set $C$.

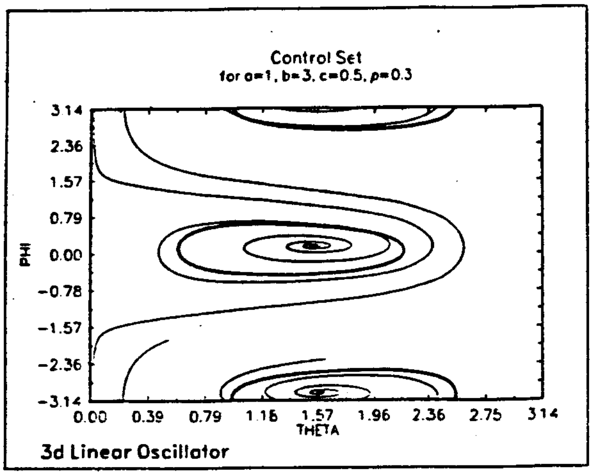


Figure 8a

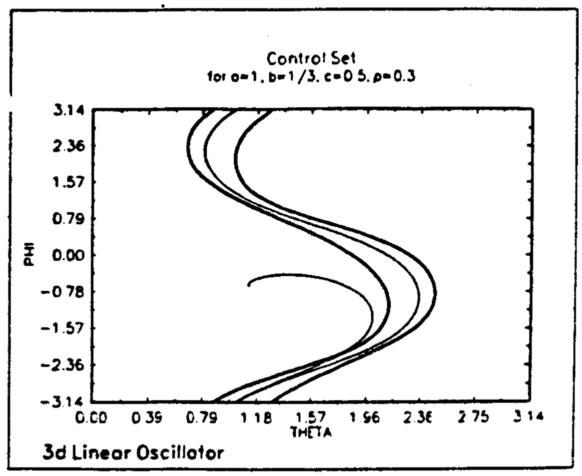


Figure 8b

Figure 8a and b show typical pictures of invariant control sets. They are produced numerically using the program 'CS' by Gerhard Häckl, U. Augsburg [Hc].

Figure 8a shows the invariant control set $C$ for $a = 1$, $b = 3$, $c = 0.5$ and $\rho = 0.3$. Observe that here $C$ is drawn on $S^2$: the lower and upper part of $C$ correspond to points in the middle part via the identification (5.12). Here for all $v \in V_\rho$ all eigenvalues of $A(a, b, c, v)$ are real. Also included is the phase portrait for $v \equiv 0$.

Figure 8b shows the invariant control set for $a = 1$, $b = \frac{1}{3}$, $c = 0.5$ and $\rho = 0.3$. Here for all $v \in V_\rho$, the maximal eigenvalues are complex conjugate. Also included is the phase portrait for $v \equiv 0$.

It may also happen in this family of systems that the maximal control set $C$ coincides with $\mathbb{P}^2$ (e.g. for $a = 1$, $b = \frac{1}{3}$, $c = 0.5$ and $\rho = 0.6$). In this case, no information about the maximal Lyapunov exponent can be gained from an analysis of the control structure.

The boundary conditions in (5.14) are either — according to (5.4) — chosen as

$$\varphi(0) = \varphi_0,\ \theta(0) = \theta_0,\ (\varphi_0, \theta_0) \in int\ C,\ \text{fixed},$$

or — according to (5.5) — as periodic in $\mathbb{P}^2$. We note the following lemma, which indicates how one can formulate this periodicity requirement in the local coordinates $\theta, \varphi$.

**5.3. Lemma:** Let $s(\cdot)$ be a trajectory corresponding to $v \in \mathcal{V}_\rho$ of equation (5.10) projected onto the sphere $S^2$, such that $s(\cdot)$ contains no pole of $S^2$. If $(s(\cdot), v)$ considered in $\mathbb{P}^2 \times \mathcal{V}_\rho$ is an optimal solution of problem (5.5) (and hence periodic in $\mathbb{P}^2$ with minimal period $T > 0$), then $s(\cdot)$ satisfies in angular (local) coordinates $\theta, \varphi$ as in (5.12) either

$$\varphi(T) = \varphi(0),\ \theta(T) = \theta(0) \tag{5.15}$$

or

$$\varphi(T) = \varphi(0) - \pi, \ \theta(T) = \pi - \theta(0). \tag{5.16}$$

**Proof:** This is a consequence of (5.12) and the Jordan curve theorem.

∎

As in the 2-d case, we first used MUSCOD in order to solve the problems with boundary conditions according to (5.4) and (5.5), respectively. The results can be described as follows:

