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ROBUSTNESS OF TIME-VARYING SYSTEMS

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Abstract.
The dynamics of many mechanical systems can be described, or approximated by smooth vector fields in d-dimensional space $\mathbb{R}^d$. External and internal excitations as well as modeling uncertainties are incorporated in the vector fields as families of (time-varying) functions, possibly with their own dynamics. The problem then is to study the response behavior of the system under the given uncertainty structure. In this paper we analyze the stability of uncertain systems at an equilibrium point, using the concept of stability radii. Roughly, a stability radius is the smallest excitation range such that a (time-varying) perturbation within this range can render the system unstable. Since we consider time-varying perturbations, the precise stability radius of the system is determined by the maximal Lyapunov exponent of the linearization at the equilibrium point. Several examples illustrate the theory and compare the precise stability radius to the one obtained via Lyapunov function techniques.

1. Introduction.

Robust stability describes the stability behavior of systems under (time varying or constant) perturbations – this concept is also known in the literature as absolute stability. For a given system with given perturbation structure the problem is to find the maximal uncertainty range such that any perturbation with a smaller range will yield a stable system. This maximal range is often called the stability radius of the system. For time invariant linear systems with time invariant uncertainties the problem basically boils down to the computation of the maximal real parts of the eigenvalues for the perturbed system matrices. The precise characterization of the stability radius for time varying uncertainties requires the knowledge of the maximal Lyapunov exponent of the system. Therefore the theory developed in Colonius and Kliemann (1993) leads to results on stability radii for systems with parametric time varying uncertainties.

Section 2 presents basic facts about stability radii of linear systems. Results on the radius based on the maximal Lyapunov exponent are mostly consequences of the theory presented in Colonius and Kliemann (1993). For comparison purposes we also discuss a Lyapunov function approach to robust stability of systems with time varying uncertainties, and the eigenvalue based approach for time invariant perturbations. Several examples illustrate the differences between these stability radii.

A stable manifold theorem from Colonius and Kliemann (1997) allows us to generalize the concept of the Lyapunov exponent based stability radius to nonlinear systems. At a singular point the nonlinear stability radius can be expressed in terms of the radius of the linearized system, and equality of the two radii holds if the maximal Lyapunov exponent is a strictly monotone function of the uncertainty range, see Section 3 for these results.

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The analysis of uncertain systems has been one of the focal points in systems theory for the last decade. Of main interest in this area are stability properties and performance design. Several approaches have been developed for the study of these problems, most notably H^∞ theory (see e.g. Francis (1990)) for operator theoretic formulations, concepts based on the gap metric (see e.g. Zames and El-Sakkray (1980) or Georgiou and Smith (1990)), Kharitonov type results on the stability of sets of polynomials (see e.g. Kharitonov (1979)) as a transfer function approach, quadratic Lyapunov function criteria (see e.g. Rotea and Khargonekar (1989)), and the stability radius concept (see e.g. Hinrichsen and Pritchard (1990)) as state space approaches. Various connections between these approaches are discussed, e.g. in Doyle et al. (1989), Rotea and Khargonekar (1989) or Townley and Ryan (1991). We refer the reader to the proceedings volume Hinrichsen and Martensson (1990) for a discussion of various topics related to uncertain systems.

It is well-known that exponential stability of the linear, time invariant differential equation $\dot{x} = Ax$ is equivalent, on the one hand, to the negativity of the real parts of the eigenvalues of A and, on the other hand, to the existence of a quadratic (time invariant) Lyapunov function. For the stability of time varying equations $\dot{x} = A(t)x$, as they arise for uncertain systems with time varying uncertainty, it is not sufficient that for each $A(t), t \geq 0$ there exists a (quadratic, time invariant) Lyapunov function, compare in the context of robust stability e.g. Hinrichsen and Pritchard (1991). Therefore the Lyapunov function approach to stability of uncertain systems is based on the existence of a common quadratic Lyapunov function for all uncertainties.

Similarly, if for all $t \geq 0$ the eigenvalues of $A(t)$ are in the left half plane, then $\dot{x} = A(t)x$ need not be (exponentially) stable, compare the example in Hahn (1967). It is then natural to use Lyapunov exponents to define stability radii for systems with time varying uncertainties. A priori, it is not clear what the relation between these two approaches is.

The response of a dynamical system to time varying perturbations shows a different structure, depending on the way in which the perturbation affects the system dynamics. If a limit set of the nominal (unperturbed) system changes under the excitation, then we talk about a regular system, and its robustness behavior can be described in terms of the control sets of an associated control system, see Colonius and Kliemann (1997). If, however, the limit set is common to all excitations, then we talk about a singular system (at this set), and its robustness behavior is studied via stability radii as described above. In Section 2 we present our results on linear systems with time varying excitations. Section 3 extends these results with the help of linearization techniques to nonlinear systems with singular equilibrium point.

2. Robustness of Linear Time-Varying Systems.

The models of uncertain systems used in this section are of the form

$$
\begin{aligned}
(1) \qquad & \dot{x} = Ax + v(t)x, \ x \in \mathbb{R}^d \\
& v \in \mathcal{V}^\rho = \{v : \mathbb{R} \to V^\rho, \text{ locally integrable}\} \\
& V \subset g\ell(d, \mathbb{R}) \text{ is a linear subspace, } V^1 \subset V \\
& \qquad \text{a compact, convex subset with } 0 \in \text{ int } V^1, \\
& V^\rho = \rho \cdot V^1, \rho \geq 0.
\end{aligned}
$$

Typical examples for the uncertainty range V include the following:

(a) $V = \{B \in g\ell(d, \mathbb{R}), ||B||^p \leq 1\}$
Here $||B||^p$ denotes the p-operator norm of the $d \times d$ matrix B, and $p = 2$ is the standard situation. Stability radii based on this range V are often called 'unstructured'.

