Topological conjugacy in affine differential equation

Fritz Colonius
Institut für Mathematik, Universität Augsburg
86135 Augsburg, Germany, fritz.colonius@math.uni-augsburg.de

Alexandre J. Santana*
Departamento de Matemática, Universidade Estadual de Maringá
Maringá, Brazil, ajsantana@uem.br

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Abstract

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1 Introduction

For linear differential equation, it was proved in [4], [6] and [7] that ... The purpose of this paper is toWe look at affine differential equation $\dot{x} = Ax$, where $A \in \mathfrak{gl}(d, \mathbb{R})$, from the point of view of (continuous time) dynamical systems, or linear flows in $\mathbb{R}^d$. Precisely, we generalize the related results of [4], [6] and [7] to affine differential equation.

The main subject in this paper are topological conjugacies of affine differential equations in $\mathbb{R}^d$. In particular we will generalize the classical result ....

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2 Affine differential equation

In this section we prove that the real part of the eigenvectors determine the exponential behavior of the solutions of a affine differential equation, described by the Lyapunov exponents. We begin recalling some facts about affine differential equation. Every differentiable function $x : \mathbb{R} \to \mathbb{R}^d$ such that $\dot{x}(t) = Ax(t)+a$ for all $t \in \mathbb{R}$ is called a solution of $\dot{x}(t) = Ax(t)+a$. The initial value problem for a linear differential equation $\dot{x} = Ax + a$ consists in finding, for a given initial value $x_0 \in \mathbb{R}^d$, a solution $x(\cdot, x_0)$ such that $x(0, x_0) = x_0$.

It is well known (see, e.g., Lecture 16 of [1]) that for each initial value problem given by $(A, a) \in \mathfrak{gl}(d, \mathbb{R}) \rtimes \mathbb{R}^d$ and $x_0 \in \mathbb{R}^d$, the solution $x(\cdot, x_0)$ is unique and given by

$$x(t, x_0) = e^{At}x_0 + \int_0^t e^{A(t-s)}ads.$$

The distinct (complex) eigenvalues of $A \in \mathfrak{gl}(d, \mathbb{R})$ will be denoted by $\mu_1, \mu_2, \ldots, \mu_r$. The real versions of the generalized eigenspaces are denoted by $E(A, \mu_k) \subset \mathbb{R}^d$ or simply E_k for $k = 1, \ldots, r \leq d$.

The real Jordan form of $A \in \mathfrak{gl}(d, \mathbb{R})$ is denoted by $J_A^{\mathbb{R}}$. And more, for any matrix A there exists a matrix $T \in \mathrm{Gl}(d, \mathbb{R})$ such that $A = T^{-1}J_A^{\mathbb{R}}T$.

Note that if $A = T^{-1}J_A^{\mathbb{R}}T$ then

$$e^{At} + \int_0^t e^{A(t-s)}ads = T^{-1}e^{J_A^{\mathbb{R}}t}T + \int_0^t T^{-1}e^{J_A^{\mathbb{R}}t}T \cdot T^{-1}e^{J_A^{\mathbb{R}}(-s)}Tads =$$

$$T^{-1}e^{J_A^{\mathbb{R}}t}T + \int_0^t T^{-1}e^{J_A^{\mathbb{R}}(t-s)}Tads.$$

Now we note that the following example will be useful. Consider B a Jordan block of dimension n associated with the complex eigenvalue $\mu = \lambda + i\nu$ of a matrix $A \in \mathfrak{gl}(d, \mathbb{R})$. Then with

$$D = \begin{pmatrix} \lambda & -\nu \\ \nu & \lambda \end{pmatrix} \text{ and } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

for

$$B = \begin{pmatrix} D & I & & & & \\ & \cdot & \cdot & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & I \\ & & & & & D \end{pmatrix} \text{ one has } e^{Bt} = e^{\lambda t} \begin{pmatrix} \hat{D} & t\hat{D} & \frac{t^2}{2!}\hat{D} & \cdot & \cdot & \frac{t^{n-1}}{(n-1)!}\hat{D} \\ & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & \cdot & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & \frac{t^2}{2!}\hat{D} \\ & & & & \cdot & t\hat{D} \\ & & & & & \hat{D} \end{pmatrix},$$

where

$$\hat{D} = \begin{pmatrix} \cos\nu t & -\sin\nu t \\ \sin\nu t & \cos\nu t \end{pmatrix}.$$

