Duality of moduli and quasiconformal mappings in metric spaces

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Research Article

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Abstract: We prove a duality relation for the moduli of the family of curves connecting two sets and the family of surfaces separating the sets, in the setting of a complete metric space equipped with a doubling measure and supporting a Poincaré inequality. Then we apply this to show that quasiconformal mappings can be characterized by the fact that they quasi-preserve the modulus of certain families of surfaces.

1 Introduction

A homeomorphism $f: X \to Y$ between two metric spaces X, Y is said to be quasiconformal if there is a constant $H \geq 1$ such that for all $x \in X$,

$$\limsup_{r \to 0^+} \frac{\sup_{y \in \overline{B}(x,r)} d_Y(f(x), f(y))}{\inf_{y \in X \setminus B(x,r)} d_Y(f(x), f(y))} \leq H.$$

In metric measure spaces satisfying suitable conditions such as Ahlfors regularity and a Poincaré inequality, the study of quasiconformal mappings was begun by Heinonen and Koskela in [12] and by now the literature is extensive, see for example [3, 11, 13, 21, 25]. As in the classical Euclidean setting, there are also other notions of quasiconformality. For Ahlfors Q-regular spaces X, Y, a homeomorphism $f: X \to Y$ is said to be *geometric quasiconformal* if there is a constant $K \geq 1$ such that whenever Γ is a family of curves in X, we have

$$\frac{1}{K} \operatorname{Mod}_Q(f\Gamma) \leq \operatorname{Mod}_Q(\Gamma) \leq K \operatorname{Mod}_Q(f\Gamma).$$

For the definition of Q-modulus and all other concepts needed in the paper, we refer to Section 2. If both X and Y are complete and also support a Q-Poincaré inequality, the two notions of quasiconformality are equivalent, see Theorem 9.8 in [13].

A fact that has received much less attention is that quasiconformal mappings also quasi-preserve the $\frac{Q}{Q-1}$-modulus of certain families of surfaces obtained as "essential boundaries" of sets of finite perimeter. This result was proved in Euclidean spaces by Kelly [18, Theorem 6.6]. In the metric space setting, the theory of functions of bounded variation (BV) and sets of finite perimeter was first developed by Ambrosio and Miranda [2, 23]. The authors of the current paper together with Shanmugalingam extended Kelly's result to metric spaces in [15].

In the current paper, our main goal is to show that the converse holds as well: if a homeomorphism f quasi-preserves the modulus of families of surfaces, then it is a quasiconformal mapping. Since the analogous

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fact is already known to hold for families of curves, we invest most of our efforts in studying the *duality* of moduli of families of curves and surfaces. Specifically, for a nonempty bounded open set $\Omega \subset X$ and two disjoint sets $E, F \subset \Omega$, we consider the family of curves Γ joining E and F in Ω, and the family of surfaces $\mathcal{L}$ separating E and F in Ω in a suitable sense. Then we prove the following theorem; the precise formulation and assumptions on the sets E and F are given in Theorem 4.5.

Theorem 1.1. *Let* $1 < p < \infty$ *and suppose X is a complete metric space equipped with a doubling measure and supporting a* 1*-Poincaré inequality. For some constant* $C \geq 1$ *depending only on p and the space X, we have*

$$1 \leq \operatorname{Mod}_{\frac{p}{p-1}}(\mathcal{L})^{\frac{p-1}{p}} \operatorname{Mod}_p(\Gamma)^{\frac{1}{p}} \leq C.$$

In Euclidean spaces, this was proved (with constant $C = 1$) by Ziemer [26], and later by Aikawa and Ohtsuka who show in [1] that the same result holds for a more general weighted modulus with weights coming from the Muckenhoupt A_p-class. Combining Theorem 1.1 with the characterization of quasiconformal mappings by means of the moduli of curve families, we get the following theorem.

Theorem 1.2. *Suppose that X and Y are complete Ahlfors p-regular metric spaces supporting a* 1*-Poincaré inequality. Suppose* $f : X \to Y$ *is a homeomorphism and there exists* $C_0 > 0$ *such that for every collection* $\mathcal{L}$ *of surfaces in X,*

$$\operatorname{Mod}_{\frac{p}{p-1}}(f\mathcal{L}) \leq C_0 \operatorname{Mod}_{\frac{p}{p-1}}(\mathcal{L}).$$

Then f is quasiconformal.

This is given, in a somewhat more general form, in Theorem 5.1. Results similar to Theorem 1.1 and Theorem 1.2 were very recently proved in the metric space setting by Lohvansuu and Rajala [22], but their viewpoint was somewhat different. In [22] (similarly to [26]) the authors understood a "surface" to be a set of finite codimension one Hausdorff measure separating E and F in a topological sense. By contrast, we understand surfaces to be sets of finite perimeter in the spirit of [18] and [15].

Moreover, we wish to study the problem under weaker assumptions: instead of Ahlfors regularity it is in fact enough to assume in Theorem 1.2 that the measures on X and Y are doubling and satisfy suitable one-sided growth bounds. Additionally, we do not assume the sets E and F to be closed, as was done in [22] and [26]. Working with more general sets makes it a rather subtle problem to find the correct definition for a "surface" that separates E and F; for this we apply the concept of *fine topology*, relying on results proved in [4, 6, 7]. Hence our arguments combine the theory of quasiconformal mappings, BV theory, and fine potential theory in metric spaces.

2 Notation and definitions

In this section we gather the definitions and assumptions that we need in the paper.

Throughout the paper, (X, d, μ) is a complete metric measure space with μ a Radon measure. We assume that X consists of at least 2 points. If a property holds outside a set with μ-measure zero, we say that it holds almost everywhere, or a.e.

Given $x \in X$ and $r > 0$, we denote an open ball by $B(x, r) := \{y \in X : d(y, x) < r\}$. Given that in a metric space a ball, as a set, could have more than one radius and more than one center, we will consider a ball to be also equipped with a radius and center; thus two different balls might correspond to the same set. We then denote $\operatorname{rad}(B) := r$ as the pre-assigned radius of the ball $B = B(x, r)$, and $aB := B(x, ar)$ for $a > 0$.

Definition 2.1. We say that μ is *doubling* if there exists a constant $C_d \geq 1$, called the *doubling constant*, such that

$$0 < \mu(2B) \leq C_d\mu(B) < \infty$$

for every ball B.

We say that (X, d, μ) is *Ahlfors Q-regular*, with $Q > 0$, if there is a constant $C_A \geq 1$ such that whenever $x \in X$ and $0 < r < \operatorname{diam}(X)$, we have

$$\frac{r^Q}{C_A} \leq \mu(B(x, r)) \leq C_A \, r^Q.$$

Throughout the paper, we always assume μ to be doubling.

Definition 2.2. Let $A \subset X$. The *codimension 1 Hausdorff measure* of A is given by

$$\mathcal{H}(A) := \lim_{r \to 0^+} \inf \left\{ \sum_{k \in I} \frac{\mu(B_k)}{\operatorname{rad}(B_k)} \,\middle|\, A \subset \bigcup_{k \in I} B_k \text{ where } \operatorname{rad}(B_k) \leq r \text{ and } I \subset \mathbb{N} \right\}.$$

Note that a complete metric space equipped with a doubling measure is always proper, that is, closed and bounded sets are compact. Given an open set $\Omega \subset X$, we write $u \in L^1_{\text{loc}}(\Omega)$ if $u \in L^1(V)$ for every open $V \Subset \Omega$; this expression means that $\overline{V}$ is a compact subset of Ω. Other local spaces are defined analogously.

A *curve* is a continuous mapping from a compact interval into X, and a *rectifiable* curve is a curve with finite length. The length of a rectifiable curve γ is denoted by ℓ_γ. Every rectifiable curve can be parametrized by arc-length, see e.g. [10, Theorem 3.2]. In the following definitions, we let $1 \leq p < \infty$; in most of the paper we will assume that $1 < p < \infty$.

