AN APPROACH TO MINIMAL BIT RATES AND ENTROPY FOR DETERMINISTIC CONTROL SYSTEMS*

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Abstract. For deterministic control systems with digital communication constraints, an approach is explained which, in particular, permits to determine minimal bit rates and entropy for exponential stabilization. In the case of linear systems, a formula for the entropy associated with exponential stabilization is provided.

Key words. digital communication, topological entropy, stabilization

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1. Introduction.

The problem to determine minimal data rates for performing control tasks has been considered for quite a while, see the survey Nair, Fagnani, Zampieri and Evans [27]. Early landmarks are the papers by Delchamps [14], who considered quantized information for stabilization and proposed to use statistical methods from ergodic theory, and Wong and Brockett [30] who discussed stabilization of linear systems via coding. From the wealth of literature on this topic we also cite Tatikonda and Mitter [28], Delvenne [15], Fagnani and Zampieri [17], Bullo and Liberzon [3] and the monograph Matveev and Savkin [24].

In the seminal paper Nair, Evans, Mareels, and Moran [26], the notion of topological feedback entropy is proposed which is a variant of the classical notion of topological entropy for dynamical systems; see Adler, Konheim, and McAndrew [1]. In non-technical terms, the basic idea for the related approach presented here (which is closer in spirit to the Bowen-Dinaburg version of topological entropy) is the following. Consider a control task on the time interval $[0,\infty)$. For example, this may be the problem to make a subset of the state space invariant or the problem to stabilize the system at an equilibrium. Then a controller device is constructed which performs the control task based on measurements of the output of the system. If successful, the controller will generate control actions on the system such that the desired behavior is achieved for all initial values in a given set K in the state space. If continual measurement of the output is not possible due to data rate constraints (in a noiseless communication channel), the controller only has a finite amount of information available on any finite interval $[0,T]$. Hence, it may appear reasonable that the controller can only generate a finite number of time-dependent control functions $u(t)$, $t\in[0,T]$, which are to guarantee the desired behavior on $[0,T]$ for every initial state in K. If time increases, the amount of information for the controller increases, and hence it may generate more controls. Looking at this from the other side, the number of control functions which are necessary for accomplishing the control task on $[0,T]$, determines the minimal bit rate. Thus, the growth rate of the minimal number of control functions as time tends to infinity is a measure for the minimal bit rate required in order to accomplish the control task on $[0,\infty)$ for all initial values in K (certainly, this provides a lower bound.)

The control functions may be obtained via quantizations of the state space, sym-

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bolic controllers or via devices like Model Predictive Control (MPC). In any case, this results in a collection of time-dependent control functions u defined on $[0,\infty)$ yielding the desired objective. Instead of concentrating on the algorithmic question how to generate these controls, we discuss the minimal bit rate needed to discern the time-dependent control functions u on any time interval $[0,T]$, $T>0$. This, in fact, is the point of view taken in Tatikonda and Mitter [28], p. 1057, who estimate the minimal bit rate for stabilization of discrete-time linear systems from below, see Proposition 3.2 in [28].

This basic idea can be formalized in different ways, depending, in particular, on the considered control objectives. The problem to keep the system in a subset of the state space is treated in detail in a forthcoming monograph by Christoph Kawan; see also Kawan [20, 21, 22, 23] and Colonius and Kawan [9], [10]. In the following we concentrate on stabilization problems.

As mentioned above, in this problem formulation a close analogy to the notion of topological entropy in the theory of dynamical systems is apparent; the monographs Katok and Hasselblatt [19], Walters [29], and Downarowicz [16] contain expositions of this theory. Here one observes, how fast trajectories of a dynamical system move apart, and hence one counts initial points. In control, the decisive parameter which determines the behavior of trajectories is the control function. Hence we will count control functions and then we use rather analogous mathematical constructions. Regrettably, many formulas in entropy theory contain repeated limit operations and, on first sight, may appear extremely technical. The same is true when this machinery is applied in control. However, these formulas have intuitive interpretations which help to guess their properties, and in my opinion this theory provides efficient mathematical tools also in our field.

In Section 2, the entropy notion for exponential stabilization is introduced and motivation for its formulation is provided. In particular, also relations to quantization of the state space are briefly discussed. Section 3 presents results for linear control systems and discusses modifications in the case where a dynamic compensator is used to generate controls. Final Section 4 mentions further work and some open problems.