(b) Let D and E be real matrices of size $d \times p$ and $q \times d$, respectively. Define
$V = \{B \text{ is real } p \times q \text{ matrix with } ||DBE||^p \leq 1\}$.
Here $|| \cdot ||^p$ is again the p-operator norm in $g\ell(d, \mathbb{R})$. Stability radii of this type are often called 'structured'.

(c) Let $B_i \in g\ell(d, \mathbb{R})$ for $i = 1 \dots m$ and let $W \subset \mathbb{R}^m$ be compact and convex with $0 \in W$. Define
$V = \{B = \sum_{i=1}^{m} w_i B_i, (w_i)_{i=1 \dots m} \in W\}$.
This model includes (i) and (ii) as special cases and corresponds to the set up in Colonius and Kliemann (1993).

We will investigate the stability behavior off (1) using the idea of stability radii, which were first defined explicitly in Hinrichsen and Pritchard (1986a). Besides the two concepts arising from common Lyapunov functions and from Lyapunov exponents we also introduce a radius based on time invariant uncertainties, i.e. on the spectrum of the matrices in $A + V^\rho$. This allows for a comparison of our results with several examples in the literature. We use the following notation

$\text{spec}(A)$ is the spectrum of a $d \times d$ matrix A,

$\varphi(t, x, v)$ denotes the solution of (1) with

$\varphi(0, x, v) = x \in \mathbb{R}^d$,

$$(2) \quad \lambda(v, x) = \limsup_{t\to\infty} \frac{1}{t} \log |\varphi(t, x, v)|$$

is its Lyapunov exponent,

$$\kappa(\rho) = \sup_{v\in V^\rho} \sup_{x\neq 0} \lambda(x, v)$$

is the maximal spectral value of (1).

With these notations we define:

Definition 1. Consider the linear, uncertain system (1).

(i) The stability radius with respect to time invariant perturbations in V^ρ is defined as

$$r_{\mathbb{R}} = \inf\{\rho \geq 0, \sup_{v\in V^\rho} \max_{\mu\in \text{ spec } (A+v)} Re\mu \geq 0\}.$$

(ii) The stability radius with respect to quadratic Lyapunov functions is

r_{Lf} = sup{$\rho \geq 0$, there exist a positive definite matrix $P \in g\ell(d, \mathbb{R})$ and $\alpha > 0$ such that for all $(v, x) \in V^\rho \times \mathbb{R}^d$ it holds that $X^T(P(A + v) + (A + v)^T P)x \leq -\alpha|x|^2$}.

The superscript T denotes transposition.

(iii) The stability radius based on asymptotic stability is given by $r = \inf\{\rho \geq 0$, there exists $v \in V^\rho$ such that $\dot{x} = (A + v(t))x$ is not exponentially stable$\}$.

If the dependence on the matrix A or on its parameters is of importance, we will use the notation $r_{\mathbb{R}}(A)$, etc.

Remark 2.

(i) For the definition of $r_{\mathbb{R}}$, also called the real stability radius, compare e.g., Hinrichsen and Pritchard (1986a, 1990). The radius r_{Lf} is implicit in much of the quadratic robust stabilization literature, see e.g. Rotea and Khargonekar (1989). For a large class of uncertainties the radius r_{Lf} coincides with the complex stability radius of Hinrichsen and Prichard, see e.g. Hinrichsen and Pritchard (1990), Peterson (1987), or Proposition 3 in Townley and Ryan (1991). The radius r was first introduced in Colonius and Kliemann (1990).

(ii) Definition 2(iii) is based on asymptotic stability of all trajectories of (1). An alternative to this concept would be the use of uniform exponential stability, i.e. the use of the supremal Bohl exponent of the system. For the model (1) the two concepts agree by Corollary 4 below. For classes of complex-valued uncertainties it is shown in Hinrichsen and Pritchard (1986b) that the complex stability radius is the same as the one obtained via Bohl exponents, compare also Hinrichsen et al. (1989).

(iii) Explicit characterizations of $r_{\mathbb{R}}$ and r_{Lf} are available for certain classes of uncertainties, e.g. for V as described in (a) and (b) above with $p = 2$. These characterizations use parametrized Riccati equations, see e.g. Hinrichsen an Prichard (1986b). One cannot expect an explicit formula for the radius r. Even for two dimensional systems an explicit expression is only available if there exists $\rho \geq 0$ with $\kappa(\rho) \geq 0$ and the projected system on $\mathbb{P}^1$ has two control sets. But this is the case where $r = r_{\mathbb{R}}$. However, for the computation of the radius r we only need to know the sign of the largest spectral value $\kappa(\rho)$, see Corollary 3 below. For two dimensional systems Joseph (1993) provides an algorithm to compare this sign. The examples below were computed using this algorithm.

(iv) The case of stochastic uncertainties has been discussed e.g., in Willems and Willems (1983), and in Colonius and Kliemann (1990) based on large deviations theory for Lyapunov exponents in Arnold and Kliemann (1987).

The following results are based on Colonius and Kliemann (1993). To exclude degeneracies, we assume the Lie algebra rank condition for the projected system on the projective space $\mathbb{P}^1$:

$$(3) \quad \begin{aligned} &\dim \mathcal{LA}\{h(v, \cdot), v \in V^\rho\}(p) = d - 1 \\ &\text{for all } \rho > 0, \text{ all } p \in \mathbb{P}^{d-1} \\ &h(v, p) = (A + v - p^T(A + v)p \cdot Id)p. \end{aligned}$$

Theorem 3. *Consider the uncertain linear system (1) under assumption (3). Then the function $\kappa(\rho)$ defined in (2) satisfies:*

$\kappa(\rho)$ is increasing and continuous for $\rho \in [0, \infty)$.

Proof. Monotonicity is obvious from the definition of V^ρ. Continuity was shown in Colonius and Kiemann (1990). □

Theorem 3 is the key to the analysis of the stability radius r. We refer the reader to Hinrichsen and Pritchard (1991) for a discussion of the continuity of perturbations of eigenvalues in the context of robust stability.