Definition 2.3. Let $\mathcal{M}$ be a collection of Borel measures on X. The admissible class of $\mathcal{M}$, denoted $\mathcal{A}(\mathcal{M})$, is the set of all nonnegative Borel functions $\rho \colon X \to [0, \infty]$ such that

$$\int_X \rho \, d\lambda \geq 1$$

for all $\lambda \in \mathcal{M}$. The *p-modulus* of the family $\mathcal{M}$ is given by

$$\operatorname{Mod}_p(\mathcal{M}) := \inf_{\rho \in \mathcal{A}(\mathcal{M})} \int_X \rho^p \, d\mu.$$

We say that a nonnegative Borel function ρ is *p-weakly admissible* for the collection $\mathcal{M}$ if ρ is admissible for all but a p-modulus zero collection of measures.

Mod_p is an outer measure on the class of all Borel measures, see [8]. There are two types of collections of measures associated with quasiconformal mappings. Firstly, given a collection Γ of curves in X, we set Γ to also denote the arc-length measures restricted to each curve in Γ; then the admissibility condition is replaced by

$$\int_\gamma \rho \, ds \geq 1$$

for every rectifiable $\gamma \in \Gamma$, where

$$\int_\gamma \rho \, ds := \int_0^{\ell_\gamma} \rho(\gamma(s)) \, ds$$

for rectifiable γ. We say that a property holds for p-almost every curve if it fails only for a curve family with zero p-modulus. Secondly, for a collection $\mathcal{L}$ of sets of finite perimeter in a set Ω, we consider the measures $P(U, \cdot)$ for each $U \in \mathcal{L}$ (see the definition given later).

Definition 2.4. Let $\Omega \subset X$ be μ-measurable. Given a function $u \colon \Omega \to \overline{\mathbb{R}}$, a Borel function $g \colon \Omega \to [0, \infty]$ is said to be an *upper gradient of u in Ω* if for every nonconstant rectifiable curve γ in Ω,

$$|u(x) - u(y)| \leq \int_\gamma g \, ds, \tag{2.1}$$

where x and y are the endpoints of γ. We interpret $|u(x) - u(y)| = \infty$ whenever either $|u(x)|$ or $|u(y)|$ is infinite. A function u is said to be in the *Newton-Sobolev class* $N^{1,p}(\Omega)$ if $u \in L^p(\Omega)$ and there is an upper gradient g of u in Ω such that $g \in L^p(\Omega)$. We let

$$\|u\|_{N^{1,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \inf \|g\|_{L^p(\Omega)},$$

where the infimum is taken over upper gradients g of u in Ω. We say that a nonnegative μ-measurable function g is a *p-weak upper gradient* of a function u in Ω if (2.1) holds for p-almost every curve in Ω.

If $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$, then there exists a *minimal p-weak upper gradient* of u in Ω, always denoted by g_u, satisfying $g_u \leq g$ a.e. in Ω for every p-weak upper gradient $g \in L^p_{\mathrm{loc}}(\Omega)$ of u in Ω; see [4, Theorem 2.25] We refer the reader to [4, 14, 24] for more details regarding mappings in $N^{1,p}_{\mathrm{loc}}(\Omega)$.

Definition 2.5. We say that the space X supports a *p-Poincaré inequality* if there exist constants $C_P > 0$ and $\lambda \geq 1$ such that for all balls B in X, all measurable functions u on X and all upper gradients g of u,

$$\fint_B |u - u_B| \, d\mu \leq C_P \operatorname{rad}(B) \left(\fint_{\lambda B} g^p \, d\mu \right)^{1/p}.$$

Here we denote the integral average of u over B by

$$u_B := \fint_B u \, d\mu := \frac{1}{\mu(B)} \int_B u \, d\mu.$$

We will assume throughout the paper that X supports a 1-Poincaré inequality.

Definition 2.6. For any disjoint sets $E, F \subset X$, we define $\Gamma(E, F; X)$ to be the collection of curves in X joining E and F. We say that X is a *Loewner space* if there is a function $\phi \colon (0, \infty) \to (0, \infty)$ such that

$$\operatorname{Mod}_p(\Gamma(E, F; X)) \geq \phi(t)$$

whenever E and F are two disjoint, nondegenerate continua (compact connected sets) such that

$$t \geq \Delta(E, F) := \frac{\operatorname{dist}(E, F)}{\min\{\operatorname{diam}(E), \operatorname{diam}(F)\}}.$$

Definition 2.7. The *p-capacity* of a set $A \subset X$ is given by

$$\operatorname{Cap}_p(A) := \inf \|u\|_{N^{1,p}(X)},$$

where the infimum is taken over functions $u \in N^{1,p}(X)$ satisfying $u \geq 1$ in A. If a property holds outside a set with p-capacity zero, we say that it holds *p-quasieverywhere*, or *p-q.e.*

We say that a set $V \subset X$ is *p-quasiopen* if for every $\varepsilon > 0$ there is an open set $G \subset X$ such that $\operatorname{Cap}_p(G) < \varepsilon$ and $V \cup G$ is open.

The *relative p-capacity* of two sets $A \subset \Omega \subset X$ is given by

$$\operatorname{cap}_p(A, \Omega) := \inf \int_X g_u^p \, d\mu$$

where the infimum is over all functions $u \in N^{1,p}(X)$ such that $u \geq 1$ p-q.e. in A and $u \leq 0$ in $X \smallsetminus \Omega$. Recall that g_u denotes the minimal p-weak upper gradient of u.

We know that Cap_p is an outer capacity in the following sense:

$$\operatorname{Cap}_p(A) = \inf\{\operatorname{Cap}_p(W) : W \supset A, \ W \text{ is open}\}$$

for any $A \subset X$, see e.g. [4, Theorem 5.31].

If $\Omega \subset X$ is μ-measurable, then

$$v = 0 \text{ p-q.e. in } \Omega \text{ implies } \|v\|_{N^{1,p}(\Omega)} = 0, \tag{2.2}$$

see [4, Proposition 1.61].

From now on, let $1 < p < \infty$.

Definition 2.8. A set $E \subset X$ is *p-thin* at $x \in X$ if

$$\int_0^1 \left(\frac{\mathrm{cap}_p(E \cap B(x,t), B(x,2t))}{\mathrm{cap}_p(B(x,t), B(x,2t))} \right)^{\frac{1}{p-1}} \frac{dt}{t} < \infty.$$

If E is not p-thin at x, we say that it is *p-thick*. We denote the collection of all points where E is p-thick by $b_p E$. If $X \smallsetminus E$ is p-thin at each point $x \in E$, we say that the set E is *p-finely open*. Then the *p-fine topology* on X is the collection of all p-finely open sets.

Definition 2.9. Given a nonempty open set Ω and two disjoint sets E, F, we define the capacity of the *condenser* $(E, F; \Omega)$ by

$$\mathrm{cap}_p(E, F; \Omega) := \inf \int_\Omega g_u^p \, d\mu,$$

where the infimum is taken over all $u \in N^{1,p}(\Omega)$ satisfying $0 \leq u \leq 1$ in Ω, $u = 1$ in $E \cap \Omega$, and $u = 0$ in $F \cap \Omega$.

Definition 2.10. A function $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ is a *p-minimizer* in an open set $\Omega \subset X$ if for all $\varphi \in \mathrm{Lip}_c(\Omega)$ we have

$$\int_{\{\varphi \neq 0\}} g_u^p \, d\mu \leq \int_{\{\varphi \neq 0\}} g_{u+\varphi}^p \, d\mu,$$

where $\mathrm{Lip}_c(\Omega)$ denotes the collection of Lipschitz functions that are compactly supported in Ω. If the above inequality holds for all nonnegative $\varphi \in \mathrm{Lip}_c(\Omega)$, we say that u is a *p-superminimizer*, and if it holds for all nonpositive $\varphi \in \mathrm{Lip}_c(\Omega)$, we say that u is a *p-subminimizer*.

