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A semi-explicit integration scheme for weakly-coupled poroelasticity with nonlinear permeability

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Within the time integration of linear elliptic-parabolic problems such as poroelasticity, large systems have to be solved in every time step. With a semi-explicit approach, these systems decouple, leading to a remarkable speed-up. This paper indicates that this idea can also be applied to nonlinear problems such as poroelasticity with nonlinear permeability. To see this, we consider a related toy problem.

1 Introduction

We consider a (possibly nonlinear) elliptic-parabolic problem

$$a(u,v) - d(v,p) = \langle f, v\rangle, \tag{1a}$$
$$d(\dot u, q) + c(\dot p, q) + b(u; p, q) = \langle g, q\rangle, \tag{1b}$$

for test functions $v \in \mathcal{V}$ and $q \in \mathcal{Q}$, with unknowns $u\colon [0,T] \to \mathcal{V}$ and $p\colon [0,T] \to \mathcal{Q}$, final time $T > 0$, and external source terms $f\colon [0,T] \to \mathcal{V}^*$ and $g\colon [0,T] \to \mathcal{Q}^*$. Hereby, $\mathcal{V}^*$ and $\mathcal{Q}^*$ denote the dual space for $\mathcal{V}$ and $\mathcal{Q}$ respectively, and we assume pivot spaces $\mathcal{H}_\mathcal{V}$ and $\mathcal{H}_\mathcal{Q}$, such that $\mathcal{V}, \mathcal{H}_\mathcal{V}, \mathcal{V}^*$ and $\mathcal{Q}, \mathcal{H}_\mathcal{Q}, \mathcal{Q}^*$ each form a Gelfand triple. Equation (1a) represents the elliptic part with elliptic bilinear form a, which is coupled via the bilinear form d to the parabolic equation (1b) with symmetric bilinear form c (elliptic in the pivot space $\mathcal{H}_\mathcal{Q}$) and nonlinear form b, which for each fixed $u \in \mathcal{V}$ is bilinear and elliptic (in $\mathcal{Q}$). Equations of the form (1) appear for instance in the field of geomechanics: in linear poroelasticity [3], i.e., if b does not depend on u, the coupled problem (1) models the deformation of porous media saturated by an incompressible viscous fluid. In several applications, the permeability, modeled via the form b, depends on the displacement u, which renders (1) nonlinear. A typical example, see for instance [4], is given by

$$b(u;p,q) = \int_\Omega \frac{\kappa(u)}{\nu}\, \nabla p \cdot \nabla q \,\mathrm{d}x.$$

Numerical approximations of (1) usually consider an implicit Euler scheme combined with finite elements, leading to a large (nonlinear) system that has to be solved in each time step. In the linear case, the coupled system can be decoupled with a semi-explicit approach as proposed in [2] (based on observations made in [1]) without decreasing the convergence order in time. To ensure convergence, however, one needs a weak coupling condition in the sense that the continuity constant of d is small compared to the ellipticity constants of a and c. This contribution aims to sound the applicability of semi-explicit discretizations also for the nonlinear model (1) since this would improve the computational efficiency dramatically.

2 Implicit and semi-explicit discretization

We first consider the temporal discretization by the implicit Euler scheme. Given a uniform partition of the time interval with step size τ, $t_n = n\,\tau$, one step is given by

$$a(u^n, v) - d(v, p^n) = \langle f(t_n), v\rangle, \tag{2a}$$
$$d(D_\tau u^n, q) + c(D_\tau p^n, q) + b(u^n; p^n, q) = \langle g(t_n), q\rangle \tag{2b}$$

for test functions $v \in \mathcal{V}$ and $q \in \mathcal{Q}$. Here, $D_\tau u^n = \tau^{-1}(u^n - u^{n-1})$ denotes the discrete time derivative and u^n, p^n are the resulting semi-discrete approximations of $u(t_n)$, $p(t_n)$, respectively. We emphasize that (2) is a coupled nonlinear system, which calls for a linearization such as a Newton or Picard iteration.

