Habilitationsschrift

Complex structures and chains of symmetric spaces

Peter Quast

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Introduction

Motivation and guideline for our work is Milnor's treatment of Bott periodicity for orthogonal groups in his seminal book [Mil-69]. Milnor iteratively constructs the inclusion chain

$$(*)\qquad \begin{array}{ccccccccc} & \mathrm{SO}_{16n} & \supset & \mathrm{SO}_{16n}/\mathrm{U}_{8n} & \supset & \mathrm{U}_{8n}/\mathrm{Sp}_{4n} & \supset & G_{2n}(\mathbb{H}^{4n}) \\ \supset & \mathrm{Sp}_{2n} & \supset & \mathrm{Sp}_{2n}/\mathrm{U}_{2n} & \supset & \mathrm{U}_{2n}/\mathrm{O}_{2n} & \supset & G_n(\mathbb{R}^{2n}) \\ \supset & \mathrm{SO}_n & & & & & & \end{array}$$

of symmetric spaces and describes it from three different viewpoints:

1. iteratively as spaces of certain orthogonal complex structures;
2. as components of the midpoint locus of certain shortest geodesic arcs [Mil-69, Lemma 24.4], following Bott's original ideas [Bo-59];
3. in geometric terms using the usual representation of SO_{16n} on $\mathbb{R}^{16n}$ [Mil-69, Lemma 24.6].

Our aim is to generalize Milnor's approach to arbitrary compact real Lie groups, still following his philosophy. Our work is organized as follows:

In Chapter 1 we introduce complex structures in matrix groups as elements that square to $-\mathrm{Id}$. Making use of the algebraic properties of complex structures we slightly generalize this notion to abstract Lie groups. If the Lie group is compact such elements can also be described geometrically. For illustration consider the symplectic group Sp_1. This is the set of unit quaternions and hence a 3-sphere. If we look at 1 as the north pole, then -1 is the south pole and any complex structure is the midpoint of a geodesic arc joining these two poles and hence a point on the equatorial 2-sphere. The geometric description of the set of complex structures as the midpoint locus of geodesic arcs from a base point to a pole extends to pointed compact symmetric spaces. Following a common nomenclature (see [CN-88]) these midpoint loci are called centrioles of a pointed symmetric space.

Chapter 2 is about centrioles, the midpoint loci of geodesic arcs joining a base point to a pole. Since poles are particular points in centers of pointed symmetric spaces of compact type, we first study these centers and the shortest geodesic arcs to it. Further, in Section 2.2, we describe all centrioles in a simply connected pointed symmetric space of compact type in terms of its root system (Theorem 2.20 and Theorem 2.28).

In Chapter 3 we present an abstract approach to Milnor's construction of $(*)$. Instead of an orthogonal group, we start our iteration with an arbitrary connected compact real Lie group $\mathfrak{G}$ and get certain inclusion chains

$$\mathfrak{G} \supset P_1 \subset P_2 \supset P_3 \supset \ldots$$

Following the first two aspects of Milnor's description, we explain our construction in terms of complex structures and also in terms of midpoints of geodesic arcs. (Section 3.1). But how long are such inclusion chains at least? This question is answered in Theorem 3.5: Under certain assumptions three interesting iteration steps can be preformed if one looks at a special class of centrioles, so-called minimal ones.

To explore Milnor's third view point, we need to start with a represented Lie group $\mathfrak{G}$. It turns out that isotropy representations of hermitian and quaternionic symmetric spaces are particularly interesting for us. We hence start our iteration with a Lie group $\mathfrak{G}$ that is either the complex linear isotropy group of a hermitian symmetric space S (Section 4.1) or the quaternionic linear isotropy group of a quaternionic symmetric space S (Section 4.2). We can describe the spaces that occur in our inclusion chain in a geometric way, as sets of certain submanifolds of S or as special Grassmannians of certain Lie subtriples of the Lie triple of S. This geometric insight also shows obstructions that force the iteration procedure of Chapter 3 to stop at some point. As a byproduct we get some quite uncommon realizations of certain symmetric spaces. We will explain this in the case of projective planes. Such realizations could be interesting for the following reason: Descriptions of classical symmetric spaces in linear algebraic terms are well known (see [Be-87, pp. 312 ff.]). But for exceptional symmetric spaces such realizations seem less well understood[1].

Bott's periodicity theorem is about homotopy. The inclusion chains from Chapter 3 can be related to homotopy, at least if one just looks at minimal centrioles. In Chapter 5 we apply a result of Mitchell [Mit-87, Mit-88] to inclusion chains of exceptional symmetric spaces, e.g.

$$\mathrm{E}_7 \supset \mathrm{E}_7/(S^1\mathrm{E}_6) \supset (S^1\mathrm{E}_6)/F_4 \supset \mathbb{O}P_2.$$

Using known results, we determine explicitly some higher homotopy groups of these spaces.

Throughout our work we use certain properties of symmetric spaces and their submanifolds. Some of them are very well-known, some of them are maybe less common. To make our work more self-contained, we summarize them in Appendix A.

[1] É. Cartan, Chevalley, Freudenthal and others described realizations of exceptional Lie groups and some exceptional symmetric spaces. Rosenfeld [Ro-62, Fr-62] also gave an outline of a description of exceptional symmetric spaces in terms of algebraic objects, but his work seems not to be entirely understood yet.

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Contents

1. Complex structures

A complex structure on a vector space is an automorphism that squares to $-\mathrm{Id}$. Sometimes one is only interested in certain types of complex structure, e.g. orthogonal or unitary ones. In this case one calls an element j of a matrix group $\mathfrak{G}$ a *complex structure* in $\mathfrak{G}$ if j squares to $-\mathrm{Id}$. Of course, a group $\mathfrak{G}$ only admits complex structures if $-\mathrm{Id}$ lies in $\mathfrak{G}$.

In this chapter we describe complex structures in Lie groups in algebraic and in geometric terms. The geometric description allows to define similar elements also in pointed symmetric spaces.

1.1. Complex structures in groups

Algebraic characterization

In order generalize the notion of a complex structure in a matrix group, we extract its algebraic properties: For this we look at the role that $-\mathrm{Id}$ plays in a matrix group. Since $-\mathrm{Id}$ commutes with any square matrix, it lies in the *center*[2] $Z(\mathfrak{G})$ of $\mathfrak{G}$. As $-\mathrm{Id}$ squares to the identity, it is an element of order two[3]. Hence, in an arbitrary group $\mathfrak{G}$ a suitable generalization of $-\mathrm{Id}$ is a center element of order two:

Lemma 1.1 (Schur Lemma). *Let ρ is an irreducible faithful representation[4] of $\mathfrak{G}$ on a finite dimensional vector space V. Then $\rho(z) = -\mathrm{Id}$ if and only if z is a center element of order two in $\mathfrak{G}$.*

Proof. If $\rho(z) = -\mathrm{Id}$, then $\rho(zg) = \rho(z)\rho(g) = -\rho(g) = \rho(g)\rho(z) = \rho(gz)$ for all $g \in \mathfrak{G}$. Since ρ is injective, we conclude $zg = gz$ for all $g \in \mathfrak{G}$, so that g lies in the center of $\mathfrak{G}$. Moreover, since $\rho(z^2) = \rho(z)\rho(z) = \mathrm{Id} = \rho(e)$, we see that $z^2 = e$, so that z has order two.

If z is a center element of order two, we have $\rho(z)^2 = \mathrm{Id}$. Thus $\rho(z)$ is diagonalizable with eigenvalues ± 1. As ρ is a faithful representation (i.e. injective) $\rho(z) \neq \mathrm{Id}$. Hence the (-1)-eigenspace of $\rho(z)$ has at least dimension 1. Since z is in the center of $\mathfrak{G}$ we have $\rho(g)\rho(z) = \rho(gz) = \rho(zg) = \rho(z)\rho(g)$ for all $g \in \mathfrak{G}$. Thus $\rho(\mathfrak{G})$ leaves the (-1)-eigenspace of $\rho(z)$ invariant. By irreducibility, the (-1)-eigenspace must be V and $\rho(z) = -\mathrm{Id}$. $\square$

[2] The *center* $Z(\mathfrak{G})$ of a group $\mathfrak{G}$ is the normal subgroup of $\mathfrak{G}$ formed by all elements of $\mathfrak{G}$ that commute with all elements of $\mathfrak{G}$.

[3] A group element $z \neq e$ is of order two if $z^2 = e$, the neutral element of $\mathfrak{G}$

[4] A *representation* ρ of $\mathfrak{G}$ on V is a Lie group homomorphism from $\mathfrak{G}$ into $\mathrm{GL}(V)$. It is called *faithful* if it is injective, and *irreducible* if the only linear subspaces of V that are invariant under the action of $\rho(\mathfrak{G})$ are $\{0\}$ and V.

Remark 1.2. As a consequence of Lemma 1.1, a group $\mathfrak{G}$ whose center contains more than just one element of order two admits no faithful irreducible representation. This is the case e.g. for the simple Lie group Spin_{4n}.

We can now naturally extend the notion of a complex structure to abstract groups: an element j of $\mathfrak{G}$ is a *complex structure* if j squares to a center element of order two.

Geometric characterization

To consider $\mathfrak{G}$ as a geometric object, we assume that $\mathfrak{G}$ is a compact real Lie group[5]. Endowed with bi-invariant Riemannian metrics, compact real Lie groups are examples of symmetric spaces (see Section A.3). Center elements of order two in $\mathfrak{G}$ can be described using the geodesic symmetries of $\mathfrak{G}$. We call an element $z \in \mathfrak{G}$ that is not the identity a *pole* of $\mathfrak{G}$, if the geodesic symmetry s_z of $\mathfrak{G}$ at the point z coincides with s_e, the geodesic symmetry of $\mathfrak{G}$ at the identity.

Observation 1.3. *Any center element z of order two in a compact real Lie group $\mathfrak{G}$ is a pole and vice-versa.*

Proof. If z is a central element of order two, then $s_z.g = zg^{-1}z = z^2g^{-1} = g^{-1} = s_e.g$ (see Equation A.8). Conversely if $z \neq e$ satisfies $s_z = s_e$, then $z = s_z.z = s_e.z = z^{-1}$, so that z has order two. Since $g^{-1} = s_e.g = s_z(g) = zg^{-1}z = zg^{-1}z^{-1}$, we see that z lies in the center of $\mathfrak{G}$. $\square$

We assume our compact Lie group $\mathfrak{G}$ to be moreover connected. Then the identity can be joined to any other point of $\mathfrak{G}$ by a geodesic. Since geodesics in $\mathfrak{G}$ that start at the identity are one-parameter subgroups of $\mathfrak{G}$ (see Equation A.9) we get:

Observation 1.4. *Any complex structure in $\mathfrak{G}$ is the midpoint of a geodesic arc joining the identity to a pole and vice versa.*

1.2. Poles and centrosomes: notions

Our next aim is to generalize the concept of complex structures to symmetric spaces of compact type using their geometric description. In contrast to groups, where the identity is an algebraically distinguished point, symmetric spaces generally do not have natural base points. Hence the choice of a base point becomes part of the setting. This leads to the notion of a *pointed symmetric space*, i.e. a tuple (P, o) consisting of a (connected) symmetric space P and a distinguished base point $o \in P$. A *pole* of (P, o) is a point $z \neq o$ whose geodesic symmetry s_z coincides with the geodesic symmetry s_o of P at the base point o. Since such points can only occur if P has a compact factor, we now restrict our attention to compact symmetric spaces. Observation 1.3 shows that we can consider poles as a generalization of center elements of order two. For pointed symmetric

[5]There is also a description of complex structures in non-compact Lie groups [Bi-98].

spaces (P, o) of compact type there is also a notion of a *center*: Each symmetric space P of compact type has an *adjoint space* $\mathrm{Ad}(P)$ (see Section A.4). This is the unique (up to isometry) symmetric space that is covered by any symmetric space that is locally isometric to P. If we denote this covering by $\pi : P \to \mathrm{Ad}(P)$, the *center*[6] of (P, o) is the set

$$Z(P, o) := \pi^{-1}(\pi(o)). \tag{1.1}$$

Notice that the adjoint space of a connected semisimple compact Lie group $\mathfrak{G}$ with base point e is exactly its adjoint group. In this case the above definition of a center coincides with the usual notion of a center of a group.

Observation 1.5. *Any pole of a pointed symmetric space* (P, o) *of compact type lies in the center of* (P, o).

Proof. Let p be a pole of (P, o). By definition the Cartan map (see Equation A.6) identifies o and p. Since $\iota^P(P)$ is a symmetric space that is covered by P, we see that the projection π of P onto the adjoint space $\mathrm{Ad}(P)$ must also identify o and p. Thus p lies in $Z(P, o)$. $\square$

Moreover, poles have order two in the sense that they are examples of midpoints of closed geodesics of (P, o) :

Observation 1.6. *An element* $z \in Z(P, o) \setminus \{o\}$, *is a pole of* (P, o) *if and only if any geodesic* γ *in* P *with* $\gamma(0) = o$ *and* $\gamma(t_0) = z$ *satisfies* $\gamma(2t_0) = o$.

The analog of a complex structure in now a group is a midpoint of some geodesic arc in a connected pointed symmetric space (P, o) joining the base point o to a pole of (P, o). By geographical intuition one could call such a point an equator point. But the set of all such points is generally not connected and in literature a term from cytology is more common[7]: Following Chen and Nagano [CN-88] we call the set $C_z(P, o)$ of midpoints of all geodesic arcs joining o to a chosen pole z of (P, o) a *centrosome* of (P, o). Connected components of centrosomes are called *centrioles*[8]. Any point in a centriole will be called a *centriole point*. This generalizes the notion of a complex structure. As the example of Spin_{4n} shows, (P, o) can have more than one pole and therefore several centrosomes corresponding to different poles.

[6] Symmetric spaces can be described using triple products (see [Lo-69-I]). Using this algebraic description of P, the center of (P, o) is the set of all points in P that commute in some sense with o (see [Lo-69-II, p. 51 f.]).

[7] However we should mention that the term equator is sometimes used e.g. in [He-07].

[8] In [Qu-06] we used the term centriole in a wider sense for a connected component of the set of midpoints between the base point o and a center element (not necessary a pole) of (P, o).

2. Centers and centrioles

Centrioles and other geometrically interesting totally geodesic submanifolds of symmetric spaces have been extensively studied, in particular by Japanese geometers[9]. The goal of this chapter is to give a complete description of centrioles of simply connected pointed symmetric spaces of compact type in terms of roots (see Theorems 2.20 and 2.28).

2.1. Shortest geodesics to center elements

We start our study of centrioles in a pointed symmetric space (P, o) of compact type with the investigation of its center $Z(P, o)$. In particular we describe shortest geodesic arcs in (P, o) joining o to an element of $Z(P, o)$. Although the results in this section are surely mostly folklore[10], we think it is still worth to present them here, since they play a role in our study later on.

I want to express my deep gratitude to A.L. Mare for his hints and many interesting and very helpful discussions about the topic of this section.

Let (P, o) be a pointed symmetric space of compact type, $\pi : P \mapsto \mathrm{Ad}(P)$ the projection onto its adjoint space and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ its Cartan decomposition (see Section A.1). We want to study shortest geodesic arcs from the base point o to elements of $Z(P, o)$. In order to reduce this problem to a closed Weyl chamber we observe:

Lemma 2.1. *Let $z \in Z(P, o)$ and k an element of the transvection group $\mathfrak{T}(P)$ of P that fixes the base point o. Then k also fixes z.*

Proof. From the description of the adjoint space given in Equation A.10 we see that $z = g.o$ for some $g \in \Delta = Z_{\mathfrak{T}(P)}(\mathfrak{I}(P))$. Since k lies in $\mathfrak{T}(P)$, it commutes with g so that $k.z = k.(g.o) = g.(k.o) = g.o = z$. □

For poles the result of Lemma 2.1 can be found in [CN-88, Prop. 2.9(vi)], [Nag-88, Prop. 1.9(6)] or [NS-91, Lemma 2.1a].

Notice that the Lie triple $\mathfrak{p}$ can be identified with both T_oP and $T_{\pi(o)}\mathrm{Ad}(P)$.

Lemma 2.2 (see e.g. [Lo-69-II], §2 of [Nag-92] or Lemma 5.6. in [NT-95])**.** *A geodesic γ in $\mathrm{Ad}(P)$ emanating from $\pi(o)$ closes at $t = \pi$, i.e. $\gamma(\pi) = \pi(o)$, if and only if its initial direction $X := \dot{\gamma}(0) \in \mathfrak{p}$ is integer[11].*

[9] see e.g. the articles of Chen, Nagano, Tanaka or Takeuchi in our bibliography. One may find further references in theses articles

[10] Our investigation is very closely related to the calculation of the fundamental group of an adjoint space (see [Ca-27], [Tak-64, p. 99], [Nag-88, Nag-92] and also Corollary 2.11). Our method is quite similar to the one in [Bu-85, p. 20].

[11] We call an element $X \in \mathfrak{g}$ *integer*, if all eigenvalues of $\frac{1}{i}\mathrm{ad}(X)$ are integer.

Proof. The geodesic γ can be written as $\gamma(t) = \exp(tX).\pi(o)$, where exp is the exponential map from $\mathfrak{g}$ into the transvection group $\mathfrak{T}(\mathrm{Ad}(P))$ of $\mathrm{Ad}(P)$ (see Sections A.1 and A.4). Observe that $\gamma(\pi) = \pi(o)$ holds if and only if $\exp(\pi X)$ lies in the isotropy group of $\pi(o)$ in $\mathfrak{T}(\mathrm{Ad}(P))$. Since this isotropy group is the fix point set in $\mathfrak{T}(\mathrm{Ad}(P))$ of the conjugation with the geodesic symmetry $s_{\pi(o)}$ of $\mathrm{Ad}(P)$ at the point $\pi(o)$ (see Lemma A.4), we can equivalently say that $\exp(\pi X) = s_{\pi(o)} \exp(\pi X) s_{\pi(o)} = \exp(\pi \mathrm{Ad}(s_{\pi(o)})X) = \exp(-\pi X)$, or $\exp(2\pi X) = e$, because $\mathfrak{p}$ is the (-1)-eigenspace of $\mathrm{Ad}(s_{\pi(o)})$. Since $\mathfrak{T}(\mathrm{Ad}(P))$ is center-free (Corollary A.3), this condition is equivalent to $\mathrm{Ad}(\exp(2\pi X)) = e^{\mathrm{ad}(2\pi X)} = e$. The latter holds if and only if the eigenvalues of $\frac{1}{i}\mathrm{ad}(X)$ are all integer. □

Since by definition a point p lies in the center of (P, o) if and only if $\pi(o) = \pi(p)$ we get:

Corollary 2.3 (see [Lo-69-II])**.** *A geodesic γ in P emanating from o satisfies $\gamma(\pi) \in Z(P, o)$ if and only if $\dot\gamma(0) \in \mathfrak{p}$ is integer.*

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ and $\mathcal{R}(P)$ the corresponding root system of P. We define the *center lattice* $\Gamma_Z(P)$ of P in $\mathfrak{a}$ by

$$\Gamma_Z(P) := \{X \in \mathfrak{a};\ \exp(X).o \in Z(P, o)\}, \tag{2.1}$$

where now exp is the exponential map from $\mathfrak{g}$ into $\mathfrak{T}(P)$. Using Corollary 2.3 we get (cf. [Lo-69-II])

$$\Gamma_Z(P) = \{X \in \mathfrak{a};\ \alpha(X) \in \pi\mathbb{Z} \text{ for all } \alpha \in \mathcal{R}(P)\}. \tag{2.2}$$

If we fix a Weyl chamber in $\mathfrak{a}$, denote by $\Sigma = \{\alpha_1, ..., \alpha_r\}$ the corresponding system of fundamental roots and consider its dual basis $\Sigma^* = \{\alpha_1^*, ..., \alpha_r^*\}$ (see Equation A.16), we can express the center lattice in terms of the fundamental root system as follows

$$\Gamma_Z(P) = \mathrm{span}_{\pi\mathbb{Z}}(\Sigma^*). \tag{2.3}$$

We first study the problem of shortest geodesics in a *simply connected* pointed symmetric space $(\tilde P, o)$ of compact type. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be its Cartan decomposition. We choose a Weyl chamber C in some maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$, denote by $\Sigma = \{\alpha_1, ..., \alpha_r\} \subset \mathcal{R}(\tilde P)$ the corresponding system of fundamental roots and by δ the corresponding highest root. Let $\mathfrak{K}$ be the identity component of the isotropy group of o and let γ be a geodesic emanating at o that satisfies $\gamma(t_0) = z \in Z(\tilde P, o)$. Lemma 2.1 shows that a curve $c := k.\gamma$ with $k \in \mathfrak{K}$ is again a geodesic starting at o that satisfies $c(t_0) = z$. Thus whenever we consider a geodesic arc γ joining o to z we can assume that, after the action of a suitable element of $\mathfrak{K}$, its initial direction $\dot\gamma$ lies in the closure $\overline{C} = \{\sum_j \lambda_j \alpha_j^*;\ \lambda_j \geq 0\}$ of our chosen Weyl chamber.

Since we look at shortest geodesic arcs between the base point and a center point, we need some information of the cut locus. Sakai [Sa-78b, p. 198] described the intersection

$\mathrm{Cut}_{\tilde{P}}(\overline{C})$ of the tangent cut locus[12] of $(\tilde{P}, o)$ with the closed Weyl chamber $\overline{C}$ as follows:

$$\mathrm{Cut}_{\tilde{P}}(\overline{C}) = \{X \in \overline{C};\ \delta(X) = \pi\}. \tag{2.4}$$

To describe the intersection of the inside[13] of $\mathrm{Cut}_{\tilde{P}}(\overline{C})$ with the center lattice, we call an element ξ in a compact semisimple Lie algebra $\mathfrak{g}$ *extrinsically symmetric*[14] if

$$\mathrm{ad}(\xi)^3 = -\mathrm{ad}(\xi) \tag{2.5}$$

or, equivalently, if the spectrum of $\mathrm{ad}(\xi)$ is $\{0, \pm i\}$. We can describe the extrinsically symmetric elements of $\mathfrak{g}$ that lie in $\mathfrak{p}$ in terms of the roots system of P. Using the linear isotropy action we can assume that our extrinsically symmetric element lies in $\overline{C}$. All extrinsically symmetric elements in $\overline{C}$ can be read off from the labelled Dynkin diagram of P :

Lemma 2.4. *A nonzero element $\xi \in \overline{C}$ is extrinsically symmetric if and only $\xi = \alpha_j^*$ and the coefficient h_j of α_j in the highest root δ is 1.*

Proof. Recall that the Lie algebra $\mathfrak{g}$ is semisimple, because P is of compact type, and has therefore trivial center. Any element $\xi \in \overline{C}$ can hence be written as $\xi = \sum_j x_j \alpha_j^*$ with non-negative coefficients x_j.

Assume that ξ is extrinsically symmetric and nonzero. Thus at least one of its coefficients x_j does not vanish. As the center of $\mathfrak{g}$ is trivial, $\pm i$ are actually eigenvalues of $\mathrm{ad}(\xi)$. Since the spectrum of $\mathrm{ad}(\xi)$ consists only of 0 and $\pm i$, we see that $\alpha_k(\xi) \in \{0, 1\}$ and hence $x_k \in \{0, 1\}$. Let $\delta = \sum_k h_k \alpha_k$ be the highest root. Then $1 = \delta(\xi) = \sum_{j,k} h_k x_j \alpha_k(\alpha_j^*) = \sum_j h_j x_j$, because at least on coefficient x_j of ξ is nonzero and $\alpha(\xi) \in \{0, 1\}$ for any positive root. If two coefficients of ξ were 1, then $\delta(\xi) \geq 2$, since all h_j are positive integers. Therefore only one coefficient x_{j_0} of X is 1, i.e. $\xi = \alpha_{j_0}^*$. Since $1 = \delta(\xi) = h_{j_0}$ we conclude that the coefficient of α_{j_0} in the highest root must be 1.

Conversely, if $\xi = \alpha_j^*$ and the coefficient $h_j = 1$, then $\delta(\xi) = 1$. Let α be a positive root w.r.t. C, then $\alpha = \sum_k c_k \alpha_k$ where the coefficients c_k are non-negative integers and $\alpha(\xi) = c_l \leq \delta(\xi) = 1$ (see Section A.6). Thus $c_l \in \{0, 1\}$. This shows that all eigenvalues of $\mathrm{ad}(\xi)$ are $\{0, \pm i\}$. □

[12] The *tangent cut locus* of a Riemannian manifold M at a point $p \in M$ consists of all vectors $X \in T_pM$, so that the geodesic γ_X emanating from p in direction X is shortest on the interval $[0, 1]$ but not shortest on $[0, 1 + \varepsilon)$ for any $\varepsilon > 0$.

[13] by this we mean the set $\{X \in \overline{C};\ \delta(X) < \pi\}$

[14] Our designation has the following reason: Ferus [Fe-80] has shown that, up products with affine subspaces and Euclidean motion, any full *extrinsically symmetric* submanifold of a Euclidean space, i.e. any submanifold that is invariant under the reflections along its (affine) normal spaces, can be realized as a connected component of an isotropy orbit of some extrinsically symmetric element X in the Lie triple of some symmetric space of compact type (see also [EH-95]). As abstract symmetric spaces these spaces are also called *symmetric R-spaces* in literature. Following Bourbaki [Bou-02], extrinsically symmetric elements are also known as *minuscule coweights* (see e.g. [Mit-88, p. 158], [Nag-92, p. 54]).

Remark 2.5. Kobayashi and Nagano [KNa-64] gave a similar characterization of extrinsically symmetric elements in $\mathfrak{p}$ using the Satake diagram of P instead of its Dynkin diagram.

Lemma 2.6. *A nonzero element $\pi X \in \overline{C} \cap \Gamma_Z(P)$ lies inside the tangent cut locus (i.e. $\delta(\pi X) \leq \pi$) of $\tilde{P}$ at o if and only if X is extrinsically symmetric. In particular $\pi X \in \mathrm{Cut}_{\tilde{P}}(\overline{C})$.*

Proof. Since $X \in \overline{C}$ we set $X = \sum_j x_j \alpha_j^*$ with $c_j \in \mathbb{N}_0$ (see Equation 2.3). As in the proof of Lemma 2.4 we conclude from $\delta(X) \leq 1$ and $X \neq 0$ that $X = \alpha_j^*$ where the coefficient h_j of α_j in δ is 1. By Lemma 2.4 the element X is extrinsically symmetric and $\delta(\pi X) = \pi$. □

Corollary 2.7. *Let $\gamma(t) = \exp(\pi t X).o$ with $X \in \overline{C}$ be a geodesic in $\tilde{P}$ satisfying $\gamma(1) \in Z(\tilde{P}, o)$. Assume that γ is length minimizing on $[0,1]$ then X is extrinsically symmetric. In particular any element of $Z(\tilde{P}, o)$ lies in the cut locus of $(\tilde{P}, o)$.*

Let $\Sigma^\delta = \Sigma \cup \{\delta\}$. Following Sakai [Sa-78a, Sa-78b] for each subset Ω of Σ^δ we define S_Ω as the set of all $X \in \overline{C}$ satisfying

$$\alpha(X) > 0 \quad \text{if} \quad \alpha \in \Sigma \cap \Omega; \tag{2.6}$$

$$\alpha(X) < \pi \quad \text{if} \quad \alpha \in \{\delta\} \cap \Omega; \tag{2.7}$$

$$\alpha(X) = 0 \quad \text{if} \quad \alpha \in \Sigma \setminus \Omega; \tag{2.8}$$

$$\alpha(X) = \pi \quad \text{if} \quad \alpha \in \{\delta\} \setminus \Omega. \tag{2.9}$$

We observe that S_Ω is a subset of the tangent cut locus if and only if Ω does not contain δ. The subset $I_\Omega := \{k.(\exp(X).o);\ k \in \mathfrak{K},\ X \in S_\Omega\}$, $\mathfrak{K}$ denotes again the identity component of the isotropy group of $(\tilde{P}, o)$, is a submanifold of $\tilde{P}$ [Sa-78b, p. 199], and we have the following Lemma (see [Sa-78a, Lemma 5.1] and [Sa-78b, Lemma 5(1)]):

Lemma 2.8 ([Sa-78a, Sa-78b])**.** *$I_\Omega \cap I_{\Omega'} \neq \emptyset$ if and only if $\Omega = \Omega'$.*

Corollary 2.9. *Let ξ and ξ' be two different extrinsically symmetric elements in $\overline{C}$ and $\gamma_{\pi\xi}$ and $\gamma_{\pi\xi'}$ the geodesics in $\tilde{P}$ that start at o in direction $\pi\xi$, respectively $\pi\xi'$. Then $z = \gamma_{\pi\xi}(1)$ and $z' = \gamma_{\pi\xi'}(1)$ are two different elements of the center of $(\tilde{P}, o)$.*

Proof. We write $\xi = \alpha_j^*$ and $\xi' = \alpha_k^*$ where the coefficients of α_j and α_k in the highest root are $h_j = h_k = 1$. In view of Lemma 2.8 it is sufficient to show that $S_{\alpha_l} = \{\pi\alpha_l^*\}$ if $h_l = 1$. Since $\delta \notin \{\alpha_l\}$, Condition (2.7) is empty for $\Omega := \{\alpha_l\}$ and (2.9) gives $\delta(X) = \pi$. Let $X = \sum_j c_j \alpha_j^*$ be an element of S_{α_l}. As $X \in \overline{C}$, we get $c_j \geq 0$. By (2.6) we have $\alpha_l(X) = c_l > 0$. From (2.8) we obtain $\alpha_m(X) = 0$ if $m \neq l$, so that $c_m = 0$ if $m \neq l$. Finally, by (2.9), we get $\delta(X) = c_l = \pi$, because $h_l = 1$. Thus $X = \pi\alpha_l^*$. □

We summarize the last results in:

Theorem 2.10. *There is a one-to-one correspondence between $Z(\tilde{P}, o)$ and the extrinsically symmetric elements (including 0) in a closed Weyl chamber $\overline{C}$ of $\mathfrak{p}$. More precisely:*

For any center element z of $(\tilde{P}, o)$ there exists precisely one shortest geodesic arc γ in $\tilde{P}$ joining $\gamma(0) = o$ to $\gamma(\pi) = z$ whose initial direction $\dot{\gamma}$ lies in $\overline{C}$, and the element $\dot{\gamma} \in \mathfrak{p}$ is extrinsically symmetric in $\mathfrak{g}$. Conversely, if ξ is an extrinsically symmetric element of $\overline{C}$ and γ_ξ the corresponding geodesic emanating from o, then γ_ξ is shortest on $[0, \pi]$ and $\gamma_\xi(\pi) \in Z(\tilde{P}, o)$.

Corollary 2.11 (see e.g. [Ca-27], [Tak-64], [Bu-85])**.** *The cardinality of $Z(\tilde{P}, o)$, this is also the order of the fundamental group of* $\mathrm{Ad}(\tilde{P})$, *is of one higher than the number of fundamental roots in $\mathcal{R}(\tilde{P})$ with coefficient 1 in the highest root.*

Let $\tilde{P}$ be an irreducible symmetric space. Then its root system $\mathcal{R}(P)$ is irreducible. If $\mathcal{R}(P)$ is non-reduced, then it is of type $(\mathfrak{bc})_r$ [He-78, p. 475]. In this case every fundamental root has coefficient 2 in the highest root. Among the irreducible reduced root systems only the exceptional root systems of type $\mathfrak{e}_8$, $\mathfrak{f}_4$ and $\mathfrak{g}_2$ have no fundamental roots with coefficient 1 in the highest root (see Table A.1). Thus we get:

Corollary 2.12. *Let $(\tilde{P}, o)$ be a simply connected irreducible pointed symmetric space of compact type. Then $Z(\tilde{P}, o) = \{o\}$ if and only if $\mathcal{R}(\tilde{P})$ is non-reduced or of type $\mathfrak{e}_8$, $\mathfrak{f}_4$ or $\mathfrak{g}_2$.*

For non simply-connected pointed symmetric spaces of compact type it still holds that the initial direction of a shortest geodesic arc joining the base point to the center is up to scaling extrinsically symmetric, but the converse is not true any further.

Corollary 2.13. *The initial direction $X := \dot{\gamma}$ of a shortest geodesic arc γ in P joining o to a pole $\gamma(\pi)$ of (P, o) is extrinsically symmetric.*

Proof. Let $\tilde{\pi} : \tilde{P} \to P$ be the universal cover of P. We fix a point $\tilde{o} \in \tilde{P}$ with the property that $\tilde{\pi}(\tilde{o}) = o$ and we lift the geodesic γ to a geodesic in $\tilde{P}$ emanating from $\tilde{o}$. Now $\mathfrak{p}$ is also canonically identified with $T_{\tilde{o}}\tilde{P}$ and the initial direction of the lifted geodesic is again X, i.e. the geodesic $\tilde{\gamma}_X$ in $\tilde{P}$ starting at $\tilde{o}$ in direction X satisfies $\tilde{\pi} \circ \tilde{\gamma}_X = \gamma$. Thus $\tilde{\gamma}_X(\pi) \in Z(\tilde{P}, \tilde{o})$ and $\tilde{\gamma}_X$ is shortest on $[0, \pi]$. The claim now follows from Corollary 2.7. $\square$

2.2. Classification of centrioles

Recall from Chapter 1 that a centriole point in a connected pointed symmetric space (P, o) of compact type is a midpoint of a geodesic arc in P joining o to a pole of (P, o). We denote by $\mathcal{P}(P, o)$ the set of all poles of (P, o). Notice that $Z(P, o) = \{o\}$ implies $\mathcal{P}(P, o) = \emptyset$, as we do not consider o as a pole of (P, o). Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of (P, o) and let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. The affine sublattice

$$\begin{aligned} \Gamma_{\mathcal{P}}(P) &:= \{X \in \mathfrak{a};\ \exp(X).o \in \mathcal{P}(P, o)\} \\ &= \{X \in \mathfrak{a};\ \exp(X).o \in Z(P, o),\ \exp(X).o \neq o \text{ and } \exp(2X).o = o\} \end{aligned} \tag{2.10}$$

(see Observation 1.6) of the center lattice of (P, o) in $\mathfrak{a}$ can also described as

$$\Gamma_{\mathcal{P}}(P) = \left(\Gamma_Z(P) \cap \frac{1}{2}\Gamma_o(P)\right) \setminus \Gamma_o(P) \tag{2.11}$$

where $\Gamma_o(P)$ is the *unit lattice* of (P, o) defined by

$$\Gamma_o(P) := \{X \in \mathfrak{a};\ \exp(X).o = o\} \tag{2.12}$$

We want to describe all centrioles of (P, o). Let $\mathfrak{K}$ be again the identity component of the isotropy group of (P, o). If j is a centriole point of (P, o), then, by Lemma 2.1, the same is true for any point in its $\mathfrak{K}$-orbit. Let $\gamma_{\pi X}$ be a geodesic emanating from o in direction $\pi X \in \mathfrak{p}$ with the property that $\gamma_{\pi X}(1)$ is a pole of (P, o). After conjugation by a suitable element of $\mathfrak{K}$, we can assume that X lies in our previously fixed closed Weyl chamber $\overline{C}$. Moreover, we suppose that $\gamma_{\pi X}$ is shortest on $\left[0, \frac{1}{2}\right]$ (until the centriole point), so that we are looking for elements X that satisfy:

1. $\pi X \in \Gamma_{\mathcal{P}}(P)$;
2. $\frac{1}{2}\pi X$ lies inside[15] or on $\mathrm{Cut}_P(\overline{C})$[16].

Remark 2.14. Since the number of such elements X is finite, centrioles are $\mathfrak{K}$-orbits in P (see [CN-88, Nag-88]). Let γ be any geodesic arc in P joining the base point $o = \gamma(0)$ to a centriole point $j = \gamma\left(\frac{1}{2}\right)$ of (P, o). Since the whole $\mathfrak{K}$-orbit consists of geodesic arcs of same length joining o to the centriole of (P, o) containing j, the first variation formula shows that the vector $\dot{\gamma}\left(\frac{1}{2}\right)$ is perpendicular to the tangent space of the centriole at j (see [Sa-96]).

It is well known that centrioles are totally geodesic (see [CN-88, Nag-88]). But Nagano stated more, namely that centrioles are always reflective, also if P is not simply connected (see [Nag-88, Prop. 2.12(ii), p. 62] and his reference to [CN-88]). For completeness we include the proof. For compact Lie groups the centrosome $C_z(\mathfrak{G}, e)$ is the fix point set of the involution $g \mapsto zg^{-1}$ and hence reflective. Hence we may now restrict our attention to connected compact pointed symmetric spaces (P, o).

Lemma 2.15 (Prop. 2.9 in [CN-88], Theorem 3.3 in [Ch-89])**.** *For any pole z of (P, o), there exists a unique fix point free involutive isometry ρ_z of P mapping o to z such that the orbit space P/Γ_z with $\Gamma_z := \{\mathrm{Id}, \rho_z\}$ is a symmetric space.*

Proof. (see the proof of Prop. 2.9 in [CN-88]). The Cartan map $\iota^P : P \mapsto \mathfrak{I}(P)$, $p \mapsto s_p$ identifies o and z, because $s_z = s_o$. Since the Cartan map is a covering and its image is again a symmetric space (see section A.2), there exists a discrete subgroup Γ of the centralizer Δ of $\mathfrak{I}(P)$ in $\mathfrak{T}(P)$ such that the image of ι^P is isomorphic to P/Γ (see Theorem A.1 and [Wo-84, p. 244]). As Γ is the deck transformation group of the covering

[15] By this we mean that $\lambda\frac{1}{2}\pi X \in \mathrm{Cut}_P(\overline{C})$ for some $\lambda > 1$.

[16] $\mathrm{Cut}_P(\overline{C})$ denotes the intersection of the tangent cut locus of P in $\mathfrak{p} \cong T_oP$ with $\overline{C}$.