Next we consider the theory of BV functions in metric spaces.

Definition 2.11. For an open set $\Omega \subset X$ and $u \in L^1_{\mathrm{loc}}(\Omega)$, the total variation of u in Ω is given by

$$\|Du\|(\Omega) := \inf \left\{ \liminf_{n \to \infty} \int_\Omega g_{u_n} \, d\mu : (u_n)_{n \in \mathbb{N}} \subset \mathrm{Lip}_{\mathrm{loc}}(\Omega), \ u_n \to u \text{ in } L^1_{\mathrm{loc}}(\Omega) \right\}.$$

We say $u \in L^1(\Omega)$ is of *bounded variation* on Ω, denoted $u \in \mathrm{BV}(\Omega)$, if $\|Du\|(\Omega) < \infty$.

It is shown in [23, Theorem 3.4] that $\|Du\|$ is a Radon measure in Ω for any $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$. We call $\|Du\|$ the variation measure of u.

Definition 2.12. A measurable set $U \subset X$ has *finite perimeter* in Ω if $\|D\chi_U\|(\Omega) < \infty$. We call $\|D\chi_U\|$ the *perimeter measure* of U and we will denote it $P(U, \cdot)$.

Definition 2.13. We say that X supports a *relative isoperimetric inequality* if there exist constants $C_I > 0$ and $\lambda \geq 1$ such that for all balls B and for all measurable sets U, we have

$$\min\{\mu(B \cap U), \mu(B \smallsetminus U)\} \leq C_I \, \mathrm{rad}(B) P(U, \lambda B).$$

We know that when μ is doubling and X supports a 1-Poincaré inequality, then it supports a relative isoperimetric inequality, see for example [20, Theorem 3.3] (in a slightly different form, this was proved earlier in [2, Theorem 4.3]).

The noncentered Hardy-Littlewood maximal function of a function $\rho \in L^1_{\text{loc}}(X)$ is defined by

$$\mathcal{M}\rho(x) := \sup_{B \ni x} \fint_B |\rho|\, d\mu, \tag{2.3}$$

where the supremum is taken over all open balls containing $x \in X$.

Finally we give the definition of quasiconformal mappings on metric spaces. Let (Y, d_Y, μ_Y) be another metric space equipped with a Radon measure μ_Y.

Definition 2.14. For a function $f \colon X \to Y$, define for all $x \in X$ and $r > 0$

$$L_f(x, r) := \sup_{y \in \overline{B}(x,r)} d_Y(f(x), f(y)) \quad \text{and} \quad l_f(x, r) := \inf_{y \in X \setminus B(x,r)} d_Y(f(x), f(y)).$$

A homeomorphism f is *(metric) quasiconformal* if there is a constant $H \geq 1$ such that for all $x \in X$ we have

$$\limsup_{r \to 0^+} \frac{L_f(x, r)}{l_f(x, r)} \leq H.$$

A homeomorphism f is *geometric quasiconformal* if there is a constant $K \geq 1$ such that whenever Γ is a family of curves in X, we have

$$\frac{1}{K} \operatorname{Mod}_p(f\Gamma) \leq \operatorname{Mod}_p(\Gamma) \leq K \operatorname{Mod}_p(f\Gamma).$$

It is known that when both X and Y are Ahlfors p-regular and support a p-Poincaré inequality, the two notions of quasiconformality are equivalent, see Theorem 9.8 in [13]. We will make use of this fact in Section 5, but we will give a self-contained proof where we only need somewhat weaker assumptions than Ahlfors regularity.

Standing assumptions: Throughout this paper we will assume that $1 < p < \infty$ and that (X, d, μ) is a complete metric measure space that supports a 1-Poincaré inequality, such that μ is doubling. We will use the letter C to denote various nonnegative constants that depend only on p and the space X, and the value of C could differ at each occurrence.

3 Background results

In this section we will gather most of the background results needed in the paper. We start with the following coarea formula for BV functions, which is stated in Remark 4.3 of [23].

Theorem 3.1. *Suppose $\Omega \subset X$ is open and $u \in \operatorname{BV}(\Omega)$. For each $t \in \mathbb{R}$, denote the super-level set of u by $\{u > t\} := \{x \in \Omega : u(x) > t\}$. Then for every nonnegative Borel function ϕ on Ω and every Borel set $A \subset \Omega$,*

$$\int_{\mathbb{R}} \left(\int_A \phi\, dP(\{u > t\}, \cdot) \right) dt = \int_A \phi\, d\|Du\|.$$

We have the following "continuity from below" for families of measures; for a proof see Lemma 2.3 in [27].

Lemma 3.2. *If $\{\mathcal{L}_j\}_{j \in \mathbb{N}}$ is a sequence of families of Borel measures such that $\mathcal{L}_j \subset \mathcal{L}_{j+1}$ for each j, then*

$$\operatorname{Mod}_p \left(\bigcup_{j \in \mathbb{N}} \mathcal{L}_j \right) = \lim_{j \to \infty} \operatorname{Mod}_p(\mathcal{L}_j).$$

By applying Fuglede's and Mazur's lemmas, see e.g. [13, p.19, p.131], we get the following.

Lemma 3.3. *Let $\mathcal{L}$ be a family of Borel measures with $\mathrm{Mod}_p(\mathcal{L}) < \infty$. Then there exists a p-weakly admissible function ρ such that*

$$\int_X \rho^p \, d\mu = \mathrm{Mod}_p(\mathcal{L}).$$

The following lemma is proved in [22, Lemma 5.2].

Lemma 3.4. *If ρ is a $p/(p-1)$-integrable, $p/(p-1)$-weakly admissible function for a family of Borel measures $\mathcal{L}$ such that*

$$\mathrm{Mod}_{\frac{p}{p-1}}(\mathcal{L}) = \int_X \rho^{\frac{p}{p-1}} \, d\mu$$

and ϕ is another $p/(p-1)$-integrable, $p/(p-1)$-weakly admissible function for $\mathcal{L}$, then

$$\mathrm{Mod}_{\frac{p}{p-1}}(\mathcal{L}) \le \int_X \phi \, \rho^{\frac{1}{p-1}} \, d\mu.$$

We note that various results that we cite, such as the following theorem, rely on assuming the space to support a p-Poincaré inequality, but this follows via Hölder's inequality from the 1-Poincaré inequality that is our standing assumption.

Theorem 3.5. *Suppose that X satisfies the lower mass bound*

$$\mu(B(x,r)) \ge cr^p$$

for all $x \in X$ and $0 < r < \mathrm{diam}(X)$, and some constant $c > 0$. Then X is a Loewner space.

Proof. See Theorem 5.7 in [12]. Note that the so-called φ-convexity assumed in this theorem holds since under our assumptions the space is *quasiconvex*, meaning that every pair of points can be joined by a curve whose length is at most a constant number times the distance between the points; see e.g. [4, Theorem 4.32]. □

The space X is *linearly locally connected* in the following sense.

Theorem 3.6. *Suppose X satisfies the upper mass bound*

$$\mu(B(x,r)) \le C_0 r^p$$

for all $x \in X$ and $r > 0$, and a constant $C_0 > 0$. Then there exists a constant $C_1 \ge 1$ such that for every ball $B = B(x,r)$, any pair of points in $B \smallsetminus \frac{1}{2}B$ can be joined by a curve in $B(x, C_1 r) \smallsetminus B(x, r/C_1)$.

Proof. See Remark 3.19 in [12]; note that there it is also assumed that the space is of Hausdorff dimension p, but this is not needed in the proof. □

Finally we give a few results concerning superminimizers; recall Definition 2.10. Let $W \subset X$ be an open set. We define the *lsc-regularization* (lower semicontinuous regularization) of a function u on W by

$$u^*(x) := \lim_{r \to 0} \operatorname*{ess\,inf}_{B(x,r)} u, \quad x \in W.$$

The following proposition is given as part of Theorem 8.22 in [4].