Corollary 4. *Under the conditions of Theorem 3 we have*

$$r = \begin{cases} \min\{\rho, \kappa(\rho) = 0\} & \text{if this set is not empty} \\ 0 & \text{iff } \kappa(0) \geq 0 \text{ i.e.} \\ & \max_{\mu \in \text{ spec } A} Re\mu \geq 0 \\ \infty & \text{otherwise.} \end{cases}$$

Proof. All we need to show is that $r > 0$ iff A is stable, the rest follows from Theorem 3. If A is stable, then $\kappa(0) = \max\{Re\mu, \mu \in \text{ spec}(A)\} < 0$. Hence by continuity of $\kappa(\rho)$ it holds that $r > 0$. Vice versa, if $r > 0$, then there exists $\rho > 0$ with $\kappa(\rho) < 0$. But $0 \in V$ implies $A \in A + V^\rho$ for all $\rho > 0$. □

The next result characterizes the systems that have infinite stability radius, i.e. that are stable for all bounded uncertainties in $\bigcup_{\rho \geq 0} V^\rho$.

Corollary 5. *Under the conditions of Theorem 3 it holds*

$r = \infty$ *iff* A *is stable and there exists a transformation matrix* $T \in g\ell(d, \mathbb{R})$ *such that* $T(A - \frac{1}{d}trA)T^{-1}$ *and* TvT^{-1} *are skew symmetric matrices for all* $v \in V$.

Proof. If $r = \infty$ holds, then all matrices $A + v, v \in V^\rho, \rho \geq 0$ are stable. Furthermore, $trv = 0$ for all $v \in V$: Since $v \in V$ implies $-v \in V$ by (1), $trv \not\equiv$ constant yields the existence of two positive constants c_1 and c_2 such that

$$\kappa(\rho) \geq \frac{1}{d} \max_{v \in V^\rho} tr(A + v) \geq c_1(\rho - c_2).$$

Therefore $\lim_{\rho \to \infty} \kappa(\rho) = \infty$ which contradicts $r < \infty$. Hence trv is constant and $0 \in V$ implies $trv = 0$ for all $v \in V$.

Recall the notation from Colonius and Kliemann (1993). If $r = \infty$ then the systems group $\mathfrak{G}^0$ is compact. $\mathfrak{G}^0$ is generated by exponentials of matrices of the form $N^0 = \{A - \frac{1}{d}trA + v, v \in V\}$. Compactness of $\mathfrak{G}^0$ means that there exists a transformation matrix $T \in g\ell(d, \mathbb{R})$ such that $TN^0T^{-1} \subset so(d, \mathbb{R})$, the skew symmetric matrices. Choosing again $v = 0$, we obtain $T(A - \frac{1}{d}trA)T^{-1} \in so(d, \mathbb{R})$ and hence $TVT^{-1} \subset so(d, \mathbb{R})$.

To show the converse, we may assume, without loss of generality that $N^0 = \{A - \frac{1}{d}trA + v, v \in V\} \subset so(d, \mathbb{R})$, and hence the group $\mathfrak{G}^0$, defined as above is compact. Furthermore, we have $\frac{1}{d}tr(A + v) = \frac{1}{d}trA$, and hence it holds that $\kappa(\rho) = \frac{1}{d}trA$ for all $\rho > 0$. Now A stable implies $\frac{1}{d}trA = \sum_{\mu_i \in \text{ spec } A} \mu_i < 0$, and therefore $r = \infty$. □

The following example shows that it is not sufficient to assume that A is stable and V consists of skew symmetric matrices to obtain $r = \infty$.

Example 6. Consider the system (1) with

$$A = \begin{pmatrix} -1 & 0 \\ 4 & -2 \end{pmatrix} \text{ and } V = \left\{ \begin{pmatrix} 0 & v \\ -v & 0 \end{pmatrix}, V \in [-1, 1] \right\}.$$

Then spec$(A+v)$ is $-\frac{3}{2} \pm \sqrt{v(4 - v) + \frac{1}{4}}$, and the maximal eigenvalue for $v = 1$ is positive. Note that for $v \to \pm\infty$ the spectrum is $-\frac{3}{2} < 0$.

Our last result in this section concerns the relation of the radius r to the other stability radii presented in Definition 1.

Corollary 7. *Under the conditions of Theorem 3 we have for stable matrices A*

(i) $0 < r_{Lf} \leq r \leq r_{\mathbb{R}}$, *all inequalities can be strict. In particular,* $r_{\mathbb{R}} = \infty$ *if* $r = \infty$ *if* $r_{Lf} = \infty$.
(ii) $r_{Lf} = \infty$ *iff* A *is stable and there exist a positive definite matrix* P *and* $\alpha > 0$ *such that* $x^T PAx \leq -\alpha|x|^2$ *and* $PV \subset so(d, \mathbb{R})$.

Proof.

(i) The inequalities follow directly from the definitions. The examples below show that the inequalities can be strict.
(ii) If $r_{Lf} = \infty$, then A is stable, i.e. there exist a positive definite matrix P and $\alpha > 0$ such that $x^T PAx = \frac{1}{2}x^T(PA + A^T P)x \leq -\alpha|x|^2$ for all $x \in \mathbb{R}^d$. Furthermore, we have by Definition 1 for all $\rho \geq 0$ and all $v \in V^\rho$

$$x^T(P(A + v) + (A + v)^T P)x = \\ 2x^T PAx + 2x^T Pvx \leq -\alpha|x|^2.$$

If there exists $x \in \mathbb{R}^d$ and $v \in V$ such that $x^T Pvx = \beta|x^2|$ for some $\beta \neq 0$, then we can find $\rho > 0$ and $v_0 = v$ or $-v$ such that $x^T P\rho v_0 x >$

$\alpha|x|^2$, which is a contradiction. Hence $x^T Pvx = 0$ for all $v \in V$, all $x \in \mathbb{R}^d$, and thus Pv is skew symmetric.