That is, for $y_0 = [y_1, z_1, \dots, y_m, z_m]^t \in E(A, \mu)$, the solution of $\dot{y}(t) = By(t) + b$, for every j-th component with $j = 1, \dots, m$, is given by

$$y_j(t, y_0) = e^{\lambda t} \sum_{k=j}^{m} \frac{t^{k-j}}{(k-j)!}(y_k\cos\nu t - z_k\sin\nu t) +$$

$$\int_0^t e^{\lambda(t-s)} \sum_{k=j}^{m} \frac{(t-s)^{k-j}}{(k-j)!}(y_k\cos\nu(t-s) - z_k\sin\nu(t-s))a_k ds,$$

$$z_j(t, y_0) = e^{\lambda t} \sum_{k=j}^{m} \frac{t^{k-j}}{(k-j)!}(z_k\cos\nu t + y_k\sin\nu t) +$$

$$\int_0^t e^{\lambda(t-s)} \sum_{k=j}^{m} \frac{(t-s)^{k-j}}{(k-j)!}(z_k\cos\nu(t-s) + y_k\sin\nu(t-s))b_k ds,$$

where $b = [a_1, b_1, \dots, a_m, b_m]^t$. For a better illustration we take $m = 2$. Then

$$y_1(t, y_0) = e^{\lambda t}((y_1\cos\nu t - z_1\sin\nu t) + t(y_2\cos\nu t - z_2\sin\nu t)) +$$
$$\int_0^t e^{\lambda(t-s)}(y_1\cos\nu(t-s) - z_1\sin\nu(t-s))a_1 ds +$$
$$\int_0^t e^{\lambda(t-s)}(t-s)(y_2\cos\nu(t-s) - z_2\sin\nu(t-s))a_2 ds,$$

$$z_1(t, y_0) = e^{\lambda t}((z_1\cos\nu t + y_1\sin\nu t) + t(z_2\cos\nu t + y_2\sin\nu t)) +$$

$$\int_0^t e^{\lambda(t-s)}(z_1\cos\nu(t-s)+y_1\sin\nu(t-s))b_1ds+$$
$$\int_0^t e^{\lambda(t-s)}(t-s)(z_2\cos\nu(t-s)+y_2\sin\nu(t-s))b_2ds,$$

$$y_2(t,y_0)=e^{\lambda t}(y_2\cos\nu t-z_2\sin\nu t)+\int_0^t e^{\lambda(t-s)}(y_2\cos\nu(t-s)-z_2\sin\nu(t-s))a_2ds,$$

$$z_2(t,y_0)=e^{\lambda t}(z_2\cos\nu t+y_2\sin\nu t)+\int_0^t e^{\lambda(t-s)}(z_2\cos\nu(t-s)+y_2\sin\nu(t-s))b_2ds.$$

Then using mathematical software, we get

$$y_1(t,y_0)=e^{\lambda t}(f_1^y(t)+tg_1^y(t))+C_1^y,\ f_1^y,g_1^y \text{ are bounded and } C_1^y \text{ is constant }.$$
$$z_1(t,y_0)=e^{\lambda t}(f_1^z(t)+tg_1^z(t))+C_1^z,\ f_1^z,g_1^z \text{ are bounded and } C_1^z \text{ is constant }.$$
$$y_2(t,y_0)=e^{\lambda t}f_2^y(t)+C_2^y,\ \text{ where } f_2^y \text{ is bounded and } C_2^y \text{ is constant }.$$
$$z_2(t,y_0)=e^{\lambda t}f_2^z(t)+C_2^z,\ \text{ where } f_2^z \text{ is bounded and } C_2^z \text{ is constant }.$$

Definition 2.1 *The element e_0 is a fixed point of the affine differential equation $\dot{x}(t)=Ax(t)+a$ if $x(t,x_0)=e_0$.*

Proposition 2.2 *Suppose A stable, that is,* $\mathrm{Re}\lambda<0$ *for all $\lambda\in\sigma(A)$. Then*
i) There exists a unique fixed point for $\dot{x}(t)=Ax(t)+a$,
ii) For all $x_0\in\mathbb{R}^d$, $\varphi(t,x_0)\to e_0$ if $t\to\infty$.