ι^P, every nontrivial element of Γ acts fix point free. Since $\iota^P(o) = \iota^P(z)$, there must be a unique element ρ_z in Γ with $\rho_z(o) = z$. Let γ be a geodesic in P satisfying $\gamma(0) = o$ and $\gamma(1) = z$, then $\gamma(2) = o$ (Observation 1.6). Let $\tau_t := s_{\gamma\left(\frac{t}{2}\right)} s_{\gamma(0)}$ be the one-parameter subgroup of transvections along γ (see also [Sa-96, p. 175]), then τ_1 maps $\gamma(0) = o$ to $\gamma(1) = z$ and squares to the identity, because $\tau_1 \circ \tau_1 = \tau_2 = s_{\gamma(1)} s_{\gamma(0)} = s_z s_o = s_o^2 = \mathrm{Id}$. Since ρ_z commutes with any transvection, we get $\rho_z^2(o) = \rho_z(z) = \rho_z(\tau_1(o)) = \tau_1(\rho_z(o)) = \tau_1(\tau_1(o)) = \tau_2(o) = o$. Hence ρ_z^2 is an element of Γ that has a fix point. Thus $\rho_z^2 = \mathrm{Id}$. This shows that $\Gamma_z := \{\mathrm{Id}, \rho_z\}$ is a subgroup of Δ that is isomorphic to $\mathbb{Z}_2$. Theorem A.1 implies that P/Γ_z is a symmetric space. □

Proposition 2.16 (see Prop. 2.12(ii) in [Nag-88]). *Centrioles of connected compact pointed symmetric spaces are reflective. More precisely, the centrosome $C_z(P, o)$ is the fix point set of the involutive automorphism $r_z := \rho_z s_o$.*

Proof. Let z be a pole of (P, o) and $j \in C_z(P, o)$ the midpoint of a geodesic arc γ in P joining $\gamma(0) = o$ to $\gamma(1) = z$. Then $\tilde{\gamma} := \rho_z \circ \gamma$ is again a geodesic in P and satisfies $\tilde{\gamma}(0) = z$ and $\tilde{\gamma}(1) = o$. Let $\pi_z : P \to P/\Gamma_z$ the canonical projection, then $\pi_z \circ \gamma = \pi_z \circ \tilde{\gamma}$. Hence $\tilde{\gamma}(t) = \gamma(t + 1)$. Thus $r_z(j) = r_z\left(\gamma\left(\frac{1}{2}\right)\right) = \rho_z\left(s_o\left(\gamma\left(\frac{1}{2}\right)\right)\right) = \rho_z\left(\gamma\left(-\frac{1}{2}\right)\right) = \tilde{\gamma}\left(\left(-\frac{1}{2}\right)\right) = \gamma\left(\frac{1}{2}\right) = j$.

Conversely, let j be a fix point of r_z. Since ρ_z is involutive we get $\rho_z(j) = s_o(j)$. Let γ be a geodesic in P satisfying $\gamma(0) = o$ and $\gamma\left(\frac{1}{2}\right) = j$. Then $\pi_z\left(\gamma\left(\frac{1}{2}\right)\right) = \pi_z(\rho_z(j)) = \pi_z(s_o(j)) = \pi_z\left(\gamma\left(-\frac{1}{2}\right)\right)$. Since a geodesics in symmetric spaces are orbits of one-parameter groups of isometries, they close at any self-intersection. Thus $(\pi_z \circ \gamma)(t) = (\pi_z \circ \gamma)(t + 1)$ and in particular $(\pi_z \circ \gamma)(0) = (\pi_z \circ \gamma)(1)$. Hence either $\gamma(1) = \gamma(0) = o$ or $\gamma(1) = z$. The first equation implies $\gamma(t) = \gamma(t + 1)$ and hence $j = \gamma\left(\frac{1}{2}\right) = \gamma\left(-\frac{1}{2}\right) = s_o(j) = \rho_z(j)$. This contradicts the fact that ρ_z has no fix point. Thus $\gamma(1) = z$ and j lies in $C_z(P, o)$.

To prove that r_z is an involution we actually show that $s_o \rho_z s_o = \rho_z$. Since ρ_z commutes with any transvection we get $(s_o \rho_z s_o)(s_p s_q) = s_o \rho_z s_o s_p s_q = s_o s_o s_p \rho_z s_q = s_p \rho_z s_q s_o s_o = (s_p s_q)(s_o \rho_z s_o)$ for all points p and q in P. As $\mathfrak{T}(P)$ is generated by the products of two geodesic symmetries, we see that $s_o \rho_z s_o$ centralizes $\mathfrak{T}(P)$. Since ρ_z is an involution without fix points, the same holds true for $s_o \rho_z s_o$. Moreover $(s_o \rho_z s_o)(o) = z$. Lemma 2.15 yields $s_o \rho_z s_o = \rho_z$ by uniqueness. □

Remark 2.17. Let $C_z(P, o)$ be a centrosome in (P, o) and r_z the corresponding reflection defined in Proposition 2.16, then $r_z(o) = z$.

While the center lattice of a symmetric space P of compact type can be read off from its root system (Equation 2.3), the same is not true for its unit lattice, e.g. for adjoint spaces the unit lattice coincides with its center lattice. Actually, a symmetric space of compact type is uniquely determined by its root system labelled with its multiplicities and its unit lattice [He-78, Lo-69-II]. For simply connected pointed symmetric spaces $(\tilde{P}, o)$ of compact type the unit lattice can still be described in terms of its root system. Following Loos (see [Lo-69-II, pp. 25, 69, 77]) the unit lattice of $(\tilde{P}, o)$ is

$$\Gamma_o(\tilde{P}) = \mathrm{span}_{\pi\mathbb{Z}}(\check{\mathcal{R}}(\tilde{P})) \tag{2.13}$$

where $\check{\mathcal{R}}(\tilde{P})$ is the system of inverse roots w.r.t. the root system $\mathcal{R}(\tilde{P})$ of $\tilde{P}$ corresponding to $\mathfrak{a}$. Let $\Sigma = \{\alpha_1, ..., \alpha_r\}$ be a fundamental root system in $\mathcal{R}(\tilde{P})$ corresponding to a Weyl chamber C. In view of Corollary 2.12 we can ignore symmetric spaces with non-reduced root systems in our further considerations. Hence we may suppose that $\mathcal{R}(\tilde{P})$ is reduced. Then $\check{\Sigma} = \{\check{\alpha}_1, ..., \check{\alpha}_r\}$ is a fundamental root system of $\check{\mathcal{R}}(\tilde{P})$ (see e.g. [Se-87]) and $\Gamma_o(\tilde{P})$ can be written as

$$\Gamma_o(\tilde{P}) = \mathrm{span}_{\pi\mathbb{Z}}(\check{\Sigma}). \tag{2.14}$$

We next want to express the vectors of the dual basis $\Sigma^* = \{\alpha_1^*, ..., \alpha_r^*\}$ in the basis $\check{\Sigma} = \{\check{\alpha}_1, ..., \check{\alpha}_r\}$ of $\mathfrak{a}$. As a first step we express the root vectors of Σ in the basis Σ^* : $H_j := H_{\alpha_j} = \sum_k s_{jk}\alpha_k^*$ with $s_{jk} = \alpha_k(H_j) = \langle H_j, H_k\rangle$. Since $\check{\alpha}_j = 2\frac{H_j}{|\alpha_j|^2}$ (see Equation A.17) we obtain $\check{\alpha}_j = \sum_k \check{s}_{jk}\alpha_k^*$ with $\check{s}_{jk} = 2\frac{\langle H_j, H_k\rangle}{|\alpha_j|^2} =: c_{kj}$, where $C = (c_{jk})$ is the Cartan matrix (see Equation A.18). Hence $\alpha_j^* = \sum_k s_{jk}^*\check{\alpha}_k$ where s_{jk}^* are the entries of $(C^{-1})^T$. We conclude:

Lemma 2.18. *Assume that $\tilde{P}$ is an irreducible simply connected symmetric space of compact type. Then the vector $\pi\alpha_j^*$ lies in $\Gamma_o(\tilde{P})$ if and only if the j-th column of C^{-1} has only integer entries.*

Equation 2.11 can be rephrased as follows: An element X lies in $\Gamma_{\mathcal{P}}(\tilde{P})$ if and only if X lies in $\Gamma_Z(\tilde{P})$ but not in $\Gamma_o(\tilde{P})$ and $2X$ lies in $\Gamma_o(\tilde{P})$. Thus we get:

Lemma 2.19. *Assume that $\tilde{P}$ is an irreducible simply connected symmetric space of compact type. Let $X = \pi\sum_j c_j\check{\alpha}_j$ be an element of $\Gamma_Z(\tilde{P})$. Then X lies also in $\Gamma_{\mathcal{P}}(\tilde{P})$ if and only if the coefficients c_j are all half-integers (elements of $\frac{1}{2}\mathbb{Z}$), but not all integers. In particular $\pi\alpha_j^*$ lies in $\Gamma_{\mathcal{P}}(\tilde{P})$ if and only if the j-th column of C^{-1} has only half-integer entries that are not all integers.*

We can now describe all centriole points in $(\tilde{P}, o)$:

Theorem 2.20. *Assume that $\tilde{P}$ is an irreducible simply connected symmetric space of compact type. There are four possible types of elements $X \in \overline{C}$ that satisfy*

1. *$\pi X \in \Gamma_{\mathcal{P}}(\tilde{P})$ and*
2. *$\frac{1}{2}\pi X$ lies inside or on $\mathrm{Cut}_{\tilde{P}}(\overline{C})$ (see Footnote 15 on p. 10),*

namely:

Type I: $X = \alpha_j^*$ where the coefficient of α_j in the highest root is $h_j = 1$ (i.e. X is extrinsically symmetric) and, moreover, the entries of the j-th column of C^{-1} are half-integers, but not all integers. The element $\frac{1}{2}\pi X$ lies in $\frac{1}{2}\mathrm{Cut}_{\tilde{P}}(\overline{C})$.

Type II: $X = \alpha_j^*$ where the coefficient of α_j in the highest root is $h_j = 2$ and, moreover, the entries of the j-th column of C^{-1} are half-integers but not all integers. The element $\frac{1}{2}\pi X$ lies on $\mathrm{Cut}_{\tilde{P}}(\overline{C})$.

Type III: $X = 2\alpha_j^*$ *where the coefficient of* α_j *in the highest root is* $h_j = 1$ *(i.e.* α_j^* *is extrinsically symmetric) and, moreover, the entries of the* j*-th column of* C^{-1} *are quarter-integers (elements of* $\frac{1}{4}\mathbb{Z}$*) but not all half-integers. The element* $\frac{1}{2}\pi X$ *lies on* $\mathrm{Cut}_{\tilde{P}}(\overline{C})$.

Type IV: $X = \alpha_j^* + \alpha_k^*$, $k \neq j$, *where the coefficients of* α_j *and* α_k *in the highest root are both 1,* $h_j = h_k = 1$ *(*X *is the sum of two extrinsically symmetric elements in* $\overline{C}$*), and, moreover, the sum of the* j*-th and the* k*-th column of* C^{-1} *has half-integer entries that are not all integers. The element* $\frac{1}{2}\pi X$ *lies on* $\mathrm{Cut}_{\tilde{P}}(\overline{C})$.

Conversely any element X *of type I, II, III or IV satisfies the requirements 1 and 2.*

Proof. The element $X \in \overline{C}$ can be written as $X = \sum_j x_j \alpha_j^*$ with $x_j \geq 0$. Since $\pi X \in \Gamma_{\mathcal{P}}(\tilde{P})$ the coefficients x_j are non-negative integers. As $0 \notin \Gamma_{\mathcal{P}}(\tilde{P})$ at least one coefficient x_j is non-zero. Since $\tilde{P}$ is an irreducible simply connected symmetric space of compact type, the tangent cut locus in $\overline{C}$ is described by Equation 2.4. Let $\delta = \sum_j h_j \alpha_j$ be the highest root, then $\frac{1}{2}\pi X \in \overline{\mathrm{Cut}_{\tilde{P}}}$ if and only if $\sum_j h_j x_j \leq 2$. We distinguish several cases:

1. Exactly one coefficient x_j does not vanish.
   a) If $h_j = 1$, there are two cases:
      i) $x_j = 1$: Then $X = \alpha_j^*$. Since $X \in \Gamma_{\mathcal{P}}(\tilde{P})$, Lemma 2.19 shows that X is of type I.
      ii) $x_j = 2$: Then $X = 2\alpha_j^*$. Since $X \in \Gamma_{\mathcal{P}}(\tilde{P})$, Lemma 2.19 shows that $X = \alpha_j^*$ is of type III.
   b) If $h_j = 2$, the only possibility is $x_j = 1$ and, by Lemma 2.19, X is of type II, since $X \in \Gamma_{\mathcal{P}}(\tilde{P})$.
2. Exactly two coefficients x_j and x_k ($j \neq k$) do not vanish. Since $h_j x_j$ and $h_k x_k$ are both greater to or equal to 1 and $h_j x_j + h_k x_k \leq 2$ we get $h_j = x_j = h_k = x_k = 1$, so that $X = \alpha_j^* + \alpha_k^*$. By Lemma 2.19 X is of type IV.
3. At least three coefficients x_j, x_k and x_l do not vanish. Since $h_j x_j$, $h_k x_k$ and $h_l x_l$ are all at least 1, we get $h_j x_j + h_k x_k + h_l x_l \geq 3$. This contradicts the requirement that $\frac{1}{2}\pi X$ lies inside or on $\mathrm{Cut}_{\tilde{P}}(\overline{C})$.

□

Remark 2.21. Given a pole z of $(\tilde{P}, o)$, one gets a symmetric space $\tilde{P}/\Gamma_z$ by identifying o and z (see Lemma 2.15 and the original results in [CN-88, Nag-88, Nag-92]). The corresponding projection is $\pi_z : \tilde{P} \to \tilde{P}/\Gamma_z$. Any centriole in the centrosome $C_z(\tilde{P}, o)$ projects to a *polar* of the pointed symmetric space $(\tilde{P}/\Gamma_z, \pi_z(o))$. A *polar* of $(\tilde{P}/\Gamma_z, \pi_z(o))$ is a connected component of the set of all midpoints of closed geodesics in $\tilde{P}/\Gamma_z$ that start in $\pi_z(o)$, or, equivalently, a connected component of the fix point set of the geodesic symmetry of $\tilde{P}/\Gamma_z$ at $\pi_z(o)$ (see [CN-78, CN-88, Nag-88, Nag-92]). Poles are singleton polars (see Observation 1.6). Lists of polars can be found in [CN-78, CN-88]. A more detailed case-by-case determination of these polars is described in [Nag-88] and further proofs can be found in [Nag-92]. It might be possible to establish case-by-case a list of all centrioles in $C_z(\tilde{P}, o)$ by looking at those polars of $(\tilde{P}/\Gamma_z, \pi_z(o))$ that are not projections of polars of $(\tilde{P}, o)$ (see also [Bu-85] or [NS-91, 1.3b]). Burns and Nagano discovered a necessary condition in terms of roots for a vector to be the initial direction of a shortest geodesic arc joining a base point to its polar (see [Bu-85, Lemma 2.1, Prop. 2.2], [Nag-88, Prop. 6.5, p. 72], [Nag-92, pp. 52 ff., in particular Prop. 2.9 and Cor. 2.13], [Bu-93, Lemma 2.1, Prop. 2.2]). Their proofs are quite similar to the above proof of Theorem 2.20[17]. In Theorem 2.20 only geodesics whose initial direction is of type I are shortest up to the pole. We are not aware that a complete description of all shortest geodesics to centrioles in an irreducible simply connected pointed symmetric space of compact type in terms of its root system has been known so far.

Remark 2.22. If $X = 2\alpha_j^*$ is of type III, then $j = \gamma_{\pi X}\left(\frac{1}{2}\right) = \gamma_{\alpha_j^*}(\pi)$ is an element of the center of $(\tilde{P}, o)$ (Theorem 2.10). One example occurs if $\tilde{P} = \mathrm{SU}_{4n}$ and $\gamma_{\alpha_j^*}(\pi)$ is an element of order four in $Z(\mathrm{SU}_{4n}) \cong Z_{4n}$.

Remark 2.23. If $(\tilde{P}, o)$ is an irreducible simply connected pointed symmetric space of compact type that has a pole z, and if $\gamma_{\pi X}$ is a shortest geodesic arc in P joining $\gamma_{\pi X}(0) = o$ to $\gamma_{\pi X}(1) = z$ where X lies in a fixed closed Weyl chamber $\overline{C}$ of $\mathfrak{p}$, then X is of type I. On the other hand, by Theorem 2.10 the element X is unique up to conjugation by the identity component of the isotropy group of $(\tilde{P}, o)$. Thus the number of poles of $(\tilde{P}, o)$ can be read off from the root system of $\tilde{P}$. To admit poles, the center of $(\tilde{P}, o)$ needs to be non-trivial (see Observation 1.5). Therefore the root system of $\tilde{P}$ must be reduced. The number of poles of $(\tilde{P}, o)$ now coincides with the number of center elements of order two in the connected simply connected compact simple Lie group $\mathfrak{G}$ that has the same root system as $\tilde{P}$. Notice that whenever the center of $(\tilde{P}, o)$ contains only one point besides o[18], the center of $\mathfrak{G}$ is isomorphic to $\mathbb{Z}_2$ and hence $(\tilde{P}, o)$ admits precisely one pole. Moreover, we observe that most simply connected simple real Lie groups have either no or just one center element of order two. The only exception is Spin_{4n} $(n \geq 2)$ whose center is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and hence contains three elements of order two. Thus poles of $(\tilde{P}, o)$ are mostly unique. The exceptions are the spaces whose the root system is of type $\mathfrak{d}_{2n}$ with $n \geq 2$, namely $\tilde{P} = \mathrm{Spin}_{4n}$ and $\tilde{P} = \tilde{G}_{2n}(\mathbb{R}^{4n}) = \mathrm{SO}_{4n}/(\mathrm{SO}_{2n} \times \mathrm{SO}_{2n})$ with $n \geq 2$. They have three poles (see also [CN-88, pp. 293 f] and [Nag-88, §2]).

We now want to show that the centrioles corresponding to two elements X and Y mentioned in Theorem 2.20 that are not conjugate under the identity component $\mathfrak{K}$ of the isotropy group of $(\tilde{P}, o)$ are distinct. For this we use again Sakai's description of the tangent cut locus of $(\tilde{P}, o)$ from [Sa-78a, Sa-78b] (see also page 8). A direct consequence of Corollary 2.7 is:

[17] This is not astonishing. All these proofs also share common aspects with the proof of the classification of inner involutions of a compact simple Lie group as presented e.g. in [Lo-69-II, p. 121].

[18] this happens for the root systems of type $\mathfrak{b}_r$, $\mathfrak{c}_r$ and $\mathfrak{e}_7$, see [He-78, Table IV, p. 516]

Corollary 2.24. *If α_j and α_k $(j \neq k)$ have both coefficient 1 in the highest root, $h_j = h_k = 1$, then the $\mathfrak{K}$-orbits of the centriole points $j = \gamma_{\pi\alpha_j^*}\left(\frac{1}{2}\right)$ and $j' = \gamma_{\pi\alpha_k^*}\left(\frac{1}{2}\right)$ are disjoint.*

To investigate the other cases we need:

Lemma 2.25. *Using the notation introduced for Lemma 2.8 we have:*

(1) If $h_j = 1$, then $S_{\{\alpha_j\}} = \{\pi\alpha_j^*\}$;

(2) If $h_j = 2$, then $S_{\{\alpha_j\}} = \left\{\frac{1}{2}\pi\alpha_j^*\right\}$;

(3) If $j \neq k$ and $h_j = h_k = 1$, then

$$S_{\{\alpha_j,\alpha_k\}} = \left\{x_j\alpha_j^* + x_k\alpha_k^*;\ x_j > 0,\ x_k > 0,\ x_j + x_k = \pi\right\}.$$

In particular $\frac{1}{2}\pi(\alpha_j^* + \alpha_k^*) \in S_{\{\alpha_j,\alpha_k\}}$.

Proof. We first observe that in all three cases Equation 2.7 is empty and that Equation 2.9 reads as $\delta(X) = \pi$. Claim 1 has been shown in the proof of Corollary 2.9. The proof of Claim 2 is similar: Let $X \in S_{\{\alpha_j\}}$. Since $X \in \overline{C}$ we set $X = \sum_k x_k\alpha_k^*$ with $x_k \geq 0$. By Equation 2.6 we have $\alpha_j(X) > 0$ and therefore $x_j > 0$. Using Equation 2.8 we get $\alpha_l(X) = 0$ if $l \neq j$. Hence $x_l = 0$ for $l \neq j$. Finally, by Equation 2.9, $\delta(X) = \pi$. If $d_j = 1$ as in (1), we get yields $x_j = \pi$ and, if $d_j = 2$ as in (2), we obtain $2x_j = \pi$. Therefore $X = \frac{1}{2}\pi\alpha_j^*$ if $d_j = 2$. To show the third claim let again $X = \sum_k x_k\alpha_k^* \in S_{\{\alpha_j,\alpha_k\}}$ with $x_k \geq 0$. Equation 2.6 yields $x_j, x_k > 0$ and, by Equation 2.8, $x_l = 0$ if $l \neq j, k$. Finally, with $d_j = d_k = 1$ Equation 2.9 implies $\delta(X) = x_j + x_k = \pi$. □

Lemmata 2.25 and 2.8 imply:

Corollary 2.26. *Let $X, Y \in \overline{C}$ be two different[19] elements mentioned in Theorem 2.20 of type II, III or IV. Then the $\mathfrak{K}$-orbits of the corresponding centriole points $j := \gamma_{\frac{1}{2}\pi X}(1)$ and $j' := \gamma_{\frac{1}{2}\pi Y}(1)$ are disjoint.*

Corollary 2.27. *Let $X \in \overline{C}$ be an element of type I and $Y \in \overline{C}$ be an element of type II, III or IV (see Theorem 2.20). Then the $\mathfrak{K}$-orbits in $\tilde{P}$ of the corresponding centriole points $j := \gamma_{\frac{1}{2}\pi X}(1)$ and $j' := \gamma_{\frac{1}{2}\pi Y}(1)$ are disjoint.*

Proof. Since $\frac{1}{2}\pi X$ lies in the interior of the tangent cut locus there is $\varepsilon > 0$ such that $\gamma_{\frac{1}{2}\pi X}$ realizes the distance (is shortest) between $o = \gamma_{\frac{1}{2}\pi X}(0)$ and $\gamma_{\frac{1}{2}\pi X}(t)$ for $t \in [0, 1+\varepsilon)$. Assume that there exists $k \in \mathfrak{K}$ with $k.\gamma_{\frac{1}{2}\pi X}(1) = \gamma_{\frac{1}{2}\pi Y}(1)$. Since k acts by isometries and leaves o fix, the geodesic $k.\gamma_{\frac{1}{2}\pi X}$ still realizes the distance between $o = k.\gamma_{\frac{1}{2}\pi X}(0)$ and $k.\gamma_{\frac{1}{2}\pi X}(t)$ for $t \in [0, 1+\varepsilon)$. But $k.\gamma_{\frac{1}{2}\pi X}(1) = \gamma_{\frac{1}{2}\pi Y}(1)$ lies in the cut locus of o, since $\frac{1}{2}\pi Y$ lies in the tangent cut locus. A contradiction. □

We can summarize Corollaries 2.24, 2.26 and 2.27 as follows:

[19] The types of X and Y may be different or not.

Theorem 2.28. *Let $\frac{1}{2}\pi X$ and $\frac{1}{2}\pi Y$ be two different elements of $\Gamma_{\mathcal{P}}(\tilde{P}) \cap \overline{C}$ lying either inside or on $\mathrm{Cut}_{\tilde{P}}(\overline{C})$ (see Footnote 15 on p. 10). Then the $\mathfrak{K}$-orbits in $\tilde{P}$ of the corresponding centriole points $j := \gamma_{\frac{1}{2}\pi X}(1)$ and $j' := \gamma_{\frac{1}{2}\pi Y}(1)$ of $(\tilde{P}, o)$ are disjoint.*

Remark 2.29. Theorem 2.20 and Theorem 2.28 show that the number of centrioles of a simply connected pointed symmetric space $(\tilde{P}, o)$ can read off from its root system.

Remark 2.30. There may well be an isometry g of $\tilde{P}$ fixing o with the property that $j' = g.j$, where j and j' are the centriole points of Theorem 2.28. A typical situation for this phenomenon is the following: The symmetric space $\tilde{P} = \mathfrak{G}$ is a simply connected compact simple Lie group, and g is an isometry of $\mathfrak{G}$ induced from a non-trivial Dynkin diagram automorphism. Then g may interchange two extrinsically symmetric elements in the Lie algebra $\mathfrak{g} \cong T_e\mathfrak{G}$ of $\mathfrak{G}$ that are not in the same $\mathfrak{K}$-Orbit. This happens e.g. for Spin_{4n}. But such an isometry g is never in the identity component $\mathfrak{K}$ of the isotropy group of $(\tilde{P}, o)$.

For any pole z of (P, o) there exists at least one[20] centriole that consists of midpoints of shortest geodesic arcs in P joining o to z. We call such centrioles *minimal*[21]. By Theorem 2.10 an Corollary 2.13 these centrioles correspond to vectors of type I.

Example 2.31 ($\mathcal{R}(\tilde{P})$ of type $\mathfrak{c}_r$)**.** To study the number of poles and centrioles of a simply connected irreducible pointed symmetric space whose root system has type $\mathfrak{c}_r$, it is sufficient to look at the symplectic group Sp_r, since these numbers depend only on the root system. As the center of Sp_1 is isomorphic to $\mathbb{Z}_2$, the symplectic group has precisely one pole, namely $-\mathrm{Id}$. Any fundamental root system of type $\mathfrak{c}_r$ contains only one fundamental root, namely α_r, with coefficient 1 in the highest root (see Table A.1, p. 65). The element α_r^* must be of type I and therefore elements of type III and IV do not occur (see Theorem 2.20). The question whether there are elements of type II remains. Instead of inverting the Cartan matrix, we rather look at the elements $\alpha_1^*, ..., \alpha_{r-1}^*$ explicitly. The Lie algebra $\mathfrak{sp}_r$ of Sp_r consists of all $2r \times 2r$ complex matrices of the form $\begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix}$ where A is a skew-hermitian and B a symmetric $r \times r$ matrix. As maximal torus in $\mathfrak{sp}_r$ we take the set formed by those matrices where B vanishes and A is a purely imaginary diagonal matrix. A fundamental root system is formed by the elements $\alpha_j = \epsilon_j - \epsilon_{j+1}$ for $1 \leq j \leq r-1$ and $\alpha_r = 2\epsilon_r$, where $\epsilon_j\left(i\begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix}\right) = d_j$, the (j,j)-entry of the real diagonal $r \times r$ matrix D. The roots α_j with $1 \leq j \leq r-1$ have coefficient 2 in the highest root, while the coefficient of α_r is the highest root is 1 [He-78, pp. 463 f. and pp. 476 f.]. If we denote by E_j the diagonal matrix $i\begin{pmatrix} D_j & 0 \\ 0 & -D_j \end{pmatrix}$, where all entries of D_j vanish except the (j,j)-entry which is 1, the corresponding dual basis is $\alpha_j^* = \sum_{k=1}^{j} E_k$ for $1 \leq j \leq r-1$ and $\alpha_r^* = \frac{1}{2}\sum_{k=1}^{r} E_k$. The geodesic in Sp_r defined

[20] if P is simply connected precisely one
[21] compare the notion of s-size in [Tan-95]

by α_j^* is $\gamma_j(t) := \exp(2t\alpha_j^*)$ (see Equation A.9). Thus $\gamma_r(\pi) = -\mathrm{Id}$, the only non-trivial center element of Sp_r, and $\gamma_j(\pi) \neq -\mathrm{Id}$ for $1 \leq j \leq r-1$. Hence elements of type II do not occur. This shows:

An irreducible simply connected pointed symmetric space $(\tilde{P}, o)$ whose root system is of type $\mathfrak{c}_r$ has exactly one pole and one centriole which is, of course, minimal.

This observation is not new, it can be found in [Bu-85] and [Nag-92, Prop. 2.23(i)].

Example 2.32 ($\mathcal{R}(\tilde{P})$ of type $\mathfrak{e}_7$)**.** The Cartan matrix of the Dynkin diagram of type $\mathfrak{e}_7$ given in Table A.1 on page 65 is

$$C = \begin{pmatrix} 2 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

and its inverse is

$$C^{-1} = \begin{pmatrix} \frac{7}{2} & 2 & 4 & 6 & \frac{9}{2} & 3 & \frac{3}{2} \\ 2 & 2 & 3 & 4 & 3 & 2 & 1 \\ 4 & 3 & 6 & 8 & 6 & 4 & 2 \\ 6 & 4 & 8 & 12 & 9 & 6 & 3 \\ \frac{9}{2} & 3 & 6 & 9 & \frac{15}{2} & 5 & \frac{5}{2} \\ 3 & 2 & 4 & 6 & 5 & 4 & 2 \\ \frac{3}{2} & 1 & 2 & 3 & \frac{5}{2} & 2 & \frac{3}{2}. \end{pmatrix}.$$

By Theorem 2.20 and 2.28 we get:

A simply connected symmetric space whose root system is of type $\mathfrak{e}_7$ has two centrioles:

- *a minimal centriole containing* $\mathrm{Exp}_o\left(\frac{\pi}{2}\alpha_7^*\right) = \exp\left(\frac{\pi}{2}\alpha_7^*\right).o$ *defined by the extrinsically symmetric element* α_7^*;
- *a non-minimal centriole containing* $\mathrm{Exp}_o\left(\frac{\pi}{2}\alpha_1^*\right) = \exp\left(\frac{\pi}{2}\alpha_1^*\right).o$ *defined by the element* α_1^* *of type II.*

Example 2.33. To find an example of elements of type IV, we consider a root system of type $\mathfrak{d}_4$ as e.g. for $\tilde{P} = \mathrm{Spin}_8$. The corresponding Dynkin diagram can be found in Table A.1 (p. 65). The roots α_1, α_3 and α_4 have coefficient one in the highest root. The Cartan matrix C and its inverse are

$$C = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix} \quad \text{and} \quad C^{-1} = \begin{pmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{2} \\ 1 & 2 & 1 & 1 \\ \frac{1}{2} & 1 & 1 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} & 1 \end{pmatrix}.$$

This shows that the elements α_1^*, α_3^* and α_4^* are of type I and $\alpha_1^* + \alpha_3^*$, $\alpha_1^* + \alpha_4^*$ and $\alpha_3^* + \alpha_4^*$ are of type IV.

Remark 2.34. We have seen that all types of elements mentioned in Theorem 2.20 occur in examples.

3. Inclusion chains of symmetric spaces

In his proof of Bott's periodicity theorem for orthogonal groups Milnor constructed the inclusion chain

$$\begin{array}{lllll} & \mathfrak{G} = \mathrm{SO}_{16n} & \supset P_1 = \mathrm{SO}_{16n}/\mathrm{U}_{8n} & \supset P_2 = \mathrm{U}_{8n}/\mathrm{Sp}_{4n} & \supset P_3 = G_{2n}(\mathbb{H}^{4n}) \\ \supset & P_4 = \mathrm{Sp}_{2n} & \supset P_5 = \mathrm{Sp}_{2n}/\mathrm{U}_{2n} & \supset P_6 = \mathrm{U}_{2n}/\mathrm{O}_{2n} & \supset P_7 = G_n(\mathbb{R}^{2n}) \\ \supset & P_8 = \mathrm{SO}_n & & & \end{array} \tag{3.1}$$

of symmetric spaces [Mil-69, § 24] in terms of complex structures:

Let P_1 be one of the two isomorphic connected components of the set of all complex structures in SO_{16n}, i.e. $P_1 \cong \mathrm{SO}_{16n}/\mathrm{U}_{8n}$. Choose an element j_1 in P_1 and denote by P_2 the set of all complex structures in P_1 that anticommute with j_1 in the sense that $j \in P_2$ satisfies $jj_1 = -j_1 j$. The space P_2 is connected (see Section 3.2) and isomorphic to $\mathrm{U}_{8n}/\mathrm{Sp}_{4n}$. In P_2 we again fix a point j_2 and consider the set of all complex structures in P_2 that anti-commute with j_2. The choice of j_1 and j_2 (together with $j_1 j_2$) induces a quaternionic structure on $\mathbb{R}^{16n}$. The space of all complex structures in P_2 that anti-commute with j_2 can be identified with the Grassmannian of all quaternionic subspaces of $\mathbb{R}^{16n}$ [Mil-69, p. 139]. This space has several connected components. The component consisting of all half-dimensional quaternionic subspaces of $\mathbb{R}^{16n}$ will be denoted by P_3. Iterating this scheme and making prudent choices of connected components, one gets the above inclusion chain.

Milnor showed that P_{k+1} can also be described as a connected component of the set of all shortest geodesic arcs in P_k joining j_k to $-j_k$ [Mil-69, Lemma 24.4]. This relates his approach with Bott's original idea in [Bo-59].

In Section 3.1 we present an abstract version of Milnor's construction starting with an arbitrary compact connected Lie group $\mathfrak{G}$. We focus on the equivalence between certain centrioles and some components of the set of all complex structures that 'anti'-commute with some chosen ones. At some steps we may, as Milnor, have to choose a connected component. Hence there may be several inclusion chains of connected spaces starting with the same connected compact real Lie group. In Section 3.2 we study inclusion chains that start with a connected simple compact real Lie group and that consist only of minimal centrioles.

3.1. Generalizing Milnor's construction

To generalize Milnor's construction we start with an arbitrary compact connected real Lie group $\mathfrak{G}$. Assume that $\mathfrak{G}$ has a pole z, i.e. a center element of order two (Observation 1.3). We call an element j of $\mathfrak{G}$ a *z-complex structure* if $j^2 = z$ or, equivalently, if j is the midpoint of a geodesic arc in $\mathfrak{G}$ joining the identity to z (Observation 1.4). We say that two z-complex structures j_1 and j_2 of $\mathfrak{G}$ *z-commute*[22], if $j_1 j_2 = z j_2 j_1$.

We now choose a connected component P_1 of the set of all z-complex structures in $\mathfrak{G}$. Since P_1 is a centriole of $(\mathfrak{G}, e)$, it is a totally geodesic conjugacy orbit of $\mathfrak{G}$ (see Remark 2.14) and hence a compact symmetric space.

Assume that we have constructed a totally geodesic inclusion chain

$$\mathfrak{G} \supset P_1 \supset \ldots \supset P_k$$

in the following way: For $2 \leq l \leq k$ the space P_l is a connected component of the set of all z-complex structures of $\mathfrak{G}$ that are contained in P_{l-1} and that z-commute with a fixed element j_{l-1} of P_{l-1}. Let us fix a point j_k in P_k. This point is of course a z-complex structure of $\mathfrak{G}$. Assume that there is a point $j \in P_k$ that z-commutes with j_k (see Assumption 3.3 below).

Lemma 3.1. *Let γ be a geodesic in P_k satisfying $\gamma(0) = j_k$ and $\gamma\left(\frac{1}{2}\right) = j$. Then $\gamma(1) = z j_k$.*

Proof. Since P_k is a totally geodesic submanifold of $\mathfrak{G}$, the curve $j_k^{-1}\gamma$ is a geodesic in $\mathfrak{G}$ starting at the identity. Hence $j_k^{-1}\gamma$ has the form $j_k^{-1}\gamma = \exp(2tX)$ for a suitable $X \in \mathfrak{g}$ (see Equation A.9). Thus $\gamma(t) = j_k \exp(2tX)$ and $j = \gamma\left(\frac{1}{2}\right) = j_k \exp(X)$ so that $\exp(X) = j_k^{-1} j = z j_k j = j j_k$. This shows that $\gamma(1) = j_k \exp(2X) = j_k \exp(X)\exp(X) = j(j j_k) = z j_k$. □

Since P_k is totally geodesic in $\mathfrak{G}$, the geodesic symmetries of P_k are just the restrictions of the geodesic symmetries of $\mathfrak{G}$ at points of P_k. The geodesic symmetry s_{j_k} of $\mathfrak{G}$ at the point j_k is given by $s_{j_k}(g) = j_k g^{-1} j_k = z j_k g^{-1} z j_k = s_{z j_k}(g)$, where $s_{z j_k}$ is the geodesic symmetry of $\mathfrak{G}$ at the point $z j_k$. This shows that $z j_k$ is a pole of (P_k, j_k). The converse of Lemma 3.1 also holds:

Lemma 3.2. *Let γ be a geodesic in P_k emanating from j_k that satisfies $\gamma(1) = z j_k$. Then $j := \gamma\left(\frac{1}{2}\right)$ z-commutes with j_k.*

Proof. As in the proof of Lemma 3.1, γ has the form $\gamma(t) = j_k \exp(2tX)$, so that $\gamma(1) = j_k \exp(2X) = z j_k$ and $j = \gamma\left(\frac{1}{2}\right) = j_k \exp(X)$. Since $\exp(X) = z j_k j$ we conclude from $z j_k = \gamma(1) = j_k \exp(X)\exp(X) = j(z j_k j)$ that $j_k = j j_k j$. Because j is a z-complex structure we get $z j j_k = j_k j$. □

We now define P_{k+1} as a connected component of the set of all elements in P_k that z-commute with j_k or, equivalently, as one centriole in the centrosome $C_{z j_k}(P_k, j_k)$. Hence P_{k+1} is a totally geodesic reflective submanifold of P_k (Proposition 2.16).

We have seen that, for a chosen point $j_k \in P_k$, the assumption

Assumption 3.3. *The point $z j_k$ lies in P_k, or, equivalently,*

[22] If $\mathfrak{G}$ is faithfully and irreducibly represented, z-commutation just means anti-commutation: $j_1 j_2 = -j_1 j_2$ (see Lemma 1.1).

- *The geodesic symmetry s_e of $\mathfrak{G}$ leaves P_k invariant.*
- *The geodesic symmetry $s_{j_{k-1}}$ of P_{k-1} leaves P_k invariant* ($k \geq 2$).

is crucial for being able to perform another iteration step in our construction. Although such an assumption is not explicitly stated in [Mil-69, §24], Milnor's careful choices of connected components ensure that it is actually satisfied whenever needed.

