The Metal-Insulator Transition in Disordered Systems

Dieter Vollhardt*

Max-Planck-Institut für Physik und Astrophysik

- Werner Heisenberg Institut für Physik -

Föhringer Ring 6

8000 München 40

Federal Republic of Germany

---

*Heisenberg-Fellow of the Deutsche Forschungsgemeinschaft

Appeared in: SOLID STATE PHYSICS, Lecture Notes of the NORDITA Spring School, Tvärminne, Finland, April 1984; Vol. I, p. 73 (Nordita, Kopenhagen)

CONTENTS

Foreword

I. Introduction

II. Noninteracting, Quantum Mechanical Particles in Disordered Systems

Theoretical Description of Disorder
The "Weakly" Localized Regime
Diffusion of Classical and Quantum Mechanical Particles
Systematic Calculation of Corrections to the Conductivity
The Influence of a Magnetic Field on Localization
The Anderson Transition
Scaling Theory
The Method of Selfconsistency

III. Interacting, Quantum Mechanical Particles in Disordered Systems

Estimating the Corrections
The Metal-Insulator Transition
Experimental Results and Theoretical Conjectures

References

Foreword

These notes on the physics of the metal-insulator transition in disordered systems are based on a series of lectures which the author gave at the "NORDITA Spring School on Condensed Matter Theory" at Tvärminne(Finland) in April 1984. Nevertheless, the detailed manuscript presented here was only worked out most recently in the course of my preparation for the "Spring School on Disordered Systems" at Jülich in March 1985.

The lectures address graduate students and, in general, non-specialists. Therefore no previous experience with the matter is assumed (except for a general familiarity with the basics of quantum mechanics and condensed matter physics). One of the focal points of the lectures is a detailed ("anschauliche") discussion of the scattering effects responsible for localization and interaction effects as worked out by the Russian school. Quite generally no knowledge of advanced theoretical methods (e.g. Green's functions, Feynman diagrams etc.) is expected. In the two instances where diagrams are actually used they appear as a "sign-language" to illustrate the use of perturbation theory and the concept of time reversal invariance. (It seemed to be important to at least mention these ideas on which a large part of the

actual research is based.) Within this task it was unfortunately not possible to discuss any of the field-theoretical methods which have proven to be of fundamental importance for the development of the subject. Instead a detailed discussion of the scaling theory by Abrahams et al. is given.

These lecture notes do not present a review of the subject. They only attempt to be an introduction — a fact which is certainly reflected in the number of references. Furthermore several topics and techniques have not been mentioned at all, while others have been discussed at greater length. This does not reflect the author's opinion about the importance of those methods and results that cannot be found here, but only the author's capability of presenting the matter within a given frame of depth and length.

Finally I would like to express my deep gratitude to all my friends and colleagues, who helped me to learn about localization and the metal-insulator transition. In particular, I am grateful to Elihu Abrahams and Peter Wölfle for many inspiring and enjoyable discussions.

I. Introduction

A metal-insulator transition separates two physical regions, which fundamentally differ in their electrical dc-conductivity $\sigma(\omega = 0)$: while a metal has a finite dc-conductivity $(\sigma(0) > 0)$, an insulator is characterized by $\sigma(0) = 0$.

Such a transition does not only occur in disordered systems. In fact, it can also take place in very clean, purely crystalline materials, in which the particles that are responsible for the current, interact via a Coulomb-interaction. In such systems an overlap of energy bands can lead to a metal-insulator transition. Yet another cause for a transition may be due to a structural change of a crystal leading to a new lattice periodicity. Indeed there exist completely different physical reasons for the occurrence of a metal-insulator transition [1]. We will here confine ourselves to the metal-insulator transition in disordered systems, i.e. where the "disorder" is ultimately responsible for the transition. By "disorder" we mean, e.g. the disturbance of a strict lattice periodicity due to impurities or defects. Our understanding of the physics of disordered systems, in particular of their transport properties and the metal-insulator transition, has greatly changed and substantially deepened in the last 5-6 years [2,3].

In the first part of these lectures we will concentrate on the behavior of non-interacting, quantum mechanical particles in a disordered environment. In the second part we will also take into account the effects of a mutual interaction between the particles.