Proposition 3.7. *If u is a p-superminimizer in W, then u^* is lower semicontinuous in W and $u = u^*$ p-q.e. in W.*

More precisely, the fact that $u = u^*$ p-q.e. in W is given in the *proof* of [4, Theorem 8.22]. By (2.2) we know that u^* is still a p-superminimizer.

It is a well known fact that *superharmonic functions* are *finely continuous*; this was shown in the metric space setting in [7] and [19]. Here we record this result in the following theorem, which follows by combining Proposition 7.12, Theorem 9.24(a,c), and Theorem 11.38 of [4].

Theorem 3.8. *Let u be a p-superminimizer in W. Then u^* is continuous with respect to the p-fine topology in W.*

4 Proof of Theorem 1.1

We will consider the following families of curves and surfaces; recall the concept of capacitary thinness from Definition 2.8.

Definition 4.1. For an open set $\Omega \subset X$ and any disjoint sets $E, F \subset X$, we define $\Gamma(E, F; \Omega)$ to be the collection of curves in Ω joining $E \cap \Omega$ and $F \cap \Omega$. We also define the collection of measures

$$\mathcal{L}(E, F; \Omega) := \{P(U, \Omega \cap \cdot) : \ U \subset X \text{ is } \mu\text{-measurable with}$$
$$b_p E \cap \Omega \subset \text{int}(U) \text{ and } b_p F \cap \Omega \subset \text{ext}(U)\},$$

where $\text{ext}(U) = X \smallsetminus \overline{U}$.

By an abuse of terminology, we will also talk about the sets U belonging to $\mathcal{L}(E, F; \Omega)$. Essentially, the boundaries of U are "surfaces" that "separate" E and F in Ω, but since we do not assume E and F to necessarily be compact subsets of Ω, the choice of the correct definition for $\mathcal{L}(E, F; \Omega)$ becomes rather subtle. If one would employ the usual definition where the surfaces need to stay at a strictly positive distance from E and F, it would be difficult to prove the lower bound of Theorem 1.1. On the other hand, if one allows the surfaces to "touch" E and F significantly, then it becomes difficult to prove the upper bound. For this reason, we allow the surfaces to "touch" E and F only at capacitary thinness points.

Throughout this section, we will abbreviate $\mathcal{L} = \mathcal{L}(E, F; \Omega)$ and $\Gamma = \Gamma(E, F; \Omega)$. We begin by proving the lower bound.

Proposition 4.2. *Let $\Omega \subset X$ be nonempty, open and bounded and let $E, F \subset X$ with $\overline{E} \cap \overline{F} = \varnothing$. Then $\text{Mod}_p(\Gamma) < \infty$, and if also $\text{Mod}_{\frac{p}{p-1}}(\mathcal{L}) < \infty$, then*

$$1 \le \text{Mod}_{\frac{p}{p-1}}(\mathcal{L})^{\frac{p-1}{p}} \text{Mod}_p(\Gamma)^{\frac{1}{p}}.$$

Proof. Since $\overline{E} \cap \overline{\Omega}$ and $\overline{F} \cap \overline{\Omega}$ are two disjoint compact sets, we have $d := \text{dist}(\overline{E} \cap \overline{\Omega}, \overline{F} \cap \overline{\Omega}) > 0$ and so $\text{Mod}_p(\Gamma) < \infty$; e.g. $d^{-1}\chi_\Omega$ is an admissible function. By [12, Proposition 2.17] and [16, Theorem 1.11] we have

$$\text{cap}_p(E, F; \Omega) = \text{Mod}_p(\Gamma); \tag{4.1}$$

recall Definition 2.9. Then by [5, Theorem 5.13] we find a capacitary potential of E and F in Ω, that is, a function $u \in N^{1,p}(\Omega)$ such that $0 \le u \le 1$ in Ω, $u = 1$ in $E \cap \Omega$, $u = 0$ in $F \cap \Omega$, and

$$\text{cap}_p(E, F; \Omega) = \int_\Omega g_u^p \, d\mu.$$

We find two disjoint open sets $W_1, W_2 \subset \Omega$ with $\overline{E} \cap \Omega \subset W_1$ and $\overline{F} \cap \Omega \subset W_2$. Since u is a capacitary potential, for any nonnegative $\varphi \in \text{Lip}_c(W_1) \subset \text{Lip}_c(\Omega)$ we have that $u + \varphi$ is admissible for $\text{cap}_p(E, F; \Omega)$ and so

$$\int_\Omega g_u^p \, d\mu \le \int_\Omega g_{u+\varphi}^p \, d\mu,$$

and so by the locality of minimal weak upper gradients (see e.g. [4, Corollary 2.21]),

$$\int_{\{\varphi \ne 0\}} g_u^p \, d\mu \le \int_{\{\varphi \ne 0\}} g_{u+\varphi}^p \, d\mu.$$

Thus u is superminimizer in W_1, and analogously a subminimizer in W_2. Let u^* be the lsc-regularization of u in W_1 and the analogously defined usc-regularization of u in W_2, and $u^* = u$ in $\Omega \smallsetminus (W_1 \cup W_2)$. Then by Proposition 3.7, u^* is lower semicontinuous in W_1 and, analogously, upper semicontinuous in W_2, and $u = u^*$ p-q.e. in Ω.

Let $\mathcal{L}^{[+]}$ be the collection of super-level sets of u^*, $U_t := \{x \in \Omega : u^*(x) > t\}$, for $t \in (0, 1)$. By Theorem 3.8 we have $u^* = 1$ in $b_p E \cap \Omega$. Thus the sets $U_t \cap W_1$, for $t \in (0, 1)$, are open and contain $b_p E \cap \Omega$, and so each set $\text{int}(U_t)$ contains $b_p E \cap \Omega$. Analogously, $b_p F \cap \Omega \subset \text{ext}(U_t)$ for all $t \in (0, 1)$. In conclusion we have $\mathcal{L}^{[+]} \subset \mathcal{L}$, or more precisely $P(U_t, \Omega \cap \cdot)$ is in $\mathcal{L}$ for every $t \in (0, 1)$. Thus $\text{Mod}_{\frac{p}{p-1}}(\mathcal{L}^{[+]}) \leq \text{Mod}_{\frac{p}{p-1}}(\mathcal{L}) < \infty$.

Let $\rho \in L^{p/(p-1)}(X)$ be any admissible function for $\text{Mod}_{\frac{p}{p-1}}(\mathcal{L}^{[+]})$. By e.g. [4, Proposition 2.44] we know that $g_{u,1} \leq g_u$ in Ω, where $g_{u,1}$ and g_u are the minimal 1-weak and p-weak upper gradients, respectively, of u in Ω. Thus also $u \in N^{1,1}(\Omega)$. Since $\text{Lip}_{\text{loc}}(\Omega)$ is dense in $N^{1,1}(\Omega)$, see [4, Theorem 5.47], it follows that $u \in \text{BV}(\Omega)$ with $d\|Du\| \leq g_{u,1}\, d\mu \leq g_u\, d\mu$ in Ω. Using also the coarea formula of Theorem 3.1, we get

$$
\begin{aligned}
1 &\leq \int_0^1 \left(\int_\Omega \rho\, dP(U_t, \cdot) \right) dt \\
&= \int_\Omega \rho\, d\|Du\| \qquad \| \text{ since } u^* = u \text{ a.e.} \\
&\leq \int_\Omega \rho\, g_u\, d\mu \\
&\leq \left(\int_\Omega \rho^{\frac{p}{p-1}}\, d\mu \right)^{\frac{p-1}{p}} \left(\int_\Omega g_u^p\, d\mu \right)^{\frac{1}{p}} \\
&= \left(\int_\Omega \rho^{\frac{p}{p-1}}\, d\mu \right)^{\frac{p-1}{p}} \text{Mod}_p(\Gamma)^{\frac{1}{p}},
\end{aligned}
$$

using also (4.1). Taking the infimum over admissible ρ, we get

$$
1 \leq \text{Mod}_{\frac{p}{p-1}}(\mathcal{L}^{[+]})^{\frac{p-1}{p}} \text{Mod}_p(\Gamma)^{\frac{1}{p}} \leq \text{Mod}_{\frac{p}{p-1}}(\mathcal{L})^{\frac{p-1}{p}} \text{Mod}_p(\Gamma)^{\frac{1}{p}}.
$$