Vice versa, if $PV \subset so(d, \mathbb{R})$, then $x^T Pvx = 0$ for all $v \in \bigcup_{\rho \geq 0} V^\rho$ and all $x \in \mathbb{R}^d$. Hence $x^T PAx \leq -\alpha|x|^2$ implies that $r_{Lf} > \rho$ for all $\rho \geq 0$. □

Remark 8. Note that $r_{Lf} = \infty$ implies $r = \infty$ and hence by Corollary 5 that $V \subset so(d, \mathbb{R})$. If P is symmetric and v skew symmetric, then $Pv = -(vP)^T$. Furthermore, $Pv \in so(d, \mathbb{R})$ means $Pv = -(Pv)^T$, and hence P and V commute. This is therefore a necessary condition for $r_{Lf} = \infty$.

The following examples illustrate the concept of stability radius and show the differences between the radii introduced in Definition 1.

Example 9. The linear oscillator with uncertain restoring force. Consider the linear oscillator

(4)
$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & -2b \end{pmatrix} x + v \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} x = A(v)x$$
$$\mathcal{V}^\rho = \{v : \mathbb{R} \to V^\rho, \text{ measurable}\}, V^\rho = [-\rho, \rho] \text{ for } \rho \geq 0.$$

Since all stability radii are zero iff $\max_{\mu \in \text{ spec } A(0)} Re\mu \geq 0$, we concentrate on the case $b > 0$. The maximal eigenvalue of (4), determining the radius $r_{\mathbb{R}}$, is $-b + \sqrt{b^2 - 1 - u}$, which is ≥ 0 iff $\rho \geq 1$. Hence we obtain

(5) $$r_{\mathbb{R}}(b) = 1 \text{ for all } b > 0.$$

In order to determine the radius r_{Lf} based on common Lyapunov functions we need to find a positive definite matrix $P = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$ such that $Q = PA(v) + A(v)^T P$ is negative definite for all $v \in [-\rho, \rho]$. Computing Sylvester's equations (compare, e.g. Hahn (1967), p. 100) for P and Q yields for $\rho \leq 1$

(a) $\alpha > 0$
(b) $\alpha\gamma - \beta^2 > 0$
(c) $2(1+v)\beta > 0$, i.e. $\beta > 0$ for $\rho < 1$,
(d) $-(\alpha - (1+v))^2 + 4b\beta(\alpha + (1+v)\gamma) - 4\beta^2((1+v) + b^2) > 0$.

Setting, without loss of generality, $\beta = 1$ (d) reads

(d') $g(\alpha, \gamma) = -\alpha^2 + 2\alpha(2b + (1+v)\gamma) - 4((1+v) + b^2) + 4b(1+v)\gamma - (1+v)^2\gamma^2 > 0$.

We solve for the zeros in α of $g(\alpha, \gamma)$ and obtain

(e) $\alpha_{1,2} = 2b + (1+v)\gamma \pm 2\sqrt{1+v}\sqrt{2b\gamma - 1}$.

Note that $2b\gamma - 1 \geq 0$ iff $b\gamma \geq \frac{1}{2}$, and for $b\gamma = \frac{1}{2}$ we have $\alpha_{1,2} = 2b + \frac{1+v}{2b}$. For each $(v, b) \in (-1, 1) \times \mathbb{R}^+$ equation (e) describes a parabola in the $\alpha - \gamma$ plane, and (d') is satisfied in the interior of this parabola. Note that for each $b \geq 0$ the parabolas are monotone in v. In order to obtain a common Lyapunov function for all $v \in [-\rho, \rho]$, we need that the parabolas corresponding to $w_1 = 1 + \rho$ and $w_2 = 1 - \rho$ intersect. Denote by $d(w_1, w_2)$ the difference between the lower branch (corresponding to α_2) for w_1 and the upper branch (corresponding to α_1) for w_2, as a function of γ:

$$d(w_1, w_2) = 2\rho\gamma - 2\sqrt{2b\gamma - 1}\left(\sqrt{1+\rho} + \sqrt{1-\rho}\right).$$

The minimum of $d(w_1, w_2)$ is attained at $\gamma = \frac{1}{2b} + \frac{b}{2\rho^2}\left(\sqrt{1+\rho} + \sqrt{1-\rho}\right)^2$ and has the value $m(\rho, b) = \frac{\rho}{b} - \frac{2b}{\rho}\left(1 + \sqrt{1-\rho^2}\right)$.

We now have to find for each $b \geq 0$ the largest $\rho(b) \in [0, 1]$ such that $m(\rho, b) \leq 0$. This value is given by

$$\rho(b) = 2b\sqrt{2 - b^2} \text{ for } 0 \leq b \leq \frac{1}{\sqrt{2}}.$$

Thus we obtain for the stability radius r_{Lf}

(6) $$r_{Lf}(b) = \begin{cases} 2b\sqrt{1-b^2} & 0 \leq b \leq \frac{1}{\sqrt{2}} \\ 1 & b \geq \frac{1}{\sqrt{2}}. \end{cases}$$

The exact stability radius for time varying uncertainties r based on Lyapunov exponents is given by $r(b) = \min\{\rho \geq 0,\ \kappa(\rho) = 0\}$, according to Corollary 4. A standard perturbation argument for $b > 0$ small shows analytically that $r_{\mathbb{R}}(b) > r(b)$ for small positive b. In fact, the explicit computations in Colonius and Kiemann (1993) show that $r(b) \sim \pi b$ around 0, and (5) implies $r_{Lf}(b) \sim 2b < \pi b$ for b around 0. Figure 1. shows the three stability radii depending on the damping b.

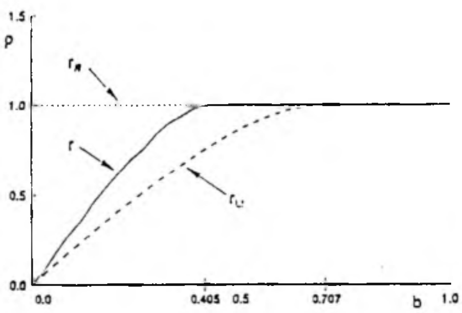

Figure 1. Stability radii of the system (4).