II. Non-Interacting, Quantum Mechanical Particles in Disordered Systems

Of three-dimensional systems (i.e. those with space dimension $d = 3$) we know, that at temperatures $T = 0$ there exist neither lattice vibrations (phonons) nor any inelastic processes. In fact, in an unbounded, perfect lattice (Fig. 1a) no scattering occurs at all. This is a consequence of quantum mechanics. Consequently such a system has an infinite dc-conductivity. One may equally well say, that the characteristic collision time τ of the particles due to scattering off defects etc. is infinitely long ($\tau = \infty$). In the case of Fermions (and those we only consider here) it follows, that the mean free path $\ell = v_F\tau$ of the particles is also infinite ($\ell = \infty$); here v_F is the Fermi-velocity. The wavefunction of a particle is then characterized by a strict spatial phase coherence.

In a lattice, which is weakly disturbed by impurities or defects (Fig. 1b), the situation is different:

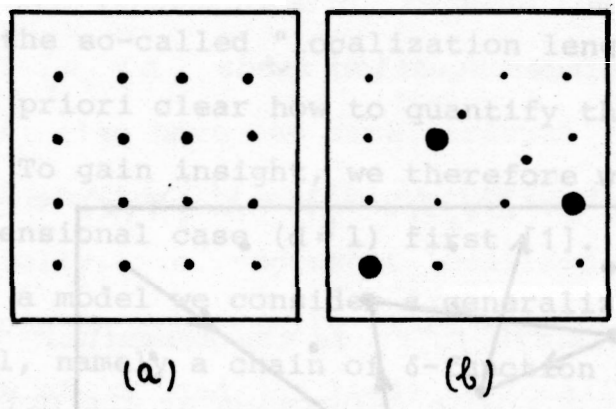


Fig. 1 : (a) perfect lattice, (b) imperfect lattice (presence of defects, impurities etc.)

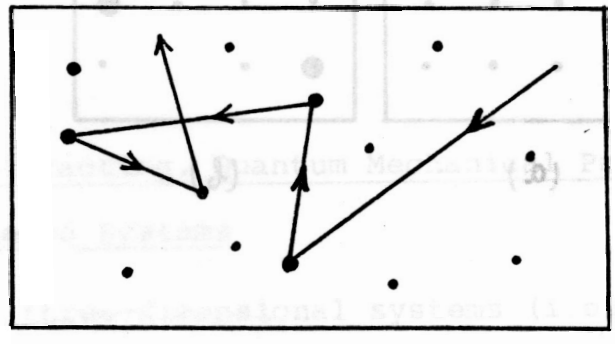
the scattering of the particles off the defects implies a finite collision time τ and thereby a finite mean free path ℓ . The dc-conductivity is therefore also finite and is given by

$$\sigma(0) = \sigma_0 = \frac{e^2 n}{m} \tau \qquad (1)$$

where e and m are the charge and the mass of the particles (e.g. electrons), respectively and n is the density. The quantity σ_0 is often called "Boltzmann - conductivity", because (1) is a direct result of the

Boltzmann transport theory.

The scattering of the particles leads to <u>diffusion</u> (Fig. 2), i.e. to a diffusive motion. The phase coherence of the wave functions is thereby limited;



<u>Fig. 2</u> : Diffusion of a particle in a disordered system

nevertheless the wavefunction $\psi(\vec{r})$ of the particle is still <u>extended</u>, i.e. one has

$$\lim_{r \to \infty} |\psi(\vec{r})|^2 \neq 0 \qquad (2)$$

When the disorder is increased (e.g. by choosing a higher impurity concentration) it may happen that the wave function becomes <u>localized</u>, such that

$$|\psi(r)|^2 \propto e^{-r/\xi}$$

where ξ is the so-called "localization length". However, it is not a priori clear how to quantify this vague statement. To gain insight, we therefore want to discuss the one-dimensional case (d = 1) first [1].

As a model we consider a generalized Kronig-Penney model, namely a chain of δ-function potentials (Fig. 3) at locations x_i with strength V_i and