Proof: Since the matrix A is invertible we have $0=Ae_0+a$ has a unique solution e_0. As $e_0=e^{At}e_0+\int_0^t e^{A(t-s)}ads$ then

$$\|\varphi(t,x_0)-e_0\|=\|e^{At}x_0+\int_0^t e^{A(t-s)}ads-e_0\|=\|e^{At}(x_0-e_0)\|\to 0$$

if $t\to\infty$. □

Definition 2.3 *Let $x(\cdot, x_0)$ be a solution of the affine differential equation $\dot{x}(t) = Ax(t) + a$ and e_0 its fixed point. Its Lyapunov exponent for x_0 is defined as $\lambda(x_0) = \limsup_{t\to\infty} \frac{1}{t}\ln\|x(t, x_0) - e_0\|$.*

Theorem 2.4 *The Lyapunov exponent $\lambda(x_0)$ of a solution $x(\cdot, x_0)$ of $\dot{x}(t) = Ax(t) + a$ satisfies $\lambda(x_0) = \lim_{t\to\infty} \frac{1}{t}\ln\|x(t, x_0) - e_0\| = \lambda_j$ if and only if $x_0 - e_0 \in L(\lambda_j)$.*

Proof: Recall that for any matrix A there is a matrix $T \in \mathrm{Gl}(d, \mathbb{R})$ such that $A = T^{-1}J_A^{\mathbb{R}}T$, where $J_A^{\mathbb{R}}$ is the real Jordan canonical form of A. Hence we can consider A in the real Jordan form. Then the assertions of the theorem follow of the solution formulas of the above example. We give an idea of these computations in the two dimensional case. Take the above solutions $y_1(t, y_0), z_1(t, y_0), y_2(t, y_0), z_2(t, y_0)$ and note that

$$\|y(t, y_0)\| = \sqrt{y_1^2 + z_1^2 + y_2^2 + z_2^2} =$$

$$\sqrt{(e^{\lambda t}(f_1^y + tg_1^y) + C_1^y)^2 + (e^{\lambda t}(f_1^z + tg_1^z) + C_1^z)^2 + (e^{\lambda t}f_2^y + C_2^y)^2 + (e^{\lambda t}f_2^z + C_2^z)^2}.$$

Then isolating $(e^{\lambda t})^2$ inside the root, the last expression can be written as

$$\|y(t, y_0)\| = \sqrt{(e^{\lambda t})^2 f(t)}.$$

Hence,

$$\frac{1}{t}\ln\|y(t, y_0)\| = \frac{1}{t}\ln\sqrt{(e^{\lambda t})^2 f(t)} = \frac{1}{t}\ln\sqrt{(e^{\lambda t})^2} + \frac{1}{t}\ln\sqrt{f(t)},$$

where $\frac{1}{t}\ln\sqrt{f(t)} \to 0$ if $t \to \infty$.

Therefore, $\lim_{t\to\infty} \frac{1}{t}\ln\|y(t, y_0)\| = \lambda$. By this computation, it is easy to see that $\lim_{t\to\infty} \frac{1}{t}\ln\|y(t, y_0) - e_0\| = \lambda$. □

With the next lemma, some of our results in affine differential equation will be a immediate consequence of the correspondent result in the linear context.

Lemma 2.5 *Let Φ the solution of the system $\dot{x}(t) = Ax(t) + a$ and e_0 its fixed point. Then $\Phi - e_0$ is solution of $\dot{x}(t) = Ax(t)$, that is, $\frac{d}{dt}(\Phi(t, y_0) - e_0) = A(\Phi(t, y_0) - e_0)$.*

Proof: Note that $\frac{d}{dt}(\Phi(t,y_0)-e_0) = A\Phi(t,y_0)+a$. On the other hand, $0 = Ae_0 + a$. Then $a = -Ae_0$. Hence $A\Phi(t,y_0)+a = A\Phi(t,y_0) - Ae_0 = A(\Phi(t,y_0)-e_0)$. □

As in case of linear differential equation (see [4]), in the following result we characterize asymptotic and exponential stability in terms of the eigenvalue of A.