Notation. The closure of a set A is $\mathrm{cl}A$ and the number of elements of a finite set A is $\#A$; if A is the empty set or if A has infinitely many elements, we set $\#A=\infty$. The limit superior and the limit inferior are denoted by $\overline{\lim}$ and $\underline{\lim}$, respectively.

2. Entropy for flows and control systems.

In this section, we start by briefly sketching the idea of topological entropy of linear autonomous differential equations. Then entropy for exponential stabilization is motivated and defined.

Topological entropy for linear autonomous differential equations answers the question: How many "different" trajectories are there? To be more precise, consider for $\lambda>0$ the scalar differential equation

$$\dot{x}(t)=\lambda x(t), t\geq 0, x(0)=x_0\in K:=[-1,1].$$

Fix $T,\varepsilon>0$. A finite set $R(T,\varepsilon)\subset[-1,1]$ of initial values is called (T,ε)-spanning if for all $x_0\in[-1,1]$ there is $y_0\in R(T,\varepsilon)$ with

$$e^{\lambda t}\left|x_0-y_0\right|=\left|e^{\lambda t}x_0-e^{\lambda t}y_0\right|<\varepsilon \text{ for all } t\in[0,T].$$

The minimal number of elements in such a set $R(T,\varepsilon)$ grows like $e^{\lambda T}$ and hence

$$\lim_{T\to\infty}\frac{1}{T}\log\#R(T,\varepsilon)=\lambda.$$

This can be interpreted in the following way: $\log_2 \# R(T,\varepsilon)$ is the number of bits (the information) generated by the system on $[0,T]$ modulo ε. If $\#\mathcal{R}(T,\varepsilon) = 2^k$, then the elements can be encoded by sequences $(s_1 \ldots s_k)$ with $s_i \in \{0,1\}$. Now consider the general case for $A \in \mathbb{R}^{n\times n}$,

$$\dot{x}(t) = Ax(t), t \geq 0, x(0) = x_0 \in K \subset \mathbb{R}^n,\ K \text{ compact.}$$

Again, fix $T, \varepsilon > 0$. A finite set $R(T,\varepsilon) \subset K$ is called (T,ε)-spanning if for all $x_0 \in K$ there is $y_0 \in R(T,\varepsilon)$ with

$$\left\| e^{At}x_0 - e^{At}y_0 \right\| < \varepsilon \text{ for all } t \in [0,T].$$

Let $R(T,\varepsilon)$ be minimal. Then the topological entropy with respect to K is defined as

$$h_{\text{top}}(K) := \lim_{\varepsilon\to 0} \overline{\lim}_{T\to\infty} \frac{1}{T} \log \# R(T,\varepsilon).$$

A classical result due to Bowen [2] states that for every set K with nonvoid interior $h_{\text{top}}(K) = \sum_{\text{Re}\,\lambda>0} \text{Re}\,\lambda$, where the natural logarithm is taken and summation is over all unstable eigenvalues λ of A counted according to their multiplicity; see also Walters [29, Theorem 8.14]. One sees that there are two ingredients in this notion: The parameter $\varepsilon > 0$ allows us to obtain on every interval $[0,T]$ a finite amount of information, and then the average value for time $T \to \infty$ is considered. At the end, the parameter ε is sent to 0 (which amounts to taking the supremum over $\varepsilon > 0$.) Bowen's formula shows up in many places in problems with communication constraints. In fact, a variant of it will also be relevant below.

Next we turn to control systems and we will use somehow analogous constructions for control problems by counting the average number of necessary time-dependent control functions. This excludes feedbacks. In fact, for a control system $\dot{x} = f(x,u)$ a stabilizing feedback $u = F(x)$ generates controls (depending on x_0)

$$u(t) := F(x(t,x_0)), x_0 \in \mathbb{R}^n,$$

where $x(t,x_0)$ solves

$$\dot{x}(t,x_0) = f(x(t,x_0), F(x(t,x_0))),\ x(0,x_0) = x_0.$$

When the initial states are in an uncountable set, in general, also uncountably many controls will have to be generated, even on a finite time interval.

We will consider exponential stabilization properties. Let $K \subset \mathbb{R}^n$ be a bounded set of initial states with $0 \in \text{int}K$ and assume that there are $\alpha > 0, M > 1$ such that for all $0 \neq x_0 \in K$ there is a control u with

$$\|x(t,x_0,u)\| < Me^{-\alpha t} \|x_0\| \text{ for all } t \geq 0.$$

Note that for stable systems it may also be of interest to increase the exponential decay rate α. The following proposition shows that finitely many controls do not suffi ce to get the exponential estimate, even on finite intervals. Hence we will have to add a parameter ε in the problem formulation.