(i) If in the problem with fixed initial point (i.e. (5.4)), MUSCOD converges, then $v_{\text{opt}} \equiv -\rho$ and hence $\kappa = \lambda_{\max}(-\rho)$. Convergence is achieved for starting values (for the whole trajectory) and initial values close enough to the corresponding projected eigenspace. We could achieve convergence if there are only maximal real eigenvalues and often if there are maximal real and maximal complex conjugate eigenvalues. In the latter case, (local) convergence to a maximal real eigenvalue may also occur if a pair of maximal complex conjugate eigenvalues yields better performance. Thus in this case the proper periodic boundary condition (5.16) (and hence the problem formulation (5.5)) gives better performance.

(ii) If in the problem formulation with periodic boundary condition (i.e. (5.15) or (5.16)), MUSCOD converges, then the obtained optimal solution is either $v_{\text{opt}} \equiv \rho$ (here the optimal solution corresponds to a real or a complex eigenvalue with maximal real part) or $v_{\text{opt}} = \begin{cases} -\rho & \text{for } 0 \leq t \leq t_s \\ \rho & \text{for } t_s < t \leq T. \end{cases}$

As in the two-dimensional case, the switching time $t_s$ can either be determined by modifying the periodic approach (assuming bang-bang structure) or by applying an indirect approach. In the first approach, one obtains the following optimization problem where $t_s \geq 0$ and $T - t_s$ are to be determined:

$$\begin{aligned}
&\max \frac{1}{T}\left[\int_0^{t_s} q(\theta_1(t), \varphi_1(t), -\rho)dt + \int_{t_s}^{T} q(\theta_2(t), \varphi_2(t), \rho)dt\right] \\
&\text{s.t. } \dot{\theta}_1(t) = h_1(\theta_1(t), \varphi_1(t), -\rho) \\
&\qquad \dot{\theta}_2(t) = h_1(\theta_2(t), \varphi_2(t), \rho) \\
&\qquad \dot{\varphi}_1(t) = h_2(\theta_1(t), \varphi_1(t)) \\
&\qquad \dot{\varphi}_2(t) = h_2(\theta_2(t), \varphi_2(t))
\end{aligned} \tag{5.17}$$

with boundary conditions

$$\begin{aligned}
&\theta_1(T) = \theta_2(0), \ \varphi_1(T) = \varphi_2(0), \ \text{(continuity)} \\
&\theta_2(T) = \pi - \theta_1(0), \ \varphi_2(T) = \varphi_1(0) - \pi, \ \text{(periodicity)}.
\end{aligned}$$

With appropriately chosen starting values in *int* $C$ for the trajectory, here convergence of MUSCOD could be achieved in all cases where a maximal complex eigenvalue (i.e. a periodic trajectory for $v \in V_\rho$) exists. However, the solution ($v_{\text{opt}} \equiv \rho$ or $v_{\text{opt}}$ of bang-bang type) may depend on the starting values. Thus the obtained local optima have to be compared in order to obtain the global optimum. Furthermore (in particular, when additionally maximal real eigenvalues are present for certain $v \in V_\rho$) also the problem formulation (5.4) has to be taken into account. Figure 9a, 9b show an example of this type. Here $a = 1$, $b = 0$, $c = 0.2$ and $\rho = 0.25$. The invariant control set together with the optimal MUSCOD trajectory, which is of bang-bang type, is displayed. For certain starting values, however, MUSCOD determines the periodic orbit corresponding to the eigenspace for the maximal (complex) eigenvalue belonging to $v \equiv \rho$ as (local) optimal solution. For other starting values, the (global) optimal solution is obtained which has the following form: the corresponding trajectory runs with $v = -\rho$ into the eigenspace for the maximal (real) eigenvalue belonging to $v \equiv -\rho$ and then with $v = \rho$ towards the eigenspace belonging to $v \equiv \rho$. Using the problem formulation (5.4), MUSCOD computes $v_{\text{opt}} \equiv -\rho$ as the (local) optimal solution.