Theorem 2.6 *For an affine differential equation $\dot{x}(t) = Ax(t) + a$ in $\mathbb{R}^d$ the following statements are equivalent:*

i) The fixed point $e_0 \in \mathbb{R}^d$ is asymptotically stable.

ii) The fixed point $e_0 \in \mathbb{R}^d$ is exponentially stable.

iii) All Lyapunov exponents (hence all real parts of the eigenvalues) are negative.

iv) The stable subspace L^- satisfies $L^- = \mathbb{R}^d$.

Proof: Take Φ as solution of the system $\dot{x}(t) = Ax(t)+a$. By above lemma, $\Phi(t,y_0)-e_0$ is a solution of the linear system $\dot{x}(t) = Ax(t)$, where $x-e_0$ is the initial value of the solution $\Phi(t,y_0)-e_0$. Then this theorem is a immediate consequence of Theorem 2.15 in [4]. □

Lemma 2.7 *For $A \in \mathfrak{gl}(d,\mathbb{R})$, the solutions of $\dot{x}(t) = Ax(t) + a$ form a continuous dynamical system with time set $\mathbb{R}$ and state space $M = \mathbb{R}^d$.*

Proof: In fact,

$$\Phi(t,x) = x(t,x) = e^{At}x + \int_0^t e^{A(t-s)}ads$$

satisfies

i) $\Phi(0,x) = x$, for all $x \in \mathbb{R}^n$,

ii) $\Phi(u+t,x) = \Phi(u,\Phi(t,x))$. Note that,

$$\Phi(u,\Phi(t,x)) = e^{A(u+t)}x + \int_0^t e^{A(u+t-s)}ads + \int_0^u e^{A(u-s)}ads.$$

But

$$\int_0^u e^{A(u-s)}ads = \int_0^{u+t} e^{A(u+t-s)}ads.$$

In fact, call $t-s=-v$ then $s=t+v$. Hence $ds=dv$ and if $s=t$, $s=u+t$ it follows that $v=0$ and $v=u$ respectively. Then, $\int_0^{u+t} e^{A(u+t-s)}ads = \int_0^u e^{A(u-v)}adv$. Therefore,

$$\Phi(u+t,x)=\Phi(u,\Phi(t,x)).$$

iii) $\Phi(t,x)$ is continuous. □

3 Conjugacy in affine differential equation

In this section we study the affine differential equation $\dot{x}(t)=Ax(t)+a$, with $(A,a)\in \mathfrak{gl}(d,\mathbb{R})\rtimes\mathbb{R}^d$, from the point of view of dynamical systems, or flows in $\mathbb{R}^d$. The results are similar to linear differential equations of the form $\dot{x}(t)=Ax(t)$ (see [4], [6] and [7]).

Theorem 3.1 *Consider the dynamical systems Φ associated with $\dot{x}(t)=Ax(t)+a$ and Ψ associated with $\dot{x}(t)=Bx(t)+b$. Then the following statements are equivalent:*

(i) Φ and Ψ are C^k conjugate for $k\geq 1$.

(ii) Φ and Ψ are linearly conjugate.

(iii) Φ and Ψ are affinely similar, that is, $A=TBT^{-1}$ and $Ta=b$ for some $T\in \mathrm{Gl}(d,\mathbb{R})$.

Proof: We have

$$\Phi(t,x)=e^{At}x+\int_0^t e^{A(t-s)}ads \text{ and } \Psi(t,x)=e^{Bt}x+\int_0^t e^{B(t-s)}bds,$$

and if h is a conjugacy map, then $h(\Phi(t,x))=\Psi(t,h(x))$, that is, for all t and all x

$$h(e^{At}x+\int_0^t e^{A(t-s)}ads)=e^{Bt}h(x)+\int_0^t e^{B(t-s)}bds.$$

First we prove that (iii) implies (ii). Note that

$$A=TBT^{-1}\Leftrightarrow e^{tA}=Te^{tB}T^{-1}\Leftrightarrow T^{-1}e^{tA}=e^{tB}T^{-1} \tag{1}$$