□

In the case where E and F are compact, we get the lower bound also for the following smaller family of surfaces:

$$
\mathcal{L}^* := \mathcal{L}^*(E, F; \Omega) := \{P(U, \Omega \cap \cdot) : E \subset \text{int}(U) \text{ and } F \subset \text{ext}(U)\}. \tag{4.2}
$$

Proposition 4.3. *Let $\Omega \subset X$ be a nonempty bounded domain and let $E, F \subset \Omega$ be disjoint nonempty compact sets. If $\text{Mod}_{\frac{p}{p-1}}(\mathcal{L}^*) < \infty$, then*

$$
1 \leq \text{Mod}_{\frac{p}{p-1}}(\mathcal{L}^*)^{\frac{p-1}{p}} \text{Mod}_p(\Gamma)^{\frac{1}{p}}.
$$

Proof. The proof is almost the same as for Proposition 4.2; we only need to note that since E and F are compact, according to Theorem 1.1 in [17] we find for every $\varepsilon > 0$ a function $u \in \text{Lip}_{\text{loc}}(\Omega)$ with $0 \leq u \leq 1$ in Ω, $u = 1$ in E, $u = 0$ in F, and

$$
\int_\Omega g_u^p\, d\mu < \text{cap}_p(E, F; \Omega) + \varepsilon.
$$

Then we can consider the super-level sets $\{x \in \Omega : u(x) > t\}$ for $t \in (0, 1)$, which all belong to $\mathcal{L}^*$. □

Now we prove the upper bound. Part of the idea for the following proof came from Lohvansuu and Rajala [22]; the authors would like to thank them for sharing an early version of their manuscript.

Proposition 4.4. *Let $\Omega \subset X$ be open and $E, F \subset X$ be disjoint sets with $\mathrm{Cap}_p(b_pE \setminus E) = 0$ and $\mathrm{Cap}_p(b_pF \setminus F) = 0$. If $\mathrm{Mod}_{\frac{p}{p-1}}(\mathcal{L}) = \infty$ then $\mathrm{Mod}_p(\Gamma) = 0$, and if $0 < \mathrm{Mod}_{\frac{p}{p-1}}(\mathcal{L}) < \infty$ then*

$$\mathrm{Mod}_{\frac{p}{p-1}}(\mathcal{L})^{\frac{p-1}{p}}\,\mathrm{Mod}_p(\Gamma)^{\frac{1}{p}} \leq C$$

for a constant C.

Note that in particular, a closed set E satisfies $\mathrm{Cap}_p(b_pE \setminus E) = 0$.

Proof. Let $x \in X \setminus b_pE$. Since $\mathrm{Cap}_p(b_pE \setminus E) = 0$, by definition of the variational capacity we get

$$\mathrm{cap}_p(B(x, t) \cap b_pE, B(x, 2t)) \leq \mathrm{cap}_p(B(x, t) \cap E, B(x, 2t))$$

for every $t > 0$. Thus

$$\int_0^1 \left(\frac{\mathrm{cap}_p(B(x, t) \cap b_pE, B(x, 2t))}{\mathrm{cap}_p(B(x, t), B(x, 2t))} \right)^{\frac{1}{p-1}} \frac{dt}{t} \leq \int_0^1 \left(\frac{\mathrm{cap}_p(B(x, t) \cap E, B(x, 2t))}{\mathrm{cap}_p(B(x, t), B(x, 2t))} \right)^{\frac{1}{p-1}} \frac{dt}{t} < \infty.$$

Thus, $X \setminus b_pE$ is p-finely open. Similarly, $X \setminus b_pF$ is p-finely open and so $X \setminus (b_pE \cup b_pF)$ is as well.

Now by Theorem 1.4 in [6], we have that $X \setminus (b_pE \cup b_pF)$ is p-quasiopen. Then for each $i \in \mathbb{N}$, we can find an open set $G_i \subset X$ with $\mathrm{Cap}_p(G_i) < 1/i$ and so that $(X \setminus (b_pE \cup b_pF)) \cup G_i$ is open. We can assume that the sets G_i form a decreasing sequence. Furthermore, we know that $\mathrm{Cap}_p(E \setminus b_pE) = 0$ and $\mathrm{Cap}_p(F \setminus b_pF) = 0$ (see Corollary 1.3 in [6]) so then since Cap_p is an outer capacity, we can choose G_i to contain $E \Delta b_pE$ and $F \Delta b_pF$, where Δ denotes the symmetric difference. We now have that $(b_pE \cup b_pF) \setminus G_i = (E \cup F) \setminus G_i$ and this is a closed set.

Take open sets $\Omega_1 \Subset \Omega_2 \Subset \ldots \Subset \Omega$ with $\Omega = \bigcup_{i=1}^{\infty} \Omega_i$. Define

$$\Gamma_i := \{\gamma \in \Gamma(E, F; \Omega) : |\gamma| \subset \Omega_i \setminus G_i\},$$

where $|\gamma|$ is the image of γ in X. Fix $i \in \mathbb{N}$ and a rectifiable curve $\gamma \in \Gamma_i$ (assume for now that $\Gamma_i \neq \varnothing$). Let

$$\mathcal{L}_{i,j} := \{P(U, \Omega \cap \cdot) : \mathrm{dist}(\overline{\Omega_i} \cap E \setminus G_i, X \setminus U) > j^{-1} \text{ and } \mathrm{dist}(\overline{\Omega_i} \cap F \setminus G_i, U) > j^{-1}\}.$$

Also fix $j \in \mathbb{N}$. We wish to construct an admissible function for $\mathcal{L}_{i,j}$. First we construct a Whitney covering of γ. Set

$$d(x) := \mathrm{dist}(x, (\overline{\Omega_i} \cap (E \cup F) \setminus G_i)) \cup (X \setminus \Omega))$$

and note that $d(x) > 0$ for all $x \in |\gamma| \setminus (E \cup F)$. For $k \in \mathbb{Z}$ set

$$|\gamma|_k := \{x \in |\gamma| : 2^{k-1} < d(x) \leq 2^k\}$$

and

$$\mathcal{F}_k := \left\{ B\left(x, \frac{d(x)}{25\lambda}\right) : x \in |\gamma|_k \right\}.$$

So $\mathcal{F}_k$ forms a cover of $|\gamma|_k$. Then by the 5-covering theorem, we can find a pairwise disjoint subcollection $\mathcal{G}_k \subset \mathcal{F}_k$ such that

$$|\gamma|_k \subset \bigcup_{B \in \mathcal{F}_k} B \subset \bigcup_{B \in \mathcal{G}_k} 5B.$$

Since $|\gamma|_k$ is bounded, $\mathcal{G}_k$ is finite for each k. Letting $\mathcal{B} := \bigcup_{k \in \mathbb{Z}} \mathcal{G}_k$, the collection of five times enlarged balls from $\mathcal{B}$ is a cover for $|\gamma| \setminus (E \cup F)$. Now for $U \in \mathcal{L}_{i,j}$, set

$$T := \sup \left\{ t \in (0, \ell_\gamma) : \frac{\mu(U \cap 5B)}{\mu(5B)} \geq \frac{1}{2} \text{ for all } B \in \mathcal{B} \text{ such that } \gamma(t) \in 5B \right\}.$$