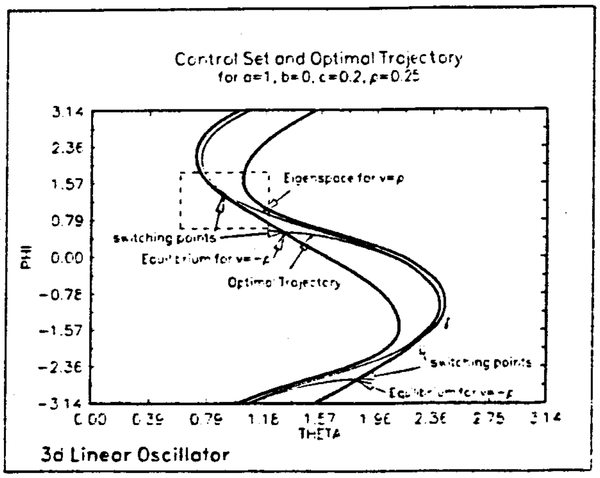


Figure 9a

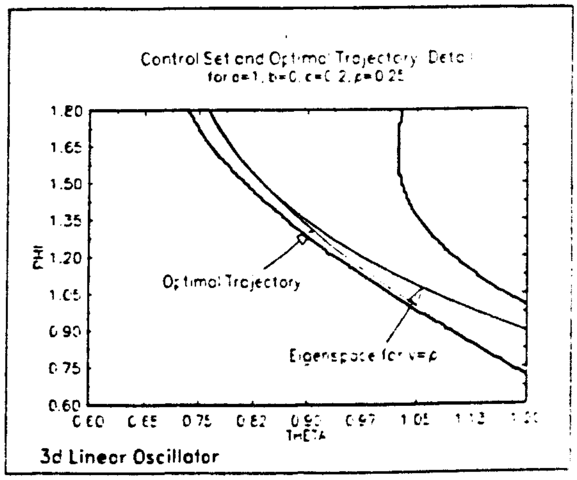


Figure 9b

We also used — instead of (5.17) — the program PARFIT in order to determine the switching time $t_s$ more accurately. From Pontryagin's maximum principle one obtains the following boundary value problem:

$$\dot{\theta} = h_1(\theta, \varphi, v) \cdot T$$
$$\dot{\varphi} = h_2(\theta, \varphi) \cdot T$$
$$\dot{T} = 0$$

with the adjoint equations

$$\dot{\lambda}_1 = 2\sin\theta\cos\theta\sin\varphi\cos\varphi - (\cos^2\theta - \sin^2\theta)\,[b_0\cos\varphi - (1-b_1)\sin\varphi]$$
$$- \left[(\cos^2\theta - \sin^2\theta)\sin\varphi\cos\varphi + 2\sin\theta\cos\theta(b_0\cos\varphi - (1-b_1)\sin\varphi)\right]\cdot T\cdot\lambda_1$$
$$+ \frac{\cos\varphi}{\sin^2\theta}\cdot T\cdot\lambda_2$$
$$\dot{\lambda}_2 = -\left[\sin^2\theta\,(\cos^2\varphi - \sin^2\varphi) + \sin\theta\cos\theta\,[b_0\sin\varphi + (1-b_1)\cos\varphi]\right]$$
$$- \left[\sin\theta\cos\theta\,(\cos^2\varphi - \sin^2\varphi) + \cos\varphi - \sin^2\theta\,[b_0\sin\varphi + (1-b_1)\cos\varphi]\right]\cdot T\cdot\lambda_1$$
$$+ \left[2\sin\varphi\cos\varphi + \frac{\cos\theta}{\sin\theta}\sin\varphi\right]\cdot T\cdot\lambda_2$$
$$\dot{\psi} = -\left[\sin\theta\cos\theta\sin\varphi\cos\varphi + \sin\varphi + \sin^2\theta\,[b_0\cos\varphi - (1-b_1)\sin\varphi]\right]\cdot\lambda_1$$
$$- \left[-\sin^2\varphi + \frac{\cos\theta}{\sin\theta}\cos\varphi\right]\cdot\lambda_2$$

and boundary conditions

$$\theta(1) = \pi - \theta(0),\ \varphi(1) = \varphi(0) - \pi$$

and transversality conditions

$$\lambda_1(1) = -\lambda_1(0),\ \lambda_2(1) = \lambda_2(0),\ \psi(0) = \psi(1) = 0.$$