(here $\Longleftarrow$ is seen by differentiating and evaluating in $t = 0$). Then define $h : \mathbb{R}^d \to \mathbb{R}^d$ as $h(z) = T^{-1}z$. Hence

$$h(e^{At}x + \int_0^t e^{A(t-s)}ads) = T^{-1}(e^{At}x + \int_0^t e^{A(t-s)}ads) =$$
$$T^{-1}e^{At}x + T^{-1}\int_0^t e^{A(t-s)}ads = e^{Bt}T^{-1}x + \int_0^t T^{-1}e^{A(t-s)}ads =$$
$$e^{Bt}T^{-1}x + \int_0^t e^{B(t-s)}T^{-1}ads = e^{Bt}h(x) + \int_0^t e^{B(t-s)}bds.$$

Supposing (ii) we prove that (iii) holds. By (ii) there is a linear conjugacy h such that for all t and all x

$$h(e^{At}x + \int_0^t e^{A(t-s)}ads) = e^{Bt}h(x) + \int_0^t e^{B(t-s)}bds. \tag{2}$$

Differentiating with respect to x, we find for all t

$$Dh(e^{At}x + \int_0^t e^{A(t-s)}ads)e^{At} = e^{Bt}Dh(x).$$

Observing that h is linear, we see with $T^{-1} := Dh(0)$

$$T^{-1}e^{At} = Dh(0)e^{At} = e^{Bt}Dh(0) = e^{Bt}T^{-1} \tag{3}$$

and hence, by (1),

$$A = TBT^{-1}.$$

Inserting into (2), we find for all t and all x

$$T^{-1}(e^{At}x + \int_0^t e^{A(t-s)}ads) = e^{Bt}T^{-1}x + \int_0^t e^{B(t-s)}bds,$$

which, with (3), implies for all t

$$T^{-1}\int_0^t e^{A(t-s)}ads = e^{Bt}T^{-1}x - T^{-1}e^{At}x + \int_0^t e^{B(t-s)}bds = \int_0^t e^{B(t-s)}bds.$$

Then for all t

$$e^{Bt}\int_0^t e^{-Bs}bds = \int_0^t e^{B(t-s)}bds = T^{-1}\int_0^t e^{A(t-s)}ads = T^{-1}e^{At}\int_0^t e^{-As}ads =$$

$$e^{Bt}T^{-1}\int_0^t e^{-As}ads = e^{Bt}\int_0^t T^{-1}e^{-As}ads = e^{Bt}\int_0^t e^{-Bs}T^{-1}ads.$$

This implies that for all t

$$e^{-Bt}b = e^{-Bt}T^{-1}a, \quad \text{i.e., } b = T^{-1}a.$$

Similar computations prove that (i) implies (iii). Now, (ii) obviously implies (i). □

Corollary 3.2 *Consider $\dot{x}(t) = Ax(t) + a$, with $A \in \mathfrak{gl}(d, \mathbb{R})$, and Φ its associated dynamical system. Take $J_A^{\mathbb{R}}$ the Jordan form of A and Ψ its associated dynamical system. Then there is a linear conjugacy h such that $h(\Phi(t,x)) = \Psi(t, h(x))$.*

Proposition 3.3 *Take $\Phi(t,x)$ a solution of $\dot{x}(t) = Ax(t) + a$, $A \in \mathfrak{gl}(d, \mathbb{R})$. Then the following properties are equivalents:*

i) There are a norm $\|\cdot\|_*$ on $\mathbb{R}^d$ and $a > 0$ such that for all $x \in \mathbb{R}^d$, $\|\Phi(t,x) - e_0\|_* \leq e^{-at}\|x - e_0\|_*$ for all $t \geq 0$.

ii) For every norm $\|\cdot\|$ on $\mathbb{R}^d$ there are $\alpha > 0$ and $C > 0$ with $\|\Phi(t,x) - e_0\| \leq Ce^{-\alpha t}\|x - e_0\|$ for all $t \geq 0$.

iii) For every eigenvalue λ of A one has $\mathrm{Re}\lambda < 0$.