Since one can choose any centriole in $C_{zj_k}(P_k, j_k)$, several inclusion chains that start with the same Lie group $\mathfrak{G}$ can be possible. It may be natural to restrict the attention to minimal centrioles as Milnor. This is done in the Section 3.2. We finish this section with an example of an 'exceptional' inclusion chain that is not entirely built by minimal centrioles:

Example 3.4. We start with $\mathfrak{G} = \mathrm{E}_7$[23] and we denote by z the unique center element of E_7 of order two. Example 2.32 shows that (E_7, e) admits two centrioles, a minimal one and a non-minimal one. As P_1 we take the centriole the contains $j_1 := \exp(\pi\alpha_1^*)$. This centriole is not minimal (see Example 2.32). We now describe the root system of $\mathfrak{e}_7$ following the notations of [E-84, p. 124]:

$$\begin{array}{rcl} \mathcal{R}(\mathrm{E}_7) & = & \mathcal{R}_1(\mathrm{E}_7) \cup \mathcal{R}_2(\mathrm{E}_7) \text{ with} \\ \mathcal{R}_1(\mathrm{E}_7) & := & \{\pm(e_i - e_j);\ 1 \leq i < j \leq 8\} \quad \text{and} \\ \mathcal{R}_2(\mathrm{E}_7) & := & \left\{ \frac{1}{2} \sum\limits_{j=1}^{8} s_j e_j; \quad s_j = \pm 1,\ \sum\limits_{j=1}^{8} s_j = 0 \right\}. \end{array}$$

As a fundamental root system $\Sigma = \{\alpha_1, ..., \alpha_7\}$ we choose with [E-84, p. 124]:

$$\begin{array}{rcl} \alpha_1 & = & \frac{1}{2}(-e_1 - e_2 - e_3 + e_4 + e_5 + e_6 + e_7 - e_8) \quad \text{and} \\ \alpha_j & = & e_{j-1} - e_j \quad \text{for} \quad 2 \leq j \leq 7. \end{array}$$

The Dynkin diagram of type $\mathfrak{e}_7$ can be found in Table A.1, p. 65. From Section A.7 we see that the tangent space of P_1 at j_1 is isomorphic to

$$\mathfrak{p}_1 = \mathfrak{g} \cap \sum_{\alpha \in \mathcal{R}_{\mathrm{odd}}} \mathfrak{g}_\alpha$$

where $\mathcal{R}_{\mathrm{odd}}$ is the set of all roots $\alpha \in \mathcal{R}(\mathrm{E}_7)$ such that $\alpha(\alpha_1^*)$ is odd. Since the coefficient of α_1 in the highest root is two, a root α lies in $\mathcal{R}_{\mathrm{odd}}$ if and only if $\alpha(\alpha_1^*) = \pm 1$. One sees that these are precisely the roots containing e_8 with coefficient $\pm\frac{1}{2}$. Thus $\mathcal{R}_{\mathrm{odd}} = \mathcal{R}_2(\mathrm{E}_7)$. Since the real dimension of $\mathfrak{p}_1$ coincides with the complex dimension of $\sum_{\alpha \in \mathcal{R}_{\mathrm{odd}}} \mathfrak{g}_\alpha$, and since all root spaces have complex dimension one, the dimension of P_1 is precisely cardinality of $\mathcal{R}_2(\mathrm{E}_7)$, namely $\binom{8}{4} = 70$. Since $\mathfrak{e}_7$ is simple, P_1 is a 70-dimensional symmetric spaces whose isometry Lie algebra is $\mathfrak{e}_7$. But the only such spaces

[23] By E_6, E_7, E_8, F_4 and G_2 we denote the (up to isomorphism unique) connected and simply connected simple compact real Lie groups whose Dynkin diagram is of type $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$, $\mathfrak{f}_4$ and $\mathfrak{g}_2$ respectively.

are $EV := \mathrm{E}_7/\mathrm{SU}_8$ and $\mathrm{Ad}(EV)$. Since the centriole of (E_7, e) containing zj_1 is not minimal, zj_1 must be a point of P_1 (see Example 2.32). Hence (P_1, j_1) admits a pole and therefore $P_1 \cong EV = \mathrm{E}_7/\mathrm{SU}_8$. Since the root system of P_1 is again of type $\mathfrak{e}_7$ (see [He-78, p. 518, p. 534]), P_1 contains again two centrioles, a minimal one and a non-minimal one. From [BCO-03, Table A.7, p. 331] we see that the minimal centriole in P_1, denoted by P_2, is the symmetric space $(\mathrm{SU}_8/\mathrm{Sp}_4)/\mathbb{Z}_2$ whose root system has type $\mathfrak{a}_3$. Let j_2 be a point of P_2. Since P_2 is the only minimal centriole in (P_1, j_1), the pointed symmetric space (P_2, j_2) has a pole, namely $j_2 z$. The centrosome $C_{zj_2}(P_2, j_2)$ contains two minimal centrioles that correspond to the roots α_1 and α_3 in the Dynkin diagram of type $\mathfrak{a}_3$ (see Table A.1, p. 65). These centrioles are both isomorphic to $\mathbb{H}P^3$ which is an adjoint space. Thus we got the inclusion chain

$$\mathrm{E}_7 \supset \mathrm{E}_7/\mathrm{SU}_8 \supset (\mathrm{SU}_8/\mathrm{Sp}_4)/\mathbb{Z}_2 \supset \mathbb{H}P^3. \tag{3.2}$$

3.2. The case of minimal centrioles

In this section we continue our investigation of the construction presented in Section 3.1. Like Milnor we now focus on inclusion chains

$$\mathfrak{G} \supset P_1 \supset ... \supset P_k$$

that start with a connected simple compact real Lie group and that consist only of minimal centrioles. A question arises now: How long are such inclusion chains at least? In this section we answer this question (Theorem 3.5 below).

First step

We start with a compact connected *simple* real Lie group $\mathfrak{G}$ and assume that it contains a center element z of order two. By exp we denote the exponential map from $\mathfrak{g}$ onto $\mathfrak{G}$. Let $\gamma_X(t) = \exp(2tX)$ be a shortest geodesic between e and $z = \gamma_X(1) = \exp(2X)$ (see Equation A.9). Corollary 2.13 shows that $X = \pi\xi_1$ where ξ_1 is extrinsically symmetric in $\mathfrak{g}$. The centriole P_1 of $(\mathfrak{G}, e)$ that contains $j_1 := \gamma_X\left(\frac{1}{2}\right) = \exp(\pi\xi_1)$ is just the conjugacy orbit of j_1, i.e. $P_1 = \{gj_1g^{-1};\ g \in \mathfrak{G}\}$. Hence P_1 is the image of the equivariant map

$$F:\ \mathfrak{g} \supset \mathrm{Ad}(\mathfrak{G})\xi_1 \to \mathfrak{G}, \quad \mathrm{Ad}(g)\xi_1 \mapsto gj_1g^{-1}.$$

Since j_1 is the midpoint of a shortest geodesic arc joining e to z, the map F is injective[24]. From Section A.8 we know that $\mathrm{Ad}(\mathfrak{G})\xi_1$ is an irreducible hermitian symmetric space of compact type. Since P_1 endowed with the submanifold metric is also a symmetric space (because P_1 is totally geodesic in $\mathfrak{G}$), the map F is an isometry up to a scaling factor and $\mathfrak{G}$ acts as the transvection group on P_1[25]. The hermitian symmetric space P_1 can

[24]if not, j_1 would be a cut point of this geodesic arc and the geodesic arc could no longer be shortest beyond j_1

[25]This means that for any transvection of P_1 we can find an element of $\mathfrak{G}$ that acts on P as this transvection. But $\mathfrak{G}$ does not act faithfully on P_1 in our case, since it has non-trivial center and any center element acts on P_1 as the identity (see also Section A.8)

be written as a coset space: $P_1 = \mathfrak{G}/\mathfrak{K}_1$, where $\mathfrak{K}_1 := \{g \in \mathfrak{G};\ g = j_1 g j_1^{-1}\} = \{g \in \mathfrak{G};\ \mathrm{Ad}(g)\xi_1 = \xi_1\}$. As any hermitian symmetric space of compact type, P_1 is simply connected.

There are two types of irreducible hermitian symmetric spaces:

- the ones whose root systems are of type $\mathfrak{c}_r$ which are called of tube type[26];
- the ones whose root systems are non-reduced and hence of type $\mathfrak{bc}_r$.

Assume that P_1 is not of tube type, then every fundamental root in $\mathcal{R}(P_1)$ has coefficient 2 in the highest root [He-78, pp. 475 f.]. Lemma 2.4 together with Corollary 2.13 show that P_1 has trivial center and hence does not admit any pole. Hence zj_1 cannot be an element of P_1 and Assumption 3.3 is not satisfied. Thus our iteration scheme stops. Nevertheless, zj_1 is also z-complex structure in $\mathfrak{G}$ and the corresponding centriole of $(\mathfrak{G}, e)$ is just zP_1 and hence isomorphic to P_1[27].

To continue our iteration, we henceforth assume that P_1 is of tube type. Looking in the list of irreducible hermitian symmetric spaces of compact type whose root system is of type $\mathfrak{c}_r$ (see [He-78, Lo-69-II, BCO-03]), we get the following examples for P_1 :

Table 3.1.: First step

	$\mathfrak{G}$	P_1	
1	Spin_n; SO'_n[28] with $n = 4m$	$\mathrm{SO}_n/\mathrm{SO}_2 \times \mathrm{SO}_{n-2}$	$n \geq 5$
2	SO_{4n}; SO'_{4n}; Spin_{4n}	$\mathrm{SO}_{4n}/\mathrm{U}_{2n}$	$n \geq 3$
3	SU_{2n}/Γ where $\Gamma < Z(\mathrm{SU}_{2n})$ with $-\mathrm{Id} \notin \Gamma$	$\mathrm{SU}_{2n}/\mathrm{S}(\mathrm{U}_n \times \mathrm{U}_n)$	
4	Sp_n	$\mathrm{Sp}_n/\mathrm{U}_n$	$n \geq 2$
5	E_7	$\mathrm{E}_7/(S^1\mathrm{E}_6)$	

The main result summarizing the effort of this section is:

Theorem 3.5. *Let $\mathfrak{G}$ be a compact connected simple real Lie group whose center contains an element z of order two. Assume that the centrosome $C_z(\mathfrak{G}, e)$ contains a minimal centriole P_1 that is of tube type and has higher rank, i.e.* $\mathrm{rank}(P_1) \geq 2$. *Then there is at least a three step inclusion chain*

$$\mathfrak{G} \supset P_1 \supset P_2 \supset P_3$$

consisting of positive dimensional minimal centrioles.

---

[26] See Section A.8 for further explication.

[27] Looking in the lists of [He-78], one sees that P_1 and zP_1 can always be identified by an isomorphism of $\mathfrak{G}$ that is induced from a Dynkin diagram automorphism of $\mathfrak{g}$.

[28] SO'_{4m} denotes the half-spin group. This group is obtained as follows: The center of Spin_{4m} is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and hence contains three subgroups of order two, namely $\Gamma_1 = \mathbb{Z}_2 \times \{0\}$, $\Gamma_2 = \{0\} \times \mathbb{Z}_2$, and $\Gamma_3 = \{(0,0),\ (1,1)\}$. There is an isomorphism of Spin_{4m} that identifies the subgroups $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ of $Z(\mathrm{Spin}_{4m})$ that are isomorphic to Γ_1 and Γ_2. The half-spin group SO'_{4m} is $\mathrm{Spin}_{4m}/\tilde{\Gamma}_1 \cong \mathrm{Spin}_{4m}/\tilde{\Gamma}_2$, while the special orthogonal group SO_{4m} is $\mathrm{Spin}_{4m}/\tilde{\Gamma}_3$.

Second step

Since (P_1, j_1) is an irreducible pointed hermitian symmetric space whose root system has type $\mathfrak{c}_r$, it has precisely one pole and one centriole (see Example 2.31). But it is not entirely clear yet that this pole is actually zj_1, so that (P_1, j_1) satisfies Assumption 3.3[29]. The main part of this section is devoted to show that zj_1 is the pole of (P_1, j_1)[30].

We denote by $\mathfrak{g} = \mathfrak{k}_1 \oplus \mathfrak{p}_1$ the Cartan decomposition of $\mathfrak{g}$ corresponding to ξ_1, i.e.

$$\begin{array}{lllll} \mathfrak{k}_1 & = & \mathfrak{k}_{\xi_1} & := & \{X \in \mathfrak{g};\ \mathrm{ad}(\xi_1)X = 0\} \qquad \text{and} \\ \mathfrak{p}_1 & = & \mathfrak{p}_{\xi_1} & := & \{X \in \mathfrak{g};\ \mathrm{ad}(\xi_1)^2 X = -X\}. \end{array} \tag{3.3}$$

This is of course also the Cartan decomposition of the pointed symmetric space (P_1, j_1). Since the root system $\mathcal{R}(P_1)$ of P_1 is of type $\mathfrak{c}_r$, any fundamental root system in $\mathcal{R}(P_1)$ contains precisely one fundamental root with coefficient 1 in the corresponding highest root. Hence there is precisely one $\mathfrak{K}_1$-conjugacy class of extrinsically symmetric elements in $\mathfrak{p}_1$. Let $\xi_2 \in \mathfrak{p}_1$ be a non-zero extrinsically symmetric element and let $\mathfrak{g} = \mathfrak{k}_{\xi_2} \oplus \mathfrak{p}_{\xi_2}$ be the corresponding Cartan decomposition of $\mathfrak{g}$ defined as in Equation 3.3 :

$$\begin{array}{lll} \mathfrak{k}_{\xi_2} & := & \{X \in \mathfrak{g};\ \mathrm{ad}(\xi_2)X = 0\} \qquad \text{and} \\ \mathfrak{p}_{\xi_2} & := & \{X \in \mathfrak{g};\ \mathrm{ad}(\xi_2)^2 X = -X\}. \end{array} \tag{3.4}$$

Since $\xi_2 \in \mathfrak{p}_1$, the Cartan relations imply that $\mathrm{ad}(\xi_2)^2$ leaves $\mathfrak{k}_1$ and $\mathfrak{p}_1$ invariant. Hence we get an orthogonal decomposition

$$\mathfrak{g} = (\mathfrak{k}_1 \cap \mathfrak{k}_{\xi_2}) \oplus (\mathfrak{k}_1 \cap \mathfrak{p}_{\xi_2}) \oplus (\mathfrak{p}_1 \cap \mathfrak{k}_{\xi_2}) \oplus (\mathfrak{p}_1 \cap \mathfrak{p}_{\xi_2}). \tag{3.5}$$

This shows that $\mathrm{ad}(\xi_1)^2$ and $\mathrm{ad}(\xi_2)^2$ commute.

Our goal is to show that ξ_2 lies in $\mathrm{Ad}(\mathfrak{G})\xi_1 \cong P_1$, or, equivalently, that ξ_1 lies in $\mathrm{Ad}(\mathfrak{G})\xi_2$. In this context the assumption that $\mathfrak{G}$ is simple is important[31]. We decompose ξ_1 according to the Cartan decomposition $\mathfrak{g} = \mathfrak{k}_{\xi_2} \oplus \mathfrak{p}_{\xi_2}$ as $\xi_1 = (\xi_1)_{\mathfrak{k}_{\xi_2}} + (\xi_1)_{\mathfrak{p}_{\xi_2}}$ and observe that $(\xi_1)_{\mathfrak{p}_{\xi_2}} = -[\xi_2, [\xi_2, \xi_1]]$. Equation 3.5 shows that $(\xi_1)_{\mathfrak{p}_{\xi_2}} \in \mathfrak{k}_1 \cap \mathfrak{p}_{\xi_2}$ and $(\xi_1)_{\mathfrak{k}_{\xi_2}} \in \mathfrak{k}_1 \cap \mathfrak{k}_{\xi_2}$. Thus $\left[\xi_1, (\xi_1)_{\mathfrak{p}_{\xi_2}}\right] = \left[\xi_1, (\xi_1)_{\mathfrak{k}_{\xi_2}}\right]$ vanish and hence $\left[(\xi_1)_{\mathfrak{p}_{\xi_2}}, (\xi_1)_{\mathfrak{k}_{\xi_2}}\right] = 0$.

Lemma 3.6. $\mathrm{ad}((\xi_1)_{\mathfrak{p}_{\xi_2}})$ *vanishes on* $\mathfrak{k}_1$.

Proof. From $0 = \mathrm{ad}(\xi_1)|_{\mathfrak{k}_1} = \mathrm{ad}\left((\xi_1)_{\mathfrak{p}_{\xi_2}}\right)\Big|_{\mathfrak{k}_1} + \mathrm{ad}\left((\xi_1)_{\mathfrak{k}_{\xi_2}}\right)\Big|_{\mathfrak{k}_1}$ we deduce

$$\mathrm{ad}\left((\xi_1)_{\mathfrak{p}_{\xi_2}}\right)\Big|_{\mathfrak{k}_1} = -\,\mathrm{ad}\left((\xi_1)_{\mathfrak{k}_{\xi_2}}\right)\Big|_{\mathfrak{k}_1}.$$

[29] Thinking e.g. of $P_1 = \mathrm{SO}_{4n}/\mathrm{U}_{2n}$, we can not a priori exclude that zj_1 lies in another centriole of $\mathfrak{G} = \mathrm{Spin}_{4n}$ that is isomorphic to P_1. In this case P_1 would contain a point p so that the geodesic symmetries of P_1 at p and j_1 coincide, while the geodesic symmetries of the ambient space $\mathfrak{G}$ at these points differ.

[30] One could prove this case-by-case, but we prefer to provide a conceptional proof, as one would have to pay particular attention at the case $\mathfrak{G} = \mathrm{Spin}_{4n}$ and $P_1 = \mathrm{SO}_{4n}/\mathrm{U}_{2n}$ mentioned before, because of the Dynkin diagram automorphism of $\mathfrak{d}_{2n}$.

[31] Indeed, we need this assumption, because if $\mathfrak{g}$ is semisimple but not simple, one could artificially produce degeneracies by choosing ξ_2 in such a way that some of its components in the simple ideals of $\mathfrak{g}$ vanish.

By Equation 3.5 it is sufficient to show that $\mathrm{ad}\Big((\xi_1)_{\mathfrak{p}_{\xi_2}}\Big)$ vanishes on $\mathfrak{k}_1 \cap \mathfrak{k}_{\xi_2}$ and on $\mathfrak{k}_1 \cap \mathfrak{p}_{\xi_2}$. Let X first be an element of $\mathfrak{k}_1 \cap \mathfrak{k}_{\xi_2}$. Then $\Big[(\xi_1)_{\mathfrak{p}_{\xi_2}}, X\Big] = -\Big[(\xi_1)_{\mathfrak{k}_{\xi_2}}, X\Big]$. The Cartan relations show that $\Big[(\xi_1)_{\mathfrak{p}_{\xi_2}}, X\Big] \in \mathfrak{p}_{\xi_2}$ and $\Big[(\xi_1)_{\mathfrak{k}_{\xi_2}}, X\Big] \in \mathfrak{k}_{\xi_2}$. Since $\mathfrak{p}_{\xi_2} \cap \mathfrak{k}_{\xi_2} = \{0\}$, we see that $\Big[(\xi_1)_{\mathfrak{p}_{\xi_2}}, X\Big] = 0$.

Similarly, if $X \in \mathfrak{k}_1 \cap \mathfrak{p}_{\xi_2}$, then again $\Big[(\xi_1)_{\mathfrak{p}_{\xi_2}}, X\Big] = -\Big[(\xi_1)_{\mathfrak{k}_{\xi_2}}, X\Big]$ and, by the Cartan relations, $\Big[(\xi_1)_{\mathfrak{p}_{\xi_2}}, X\Big]$ lies in $\mathfrak{k}_{\xi_2}$ and $\Big[(\xi_1)_{\mathfrak{k}_{\xi_2}}, X\Big]$ in $\mathfrak{p}_{\xi_2}$. Thus $\Big[(\xi_1)_{\mathfrak{p}_{\xi_2}}, X\Big] = 0$. □

Lemma 3.7. $(\xi_1)_{\mathfrak{p}_{\xi_2}}$ *is a non-zero extrinsically symmetric element of* $\mathfrak{g}$.

Proof. Since the complexification of $\mathfrak{p}_1$ is the direct sum of the $(\pm i)$-eigenspaces of $\mathrm{ad}(\xi_1)$, we firstly notice that

$$\begin{array}{rclcll} \mathrm{ad}(\xi_1)|_{\mathfrak{p}_1} & = & \mathrm{Ad}\left(\exp\left(\frac{\pi}{2}\xi_1\right)\right)\Big|_{\mathfrak{p}_1} & = & e^{\frac{\pi}{2}\mathrm{ad}(\xi_1)}\Big|_{\mathfrak{p}_1} & \text{and, similarly,} \\ \mathrm{ad}(\xi_2)|_{\mathfrak{p}_{\xi_2}} & = & \mathrm{Ad}\left(\exp\left(\frac{\pi}{2}\xi_2\right)\right)\Big|_{\mathfrak{p}_{\xi_2}} & = & e^{\frac{\pi}{2}\mathrm{ad}(\xi_2)}\Big|_{\mathfrak{p}_{\xi_2}} . & \end{array}$$

Secondly, we see that $[\xi_1, \xi_2] = \Big[(\xi_1)_{\mathfrak{p}_{\xi_2}}, \xi_2\Big] + \Big[(\xi_1)_{\mathfrak{k}_{\xi_2}}, \xi_2\Big] = \Big[(\xi_1)_{\mathfrak{p}_{\xi_2}}, \xi_2\Big]$ lies in $\mathfrak{p}_{\xi_2}$. Since $(\xi_1)_{\mathfrak{p}_{\xi_2}} = -[\xi_2, [\xi_2, \xi_1]] = [\xi_2, [\xi_1, \xi_2]]$ we get with $\xi_2 \in \mathfrak{p}_1$:

$$\begin{array}{rcl} (\xi_1)_{\mathfrak{p}_{\xi_2}} & = & \mathrm{Ad}\left(\exp\left(\frac{\pi}{2}\xi_2\right)\right)\left(\mathrm{Ad}\left(\exp\left(\frac{\pi}{2}\xi_1\right)\right)\xi_2\right) \\ & = & \mathrm{Ad}\left(\exp\left(\frac{\pi}{2}\xi_2\right)\exp\left(\frac{\pi}{2}\xi_1\right)\right)\xi_2. \end{array} \tag{3.6}$$

Since the adjoint action preserves eigenvalues, $\mathrm{ad}\Big((\xi_1)_{\mathfrak{p}_{\xi_2}}\Big)$ has eigenvalues $\pm i$ and 0. Hence $(\xi_1)_{\mathfrak{p}_{\xi_2}}$ is a non-zero extrinsically symmetric element of $\mathfrak{g}$. □

Lemma 3.8. ξ_1 *lies in* $\mathfrak{p}_{\xi_2}$, *i.e.* $\xi_1 = (\xi_1)_{\mathfrak{p}_{\xi_2}}$.

Proof. Since $\mathfrak{G}$ is simple, the hermitian symmetric space P_1 is irreducible and the center of $\mathfrak{k}_1$ is $\mathfrak{z}(\mathfrak{k}_1) = \mathbb{R}\xi_1$. (see Section A.8). By Lemma 3.6 the element $(\xi_1)_{\mathfrak{p}_{\xi_2}}$ also lies in $\mathfrak{z}(\mathfrak{k}_1)$. Hence $(\xi_1)_{\mathfrak{p}_{\xi_2}} = \lambda\xi_1$ for some real scalar λ. Since $\mathrm{ad}(\xi_1)$ and $\mathrm{ad}\Big((\xi)_{\mathfrak{p}_{\xi_2}}\Big)$ have the same eigenvalues, namely $\pm i$ and 0 (Lemma 3.7) we see that λ can only be ± 1. But $\lambda = -1$ is impossible. □

Equation 3.6 shows:

Lemma 3.9. ξ_1 *is lies in* $\mathrm{Ad}(\mathfrak{G})\xi_2$.

We are now able to show that P_1 satisfies Assumption 3.3:

Lemma 3.10. *The element* zj_1 *lies in* P_1 *and is a pole of* (P_1, j_1).

Proof. We only need to see that zj_1 lies in P_1. Since, by Lemma 3.9, ξ_2 lies in $\mathrm{Ad}(\mathfrak{G})\xi_1$, we get $\exp(2\pi\xi_2) = \exp(2\pi\xi_1) = z$ (see Lemma 2.1). Because $\xi_2 \in \mathfrak{p}_1$ and P_1 is a totally geodesic submanifold of $\mathfrak{G}$, the geodesic $t \mapsto \exp(2t\pi\xi_1)j_1$[32] of $\mathfrak{G}$ starting at j_1 is also a geodesic in P_1. Thus $zj_1 = \exp(2\pi\xi_1)j_1$ lies in P_1. □

[32] notice that $\exp(2t\pi\xi_1)j_1 = \exp(t\pi\xi_1)j_1\exp(-t\pi\xi_1)$, since $j_1\exp(-t\pi\xi_1)j_1^{-1} = \exp(t\pi\xi_1)$ as $\xi_1 \in \mathfrak{p}_1$.

By Example 2.31 the only centriole P_2 of (P_1, j_1) is the symmetric space

$$P_2 = \mathrm{Int}_{\mathfrak{G}}(\mathfrak{K}_1)j_2 \cong \mathrm{Ad}_{\mathfrak{G}}(\mathfrak{K}_1)\xi_2$$

where $j_2 = \exp(\pi\xi_2)j_1 \in P_1 \subset \mathfrak{G}$ and $\mathrm{Int}_{\mathfrak{G}}$ denotes the conjugation in $\mathfrak{G}$. We can also considered P_2 as the quotient space[33]

$$P_2 \cong \mathfrak{G}_2/\mathfrak{K}_2,$$

where $\mathfrak{G}_2 := \mathfrak{K}_1$ and $\mathfrak{K}_2 := \{k \in \mathfrak{K}_1;\ \mathrm{Ad}_{\mathfrak{G}}(k)\xi_2 = \xi_2\}$[34]. The corresponding Cartan decomposition of the Lie algebra $\mathfrak{g}_2 = \mathfrak{k}_1$ is

$$\mathfrak{g}_2 = \mathfrak{k}_2 \oplus \mathfrak{p}_2$$

with $\mathfrak{k}_2 = \mathfrak{k}_1 \cap \mathfrak{k}_{\xi_2}$ and $\mathfrak{p}_2 = \mathfrak{k}_1 \cap \mathfrak{p}_{\xi_2}$. Since the center[35] of $\mathfrak{K}_1$ is isomorphic to the circle group S^1 [He-78, p. 382], the group $\mathfrak{G}_2$ is not semisimple and the Lie algebra $\mathfrak{k}_1$ splits orthogonally (w.r.t. the Killing form of $\mathfrak{g}$)[36] into two Lie subalgebras:

$$\mathfrak{g}_2 = \mathfrak{k}_1 = \mathfrak{c}(\mathfrak{k}_1) \oplus \hat{\mathfrak{k}}_1 \tag{3.7}$$

where $\hat{\mathfrak{k}}_1 = \hat{\mathfrak{g}}_2 = [\mathfrak{k}_1, \mathfrak{k}_1] = [\mathfrak{g}_2, \mathfrak{g}_2]$, the ideal in $\mathfrak{k}_1$ spanned by $[\mathfrak{k}_1, \mathfrak{k}_1]$, is a semisimple compact Lie algebra [He-78, p. 132]. Recall that $\mathfrak{c}(\mathfrak{k}_1) = \mathbb{R}\xi_1$ is a subspace of $\mathfrak{p}_2 \cong T_{j_2}P_2$ (Lemma 3.8). Thus $\mathfrak{p}_2$ splits orthogonally (w.r.t. the Killing form of $\mathfrak{g}$) as

$$\mathfrak{p}_2 = \mathbb{R}\xi_1 \oplus \hat{\mathfrak{p}}_2 \tag{3.8}$$

where $\hat{\mathfrak{p}}_2 = \mathfrak{p}_2 \cap \hat{\mathfrak{k}}_1$ is a Lie subtriple of $\mathfrak{p}_2$. Thus we get an orthogonal decomposition

$$\mathfrak{g}_2 = (\mathfrak{k}_2 \oplus \hat{\mathfrak{p}}_2) \oplus \mathfrak{c}(\mathfrak{k}_1)$$

where $\mathfrak{k}_2 \oplus \hat{\mathfrak{p}}_2 = \hat{\mathfrak{g}}_2$.

Therefore the compact symmetric space P_2 is locally isomorphic to a product of $\mathrm{Exp}^{P_1}_{j_2}(\mathbb{R}\xi) = \exp(\mathbb{R}\xi)j_2 \cong S^1$ and $\hat{P}_2 := \mathrm{Exp}^{P_1}_{j_2}(\hat{\mathfrak{p}}_2) = \exp(\hat{\mathfrak{p}}_2)j_2$ and hence not of compact type[37]. This has already been proved in [Nag-92, Prop. 2.23(iv)]. The Cartan decomposition corresponding to the symmetric space $\hat{P}_2$ which is of compact type is

$$\hat{\mathfrak{k}}_1 = \hat{\mathfrak{g}}_2 = \mathfrak{k}_2 \oplus \hat{\mathfrak{p}}_2.$$

[33] Recall that $\mathfrak{G}$ acts almost effectively as the transvection group on the irreducible symmetric space P_1, i.e. only a finite subset of elements of $\mathfrak{G}$ act trivially on P_1. Since $\mathrm{Ad}_{\mathfrak{G}}(\mathfrak{K}_1)\xi_2$ is a non-trivial isotropy orbit of P_1, it is full in $\mathfrak{p}_1$. The elements of $\mathfrak{K}_1$ act as the identity component of the isotropy group of j_1 on $\mathfrak{p}_1$. Since the P_1 is strongly isotropy irreducible, the action of $\mathfrak{K}_1$ on $\mathfrak{p}_1$ is irreducible. Hence, by the fullness of $\mathrm{Ad}_{\mathfrak{G}}(\mathfrak{K}_1)\xi_2$, any element of $\mathfrak{K}_1$ that acts as the identity on $\mathrm{Ad}_{\mathfrak{G}}(\mathfrak{K}_1)\xi_2$ also acts as the identity on $\mathfrak{p}_1$. But then it also acts as the identity on P_1. Hence $\mathfrak{K}_1 = \mathfrak{G}_2$ acts almost effectively on P_2.

[34] Notice that $\mathfrak{K}_1$ is connected, because $\mathfrak{G}$ is connected and $P_1 = \mathfrak{G}/\mathfrak{K}_1$ is simply connected.

[35] Recall that the Lie algebra of the center of $\mathfrak{K}_1$ is $\mathfrak{c}(\mathfrak{k}_1) = \mathbb{R}\xi_1$.

[36] The bi-invariance of the Killing form on $\mathfrak{g}$ yields $\kappa(\xi_1, [X, Y]) = -\kappa([X, \xi_1], Y) = 0$ for $X, Y \in \mathfrak{k}_1$.

[37] $\mathrm{Exp}^{P_1}_{j_2}$ denotes the Riemannian exponential map of P_1 at the point j_2.

Notice that $\hat{P}_2$ is totally geodesic in P_1 and therefore totally geodesic in $\mathfrak{G}$, too. Thus the subgroup $\mathfrak{L}$ of $\mathfrak{G}$ generated by those elements of $\mathfrak{G}$ that act on P_1 as transvections along geodesics in P_2 acts (almost effectively) on P_2 as its transvection group[38]. Its Lie algebra $\mathfrak{l}$ is therefore the same as the Lie algebra of the transvection group of P_2, i.e. $\mathfrak{l} = [\mathfrak{p}_2, \mathfrak{p}_2] \oplus \mathfrak{p}_2$. By $\hat{\mathfrak{L}}$ we denote the subgroup of $\mathfrak{G}$ generated by those elements of $\mathfrak{G}$ that act on P_1 as transvections along geodesics in $\hat{P}_2$. This group $\hat{\mathfrak{L}}$ is of course a subgroup of $\mathfrak{L}$ and acts on $\hat{P}_2$ almost effectively as its transvection group. Hence its Lie algebra $\hat{\mathfrak{l}}$ coincides with the one of the transvection group of $\hat{P}_2$, namely $\hat{\mathfrak{l}} = [\hat{\mathfrak{p}}_2, \hat{\mathfrak{p}}_2] \oplus \hat{\mathfrak{p}}_2$. Since $\hat{\mathfrak{g}}_2$ is semisimple, we get $\mathfrak{k}_2 = [\hat{\mathfrak{p}}_2, \hat{\mathfrak{p}}_2] = [\mathfrak{p}_2, \mathfrak{p}_2]$ and therefore $\hat{\mathfrak{l}} = \hat{\mathfrak{g}}_2$ and $\mathfrak{l} = \mathfrak{g}_2$. This shows that the identity component of $\mathfrak{L}$ is the connected subgroup $\mathfrak{G}_2 = \mathfrak{K}_1$ of $\mathfrak{G}$ and that the connected component of $\hat{\mathfrak{L}}$ is $\exp(\hat{\mathfrak{g}}_2)$[39]. Moreover, the isotopy subgroups of j_2 in $\mathfrak{L}$ and in $\hat{\mathfrak{L}}$ have the same Lie algebra, namely $\mathfrak{k}_2$. Hence the identity component of these isotropy subgroups is just $\exp(\mathfrak{k}_2)$. This also shows that the isotropy action of the connected component of the stabilizer of j_2 in $\mathfrak{L}$ fixes ξ_1.

Using [BCO-03, Table A.7] we can continue Table 3.1, page 23:

Table 3.2.: Second step

	$\mathfrak{G}$	P_1	P_2	
1	$\mathrm{Spin}_n,\ \mathrm{SO}'_n\ (n = 4m)$	$\mathrm{SO}_n/\mathrm{SO}_2 \times \mathrm{SO}_{n-2}$	$(S^1 \times S^{n-3})/\Delta \mathrm{Z}_2$	$n \geq 5$
2	$\mathrm{SO}_{4n},\ \mathrm{SO}'_{4n},\ \mathrm{Spin}_{4n}$	$\mathrm{SO}_{4n}/\mathrm{U}_{2n}$	$\mathrm{U}_{2n}/\mathrm{Sp}_n$	$n \geq 3$
3	$\mathrm{SU}_{2n}/\Gamma\ (\Gamma < Z(\mathrm{SU}_{2n}),\ -\mathrm{Id} \notin \Gamma)$	$\mathrm{SU}_{2n}/\mathrm{S}(\mathrm{U}_n \times \mathrm{U}_n)$	U_n[40]	
4	Sp_n	$\mathrm{Sp}_n/\mathrm{U}_n$	$\mathrm{U}_n/\mathrm{SO}_n$	$n \geq 2$
5	E_7	$\mathrm{E}_7/(S^1\mathrm{E}_6)$	$(S^1\mathrm{E}_6)/\mathrm{F}_4$	

Third step

Let γ be the geodesic in P_2 emanating from j_2 in direction ξ_1 (recall that $\xi_1 \in \mathfrak{p}_2$), i.e.

$$\gamma(t) = \exp(2t\xi_1)j_2 \tag{3.9}$$

(cf. Footnote 32, page 25). Since $\exp(2\pi\xi_1) = z$ we get $\gamma(\pi) = zj_2$ and (P_2, j_2) satisfies Assumption 3.3. The geodesic γ of Equation 3.9 lies entirely in the local S^1-factor of P_2 and the z-complex structure $\gamma(\pi/2) = \exp(\pi\xi_1)j_2 = j_1j_2$ of $\mathfrak{G}$ is not very interesting for our purposes[41].

As P_2 is a totally geodesic submanifold of P_1, γ is also a geodesic in P_1. Because ξ_1 is an extrinsically symmetric element in $\mathfrak{g}$, γ is also a shortest geodesic in P_1 between $\gamma(0) = j_2$

[38] Recall from Footnote 33 that $\mathfrak{G}$ acts almost effectively on P_1 as the transvection group of P_1.

[39] exp denotes the Lie theoretic exponential map from $\mathfrak{g}$ to $\mathfrak{G}$.

[40] To be considered as a symmetric space only, rather than a Lie group.

[41] Indeed, whenever we have fixed two z-complex structures j_1 and j_1 in $\mathfrak{G}$ that z-commute, we automatically have a third one, namely j_1j_2 (and, of course, j_2j_1), that z-commutes with j_1 and j_2. We will not pay particular attention at this z-complex structure, because the corresponding centriole in P_2 is a singleton. We rather look for z-complex structures in P_2 that z-commute with j_1 and j_2, but are not algebraically dependent in the above sense.

and $\gamma(\pi) = zj_2$. Moreover, any other shortest geodesic $\tilde{\gamma}$ in P_1 between $\tilde{\gamma}(0) = j_2$ and $\tilde{\gamma}(\pi) = zj_2$ lies in the orbit of γ under the action of the identity component of the isotropy subgroup of j_2 of the isometry group of P_1 (see Example 2.31). Thus:

Observation 3.11. *The distances between j_2 and zj_2 in P_1 and P_2 coincide*[42].

We now want to determine all minimal centrioles in the centrosome $C_{zj_2}(P_2, j_2)$. By the preceding discussion a shortest geodesic arc c in P_2 satisfying $c(0) = j_2$ and $c(\pi) = zj_2$ must lie in the orbit of $\gamma|_{[0,\pi]}$, the geodesic defined in Equation 3.9, under the action of the isotropy subgroup of j_2 in the isometry group of P_1. The only geodesic arcs of this kind that lie entirely in the S^1-factor are $\gamma(\pm t)$ for $t \in [0, \pi]$. The centrioles of (P_2, j_2) containing $\gamma\left(\pm\frac{\pi}{2}\right)$ are actually singletons (see Footnote 41). We therefore restrict our attention hereafter to those shortest geodesic arcs in P_2 that do not lie in the circle factor of P_2.

To proceed our iteration, we need to take a closer look at the relations between the root systems $\mathcal{R}(P_1)$, $\mathcal{R}(P_2)$ and $\mathcal{R}(\hat{P}_2)$.

Observation 3.12. *Since ξ_1 lies in $\mathfrak{p}_2$ and centralizes $\mathfrak{k}_1$, it is contained in any maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}_2$. Thus $\mathfrak{a}$ splits orthogonally according to Equation 3.8:*

$$\mathfrak{a} = \mathbb{R}\xi_1 \oplus \hat{\mathfrak{a}}, \tag{3.10}$$

where $\hat{\mathfrak{a}} = \mathfrak{a} \cap \hat{\mathfrak{p}}_2$ is a maximal abelian subspace of $\hat{\mathfrak{p}}_2$. Conversely, any maximal abelian subspace of $\mathfrak{p}_2$ has this form.