PROPOSITION 2.1. *Consider a linear control system of the form*

$$\dot{x} = Ax + Bu$$

with $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$ and controls $u : [0,\infty) \to \mathbb{R}^m$ which are integrable on every bounded interval. Assume that there is an eigenvalue λ of A with $\operatorname{Re}\lambda \geq 0$. Let $\alpha > 0, M > 1$ and consider a neighborhood K of the origin. Then, for $T > 0$ large enough, there is no finite set $\mathcal{R}$ of control functions such that for every $0 \neq x_0 \in K$ there is $u \in \mathcal{R}$ with

$$\|x(t,x_0,u)\| < Me^{-\alpha t}\|x_0\| \text{ for all } t \in [0,T]. \tag{2.1}$$

Proof. We proceed by contradiction. Suppose that a finite set $\mathcal{R} = \{u_1, ..., u_r\}$ of control functions with the stated property exists and define

$$K_j := \{x_0 \in K \mid \|x(t,x_0,u_j)\| < Me^{-\alpha t}\|x_0\| \text{ for all } t \in [0,T]\}.$$

Consider $x_0 \neq 0$ in the real eigenspace $E(\lambda)$ for λ and suppose that T is large enough such that $Me^{-\alpha T} < 1$. Then the variations-of-constants formula shows that no control u_0 with $\int_0^t e^{A(t-s)}Bu_0(s)ds \equiv 0$ will yield estimate (2.1) for x_0. There are controls $u_j \in \mathcal{R}, j \in J \subset \{1, ..., r\}$ which for all $0 \neq x_0 \in K \cap E(\lambda)$ yield estimate (2.1). Then we may assume that for every control $u_j, j \in J$, there is $t_j \in [0,T]$ with

$$c_j := \max_{t\in[0,T]} \left\| \int_0^t e^{A(t-s)}Bu_j(s)ds \right\| = \left\| \int_0^{t_j} e^{A(t_j-s)}Bu_j(s)ds \right\| > 0.$$

Choose $0 \neq x_0 \in K \cap E(\lambda)$ with $\|x_0\| < e^{-T\operatorname{Re}\lambda} \min_{j\in J} \frac{c_j}{2M}$. We find for every $j \in J$ the contradiction

$$\begin{aligned}\|x(t_j,x_0,u_j)\| &= \left\| e^{\lambda t_j}x_0 + \int_0^{t_j} e^{A(t_j-s)}Bu_j(s)ds \right\| \\ &\geq \left\| \int_0^{t_j} e^{A(t_j-s)}Bu_j(s)ds \right\| - e^{t_j\operatorname{Re}\lambda}\|x_0\| \\ &\geq c_j - \frac{c_j}{2} = \frac{c_j}{2} \geq e^{-\alpha t_j}\frac{c_j}{2} > Me^{-\alpha t_j}\|x_0\|.\end{aligned}$$

□

In contrast to linear control systems, the scalar bilinear system

$$\dot{x} = (1+u)x, u \in U = \mathbb{R},$$

can be stabilized by the single constant control $u(t) \equiv -2$. Thus a single bit is suffi cient. See also de Persis [13] for other situations where finitely many bits are suffi cient. While it might be worthwhile to study bilinear control systems in this context, we follow a different path in the rest of this paper and relax the exponential controllability property by introducing a small additive term.

Let us formulate our problem of exponential stabilization about the equilibrium $x^* = 0$ corresponding to the control $u^* = 0$ more formally. Consider a nonlinear control system of the form

$$\dot{x}(t) = f(x(t),u(t)), \ u \in \mathcal{U}, \tag{2.2}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous and Lipschitz continuous with respect to the first argument; the admissible controls are given by

$$\mathcal{U} = \{u : [0,\infty) \to \mathbb{R}^m \mid u(t) \in U \text{ for almost all } t \geq 0\},$$

where we assume that the controls u are integrable on every bounded interval and the control range U is a subset of $\mathbb{R}^m$. We assume that (i) unique global solutions $x(t,x_0,u), t \geq 0$, of the differential equation with initial condition $x(0) = x_0 \in \mathbb{R}^n$ and control $u \in \mathcal{U}$ exist and (ii) on compact intervals, the solutions depend continuously on the initial value.