We see that

$$
\begin{aligned}
1 = r_{\mathbb{R}}(b) > r(b) > r_{Lf}(b) &\quad \text{for } 0 < b < b_0 \sim 0.405 \\
1 = r_{\mathbb{R}}(b) = r(b) > r_{Lf}(b) &\quad \text{for } b_0 < b < b_1 = \frac{1}{\sqrt{2}}, \\
1 = r_{\mathbb{R}}(b) = r(b) = r_{Lf}(b) &\quad \text{for } b \geq b_1.
\end{aligned}
$$

Therefore, if $b > b_0$, we have $r(b) = 1$ and the destabilizing uncertainty for $\rho = r(b)$ can be chosen to be a constant, real function. For $0 < b < b_0$ a destabilizing function for $\rho = r(b)$ can be found which is piecewise constant with two switches. This destabilizing uncertainty is adapted to the system dynamics. Faster switchings may even stabilize a system, compare e.g. Bellman et al. (1986). □

The next example shows that the stability radius r need not depend on the system parameters in a monotone way, and may actually be positive only on bounded parameter sets.

Example 10. Consider the system

$$
\begin{aligned}
\dot{x} &= \begin{pmatrix} \alpha & 1 \\ -1 & -2b+\alpha \end{pmatrix} x + v \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} x \\
&=: \left(A(v) + \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \right) x \qquad (7)\\
\mathcal{V}^\rho &= \{v : \mathbb{R} \to V^\rho, \text{ measurable}\}, V^\rho = [-\rho, \rho] \\
&\quad \text{for } \rho \geq 0,
\end{aligned}
$$

which is equation (4) with an added diagonal term $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} x$. For the following computations we use $\alpha = \frac{1}{4}$.

Eigenvalue computations result in the stability radius $r_{\mathbb{R}}$ as

$$
r_{\mathbb{R}}(b) = \begin{cases} 0 & \text{for } 0 \leq b \leq \frac{1}{4}, \ \frac{17}{8} \leq b \\ \frac{1}{2}\left(\frac{17}{8} - b\right) & \text{for } \frac{1}{4} < b < \frac{17}{8}. \end{cases}
$$

Computations for the radius r_{Lf} are similar to the ones shown in the previous example. $r_{Lf}(b)$ has a unique maximum at $(0.77, 0.67)$ and decreases along the line $\frac{1}{2}\left(\frac{17}{8} - b\right)$ as $b \uparrow \frac{17}{8}$.

To compute the Lyapunov exponents note that the matrix $A_\alpha = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$ commutes with all matrices. Denote by $S_\alpha(\tau) = S_\alpha(\tau, S_0, A_\alpha + A(v))$ the solution of the projected system, then we obtain

$$
\begin{aligned}
\lambda(v, x; \alpha) &= \limsup_{t \to \infty} \frac{1}{t} \int_0^t S_\alpha^T(\tau)(A(v) + A_\alpha) S_\alpha(\tau) d\tau \\
&= \limsup_{t \to \infty} \frac{1}{t} \int_0^t S_\alpha^T(\tau) A_\alpha S_\alpha(\tau) A_\alpha S_\alpha(\tau) d\tau + \\
&\quad \limsup_{t \to \infty} \frac{1}{t} \int_0^t S_\alpha^T(\tau) A(v) S_\alpha(\tau) d\tau \\
&= \alpha + \lambda(v, x),
\end{aligned}
$$

where $\lambda(v, x)$ are the Lyapunov exponents of (4). Hence $\kappa(\rho; \alpha) = \alpha + \kappa(\rho)$ and $r(b; \alpha) = \min\{\rho \geq 0, \kappa(\rho, b) \geq -\alpha\}$.

For the value $\alpha = \frac{1}{4}$ the three stability radii are shown in Figure 2.

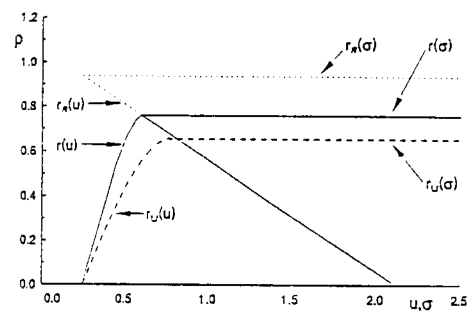

Figure 2. Stability radii of the system (7).

Note that the maxima (in b) of the radii are different, and they are attained at different b-values. □

The next two examples show that the stability radii need not be continuous in parameters of the system. In particular, a jump from 0 to ∞ is possible.

Example 11. Consider the system

$$
\begin{aligned}
&(8) \\
&\dot{x} = \begin{pmatrix} 1+\alpha & 0 \\ 0 & 1+\alpha \end{pmatrix} x + v \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} x \\
&v \in \mathcal{V}^\rho = \{v : \mathbb{R} \to V^\rho, \text{ measurable}\}, V^\rho = [-\rho, \rho].
\end{aligned}
$$

This system satisfies the conditions of Corollary 5, and hence

$$
r(\alpha) = \begin{cases} 0 & \text{for} \quad \alpha \geq -1 \\ \infty & \text{for} \quad \alpha < -1. \end{cases}
$$

The system also satisfies the conditions of Corollary 7 (ii) and hence all three radii have the same behavior. □

Example 12. The linear oscillator with uncertain damping. Consider the oscillator $\ddot{y}+2(b+v(t))\dot{y}+(1+c)y=0$ with $v(t)\in[-\rho,\rho]$, and $c\in\mathbb{R}$. In equivalent first order form the system reads

$$(9)\qquad \dot{x}=\begin{pmatrix}0 & 2\\ -1-c & -2b\end{pmatrix}x+v\begin{pmatrix}0 & 0\\ 0 & -2\end{pmatrix}x.$$

Note that the Lie algebra rank condition (3) is satisfied for this system, except at the value $c=-1$. Clearly, the system is not exponentially stable for $c\leq -1$ with $\rho=0$, and for $c>-1$ with $\rho\geq b$. It remains to consider the case $c>-1$ and $\rho\in(0,b)$. This leads to the optimal strategy $v=-b$, which is a constant. The stabilization radius therefore is given by

$$r_{\mathbb{R}}(c)=r(c)=\begin{cases}0 & \text{for } c\leq -1\\ b & \text{for } c>-1.\end{cases}\qquad\Box$$

3. Robustness of Nonlinear Time-Varying Systems.

In this section we extend the discussion of stability radii to nonlinear systems with time varying parametric uncertainties. We restrict our attention to systems with singular point, i.e. to a common fixed point of the dynamics and the perturbation vector fields. A stability like concept can also be defined for regular systems using the global theory developed Colonius and Kliemann (1997).