Lemma 3.13. $\mathrm{rank}(P_1) = \mathrm{rank}(P_2)$.

Proof. Take a maximal abelian subset $\mathfrak{a}$ of $\mathfrak{p}_2$ and enlarge it to a maximal abelian subspace $\mathfrak{a}' \supset \mathfrak{a}$ of $\mathfrak{p}_{\xi_2}$. Recall that $\mathfrak{p}_{\xi_2}$ can be identified with the tangent space of $\mathrm{Ad}(\mathfrak{G})\xi_1 \cong P_1$ at the point ξ_2 (see Lemma 3.9). We have to show that $\mathfrak{a}' = \mathfrak{a}$. Let X be any element of $\mathfrak{a}'$. Since $\mathfrak{p}_{\xi_2}$ splits as $\mathfrak{p}_{\xi_2} = \mathfrak{p}_2 \oplus (\mathfrak{p}_1 \cap \mathfrak{p}_{\xi_2})$, the element X can be written as $X = X' + X''$ with $X' \in \mathfrak{p}_2 \subset \mathfrak{k}_1$ and $X'' \in \mathfrak{p}_1 \cap \mathfrak{p}_{\xi_2}$. For any $A \in \mathfrak{a}$ the Cartan relations yield $[A, X'] \in \mathfrak{k}_2 = \mathfrak{k}_1 \cap \mathfrak{k}_{\xi_2}$ and $[A, X''] \in \mathfrak{p}_1 \cap \mathfrak{k}_{\xi_2}$. Thus $0 = [A, X] = [A, X'] + [A, X'']$ implies $[A, X'] = [A, X''] = 0$. Since $\mathfrak{a}$ is maximal abelian in $\mathfrak{p}_2$ we conclude $X' \in \mathfrak{a}$. If we take in particular $A = \xi_1$, we get $[\xi_1, X''] = 0$ and therefore $X'' \in \mathfrak{k}_1$. Hence $X'' \in \mathfrak{k}_1 \cap \mathfrak{p}_1 = \{0\}$. Thus $X = X'$ lies in $\mathfrak{a}$. □

We choose a fundamental root system $\Sigma(P_1) = \{\alpha_1, ..., \alpha_r\}$ in $\mathcal{R}(P_1)$, w.r.t. a fixed maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}_2$. After a suitable renumeration of the fundamental roots we can assume that $\xi_1 = \alpha_r^*$.

Lemma 3.14. *If the rank r of P_1 is not one, then $\mathfrak{a}$ contains an extrinsically symmetric element ξ_3 that is not collinear to ξ_1.*

[42] Such an observation can be found in [NS-91, Remark 3.2b] for centrioles of arbitrary irreducible pointed symmetric spaces, where a case-by-case verification is suggested.

Proof. Observe that ξ_1 is not collinear to the root vector H_r of α_r. Indeed, assume the converse, namely $\lambda\xi_1 = H_r$. Since P_1 is irreducible, the Dynkin diagram of P_1 is connected. As $r \geq 2$, there is a root in $\Sigma(P_1)$ which is joined to α_r in the Dynkin diagram (we may assume that this root is α_{r-1}). Hence the angle between these two roots is nonzero, i.e. $\langle H_r, H_{r-1}\rangle \neq 0$, where H_{r-1} is the root vector of α_{r-1}. This leads to a contradiction, because $0 \neq \langle H_r, H_{r-1}\rangle = \lambda\langle \xi_1, H_{r-1}\rangle = \lambda\alpha_{r-1}(\xi_1) = \lambda\alpha_{r-1}(\alpha_r^*) = 0$.

Let now S_r be the reflection of $\mathfrak{a}$ along the kernel of α_r. Since the orthogonal complement of this kernel is spanned by H_r we see that $\xi_3 = S_r(\xi_1)$ is not collinear to ξ_1. On the other hand this reflection is an element of the Weyl group of P_1 and can hence be realized as the restriction of $\mathrm{Ad}_{\mathfrak{G}}(k)$ to $\mathfrak{a}$ for a suitable element $k \in \mathfrak{K}_1$ (see e.g. [He-78]). Thus $\xi_3 = \mathrm{Ad}_{\mathfrak{G}}(k)\xi_1$ is extrinsically symmetric and not collinear to ξ_1. $\square$

For the rest of this section we assume that P_1 is not a rank-one symmetric space.

Lemma 3.15. *For any root $\alpha \in \mathcal{R}(P_1)$ the following three statements are equivalent:*

(i) $\mathfrak{g}_\alpha \cap \mathfrak{k}_1^c \neq \{0\}$;

(ii) $\alpha(\xi_1) = 0$;

(iii) $\mathfrak{g}_\alpha \subset \hat{\mathfrak{g}}_2^c$.

Proof. By the superscript 'c' we denote complexifications. Assume that X is a non-zero element of $\mathfrak{g}_\alpha \cap \mathfrak{k}_1^c$. Since $\xi_1 \in \mathfrak{a}$ centralizes $\mathfrak{k}_1$ and therefore $\mathfrak{k}_1^c$, too, we get $0 = [\xi_1, X] = i\alpha(\xi_1)X$. Thus $\alpha(\xi_1) = 0$. This shows that (i) implies (ii). If $\alpha(\xi_1)$ vanishes, then ξ_1 centralizes $\mathfrak{g}_\alpha$. Thus $\mathfrak{g}_\alpha \subset \mathfrak{k}_1^c$, because, by definition, $\mathfrak{k}_1 = \{X \in \mathfrak{g};\ [\xi_1, X] = 0\}$. Since $\xi \in \mathfrak{a}$ and therefore $\mathfrak{c}(\mathfrak{k}_1)^c = \mathbb{C}\xi_1 \subset \mathfrak{g}_0$, and because the root space decomposition of $\mathfrak{g}^c$ is orthogonal w.r.t. a scalar product defined in Footnote 92 on page 62, we see that $\mathfrak{g}_\alpha$ is actually contained in $\hat{\mathfrak{g}}_2^c = \hat{\mathfrak{k}}_1^c$. Hence (ii) implies (iii), and (iii) implies (i) trivially. $\square$

Corollary 3.16. *Let $\alpha \in \mathcal{R}(P_1)$. Then $\alpha(\xi_1) \neq 0$ if and only if $\mathfrak{g}_\alpha \subset \mathfrak{p}_1^c$.*

Proof. Since ξ_1 is extrinsically symmetric we have $\alpha(\xi_1) \in \{\pm 1, 0\}$. If $\alpha(\xi_1) \neq 0$, then $\alpha(\xi_1)^2 = 1$. Hence for all $X \in \mathfrak{g}_\alpha$ we have $[\xi_1, [\xi_1, X]] = -X$. Thus $\mathfrak{g}_\alpha \in \mathfrak{p}_1^c$. Conversely, if $\mathfrak{g}_\alpha \subset \mathfrak{p}_1^c$, then $\mathfrak{g}_\alpha \cap \mathfrak{k}_1^c = \{0\}$. Lemma 3.15 implies $\alpha(\xi_1) \neq 0$. $\square$

Observation 3.17. *Since $\alpha_j(\xi_1) = 0$ for $j \in \{1, ..., r-1\}$, the corresponding root spaces $\mathfrak{g}_{\alpha_j}$ are contained in $\mathfrak{k}_1^c$.*

Lemma 3.18. *The root system $\mathcal{R}(P_2)$ of P_2 corresponding to $\mathfrak{a}$ is*

$$\mathcal{R}(P_2) = \{\alpha \in \mathcal{R}(P_1);\ \alpha(\xi_1) = 0\}.$$

Proof. Recall that the Lie algebra of infinitesimal transvections of P_2 is $\mathfrak{l} = \mathfrak{g}_2 = \mathfrak{k}_1$. Let $\mathfrak{g}^c = \mathfrak{g}_0 \oplus \sum_{\alpha \in \mathcal{R}(P_1)} \mathfrak{g}_\alpha$ be the root space decomposition of $\mathfrak{g}^c$. The complexification of the involution σ_1 corresponding to (P_1, j_1) leaves $\mathfrak{g}_0 = \{X \in \mathfrak{g}^c;\ [\mathfrak{a}, X] = \{0\}\}$ invariant[43].

[43]Indeed, let $X \in \mathfrak{g}_0$, then for all $A \in \mathfrak{a}$ we have $[A, \sigma(X)] = [-\sigma(A), \sigma(X)] = -\sigma[A, X] = 0$.

Thus $\mathfrak{g}_0$ splits as $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$. where $\mathfrak{p}_0 = \mathfrak{p}_1^c \cap \mathfrak{g}_0 = \mathfrak{a}^c$ and $\mathfrak{k}_0 = \mathfrak{k}_1^c \cap \mathfrak{g}_0$. Using Lemma 3.15 and Corollary 3.16 we get

$$\begin{aligned} \mathfrak{g}_2^c = \mathfrak{k}_1^c &= \mathfrak{k}_0 \oplus \sum_{\alpha\in\mathcal{R}(P_1),\,\alpha(\xi_1)=0} \mathfrak{g}_\alpha \qquad \text{and} \\ \mathfrak{p}_1^c &= \mathfrak{p}_0 \oplus \sum_{\alpha\in\mathcal{R}(P_1),\,\alpha(\xi_1)\neq 0} \mathfrak{g}_\alpha . \end{aligned} \tag{3.11}$$

Lemma 3.15 implies that any root $\alpha \in \mathcal{R}(P_1)$ with $\alpha(\xi_1) = 0$ is also an element of $\mathcal{R}(P_2)$. On the other hand the first line of Equation 3.11 shows that there is no room for further roots in $\mathcal{R}(P_2)$. □

Since $\mathfrak{a} = \mathbb{R}\xi_1 \oplus \hat{\mathfrak{a}}$ where $\hat{\mathfrak{a}}$ is a maximal abelian subset of $\hat{\mathfrak{p}}_2$, there is a one-to-one correspondence between the root system $\mathcal{R}(P_2)$ of P_2 w.r.t. $\mathfrak{a}$ and the root system $\mathcal{R}(\hat{P}_2)$ of $\hat{P}_2$ w.r.t. $\hat{\mathfrak{a}}$, namely

$$\mathcal{R}(P_2) \xrightarrow{\cong} \mathcal{R}(\hat{P}_2), \quad \alpha \longmapsto \hat{\alpha} := \alpha|_{\hat{\mathfrak{a}}}. \tag{3.12}$$ [44]

Observation 3.19. *If $\alpha \in \mathcal{R}(P_1)$ with $\alpha(\xi_1) = 0$, then its root space lies in $\hat{\mathfrak{g}}_2^c$ (Lemma 3.15) and this space is also the root space of $\hat{\alpha} = \alpha|_{\hat{\mathfrak{a}}}$. Hence the multiplicities of α and of $\hat{\alpha}$ coincide, i.e. $m_\alpha = m_{\hat{\alpha}}$.*

Lemma 3.20. *The kernel of α_r is not contained in $\hat{\mathfrak{a}}$.*

Proof. In the proof of Lemma 3.14 we have shown that $H_r \notin \mathbb{R}\xi_1$. Since the kernel of α_r is the orthogonal complement of H_r in $\mathfrak{a}$, we see that the kernel of α_r is not contained in the orthogonal complement of ξ_1 in $\mathfrak{a}$. The latter space is nothing else than $\hat{\mathfrak{a}}$. □

Observation 3.21. *Since $\xi_1 = \alpha_r^*$, we see that $\mathbb{R}\xi_1 \subset \bigcap_{j=1}^{r-1} \ker(\alpha_j)$.*

Lemma 3.22. *$\Sigma(\hat{P}_2) := \{\hat{\alpha}_1, ..., \hat{\alpha}_{r-1}\}$ is a fundamental root system of $\mathcal{R}(\hat{P}_2)$.*

Proof. We denote by $\mathcal{R}^+(P_1)$ the system of positive roots in $\mathcal{R}(P_1)$ corresponding to $\Sigma(P_1)$. A positive root $\alpha \in \mathcal{R}^+(P_1)$ can be written as $\alpha = \sum_{j=1}^r c_j\alpha_j$ with non-negative integer coefficients c_j. It satisfies $\alpha(\xi_1) = 0$ if and only if $c_r = 0$. This shows that $\mathcal{R}^+(\hat{P}_2) := \{\hat{\alpha} = \alpha|_{\hat{\mathfrak{a}}};\ \alpha \in \mathcal{R}^+(P_1),\ \alpha(\xi_1) = 0\}$ is a positive root system in $\mathcal{R}(\hat{P}_2)$ which is spanned by $\Sigma(\hat{P}_2)$. To show that $\Sigma(\hat{P}_2)$ is a fundamental root system, we have to verify that all elements $\hat{\alpha}_j$ of $\Sigma(\hat{P}_2) \subset \mathcal{R}^+(\hat{P}_2)$ are indecomposable within $\mathcal{R}^+(\hat{P}_2)$. Let α and β be two elements of $\mathcal{R}^+(P_1)$ satisfying $\alpha(\xi_1) = \beta(\xi_1) = 0$, and assume that $\hat{\alpha}_j = \hat{\alpha} + \hat{\beta}$. Since $\alpha_j(\xi_1) = \alpha(\xi_1) = \beta(\xi_1) = 0$ we see that $\alpha_j = \alpha + \beta$. Thus α_j is not indecomposable within $\mathcal{R}^+(P_1)$, a contradiction. □

We denote by $\{\hat{\alpha}_1^*, ..., \hat{\alpha}_{r-1}^*\}$ the basis of $\hat{\mathfrak{a}}$ that is dual to $\Sigma(\hat{P}_2)$.

Lemma 3.23. *$\hat{\alpha}_j^*$, $j \neq r$, is the orthogonal projection of α_j^* onto $\hat{\mathfrak{a}}$.*

[44]Notice that $\alpha(\xi_1) = 0$ implies that α does not vanish on $\hat{\mathfrak{a}}$ since α does not vanish on $\mathfrak{a}$.

Proof. Let $\pi_{\hat{\mathfrak{a}}}$ denote the orthogonal projection of $\mathfrak{a}$ onto $\hat{\mathfrak{a}}$, then $\alpha_j^* = \pi_{\hat{\mathfrak{a}}}(\alpha_j^*) + \lambda_j \xi_1$ for some real number λ_j. For $k \neq r$ we get $\delta_{jk} = \alpha_k(\alpha_j^*) = \alpha_k(\pi_{\hat{\mathfrak{a}}}(\alpha_j^*)) + \alpha_k(\lambda_j \xi_1) = \hat{\alpha}_k(\pi_{\hat{\mathfrak{a}}}(\alpha_j^*)) + \lambda_j \alpha_k(\alpha_r^*) = \hat{\alpha}_k(\pi_{\hat{\mathfrak{a}}}(\alpha_j^*))$. This shows the claim. □

Since $0 = \alpha_j(\xi_1) = \langle H_j, \xi_1 \rangle$ for $j \neq r$, the root vectors $H_1, ..., H_{r-1}$ of $\alpha_1, ..., \alpha_{r-1}$ are contained in $\hat{\mathfrak{a}}$. For any element $\hat{A} \in \hat{\mathfrak{a}}$ we have $\hat{\alpha}_j(\hat{A}) = \alpha_j(\hat{A}) = \langle H_j, \hat{A} \rangle$, if $j \neq r$, so that H_j is also the root vector of $\hat{\alpha}_j$. i.e. $H_j = \hat{H}_j$. Therefore α_j and $\hat{\alpha}_j$ have the same length and the angle between α_j and α_k equals the angle between $\hat{\alpha}_j$ and $\hat{\alpha}_k$ if j and k are both not r. This shows that the Dynkin diagram of the root system $\mathcal{R}(\hat{P}_2)$ can be obtained from the Dynkin diagram of the root system $\mathcal{R}(P_1)$ which is of type $\mathfrak{c}_r$, by removing the vertex representing the unique fundamental root with coefficient 1 in the highest root. Having a look at the Dynkin diagram of type $\mathfrak{c}_r$ (see Table A.1, p. 65) we realize:

Lemma 3.24 (cf. also Prop. 2.23(iv) in Na-92)**.** *The root system $\mathcal{R}(\hat{P}_2)$ is of type $\mathfrak{a}_{r-1}$. In particular all fundamental roots $\hat{\alpha}_1, ..., \hat{\alpha}_{r-1}$ have coefficient 1 in the corresponding highest root.*

Observation 3.25. *If $X \in \mathfrak{a}$ is extrinsically symmetric in $\mathfrak{g}$ and $X \neq \pm\xi_1$, then $\pi_{\hat{\mathfrak{a}}}(X)$ is a nonzero extrinsically symmetric element of $\hat{\mathfrak{g}}_2$*[45]*.*

Proof. Since $X \in \mathfrak{a} \subset \mathfrak{k}_1$ and $\hat{\mathfrak{g}}_2 = [\mathfrak{k}_1, \mathfrak{k}_1]$ is a Lie algebra, $\mathrm{ad}(X)$ leaves $\hat{\mathfrak{g}}_2$ invariant. The only possible eigenvalues of $\mathrm{ad}(X)$ on $\hat{\mathfrak{g}}_2$ are $\pm i$ and 0, because X is extrinsically symmetric in $\mathfrak{g}$, and $\mathfrak{g}$ contains $\hat{\mathfrak{g}}_2$ as a subalgebra. There is a real number q such that $X = \pi_{\hat{\mathfrak{a}}}(X) + q\xi_1$. Since ξ_1 centralizes $\hat{\mathfrak{g}}_2 \subset \mathfrak{k}_1$ we get $\mathrm{ad}(X)|_{\hat{\mathfrak{g}}_2} = \mathrm{ad}(\pi_{\hat{\mathfrak{a}}}(X))|_{\hat{\mathfrak{g}}_2}$. Hence the only possible eigenvalues of $\mathrm{ad}(\pi_{\hat{\mathfrak{a}}}(X))|_{\hat{\mathfrak{g}}_2}$ are $\pm i$ and 0.

Assume that $\mathrm{ad}(\pi_{\hat{\mathfrak{a}}}(X))|_{\hat{\mathfrak{g}}_2}$ is the zero map. Since $\hat{\mathfrak{g}}_2$ is semi-simple, its center is trivial. Therefore $\pi_{\hat{\mathfrak{a}}}(X) = 0$ and $X \in \mathbb{R}\xi_1$. But the only extrinsically symmetric elements in $\mathbb{R}\xi_1$ are $\pm\xi_1$. Hence $\mathrm{ad}(\pi_{\hat{\mathfrak{a}}}(X))|_{\hat{\mathfrak{g}}_2}$ is not the zero map, and $\pi_{\hat{\mathfrak{a}}}(X)$ is nonzero. □

We next want to show the converse, namely, if $X \in \hat{\mathfrak{a}}$ is extrinsically symmetric in $\hat{\mathfrak{g}}_2$, then there exists a unique real number μ such that $X + \mu\xi_1$ is extrinsically symmetric in $\mathfrak{g}$. After conjugation with an suitable element of $\exp(\mathfrak{k}_2)$, the identity component of the stabilizer of j_2 in $\hat{\mathfrak{L}}$, we can assume that X lies in closure of the positive Weyl in $\hat{\mathfrak{a}}$ chamber defined by the fundamental root system $\Sigma(\hat{P}_2)$ of $\mathcal{R}(\hat{P}_2)$[46]. Lemmata 3.24 and 2.4 show that $X = \hat{\alpha}_j^*$ for some $j \in \{2, ..., r\}$.

Proposition 3.26. *Let $j \in \{1, ..., r-1\}$. Then there is exactly one real number μ (depending on j) such that $\chi_j := \hat{\alpha}_j^* + \mu\xi_1$ is extrinsically symmetric in $\mathfrak{g}$, namely $\mu = -1 - \alpha_r(\hat{\alpha}_j^*)$.*

Proof. We have to show that there is exactly one choice for μ such that $\alpha(\chi_j) \in \{\pm 1, 0\}$ for any root $\alpha \in \mathcal{R}(P_1)$. Since $\hat{\alpha}_j^*$ is extrinsically symmetric in $\hat{\mathfrak{g}}_2$ (Lemma 3.24), this is true for any real value of μ if $\alpha(\xi_1) = 0$ (see Lemma 3.18 and Equation 3.12). Let α be

[45] i.e. the eigenvalues of $\mathrm{ad}(\pi_{\hat{\mathfrak{a}}}(X))$ on $\hat{\mathfrak{g}}_2$ are $\pm i$ and 0

[46] Recall that the conjugation with an element of $\mathfrak{K}_2$ fixes ξ_1.

a root in $\mathcal{R}(P_1)$ with $\alpha(\xi_1) \neq 0$. We can assume that α is positive w.r.t. $\Sigma(P_1)$ and write $\alpha = \sum_{k=1}^{r} c_k \alpha_k$ with $c_k \geq 0$ and $c_r > 0$. Since the coefficient of α_r in the highest root is 1, we have $c_r = 1$. We now set $\mu := -s - \alpha_r(\hat{\alpha}_j^*)$. With $\alpha_k(\hat{\alpha}_j^*) = \hat{\alpha}_k(\hat{\alpha}_j^*) = \delta_{kj}$ for $k \neq r$ (see Lemma 3.23) we get

$$\begin{aligned}
\alpha(\chi_j) &= \alpha\left(\hat{\alpha}_j^*\right) - \left(s + \alpha_r(\hat{\alpha}_j^*)\right)\alpha(\xi_1) \\
&= \sum_{k=1}^{r} c_k \alpha_k\left(\hat{\alpha}_j^*\right) - \left(s + \alpha_r(\hat{\alpha}_j^*)\right)\sum_{k=1}^{r} c_k \alpha_k(\xi_1) \\
&= c_j + c_r \alpha_r\left(\hat{\alpha}_j^*\right) - \left(s + \alpha_r(\hat{\alpha}_j^*)\right) c_r \\
&= c_j + \alpha_r\left(\hat{\alpha}_j^*\right) - \left(s + \alpha_r(\hat{\alpha}_j^*)\right) \\
&= c_j - s.
\end{aligned}$$

Since $j \neq r$ and $\mathcal{R}(P_1)$ is of type $\mathfrak{c}_r$ with $r \geq 2$ we see that the coefficient of α_j in the highest root is 2, so that $c_j \in \{0, 1, 2\}$ (see the Dynkin diagram of type $\mathfrak{c}_r$ in Table A.1).

Take $\alpha = \alpha_r$, then $c_j = 0$ and we get $\alpha(\chi_j) = -s \in \{\pm 1, 0\}$. If we take $\alpha = \delta$, the highest root, then $c_j = 2$ and $\alpha(\chi_j) = 2 - s \in \{\pm 1, 0\}$, so that $s \in \{1, 2, 3\}$. Hence $s = 1$, or $\mu = -1 - \alpha_r(\hat{\alpha}_j^*)$, as desired. If finally $c_j = 1$, then $\alpha(\chi_j) = 1 - 1 = 0$. □

Corollary 3.27. *If X and Y are two distinct elements of $\mathfrak{a} \setminus \{\pm\xi\}$ that are both extrinsically symmetric in $\mathfrak{g}$, then $\pi_{\hat{\mathfrak{a}}}(X)$ and $\pi_{\hat{\mathfrak{a}}}(Y)$ are two distinct elements which are both extrinsically symmetric in $\mathfrak{g}_2'$.*

A shortest geodesic arc $\gamma(t) = \exp(2tX)j_2$ in P_2 joining $\gamma(0) = j_2$ to $\gamma(\pi) = zj_2$ is also length minimizing in P_1 by Observation 3.11. Hence $X \in \mathfrak{p}_2$ is extrinsically symmetric in $\mathfrak{g}$ (Corollary 2.13). If we assume that γ does not lie in the local S^1-factor of P_2, then $X \in \mathfrak{p}_2 \setminus \{\pm\xi_1\}$. After a suitable conjugation with an element of $\exp(\mathfrak{k}_2)$[47], we can assume that X lies in $\mathfrak{a}$. Then $X = \pi_{\hat{\mathfrak{a}}}(X) + r\xi_1$.

Since $\exp(\mathfrak{k}_2)$ is also the identity component of the isotropy group of j_2 in $\hat{\mathfrak{L}}$, we can suppose that $\pi_{\hat{\mathfrak{a}}}(X)$ lies in the closure of the Weyl chamber in $\hat{\mathfrak{a}}$ defined by $\Sigma(\hat{P}_2)$. The element of $\mathfrak{a}$ thus obtained is of course still extrinsically symmetric in $\mathfrak{g}$ and $\pi_{\hat{\mathfrak{a}}}(X)$ is extrinsically symmetric in $\hat{\mathfrak{g}}_2$ by Observation 3.25. Hence there is some $j \in \{1, ..., r-1\}$ such that $\pi_{\hat{\mathfrak{a}}}(X) = \hat{\alpha}_j^*$ (Lemma 2.4) and $X = \hat{\alpha}_j^* - (1 + \alpha_r(\hat{\alpha}_j^*))\xi_1$ by Proposition 3.26. The $\exp(\mathfrak{k}_2)$-orbit of $X = \hat{\alpha}_j^* - (1 + \alpha_r(\hat{\alpha}_j^*))\xi_1$ is equivariantly diffeomorphic to the $\exp(\mathfrak{k}_2)$-orbit of $\hat{\alpha}_j^*$, just by forgetting the last summand $-(1 + \alpha_r(\hat{\alpha}_j^*))\xi_1$.

Conversely, let $Y \in \hat{\mathfrak{p}}_2$ be an extrinsically symmetric element in $\hat{\mathfrak{g}}_2$. We can again assume that this element lies in the closure of the Weyl chamber of $\hat{\mathfrak{a}}$ defined by $\Sigma(\hat{P}_2)$, so that $Y = \hat{\alpha}_j^*$. By Proposition 3.26 the element $X = \hat{\alpha}_j^* - (1 + \alpha_1(\hat{\alpha}_j^*))\xi_1$ is extrinsically symmetric in $\mathfrak{g}$. Since P_1 is simply connected $t \mapsto \exp(2tX)j_2, t \in [0, \pi]$, is a shortest geodesic arc in P_1 joining j_2 to zj_2. But this geodesic arc lies entirely in P_2 and is also length minimizing in P_2. Hence $\exp(\pi X)j_2$ is an element of a minimal centriole in the centrosome $C_{zj_2}(P_2, j_2)$. Summing up:

[47]Recall that $\exp(\mathfrak{k}_2)$ is the identity component of the isotropy group of j_2 in $\mathfrak{L}$ and that it fixes ξ_1.

Proposition 3.28. *The set of all minimal centrioles in the centrosome $C_{zj_2}(P_2, j_2)$ that are not the singletons $\{j_1 j_2\}$ and $\{j_2 j_1\}$ is in one-to-one correspondence with the set of all non-zero s-orbits in $\hat{\mathfrak{p}}_2$ formed by elements that are extrinsically symmetric in $\hat{\mathfrak{g}}_2$.*

Looking again at [BCO-03, Table A.7] we can continue Table 3.2 by listing for P_3 all possible types of extrinsically symmetric s-orbits in $\hat{\mathfrak{p}}_2$:

Table 3.3.: Third step

	$\mathfrak{G}$	P_1	P_2	P_3	
1	...	$\mathrm{SO}_n/\mathrm{SO}_2 \times \mathrm{SO}_{n-2}$	$(S^1 \times S^{n-3})/\Delta \mathrm{Z}_2$	S^{n-4}	$n \geq 5$
2	...	$\mathrm{SO}_{4n}/\mathrm{U}_{2n}$	$\mathrm{U}_{2n}/\mathrm{Sp}_n$	$\mathrm{Sp}_n/\mathrm{Sp}_p \times \mathrm{Sp}_{n-p}$	$n \geq 3$
3	...	$\mathrm{SU}_{2n}/\mathrm{S}(\mathrm{U}_n \times \mathrm{U}_n)$	U_n	$\mathrm{SU}_n/\mathrm{S}(\mathrm{U}_p \times \mathrm{U}_{n-p})$	
4	...	$\mathrm{Sp}_n/\mathrm{U}_n$	$\mathrm{U}_n/\mathrm{SO}_n$	$\mathrm{SO}_n/\mathrm{S}(\mathrm{O}_p \times \mathrm{O}_{n-p})$	$n \geq 2$
5	...	$\mathrm{E}_7/(S^1\mathrm{E}_6)$	$(S^1\mathrm{E}_6)/\mathrm{F}_4$	$\mathbb{O}P^2 = \mathrm{F}_4/\mathrm{Spin}_9$	

Concluding remarks

1. The inclusion chains

$$\begin{array}{ccccccc} \mathfrak{G} & \supset & P_1 & \supset & P_2 & \supset & P_3 \\ \hline \hline \mathrm{Sp}_3 & \supset & \mathrm{Sp}_3/\mathrm{U}_3 & \supset & \mathrm{U}_3/\mathrm{SO}_3 & \supset & \mathbb{R}P_2 \\ \mathrm{SU}_6 & \supset & G_3(\mathbb{C}^6) & \supset & \mathrm{U}_3 & \supset & \mathbb{C}P_2 \\ \mathrm{SO}_{12} & \supset & \mathrm{SO}_{12}/\mathrm{U}_6 & \supset & \mathrm{U}_6/\mathrm{Sp}_3 & \supset & \mathbb{H}P_2 \\ \mathrm{E}_7 & \supset & \mathrm{E}_7/(S^1\mathrm{E}_6) & \supset & (S^1\mathrm{E}_6)/\mathrm{F}_4 & \supset & \mathbb{O}P_2 \end{array} \tag{3.13}$$

   show that the bound of three steps in Theorem 3.5 is optimal. Indeed, the projective planes P_3 are all adjoint spaces and therefore do not contain any pole. But P_2 always contains two isomorphic copies of P_3 as centrioles.

2. If P_3 is not already a circle, we can continue the inclusion chain in the first line of Table 3.3 by the usual inclusions of standard $(d-1)$-spheres as equators in standard d-spheres.

3. Though less known, the inclusion chain $\mathrm{E}_7 \supset \mathrm{E}_7/(S^1\mathrm{E}_6) \supset (S^1\mathrm{E}_6)/\mathrm{F}_4 \supset \mathbb{O}P_2$ is not entirely new. It has already been observed in [NS-91, p. 346].

4. Some of the inclusion chains in Table 4.12 also arise as subchains of the inclusion chains in [NT-95, Table I, p. 201].

5. Assume that the rank of $\hat{P}_2$ is odd, or, equivalently, that the rank r of P_1 is even. Then one can continue the construction as follows: For P_3 choose the centriole that corresponds to the dual of some fundamental root that is in the middle of the Dynkin diagram of $\hat{P}_2$ which has type $\mathfrak{a}_{r-1}$. We will explain this for lines two, three and four of Table 3.3:

- In the second line we get $P_3 = \mathrm{Sp}_r/\mathrm{Sp}_{\frac{r}{2}} \times \mathrm{Sp}_{\frac{r}{2}}$ and in a subsequent step we would get $P_4 = \mathrm{Sp}_{\frac{r}{2}}$ as a symmetric space. We could then proceed as in line four.
- In the fourth line we obtain $P_3 = \mathrm{SO}_r/\mathrm{S}(\mathrm{O}_{\frac{r}{2}} \times \mathrm{O}_{\frac{r}{2}})$ and P_4 would be $\mathrm{SO}_{\frac{r}{2}}$. If r is a multiple of eight, we could continue as in line two (see also [NS-91, pp. 342ff.]). In this way we recover Milnor's inclusion chain from Equation 3.1 and we can guess the period eight of Bott's periodicity for the orthogonal and symplectic groups (see [Mil-69][§24]).
- In the third line of Table 3.3 we get $P_3 = \mathrm{SU}_r/\mathrm{S}(\mathrm{U}_{\frac{1}{2}} \times \mathrm{U}_{\frac{1}{2}})$. One can continue the iteration scheme as in the second step (see also [NS-91, pp. 342ff.]). This observation in used to show the 2-periodicity of the stable homotopy groups of the (special) unitary group (see [Bo-59] and [Mil-69, § 23]).

6. Looking at [BCO-03, p. 311] we notice that the only compact simple Lie groups that can arise as minimal centrioles in irreducible symmetric spaces are Sp_n and SO_n. Hence lines two and four of Table 3.3 can be continued to the left hand side. But U_n also appears as a minimal centriole in $\mathrm{SU}_{2n}/\mathrm{S}(\mathrm{U}_n \times \mathrm{U}_n)$. So one can also continue the third line of Table 3.3 to the left hand side by including SU_{2n} into U_{2n}.

4. Complex structures in representations

In [Mil-69, Lemma 24.6] Milnor described his inclusion chain (see Equation 3.1) also in geometric and linear algebraic terms using the usual action of SO_{16n} on $\mathbb{R}^{16n}$. This is actually the linear isotropy representation of $\mathbb{R}P^{16n}$. Following his example, we assume that the Lie group $\mathfrak{G}$ we start our construction with (Section 3.1) is also suitably represented. Using this representation we describe the sets of complex structures that occur in our construction in geometric terms. It turns out that particularly interesting groups to consider as starting points are either complex linear isotropy groups of irreducible hermitian symmetric spaces of compact type or quaternionic linear isotropy groups of quaternionic symmetric spaces of compact type[48]. By nature, these groups are not always connected. We hence slightly enlarge our view point, and replace the centrioles P_k by certain non-connected subsets that will be denoted by Ω_k as in [Mil-69, §24]. Thus we get inclusion chains

$$\mathfrak{G} \supset \Omega_1 \supset \Omega_2 \supset \ldots .$$

The sets Ω_k will be described in terms of certain Lie subtriples of T_oS and sometimes also in terms of special submanifolds. Our method is particularly inspired by [Mil-69, Lemma 24.6(5-8)]. As a byproduct we get uncommon realizations of some symmetric spaces. This will be illustrated in the case of projective planes.

4.1. Complex linear isotropy groups of hermitian symmetric spaces

Let (S, o) be a pointed irreducible hermitian symmetric space of compact type and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ its Cartan decomposition where $\mathfrak{s} \cong T_oS$. The isotropy group of o in $\mathfrak{I}(P)$ contains the subgroup $\mathfrak{H}$ formed by all elements h of the full isotropy group of (S, o) that are $\mathbb{C}$-linear in the sense that they commute with the Kähler structure J of S on $\mathfrak{s} \cong T_oS$, i.e. $hJ = Jh$.

Remark 4.1. Any isometry of an irreducible hermitian symmetric space is either holomorphic or anti-holomorphic[49]. The full isotropy group of a pointed irreducible hermitian symmetric space S always contains involutions τ that are anti-holomorphic, i.e.

[48] In the context of this work we concentrate on irreducible hermitian or quaternionic symmetric spaces of compact type. But our results pass straight over to irreducible hermitian or quaternionic symmetric spaces of non-compact type.

[49] See [Sz-06] for a general statement. The reason can be shortly explained: Any isometry of S would

that anti-commute with J. The connected components of the fixed space of such τ in S are totally real totally geodesic submanifolds. Particulary interesting are the totally real totally geodesic submanifolds of half dimension, called real forms of S. Real forms of irreducible hermitian symmetric spaces have independently been classified by Leung [Le-79b] and Takeuchi [Tak-84] and they are deeply related with symmetric R-spaces. The classification of real forms in particular shows that any isotropy group of a hermitian symmetric space of compact type must contain anti-holomorphic elements. Therefore the number of connected components of the isotropy group of S is even. Since $\mathfrak{H}$ contains the identity component of the full isotropy group of (S, o) (see [He-78, p. 382]), we conclude that $\mathfrak{H}$ is connected if and only if the number of connected components of the full isotropy group of (S, o) is two. Since S is simply connected, the number of connected components of the isometry group of S and of the isotropy group of (S, o) coincide. This number has been determined in [Lo-69-II, Chap. VII, §4] for any irreducible symmetric space of compact type and the results are presented in [Lo-69-II, Table 10, p. 156]. Using Loos' table and the description of the connected components of isotropy groups of simply connected symmetric spaces in [WZ-93, p. 324] we get the following list of irreducible hermitian symmetric spaces of compact type with connected $\mathfrak{H}$:

Table 4.1.: Irreducible hermitian symmetric spaces with connected $\mathfrak{H}$

S	$\mathfrak{H}$	
$G_p(\mathbb{C}^{p+q})$	$\mathrm{S}(\mathrm{U}_p \times \mathrm{U}_q)/\Delta\mathbb{Z}_{p+q}$	$p \neq q$
$\tilde{G}_2(\mathbb{R}^{2n+1})$	$\mathrm{SO}_2 \times \mathrm{SO}_{2n-1}$	
$\mathrm{Sp}_n/\mathrm{U}_n$	$\mathrm{U}_n/\mathbb{Z}_2$	
$\mathrm{SO}_{2n}/\mathrm{U}_n$	$\mathrm{U}_n/\mathbb{Z}_2$	$n \geq 3$ odd
$\mathrm{SO}_{4n}/\mathrm{U}_{2n}$	$\mathrm{U}_{2n}/\mathbb{Z}_2$	$n \geq 3$
$\mathrm{E}_6/(\mathrm{Spin}_{10} \times S^1)$	$(\mathrm{Spin}_{10} \times S^1)/\Delta\mathbb{Z}_4$	
$\mathrm{E}_7/(\mathrm{E}_6 \times S^1)$	$(\mathrm{E}_6 \times S^1)/\Delta\mathbb{Z}_3$	

Remark 4.2. Since irreducible symmetric spaces of compact type are strongly isotropy irreducible[50], the representation of $\mathfrak{H}$ on $\mathfrak{s}$ is irreducible. As hermitian symmetric spaces are inner[51], $\mathfrak{H}$ contains $-\mathrm{Id}$.

We call a normal Lie subtriple $\mathfrak{m}$ of $\mathfrak{s}$ a *normal complex Lie subtriple* if its is invariant under J. Consequently, the orthogonal complement $\mathfrak{m}^\perp$ of a normal complex Lie subtriple

map the Kähler structure J on another Kähler structure on S. But, since irreducible hermitian symmetric spaces are never hyper-kählerian (cf. [Be-87, Thm. 14.19, p. 399] and the classification of hermitian symmetric spaces) the only Kähler structures on S are $\pm J$. Hence an isometry either preserves J or maps J to $-J$. Recall also that the restricted holonomy group of a simply connected pointed symmetric space coincides with the identity component of its isotropy group (see e.g. [BCO-03, p. 47]).