We know that the above supremum is attained and $T \in (0, \ell_\gamma)$ since $\overline{\Omega_i} \cap b_pE \setminus G_i = \overline{\Omega_i} \cap E \setminus G_i$ is a compact subset of the open set $\mathrm{int}(U)$ and $\overline{\Omega_i} \cap b_pF \setminus G_i = \overline{\Omega_i} \cap F \setminus G_i$ is a compact subset of the open set $\mathrm{ext}(U)$, and

by the definition of $\mathcal{B}$. Let $B_1 \in \mathcal{B}$ such that $\gamma(T) \in 5B_1$. By continuity of γ and definition of T, there exists $\kappa > 0$ such that $\gamma(T - \kappa) \in 5B_1$ and $\frac{\mu(U \cap 5B)}{\mu(5B)} \geq \frac{1}{2}$ for all $B \in \mathcal{B}$ such that $\gamma(T - \kappa) \in 5B$. Thus $\frac{\mu(U \cap 5B_1)}{\mu(5B_1)} \geq \frac{1}{2}$. Again by continuity of γ, there exists $\delta > 0$ such that $\gamma(T + \delta) \in 5B_1$. Then since $T + \delta > T$, we know that there exists a ball $B_2 \in \mathcal{B}$ with $\gamma(T + \delta) \in 5B_2$ such that $\frac{\mu(U \cap 5B_2)}{\mu(5B_2)} < \frac{1}{2}$. So we have

$$\frac{\mu(U \cap 5B_2)}{\mu(5B_2)} < \frac{1}{2} \leq \frac{\mu(U \cap 5B_1)}{\mu(5B_1)}. \tag{4.3}$$

By the fact that $5B_1 \cap 5B_2$ is nonempty (since it contains $\gamma(T + \delta)$), it is easy to check that $\mathrm{rad}(B_1) \leq 2\,\mathrm{rad}(B_2) \leq 4\,\mathrm{rad}(B_1)$. Hence $5B_2 \subset 25B_1$. By using first (4.3) and the doubling property and then the relative isoperimetric inequality of Definition 2.13, we get for some constant $\widetilde{C} > 0$ (depending only on the doubling constant)

$$\frac{1}{\widetilde{C}} \leq \min\left\{ \frac{\mu(25B_1 \cap U)}{\mu(25B_1)}, \frac{\mu(25B_1 \setminus U)}{\mu(25B_1)} \right\} \leq \frac{25 C_I \,\mathrm{rad}(B_1)}{\mu(B_1)} P(U, 25\lambda B_1). \tag{4.4}$$

Choose $K_j \in \mathbb{Z}$ so that $2^{K_j} < \frac{1}{2}\min\left\{\frac{1}{j}, \mathrm{dist}(\Omega_i^{[*]}, X \setminus \Omega)\right\}$. Then for any $k \leq K_j$ and $B \in \mathcal{G}_k$, either $25B \subset U$ or $25B$ and U are disjoint, which implies that $\frac{\mu(U \cap 25B)}{\mu(25B)} \in \{0, 1\}$. Therefore we know that $B_1 \in \mathcal{G}_k$ for some $k \geq K_j$. Define

$$\phi_{i,j} := 25 C_I \widetilde{C} \sum_{k \geq K_j} \sum_{B \in \mathcal{G}_k} \frac{\mathrm{rad}(B)}{\mu(B)} \chi_{25\lambda B}.$$

Recall that $\mathcal{G}_k$ is finite for each k. Also $|\gamma|$ is bounded, so there exists a $K_0 \in \mathbb{Z}$ such that $\mathcal{G}_k$ is empty for all $k \geq K_0$. Hence the function $\phi_{i,j}$ is $p/(p-1)$-integrable. Furthermore, $\phi_{i,j}$ is admissible for $\mathcal{L}_{i,j}$, since for any $U \in \mathcal{L}_{i,j}$, by (4.4) we have

$$\int_\Omega \phi_{i,j} \, dP(U, \cdot) = 25 C_I \widetilde{C} \int_\Omega \sum_{k \geq K_j} \sum_{B \in \mathcal{G}_k} \frac{\mathrm{rad}(B)}{\mu(B)} \chi_{25\lambda B} \, dP(U, \cdot)$$

$$= 25 C_I \widetilde{C} \sum_{k \geq K_j} \sum_{B \in \mathcal{G}_k} \frac{\mathrm{rad}(B)}{\mu(B)} P(U, 25\lambda B)$$

$$\geq 1.$$

Using Lemma 3.3, pick a $p/(p-1)$-weakly admissible function $\rho_{i,j}$ such that

$$\mathrm{Mod}_{\frac{p}{p-1}}(\mathcal{L}_{i,j}) = \int_X \rho_{i,j}^{\frac{p}{p-1}} \, d\mu.$$

Recall the definition of the noncentered Hardy-Littlewood maximal function from (2.3). We now apply Lemma 3.4 which gives

$$\mathrm{Mod}_{\frac{p}{p-1}}(\mathcal{L}_{i,j}) \leq \int_X \phi_{i,j} \rho_{i,j}^{\frac{1}{p-1}} \, d\mu$$

$$\leq C \int_X \sum_{k \geq K_j} \sum_{B \in \mathcal{G}_k} \frac{\mathrm{rad}(B)}{\mu(B)} \chi_{25\lambda B} \, \rho_{i,j}^{\frac{1}{p-1}} \, d\mu$$

$$\leq C \sum_{k \in \mathbb{Z}} \sum_{B \in \mathcal{G}_k} \frac{\mathrm{rad}(B)}{\mu(B)} \int_{25\lambda B} \rho_{i,j}^{\frac{1}{p-1}} \, d\mu$$

$$\leq C \sum_{k \in \mathbb{Z}} \sum_{B \in \mathcal{G}_k} \mathrm{rad}(B) \fint_{25\lambda B} \rho_{i,j}^{\frac{1}{p-1}} \, d\mu \tag{4.5}$$

$$\leq C \sum_{k \in \mathbb{Z}} \sum_{B \in \mathcal{G}_k} \mathrm{rad}(B) \inf_{x \in B} \mathcal{M}\rho_{i,j}^{\frac{1}{p-1}}(x)$$

$$\leq C \int_\gamma \mathcal{M}\rho_{i,j}^{\frac{1}{p-1}} \, ds;$$

the last inequality holds because the curve γ travels at least the length $\mathrm{rad}(B)$ inside B, and the balls in each $\mathcal{G}_k$ are pairwise disjoint and clearly two balls $B_1 \in \mathcal{G}_k$ and $B_2 \in \mathcal{G}_l$ can only intersect if $|k - l| = 1$.

Now we show that $\bigcup_j \mathcal{L}_{i,j} = \mathcal{L}$. First note that

$$\bigcup_j \mathcal{L}_{i,j} = \{U \in \mathcal{L} : \mathrm{dist}(\overline{\Omega_i} \cap E \smallsetminus G_i, X \smallsetminus U) > 0 \text{ and } \mathrm{dist}(\overline{\Omega_i} \cap F \smallsetminus G_i, U) > 0\}.$$

Now if we take $U \in \mathcal{L}$, then $\overline{\Omega_i} \cap b_p E \smallsetminus G_i \subset \Omega \cap b_p E \subset \mathrm{int}(U)$, i.e. $\overline{\Omega_i} \cap b_p E \smallsetminus G_i$ is a compact subset of the open set $\mathrm{int}(U)$. Hence there is a strictly positive distance between $\overline{\Omega_i} \cap b_p E \smallsetminus G_i = \overline{\Omega_i} \cap E \smallsetminus G_i$ and $X \smallsetminus U$. A similar argument shows that there is a strictly positive distance between $\overline{\Omega_i} \cap F \smallsetminus G_i$ and U. Therefore $U \in \bigcup_j \mathcal{L}_{i,j}$, proving that $\bigcup_j \mathcal{L}_{i,j} = \mathcal{L}$.