The following simple, but basic lemma shows that for a relaxed version of the exponential controllability property only finitely many bits are required on a finite interval.

LEMMA 2.2. *Consider a control system of the form (2.2) and let K be a compact subset of $\mathbb{R}^n$. Assume that there are constants $\alpha > 0$ and $M > 1$ such that for all $0 \neq x_0 \in K$ there is $u \in \mathcal{U}$ with*

$$\|x(t,x_0,u)\| < Me^{-\alpha t}\|x_0\| \text{ for all } t \geq 0. \tag{2.3}$$

Let $\varepsilon > 0$. Then for every $T > 0$ there is a finite set $\mathcal{R} = \{u_1, \ldots, u_r\} \subset \mathcal{U}$ such that for every $x_0 \in K$ there is $u_j \in \mathcal{S}$ with

$$\|x(t,x_0,u_j)\| < e^{-\alpha t}(\varepsilon + M\|x_0\|). \tag{2.4}$$

Proof. For every $x_0 \in K$ choose a control $u \in \mathcal{U}$ with

$$\|x(t,x_0,u)\| < Me^{-\alpha t}\|x_0\| \text{ for all } t \in [0,T].$$

By continuous dependence on initial values (as assumed for (2.2)) there is δ with $0 < \delta < \varepsilon/M$ such that for all $y_0 \in \mathbb{R}^n$ with $\|x_0 - y_0\| < \delta$ one has for all $t \in [0,T]$

$$\begin{aligned}\|x(t,y_0,u)\| &< Me^{-\alpha t}\|x_0\| \leq Me^{-\alpha t}(\|x_0 - y_0\| + \|y_0\|)\\ &< Me^{-\alpha t}(\delta + \|y_0\|)\\ &< e^{-\alpha t}(\varepsilon + M\|y_0\|).\end{aligned}$$

Now compactness of K shows that there is a finite set $\mathcal{R} = \{u_1, \ldots, u_r\} \subset \mathcal{U}$ such that for each $y_0 \in K$ there is $u_j \in \mathcal{R}$ satisfying for all $t \in [0,T]$

$$\|x(t,y_0,u_j)\| < e^{-\alpha t}(\varepsilon + M\|y_0\|).$$

□

We will introduce two ways to measure the information needed for stabilization and begin with an entropy-like notion. Consider a compact set $K \subset \mathbb{R}^n$ of initial states, and let $\alpha > 0, M > 1$ and $\varepsilon > 0$. For a time $T > 0$ we call $\mathcal{R} \subset \mathcal{U}$ a (T,ε)-spanning set of controls if for all $x_0 \in K$ there is $u \in \mathcal{R}$ with

$$\|x(t,x_0,u)\| < e^{-\alpha t}(\varepsilon + M\|x_0\|) \text{ for all } t \in [0,T]. \tag{2.5}$$

The minimal number of elements in such a set is

$$r_{\text{stab}}(T,\varepsilon) := \min\{\#\mathcal{R} \mid \mathcal{R} \text{ is } (T,\varepsilon)\text{-spanning}\}. \tag{2.6}$$

(Recall the convention at the end of the introduction.) Note that for $\varepsilon_1 > \varepsilon_2 > 0$, any (T,ε_2)-spanning set is also (T,ε_1)-spanning. Lemma 2.2 shows that exponential controllability condition (2.3) implies the existence of finite (T,ε)-spanning sets. We want to determine which information has to be transmitted through a digital communication channel in order to identify a control function in such a finite set $\mathcal{S}$. The elements can be encoded by symbols given by finite sequences of 0's and 1's in the set

$$\Sigma_k := \{(s_0 s_1 s_2 ... s_{k-1}) \mid s_i \in \{0,1\} \text{ for } i = 0,1,...,k-1\},$$

where $k \in \mathbb{N}$ is the least integer greater than or equal to $\log_2 \#\mathcal{R}$. Thus $\#\mathcal{R}$ is bounded above by 2^k. Equivalently, the number of bits determining an element of $\mathcal{R}$ is $\log_2(2^k) = k$. It will be convenient to use the natural logarithm instead of the logarithm with base 2.