[50] i.e. the linear isotropy representations of the identity component of the full isotropy group is still irreducible

[51] i.e. the geodesic symmetry s_o of S at o is contained in the identity component of the isotropy group of (S, o)

of $\mathfrak{s}$ is again a normal complex Lie subtriple of $\mathfrak{s}$. Of course, $\mathfrak{s}$ and $\{0\}$ are trivial examples of normal complex Lie subtriples of $\mathfrak{s}$. In view of Lemma A.6 and Section A.5, a normal complex Lie subtriple $\mathfrak{m}$ of $\mathfrak{p}$ corresponds to a totally complex reflective submanifold $M := \mathrm{Exp}_o^S(\mathfrak{m})$ of S.

First step

Theorem 4.3. *The set Ω_1 of all complex structures in $\mathfrak{H}$ is in one-to-one correspondence with*

- *the Grassmannian of all normal complex Lie subtriples of $\mathfrak{s}$;*
- *the Grassmannian of all complex reflective submanifolds of S containing o.*

Proof. Let j be a complex structure in $\mathfrak{H}$, then $\rho := jJ$ squares to the identity. Hence $\mathfrak{s}$ splits orthogonally as $\mathfrak{s} = \mathfrak{m} \oplus \mathfrak{m}^\perp$ into the fix space $\mathfrak{m}$ and the (-1)-eigenspace $\mathfrak{m}^\perp$ of ρ. Since the linear isometry ρ is the differential of an isometry of S, it preserves the Lie triple structure. Therefore $\mathfrak{m}$ and $\mathfrak{m}^\perp$ are both Lie subtriples of $\mathfrak{s}$. Since J and ρ commute[52], $\mathfrak{m}$ and $\mathfrak{m}^\perp$ are both J-invariant.

Conversely, given a normal complex Lie subtriple $\mathfrak{m}$ of $\mathfrak{s}$ we denote by $\mathfrak{m}^\perp$ its orthogonal complement in $\mathfrak{s}$. Then $\mathfrak{s} = \mathfrak{m} \oplus \mathfrak{m}^\perp$ satisfies the relations of Equation A.11 (see page 61). Let ρ be the linear isometry of $\mathfrak{s}$ that is Id on $\mathfrak{m}$ and $-\mathrm{Id}$ on $\mathfrak{m}^\perp$, i.e. ρ is the orthogonal reflection of $\mathfrak{s}$ in $\mathfrak{m}$. Since $\mathfrak{m}$ and $\mathfrak{m}^\perp$ are Lie triples satisfying the relations of Equation A.11, we see that ρ preserves the Lie triple structure, i.e. $[\rho(X), [\rho(Y), \rho(Z)]] = \rho([X, [Y, Z]])$ and therefore the curvature tensor R on $\mathfrak{s}$. Since S is a simply connected symmetric space, ρ is the differential of an isometry of S that fixes o. (see Section A.1). Moreover, since $\mathfrak{m}$ and $\mathfrak{m}^\perp$ are both J-invariant, we have $\rho J = J\rho$. Hence $\rho \in \mathfrak{H}$ and, by construction, ρ^2 squares to the identity. Therefore $j := -\rho J$ is a complex structure in $\mathfrak{H}$. Indeed, $j^2 = \rho J \rho J = \rho^2 J^2 = -\mathrm{Id}$. □

Second step

We now fix a complex structure $j_1 \in \mathfrak{H}$ and denote by $\rho_1 := j_1 J$ the corresponding $\mathbb{C}$-linear involutive orthogonal Lie triple automorphism of $\mathfrak{s}$ and by $\mathfrak{s} = \mathfrak{m} \oplus \mathfrak{m}^\perp$ the corresponding eigenspace decomposition, where $\mathfrak{m}$ is the fix space of ρ_1. Assume that there exists another element $j \in \Omega_1$ that anti-commutes with j_1, i.e. $j_1 j = -j j_1$ on $\mathfrak{s}$. Then j anti-commutes with ρ_1 and therefore maps $\mathfrak{m}$ onto $\mathfrak{m}^\perp$ and vice-versa.

Observation 4.4. *If a complex structure $j_1 \in \Omega_1$ admits another complex structure $j \in \Omega_1$ that anti-commutes with it, then the two eigenspaces $\mathfrak{m}$ and $\mathfrak{m}^\perp$ of ρ_1 have the same dimension.*

Conversely, let j be a complex structure in Ω_1 that interchanges $\mathfrak{m}$ and $\mathfrak{m}^\perp$, then it anti-commutes with ρ_1 and therefore with j_1, too. Thus

$$\Omega_2 := \{j \in \Omega_1;\ j^2 = -\mathrm{Id}, jj_1 = -j_1 j\}$$

[52] because $\rho J = jJJ = -j$ and $J\rho = JjJ = JJj = -j$.

is isomorphic to the set of all complex structures in $\mathfrak{H}$ that map $\mathfrak{m}$ to $\mathfrak{m}^\perp$. Since $j = -j^{-1}$, j is entirely determined by its action on $\mathfrak{m}$.

If we fix a complex structure $j_2 \in \Omega_2$, we can include Ω_2 in the group $\mathfrak{H}(\mathfrak{m})$ of all $\mathbb{C}$-linear orthogonal Lie triple automorphisms of the complex Lie triple $\mathfrak{m}$ as follows:

$$\eta_{j_2} : \Omega \hookrightarrow \mathfrak{H}(\mathfrak{m}), \quad j \mapsto jj_2|_{\mathfrak{m}}. \tag{4.1}$$

Example 4.5 (Complex projective spaces)**.** Let $S = \mathbb{C}P^{2n}$ with base point $o = \mathbb{C}e_{2n+1} \subset \mathbb{C}^{2n+1}$ and let $\mathfrak{s}$ be the corresponding Lie triple. The Lie triple structure of $\mathbb{C}P^{2n}$ (with the Fubini Study metric of constant holomorphic sectional curvature 4) can be written in terms of the Riemannian metric and the complex structure J

$$[[X,Y],Z] = \langle X,Z\rangle Y - \langle Y,Z\rangle X - \langle JY,Z\rangle JX + \langle JX,Z\rangle JY + 2\langle JX,Y\rangle JZ \tag{4.2}$$

(see e.g. [Ok-78, p. 511]). Any orthogonal transformation of $\mathfrak{s}$ that commutes with J preserves automatically the curvature tensor on $\mathfrak{s}$ and is hence an element of the isotropy group. If we identify $\mathfrak{s}$ canonically with $\mathbb{C}^{2n}$, the isotropy action of $\mathfrak{H}$ on $\mathfrak{s}$ becomes the usual action of U_{2n} on $\mathbb{C}^{2n}$. Theorem 4.3 shows that Ω_1 can be identified with the Grassmannian of all complex linear subspace of $\mathbb{C}^{2n}$ (cf. also [Mil-69, §23]). This Grassmannian has several connected components depending on the dimension of the subspaces. Only the Grassmannian of half-dimensional subspaces of $\mathbb{C}^{2n}$ is not an adjoint space and contains precisely one pole. A complex structure j_1 in Ω_1 therefore admits another complex structure j that anti-commutes with j_1 if and only if the complex subspace $\mathfrak{m}$ of $\mathbb{C}^{2n}$ corresponding to j_1 is half-dimensional. Take j_1 such that $\mathfrak{m}$ is the usual $\mathbb{C}^n$ in $\mathbb{C}^{2n}$, i.e. $\mathfrak{m} \cong \mathrm{span}_{\mathbb{C}}(e_1, ..., e_n)$. We now fix an element $j_2 \in \Omega_2$ and observe that the map η_{j_2} takes values in U_n which is identified with the unitary group of $\mathfrak{m}$. We claim that η_{j_2} is surjective. Given an element $f \in \mathrm{U}_n$, the unique element j in Ω_2 that satisfies $\eta_{j_2}(j) = f$ is the $\mathbb{C}$-linear extension of the map that coincides with $j_2^{-1}f^{-1}$ on $\mathfrak{m}$ and with fj_2^{-1} on $\mathfrak{m}^\perp \cong \mathrm{span}_{\mathbb{C}}(e_{n+1}, ..., e_{2n})$. Thus we get an interpretation of the inclusion chain $\mathrm{U}_{2n} \supset G_n(\mathbb{C}^{2n}) \supset \mathrm{U}_n$ (cf. third line of Table 3.3, p. 33, and [Mil-69, §23]).

4.2. Quaternionic linear isotropy groups of quaternionic symmetric spaces

This section is inspired by the methods of [Mil-69, Lemma 24.6(5-8)]. Example 4.15 shows that the results in this section can actually be considered as a generalization of [Mil-69, Lemma 24.6(5-8)].

Let (S, o) be a pointed irreducible quaternionic symmetric space of compact type with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$. Since S is quaternionic, the identity component $\mathfrak{K}$ of the full isotropy group of (S, o) can be written as $\mathfrak{K} = (\mathfrak{K}' \times \mathrm{Sp}_1)/\Delta\mathbb{Z}_2$. The group Sp_1 can be considered as the group of unit quaternions and its linear isotropy action defines a scalar multiplication by unit quaternions on $\mathfrak{s}$, making $\mathfrak{s}$ a quaternionic vector space. The Lie group $\mathfrak{K}'$ is the identity component of the group $\mathfrak{H}$ of all elements in the full isotropy group of o that commute with the action of all elements of Sp_1, i.e. these elements are $\mathbb{H}$-linear. Since S is simply connected, $\mathfrak{H}$ can be considered as the set of all quaternionic linear isometries of $\mathfrak{s}$ that preserve the Lie triple structure. Notice that the action of the identity component of $\mathfrak{H}$ on $\mathfrak{s}$ may not be irreducible as the example of the Grassmannian of complex two-planes shows. Since quaternionic symmetric spaces are inner, $\mathfrak{H}$ contains $-\mathrm{Id}$.

We now list all quaternionic symmetric spaces and we give $\mathfrak{H}$ in those cases where the full isotropy group and hence the full isometry group is connected (see [Lo-69-II, p. 156]). The description of the identity components $\mathfrak{K}$ of the isotropy groups can be found in [WZ-93, p. 324]:

Table 4.2.: Quaternionic symmetric spaces

S	$\mathcal{R}(S)$	$\mathfrak{K}$	$\mathfrak{H}$ (if $\mathfrak{I}_o(S)$ conn.)	
$\mathbb{C}P^2$	$\mathfrak{bc}_1$	$\mathrm{P}(\mathrm{U}_1 \times \mathrm{U}_2)$		
$G_2(\mathbb{C}^4)$	$\mathfrak{c}_2$	$\mathrm{P}(\mathrm{U}_2 \times \mathrm{U}_2)$		
$G_2(\mathbb{C}^{n+2})$	$\mathfrak{bc}_2$	$\mathrm{P}(\mathrm{U}_n \times \mathrm{U}_2)$		$n \geq 3$
$\tilde{G}_4(\mathbb{R}^7)$	$\mathfrak{b}_3$	$\mathrm{SO}_3 \times \mathrm{SO}_4$		
$\tilde{G}_4(\mathbb{R}^8)$	$\mathfrak{d}_4$	$P(\mathrm{SO}_4 \times \mathrm{SO}_4)$		
$\tilde{G}_4(\mathbb{R}^{2n+4})$	$\mathfrak{b}_4$	$P(\mathrm{SO}_{2n} \times \mathrm{SO}_4)$		$n \geq 3$
$\tilde{G}_4(\mathbb{R}^{2n+5})$	$\mathfrak{b}_4$	$\mathrm{SO}_{2n+1} \times \mathrm{SO}_4$		$n \geq 2$
$\mathbb{H}P^n$	$\mathfrak{bc}_1$	$P(\mathrm{Sp}_n \times \mathrm{Sp}_1)$	Sp_n	$n \neq 1$
$EII = \mathrm{E}_6/(\mathrm{SU}_6 \times \mathrm{Sp}_1)$	$\mathfrak{f}_4$	$(\mathrm{SU}_6/\mathbb{Z}_3 \times \mathrm{Sp}_1)/\Delta\mathbb{Z}_2$		
$EVI = \mathrm{E}_7/(\mathrm{SO}_{12} \times \mathrm{Sp}_1)$	$\mathfrak{f}_4$	$(\mathrm{SO}'_{12} \times \mathrm{Sp}_1)\Delta\mathbb{Z}_2$	SO'_{12}	
$EIX = \mathrm{E}_8/(\mathrm{E}_7 \times \mathrm{Sp}_1)$	$\mathfrak{f}_4$	$(\mathrm{E}_7 \times \mathrm{Sp}_1)\Delta\mathbb{Z}_2$	E_7	
$FI = \mathrm{F}_4/(\mathrm{Sp}_3 \times \mathrm{Sp}_1)$	$\mathfrak{f}_4$	$(\mathrm{Sp}_3 \times \mathrm{Sp}_1)/\Delta\mathbb{Z}_2$	Sp_3	
$G = \mathrm{G}_2/\mathrm{SO}_4$	$\mathfrak{g}_2$	$(\mathrm{Sp}_1 \times \mathrm{Sp}_1)/\Delta\mathbb{Z}_2$	$\mathrm{Sp}_1 \cong \mathrm{SU}_2$	

First step

We now fix:

- an imaginary unit quaternion $J_1 \in \mathrm{Sp}_1$, i.e. an element $J_1 \in \mathrm{Sp}_1$ that satisfies $J_1^2 = -\mathrm{Id}$[53].

Further we take a complex structure $j \in \mathfrak{H}$. Since j commutes with J_1, the map $\rho := jJ_1$ is an involution on $\mathfrak{s}$. Hence $\mathfrak{s}$ splits orthogonally as

$$\mathfrak{s} = \mathfrak{m} \oplus \mathfrak{m}^\perp$$

into the fix space $\mathfrak{m}$ and the (-1)-eigenspace $\mathfrak{m}^\perp$ of ρ. Both $\mathfrak{m}$ and $\mathfrak{m}^\perp$ are J_1-invariant subtriples of $\mathfrak{s}$. Let now J_2 be another imaginary unit quaternion in Sp_1 that anti-

[53] For our considerations the actual choice of J_1 does not really matter, since the 2-sphere of all imaginary unit quaternions in Sp_1 is a Sp_1-conjugacy orbit. Actually, the conjugation of Sp_1 on the imaginary unit quaternions is just the usual action of SO_3 on the 2-sphere.

commutes with J_1[54], i.e. $J_1 J_2 = -J_2 J_1$. Since j still commutes with J_2, ρ anti-commutes with J_2. Therefore J_2 maps $\mathfrak{m}$ onto $\mathfrak{m}^\perp$ and vice-versa, i.e. J_2 is an isometric automorphism of $\mathfrak{s}$ identifying $\mathfrak{m}$ and $\mathfrak{m}^\perp$. In particular $\dim(\mathfrak{m}) = \dim(\mathfrak{m}^\perp) = \frac{1}{2}\dim(\mathfrak{s})$. We call a J_1-totally complex Lie subtriple $\mathfrak{m}$ of $\mathfrak{s}$ *self-complementary* , if J_1 leaves $\mathfrak{m}$ invariant and any imaginary unit quaternion $J \in \mathrm{Sp}_1$ that anti-commutes with J_1 maps $\mathfrak{m}$ onto its orthogonal complement $\mathfrak{m}^\perp$ in $\mathfrak{s}$ and vice-versa (see also [Tak-86, p. 167]). The subtriple $\mathfrak{m}^\perp$ is again a self-complementary J_1-totally complex Lie subtriple of $\mathfrak{s}$.

Conversely, let now $\mathfrak{m}$ be a self-complementary J_1-totally complex Lie subtriple of $\mathfrak{s}$. Then its orthogonal complement $\mathfrak{m}^\perp$ in $\mathfrak{s}$ is again a Lie subtriple of $\mathfrak{s}$. Let ρ be the linear transformation of $\mathfrak{s}$ that is the identity on $\mathfrak{m}$ and $-\mathrm{Id}$ on $\mathfrak{m}^\perp$. In other words ρ is the orthogonal reflection of $\mathfrak{s}$ in $\mathfrak{m}$. In particular ρ is involutive. Since $\mathfrak{m}$ and $\mathfrak{m}^\perp$ are Lie subtriples of $\mathfrak{s}$ satisfying the relations in Equation A.11, we see that ρ is a linear isometry of $\mathfrak{s}$ that preserves the Lie triple structure, i.e. $[\rho(X), [\rho(Y), \rho(Z)]] = \rho([X, [Y, Z]])$ and therefore the curvature tensor on $\mathfrak{s}$. Since S is a simply connected symmetric space, ρ is the differential of an isometry of S fixing o (see again Section A.1). Moreover, since $\mathfrak{m}$ and $\mathfrak{m}^\perp$ are J_1-invariant, ρ commutes with J_1. Let now $j := \rho J_1^{-1}$, then j is a complex structure in the isotropy group of (S, o) that commutes with J_1. But j also commutes with any other imaginary unit quaternion J_2 in Sp_1 that anti-commutes with J_1[55]. This shows that j commutes with any element of Sp_1, i.e. $j \in \mathfrak{H}$. Together with Lemma A.7 we get:

Theorem 4.6. *With the above choice of J_1 the space Ω_1 of all complex structures in $\mathfrak{H}$ can be identified with*

- *the Grassmannian of all self-complementary J_1-totally complex Lie subtriples of* $\mathfrak{s}$;
- *the Grassmannian of all self-complementary totally complex reflective submanifolds of S that contain o and whose tangent space at o is J_1-invariant.*[56]

Second step

We now further choose

- an imaginary unit quaternion $J_2 \in \mathrm{Sp}_1$ that anti-commutes with J_1 and we set $J_3 := J_1 J_2$[57];

[54] Any such J_2 is orthogonal to J_1 if we consider Sp_1 as the set of unit quaternions.

[55] Indeed, if $X \in \mathfrak{m}$, then $J_1(X) \in \mathfrak{m}$ and $J_2(X)$ as well as $J_1 J_2(X)$ are in $\mathfrak{m}^\perp$. Hence $jJ_2(X) = -\rho J_1 J_2(X) = J_1 J_2(X)$ and $J_2 j(X) = -J_2 \rho J_1(X) = -J_2 J_1(X) = J_1 J_2(X) = jJ_2(X)$. Similarly, if $X \in \mathfrak{m}^\perp$, then $J_1(X) \in \mathfrak{m}^\perp$ and $J_2(X)$ as well as $J_1 J_2(X)$ are in $\mathfrak{m}$. Hence $jJ_2(X) = -\rho J_1 J_2(X) = -J_1 J_2(X)$ and $J_2 j(X) = -J_2 \rho J_1(X) = J_2 J_1(X) = -J_1 J_2(X) = jJ_2(X)$.

[56] These submanifolds are complex forms of S.

[57] The three anti-commuting imaginary unit quaternions J_1, J_2 and J_3 form an orthonormal basis of the set of imaginary quaternions. In particular any other imaginary unit quaternion in Sp_1 is a linear combination them. Since conjugation by Sp_1 on the set of imaginary unit quaternions coincides with the usual action of SO_3 on S^2, any two pairs of orthogonal imaginary unit quaternions are conjugate within Sp_1.

- an element $j_1 \in \Omega_1$.

The previously chosen j_1 yielded an involution $\rho_1 := j_1 J_1$. As seen before, the fix space $\mathfrak{m}$ of ρ_1 as well as its (-1)-eigenspace $\mathfrak{m}^\perp$ are self-complementary J_1-totally complex Lie subtriples of $\mathfrak{s}$. Assume that Ω_1 contains a complex structure j_2 that anti-commutes with j_1. Then the involution $\rho_2 := j_2 J_2$ commutes with ρ_1. Hence $\mathfrak{m}$ and $\mathfrak{m}^\perp$ split orthogonally into Lie subtriples as

$$\mathfrak{m} = \mathfrak{r} \oplus \mathfrak{t} \quad \text{and} \quad \mathfrak{m}^\perp = \mathfrak{r}^\perp \oplus \mathfrak{t}^\perp,$$

so that

$$\mathfrak{s} = \mathfrak{r} \oplus \mathfrak{t} \oplus \mathfrak{r}^\perp \oplus \mathfrak{t}^\perp. \tag{4.3}$$

The fix space $\mathfrak{r} \oplus \mathfrak{r}^\perp$ and the (-1)-eigenspace $\mathfrak{t} \oplus \mathfrak{t}^\perp$ of ρ_2 are self-complementary J_2-totally complex Lie subtriples of $\mathfrak{s}$. Since J_1 preserves the two subspaces $\mathfrak{r}\oplus\mathfrak{t}$ and $\mathfrak{r}^\perp \oplus \mathfrak{t}^\perp$ and interchanges $\mathfrak{r} \oplus \mathfrak{r}^\perp$ and $\mathfrak{t} \oplus \mathfrak{t}^\perp$ we get

$$J_1(\mathfrak{r}) = \mathfrak{t} \quad \text{and} \quad J_1(\mathfrak{r}^\perp) = \mathfrak{t}^\perp.$$

As J_2 preserves the two subspaces $\mathfrak{r} \oplus \mathfrak{r}^\perp$ and $\mathfrak{t} \oplus \mathfrak{t}^\perp$ and interchanges $\mathfrak{r} \oplus \mathfrak{t}$ and $\mathfrak{r}^\perp \oplus \mathfrak{t}^\perp$ we see that

$$J_2(\mathfrak{r}) = \mathfrak{r}^\perp \quad \text{and} \quad J_2(\mathfrak{t}) = \mathfrak{t}^\perp.$$

With $J_3 = J_1 J_2$ we conclude

$$J_3(\mathfrak{r}) = \mathfrak{t}^\perp \quad \text{and} \quad J_3(\mathfrak{t}) = \mathfrak{r}^\perp.$$

Thus the decomposition of $\mathfrak{s}$ into common eigenspaces of ρ_1 and ρ_2 is

$$\mathfrak{s} = \mathfrak{r} \oplus J_1(\mathfrak{r}) \oplus J_2(\mathfrak{r}) \oplus J_3(\mathfrak{r}).$$

The corresponding eigenvalues are summarized in the following table

	$\mathfrak{r}$	$J_1(\mathfrak{r})$	$J_2(\mathfrak{r})$	$J_3(\mathfrak{r})$
ρ_1	$+$	$+$	$-$	$-$
ρ_2	$+$	$-$	$+$	$-$

('$\pm$' denotes the eigenvalue ± 1). Since ρ_1 and ρ_2 preserve the Lie triple product of $\mathfrak{s}$, one can check that

$$\big[\,[J_{\alpha_1}(\mathfrak{r}),\ J_{\alpha_2}(\mathfrak{r})]\,,\ J_{\alpha_3}(\mathfrak{r})\big] \subset J_{\alpha_4}(\mathfrak{r}) \tag{4.4}$$

holds with $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} = \{0, 1, 2, 3\}$ and $J_0 := \mathrm{Id}$.

A Lie subtriple $\mathfrak{p}$ of $\mathfrak{s}$ is called *totally real* if any imaginary unit quaternion in Sp_1 maps $\mathfrak{p}$ to its orthogonal complement. We call a totally real subtriple $\mathfrak{p}$ of $\mathfrak{s}$ *full*, if it is maximal in the sense that $\mathfrak{s}$ is the orthogonal direct sum $\mathfrak{s} = \mathfrak{p} \oplus J_1(\mathfrak{p}) \oplus J_2(\mathfrak{p}) \oplus J_3(\mathfrak{p})$. A full totally real subtriple $\mathfrak{p}$ that satisfies Equation 4.4 is called (J_1, J_2)-*strongly totally real*.

Observation 4.7. *$\mathfrak{r}$, $\mathfrak{r}^\perp$, $\mathfrak{t}$ and $\mathfrak{t}^\perp$ are isomorphic (J_1, J_2)-strongly totally real subtriples of $\mathfrak{s}$.*

Lemma 4.8. *Let $\mathfrak{p}$ be a (J_1, J_2)-strongly totally real subtriple of $\mathfrak{s}$ and $\alpha \in \{1, 2, 3\}$. Then $\mathfrak{p} \oplus J_\alpha(\mathfrak{p})$ is a J_α-totally complex self-complementary Lie subtriple of $\mathfrak{s}$.*

Proof. To prove that $\mathfrak{p} \oplus J_\alpha(\mathfrak{p})$ is a Lie subtriple of $\mathfrak{s}$ we have to show that it is invariant under the Lie triple product. Since $\mathfrak{p}$ and $J_\alpha(\mathfrak{p})$ are Lie subtriples of $\mathfrak{s}$, we only need to care about the following mixed terms

$$[[X, Y], J_\alpha(Z)], \qquad [[X, J_\alpha(Y)], Z] \tag{4.5}$$

and

$$[[J_\alpha(X), J_\alpha(Y)], Z], \qquad [[J_\alpha(X), Y], J_\alpha(Z)] \tag{4.6}$$

where $X, Y, Z \in \mathfrak{p}$. Notice that (4.6) follows from (4.5), because J_α preserves the Lie triple product. Hence we only need to deal with Equation 4.5. From a well-known property of the curvature tensor of a quaternionic Kähler manifold (see [Alek-68], [Tak-86, pp. 167 f.], [Be-87, pp. 403 f.], [ADM-05, p. 526]) we deduce that the Lie triple product in $\mathfrak{s}$ has the following symmetry:

$$[[A, B], J_r(C)] = J_r([[A, B], C]) + \frac{\mathrm{scal}}{4n(n+2)}\big(\langle J_t(A), B\rangle J_s(C) - \langle J_s(A), B\rangle J_t(C)\big), \tag{4.7}$$

for all $A, B, C \in \mathfrak{s}$, where scal denotes the scalar curvature of S, $4n$ is the real dimension of S, and (r, s, t) is a cyclic permutation of $(1, 2, 3)$. Since $\mathfrak{p}$ is a totally real Lie triple, we get with $\alpha_1 := \alpha$

$$\begin{aligned}
[[X, Y], J_{\alpha_1}(Z)] &= J_{\alpha_1}([[X, Y], Z]) \\
&\quad + \frac{\mathrm{scal}}{4n(n+2)}\big(\langle J_{\alpha_3}(X), Y\rangle J_{\alpha_2}(Z) - \langle J_{\alpha_2}(X), Y\rangle J_{\alpha_3}(Z)\big) \\
&= J_{\alpha_1}([[X, Y], Z]) \subset J_{\alpha_1}(\mathfrak{p}),
\end{aligned}$$

where $(\alpha_1, \alpha_2, \alpha_3)$ is a cyclic permutation of $(1, 2, 3)$. This shows the first relation desired in Equation 4.5. To verify the second relation of Equation 4.5 we use that $\mathrm{ad}(X)$ is skew-symmetric and that $\mathfrak{p}$ satisfies Equation 4.4. For any element $V \in \mathfrak{p}$ we get

$$\begin{aligned}
\langle [[X, J_{\alpha_1}(Y)], Z],\ J_{\alpha_2}(V)\rangle &= \langle X,\ \underbrace{[[J_{\alpha_2}(V), Z], J_{\alpha_1}(Y)]}_{\subset J_{\alpha_3}(\mathfrak{p})}\rangle \\
&= 0
\end{aligned}$$

and, similarly, $\langle [[X, J_{\alpha_1}(Y)], Z],\ J_{\alpha_3}(V)\rangle = 0$. Since $\mathfrak{s}$ is the orthogonal direct sum $\mathfrak{s} = \mathfrak{p} \oplus J_{\alpha_1}(\mathfrak{p}) \oplus J_{\alpha_2}(\mathfrak{p}) \oplus J_{\alpha_3}(\mathfrak{p})$, we conclude that $[[X, J_{\alpha_1}(Y)], Z] \subset \mathfrak{p} \oplus J_{\alpha_1}(\mathfrak{p})$.

As J_{α_2} is a Lie triple automorphism of $\mathfrak{s}$, the orthogonal complement $J_{\alpha_2}(\mathfrak{p}) \oplus J_{\alpha_3}(\mathfrak{p})$ of $\mathfrak{p} \oplus J_{\alpha_1}(\mathfrak{p})$ coincides with $J_{\alpha_2}(\mathfrak{p} \oplus J_{\alpha_1}(\mathfrak{p}))$. Hence $\mathfrak{p} \oplus J_{\alpha_1}(\mathfrak{p})$ is self-complementary. □

Theorem 4.9. *With the above choices the space Ω_2 of all elements in Ω_1 that anti-commute with j_1 can be identified with the Grassmannian of all (J_1, J_2)-strongly totally real Lie subtriples of $\mathfrak{s}$ that are contained in $\mathfrak{m}$.*

Proof. Let j be an element of Ω_2. As seen above, the common fix space $\mathfrak{r}$ of the involutions $\rho_1 := j_1 J_1$ and $\rho_2 := jJ_2$ is a (J_1, J_2)-strongly totally real Lie subtriple of $\mathfrak{s}$ that is contained in $\mathfrak{m}$.

Conversely, let $\mathfrak{r}$ be a (J_1, J_2)-strongly totally real Lie subtriple of $\mathfrak{s}$ that lies in $\mathfrak{m}$. Then $\mathfrak{s} = \mathfrak{r} \oplus J_1(\mathfrak{r}) \oplus J_2(\mathfrak{r}) \oplus J_3(\mathfrak{r})$. Since $\mathfrak{m}$ is a J_1-totally complex Lie subtriple of $\mathfrak{s}$ we get $\mathfrak{m} = \mathfrak{r} \oplus J_1(\mathfrak{r})$ and $\mathfrak{m}^\perp = J_2(\mathfrak{r}) \oplus J_3(\mathfrak{r})$. By Lemma 4.8 $\mathfrak{r} \oplus J_2(\mathfrak{r})$ is a J_2-totally complex self-complementary Lie subtriple of $\mathfrak{s}$. Let ρ_2 be the orthogonal involution of $\mathfrak{s}$ whose fix space is $\mathfrak{r} \oplus J_2(\mathfrak{r})$ and whose (-1)-eigenspace is $J_1(\mathfrak{r} \oplus J_2(\mathfrak{r})) = J_1(\mathfrak{r}) \oplus J_3(\mathfrak{r})$. As in the proof of Theorem 4.6 we conclude that ρ_2 is a Lie triple automorphism of $\mathfrak{s}$ that commutes with J_2 and that $j_2 := -\rho_2 J_2 = \rho_2 J_2^{-1}$ is an element of Ω_1. Since $\mathfrak{m}$ is J_1-invariant and J_2 maps $\mathfrak{m}$ onto $\mathfrak{m}^\perp$ and vice-versa, we see that $\mathfrak{m} = \mathfrak{r} \oplus J_1(\mathfrak{r})$ and $\mathfrak{m}^\perp = J_2(\mathfrak{r}) \oplus J_3(\mathfrak{r})$. Since ρ_2 leaves $\mathfrak{m}$ and $\mathfrak{m}^\perp$ invariant, it commutes with $\rho_1 := j_1 J_1$. From $j_1 j_2 J_3 = j_1 J_1 j_2 J_2 = \rho_1 \rho_2 = \rho_2 \rho_1 = j_2 J_2 j_1 J_1 = -j_2 j_1 J_3$ we see that j_1 anti-commutes with j_2. $\square$

Remark 4.10. Using the construction in the proof of Theorem 4.9 one can show that the converse implication of Lemma 4.8 is also true.

Third step

We further fix

- an element $j_2 \in \Omega_2$

and consider the space Ω_3 consisting of all elements in Ω_2 that anti-commute with j_2. Let $\mathfrak{r}$ be the totally real Lie subtriple corresponding to the choice of j_2, i.e. $\mathfrak{r}$ is the intersection of the fix spaces of the orthogonal involutive Lie triple automorphisms $\rho_1 := j_1 J_1$ and $\rho_2 := j_2 J_2$. Take now an element j_3 of Ω_3. Since j_3 commutes with $J_1,\ J_2$ and J_3 and anti-commutes with j_1 and j_2, the orthogonal involutive Lie triple automorphism $\rho_3 := j_3 J_3$ of $\mathfrak{s}$ commutes with ρ_1 and with ρ_2. Hence ρ_3 acts on each common eigenspace of ρ_1 and ρ_2. Thus the common fix space $\mathfrak{r}$ of ρ_1 and ρ_2 splits orthogonally into two Lie triples

$$\mathfrak{r} = \mathfrak{r}_+ \oplus \mathfrak{r}_-,$$

where $\mathfrak{r}_+$ is the fix space and $\mathfrak{r}_-$ the (-1)-eigenspace of ρ_3 in $\mathfrak{r}$. The choice $j_3 = j_1 j_2$ corresponds to the trivial splitting $\mathfrak{r}_+ = \mathfrak{r}$. Recall from Equation 4.3 that the decomposition of $\mathfrak{g}$ in common eigenspaces of ρ_1 and ρ_2 is $\mathfrak{s} = \mathfrak{r} \oplus \mathfrak{r}^\perp \oplus \mathfrak{t} \oplus \mathfrak{t}^\perp$ with $\mathfrak{t} = J_1(\mathfrak{r}),\ \mathfrak{r}^\perp = J_2(\mathfrak{r})$ and $\mathfrak{t}^\perp = J_3(\mathfrak{r})$. Thus the remaining common eigenspaces of $\rho_1,\ \rho_2$ and ρ_3 are:

$$\begin{aligned} \mathfrak{t}_\pm &= J_1(\mathfrak{r}_\mp), \\ \mathfrak{r}^\perp_\pm &= J_2(\mathfrak{r}_\mp), \\ \mathfrak{t}^\perp_\pm &= J_3(\mathfrak{r}_\pm), \end{aligned} \tag{4.8}$$

where the subscripts '+' and '−' indicate the eigenvalue of ρ_3 on these spaces. Any of these spaces is a Lie subtriple of $\mathfrak{s}$, because it is an intersections of Lie subtriples. As a synopsis we list the eigenvalues ('±' denotes the eigenvalue ± 1) in a table:

(4.9)

	$\mathfrak{r}_+$	$\mathfrak{t}_+$ $J_1(\mathfrak{r}_-)$	$\mathfrak{r}_+^\perp$ $J_2(\mathfrak{r}_-)$	$\mathfrak{t}_+^\perp$ $J_3(\mathfrak{r}_+)$	$\mathfrak{r}_-$	$\mathfrak{t}_-$ $J_1(\mathfrak{r}_+)$	$\mathfrak{r}_-^\perp$ $J_2(\mathfrak{r}_+)$	$\mathfrak{t}_-^\perp$ $J_3(\mathfrak{r}_-)$
ρ_1	+	+	−	−	+	+	−	−
ρ_2	+	−	+	−	+	−	+	−
ρ_3	+	+	+	+	−	−	−	−

Notice that

$$\mathfrak{s} = \mathfrak{r}_+ \oplus J_1(\mathfrak{r}_-) \oplus J_2(\mathfrak{r}_-) \oplus J_3(\mathfrak{r}_+) \oplus \mathfrak{r}_- \oplus J_1(\mathfrak{r}_+) \oplus J_2(\mathfrak{r}_+) \oplus J_3(\mathfrak{r}_-). \tag{4.10}$$

Lemma 4.11. *The common fix space $\mathfrak{n} := \mathfrak{r}_+ \oplus J_1(\mathfrak{r}_-)$ of ρ_1 and ρ_3 is a (J_1, J_2)-strongly totally real Lie subtriple of $\mathfrak{s}$.*

Proof. As an intersection of two Lie triples, the fix spaces of ρ_1 and ρ_3 respectively, $\mathfrak{n}$ is a Lie triple. With

$$J_1(\mathfrak{n}) = J_1(\mathfrak{r}_+) \oplus \mathfrak{r}_- \tag{4.11}$$
$$J_2(\mathfrak{n}) = J_2(\mathfrak{r}_+) \oplus J_3(\mathfrak{r}_-) \tag{4.12}$$
$$J_3(\mathfrak{n}) = J_3(\mathfrak{r}_+) \oplus J_2(\mathfrak{r}_-). \tag{4.13}$$

Equation 4.10 implies that $\mathfrak{n}$ is a full totally real Lie triple. Since ρ_1 and ρ_3 are Lie triple automorphisms of $\mathfrak{s}$, one checks using

	$\mathfrak{n}$	$J_1(\mathfrak{n})$	$J_2(\mathfrak{n})$	$J_3(\mathfrak{n})$
ρ_1	+	+	−	−
ρ_3	+	−	−	+

that $\mathfrak{n}$ satisfies Equation 4.4, i.e. $\mathfrak{n}$ is a (J_1, J_2)-strongly totally real Lie subtriple of $\mathfrak{s}$. □

Theorem 4.12. *The space Ω_3 can be identified with the Grassmannian of all normal Lie subtriples $\mathfrak{r}_+$ of $\mathfrak{r}$ with the property that $\mathfrak{r}_+ \oplus J_1(\mathfrak{r}_-)$ is a (J_1, J_2)-strongly totally real Lie subtriple of $\mathfrak{s}$, where $\mathfrak{r}_-$ is the orthogonal complement of $\mathfrak{r}_+$ in $\mathfrak{r}$.*

Proof. If j_3 is an element of Ω_3, then $\rho_3 := j_3 J_3$ commutes with $\rho_1 := j_1 J_1$ and $\rho_2 := j_2 J_2$. We have seen above that the common fix space of these three Lie triple automorphisms has the desired properties.

Conversely, let $\mathfrak{r}_+$ be a normal Lie subtriple of $\mathfrak{r}$ such that $\mathfrak{n} := \mathfrak{r}_+ \oplus J_1(\mathfrak{r}_-)$ is a (J_1, J_2)-strongly totally real Lie subtriple of $\mathfrak{s}$. Then the Lie subtriples

$$\begin{array}{rclcl} \mathfrak{s}_+ & := & \mathfrak{n} \oplus J_3(\mathfrak{n}) & = & \mathfrak{r}_+ \oplus J_1(\mathfrak{r}_-) \oplus J_3(\mathfrak{r}_+) \oplus J_2(\mathfrak{r}_-) \quad \text{and} \\ \mathfrak{s}_- & := & J_1(\mathfrak{n} \oplus J_3(\mathfrak{n})) & = & J_1(\mathfrak{r}_+) \oplus \mathfrak{r}_- \oplus J_2(\mathfrak{r}_+) \oplus J_3(\mathfrak{r}_-) \end{array} \tag{4.14}$$

are self-complementary J_3-totally complex Lie subtriples of $\mathfrak{s}$ (Lemma 4.11). We define ρ_3 to be the orthogonal linear transformation of $\mathfrak{s}$ defined by $\rho_3 := \mathrm{Id}$ on $\mathfrak{s}_+$ and $\rho_3 := -\mathrm{Id}$ on $\mathfrak{s}_-$. As in the proof of Theorem 4.6 one proves that ρ_3 is an involutive Lie triple automorphism of $\mathfrak{s}$ and that $j_3 := \rho_2 J_3^{-1} = -\rho_3 J_3$ is a complex structure in $\mathfrak{H}$. By Equation 4.14 $\mathfrak{s}$ splits into common eigenspaces of ρ_1, ρ_2 and ρ_3. Hence ρ_1 commutes with ρ_2 and ρ_3. Therefore j_3 anti-commutes with j_1 and j_2, i.e. j_3 lies in Ω_3. □

Remark 4.13. A look at the list suggests to conjecture that this three step construction is possible for any quaternionic symmetric space of compact type. In the case $\mathbb{C}P^2$ one can only choose $j_3 = \pm j_1 j_2$.