The switching function (obtained from the maximum condition) is

$$s(\theta,\varphi,T,\lambda_1,\lambda_2,\psi) = -\sin\theta\cos\varphi(\cos\theta - \sin\theta\cdot T\cdot\lambda_1).$$

Then the optimal value $\kappa$ is obtained by integrating $q$ along the solution $\theta(\cdot),\varphi(\cdot)$ of the above boundary value problem, $\kappa = \int_0^1 q(\theta(t),\varphi(t),v(t))dt$, where $v$ is determined by PARFIT using the switching function.

In general, we take the result of a MUSCOD run as starting values for PARFIT (MUSCOD yields also starting values for the adjoint functions $\lambda_1(\cdot)$ and $\lambda_2(\cdot)$). Sometimes (for the computation of $r(a,b)$ where we determine the zero level curve of $\kappa(\rho,a,b)$), these starting values were not good enough to ensure convergence of PARFIT. Here we used a 'homotopy approach': Suppose $\rho_{crit}$ is a critical value of $\rho$, where the MUSCOD result taken as starting value of PARFIT does not lead to convergence. Then take $\rho^0$ 'near' to $\rho_{crit}$, where PARFIT converges (with the corresponding MUSCOD result as starting value). Define $\rho^{n+1} := \rho^n + \varepsilon^n \mathrm{sgn}(\rho_{crit} - \rho^0)$, $\varepsilon^n > 0$, small, $n \geq 0$, and use PARFIT results for $\rho^n$ as starting values for $\rho^{n+1}$. Proceeding this way, we could obtain PARFIT convergence for $\rho_{crit}$.

Figures 10–13 present the obtained PARFIT results.

Figure 10, 11 show the maximal Lyapunov exponent $\kappa(\rho,b)$ with $a = 1$, $c = 0.5$. Here $\kappa(\rho,b)$ has been computed for $\rho = 0, 0.1, \ldots 1$ and $b = 0, \frac{1}{3}, \frac{2}{3}, \ldots 3$. The zero level curve $\kappa(\rho,b) = 0$ gives an estimate for the stability radius $r(b)$. Observe that the stability reserve gets smaller, if $b$ is increased above a critical value. (In these computations the difficulties discussed above occured and the homotopy approach was taken.)

Figure 12 presents the stability radius $r$ as a function of $a$ and $b$, where $c = 0.5$ is kept fixed. It is obtained by calculating the stability radii for $a = 1, 1.3, 1.6, \ldots$ and $b = 0, 0.15, 0.30, \ldots$ Note that the undisturbed system with $\rho = 0$ is stable only if $b > \frac{1}{2}a$. Hence $r(a, b)$ is only computed for these values of $a$, $b$.
Figure 13 shows the stability radii $r(b)$ for $a = 1.0$ and $a = 2.5$, based on these more accurate computations (cp. with $r(b)$ from Figure VI).

**Conclusions:** The numerical findings in this section indicate that the supremal exponential growth rate $\kappa(\rho)$ and hence the stability radius $r$ can be determined numerically, using good optimal control software based on a direct optimization approach or Pontryagin's maximum principle. At least in the examples above the usual devices (like homotopy techniques) in order to overcome the difficulties of small convergence regions and only local optima yield satisfactory results. Thus, although no explicit formulae for the exact stability radius are available (and cannot be expected), they appear to be numerically feasible. However, difficulties may occur if the optimal period lengths $T$ are large or if even no optimal periodic solution exists (i.e. if the optimal values are increasing for $T \to \infty$).