Specifically, we consider the following kind of systems in $\mathbb{R}^d$

(10)

$$\dot{y}(t)=f_0(y(t))+\sum_{i=1}^{m}v_i(t)f_i(y(t))=f(y(t),v(t)),$$
$$v\in\mathcal{V}^\rho=\{v:\mathbb{R}\to V^\rho,\ \text{locally integrable}\}$$
$$V\subset\mathbb{R}^m,\ \text{compact and convex}, 0\in\ \text{int}\ V,$$
$$V^\rho=\rho\cdot V,\rho\geq 0.$$

We assume that the maps $f_i:\mathbb{R}^d\to\mathbb{R}^d$ are differentiable with Lipschitz continuous first derivative for $i=0\ldots m$. Let $y^0\in\mathbb{R}^d$ be a common fixed point, i.e. $f_i(y^0)=0$ for $i=0\ldots m$, which is asymptotically stable for the nominal system, i.e. for the differential equation with $v(t)\equiv 0$. Then it is of interest to determine the maximal number $\rho\geq 0$ such that y^0 is a locally asymptotically stable equilibrium for all parametric time varying uncertainties $v\in\mathcal{V}^\rho$.

For constant uncertainties the theory of stable and unstable manifolds implies that the local stability behavior of the nonlinear system around a fixed point y^0 can be analyzed using the linearized system at y^0: If y^0 is a hyperbolic point then the nonlinear system is stable iff all eigenvalues of the linearized system lie in the left half plane $\mathbb{C}^-$. The computation of a nonlinear stability radius for (10) with $v\in V^\rho$ hence reduces to the analysis of the linear radius $r_{\mathbb{R}}$ (as defined in Definition 1) of the linearized system.

For differential equations with time varying perturbations a classical result due to Lyapunov states that only in the case of Lyapunov regularity negative Lyapunov exponents imply local stability of the nonlinear system, see e.g. Hahn (1967). However, in the theory of stability radii stability is required for all disturbances within a given range. The regularity results from Section 2 and the uniform stable manifold theorem from Colonius and Kliemann (1997) are the key to overcome this difficulty.

The linearization of (10) at the singular point y^0 has the form

(11)

$$\dot{x}(t)=\frac{\partial}{\partial y}f(y^0,v(t))x(t)=\left(A_0+\sum_{i=1}^{m}v_i(t)A_i\right)x(t)$$
$$\text{in } \mathbb{R}^d v\in\mathcal{V}^\rho,$$

where $A_i=D_yf_i(y)|_{y=y^0}$ for $i=0\ldots m$ denotes the Jacobian at y^0. Note that this system lives in the tangent space at y^0, which is here identified with $\mathbb{R}^d$. We assume that the projection of (11) onto the projective space $\mathbb{P}^{d-1}$ satisfies the Lie algebra rank condition (3).

Definition 13. The nonlinear stability radius $r_{n\ell}$ of the system (10) at the singular point $y^0\in\mathbb{R}^d$ is defined as

$$r_{n\ell}(y^0)=\inf\{\rho\geq 0,\ \text{there exists } v\in\mathcal{V}^\rho \text{ such that}$$
$$y^0 \text{ is not locally asymptotically stable for}$$
$$\dot{y}=f(y,v)\}.$$

The following theorem shows that this stability radius is determined by the Lyapunov exponents of the linearized system.

Theorem 14. *Consider the system (10) with singular point y^0 and the linearized system (11). Assume that y^0 is locally asymptotically stable for the nominal system $\dot{y}(t)=f(y(t),0),t\geq 0$. Then*

$$\sup\{\rho\geq 0,\kappa(\rho)<0\}\leq r_{n\ell}(y^0)\leq\inf\{\rho\geq 0,\kappa(\rho)>0\},$$

where $\kappa(\rho)$ is defined as in (2). In particular we obtain

$r \leq r_{n\ell}(y^0)$ and

$r = r_{n\ell}(y^0)$ if $\kappa(\rho)$ is strictly increasing at $\rho = r$.

Here r is the stability radius of the linearized system (11) as introduced in Definition 1.

Proof. If $\kappa(\rho) > 0$, then there exists a pair $(v, x) \in \mathcal{V}^\rho \times \mathbb{R}^d \setminus \{0\}$ with Lyapunov exponent $\lambda(v, x) > 0$ and v is periodic. Hence by the theorem on stability in the first approximation (compare Hahn (1967), Theorem 65.4) the equilibrium y^0 is unstable, i.e. there exist a number $\varepsilon > 0$, a sequence $y^n \to y^0$ and a sequence $t^n \uparrow \infty$ such that for all $n \in \mathbb{N}$ it holds that $|\varphi(t^n, y^n, v) - \varphi(t^n, y^0, v)| \geq \varepsilon$. (Here $\varphi(\cdot, y, v)$ denotes the solution of (10) with $\varphi(0, y, v) = y$.) In particular, for $\delta > 0$ sufficiently small the set $\{y \in \mathbb{R}^d, |\varphi(t, y, v) - y^0| < \delta$ for all $t \geq 0\}$ is not a neighborhood of the equilibrium y^0, and y^0 is not locally asymptotically stable. Therefore, $r_{n\ell} \leq \inf\{\rho \geq 0, \kappa(\rho) > 0\}$.