Fourth step

We further fix

- an element $j_3 \in \Omega_3$,

and consider the space Ω_4 of all elements in Ω_3 that anti-commute with j_3. Any element j_4 in Ω_4 anti-commutes with j_1, j_2 and j_3 and hence also with ρ_1, ρ_2 and ρ_3. Thus j_4 maps $\mathfrak{r}_+$, the common fix space of ρ_1, ρ_2 and ρ_3, onto $J_3(\mathfrak{r}_-)$, the common (-1)-eigenspace of ρ_1, ρ_2 and ρ_3 and vice-versa. This shows that the dimensions of $\mathfrak{r}_+$ and $\mathfrak{r}_-$ must coincide.

Observation 4.14 (cf. Observation 4.4)**.** *If the space Ω_4 is non-empty, then $\rho_3 = j_3 J_3$ splits the common fix space $\mathfrak{r}$ of ρ_1 and ρ_2 into two equal dimensional normal Lie subtriples $\mathfrak{r}_+$ and $\mathfrak{r}_-$ of $\mathfrak{r}$.*

Example 4.15 (Quaternionic projective spaces)**.** Let S be the quaternionic projective space $\mathbb{H}P^{2n}$ with base point $o = \mathrm{span}_{\mathbb{H}}(e_{n+1})$. The corresponding Lie triple $\mathfrak{s}$ can be canonically identified with $\mathbb{H}^{2n}$ (with scalar multiplication from the right). We choose the imaginary unit quaternions J_1 and J_2 as follows: J_1 is the right multiplication with i and J_2 the right multiplication with j. Similarly to the case of complex projective spaces (see Example 4.5), the Lie triple product on $\mathfrak{s}$ can be written in terms of the quaternionic structures J_1, J_2, J_3 and the Riemannian metric (see [Be-87, p. 406]):

$$\begin{aligned}[][[X,Y],Z] \;=\; & \tfrac{1}{4}\big(\langle X,Z\rangle Y - \langle Y,Z\rangle X \\ & +2\langle J_1(X),Y\rangle J_1(Z) + \langle J_1(X),Z\rangle J_1(Y) - \langle J_1(Y),Z\rangle J_1(X) \\ & +2\langle J_2(X),Y\rangle J_2(Z) + \langle J_2(X),Z\rangle J_2(Y) - \langle J_2(Y),Z\rangle J_2(X) \\ & +2\langle J_3(X),Y\rangle J_3(Z) + \langle J_3(X),Z\rangle J_3(Y) - \langle J_3(Y),Z\rangle J_3(X)\big).\end{aligned} \tag{4.15}$$

Using this formula one can check that

- any J_1-totally complex subspace[58] of $\mathfrak{s}$ is a Lie subtriple;
- any totally real subspace of $\mathfrak{s}$ is a Lie subtriple;

[58] This is a subspace $\mathfrak{m}$ of $\mathfrak{s}$ that is invariant under J_1 with the property that $J_2(\mathfrak{m})$ and $J_3(\mathfrak{m})$ are perpendicular to $\mathfrak{m}$.

- any full totally real Lie subtriple in $\mathbb{H}P^{2n}$ is also (J_1, J_2)-strongly totally real[59]

Any $\mathbb{H}$-linear orthogonal transformation of $\mathfrak{s}$ is automatically a Lie triple automorphism. Using the canonical identification of $\mathfrak{s}$ with $\mathbb{H}^{2n}$, the action of $\mathfrak{H}$ on $\mathfrak{s}$ becomes the usual action of Sp_{2n} on $\mathbb{H}^{2n}$ by matrix multiplication from the left. We now fix successively:

- the complex structure j_1 in Sp_{2n} such that the fix space $\mathfrak{m}$ of $\rho_1 := j_1 J_1$ is the real span of $\{e_1, ..., e_{2n}, e_1 i, ..., e_{2n} i\}$, where $\{e_1, ..., e_{2n}\}$ is the standard basis of $\mathbb{H}^{2n}$ as a quaternionic vector space;

- the element $j_2 \in \Omega_2$ such that the fix space $\mathfrak{r}$ of $\rho_2 := j_2 J_2$ in $\mathfrak{m}$ is the real span of $\{e_1, ..., e_{2n}\}$;

- the element $j_3 \in \Omega_3$ such that the fix space $\mathfrak{r}_+$ of $\rho_3 := j_3 J_3$ in $\mathfrak{r}$ is the real vector space spanned by $\{e_1, ..., e_n\}$.

Fixing an element $j_4 \in \Omega_4$[60] we get a bijective map

$$\eta_{j_4} : \Omega_4 \to \mathrm{O}(\mathfrak{r}_+) \cong \mathrm{O}_n, \quad j \mapsto j j_4|_{\mathfrak{r}_+}.$$

Indeed, let f be an element $\mathrm{O}(\mathfrak{r}_+)$. The unique element j in Ω_4 that satisfies $\eta_{j_4}(j) = f$ can be constructed as follows: Let $\tilde{\jmath}$ be $\mathbb{R}$-linear orthogonal map on $\mathfrak{w} := \mathfrak{r}_+ \oplus J_3(\mathfrak{r}_-)$ that coincides with $-j_4 f^{-1}$ on $\mathfrak{r}_+$ and with $-f j_4$ on $J_3(\mathfrak{r}_-)$. Since $\mathfrak{w}$ is a totally real subspace of $\mathfrak{s}$ of real dimension $2n$, we see that $\mathfrak{s}$ is the $\mathbb{H}$-linear span of $\mathfrak{w}$. The $\mathbb{H}$-linear extension j of $\tilde{\jmath}$ is a $\mathbb{H}$-linear orthogonal map that squares to $-\mathrm{Id}$. The eigenspace decomposition (Equation 4.10) shows that j interchanges the eigenspaces of ρ_1, ρ_2 and ρ_3 and therefore anti-commutes with these maps. Hence j anti-commutes with j_1, j_2 and j_3, i.e. $j \in \Omega_4$.

Since the linear isotropy representation of $\mathfrak{H}$ on $\mathfrak{s}$ in this example is the usual action of Sp_{2n} on $\mathbb{H}^{2n}$, our above construction is the same as Milnor's construction in [Mil-69, Lemma 24.6(5-8)].

Remark 4.16. The Grassmannian of complex 2-planes is both, hermitian and quaternionic symmetric, but its Kähler structure is not one of the quaternionic structures. These spaces are still pretty similar to the complex and quaternionic projective spaces, because its curvature tensor can locally be expressed by its metric, its Kähler and its quaternionic structure [Ber-97, p. 42] (see also Footnote 59). Thus one may join the methods of Sections 4.1 and 4.2 in this case.

Uncommon realizations of symmetric spaces

The construction in this section provides rather uncommon realizations of symmetric space. To explain this, we can look for example at the inclusion chains (see Equation

[59] We do not know if this still holds for full totally real Lie subtriples of $\mathfrak{s}$ if S is an arbitrary quaternionic symmetric space. Sebastian Klein told us that his classification of totally geodesic submanifolds in [Kl-09] shows that this property is also true for the Grassmannians of complex 2-planes.

[60] The set Ω_4 is not empty in this case (see [Mil-69, §24]).

3.13)[61]:

S	$\mathfrak{G} = \mathfrak{H}^e$	$\supset$	P_1	$\supset$	P_2	$\supset$	P_3
FI	Sp_3	$\supset$	$\mathrm{Sp}_3/\mathrm{U}_3$	$\supset$	$\mathrm{U}_3/\mathrm{SO}_3$	$\supset$	$\mathbb{R}P_2$
EII	SU_6	$\supset$	$G_3(\mathbb{C}^6)$	$\supset$	U_3	$\supset$	$\mathbb{C}P_2$
EVI	SO'_{12}	$\supset$	$\mathrm{SO}_{12}/\mathrm{U}_6$	$\supset$	$\mathrm{U}_6/\mathrm{Sp}_3$	$\supset$	$\mathbb{H}P_2$
EIX	E_7	$\supset$	$\mathrm{E}_7/(S^1\mathrm{E}_6)$	$\supset$	$(S^1\mathrm{E}_6)/\mathrm{F}_4$	$\supset$	$\mathbb{O}P_2$

From Table 4.2 we see that Sp_3, SO'_{12} and E_7 arise as the group $\mathfrak{H}$ of some pointed quaternionic symmetric spaces whose root system has type $\mathfrak{f}_4$. The remaining group SU_6 is still the identity component of $\mathfrak{H}$ for $S = EII$. The symmetric spaces P_1, P_2 and P_3 are connected components of the spaces Ω_1, Ω_2 and Ω_3 for suitable choices of j_1, j_2 and j_3. Theorem 4.12 shows that any projective plane arises as a connected component of the Grassmannian of certain normal Lie subtriples of particular totally real subtriples of the Lie triple $\mathfrak{s}$ of a symmetric space P of compact type whose root system has type $\mathfrak{f}_4$[62]. Unfortunately, we do not know how to identify a priori the corresponding connected components.

[61] However, our arguments can be adopted for any inclusion chain constructed by the method described in Section 3.1 that start with the identity component of the group of $\mathbb{H}$-linear isotropies of a pointed quaternionic symmetric space.

[62] EII, EVI, EIX and FII are the only symmetric spaces of compact type whose root system has type $\mathfrak{f}_4$ (see [He-78, pp. 532 ff])

5. Applications to homotopy

5.1. Minimal centrioles and homotopy groups

Bott's periodicity theorem is a result about (stable) homotopy groups. Already in his original paper [Bo-59] Bott considered sets of minimal geodesics joining certain points. Milnor [Mil-69] gave a particularly geometric proof for the Bott' periodicity theorem. A more general investigation of the relations between centrioles and homotopy is due to Burns [Bu-85, Bu-92] (see also [Nag-88, p. 74]). The relation between the inclusion chain of Equation 3.1, the inclusion chain $\mathrm{SU}_{4n} \supset G_{2n}(\mathbb{C}^{4n}) \supset \mathrm{U}_{2n} \supset G_n(\mathbb{C}^{2n})$ and Bott's periodicity theorem in connection with Milnor's approach has also been mentioned in [NS-91, p. 334] and [NT-99, 4.3a].

It seems that the most practical result in this direction for our considerations is Theorem 5.1 below that is due to Mitchell [Mit-88, Theorem 7.1].[63] The advantage of Mitchell's theorem is that it gives an explicit upper bound for the degree of the homotopy groups such that Equation 5.2 holds. This upper bound can be directly read off from the extended Dynkin diagram labelled with the multiplicities. Mitchell's statement is not only about minimal centrioles, but about extrinsically symmetric s-orbits in general. As we have seen in Chapter 2, minimal centrioles are homeomorphic to some of these s-orbits.

In Section 5.2 we use Mitchell's result and its direct corollaries together with some known homotopy groups of exceptional Lie groups and symmetric spaces to determine explicitly some further homotopy groups of certain exceptional Lie groups and symmetric spaces. For example, we explore the information we get from the inclusion chain

$$\mathrm{E}_7 \supset \mathrm{E}_7/(S^1\mathrm{E}_6) \supset (S^1\mathrm{E}_6)/\mathrm{F}_4 \supset \mathbb{O}P_2. \tag{5.1}$$

Theorem 5.1 (Theorem 7.1 in [Mit-88]). *Let $\tilde{P} = \mathfrak{G}/\mathfrak{K}$[64] be an irreducible simply connected symmetric space of compact type. Assume that $\xi \in \mathfrak{p} \cong T_o\tilde{P}$ is extrinsically symmetric. Then*

$$\pi_{i+1}(\tilde{P}) \cong \pi_i(\mathfrak{K}.\xi) = \pi_i(\mathrm{Ad}(\mathfrak{K})\xi) \qquad \text{for} \quad 0 \le i \le d_\xi - 2. \tag{5.2}$$

The number d_ξ is obtained as follows: Take a simple root system Σ in $\mathcal{R}(\tilde{P})$ such that $\xi = \alpha_\xi^*$ for some $\alpha_\xi \in \Sigma$. Let δ be the highest root corresponding to Σ. To each path γ in the extended Dynkin diagram[65] of $\mathcal{R}(\tilde{P})$ joining α_ξ to $-\delta$ we associate the sum d_γ of

[63] Here I owe special thanks to A.L. Mare. He made me aware of Mitchell's result.

[64] $\mathfrak{G}$ is the identity component of the isometry group of $\tilde{P}$. Then $\mathfrak{K}$ is connected.

[65] For explications see Section A.6. The extended Dynkin diagrams can be found in Table A.1 on page 65.

the multiplicities of all vertices on γ (including α_ξ and $-\delta$). The minimum of d_γ over all such paths is d_ξ.

If $\tilde{P}$ is a compact connected and simply connected simple Lie group, then the result can already be found in [Mit-87, Prop. 2.6]. Mitchell applied his theorem to reinterpret Bott's periodicity theorem [Bo-59] for the homotopy of classical Lie groups. We now apply the above result to exceptional Lie groups and symmetric spaces. The symmetric spaces appearing in this methods are

Table 5.1.: Symmetric s-orbits of exceptional symmetric spaces

$\tilde{P} = \mathfrak{G}/\mathfrak{K}$	type of $\mathcal{R}(\tilde{P})$	Multiplicities	$\mathfrak{K}.\xi = \mathrm{Ad}(\mathfrak{K})\xi$	d_ξ
E_6	$\mathfrak{e}_6$	2	$EIII = \mathrm{E}_6/(S^1\mathrm{Spin}_{10})$	10
E_7	$\mathfrak{e}_7$	2	$EVII = \mathrm{E}_7/(S^1\mathrm{E}_6)$	14
$EI = \mathrm{E}_6/\mathrm{Sp}_4$	$\mathfrak{e}_6$	1	$\mathrm{Ad}(\mathfrak{G}_2(\mathbb{H}^4))$	5
$EIV = \mathrm{E}_6/\mathrm{F}_4$	$\mathfrak{a}_2$	8	$\mathbb{O}P_2 = FII = \mathrm{F}_4/\mathrm{Spin}_9$	16
$EV = \mathrm{E}_7/\mathrm{SU}_8$	$\mathfrak{e}_7$	1	$(\mathrm{SU}_8/\mathrm{Sp}_4)/\mathbb{Z}_2$	7
$EVII = \mathrm{E}_7/(S^1\mathrm{E}_6)$	$\mathfrak{c}_3$	$m_{\alpha_1} = m_{\alpha_2} = 8$ $m_{\alpha_\xi} = m_{-\delta} = 1$	$(S^1\mathrm{E}_6)/\mathrm{F}_4$	18

The description of $\mathfrak{K}.\xi = \mathrm{Ad}(\mathfrak{K})\xi$ is taken from [BCO-03, p. 311], the Dynkin diagram type of $\mathcal{R}(\tilde{P})$ and the multiplicities can be found in [He-78, p. 534]. The values of d_ξ can now be read off from the extended Dynkin diagrams given in Table A.1, p. 65.

Corollary 5.2.

$$\pi_{i+1}(\mathrm{E}_6) \cong \pi_i(\mathrm{E}_6/(S^1\mathrm{Spin}_{10})), \qquad 0 \le i \le 8; \tag{5.3}$$

$$\pi_{i+1}(\mathrm{E}_7) \cong \pi_i(\mathrm{E}_7/(S^1\mathrm{E}_6)), \qquad 0 \le i \le 12; \tag{5.4}$$

$$\pi_{i+1}(\mathrm{E}_6/\mathrm{Sp}_4) \cong \pi_i(\mathrm{Ad}(\mathfrak{G}_2(\mathbb{H}^4))), \qquad 0 \le i \le 3; \tag{5.5}$$

$$\pi_{i+1}(\mathrm{E}_6/\mathrm{F}_4) \cong \pi_i(\mathbb{O}P_2), \qquad 0 \le i \le 14; \tag{5.6}$$

$$\pi_{i+1}(\mathrm{E}_7/\mathrm{SU}_8) \cong \pi_i((\mathrm{SU}_8/\mathrm{Sp}_4)/\mathbb{Z}_2), \qquad 0 \le i \le 5; \tag{5.7}$$

$$\pi_{i+1}(\mathrm{E}_7/(S^1\mathrm{E}_6)) \cong \pi_i((S^1\mathrm{E}_6)/\mathrm{F}_4), \qquad 0 \le i \le 16; \tag{5.8}$$

Equation 5.7 coincides with [Bu-92, Prop. 2.4] and Equation 5.8 is also stated in [Nag-88, p. 74] with reference to Burns' thesis. Since the homotopy groups $\pi_i(M)$ of a space M are the same as the homotopy group of is universal covering $\tilde{M}$ for $i \ge 2$ and as $\pi_i(M_1 \times M_2) \cong \pi_i(M_1) \times \pi_i(M_2)$ we get with $\pi_i(\mathbb{R}) \cong 0$:

Corollary 5.3.

$$\pi_{i+1}(\mathrm{E}_6/\mathrm{Sp}_4) \cong \pi_i(\mathrm{G}_2(\mathbb{H}^4)), \qquad i = 2, 3; \tag{5.9}$$

$$\pi_{i+1}(\mathrm{E}_7/\mathrm{SU}_8) \cong \pi_i(\mathrm{SU}_8/\mathrm{Sp}_4), \qquad 2 \le i \le 5; \tag{5.10}$$

$$\pi_{i+1}(\mathrm{E}_7/(S^1\mathrm{E}_6)) \cong \pi_i(\mathrm{E}_6/\mathrm{F}_4), \qquad 2 \le i \le 16. \tag{5.11}$$

As a consequence the information on homotopy that we get by Theorem 5.1 from the inclusion chain of Equation 5.1 is.

Corollary 5.4.

$$\pi_{i+3}(\mathrm{E}_7) \cong \pi_{i+2}(\mathrm{E}_7/(S^1\mathrm{E}_6)) \cong \pi_{i+1}(\mathrm{E}_6/\mathrm{F}_4) \cong \pi_i(\mathbb{O}P^2), \qquad 1 \leq i \leq 10. \tag{5.12}$$

5.2. Higher homotopy groups of certain exceptional symmetric spaces

Using Mitchell's theorem (Theorem 5.1) and its corollaries we want to determine explicitly some higher homotopy groups of certain irreducible compact exceptional symmetric spaces. The fundamental groups of adjoint spaces have already been calculated by É. Cartan [Ca-27] (see also [Tak-64]). The second homotopy groups of irreducible compact symmetric spaces that are not Lie groups can be found in [Tak-64, p. 122]. The second homotopy group of a compact connected simple non-abelian real Lie group $\mathfrak{G}$ vanishes (É. Cartan) and the third homotopy group of $\mathfrak{G}$ is isomorphic to $\mathbb{Z}$ (Bott) (see [Mim-95, p. 969]).

We first state some known results for the homotopy of the exceptional Lie groups E_6 and E_7 that can be found in [MT-91, p. 363], [Mim-95, p. 968 - 971] and the review Zbl. 0101.39702 of [BS-58]

Table 5.2.: Homotopy groups of E6 and E7

	$\pi_4 - \pi_8$	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
E_6	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}_{12}$	0	0	$\mathbb{Z}$
E_7	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$

Using the methods of polars and meridians introduced by Chen and Nagano [CN-78], Burns [Bu-92] (see also [Bu-85]) found relations among homotopy groups of various symmetric spaces. Here we only list the relations of the homotopy groups of those spaces we are interested in:

$$\pi_{i+1}(\mathrm{E}_6/(S^1\mathrm{Spin}_{10})) \cong \pi_i(S^1), \qquad 1 \leq i \leq 5 \qquad \text{[Bu-92, Prop. 2.3];} \tag{5.13}$$

$$\pi_{i+1}(\mathrm{E}_6/\mathrm{F}_4) \cong \pi_i(S^8), \qquad 1 \leq i \leq 14 \qquad \text{[Bu-92, Prop. 2.2];} \tag{5.14}$$

$$\pi_{i+1}(\mathrm{E}_7/(S^1\mathrm{E}_6)) \cong \pi_i(S^1), \qquad 1 \leq i \leq 7 \qquad \text{[Bu-92, Prop. 2.5];} \tag{5.15}$$

$$\pi_{i+1}(\mathbb{O}P^2) \cong \pi_i(S^7), \qquad 1 \leq i \leq 13 \qquad \text{[Bu-92, Prop. 2.1].} \tag{5.16}$$

Using the tables in [To-62, p. 186] and [Ha-02, p. 339] we get:

Table 5.3.: Homotopy groups (continued)

	π_2	$\pi_3-\pi_6$	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
$\mathrm{E}_6/(S^1\mathrm{Spin}_{10})$	$\mathbb{Z}$	0									
$\mathrm{E}_6/\mathrm{F}_4$	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$
$\mathrm{E}_7/(S^1\mathrm{E}_6)$	$\mathbb{Z}$	0	0	0							
$\mathbb{O}P^2$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	

In the 1960's Conlon [Co-65, Co-66] described the homotopy of the exceptional symmetric spaces $\mathrm{E}_6/(S^1\mathrm{Spin}_{10})$ and $\mathrm{E}_6/\mathrm{F}_4$. Here we follow him up to the degrees we need in view of Corollary 5.2 and 5.3:

$$\pi_{i+1}(\mathrm{E}_6/(S^1\mathrm{Spin}_{10})) \cong \pi_i(S^7), \quad 2 \le i \le 14 \qquad \text{[Co-65];} \tag{5.17}$$

$$\pi_i(\mathrm{E}_6/\mathrm{F}_4) \cong \pi_i(S^9), \quad 1 \le i \le 15 \qquad \text{[Co-66];} \tag{5.18}$$

$$\pi_{16}(\mathrm{E}_6/\mathrm{F}_4) \cong 0. \qquad \text{[Co-66].} \tag{5.19}$$

Using again the tables of [To-62, p. 186] and [Ha-02, p. 339] we complete Table 5.3 for $\mathrm{E}_6/(S^1\mathrm{Spin}_{10})$ and $\mathrm{E}_6/\mathrm{F}_4$ as far as we need:

Table 5.4.: Homotopy groups of EIII

	π_2	$\pi_3-\pi_7$	π_8
$\mathrm{E}_6/(S^1\mathrm{Spin}_{10})$	$\mathbb{Z}$	0	$\mathbb{Z}$

Table 5.5.: Homotopy groups of EI

	$\pi_2-\pi_8$	π_9	$\pi_{10}-\pi_{11}$	π_{12}	$\pi_{13}-\pi_{14}$	π_{15}	π_{16}
$\mathrm{E}_6/\mathrm{F}_4$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}_2$	0

The Tables 5.3, 5.4 and 5.5 may at least partially follow directly from the homotopy groups of Lie groups given in [Mim-95, p. 969 -970] and the long exact sequence[66] for the homotopy groups of coset spaces

$$\ldots \to \pi_{n+1}(\mathfrak{K}) \to \pi_{n+1}(\mathfrak{G}) \to \pi_{n+1}(\mathfrak{G}/\mathfrak{K}) \to \pi_n(\mathfrak{K}) \to \ldots \quad . \tag{5.20}$$

We now use Corollary 5.2 and 5.3 to continue Table 5.3. We first notice that Equation 5.6 can also be verified using Table 5.3, but gives no further information. To use Equations 5.9 and 5.10 we need to know the homotopy of $G_2(\mathbb{H}^4) = \mathrm{Sp}_4/\mathrm{Sp}_2 \times \mathrm{Sp}_2$ and $\mathrm{SU}_8/\mathrm{Sp}_4$. For this we could use Bott periodicity (see [Bo-59], [Mil-69, §24], [Mit-88, Cor. 7.2]):

$$\begin{aligned} \pi_i(G_2(\mathbb{H}^4)) &\cong \pi_{i+1}(\mathrm{SU}_8/\mathrm{Sp}_4) \cong \pi_{i+1}(\mathrm{U}_8/\mathrm{Sp}_4) \\ &\cong \pi_{i+2}(\mathrm{SO}_{16}/\mathrm{U}_8) \cong \pi_{i+3}(\mathrm{SO}_{16}), \qquad 2 \le i \le 5. \end{aligned} \tag{5.21}$$

[66] A sequence $A_3 \xrightarrow{f} A_2 \xrightarrow{g} A_1$ is *exact* if the kernel of g is the image of f, $\ker(g) = \mathrm{im}(f)$. A long sequence $\ldots \to A_{i+1} \to A_i \to A_{i-1} \to \ldots$ is *exact* if each subsequence $A_{i+1} \to A_i \to A_{i-1}$ is exact.

Since $\pi_4(\mathrm{SO}_{16}) \cong \pi_5(\mathrm{SO}_{16}) \cong \pi_6(\mathrm{SO}_{16}) \cong 0$ and $\pi_7(\mathrm{SO}_{16}) \cong \mathbb{Z}$ [Mim-95, p. 969 - 970] we get $\pi_2(G_2(\mathbb{H}^4)) \cong \pi_3(G_2(\mathbb{H}^4)) \cong \pi_2(\mathrm{SU}_8/\mathrm{Sp}_4) \cong \pi_3(\mathrm{SU}_8/\mathrm{Sp}_4) \cong \pi_4(\mathrm{SU}_8/\mathrm{Sp}_4) \cong 0$ and $\pi_5(\mathrm{SU}_8/\mathrm{Sp}_4) \cong \mathbb{Z}$. Equations 5.9 and 5.10 imply

$$\pi_4(\mathrm{E}_6/\mathrm{Sp}_4) \cong 0, \tag{5.22}$$

$$\pi_4(\mathrm{E}_7/\mathrm{SU}_8) = \pi_5(\mathrm{E}_7/\mathrm{SU}_8) \cong 0. \tag{5.23}$$

and

$$\pi_6(\mathrm{E}_7/\mathrm{SU}_8) \cong \mathbb{Z}. \tag{5.24}$$

Equations 5.22, 5.23 and 5.24 can also be obtained from the long exact sequence (see Equation 5.20), since $\pi_4(\mathrm{SU}_8) \cong 0$ and $\pi_5(\mathrm{SU}_8) \cong \mathbb{Z}$ [Mim-95, p. 969 - 970].

Applying Equation 5.11 to Table 5.5 we can continue Table 5.3 for $\mathrm{E}_7/(S^1\mathrm{E}_6)$:

Table 5.6.: Homotopy groups of EVII

	π_9	π_{10}	$\pi_{11} - \pi_{12}$	π_{13}	$\pi_{14} - \pi_{15}$	π_{16}	π_{17}
$\mathrm{E}_7/(S^1\mathrm{E}_6)$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}_2$	0

Remark 5.5. (i) Table 5.6 also fits with Equation 5.4.

(ii) The homotopy groups $\pi_9(\mathrm{E}_7/(S^1\mathrm{E}_6)) \cong \pi_{14}(\mathrm{E}_7/(S^1\mathrm{E}_6)) \cong 0$ and $\pi_{10}(\mathrm{E}_7/(S^1\mathrm{E}_6)) \cong \mathbb{Z}$ in Table 5.6 can also be directly read off from the long exact sequence (Equation 5.20).

(iii) We do not know if at least some the few remaining homotopy groups of Table 5.6 are new in this concreteness.

A. Symmetric spaces

In this chapter we summarize some mostly well known aspects of the theory of symmetric spaces and their submanifolds that we need in this work. We focus particularly on symmetric spaces of compact type. Symmetric spaces have been extensively studied by Élie Cartan in the 1920th. Standard references for symmetric spaces include Helgason's book [He-78], Loos' two volumes [Lo-69-I, Lo-69-II], Chapter 8 of Wolf's book [Wo-84] and section IV(6) in Sakai's book [Sa-96]. For submanifolds of symmetric spaces we recommend [BCO-03]. A very readable short description of root systems can be found in [BR-90]. Whenever no further reference is given our statement can be found in at least one of the books mentioned above.

A.1. Basic notions and properties

A *symmetric space* P is a Riemannian manifold with the following property: For each point $p \in P$ there exists an isometry s_p of P, called *(geodesic) symmetry* of P at p, that reverses all geodesics that start at p, i.e. s_p fixes p and the derivative of s_p at p is $-\mathrm{Id}$ on T_pP. A symmetric space that is not a compact Lie group is always assumed connected[67]. If we fix a base point $o \in P$, we call the pair (P, o) a *pointed symmetric space*.

We can extend any geodesic $\gamma : (-2\varepsilon, 2\varepsilon) \to P$ to the interval $(-3\varepsilon, 3\varepsilon)$ by setting $\gamma(2\varepsilon + t) := s_{\gamma(\varepsilon)}.\gamma(-t)$ and $\gamma(-2\varepsilon - t) := s_{\gamma(-\varepsilon)}.\gamma(t)$ for $t \in (-\varepsilon, \varepsilon)$. The resulting curve is still a geodesic. Hence all geodesics in symmetric spaces are defined entirely on $\mathbb{R}$. The Hopf-Rinow theorem implies that any two points in a (connected) symmetric space can be joint by a geodesic.

Take now two points in a connected symmetric space and consider a geodesic arc joining them, then the symmetry of the midpoint of this geodesic arc interchanges these two points. Therefore connected symmetric spaces are *homogeneous*, i.e. the isometry group acts transitively on them.

The universal Riemannian cover of a symmetric space is again a symmetric space. We call a symmetric space P *irreducible*, if its universal cover is not the Riemannian product of several Riemannian manifolds.

The (full) isometry group $\mathfrak{I}(P)$ of a symmetric space P contains two interesting Lie subgroups:

- the *symmetry group* $\mathfrak{S}(P)$ of P. This is the closed subgroup of $\mathfrak{I}(P)$ generated by all geodesic symmetries of P,

[67]We allow a compact Lie group $\mathfrak{G}$ (with bi-invariant metric) to be non-connected, because a Lie group is by definition homogeneous. The geodesic symmetry s_h of $\mathfrak{G}$ at a point $h \in \mathfrak{G}$ is then defined by $s_h(g) := hg^{-1}h$ (see also Section A.3).

- the *transvection group* $\mathfrak{T}(P)$ of P. This is the closed subgroup generated by all products of two geodesic symmetries of P.

Generally, a *transvection* in P is an isometry of P that induces parallel translation along some geodesic γ, we also speak about a transvection along γ. To a geodesic γ of P we associate a one-parameter subgroup of transvections $p_t^\gamma := s_{\gamma(t/2)} \circ s_{\gamma(0)}$. We observe that $p_s^\gamma(\gamma(t)) = \gamma(t+s)$ and $p_{s*}^\gamma : T_{\gamma(t)}P \to T_{\gamma(t+s)}P$ coincides with the parallel transport along γ. This shows that $\mathfrak{T}(P)$ is also the closed subgroup of $\mathfrak{I}(P)$ that is generated by all transvections of P.

Let $\mathfrak{G}$ be a Lie group that acts transitively and by isometries on P. We fix a base point $o \in P$ and assume that the isotropy subgroup $\mathfrak{K} = \{g \in \mathfrak{G};\ g.o = o\}$ of o in $\mathfrak{G}$ is compact. Suppose moreover that there is an involution[68] σ of $\mathfrak{G}$ such that $\mathfrak{K}$ lies between the identity component of the fix point set $\mathfrak{G}^\sigma$ of σ and $\mathfrak{G}^\sigma$ itself. Let $\mathfrak{g}$ be the Lie algebra of $\mathfrak{G}$ and σ_* the differential of σ at $e \in \mathfrak{G}$. Then σ_* is an involution of $\mathfrak{g}$ and its fix point set $\mathfrak{k}$ is the Lie algebra of $\mathfrak{K}$. The eigenspace decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \tag{A.1}$$

of $\mathfrak{g}$ corresponding to σ_* is called the *Cartan decomposition* of the symmetric pair $(\mathfrak{G}, \mathfrak{K})$. The space $\mathfrak{p}$ here denotes the (-1)-eigenspace of σ_*. This decomposition satisfies the *Cartan relations*

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p} \quad \text{and} \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}. \tag{A.2}$$

If $\mathfrak{g}$ is semisimple[69], then $\mathfrak{k}$ is the Lie subalgebra generated by all Lie brackets of two elements in $\mathfrak{p}$, denoted $\mathfrak{k} = [\mathfrak{p}, \mathfrak{p}]$.

The (-1)-eigenspace can be identified with T_oP as follows: We have a $\mathfrak{K}$-principal bundle

$$\pi : \mathfrak{G} \to P,\ g \mapsto g.o.$$

Its differential $d\pi|_e$ at the identity has kernel $\mathfrak{k}$ and defines an isomorphism between $\mathfrak{p}$ and T_oP. By this isomorphism geodesics of P are images of 1-parameter subgroups whose generator lies in $\mathfrak{p}$. More precisely, let $\tilde{X} = \mathrm{d}\pi(X)$ for some X in $\mathfrak{p}$, then the geodesic γ_X of P emanating from o with $\dot{\gamma}(0) = \tilde{X}$ is given by

$$\gamma_X(t) = \exp(tX).o \tag{A.3}$$

where exp is the exponential map from $\mathfrak{g}$ to $\mathfrak{G}$. As a consequence any geodesic loop γ in P is actually a closed geodesic. More precisely, if γ is a geodesic in P with $\gamma(t_0) = \gamma(t_1)$, then $\dot{\gamma}(t_0) = \dot{\gamma}(t_1)$.

Assume now that $\mathfrak{G}$ is the isometry group $\mathfrak{I}(P)$ of P, or a Lie subgroup of it that contains its identity component. Then σ is the conjugation with the geodesic symmetry s_o of P at o and $\sigma_* = \mathrm{Ad}(s_o)$. In this case we call the decomposition (A.1) of the isometry

[68] An *involution* is an automorphism of order two.

[69] This means that the $\mathfrak{g}$ is a direct sum of simple ideals. A Lie algebra is simple, if its only ideals are $\{0\}$ and the full Lie algebra itself.

Lie algebra just the *Cartan decomposition of* (P,o). The Lie algebra of the transvection group of P is the smallest subalgebra of the Lie algebra $\mathfrak{g}$ of $\mathfrak{I}(P)$ that contains $\mathfrak{p}$, namely $[\mathfrak{p},\mathfrak{p}] \oplus \mathfrak{p}$.

The *isotropy group* $\mathfrak{I}_o(P) := \{g \in \mathfrak{I}(P);\ g.o = o\}$ of (P,o) can be considered as a closed subgroup of the orthogonal group of $T_oP \cong \mathfrak{p}$: To an element of $\mathfrak{I}_o(P)$ we associate its differential at the point o. The action of $\mathfrak{I}_o(P)$ on $\mathfrak{p}$ thus obtained is called the *(linear) isotropy action* on T_oP, or the *(linear) isotropy representation* of $\mathfrak{I}_o(P)$. If we identify T_oP with $\mathfrak{p}$ as above, this action becomes the restriction of the adjoint action:

$$\mathfrak{I}_o(P) \times \mathfrak{p} \mapsto \mathfrak{p}, \quad (k,X) \mapsto k.X = \mathrm{Ad}_{\mathfrak{G}}(k)X. \tag{A.4}$$

The orbits of the (linear) isotropy action of the identity component $\mathfrak{K}$ of $\mathfrak{I}_o(P)$ are called *s-orbits*. Notice that s-orbits only depend on the local isometry class of P.

In this work we mostly deal with symmetric spaces of *compact type*. These are compact symmetric spaces whose universal cover is still compact. In this case its isometry group $\mathfrak{I}(P)$ is a semisimple compact Lie group and we see that its transvection group $\mathfrak{T}(P)$ is the identity component of $\mathfrak{I}(P)$. The Cartan decomposition associated with (P,o) is orthogonal w.r.t. the *Killing form* κ[70] of $\mathfrak{g}$. Irreducible symmetric spaces of compact type are also *strongly isotropy irreducible*. This means that the linear isotropy representation of the identity component $\mathfrak{K}$ of the isotropy group is still irreducible. Hence, in the irreducible case, the non-trivial s-orbits are full submanifolds of $\mathfrak{p}$[71] and the (identity component) of the isotropy group acts effectively on them.

The curvature tensor[72] R of a symmetric space P is parallel and its restriction R_o to T_oP can be expressed using $\mathfrak{p}$: Let $X_1, X_2, X_3 \in \mathfrak{p}$ and $\tilde{X}_j = \mathrm{d}\pi(X_j)$, then

$$R_o(\tilde{X}_1, \tilde{X}_2)\tilde{X}_3 = -[[X_1,X_2],X_3] =: r(X_1,X_2,X_3). \tag{A.5}$$

The triple product r on $\mathfrak{p}$ therefore enjoys the algebraic properties of a curvature tensor. Endowed with this product we call the Euclidean vector space $\mathfrak{p}$ an *(orthogonal) Lie triple*[73] and r a *Lie triple structure*. The role of (orthogonal) Lie triples in the theory of symmetric spaces is somehow similar to the role of Lie algebras in the theory of Lie groups. Any orthogonal Lie triple describes a symmetric space uniquely up to local isometry.

[70] The Killing from is a negative definite symmetric bilinear form on the compact Lie algebra $\mathfrak{g}$ defined by $\kappa(X,Y) := \mathrm{trace}(\mathrm{ad}(X) \circ \mathrm{ad}(Y))$

[71] i.e. these submanifolds are not contained in any proper affine subspace of $\mathfrak{p}$, or, in other words, $\mathfrak{p}$ is the linear hull of any non-trivial s-orbit.

[72] For the curvature tensor we use the convention $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ where X,Y,Z are vector fields on P and ∇ is the Levi-Civita connection of P.