Definition 2.3. *Let K be a compact set in $\mathbb{R}^n$ and $\alpha > 0, M > 1$. Then the (α, M)-stabilization entropy $h_{\text{stab}}(\alpha, M, K)$ is defined by*

$$h_{\text{stab}}(\alpha, M, K) = \lim_{\varepsilon \searrow 0} \overline{\lim_{T \to \infty}} \frac{1}{T} \log r_{\text{stab}}(T, \varepsilon).$$

Remark 2.4. *Naturally, the number $r_{\text{stab}}(T, \varepsilon)$ depends also on α, M, and K and on the considered control system. We omit these arguments.*

The existence of the limit for $\varepsilon \searrow 0$ is obvious, since it equals the supremum over all $\varepsilon > 0$. (The value $+\infty$ is allowed.) Furthermore, the inequality $h_{\text{stab}}(\alpha', M') \le h_{\text{stab}}(\alpha, M)$ holds for $\alpha \ge \alpha' > 0$ and $M' \ge M > 1$. If one would consider $\alpha = 0$, condition (2.5) just means that every trajectory starting in K remains in the ball around the origin with radius $\varepsilon + M \max_{x \in K} \|x\|$. In this case, the results on invariance entropy from Kawan [21, 22, 23] would apply.

A second way of counting bits, different from entropy, is the following. Consider a set of control functions defined on $[0, \infty)$ which allow us to steer the system asymptotically to the equilibrium $x^* = 0$ satisfying the following conditions. Let $\alpha > 0, M > 1, \varepsilon > 0$ and let γ be a decreasing function on $[0, \infty)$ with $\gamma(0) = \varepsilon$ and $\lim_{t \to \infty} \gamma(t) = 0$. For brevity, we call γ an $\mathcal{L}_\varepsilon$-function (note that continuity of γ is not required.) Let $\mathcal{S} \subset \mathcal{U}$ be a set of of control functions such that for all $x_0 \in K$ there is $u \in \mathcal{S}$ with

$$\|x(t, x_0, u)\| < \gamma(t) + M e^{-\alpha t} \|x_0\| \text{ for all } t \ge 0. \tag{2.7}$$

Then $\mathcal{S}$ is called (γ, ε)-stabilizing for K. Thus in the ε-neighborhood of the equilibrium, the decay given by the exponential rate α may slow down, but still convergence holds for $t \to \infty$. Let $\mathcal{S}_T := \{u_{|[0,T]} \mid u \in \mathcal{S}\}$ be the corresponding restrictions of the controls in $\mathcal{S}$. The bit rate on the time interval $[0, T]$ is defined as $\frac{1}{T} \log \#\mathcal{S}_T$ and the required bit rate for stabilization using controls in $\mathcal{S}$ is

$$b(\mathcal{S}) := \underline{\lim}_{T \to \infty} \frac{1}{T} \log \#\mathcal{S}_T.$$

Definition 2.5. *With the notions introduced above, the minimal bit rate for (α, M)-stabilization at $x^* = 0$ of a compact set $K \subset \mathbb{R}^n$ is*

$$b_{\text{stab}}(\alpha, M, K) := \lim_{\varepsilon \searrow 0} \inf_{\gamma \in \mathcal{L}_\varepsilon} \inf_{\mathcal{S}} b(\mathcal{S}),$$

where the inner infimum is taken over all (γ, ε)-stabilizing sets $\mathcal{S} \subset \mathcal{U}$ of controls and the outer infimum is taken over all $\mathcal{L}_\varepsilon$-functions γ.

If we would define a minimal stabilization bit rate using the fixed $\mathcal{L}_\varepsilon$-function $\gamma(t) := \varepsilon e^{-\alpha t}, t \ge 0$, any (γ, ε)-stabilizing set $\mathcal{S}$ would lead to (T, ε)-spanning sets

$\mathcal{S}_T, T > 0$, and hence one could derive that the stabilization entropy is a lower bound. However, in this case we cannot prove that the stabilization entropy h_{stab} is an upper bound (or merely that there exists a finite upper bound); see Theorem 3.1.