In the semi-explicit approach, we replace p^n in (2a) by p^{n-1}, leading to

$$a(u^n, v) - d(v, p^{n-1}) = \langle f(t_n), v\rangle, \tag{3a}$$
$$d(D_\tau u^n, q) + c(D_\tau p^n, q) + b(u^n; p^n, q) = \langle g(t_n), q\rangle \tag{3b}$$

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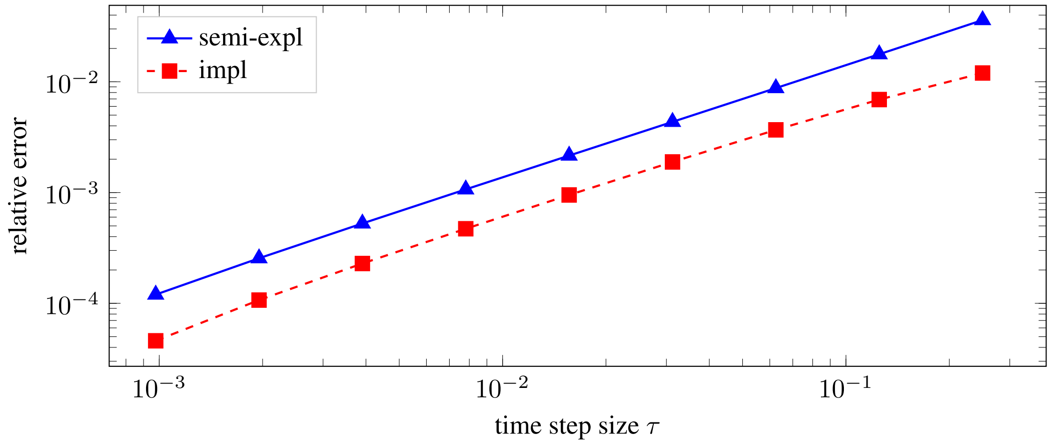

Fig. 1: Illustration of first-order convergence of implicit and semi-explicit discretization.

for $v \in \mathcal{V}$ and $q \in \mathcal{Q}$. Already this minor change has a large influence on the needed computations in each time step. First, the system decouples, since equation (3a) can be solved for u^n without the second equation. Second, since u^n is already known, equation (3b) represents a linear equation for p^n. Thus, the semi-explicit discretization does not only decouple the system but also includes a linearization.

In the following numerical example, we illustrate the potential of this approach. Further research is required to prove convergence and identify the precise requirements on b.

3 Numerical toy example

We consider a 3×3 system in order to test the semi-explicit discretization for a nonlinear system, which is motivated by the nonlinear permeability considered in [4]. In the linear setting, we consider finite-dimensional spaces $\mathcal{V} = \mathcal{H}_\mathcal{V} = \mathbb{R}^3$ (for u), $\mathcal{Q} = \mathcal{H}_\mathcal{Q} = \mathbb{R}^1$ (for p) together with

$$a(u,v) = v^T A u, \qquad d(v,p) = \tfrac{1}{10}\, p^T D v, \qquad c(p,q) = q^T C p, \qquad b(p,q) = q^T B p,$$

defined through

$$A := \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \qquad D := \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \qquad C := 1, \qquad \text{and} \qquad B := 1.$$

This example satisfies the weak coupling condition proposed in [2] and we refer to this paper for a numerical study. In the linear case, the semi-explicit Euler schemes approximates the true solution well with the same order as the fully implicit scheme.

For the nonlinear case, we adjust b similar to [4] and define $b(p,q) = \kappa(u_1 + u_2 + u_3)\, q^T B p$ with $\kappa(s) = 1/2$ for $s \leq 0$, $\kappa(s) = 27/4$ for $s \geq 1/2$, and

$$\kappa(s) := \frac{\rho^3(s)}{(1-\rho(s))^2}, \qquad \rho(s) = \tfrac{1}{2}\,(1+s), \qquad 0 < s < 1/2.$$

Figure 1 shows the numerical results for a time horizon $T = 2$ and forcing functions $f \equiv 0$ and $g(t) = \sin(t)$. It can be observed that the semi-discrete approach converges with order 1 as the fully implicit discretization. Each step, however, involves two decoupled linear systems rather than one coupled nonlinear system.

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