[73] An orthogonal Lie triple is a finite dimensional real vector space $\mathfrak{p}$ endowed with a scalar product and a trilinear multiplication $(X,Y,Z) \mapsto R(X,Y)Z =: r(X,Y,Z)$ that has the algebraic properties of a curvature tensor, namely $R(X,Y)Z = -R(Y,X)Z$, $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$ and $\langle R(X,Y)Z,W\rangle = \langle R(Z,W)X,Y\rangle = -\langle R(X,Y)W,Z\rangle$ for all $X,Y,Z,W \in \mathfrak{p}$. Moreover, the corresponding linear operators $R(A,B)$ of $\mathfrak{p}$ are derivations of the triple product, i.e. $R(A,B)(R(X,Y)Z) = R(R(A,B)X,Y)Z + R(X,R(A,B)Y)Z + R(X,Y)(R(A,B)Z)$ for all $A,B,X,Y,Z \in \mathfrak{p}$.

The full isotropy group $\mathfrak{I}_o(\tilde{P})$ of a simply connected pointed symmetric space $(\tilde{P}, o)$ can be identified with the group of all orthogonal transformations that preserve the Lie triple structure. Indeed, the differential of any isotropy is of this form. Conversely any orthogonal transformation of $\mathfrak{p}$ that preserves the Lie triple structure (curvature tensor) is the differential of a local isometry of $\tilde{P}$ preserving o [He-78, pp. 200 f.]. Since $\tilde{P}$ is a complete connected simply connected and real analytic manifold this local isometry can be extended to a global one.

A symmetric space P is called *inner* if its geodesic symmetries are contained in the identity component of its isometry group. In this case the corresponding involutions $\mathrm{Ad}(s_o)$ of the isometry Lie algebra are inner automorphisms. A symmetric space of compact type is inner if and only if the ranks of its isometry Lie algebra and its isotropy Lie algebra coincide.

A.2. Cartan maps

An interesting tool to study a symmetric space P is the corresponding *Cartan map* into its symmetry group:

$$\iota^P : P \to \mathfrak{S}(P), \quad p \mapsto s_p \tag{A.6}$$

The Cartan map is equivariant: Let g be an isometry of P, then $\iota^P(g.p) = g s_p g^{-1} = g\iota(p)g^{-1}$. If P is a compact symmetric space, then $\mathfrak{S}(P)$ is compact, too, and we can equip $\mathfrak{S}(P)$ with a bi-invariant metric such that the Cartan map (A.6) is a Riemannian covering[74]. The image of the Cartan map is a totally geodesic submanifold[75] of $\mathfrak{S}(P)$ and hence itself a symmetric space. Given a base point $o \in P$, one sometimes considers the s_o-left translate of the Cartan map

$$\iota^P_o : P \to \mathfrak{T}(P), \quad p \mapsto s_o s_p, \tag{A.7}$$

called the *pointed Cartan map*. The image of the base point is the identity.

A.3. Compact Lie groups as symmetric spaces

Let $\mathfrak{G}$ be a compact real Lie group endowed with a bi-invariant metric. Then $\mathfrak{G}$ acts transitively on itself by left and right translations:

$$\begin{aligned} L_h &: \mathfrak{G} \to \mathfrak{G}, \qquad g \mapsto hg \\ R_h &: \mathfrak{G} \to \mathfrak{G}, \qquad g \mapsto gh^{-1}. \end{aligned}$$

The geodesic symmetry of $\mathfrak{G}$ at a point h is defined by

$$s_h(g) := hg^{-1}h. \tag{A.8}$$

[74] Recall that the p is an isolated fix point of s_p.

[75] Because the image of ι^P is a connected component of the fix point set of the isometry $g \mapsto g^{-1}$ of $\mathfrak{S}(P)$.

One verifies that $s_h(h) = h$ and that the differential of s_h at h is $-\mathrm{Id}$ on $T_h\mathfrak{G}$.

The product of two geodesic symmetries is $s_{h_1} \circ s_{h_2} = L_{h_1 h_2^{-1}} \circ R_{h_1^{-1} h_2}$. This shows that the product $\mathfrak{G} \times \mathfrak{G}$ acts transitively and by transvections on $\mathfrak{G}$ by $(g_1, g_2) \mapsto L_{g_1} \circ R_{g_2^{-1}}$[76]. The isotropy group of the neutral element e in $\mathfrak{G} \times \mathfrak{G}$ is the diagonal of $\mathfrak{G} \times \mathfrak{G}$. The involution σ_* on $\mathfrak{g} \times \mathfrak{g}$ induced by the conjugation with the geodesic symmetry of $\mathfrak{G}$ at e just interchanges the two factors. This yields the Cartan decomposition $\mathfrak{g}\times\mathfrak{g} = \Delta\mathfrak{g} \oplus \Delta^-\mathfrak{g}$ where $\Delta\mathfrak{g}$ is the diagonal and $\Delta^-\mathfrak{g}$, the anti-diagonal in $\mathfrak{g} \times \mathfrak{g}$, is identified with $T_e\mathfrak{G}$. We further identify the anti-diagonal $\Delta^-\mathfrak{g}$ just with $\mathfrak{g}$ by forgetting the second entry. The geodesic in $\mathfrak{G}$ emanating from e with initial direction $X \in \mathfrak{g}$ is then the one-parameter subgroup

$$t \mapsto (L_{\exp(tX)} \circ R_{\exp(-tX)})e = \exp(2tX). \tag{A.9}$$

The adjoint action of $\mathfrak{G}$ on its Lie algebra $\mathfrak{g}$ can be considered as the isotropy action of the symmetric space $\mathfrak{G}$ at the identity.

A.4. Adjoint spaces

The set of all symmetric spaces that are locally isometric to a given symmetric space P of compact type forms a lattice in the sense of a partially ordered set[77] having a unique (up to isometry) supremum, the universal Riemannian cover $\tilde{P}$ of P, and a unique (up to isometry) infimum, called the adjoint space (see [He-78, Chap. VII] and [Lo-69-II])[78]. Hence the *adjoint space*[79] $\mathrm{Ad}(P)$ of P is characterized by the property that it is covered by any symmetric space that is locally isometric to P. For semisimple compact connected Lie groups the adjoint space coincides with its usual adjoint group, the group of its inner automorphisms. In this section we want to give a geometric description of the adjoint spaces. The group theoretic description as a coset space can be found in [He-78, pp. 326 f.]. As a key result we use:

Theorem A.1 (p. 244 in [Wo-84])**.** *If a connected symmetric space P covers another symmetric space P', then P' is isometric to an orbit space $P/\Gamma = \{\Gamma.p;\ p \in P\}$*[80] *where Γ is a discrete subgroup of the centralizer Δ of $\mathfrak{T}(P)$ in $\mathfrak{I}(P)$, i.e. $\Delta := Z_{\mathfrak{T}(P)}(\mathfrak{I}(P)) := \{g \in \mathfrak{I}(P);\ gh = hg \text{ for all } h \in \mathfrak{T}(P)\}$. Conversely, any such orbit space P/Γ is a symmetric space.*

If P is of compact type, then $\Delta = Z_{\mathfrak{T}(P)}(\mathfrak{I}(P))$ if finite. Therefore the adjoint space of a symmetric space P of compact type is isometric to the orbit space

$$\mathrm{Ad}(P) \cong P/\Delta = \{\Delta.p;\ p \in P\}. \tag{A.10}$$

[76] This action is not effective if the center of $\mathfrak{G}$ is not trivial.
[77] The ordering is $P > P'$ if P covers P' as a symmetric space.
[78] Both spaces, the universal cover and the adjoint space of P can be constructed from the Lie triple of P without any further information.
[79] also known as bottom space
[80] with the metric induced from P

Notice that Δ is a normal subgroup of $\mathfrak{S}(P)$. In fact, since $s_p\mathfrak{T}(P)s_p = \mathfrak{T}(P)$ we have $(s_p\Delta s_p)g = s_p\Delta(s_pgs_p)s_p = s_p(s_pgs_p)\Delta s_p = g(s_p\Delta s_p)$ for any element $g \in \mathfrak{T}(P)$. Hence $s_p\Delta s_p \subset \Delta$. Therefore $\Delta_{\mathfrak{S}} := \Delta \cap \mathfrak{S}(P)$ and $\Delta_{\mathfrak{T}} := \Delta \cap \mathfrak{T}(P) = Z(\mathfrak{T}(P))$ are normal subgroups of $\mathfrak{S}(P)$ and $\mathfrak{T}(P)$.

Lemma A.2. *$\mathfrak{S}(P)$ and $\mathfrak{T}(P)$ act by isometries on P/Δ and the induced actions of $\mathfrak{S}(P)/\Delta_{\mathfrak{S}}$ and $\mathfrak{T}(P)/\Delta_{\mathfrak{T}}$ on $\mathrm{Ad}(P) \cong P/\Delta$ are effective.*

Proof. Since Δ is a normal subgroup of $\mathfrak{S}(P)$, the action $\mathfrak{S}(P)\times P/\Delta \to P/\Delta$, $(g,[p]) \mapsto [gp]$ is well defined and by isometries[81]. If the actions on P/Δ of two elements g and h of $\mathfrak{S}(P)$ coincide, then $\pi\circ(h^{-1}g) = \pi$, i.e. $h^{-1}g$ is a deck transformation of the Riemannian covering $\pi : P \to \mathrm{Ad}(P)$. But Δ is the group of deck transformations of π. Hence $h^{-1}g$ is contained in $\Delta\cap\mathfrak{S}(P) = \Delta_{\mathfrak{S}}$. Thus $\mathfrak{S}(P)/\Delta_S$ acts effectively on P/Δ. The arguments for $\mathfrak{T}(P)$ are similar. $\square$

Since the Riemannian covering $\pi : P \to \mathrm{Ad}(P)$ maps geodesics of P onto geodesics of $\mathrm{Ad}(P)$ and since s_p reverses geodesics in P through p, the coset $s_p\Delta_{\mathfrak{S}} \in \mathfrak{S}(P)/\Delta_S$ can be identified with the geodesic symmetry $s_{\pi(p)}$ of $\mathrm{Ad}(P)$. Thus:

Corollary A.3 (see p. 327 in [He-78])**.** *The symmetry group $\mathfrak{S}(\mathrm{Ad}(P))$ and the transvection group $\mathfrak{T}(\mathrm{Ad}(P))$ of $\mathrm{Ad}(P)$ are isomorphic to $\mathfrak{S}(P)/\Delta_{\mathfrak{S}}$ and $\mathfrak{T}(P)/\Delta_{\mathfrak{T}} \cong \mathrm{Ad}(\mathfrak{T}(P))$ respectively*[82].

The Cartan map is a Riemannian covering. Therefore the Cartan map of an adjoint space is injective and isometric if one takes a suitable metric on $\mathfrak{S}(\mathrm{Ad}(P) \cong \mathfrak{S}(P)/\Delta_{\mathfrak{S}}$. In particular, the geodesic symmetries of two distinct points in $\mathrm{Ad}(P)$ are different.

Let us fix a base point $o \in P$. Since $\mathrm{Ad}(P)$ coincides with its image under the Cartan map, the equivariance of the Cartan map shows that the full isotropy group of $[o] = \pi(o)$ is $\{g \in \mathfrak{I}(\mathrm{Ad}(P));\ gs_{[o]}g^{-1} = s_{[o]}\}$ and hence the fix point set of the conjugation $\sigma_{[o]}$ with $s_{[o]}$. As a consequence we get[83]:

Lemma A.4 (see p. 327 in [He-78])**.** *The isotropy group of $[o]$ in $\mathfrak{T}(\mathrm{Ad}(P))$ is the fix point set of the involution $\sigma_{[o]}$ in $\mathfrak{T}(\mathrm{Ad}(P))$.*

A.5. Totally geodesic and reflective submanifolds

Any complete connected totally geodesic submanifold[84] M of P is again a symmetric space. Indeed, take an arbitrary point $m \in M$, then the geodesic symmetry s_m of P leaves M invariant and reverses all geodesics of M that start at m. The subgroup of $\mathfrak{I}(P)$

[81]recall that the projection $\pi : P \to P/\Delta$ is a Riemannian covering

[82] In particular the identity component of $\mathfrak{I}(\mathrm{Ad}(P))$ is center-free and hence an adjoint group.

[83]This Lemma can also be deduced from [He-78, Prop. 3.5, p. 212].

[84]A submanifold M of P is *totally geodesic* if any geodesic in M (w.r.t. the induced metric) is also a geodesic in P. This implies that the second fundamental form of M vanishes. Therefore the parallel transports of tangent vectors of M along any curve c in M coincides with their parallel transports along c in the abient space P.

generated by the transvections of P along geodesics in M also acts by transvections on M. But this action is not necessarily effective[85], so that this group is not the transvection group of M, but rather a cover of it[86]. There is a one-to-one correspondence between complete connected totally geodesic submanifolds of P containing o and Lie subtriples[87] of $\mathfrak{p}$. Any complete connected totally geodesic submanifold M of P is of the form $\mathrm{Exp}_o(\mathfrak{m})$ where $\mathrm{Exp}_o = \mathrm{Exp}_o^P$ denotes the Riemannian exponential map of P at o and $\mathfrak{m}$ is a Lie subtriple of $\mathfrak{p} \cong T_oP$.

A distinguished class of totally geodesic submanifolds are *reflective* ones. These are connected components of fix point sets of involutive[88] isometries of symmetric spaces. Reflective submanifolds in simply connected symmetric spaces have been thoroughly studied by Leung in a series of papers [Le-73, Le-74, Le-79a, Le-79b]. If P is simply connected, there exists a one-to-one correspondence between reflective submanifolds M of P containing the base point $o \in P$ and orthogonal splittings $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{m}^\perp$ of $\mathfrak{p}$ into two Lie subtriples satisfying

$$\begin{array}{lcl} [[\mathfrak{m},\mathfrak{m}],\mathfrak{m}^\perp] \subseteq \mathfrak{m}^\perp & , & [[\mathfrak{m}^\perp,\mathfrak{m}],\mathfrak{m}] \subseteq \mathfrak{m}^\perp, \\ [[\mathfrak{m}^\perp,\mathfrak{m}^\perp],\mathfrak{m}] \subseteq \mathfrak{m} & , & [[\mathfrak{m},\mathfrak{m}^\perp],\mathfrak{m}^\perp] \subseteq \mathfrak{m} \end{array} \tag{A.11}$$

which is given by $\mathfrak{m} = T_oM$ [Le-73]. If P is a compact symmetric space, then any orthogonal splitting $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{m}^\perp$ into two Lie subtriples satisfies the relations of Equation A.11[89]. We call a Lie subtriple $\mathfrak{m}$ of an orthogonal Lie triple $\mathfrak{p}$ a *normal Lie subtriple* if its orthogonal complement $\mathfrak{m}^\perp \subset \mathfrak{p}$ is again a Lie triple.

A reflective submanifold $M = \mathrm{Exp}_o(\mathfrak{m})$ in P is said to be *self complementary* if the Lie triple $\mathfrak{m}^\perp$ is isomorphic to $\mathfrak{m}$. In this case $M^\perp := \mathrm{Exp}_o(\mathfrak{m}^\perp)$ is isomorphic to M if P is simply connected.

The involutive isometry ρ of P defining M commutes with s_o. Thus the automorphism of $\mathfrak{T}(P)$ given by the conjugation with ρ commutes with the automorphism σ of $\mathfrak{T}(P)$ given by the conjugation with s_o [Le-73, p. 156]. Hence the classification of reflective submanifolds in irreducible simply connected symmetric spaces of compact type is intimately related to the classification of commuting automorphisms of simple compact Lie groups (see also [Nag-88, pp. 66 ff.] and [Nag-92]) which in turn are related with $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric spaces classified in [BG-08, Kol-09]. Another feature of a reflective submanifold $M \subset P$ is that M is *extrinsically symmetric*, i.e. for any point $m \in M$

[85] There may be transvections of P along closed geodesics in M that leave M pointwise fix, but they are not the identity on P.

[86] The Lie algebra of the group generated by transvections of P along geodesics in M coincides with the Lie algebra of the transvection group of M, since a transvection of P_1 along sufficiently short geodesic arcs in M cannot act trivially on M.

[87] A *Lie subtriple* of a Lie triple is a linear subspace which is invariant under the Lie triple structure.

[88] An isometry f is called involutive if $f^2 := f \circ f$ is the identity.

[89] As we can interchange the roles of $\mathfrak{m}$ and $\mathfrak{m}^\perp$ it is sufficient to verify the first line of Equation A.11. The Riemannian metric on P is induced from a bi-invariant metric on $\mathfrak{G}$ coming from an $\mathrm{Ad}(\mathfrak{G})$-invariant metric $\langle .,. \rangle$ on $\mathfrak{g}$. With respect to this metric the linear transformation $\mathrm{ad}(X)$, $X \in \mathfrak{g}$, is skew-symmetric. Let X, Y and Z be three elements of $\mathfrak{m}$ and take $W \in \mathfrak{m}^\perp$. Then $\langle [[X,Y],W],Z\rangle = -\langle W,[[X,Y]Z]\rangle = 0$, because $\mathfrak{m}$ is a Lie triple. Thus $[[\mathfrak{m},\mathfrak{m}],\mathfrak{m}^\perp] \subseteq \mathfrak{m}^\perp$. Similarly $\langle [[W,X],Y],Z\rangle = \langle [W,X],[Y,Z]\rangle = \langle W,[X,[Y,Z]]\rangle = 0$ and therefore $[[\mathfrak{m}^\perp,\mathfrak{m}],\mathfrak{m}] \subseteq \mathfrak{m}^\perp$ (see also [Nai-86, p. 218]).

there exists an isometry of P which fixes m, leaves M invariant and whose differential at m is $(-\mathrm{Id})$ on the tangent space of M at m and Id on the normal space of M at m. Actually, for complete totally geodesic submanifolds these two notions, reflective and (extrinsically) symmetric submanifold, coincide at least if P is simply connected (see [Nai-86] and [BCO-03, p. 257]).

A.6. Fine structure

Root systems

Let (P, o) be a pointed compact symmetric space, $\mathfrak{G}$ the identity component of its isometry group, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ its Cartan decomposition and $\sigma_* = \mathrm{Ad}(s_o)$ the corresponding involution of $\mathfrak{g}$. Since P is compact, so is $\mathfrak{G}$ (see e.g. [KNo-63, p. 239]). Thus we can choose an $\mathrm{Ad}(\mathfrak{G})$-invariant scalar product $\langle .,. \rangle$ on $\mathfrak{g}$[90]. Since the endomorphisms $\mathrm{ad}(X)$, $X \in \mathfrak{g}$, are skew-symmetric w.r.t. this scalar product, they are diagonalizable with purely imaginary eigenvalues. Let us choose a maximal abelian subspace[91] $\mathfrak{a}$ of $\mathfrak{p}$. Then the endomorphisms $\mathrm{ad}(X)$ and $\mathrm{ad}(Y)$ with $X, Y \in \mathfrak{a}$ are simultaneously diagonalizable. For each element α of $\mathfrak{a}^*$, the set of all linear maps from $\mathfrak{a}$ into $\mathbb{R}$, we consider the subspace

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g}^c;\ \mathrm{ad}(A)X = i\alpha(A)X \text{ for all } A \in \mathfrak{a}\} \tag{A.12}$$

of $\mathfrak{g}^c$, the complexification of $\mathfrak{g}$, i.e. $\mathfrak{g}^c = \mathfrak{g} \otimes \mathbb{C} = \mathfrak{g} \oplus i\mathfrak{g}$. A non-zero element α of $\mathfrak{a}^*$ is called a *root* of P (w.r.t. $\mathfrak{a}$) if $\mathfrak{g}_\alpha \neq \{0\}$. The corresponding space $\mathfrak{g}_\alpha$ is then called the *root space* of the root α. The set of all roots of P, called the *root system* of P, is denoted by $\mathcal{R}(P)$. Notice that if $\alpha \in \mathcal{R}(P)$, then $-\alpha \in \mathcal{R}(P)$ and $\mathfrak{g}_{-\alpha} = \overline{\mathfrak{g}_\alpha}$. The complexified Lie algebra $\mathfrak{g}^c$ can be decomposed as a direct sum

$$\mathfrak{g}^c = \mathfrak{g}_0 \oplus \sum_{\alpha \in \mathcal{R}(P)} \mathfrak{g}_\alpha, \tag{A.13}$$

where $\mathfrak{g}_0 := \{X \in \mathfrak{g}^c;\ [X, \mathfrak{a}] = \{0\}\}$. The decomposition A.13 is called *root space decomposition*. It is orthogonal w.r.t. any $\mathrm{Ad}(\mathfrak{G})$-invariant scalar product on $\mathfrak{g}^c$[92]. Since $\mathfrak{a}$ lies in $\mathfrak{p}$ and since the Cartan relations show that any linear transformation $\mathrm{ad}(X)^2$ of $\mathfrak{g}$ with $X \in \mathfrak{p}$ preserves the Cartan decomposition, the root space decomposition fits well with the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: Let α be a root, then $(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{g}$ is σ_*-invariant and therefore splits as

$$(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{g} = \mathfrak{k}_\alpha \oplus \mathfrak{p}_\alpha \tag{A.14}$$

[90] If P is of compact type, the Lie algebra $\mathfrak{g}$ is semisimple, and we can take $\langle .,. \rangle = -\kappa$, where κ is the Killing from of $\mathfrak{g}$.

[91] Any two maximal abelian subspaces are conjugate under the linear isotropy action of the identity component $\mathfrak{K}$ of the isotropy group of (P, o), so that all maximal abelian subspaces of $\mathfrak{p}$ have the same dimension. This dimension is called the *rank* of P, denoted by $\mathrm{rank}(P)$.

[92] One can for example choose $(X, Y) := \langle X, \overline{Y} \rangle_c$, where $\langle .,. \rangle_c$ denotes the $\mathbb{C}$-linear extension of $\langle .,. \rangle$.

where $\mathfrak{k}_\alpha := \mathfrak{k} \cap (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$ and $\mathfrak{p}_\alpha := \mathfrak{p} \cap (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$. Of course $\mathfrak{k}_\alpha = \mathfrak{k}_{-\alpha}$ and $\mathfrak{p}_\alpha = \mathfrak{p}_{-\alpha}$ Moreover, $\dim_{\mathbb{C}} \mathfrak{g}_\alpha = \dim_{\mathbb{R}} \mathfrak{k}_\alpha = \dim_{\mathbb{R}} \mathfrak{p}_\alpha =: m_\alpha$. The number m_α is called the *multiplicity* of α.

Using the $\mathrm{Ad}(\mathfrak{G})$-invariant scalar product on $\mathfrak{g}$ we can associate to any root α a *root vector* $H_\alpha \in \mathfrak{a}$ defined by $\alpha(A) = i\langle H_\alpha, A\rangle$ for $A \in \mathfrak{a}$. By this we can define the angle and the length between two roots: The angle $\angle(\alpha, \beta)$ between two roots α and β is just the angle between H_α and H_β and the length $|\alpha|$ of α is just the length of H_α.

Weyl chambers

We now assume that P is irreducible and of compact type. Then $\mathfrak{g}$ is semisimple and $\mathcal{R}(P)$ is also irreducible[93]. The kernel $\alpha^\perp$ in $\mathfrak{a}$ of a root α is a hyperplane, called the *root hyperplane* of α. Notice that $\alpha^\perp$ is the orthogonal complement of H_α. The union of the kernels of all roots of $\mathcal{R}(P)$ decomposes $\mathfrak{a}$ in different connected components, the connected components of $\mathfrak{a} \setminus \bigcup_{\alpha \in \mathcal{R}(P)} \alpha^\perp$, called *Weyl chambers*. Any two Weyl chambers in $\mathfrak{p}$ are conjugate under the linear isotropy action of the identity component $\mathfrak{K}$ of the isotropy group of (P, o). Let us fix a Weyl chamber C. This choice provides a splitting of $\mathcal{R}(P)$ into two disjoint equicardinal subsets, the set of positive roots C given by

$$\mathcal{R}^+(P) := \{\alpha \in \mathcal{R}(P);\ \alpha(X) > 0 \text{ for all } X \in C\} \tag{A.15}$$

and the set of negative roots on C given by $\mathcal{R}^-(P) := -\mathcal{R}^+(P)$. A Weyl chamber is a convex simplicial cone and therefore bounded by $r = \mathrm{rank}(P)$ many root hyperplanes corresponding to the positive roots $\alpha_1, ..., \alpha_r$. If $\mathcal{R}(P)$ is not reduced[94], we further assume that $\frac{1}{2}\alpha_j$ is not a root. The set Σ of these roots, is a *fundamental root system*, i.e. Σ is a basis of $\mathfrak{a}^*$ and any root $\alpha \in \mathcal{R}(P)$ can be written as a linear combination of roots in Σ with either only non-negative[95] or non-positive integer[96] coefficients.

Each fundamental root system can also be obtained in a purely algebraic way. We first choose a positive root system $\mathcal{R}^+(P)$ in $\mathcal{R}(P)$. This is a subset of $\mathcal{R}(P)$ enjoying the following three properties:

- $\mathcal{R}(P) = \mathcal{R}^+(P) \cup (-\mathcal{R}^+(P))$;
- $\mathcal{R}^+(P)$ and $-\mathcal{R}^+(P)$ are disjoint;
- If α and β are roots in $\mathcal{R}^+(P)$ such that $\alpha + \beta \in \mathcal{R}(P)$, then $\alpha + \beta \in \mathcal{R}^+(P)$.

A root α in $\mathcal{R}^+(P)$ is called *indecomposable* (or sometimes *simple*) within $\mathcal{R}^+(P)$, if it cannot be written as a sum of two other roots in $\mathcal{R}^+(P)$. The set of all indecomposable

[93] $\mathcal{R}(P)$ is *irreducible* if we cannot split it into two complementary subsets $\mathcal{R}_1$ and $\mathcal{R}_2$ with the property that α_1 is perpendicular to α_2 for any choice of $\alpha_1 \in \mathcal{R}_1$ and $\alpha_2 \in \mathcal{R}_2$.

[94] $\mathcal{R}(P)$ is called *reduced*, if the only roots in $\mathcal{R}(P)$ that are collinear with a given root $\alpha \in \mathcal{R}(P)$ are $\pm\alpha$.

[95] These are precisely the positive roots.

[96] These are precisely the negative roots.

roots in $\mathcal{R}^+(P)$ is a fundamental root system and any fundamental root system arises in this way.

The dual basis of Σ, denoted by $\Sigma^* = \{\alpha_1^*, ..., \alpha_r^*\}$ is defined by

$$\alpha_j(\alpha_k^*) = \delta_{jk} = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases} \tag{A.16}$$

The Weyl chamber C can be described as $C = \{X \in \mathfrak{a};\ X = \sum_j x_j \alpha_j^*,\ x_j > 0\}$. We call its closure $\overline{C} = \{X \in \mathfrak{a};\ X = \sum_j x_j \alpha_j^*,\ x_j \geq 0\}$ a *closed Weyl chamber*. Among all roots there is one root $\delta = \sum_{j=1}^r h_j \alpha_j$, called the *highest root* w.r.t. Σ, with the property that the coefficients of any other root $\alpha = \sum_{j=1}^r c_j \alpha_j$ satisfy $c_j \leq h_j$. The diagrams of the irreducible reduced fundamental root systems in Table A.1 are indexed with the coefficients h_j of δ.

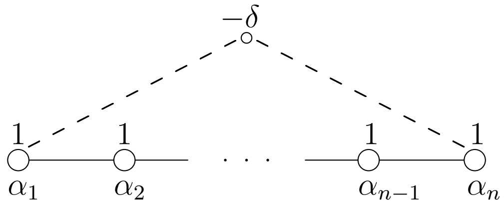
Dynkin diagrams

The geometry of the fundamental root system can be encoded in a graph, called the *Dynkin diagram*. It is obtained as follows: The vertices (circles) represent the fundamental roots[97]. The angle $\angle(\alpha_j, \alpha_k)$ between α_j and α_k is represented by the number of edges joining the vertices (circle) representing α_j and α_k. If α_j and α_k are perpendicular, their vertices are not joined. If $\angle(\alpha_j, \alpha_k) = 120°$, their vertices are linked by a single edge, if $\angle(\alpha_j, \alpha_k) = 135°$, their vertices are linked by a double edge. In the latter case one further adds an arrow pointing towards the shorter root. Finally, if $\angle(\alpha_j, \alpha_k)$ is of 150 degree, the vertices are joined by a triple edge directed towards the shorter root. The Dynkin diagram of an irreducible root system is a connected graph. It determines the root system $\mathcal{R}(P)$ up to isomorphisms. Together with the multiplicities it also determines the symmetric space of compact type up to local isometry (see [Ar-62] and [Nag-92, p. 67]).

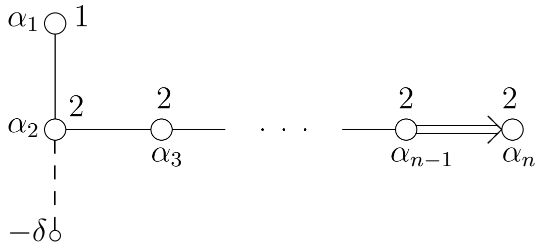
Below we present the (extended) Dynkin diagrams of all irreducible reduced root systems (see e.g. [He-78, p. 503]). The extension is obtained by adding a small circle representing $-\delta$ to the Dynkin diagram and by linking it (we indicate these links by dashed lines) to the fundamental roots according to the rules described above. For the classical root systems $\mathfrak{a}_n$, $\mathfrak{b}_n$, $\mathfrak{c}_n$ and $\mathfrak{d}_n$ and for the exceptional root systems $\mathfrak{f}_4$ and $\mathfrak{g}_2$ we take the enumeration of the fundamental roots from [He-78, Chap. X]. For $\mathfrak{e}_6$, $\mathfrak{e}_7$ and $\mathfrak{e}_8$ it is more convenient for us to take the enumeration of [E-84, pp. 128 ff.]. As an additional information we add the coefficients of each fundamental root in highest root δ.

[97] If $\mathcal{R}(P)$ is not reduced, the vertex representing a fundamental root α_j with the property $2\alpha_j \in \mathcal{R}(P)$ is sometimes denoted by a double circle.

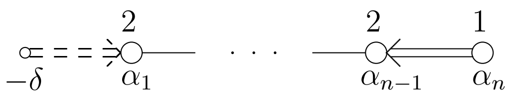
Table A.1.: Extended Dynkin diagrams of irreducible reduced root systems



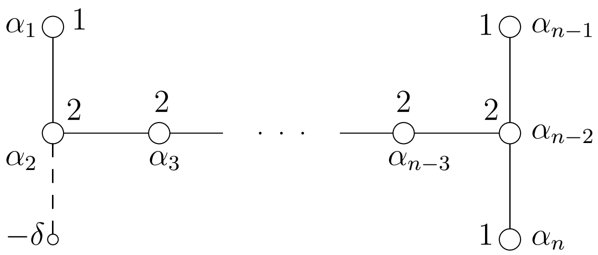
Type $\mathfrak{a}_n$



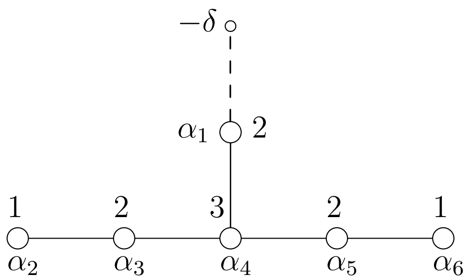
Type $\mathfrak{b}_n$



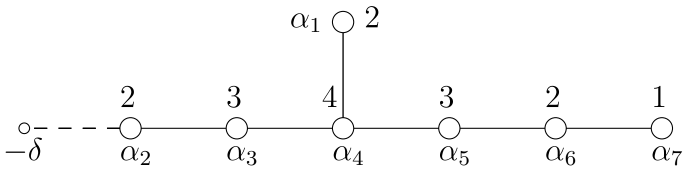
Type $\mathfrak{c}_n$



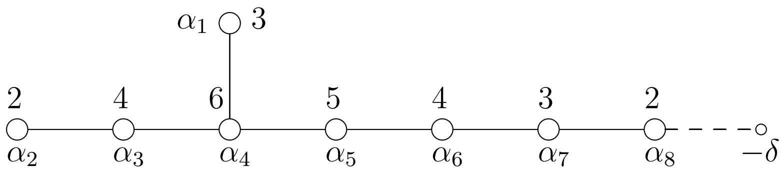
Type $\mathfrak{d}_n$



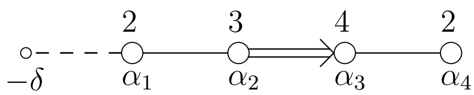
Type $\mathfrak{e}_6$



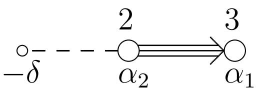
Type $\mathfrak{e}_7$



Type $\mathfrak{e}_8$



Type $\mathfrak{f}_4$



Type $\mathfrak{g}_2$

Cartan matrices

If $\mathcal{R}(P)$ is reduced, we may also encode the information of the Dynkin diagram in a matrix $C = (c_{jk})$, called the *Cartan matrix* of $\mathcal{R}(P)$. For our convenience we introduce the system $\check{\mathcal{R}}(P) \subset \mathfrak{a}$ of all *inverse roots* of $\mathcal{R}(P)$. The *inverse root* $\check{\alpha}$ of α is defined by

$$\check{\alpha} := 2\frac{H_\alpha}{|\alpha|^2}. \tag{A.17}$$

If $\mathcal{R}(P)$ is reduced, then $\check{\Sigma} := \{\check{\alpha}_1, ..., \check{\alpha}_r\}$ is a fundamental system of $\check{\mathcal{R}}_P$ (see [Se-87, pp. 32 f.]). The coefficients of the Cartan matrix are now defined by

$$c_{jk} := \alpha_j(\check{\alpha}_k) = 2\frac{\langle H_{\alpha_j}, H_{\alpha_k}\rangle}{|\alpha_k|^2}. \tag{A.18}$$

The number of edges linking the vertices of α_j and α_k in the Dynkin diagram is $c_{jk}c_{kj}$.

A.7. Integer s-orbits

Let (P, o) be a pointed symmetric space of compact type, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ its Cartan decomposition and $\sigma_* = \mathrm{Ad}(s_o)$ the corresponding involution of $\mathfrak{g}$. We choose a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ and denote by $\mathcal{R}(P)$ the corresponding root system. Let $X \in \mathfrak{p}$ be a nonzero *integer* element, i.e. all eigenvalues of $\frac{1}{i}\mathrm{ad}(X)$ are integers. We denote by $\mathrm{spec}(\mathrm{ad}^2(X)) = \{0, -\nu_1, ..., -\nu_n\}$ the set of eigenvalues (spectrum) of $\mathrm{ad}^2(X)$. The numbers ν_j are prefect squares. Let

$$\mathfrak{g} = \mathfrak{g}_0^X \oplus \sum_{j=1}^{n} \mathfrak{g}_{\nu_j}^X \tag{A.19}$$

be the decomposition of $\mathfrak{g}$ into the eigenspaces of $\mathrm{ad}(X)^2$, where $\mathfrak{g}_0^X$ denotes the 0-eigenspace and $\mathfrak{g}_{\nu_j}^X$ the $(-\nu_j)$-eigenspace of $\mathrm{ad}(X)^2$. These eigenspaces can be described by root spaces:

$$\mathfrak{g}_0^X = \mathfrak{g} \cap \left(\mathfrak{g}_0 \oplus \sum_{\alpha \in \mathcal{R}_0^X(P)} \mathfrak{g}_\alpha \right) \quad \text{and} \quad \mathfrak{g}_{\nu_j}^X = \mathfrak{g} \cap \left(\sum_{\alpha \in \mathcal{R}_{\nu_j}^X(P)} \mathfrak{g}_\alpha \right), \tag{A.20}$$

where $\mathcal{R}_0^X(P) = \{\alpha \in \mathcal{R}(P);\ \alpha(X) = 0\}$ and $\mathcal{R}_{\nu_j}^X(P) = \{\alpha \in \mathcal{R}(P);\ \alpha(X) = \pm\sqrt{\nu_j}\}$.

Since $X \in \mathfrak{p}$, the decomposition (A.19) is compatible with the Cartan decomposition of (P, o) and we get

$$\mathfrak{k} = \mathfrak{k}_0^X \oplus \sum_{j=1}^{n} \mathfrak{k}_{\nu_j}^X \qquad \text{and} \qquad \mathfrak{p} = \mathfrak{p}_0^X \oplus \sum_{j=1}^{n} \mathfrak{p}_{\nu_j}^X, \tag{A.21}$$

where $\mathfrak{k}_0^X := \mathfrak{k} \cap \mathfrak{g}_0^X$, $\mathfrak{p}_0^X := \mathfrak{p} \cap \mathfrak{g}_0^X$, $\mathfrak{k}_{\nu_j}^X := \mathfrak{k} \cap \mathfrak{g}_{\nu_j}^X$ and $\mathfrak{p}_{\nu_j}^X := \mathfrak{p} \cap \mathfrak{g}_{\nu_j}^X$. The Cartan relations (A.2) show that $\mathrm{ad}(X)$ maps $\mathfrak{k}_{\nu_j}^X$ to $\mathfrak{p}_{\nu_j}^X$ and vice-versa.

Let M_X be the s-orbit of X, i.e. $M^X := \mathrm{Ad}_{\mathfrak{G}}(\mathfrak{K})X \subset \mathfrak{p}$, where $\mathfrak{K}$ is the identity component of the isotropy subgroup of (P, o). This orbit can be identified with the coset space $M^X \cong \mathfrak{K}/\mathfrak{K}_X$, where $\mathfrak{K}_X := \{k \in \mathfrak{K};\ \mathrm{Ad}(k)X = X\}$, by the map

$$\phi : \mathfrak{K}/\mathfrak{K}_X \to \mathfrak{p}, \quad k\mathfrak{K}_X \mapsto k.X = \mathrm{Ad}_{\mathfrak{G}}(k)X. \tag{A.22}$$

The Lie algebra of $\mathfrak{K}_X$ is $\mathfrak{k}_X = \mathfrak{k}_0^X$. Hence its orthogonal complement $\mathfrak{k}_+^X = \sum_{j=1}^{n} \mathfrak{k}_{\nu_j}^X$ in $\mathfrak{k}$ can be identified with the tangent space of the coset space $\mathfrak{K}/\mathfrak{K}_X$ at the coset $\mathfrak{K}_X$. The derivative of ϕ at $\mathfrak{K}_X$ is the map $-\mathrm{ad}(X) : \mathfrak{k}_+^X \to \mathfrak{p}$. Its image $\mathfrak{m}^X = \sum_{j=1}^{n} \mathfrak{p}_{\nu_j}^X$ is the tangent space of $M^X \subset \mathfrak{p}$ at the point X. Hence the normal space of $M^X \subset \mathfrak{p}$ at X is $\mathfrak{p}_0^X$.