Now note that since we are assuming $\mathrm{Mod}_{\frac{p}{p-1}}(\mathcal{L}) \neq 0$, and the families $\mathcal{L}_{i,j}$ are increasing in j, by Lemma 3.2 we have $\mathrm{Mod}_{\frac{p}{p-1}}(\mathcal{L}_{i,j}) > 0$ for all sufficiently large $j \in \mathbb{N}$ (with i still fixed). Thus by (4.5), $C \, \mathrm{Mod}_{\frac{p}{p-1}}(\mathcal{L}_{i,j})^{-1} \mathcal{M} \rho_{i,j}^{\frac{1}{p-1}}$ is admissible for Γ_i. Therefore

$$\mathrm{Mod}_p(\Gamma_i) \leq C \, \mathrm{Mod}_{\frac{p}{p-1}}(\mathcal{L}_{i,j})^{-p} \int_X \left(\mathcal{M}\rho_{i,j}^{\frac{1}{p-1}}\right)^p d\mu.$$

Since the maximal function is a bounded operator from $L^p(X)$ to $L^p(X)$ when $1 < p < \infty$, see e.g. [4, Theorem 3.13], we get

$$\mathrm{Mod}_p(\Gamma_i)^{\frac{1}{p}} \leq C \, \mathrm{Mod}_{\frac{p}{p-1}}(\mathcal{L}_{i,j})^{-1} \left\| \mathcal{M}\rho_{i,j}^{\frac{1}{p-1}} \right\|_{L^p(X)}$$

$$\leq C \, \mathrm{Mod}_{\frac{p}{p-1}}(\mathcal{L}_{i,j})^{-1} \left\| \rho_{i,j}^{\frac{1}{p-1}} \right\|_{L^p(X)}$$

$$= C \, \mathrm{Mod}_{\frac{p}{p-1}}(\mathcal{L}_{i,j})^{-1} \left(\int_X \rho_{i,j}^{\frac{p}{p-1}} d\mu \right)^{\frac{1}{p}}$$

$$= C \, \mathrm{Mod}_{\frac{p}{p-1}}(\mathcal{L}_{i,j})^{\frac{1-p}{p}}.$$

Recalling that $\lim_{j \to \infty} \mathrm{Mod}_{\frac{p}{p-1}}(\mathcal{L}_{i,j}) = \mathrm{Mod}_{\frac{p}{p-1}}(\mathcal{L})$, we get

$$\mathrm{Mod}_p(\Gamma_i)^{\frac{1}{p}} \leq C \, \mathrm{Mod}_{\frac{p}{p-1}}(\mathcal{L})^{\frac{1-p}{p}}, \tag{4.6}$$

and in particular $\mathrm{Mod}_p(\Gamma_i) = 0$ if $\mathrm{Mod}_{\frac{p}{p-1}}(\mathcal{L}) = \infty$. Note that (4.6) holds also if $\Gamma_i = \emptyset$.

Finally note that the sequence Γ_i is increasing with $\bigcup_{i \in \mathbb{N}} \Gamma_i = \Gamma \smallsetminus N$, where

$$N := \left\{ \gamma \in \Gamma : |\gamma| \cap \bigcap_i G_i \neq \emptyset \right\}.$$

But $\mathrm{Cap}_p(\bigcap_i G_i) = 0$, and so $\mathrm{Mod}_p(N) = 0$, see e.g. [4, Proposition 1.48]. So by Lemma 3.2 we have $\lim_{i \to \infty} \mathrm{Mod}_p(\Gamma_i) = \mathrm{Mod}_p(\Gamma \smallsetminus N) = \mathrm{Mod}_p(\Gamma)$. Combining this with (4.6) above, we get

$$\mathrm{Mod}_p(\Gamma)^{\frac{1}{p}} = \lim_{i \to \infty} \mathrm{Mod}_p(\Gamma_i)^{\frac{1}{p}} \leq C \, \mathrm{Mod}_{\frac{p}{p-1}}(\mathcal{L})^{\frac{1-p}{p}},$$

and in particular $\mathrm{Mod}_p(\Gamma) = 0$ if $\mathrm{Mod}_{\frac{p}{p-1}}(\mathcal{L}) = \infty$. □

Now Theorem 1.1 from the introduction follows from Proposition 4.2 and Proposition 4.4. We give the theorem in the following more precise form.

Theorem 4.5. *Let $\Omega \subset X$ be nonempty, open and bounded and let $E, F \subset X$ be such that $\overline{E} \cap \overline{F} = \emptyset$ and $\mathrm{Cap}_p(b_p E \smallsetminus E) = 0$ and $\mathrm{Cap}_p(b_p F \smallsetminus F) = 0$. If $\mathrm{Mod}_{\frac{p}{p-1}}(\mathcal{L}) = \infty$ then $\mathrm{Mod}_p(\Gamma) = 0$, and else*

$$1 \leq \mathrm{Mod}_{\frac{p}{p-1}}(\mathcal{L})^{\frac{p-1}{p}} \mathrm{Mod}_p(\Gamma)^{\frac{1}{p}} \leq C$$

for some constant $C \geq 1$.

Proof. If $\text{Mod}_{\frac{p}{p-1}}(\mathcal{L}) = \infty$ then $\text{Mod}_p(\Gamma) = 0$ by Proposition 4.4. Else Proposition 4.2 gives $\text{Mod}_p(\Gamma) < \infty$ as well as the lower bound of the theorem, and in particular this guarantees $0 < \text{Mod}_p(\Gamma) < \infty$. Then the upper bound follows from Proposition 4.4. $\qquad\square$

5 Proof of Theorem 1.2

Now we can prove Theorem 1.2 given in the introduction. We give it in the following somewhat more general form.

Theorem 5.1. *Suppose that* (Y, d_Y, μ_Y) *is another complete metric space that supports a* 1-*Poincaré inequality, such that* μ_Y *is doubling and*

$$\mu(B(x,r)) \geq C_0^{-1} r^p \quad and \quad \mu_Y(B(y,r)) \leq C_0 r^p \tag{5.1}$$

for all $x \in X$, $y \in Y$, $r > 0$, *and a constant* $C_0 > 0$. *Suppose* $f: X \to Y$ *is a homeomorphism such that for every collection of surfaces* $\mathcal{L} = \mathcal{L}(E, F; \Omega)$ *with* $\Omega \subset X$ *nonempty, open and bounded and* $E, F \subset \Omega$ *compact, we have*

$$\text{Mod}_{\frac{p}{p-1}}(f\mathcal{L}) \leq C_0 \text{Mod}_{\frac{p}{p-1}}(\mathcal{L}),$$

where

$$f\mathcal{L} = \{P(f(U), f(\Omega) \cap \cdot) : b_p E \cap \Omega \subset \text{int}(U) \text{ and } b_p F \cap \Omega \subset \text{ext}(U)\}.$$

Then f *is quasiconformal with a constant depending only on* C_0, p, *and the space* X.

Of course, (5.1) is satisfied in particular if X and Y are both Ahlfors p-regular; recall Definition 2.1. Also recall the definitions of L_f and l_f from Definition 2.14.

Proof. As complete metric spaces equipped with a doubling measure and supporting a Poincaré inequality, X and Y are *quasiconvex*, see e.g. [4, Theorem 4.32], and so for each of them a biLipschitz change in the metric gives a geodesic space (see Section 4.7 in [4]). Since the theorem is easily seen to be invariant under biLipschitz changes in the metrics, we can assume that X and Y are geodesic.