The stabilization entropy h_{stab} indicates how much the number of required control functions increases, when time increases. Here minimization is performed on each interval $[0, T]$ separately. If one wants to enlarge the time interval where the exponential decay holds, one may have to consider controls which, when restricted to the smaller interval, are different from the earlier ones. This is in contrast to minimal bit rates b_{stab}, where restrictions to $[0, T]$ are considered for control functions defined on $[0, \infty)$. Thus, while stabilization entropy certainly merits its own interest, the minimal bit rate b_{stab} might appear more appealing from this point of view. The difference between these two concepts can also be seen by looking at them from a quantization point of view. Let $\mathcal{S}$ be a (γ, ε)-stabilizing set of controls such that for every $T > 0$ the set $\mathcal{S}_T$ of restrictions to $[0, T]$ is finite. Then define for every $u \in \mathcal{S}_T$

$$K(u, T) := \{x_0 \in K \mid \|x(t, x_0, u)\| < \gamma(t) + Me^{-\alpha t} \|x_0\| \text{ for all } t \in [0, T]\}.$$

The sets $K(u, T)$ form an open cover of K which may be viewed as a finite quantization. For $T' > T$, the same construction for $\mathcal{S}_{T'}$ again yields a finite quantization of K which is obtained by refining the quantization at time T, since both are obtained by restrictions of controls in $\mathcal{S}$. In contrast, the quantization for $T' > T$ obtained by a (T', ε)-spanning set of controls used for defining the entropy h_{stab} is not related to the quantization associated with a (T, ε)-spanning set.

For general nonlinear control systems described by ordinary differential equations, a number of estimates from above and from below are available in Colonius [5]. In the next section, only results in the linear case are cited.

3. Entropy for linear control systems. In this section several results on entropy and minimal bit rates for stabilization are presented. In particular, entropy in the presence of a compensator is discussed.

Consider linear control systems in $\mathbb{R}^n$ of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \quad u(t) \in \mathbb{R}^m, \tag{3.1}$$

with matrices $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$.

The next theorem characterizes the stabilization entropy about the equilibrium $x^* = 0$ for linear control systems and gives an estimate for the stabilization bit rate. Proofs are given in [5].

THEOREM 3.1. *Consider a linear control system of the form (3.1). Assume that there are $\alpha > 0, M > 1$ such that for every initial value $0 \neq x_0 \in \mathbb{R}^n$ there is a control $u \in \mathcal{U}$ with*

$$\|x(t, x_0, u)\| < Me^{-\alpha t} \|x_0\| \text{ for all } t \geq 0. \tag{3.2}$$

For every compact set $K \subset \mathbb{R}^n$ with nonvoid interior the (α, M)-stabilization entropy h_{stab} and the minimal bit rate b_{stab} of system (3.1) satisfy for $\alpha > \alpha' > 0$

$$b_{\text{stab}}(\alpha', M, K) \leq h_{\text{stab}}(\alpha, M, K) = \sum_{\operatorname{Re} \lambda_i > -\alpha} (\alpha + \operatorname{Re} \lambda_i); \tag{3.3}$$

here summation is over all eigenvalues λ_i of A, counted according to their multiplicity, with $\operatorname{Re} \lambda_i > -\alpha$. Furthermore,

$$\inf_{\alpha > 0} b_{\text{stab}}(\alpha, M, K) = \inf_{\alpha > 0} h_{\text{stab}}(\alpha, M, K) = \sum_{\operatorname{Re} \lambda_i > 0} \operatorname{Re} \lambda_i.$$

REMARK 3.2. *Formula (3.3) shows that in the linear case* $h_{\mathrm{stab}}(\alpha, M, K)$ *is independent of* K *and of* $M > 1$*, large enough, and hence we may just write it as* $h_{\mathrm{stab}}(\alpha)$.

REMARK 3.3. *In a discrete time setting, a formula similar to (3.3) shows up in Nair and Evans [25, Theorem 1] for a problem with random initial states.*

Theorem 3.1 characterizes the stabilization entropy without any assumption on how the stabilizing controls are generated. Now, a standard way to stabilize a system relies on dynamic compensators. So suppose that the system is connected with a dynamic compensator and the average bit rate for the communication from the system to the compensator is restricted. What can we say about the entropy in this case?

More specifically, consider the problem to stabilize a system of the form

$$\dot{x} = Ax + Bu, y = Cx$$

using $u = Fz$ with z generated by a compensator on the form

$$\dot{z} = Jz + Nu - Ky,$$

where J, K and N are matrices of appropriate dimensions. Suppose that there are a feedback matrix F and a compensator such that the closed loop system given by

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A & BF \\ -KC & J + NF \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}$$

is exponentially stable (this holds, if (A, B) is stabilizable and (C, A) is detectable.) In particular, this system can be constructed from a stabilizing state feedback F and an observer with $J = A + GC, K = G, N = B$.