We now look at the $\mathfrak{K}$-equivariant map

$$\psi : M^X \to P, \quad Z \mapsto \gamma_{\frac{\pi}{2}Z}(1),$$

where $\gamma_{\frac{\pi}{2}Z}$ is the geodesic in P emanating at o with initial direction $\frac{\pi}{2}Z$. Let π be the projection of P onto $\mathrm{Ad}(P)$. Since X is integer, the geodesic $\pi \circ \gamma_{\frac{\pi}{2}X}$ in $\mathrm{Ad}(P)$ closes at $t = 2$ (Lemma 2.2). Thus the image of $\pi \circ \psi$ is a *polar*[98] of $(\mathrm{Ad}(P), \pi(o))$ and therefore totally geodesic (see e.g. [CN-78, p. 406]). Since π is a Riemannian covering, the same holds true for the image of ψ : it is a totally geodesic submanifold of P.

To determine the kernel of the differential of ψ at X, we observe that the point ψ is the restriction to M of the map $\mathrm{Exp}_o \circ \mu_{\frac{\pi}{2}}$, where $\mu_{\frac{\pi}{2}}$ denotes the scaling (scalar multiplication) on $\mathfrak{p}$ by the factor $\frac{\pi}{2}$ and Exp_o is the Riemannian exponential map[99] of P at o. For a point $Y \in \mathfrak{m}^X$ we obtain by the chain rule

$$\mathrm{d}\psi|_X Y = \mathrm{dExp}_o|_{\frac{\pi}{2}X}\left(\frac{\pi}{2}Y\right) = J\left(\frac{\pi}{2}\right), \tag{A.23}$$

where J is the *Jacobi field*[100] along the geodesic γ_X in P satisfying $J(0) = 0$ and $\dot{J}(0) = Y$ (see [GHL-04, Cor. 3.46, p. 146]). Since the Riemannian curvature tensor R of P is parallel and its restriction to $T_oP \cong \mathfrak{p}$ is given by Equation A.5, we can describe the Jacobi field J as follows (see e.g. [Sa-78a, p. 131 f.]): Decompose Y as $Y = \sum_{j=1}^{n} Y_j$ with $Y_j \in \mathfrak{p}^X_{\nu_j}$, then the Jacobi field J along γ_X satisfying $J(0) = 0$ and $\dot{J}(0) = Y$ is

$$J(t) = \sum_{j=1}^{n} \frac{1}{\nu_j} \sin(\nu_j t) J_j(t), \tag{A.24}$$

where $J_j(t)$ is the parallel vector fields along γ_X satisfying $J_j(0) = Y_j$. In particular

$$\mathrm{d}\psi|_X Y_j = \frac{1}{\nu_j} \sin\left(\frac{\pi}{2}\nu_j\right) J_j\left(\frac{\pi}{2}\right). \tag{A.25}$$

If Y_j is non-zero, the parallel vector field $J_j(t)$ never vanishes. Hence the kernel of $\mathrm{d}\psi|_X$ is

$$\ker(\mathrm{d}\psi|_X) = \sum_{\nu_j \text{ even}} \mathfrak{p}^X_{\nu_j}. \tag{A.26}$$

Lemma A.5. *If P is simply connected and $X \in \mathfrak{p}$ integer, then the submanifold $\psi(M^X) = \mathfrak{K}.\gamma_{\frac{\pi}{2}X}(1)$ of P is reflective.*

Proof. Equations A.23, A.24 and A.26 show that the tangent space $T_{\gamma_{\frac{\pi}{2}X}(1)}\psi(M^X)$ is the parallel transport of $\mathfrak{s}_X := \sum_{\nu_j \text{ odd}} \mathfrak{p}^X_{\nu_j}$, and that the normal space $N_{\gamma_{\frac{\pi}{2}X}(1)}\psi(M^X)$ is the parallel transport of $\mathfrak{n}_X := \mathfrak{p}_0 \oplus \sum_{\nu_j \text{ even}} \mathfrak{p}^X_{\nu_j}$ along γ_X. Since the curvature tensor of P

[98] This is a connected component of the set of all midpoints of closed geodesics emanating from o, or, equivalently a connected component of the fix point set of the geodesic symmetry of $\mathrm{Ad}(P)$ at $\pi(o)$ (see e.g. [CN-78, CN-88]).

[99] This map maps a point $X \in \mathfrak{p} \cong T_oP$ to $\gamma_X(1) \in P$.

[100] A Jacobi field along a geodesic γ is a vector field J along γ satisfying $\nabla_{\gamma'}\nabla_{\gamma'}J = R(\gamma', J)\gamma'$ where ∇ is the Levi-Civita connection.

is parallel, these two subspaces of $T_{\gamma_{\frac{\pi}{2}X}(1)}P$ are Lie subtriples if and only if $\mathfrak{s}_X$ and $\mathfrak{n}_X$ are Lie subtriples. Since $\alpha^2(X)$ is even if and only if $\alpha(X)$ is even we get

$$\mathfrak{s}_X = \mathfrak{p} \cap \left(\sum_{\alpha \in \mathcal{R}^X_{\mathrm{odd}}(P)} \mathfrak{g}_\alpha \right) \qquad \text{and} \qquad \mathfrak{n}_X = \mathfrak{p} \cap \left(\mathfrak{g}_0 \oplus \sum_{\alpha \in \mathcal{R}^X_{\mathrm{even}}(P)} \mathfrak{g}_\alpha \right)$$

where $\mathcal{R}^X_{\mathrm{even}}(P) := \{\alpha \in \mathcal{R}(P);\ \alpha(X) \in 2\mathbb{Z}\}$ and $\mathcal{R}^X_{\mathrm{odd}}(P) := \{\alpha \in \mathcal{R}(P);\ \alpha(X) \in 2\mathbb{Z}+1\}$. The Jacobi-identity[101] implies $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ (where $\mathfrak{g}_{\alpha+\beta} = \{0\}$ if $\alpha + \beta \notin \mathcal{R}(P) \cup \{0\}$). Thus $\mathfrak{g}^X_{\mathrm{odd}} := \sum_{\alpha \in \mathcal{R}^X_{\mathrm{odd}}(P)} \mathfrak{g}_\alpha$ and $\mathfrak{g}^X_{\mathrm{even}} := \mathfrak{g}_0 \oplus \sum_{\alpha \in \mathcal{R}^X_{\mathrm{even}}(P)} \mathfrak{g}_\alpha$ are Lie subtriples of $\mathfrak{g}^c$ satisfying $[[\mathfrak{g}^X_{\mathrm{odd}}, \mathfrak{g}^X_{\mathrm{odd}}], \mathfrak{g}^X_{\mathrm{even}}] \subset \mathfrak{g}^X_{\mathrm{even}}$, $[[\mathfrak{g}^X_{\mathrm{odd}}, \mathfrak{g}^X_{\mathrm{even}}], \mathfrak{g}^X_{\mathrm{odd}}] \subset \mathfrak{g}^X_{\mathrm{even}}$, $[[\mathfrak{g}^X_{\mathrm{even}}, \mathfrak{g}^X_{\mathrm{even}}], \mathfrak{g}^X_{\mathrm{odd}}] \subset \mathfrak{g}^X_{\mathrm{odd}}$ and $[[\mathfrak{g}^X_{\mathrm{odd}}, \mathfrak{g}^X_{\mathrm{even}}], \mathfrak{g}^X_{\mathrm{even}}] \subset \mathfrak{g}^X_{\mathrm{odd}}$. As $\mathfrak{p}$ is also a Lie subtriple of $\mathfrak{g}^c$, we see that $\mathfrak{s}_X = \mathfrak{p} \cap \mathfrak{g}^X_{\mathrm{odd}}$ and $\mathfrak{n}_X = \mathfrak{p} \cap \mathfrak{g}^X_{\mathrm{even}}$ are Lie subtriples of $\mathfrak{p}$ satisfying $[[\mathfrak{s}^X, \mathfrak{s}^X], \mathfrak{n}^X] \subset \mathfrak{n}^X$, $[[\mathfrak{s}^X, \mathfrak{n}^X], \mathfrak{s}^X] \subset \mathfrak{n}^X$, $[[\mathfrak{n}^X, \mathfrak{n}^X], \mathfrak{s}^X] \subset \mathfrak{s}^X$ and $[[\mathfrak{s}^X, \mathfrak{n}^X], \mathfrak{n}^X] \subset \mathfrak{s}^X$. The parallelism of the curvature tensor of P shows that the corresponding relations also hold for the subtriples $T_{\gamma_{\frac{\pi}{2}X}(1)}\psi(M^X)$ and $N_{\gamma_{\frac{\pi}{2}X}(1)}\psi(M^X)$ of $T_{\gamma_{\frac{\pi}{2}X}(1)}P$. Since P is simply connected, the claim follows eventually by [Le-73, Theorem 3, p. 156]. □

A.8. Hermitian symmetric spaces

We are now concerned with *hermitian symmetric spaces* of compact type. These are symmetric spaces P of compact type endowed with an almost hermitian structure[102] J such that the geodesic symmetries are almost complex maps[103], i.e. $(s_p)_* \circ J = J \circ (s_p)_*$ for all $p \in P$. Hence J is invariant under $\mathfrak{S}(P)$. Hermitian symmetric spaces of compact type are automatically simply connected [He-78, p. 376]. Irreducible hermitian symmetric spaces of compact type are characterized among all irreducible symmetric spaces of compact type by the fact that the isotropy Lie algebra $\mathfrak{k}$ has a one-dimensional center [He-78, pp. 381 f.].

Hermitian symmetric spaces of compact type arise as adjoint orbits of extrinsically symmetric elements in their isometry Lie algebra $\mathfrak{g}$ (see [Li-58, pp. 165 ff.] and [Hi-70]). Adjoint orbits are s-orbits of connected compact Lie groups. Embedded in this way, hermitian symmetric spaces of compact type are examples of extrinsically symmetric submanifolds in Euclidean spaces [Fe-80, EH-95]:

Let (P, o) be a pointed hermitian symmetric space of compact type with complex structure J and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition. Using the identification $\mathfrak{p} \cong T_oP$ we consider J_o as a linear isomorphism of $\mathfrak{p}$. By setting $J_o := 0$ on $\mathfrak{k}$, we enlarge J_o to a linear map defined on $\mathfrak{g}$. This map is actually a derivation[104]

[101] $[[X,Y],Z] + [[Z,X],Y] + [[Y,Z],X] = 0$

[102] This is a tensor field J of type (1,1) that satisfies $J(J(X)) = -X$ and that is compatible with the Riemannian metric, $g(JX, JY) = g(X,Y)$.

[103] It follows that J is an integrable Kähler structure (see [KNo-69, pp. 259 ff.], [He-78, pp. 372 ff.]).

[104] A *derivation* f of a Lie algebra $\mathfrak{g}$ is a linear map $f : \mathfrak{g} \to \mathfrak{g}$ that satisfies $f([X,Y]) = [f(X), Y] + [X, f(Y)]$.

of $\mathfrak{g}$. Since $\mathfrak{g}$ is semisimple, all its derivations are inner. Thus our extended J_o can be considered as an element of $\mathfrak{g}$ acting on $\mathfrak{g}$ by the adjoint representation. This yields a map

$$\rho : P \to \mathfrak{g},\ o \mapsto J_o. \tag{A.27}$$

By construction $\mathrm{ad}(J_o)$ has eigenvalues $\pm i$ and 0, so that $J_o \in \mathfrak{g}$ is by definition extrinsically symmetric. Since J is $\mathfrak{T}(P)$-invariant, the image of ρ is the adjoint orbit $\mathrm{Ad}(\mathfrak{T}(P))J_o$ in $\mathfrak{g}$. We see that ρ is an equivariant covering map. But since both P and $\mathrm{Ad}(\mathfrak{T}(P))J_o$ are simply connected[105], ρ is bijective and hence a $\mathfrak{G}$-equivariant embedding of P into $\mathfrak{g}$, called the *standard embedding* of P.

Conversely, the orbit $P \subset \mathfrak{g}$ of any non-zero extrinsically symmetric element ξ under the group of inner automorphisms of a simple compact Lie algebra $\mathfrak{g}$ (with a scalar product that is invariant under all inner automorphisms of $\mathfrak{g}$) is a hermitian symmetric space of compact type (w.r.t. the induced metric): The orbit P is of course homogeneous, and its tangent space $\mathfrak{p}$ at ξ is the (-1)-eigenspace of $\mathrm{ad}(\xi)^2$. Its normal space $\mathfrak{k}$ at ξ is therefore the 0-eigenspace of $\mathrm{ad}(\xi)$. The decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ thus obtained satisfies the Cartan relations (see Equation A.2). It turns out that this is also the Cartan decomposition[106] of (P, ξ). The orthogonal reflection of $\mathfrak{g}$ along $\mathfrak{k}$ is the inner automorphism $e^{\mathrm{ad}(\pi\xi)}$, leaves P invariant and induces the geodesic symmetry of P at ξ. Finally, $\mathrm{ad}(\xi)$ defines a complex structure on $\mathfrak{p}$ which coincides on $\mathfrak{p}$ with the inner automorphism $e^{\mathrm{ad}\left(\frac{\pi}{2}\xi\right)}$. One sees that the geodesic symmetries of P are holomorphic. Hence P is a hermitian symmetric space of compact type. Moreover, the group of inner automorphisms of $\mathfrak{g}$ acts faithfully on P[107]. The Lie algebra $\mathfrak{g}$ coincides with $[\mathfrak{p}, \mathfrak{p}] \oplus \mathfrak{p}$ by the simplicity of $\mathfrak{g}$. Hence the group of inner automorphisms of $\mathfrak{g}$ is the transvection group of P. This shows: *The transvection group of an irreducible hermitian symmetric space of compact type has trivial center.*

Let $\mathfrak{G}$ be a compact connected simple real Lie group with Lie algebra $\mathfrak{g}$ and ξ an extrinsically symmetric element of $\mathfrak{g}$. If the center element $\mathrm{Exp}_e(\pi\xi) = \exp(2\pi\xi)$ of $\mathfrak{G}$ is a pole of e, we can realize the hermitian symmetric space $\mathrm{Ad}(\mathfrak{G})\xi$ as a connected component of the set of complex structures (centriole) in $\mathfrak{G}$ by the map $\mathrm{Ad}(g)\xi \mapsto \exp(\pi \mathrm{Ad}(g)\xi)$. If the center element $\exp(2\pi\xi)$ has not order 2, as it may occur e.g. if $\mathfrak{G}$ is SU_n, Spin_{4n+2} or E_6, one can 'add' a (local) S^1-factor to $\mathfrak{G}$ and consider the exponential image of $\pi(\xi + X)$ for a suitable X in the Lie algebra of the local S^1-factor[108]. In this way any irreducible hermitian symmetric space of compact type can be realized as a centriole in a Lie group. This has been observed by Nagano and Tanaka (see [NT-95, pp. 198 f.] and [NT-00, pp. 414 f.]).

There are two types of irreducible hermitian symmetric spaces (of compact type):

[105] All hermitian symmetric spaces of compact type and all adjoint orbits in compact Lie groups are simply connected (see e.g. [He-78]).

[106] The identification of $\mathfrak{p}$ with $T_\xi P$ as the tangent space of a submanifold of $\mathfrak{g}$ is slightly different to the identification of $\mathfrak{p}$ with $T_\xi P$ described in Section A.1 using the principal bundle.

[107] Notice that this is the isotropy representation of a pointed symmetric space $(\mathfrak{G}, e)$, where $\mathfrak{G}$ is any compact Lie group with Lie algebra $\mathfrak{g}$.

[108] This is quite similar to the construction in the third step of Section 3.2.

- The ones whose root system is reduced. In this case the root system has type $\mathfrak{c}_r$ (see [KW-65, Lo-69-II]). Theorem 2.10 shows that $P \to \mathrm{Ad}(P)$ is a two-fold cover. Since the noncompact dual symmetric spaces of such hermitian symmetric spaces can be realized as tube domains[109], we call these hermitian symmetric space of *tube type* .
- The ones whose root system is reduced and therefore of type $\mathfrak{bc}_r$ (see [He-78, p. 475]). These spaces are adjoint spaces.

A submanifold M of a hermitian symmetric space (P, J) is called *complex*, if for any $m \in M$ the complex structure J_m on T_mP leaves T_mM invariant.

Lemma A.6. *Let (P, o) be a pointed hermitian symmetric space and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the corresponding Cartan decomposition. Let $\mathfrak{m}$ be a Lie subtriple of $\mathfrak{p}$ that is invariant under J_o. Then the totally geodesic submanifold $M := \mathrm{Exp}_o(\mathfrak{m})$ of P is a complex submanifold of P. Every complete connected complex totally geodesic submanifold M of P containing o is obtained in this way.*

Proof. Recall that T_oM is identified with $\mathfrak{m}$. Let m be a point in M and let γ be a geodesic arc in M satisfying $\gamma(0) = o$ and $\gamma(1) = m$. Let g be the transvection of P along γ with $g(o) = m$. The differential g_* of g at o coincides with the parallel translation along γ from $\gamma(0) = o$ to $\gamma(1) = m$. Since M is totally geodesic, g leaves M invariant and $T_mM = g_*(T_oM)$. Since J is parallel, we get $J_m = g_* J_o g_*^{-1}$. Thus J_m leaves T_mM invariant. Hence M is a complex submanifold of P. Since any complete connected totally geodesic submanifold M through o is of the form $M = \mathrm{Exp}_o(T_oM)$ the last assertion of Lemma A.6 is immediate. □

We observe that a complex totally geodesic submanifold M of a hermitian symmetric space P is again a hermitian symmetric space, because the complex structure J of P induces a Kähler structure on M and the geodesic symmetries of M which are restrictions of geodesic symmetries of P are holomorphic.

A submanifold M of a hermitian symmetric space P is called *totally real* if at any point of M the complex structure of P maps the tangent space of M on its orthogonal complement.

A.9. Quaternionic symmetric spaces

In this short exposition we follow Takeuchi [Tak-86]. More about quaternionic symmetric spaces can also be found in [Wo-65] and [Be-87, pp. 408 ff.]. Let $S(\mathbb{H})$ denote the unit sphere in the algebra $\mathbb{H}$ of (real) quaternions[110]. The 2-sphere $S(\Im(\mathbb{H}))$ formed by

[109] Every hermitian symmetric space of noncompact type can be realized as a bounded symmetric domain (see e.g. [He-78]). Sometimes this bounded symmetric domain is equivalent to a half-space, like the disc model of the hyperbolic plane is equivalent to its upper half-plane model.

[110] The underlying real vector space of $\mathbb{H}$ is $\mathbb{R}^4$ which we endow with the standard scalar product. Considering the action of $S(\mathbb{H})$ on $\mathbb{H}$ by right multiplication, we can identify $S(\mathbb{H})$ with the group Sp_1 of those invertible (1×1)-matrices with entries in $\mathbb{H}$ that preserve the standard hermitian form on $\mathbb{H}$ which acts from the left by usual matrix multiplication.

the imaginary unit quaternions, i.e. those $q \in S(\mathbb{H})$ that satisfy $q^2 = -1$, is an equator in the 3-sphere $S(\mathbb{H})$. A *quaternionic Kähler structure* on a connected Riemannian manifold M is a parallel (w.r.t. the Levi-Civita connection) subalgebra bundle H of the endomorphism bundle $TM \otimes T^*M$ of TM such that each point of M has a neighborhood U so that the restriction of H to U is isomorphic (as an algebra bundle) to $U \times \mathbb{H}$ and the elements of $U \times S(\mathbb{H})$ act as linear isometries. We can describe the set $S(\Im(H_m))$ of all elements of the fiber H_m of H over $m \in M$ that lie in $S(\Im(\mathbb{H}))$ (under the local trivialization above) as $S(\Im(H_m)) = \{h \in H_m;\ h^2 = -\mathrm{Id}_{TM}\}$, where Id_{TM} is the identity transformation of TM. A symmetric space P carrying a quaternionic Kähler structure H is called *quaternionic symmetric*, if $S(H_p)$ lies in the identity component of the isotropy group of $p \in P$ (acting on T_pP by the linear isotropy representation). The transvection group of a quaternionic symmetric space of compact type leaves its quaternionic Kähler structure invariant ([Tak-86, Remark 3, p. 166]. The identity component $\mathfrak{K}$ of the isotropy group of P at p splits as a product $\mathfrak{K} = \mathfrak{K}' \cdot \mathrm{Sp}_1$ of normal subgroups such that $\mathfrak{K}' \cap \mathrm{Sp}_1 \subseteq \{\pm\mathrm{Id}\} = Z(\mathrm{Sp}_1)$, i.e. $\mathfrak{K} = (\mathfrak{K}' \times \mathrm{Sp}_1)/\Delta\mathbb{Z}_2$. The linear isotropy action of Sp_1 coincides with the action of $S(H_p)$, the set of all elements of H_p that lie in $\{p\} \times S(\mathbb{H})$ under the local trivialization of H. We therefore identity $S(H_p)$ with the Sp_1 factor of the identity component of the isotropy group of P at p. Wolf [Wo-65] has shown that quaternionic symmetric spaces of compact type are irreducible and simply connected.

Following Tsukada [Ts-85] and Takeuchi [Tak-86] we call a submanifold M of a quaternionic symmetric space P *totally complex*, if for each point m in M there is an element $I_m \in S(\Im(H_m))$ that leaves T_mM invariant and has the following further feature: Each $h \in S(\Im(H_m))$ which is perpendicular to I_m, or, equivalently, that anti-commutes with I_m, maps T_mM to the normal space N_mM of M at m. The elements element $I_m,\ m \in M$, are uniquely determined up to sign. They therefore define locally an almost complex structure on M. If the ambient space P is moreover of compact type, then, since P has positive scalar curvature, M is actually locally Kähler, i.e. the locally defined complex structure is parallel (see [Ts-85, p. 192]). Complete connected half-dimensional totally geodesic totally complex submanifolds (complex forms) of quaternionic symmetric spaces of compact type are reflective (see [Tak-86, p. 169]). Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of a pointed quaternionic symmetric space of compact type (P, o) and let I_o be an element of $S(\Im(H_o))$. An I_o-invariant Lie triple $\mathfrak{m}$ of $\mathfrak{p}$ is called *I_o-totally complex*, if for any $J_o \in S(\Im(H_o))$ that is perpendicular to I_o we have $J_o(\mathfrak{m})$ is orthogonal to $\mathfrak{m}$.

Lemma A.7. *Let (P, o) be a pointed quaternionic symmetric space and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the corresponding Cartan decomposition and let $\mathfrak{m}$ be an I_o-totally complex Lie subtriple of $\mathfrak{p}$. Then the totally geodesic submanifold $M := \mathrm{Exp}_o(\mathfrak{m})$ of P is totally complex and any complete connected totally geodesic totally complex submanifold M of P containing o arises in this way.*

Proof. The key argument for this proof is sketched in [Tak-86, p. 172]]. Recall that T_oM is identified with $\mathfrak{m}$. Let m be a point in M and let γ be a geodesic arc in M satisfying $\gamma(0) = o$ and $\gamma(1) = m$. Let g be the transvection of P along γ with $g(o) = m$. The differential g_* of g at o coincides with the parallel translation along γ from $\gamma(0) = o$

to $\gamma(1) = m$. Since M is totally geodesic, g leaves M invariant, $T_mM = g_*(T_oM)$ and $N_mM = g_*(N_oM)$. Since parallel translation leaves H invariant, we see that $I_m := g_* \circ I_o \circ g_*^{-1}$ is an element of H_m that squares to $-\mathrm{Id}$. Moreover, every element $J_m \in S(\Im(H_m))$ that is perpendicular to I_m has the form $J_m = g_* \circ J_o \circ g_*^{-1}$ for some $J_o \in S(\Im(H_o))$ that is perpendicular to I_o. Hence $I_m(T_mM) = T_mM$ and $J_m(T_mM) \subset N_mM$. This shows that M is totally complex. It is not difficult to see that any totally geodesic totally complex submanifold M of P containing o can be obtained in this manner. □

Finally, a submanifold M of a quaternionic symmetric space P is said to be *totally real*, if for any $m \in M$ we have $S(\Im(H_m))T_mM \subset N_mM$. A Lie subtriple $\mathfrak{m}$ of $\mathfrak{p} \cong T_oP$ is called *totally real*, if $S(\Im(H_m))\mathfrak{m} \subset \mathfrak{m}^\perp$, the orthogonal complement of $\mathfrak{m}$ in $\mathfrak{p}$.

Lemma A.8. *Let (P, o) be a pointed quaternionic symmetric space and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the corresponding Cartan decomposition. Let $\mathfrak{m}$ be a totally real Lie subtriple of $\mathfrak{p}$, then the totally geodesic submanifold $M := \mathrm{Exp}_o(\mathfrak{m})$ of P is totally real. Any complete connected totally geodesic totally real submanifold M of P containing o arises in this way.*

Proof. This proof is similar to the one of Lemma A.7. Recall that T_oM is identified with $\mathfrak{m}$. Let m be a point in M and let γ be again a geodesic arc in M satisfying $\gamma(0) = o$ and $\gamma(1) = m$ and g the transvection of P along γ with $g(o) = m$. The differential g_* of g at o coincides with the parallel translation along γ from $\gamma(0) = o$ to $\gamma(1) = m$. Since M is totally geodesic, g leaves M invariant and $T_mM = g_*(T_oM)$ and $N_mM = g_*(N_oM)$. Since the transvection g leaves H invariant we see that $S(\Im(H_m)) = g_* \circ S(\Im(H_o)) \circ g_*^{-1}$ Hence any element of $S(\Im(H_m))$ maps T_mM to N_mM. This shows that M is totally real. It is not hard to see that every any totally geodesic totally complex submanifold M of P containing o is obtained like this. □

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[Alek-68] Alekseevskii, D.V., Riemannian spaces with exceptional holonomy groups, *Funct. Anal. Appl.* **2**(2) (1968), 97 - 105 [Russian original in: *Funkts. Anal. Prilozh.* **2**(2) (1968), 1-10]

[ADM-05] Alekseevskii, D.V.; Di Scala, A.J. & Marchiafava, S., Parallel Kähler submanifolds of quaternionic Kähler symmetric spaces, *Tôhoku Math. J.* **57**(4) (2005), 521 - 540

[Ar-62] Araki, S., On root systems and an infinitesimal classification of irreducible symmetric spaces, *J. Math. Osaka City Univ., Ser. A* **13** (1962), 1 - 34

[BG-08] Bahturin, Y. & Goze, M., $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric spaces, *Pacific J. Math.* **236**(1) (2008), 1 - 21

[Ber-97] Berndt, J., Riemannian geometry of complex two-plane Grassmannians, *Rend. Semin. Mat., Torino* **55**(1) (1997), 19 - 83

[BCO-03] Berndt, J.; Console, S. & Olmos, C., *Submanifolds and holonomy,* CRC Research Notes in Mathematics 434, Chapman & Hall, Boca Raton 2003

[Be-87] Besse, A.L., *Einstein manifolds,* Springer-Verlag, Berlin 1987

[Bi-98] Birkenhake, C., Complex structure contained in classical groups, *J. Lie Theory* **8**(1) (1998), 139 - 152

[Bou-02] Bourbaki, N., *Elements of Mathematics, Lie groups and Lie algebras, Chap. 4-6,* Springer-Verlag, Berlin 2002

[Bo-59] Bott, R., The stable homotopy of the classical groups, *Ann. Math. (2)* **70**(2) (1959), 313 - 337

[BS-58] Bott, R. & Samelson, H., Applications of the theory of Morse to symmetric spaces, *Amer. J. Math.* **80** (1958), 964 - 1029

[Bu-85] Burns, J. M., *Conjugate loci in compact symmetric spaces,* Ph.D. Thesis, University of Notre Dame, Notre Dame (Indiana) 1985

[Bu-92] Burns, J. M., Homotopy of compact symmetric spaces, *Glasg. Math. J.* **34** (1992), 221 - 228

[Bu-93] BURNS, J. M., Conjugate loci of totally geodesic submanifolds of symmetric spaces, *Trans. Am. Math. Soc.* **337**(1) (1993), 411 - 425

[BR-90] BURSTALL, F.E. & RAWNSLEY, J.H., *Twistor Theory for Riemannian Symmetric Spaces. With applications to harmonic maps of Riemann surfaces,* Lect. Notes Math. **1424**, Springer-Verlag, Berlin 1990

[Ca-27] CARTAN, É., Sur certains formes riemanniennes remarquables des géométries à groupe fondamental simple, *Ann. Sci. École Norm. Sup.,* 3*ième* *sér.* **44** (1927), 345 - 467

[CN-78] CHEN, B.-Y. & NAGANO, T., Totally geodesic submanifolds of symmetric spaces II, *Duke Math. J.* **45** (1978), 405 - 425

[CN-88] CHEN, B.-Y. & NAGANO, T., A Riemannian geometric invariant and its applications to a problem of Borel and Serre, *Trans. Am. Math. Soc.* **308**(1) (1988), 273 - 297

[Ch-89] CHEN, B.-Y., Symmetries of compact symmetric spaces, in *Geometry and topology of submanifolds, Proc. Meet. Luminy, Marseille 1987,* 38 - 48, World Scientific, Singapore 1989

[Co-65] CONLON, L., On the topology of EIII and EIV, *Proc. Am. Math. Soc.* **16**(4) (1965), 575 - 581

[Co-66] CONLON, L., An application of the Bott suspension map to the topology of EIV, *Pac. J. Math.* **19**(3) (1966), 411 - 428

[E-84] ESCHENBURG, J.-H., *Freie isometrische Aktionen auf kompakten Lie-Gruppen mit positiv gekrümmten Orbiträumen,* Schriftenr. Math. Inst. Univ. Münster, II. Ser. **32**, Münster 1984

[EH-95] ESCHENBURG, J.-H. & HEINTZE, E., Extrinsic symmetric spaces and orbits of s-representations, *Manuscr. Math.* **88**(4) (1995), 517 -524

[Fe-80] FERUS, D., Symmetric submanifolds of euclidean space, *Math. Ann.* **247** (1980), 81 - 93

[Fr-62] FREUDENTHAL, H., Bericht über die Theorie der Rosenfeldschen elliptischen Ebenen, in *Algebr. Topol. Foundations Geom., Proc. Colloq. Utrecht, August 1959*, 35 - 37, Utrecht 1962

[GHL-04] GALLOT, A.; HULIN, D. & LAFONTAINE, J., *Riemannian geometry,* 3rd edition, Springer, Berlin 2004

[Ha-02] HATCHER, A., *Algebraic topology,* Cambridge University Press, Cambridge 2002

[He-78] HELGASON, S., *Differenital Geometry, Lie groups and symmetric space,* Academic Press, San Diego 1978

[He-07] HELGASON, S., The inversion of the X-ray transform on a compact symmetric space, *J. Lie Theory* **17**(2) (2007), 307 - 315

[Hi-70] HIRZEBRUCH, U., Über eine Realisierung der hermiteschen, symmetrischen Räume, *Math. Z.* **115** (1970), 371 - 382

[Kl-09] KLEIN, S., Totally geodesic submanifolds of the complex and the quaternionic 2-Grassmannians., *Trans. Am. Math. Soc.* **361(9)** (2009), 4927 - 4967

[KNa-64] KOBAYASHI, S. & NAGANO, T., On filtered Lie algebras and geometric structures I, *J. Math. Mech.* **13(5)** (1964), 875 - 907

[KNo-63] KOBAYASHI, S. & NOMIZU, K., *Foundations of differential geometry, Vol. I,* Interscience Publishers, New York 1963

[KNo-69] KOBAYASHI, S. & NOMIZU, K., *Foundations of differential geometry, Vol. II,* Interscience Publishers, New York 1969

[Kol-09] KOLLROSS, A., Exceptional $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric spaces, *Pacific J. Math.* **242**(1) (2009), 113 - 130

[KW-65] KORÁNYI, A. & WOLF, J.A., Realization of hermitian symmetric spaces as generalized half-planes, *Ann. Math. (2)* **81**(2) (1965), 265 - 288

[Le-73] LEUNG, D.S.P., The reflection principle for minimal submanifolds of Riemannian symmetric spaces, *J. Differ. Geom.* **8** (1973), 153 - 160

[Le-74] LEUNG, D.S.P., On the classification of reflective submanifolds of Riemannian symmetric spaces, *Indiana Univ. Math. J.* **24**(4) (1974), 327 - 339

[Le-79a] LEUNG, D.S.P., Reflective submanifolds. III. Congruency of isometric reflective submanifolds and corrigenda to the classification of reflective submanifolds, *J. Differ. Geom.* **14** (1979), 167 - 177

[Le-79b] LEUNG, D.S.P., Reflective submanifolds. IV. Classification of real forms of hermitian symmetric spaces, *J. Differ. Geom.* **14** (1979), 179 - 185

[Li-58] LICHNÉROWICZ, A., *Géométrie des groupes de transformations,* Dunod, Paris 1958

[Lo-69-I] LOOS, O., *Symmetric spaces I: General theory,* W. A. Benjamin, New York 1969

[Lo-69-II] LOOS, O., *Symmetric spaces II: Compact Spaces and Classification,* W. A. Benjamin, New York 1969

[Mil-69] MILNOR, J., *Morse Theory,* Annals of mathematical studies **51**, 3^{rd} printing, Princeton University Press, Princeton 1969

[Mim-95] MIMURA, M., Homotopy theory of Lie groups, in *Handbook of algebraic topology* (ed. James, I.M.), 951 - 991, North-Holland, Amsterdam 1995

[MT-91] MIMURA, M. & TODA, H., *Topology of Lie groups I and II,*, Translations of Mathematical Monographs. **91**, American Mathematical Society, Providence 1991

[Mit-87] MITCHELL, S. A., The Bott filtration of a loop group, in *Lect. Notes Math.* **1298**, 215 - 226, Springer, Berlin 1987

[Mit-88] MITCHELL, S. A., Quillen's theorem on buildings and the loops on a symmetric space, *Enseign. Math., II. Sér.* **34** (1988), 123 - 166

[Nag-88] NAGANO, T., The involutions of compact symmetric spaces, *Tokyo J. Math.* **11**(1) (1988), 57 - 79

[Nag-92] NAGANO, T., The involutions of compact symmetric spaces. II, *Tokyo J. Math.* **15**(1) (1992), 39 - 82

[NS-91] NAGANO, T. & SUMI, M., The spheres in symmetric spaces, *Hokkaido Math. J.* **20**(2) (1991), 331 - 352

[NT-95] NAGANO, T. & TANAKA, M.S., The involutions of compact symmetric spaces. III, *Tokyo J. Math.* **18**(1) (1995), 193 - 212

[NT-99] NAGANO, T. & TANAKA, M.S., The involutions of compact symmetric spaces. IV, *Tokyo J. Math.* **22**(1) (1999), 193 - 211

[NT-00] NAGANO, T. & TANAKA, M.S., The involutions of compact symmetric spaces. V, *Tokyo J. Math.* **23**(2) (2000), 403 - 416

[Nai-86] NAITOH, H., Symmetric submanifolds of compact symmetric spaces, *Tsukuba J. Math.* **10** (1986), 215 - 242

[Ok-78] OKUMURA, M., Normal curvature and real submanifold of the complex projective space, *Geom. Dedicata* **7** (1978), 509 - 517

[Qu-06] QUAST, P. 'Spindles' in symmetric spaces, *J. Math. Soc. Japan* **58**(4) (2006), 985 - 994

[Ro-62] ROSENFELD, B.A., Einfache Lie-Gruppen und nichteuklidische Geometrien, in *Algebr. Topol. Foundations Geom., Proc. Colloq. Utrecht, August 1959*, 135 - 155, Utrecht 1962

[Sa-78a] SAKAI, T., On the structure of cut loci in compact Riemannian symmetric spaces, *Math. Ann.* **235** (1978), 129 - 148

[Sa-78b] SAKAI, T., Cut loci of compact symmetric spaces, in *Minimal submanifolds and geodesics*, Kaigai Publications, Tokyo 1978, 193 - 207

[Sa-96] SAKAI, T., *Riemannian Geometry,* Translations of Mathematical Monographs. **149**, American Mathematical Society, Procidence 1996

[Se-87] SERRE, J.-P., *Complex semisimple Lie algebras,* Springer-Verlag, Berlin 1987

[Sz-06] SZÖKE, R., On isometries of Kähler manifolds, *Acta Math. Hung.* **111** (2006), 77 - 79

[Tak-64] TAKEUCHI, M., On the fundamental group and the group of isometries of a symmetric space, *J. Fac. Sci., Univ. Tokyo, Sect. I* **10** (1964), 88 - 123

[Tak-84] TAKEUCHI, M., Stability of certain minimal submanifolds of compact hermitian symmetric spaces, *Tôhoku Math. J., II. Ser.* **36** (1984), 293 - 314

[Tak-86] TAKEUCHI, M., Totally complex submanifolds of quaternionic symmetric spaces, *Jap. J. Math., New Ser.* **12** (1986), 161 - 189

[Tan-95] TANAKA, M., Stability of minimal submanifolds in symmetric spaces, *Tsukuba J. Math.* **19**(1) (1995), 27 - 56

[To-62] TODA, H., *Composition methods in homotopy groups of spheres,* Annals of Mathematics Studies **49**, Princeton University Press, Princeton 1962

[Ts-85] TSUKADA, K., Parallel submanifolds in a quaternion projective space, *Osaka J. Math.* **22**(1) (1985), 187 - 241

[WZ-93] WANG, M. & ZILLER, W., Symmetric spaces and strongly isotropy irreducible spaces, *Math. Ann.* **296**(2) (1993), 285 - 326

[Wo-65] WOLF, J.A., Complex homogeneous contact manifolds and quaternionic symmetric spaces, *J. Math. Mech.* **14**(6) (1965), 1033 - 1047

[Wo-84] WOLF, J.A.,*Spaces of constant curvature,* 5^{th} edition, Publish or Perish, Wilmington (Delaware) 1984