We want to apply Proposition 4.3 and Proposition 4.4 to suitable sets defined via the homeomorphism f. Fix $x \in X$ and $r > 0$ and let $L := L_f(x, r)$ and $l := l_f(x, r) > 0$. Suppose also that $L > 2C_1 l$, where C_1 is the constant from Theorem 3.6. By choosing r sufficiently small, we have $L < \text{diam}(Y)/4$ (we can assume that $\text{diam}(Y) > 0$). Since $f(\overline{B}(x, r))$ is compact, there exists $y \in f(\overline{B}(x, r))$ such that $d_Y(f(x), y) = L$.

Let $E := f^{-1}(\overline{B}(f(x), l))$, and $F := f^{-1}(F_*)$ where F_* is the maximal connected set containing y and contained in $\overline{B}(f(x), M) \setminus B(f(x), L/C_1)$, for some fixed $M \geq 2C_1 L$. By Theorem 3.6 (note that here we use the upper bound in (5.1)) we have

$$F_* \supset B(f(x), M/C_1) \setminus B(f(x), L). \tag{5.2}$$

Note that balls are connected in geodesic spaces, and f is a homeomorphism, so E and F are connected. Both E and F are moreover closed, and since X and Y are proper, f and f^{-1} map bounded sets to bounded sets, and so E and F are also bounded and thus compact. Since Y is connected, the set F_* and thus also the set F consists of at least 2 points and so $\text{diam}(F) > 0$. If $r \to 0$ then $\text{diam}(E) \to 0$, and thus by choosing r even smaller if necessary, we can assume that $\text{diam}(E)$ is less than $\text{diam}(F)$. Note that $\Omega := f^{-1}(B(f(x), M + 1))$ is also bounded. Let $\Gamma = \Gamma(E, F; \Omega)$ and $\mathcal{L} = \mathcal{L}(E, F; \Omega)$.

For the family $f\Gamma$ consisting of the curves $f \circ \gamma$, with $\gamma \in \Gamma$, we have

$$f\Gamma = \Gamma(\overline{B}(f(x), l), F_*; B(f(x), M + 1)).$$

From this it is easy to see that every curve in $\Gamma(E, F; X)$ has a subcurve in Γ, and so $\text{Mod}_p(\Gamma) = \text{Mod}_p(\Gamma(E, F; X))$; see e.g. [4, Lemma 1.34(c)]. Notice that $f^{-1}(y) \in F \cap \overline{B}(x, r)$, and we know that $x \in E$,

so $\text{dist}(E, F) \le r$. It is straightforward to show that there is some $z \in X \setminus B(x, r)$ with $d_Y(f(x), f(z)) = l$. Thus $r \le \text{diam}(E)$, which we noted to be less than $\text{diam}(F)$, and so

$$\frac{\text{dist}(E, F)}{\min\{\text{diam}(E), \text{diam}(F)\}} \le \frac{r}{r} = 1.$$

By Theorem 3.5 we know that X is a Loewner space (note that here we need the lower mass bound in (5.1)), and so

$$\text{Mod}_p(\Gamma) = \text{Mod}_p(\Gamma(E, F; X)) \ge \phi(1) > 0, \tag{5.3}$$

where ϕ is the Loewner function for X. We observe that every curve in $f\Gamma$ has a subcurve in the family $\Gamma(\overline{B}(f(x), l), Y \setminus B(f(x), L/C_1); Y)$. Thus

$$\text{Mod}_p(f\Gamma) \le \text{Mod}_p(\Gamma(\overline{B}(f(x), l), Y \setminus B(f(x), L/C_1)); Y).$$

Now by Proposition 5.3.9 in [14] we have

$$\text{Mod}_p(f\Gamma) \le C_2 \left(\log \frac{L}{C_1 l}\right)^{1-p} \tag{5.4}$$

for a constant C_2 depending only on C_0 and p (here we need the upper mass bound in (5.1)).

Since $L < \text{diam}(Y)/4$ and Y is connected, by (5.2) there exists a ball $B \subset F_*$ with $\text{rad}(B) = L/2$. Then from the relative isoperimetric inequality of Definition 2.13 and the doubling property of μ_Y we see that for every $U \in f\mathcal{L}$,

$$P(U, B(f(x), 2\lambda M)) \ge (2C_I M)^{-1} \min\{\mu_Y(B(f(x), l)), \mu_Y(B)\} =: c > 0.$$

It follows that $\text{Mod}_{p/(p-1)}(f\mathcal{L}) < \infty$, as $c^{-1}\chi_{B(f(x), 2\lambda M)}$ is an admissible test function. Recall the definition of the family $\mathcal{L}^* \subset \mathcal{L}$ from (4.2). For this family it is easy to verify (since f is a homeomorphism) that

$$f\mathcal{L}^* = \mathcal{L}^*(\overline{B}(f(x), l), F^*; B(f(x), M+1)).$$

Thus by Proposition 4.3 and the assumption of the quasi-preservation of modulus of surfaces,

$$1 \le \text{Mod}_{\frac{p}{p-1}}(f\mathcal{L}^*)^{\frac{p-1}{p}} \text{Mod}_p(f\Gamma)^{\frac{1}{p}} \le C_0^{\frac{p-1}{p}} \text{Mod}_{\frac{p}{p-1}}(\mathcal{L})^{\frac{p-1}{p}} \text{Mod}_p(f\Gamma)^{\frac{1}{p}}.$$

Therefore, by Proposition 4.4 and since we had $\text{Mod}_p(\Gamma) \ne 0$ (recall (5.3))

$$\text{Mod}_p(\Gamma)^{\frac{1}{p}} \le C_0^{\frac{p-1}{p}} \text{Mod}_p(\Gamma)^{\frac{1}{p}} \text{Mod}_{\frac{p}{p-1}}(\mathcal{L})^{\frac{p-1}{p}} \text{Mod}_p(f\Gamma)^{\frac{1}{p}} \le C_0^{\frac{p-1}{p}} C \text{Mod}_p(f\Gamma)^{\frac{1}{p}}.$$

Combining this with (5.3) and (5.4), we have

$$\phi(1) \le \text{Mod}_p(\Gamma) \le C_0^{p-1} C^p C_2 \left(\log \frac{L}{C_1 l}\right)^{1-p}.$$

Thus

$$\frac{L}{l} \le C_1 \exp\left(C_0 C^{\frac{p}{p-1}} C_2^{\frac{1}{p-1}} \phi(1)^{\frac{1}{1-p}}\right).$$

Recall that we were assuming $L > 2C_1 l$; in conclusion

$$\limsup_{r \to 0^+} \frac{L_f(x, r)}{l_f(x, r)} \le C_1 \max\left\{2, \exp\left(C_0 C^{\frac{p}{p-1}} C_2^{\frac{1}{p-1}} \phi(1)^{\frac{1}{1-p}}\right)\right\}$$

for every $x \in X$. Therefore f is quasiconformal. □

Remark 5.2. Note that in the above proof we employed the family $\mathcal{L}^*$ because it is not clear that

$$f\mathcal{L} = \mathcal{L}(\overline{B}(f(x), l), F_*; B(f(x), M+1)).$$

This is the case because it is not clear that

$$b_p E \subset \text{int}(f^{-1}(U)) \quad \text{and} \quad b_p F \subset \text{ext}(f^{-1}(U))$$

for every $U \in \mathcal{L}(\overline{B}(f(x), l), F_*; B(f(x), M + 1))$, as would be required in the definition of $\mathcal{L}(E, F; \Omega)$. In other words, the image under f or f^{-1} of every "separating surface" might not be a "separating surface". It is known, at least in Ahlfors regular spaces, that a quasiconformal mapping (whose inverse is also quasiconformal) preserves the *measure-theoretic* interior, exterior, and boundary, see [9], [21, Theorem 6.1], and [15, Lemma 4.8]. If we knew a similar property to hold for capacitary thickness points, then the above problem would not arise. Thus we ask:

- If $f: X \to Y$ is a quasiconformal mapping, do we have $f(b_p E) = b_p f(E)$ for every (closed) set $E \subset X$?

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