Recall the local stable manifold from Colonius and Kliemann (1997): For the singular point $y^0 \in \mathbb{R}^d$, $\delta > 0$ and $v \in \mathcal{V}^\rho$ we set

$$\mathcal{W}_v^{\delta,t} = \{y \in \mathbb{R}^d, |\varphi(t, y, v) - y^0| < \delta \text{ for all } t \geq 0 \text{ and } \lim_{t\to\infty} |\varphi(t, y, v) - y^0| = 0\}.$$

If $\kappa(\rho) < 0$, then for $\delta > 0$ sufficiently small we have $y^0 \in \text{ int } \mathcal{W}_v^{\delta,t} \subset \mathbb{R}^d$ for all $v \in \mathcal{V}^\rho$. Hence y^0 is locally asymptotically stable for all $v \in \mathcal{V}^\rho$, and therefore $\sup\{\rho \geq 0, \kappa(\rho) < 0\} \leq r_{n\ell}(y^0)$.

The other statements follow from the definitions and continuity of $\kappa(\rho)$. □

The following examples apply Theorem 14 to some two dimensional uncertain systems.

Example 15. Consider the van-der-Pol oscillator

$$\ddot{y} - 2b(y^2 - 1)\dot{y} + (1 + v)y = 0$$
$$v \in \mathcal{V}^\rho, \; V^\rho = [-\rho, \rho], \; \rho \geq 0.$$

The nominal system (with $v \equiv 0$) admits a Hopf bifurcation at $b = 0$. In the coordinates $(y, \dot{y}) = (y_1, y_2)$ the system reads

$$(12) \qquad \dot{y} = \begin{pmatrix} y_2 \\ -y_1 + 2b(y_1^2 - 1)y_2 \end{pmatrix} + v\begin{pmatrix} 0 \\ -y_1 \end{pmatrix}.$$

The origin is a fixed point for all $v \in V$, and linearization at 0 yields

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & -2b \end{pmatrix} x + v \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} x.$$

This is the linear oscillator, which was analyzed in Example 9. The stability radius $r_{n\ell}$ of (12) is, therefore, given by the linear radius r, as shown in Figure 1.

Example 16. A model for the roll motion of a ship is given by the uncertain system

$$(13) \qquad \dot{y} = \begin{pmatrix} y_2 \\ -y_1 + \alpha y_1^3 - \delta_1 y_2 - \delta_2 y_2 |y_2| \end{pmatrix} + v\begin{pmatrix} 0 \\ -y_1 \end{pmatrix}$$
$$v \in \mathcal{V}^\rho, \; V^\rho = [-\rho, \rho], \; \rho \geq 0.$$

Here we consider the robust the robust stability of the singular point $y^0 = (0, 0)$. Linearization around this point yields

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & -\delta_1 \end{pmatrix} x + v \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} x.$$

Again, one obtains the linear oscillator as the linearized system, here with damping $2b = \delta_1$. The stability radius $r_{n\ell}$ of (13) is given again by the linear radius r in Figure 1., with proper scaling of the b-axis. □

The last example in this section explores a connection between nonlinear stability radii and the global behavior of systems.

Example 17. Consider the model

$$(14) \qquad \begin{aligned} \dot{y}_1 &= \alpha y_1 (1 - \frac{1}{K} y_1) - \beta y_1 y_2 \\ \dot{y}_2 &= -\beta y_1 y_2 + \gamma(L - y^2) \end{aligned}$$

with positive constants K, L, α and uncertainty $\beta(t) \in [\beta_0 - \rho, \beta_0 + \rho]$, i.e. $v(t) \in [-\rho, \rho]$ and $\beta(t) = \beta_0 + v(t)$. Furthermore, we have $\frac{1}{\beta} + \frac{1}{\gamma} = 1$ with $\beta > 1$ and $\gamma > 1$.

We first analyze the global behavior of this system with the methods described in Colonius and Kliemanan (1997). Note that the set $M = [0, K] \times [0, L]$ is a compact, forward invariant set of (14), and the Lie algebra rank condition holds in the interior of M. For the following figures we use the parameter values

(15)
$K = 0.5, \; L = 1, \alpha = 4, \beta_0 > 1, v(t) \in [-\rho, \rho]$ for $\rho \geq 0$.

$\beta_0 = 4.15, \; \rho = 0.05$

The nominal system (with $\rho = 0$) has two fixed points in int M for $\beta_0 > 4$. Figure 3. shows the corresponding control sets, with C being invariant (around the stable fixed point), and D being variant. Figure 4. depicts also the domain of attraction $\mathcal{A}(D)$, which is the 'bistability' region for C and the singular point $y^0 = (0, L)$, i.e.

$$\mathcal{A}(d) = \{y \in M, \text{ there exist } v_1, v_1 \in \mathcal{V}^\rho \text{ such that } \varphi(t, y, v_1) \to C \text{ and } \varphi(t, y, v_2) \to y^0 \text{ for } t \to \infty\}.$$

Here $\varphi(\cdot, y, v)$ denotes the solution of (14) with $\varphi(0, y, v) = 0$. Note that the left boundary of $\mathcal{A}(D)$ is the stable manifold of the point

$$y(-\rho) = \left(\frac{1}{2}\left(3 - \frac{\beta_0 - \rho}{\beta_0 - \rho - 1}\right) - \sqrt{\frac{1}{2}\left(\frac{3(\beta_0-\rho)}{\beta_0-\rho-1} - 4\right) + \frac{1}{4}\left(3 - \frac{\beta_0-\rho}{\beta_0-\rho-1}\right)^2}, \frac{4}{\beta_0-\rho}(1 - 2y_1(-\rho)) \right) \tag{16}$$

(where $y_1(-\rho)$ is the first component of $y(-\rho)$) for the system (14) with $\beta = \beta_0 - \rho$.