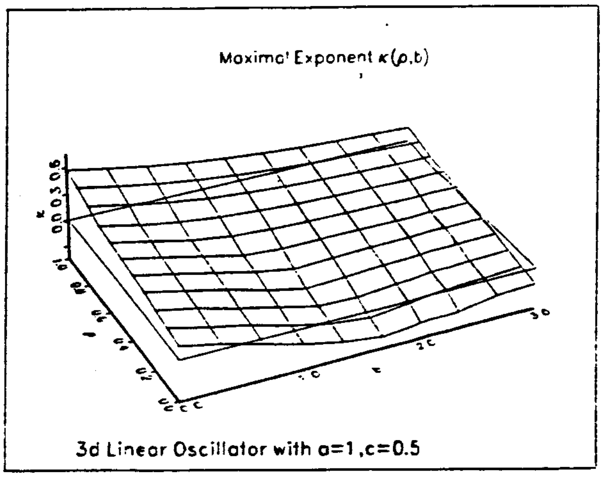


Figure 10

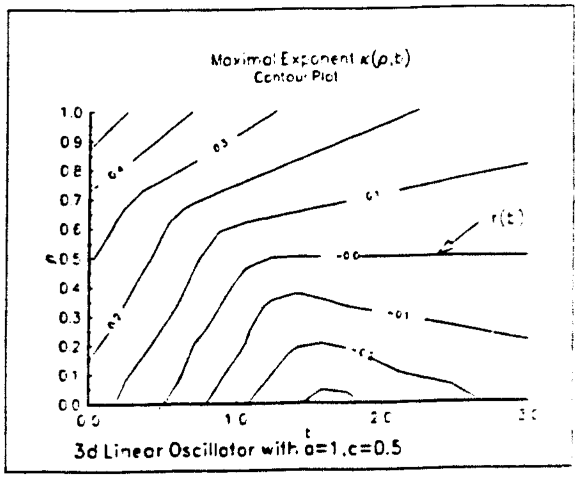


Figure 11

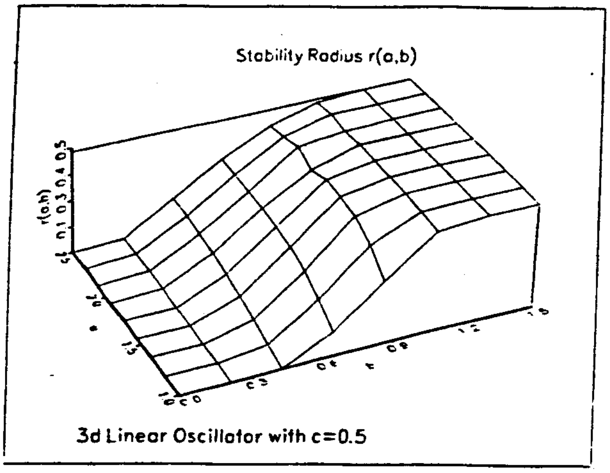


Figure 12

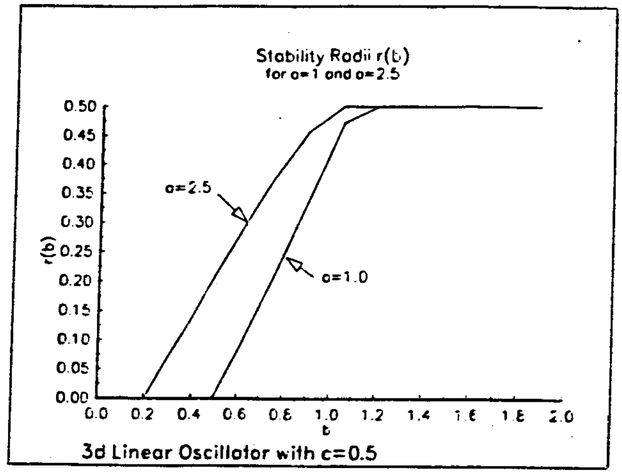


Figure 13

**Acknowledgements:** The numerical computation of the maximal Lyapunov exponents is based on optimal control packages developed by Hans Georg Bock and his coworkers. We thank for the permission to use this software and Marc Steinbach and Johannes Schlöder for their advice in using it. Naturally, any misuse is ours. The numerical computation of the control sets is performed with software by Gerhard Häckl. We also thank him for his help.

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