Proof: The item i) implies ii) since all norms on $\mathbb{R}^d$ are equivalents. The item ii) are equivalent to iii) by Theorem 2.6. It remains to show that ii) implies i). As $\Phi(t, y_0) - e_0$ is a particular solution for $\dot{x}(t) = Ax(t)$ then by Proposition 3.17 in [4] there exists a norm $\|\cdot\|_*$ on $\mathbb{R}^d$ and $a > 0$ satisfying the item i). Note that here $x - e_0$ is the initial value of the solution $\Phi(t, y_0) - e_0$, i. e., $\Phi(o, y_0) - e_0 = x - e_0$. □

Proposition 3.4 *Let $A, B \in \mathfrak{gl}(d, \mathbb{R})$. If all eigenvalues of A and of B have negative real parts, then the respective flows*

$$\Phi(t,x) = e^{At}x + \int_0^t e^{A(t-s)}ads \text{ and } \Psi(t,x) = e^{Bt}x + \int_0^t e^{B(t-s)}bds$$

are topologically conjugate.

Proof: If $A, B \in \mathfrak{gl}(d, \mathbb{R})$ and if all eigenvalues of A and B have negative real parts, then by Proposition 3.19 in [4] there exists a topological map h with $h(e^{At}x) = e^{Bt}(h(x))$. Now note that $\Phi(t,x) - e_A$ $(\Psi(t,x) - e_B)$ is a solution of $\dot{x}(t) = Ax(t)$ $(\dot{x}(t) = Bx(t))$ with initial value $x - e_A$ $(x - e_B)$ where e_A (e_B) is the fixed point of $\dot{x}(t) = Ax(t) + a$ $(\dot{x}(t) = Bx(t) + b)$. Then

$$\Phi(t,x) - e_A = e^{At}(x - e_A) \qquad (\Psi(t,x) - e_B = e^{Bt}(x - e_B)). \tag{4}$$

Furthermore, $h(e^{At}(x-e_A)) = e^{Bt}h(x-e_A) = e^{Bt}(h(x-e_A)+e_B-e_B)$. Then by (4) we have $h(e^{At}(x-e_A)) = \Psi(t, h(x-e_A)+e_B)-e_B$. But $h(e^{At}(x-e_A)) = h(\Phi(t,x)-e_A)$ then $h(\Phi(t,x)-e_A) = \Psi(t, h(x-e_A)+e_B)-e_B$, or equivalently,

$$h(\Phi(t,x) - e_A) + e_B = \Psi(t, h(x - e_A) + e_B). \tag{5}$$

Define $H(x) = h(x-e_A)+e_B$. Then by (5) we have $H(\Phi(t,x)) = \Psi(t, H(x))$.

Now, as h is bijective, continuous, invertible and h^{-1} is continuous the same is true to H. Therefore H is a topological conjugacy. $\square$

Theorem 3.5 *Suppose A and B are hyperbolic and take their associated linear flows Φ and Ψ in $\mathbb{R}^d$. Then Φ and Ψ are conjugate if and only if the dimensions of the stable subspaces (and hence the dimensions of the unstable subspaces) of A and B agree.*

Proof: If A and B are hyperbolic then A and B has no eigenvalues with null real parts, that is, every eigenvalue has no null real part. Then we can decompose $\mathbb{R}^n$ as $\mathbb{R}^n = \mathbb{E}^s_A \oplus \mathbb{E}^u_A$ and $\mathbb{R}^n = \mathbb{E}^s_B \oplus \mathbb{E}^u_B$, where $\mathbb{E}^s_A$ $(\mathbb{E}^s_B)$ and $\mathbb{E}^u_A$ $(\mathbb{E}^u_B)$ denote the stable and unstable subspace associated with A (B). As the stable subspaces have the same dimension, by last proposition there is a conjugation

$$H^s : \mathbb{E}^s_A \to \mathbb{E}^s_B.$$

Considering negative time there is also a conjugation

$$H^u : \mathbb{E}^u_A \to \mathbb{E}^u_B.$$

Hence with the natural projections $\pi^s : \mathbb{R}^n \to \mathbb{E}^s_A$ and $\pi^u : \mathbb{R}^n \to \mathbb{E}^u_A$ we define a topological conjugation as

$$H(x) = H^s(\pi^s(x)) + H^u(\pi^u(x)).$$

Remark 3.6 *As in case of linear differential equation, if $A \in \mathfrak{gl}(n, \mathbb{R})$ is hyperbolic and B is close enough to A, then the corresponding affine flows are topologically conjugate.*

□

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