$\underline{\beta_0 = 4.0, \rho = 1.0}$

In this case the system has only one (invariant) control set C in int M, shown in Figure 5. This set is globally attractive from int$M$. The singular point $y^0 = (0, L)$ is in the boundary of C, and hence C is not closed.

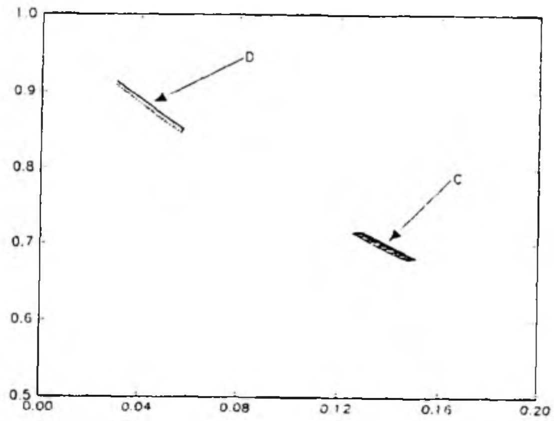

Figure 3. Control sets of (14) with $\beta_0 = 4.15, \rho = 0.05$

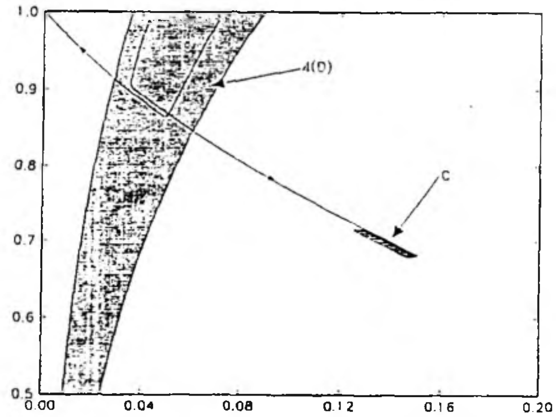

Figure 4. Control sets and bistability region of (14)

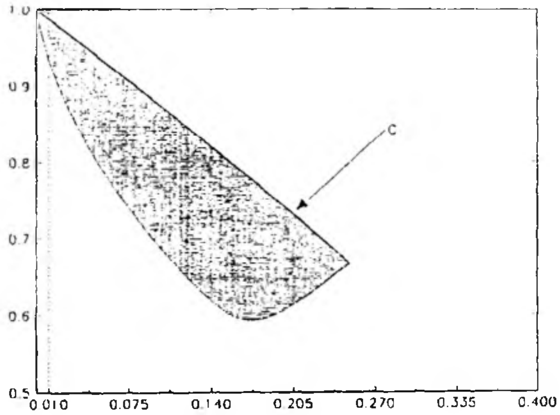

Figure 5. Control sets of (14) with $\beta_0 = 4.0, \rho = 1.0$

with $\beta_0 = 4.15, \rho = 0.05$

We analyze the transition between these two cases using the nonlinear stability radius at the singular point $y^0 = (0, L)$. Linearization of (14) at y^0 yields

$$\dot{x} = \begin{pmatrix} \alpha - \beta(t)L & 0 \\ -\beta(t)L & -\gamma(t) \end{pmatrix} x \tag{17}$$

$$\beta(t) = \beta_0 + v(t), \quad v \in \mathcal{V}^\rho, \quad V^\rho = [-\rho, \rho], \quad \rho \geq 0.$$

The Lyapunov exponents of (17) can be computed directly from the explicit solution, which for the initial value $x^0 = (x_1^0, x_2^0)$ is given by

$$x_1(t) = x_1^0 \exp\left(\int_0^t (\alpha - \beta(s)L)ds\right)$$

$$x_2(t) = x_2^0 \exp\left(\int_0^t -\gamma(s)ds\right) + \exp\left(\int_0^t -\gamma(s)ds\right) \times$$

$$\times \int_0^t \left(\exp\left(\int_0^s \gamma(\tau)d\tau\right) \times (-\beta(s)Lx_1^0 \times \times \exp\left(\int_0^s (\alpha - \beta(\tau)L)d\tau\right)\right) ds.$$

This yields the Lyapunov exponents

$$\lambda_1(v) = \alpha - \limsup_{t\to\infty} \frac{1}{t}\int_0^t L(\beta_0 + v(s))ds$$

$$\lambda_2(v) = -\limsup_{t\to\infty} \frac{1}{t}\int_0^t \gamma(s)ds.$$

Using $\gamma(t) = \frac{\beta(t)}{\beta(t)-1} > 0$, we obtain

$$\kappa(\rho) = \alpha - L(\beta_0 - \rho)$$

and therefore

$$r_{n\ell}(y^0) = \begin{cases} 0 & \text{if } \beta_0 \leq \frac{\alpha}{L} \\ \frac{L\beta_0-\alpha}{L} & \text{if } \beta_0 > \frac{\alpha}{L}. \end{cases}$$

Note that the projection onto $\mathbb{P}^1$ of the linearized system does not satisfy the Lie algebra rank condition (3). However, the Lyapunov exponents of (17) are computed explicitly and $\kappa(\rho)$ is attained at a constant uncertainty $v(t) \equiv -\rho$. Thus Theorem 14 remains valid.

For the parameter settings (15) we obtain

$$r_{n\ell}(y^0) = \begin{cases} 0 & \text{if } \beta_0 \leq 4 \\ \beta_0 - 4 & \text{if } \beta_0 > 4. \end{cases}$$

Comparing the nonlinear stability radius with (16) we make the following observation if $y^0 = (0, L)$ is symptotically stable: The radius $r_{n\ell}(y^0)$ coincides with the ρ-value for which $y(-\rho) = y^0$, and the stable manifold of $y(-\rho)$ becomes the center manifold of y^0. In other words, the left boundary of the 'bistability' region collides at $\rho = r_{n\ell}(y^0)$ with the singular point y^0. For $\beta_0 = 4.0, \rho = 1.0$ the maximal spectral interval of (14) at y^0 hence contains zero in its interior.

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