Then, what is the entropy for the transfer from the system to the compensator? When the implementation of $y = Cx$ is not possible, we have to replace $y = Cx$ by an input $v(t)$ for the compensator. This yields the extended system

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A & BF \\ 0 & J + NF \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ -K \end{bmatrix} v(t). \tag{3.4}$$

By stability of the closed loop system, this system satisfies for appropriate $\alpha > 0$ the exponential controllability condition (3.2). Hence, by Theorem 3.1, the entropy $\hat{h}_{\mathrm{stab}}$ for α-stabilization of the extended system is given by

$$\hat{h}_{\mathrm{stab}}(\alpha) = \sum_{\mathrm{Re}\,\lambda_i > -\alpha} (\alpha + \mathrm{Re}\,\lambda_i) + \sum_{\mathrm{Re}\,\mu_i > -\alpha} (\alpha + \mathrm{Re}\,\mu_i),$$

where summation is over the eigenvalues λ_i of A with $\mathrm{Re}\,\lambda_i > -\alpha$ and over the eigenvalues μ_i of $J + NF$ with $\mathrm{Re}\,\mu_i > -\alpha$, respectively.

This entropy may be strictly larger than the stabilization entropy of the system given by $\dot{x} = Ax + Bu$ as shown by the following simple example.

EXAMPLE 3.4. *Consider the system given by*

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C = [1, 0].$$

The system is controllable and observable with eigenvalues ± 1 *and hence* $h_{\mathrm{stab}}(\alpha) = \alpha + 1$ *for* $\alpha \in (0, 1)$. *Stabilization with a dynamic observer and a stabilizing state feedback gives rise to an extended system (3.4) of the form*

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A & BF \\ 0 & A + BF + GC \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ -G \end{bmatrix} v(t).$$

A direct computation shows that $A + BF, A + GC$*, and* $A + BF + GC$ *cannot all be stable for any* F *and* G *and hence the entropy satisfies* $\hat{h}_{\text{stab}}(\alpha) > h_{\text{stab}}(\alpha)$.

In fact, the problem to determine the minimal entropy of the extended system is closely related to a classical problem in linear control: In general, one cannot find a stable controller (of arbitrary order) for a stabilizable system. For single-input-single-output systems, the paper Youla, Bongiorno and Lu [31] has shown that this is possible if and only if the so-called parity interlacing condition on the zeros and poles of the system holds.

This discussion illustrates that the stabilization entropy defined above is only a lower bound for data rates. If additional conditions are imposed on how the control functions are generated the entropy will, in general, increase.

4. Further work and open problems. It seems that the basic idea for entropy, sketched in Section 1, can be brought to bear for many control problems. However, for the precise problem formulation (the "entropy-zation") as well as for its analysis, considerable further work may have to be invested. In addition to the exponential stabilization problem discussed above and invariance entropy, an entropy notion for controlled invariant subspaces has been studied in Colonius and Helmke [8], Colonius [6]. Here also extensions of Bowen's classical results (Bowen [2]) on topological entropy of linear flows are needed which are given in Colonius, San Martin, da Silva [11]. In the presence of an exosystem, entropy for stabilization problems is discussed in Colonius [4]. Da Silva [12] has introduced invariance entropy for random control systems. Invariance entropy for linear infinite-dimensional systems (with finite-dimensional unstable subspace) has been treated in Hoock [18]; again a characterization in terms of the sum of the real parts of the unstable eigenvalues can be obtained.

Topological entropy of flows is particularly relevant for nonlinear systems where its positivity indicates complicated dynamics, and Kawan [21, 22, 23] could use related techniques in the context of invariance entropy. For exponential stabilization entropy of nonlinear control systems only first results are available in [5]. A further challenge is to analyze minimal data rates and entropy associated with control systems comprised of many subsystems.

As is well known, control systems may be viewed in the general framework of semigroup actions. For example, if one considers piecewise constant controls the corresponding solution maps form a semigroup acting on the state space. The notion of invariance entropy can appropriately be modified to apply to general topological semigroup actions and some results are available in Colonius, Fukuoka, and Santana [7].

Open problems also include the question if similar constructions may help in finding minimal bit rates and entropy for state estimation. Furthermore, a great deal of interest in the concept of topological entropy for flows comes from the fact that it coincides with the supremum of the measure-theoretic entropies (with respect to all invariant measures). An analogous construction for a control theoretic entropy is not available.

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