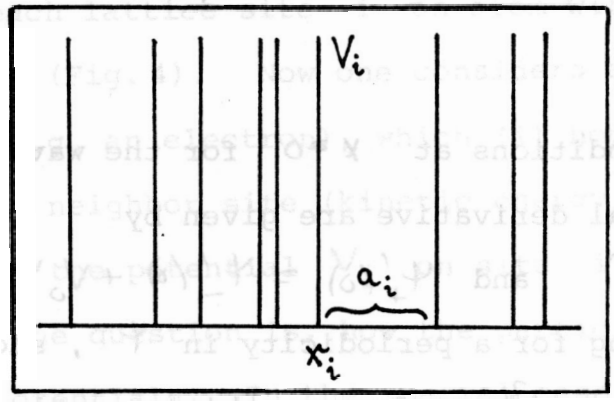


Fig. 3 : One-dimensional model of a disordered system

separations a_i . The potential is then given by

from which $\tilde{k}$ and a can be obtained as functions of k. For a chain of potentials we may conclude from (6) that a wave function will only be periodic (and the state extended) if the separations a_i between the potentials are <u>all</u> equal $(a_i = a)$ and if, at the same time, <u>all</u> potentials V_i also have the same strength $(V_i = V)$. Otherwise all physically sensible wave functions fall off exponentially, i.e. represent localized states. This means that even an arbitrarily small statistical spread of the values of a_i and V_i leads to localized solutions of the Schrödinger equation only.

The first quantitative definition of "localization" dates back to Anderson in 1958 [4]. He investigated a three-dimensional model, namely a regular point lattice, where on each lattice site i an atom with an energy V_i was located (Fig. 4). Now one considers a quantum mechanical particle (e.g. an electron), which (i) hops from one site to the next neighbor site (kinetic energy) and (ii) experiences the potential V_i on site i (potential energy). The question is, how the particles are influenced by these potentials. In the special case that all V_i are equal $(V_i = V)$ one of course obtains a sharply bounded energy <u>band</u>, whose width we characterize by an energy B. This situation is changed when the V_i are statistically distributed (Fig. 5), e.g. with a rectangular distribution

$$P(V_i) = \begin{cases} 1/W, & |V_i| \leq W/2 \\ 0, & \text{else} \end{cases} \tag{7}$$

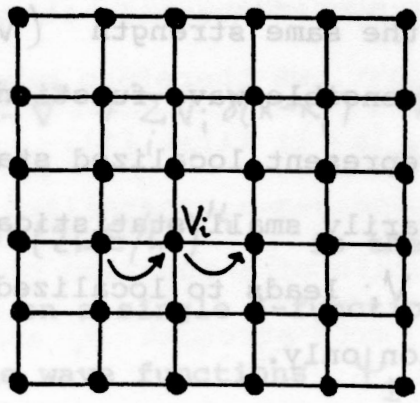


Fig. 4 : Disorder model due to Anderson

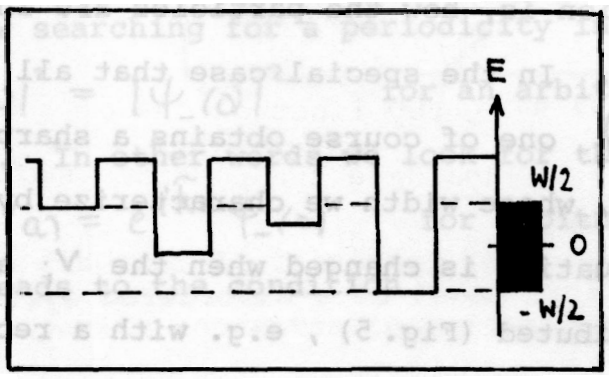


Fig. 5 : Example of the energy distribution of the atoms

For an unbounded system one may now ask: How large is the probability P for a particle to return to a particular lattice site in the limit of long times $(t \to \infty)$? In the case $P = 0$ the particle has "disappeared" in the system; hence it will be characterized by an extended state. In the case $P > 0$, i.e. finite return probability, we speak of "localization", i.e. the particle is described by a localized state.

The answer to the above question depends on the ratio of the maximal energy difference of the atoms, W, to the band width B of the undisturbed system, i.e. on W/B. This number (!) is a measure of the strength of energy fluctuations in the system, i.e. a measure of the <u>disorder</u>. The quantitative answer [4] is, that for $W/B \lesssim 5$ one finds $P = 0$ while for $W/B \gtrsim 5$ one obtains $P > 0$. In between there is a <u>sharp</u> transition, the "Anderson transition". In other words: if the energy E of the particle lies below a certain critical energy E_c (which is determined by the strength of the disorder, $(W/B)_c \approx 5$), it is localized, while for $E > E_c$ the energy fluctuations of the system will not be able to dominate the particle such that it is described by an extended state. In the first case one deals with an insulator, in the second one with a metal. Since electrons in a metal have a characteristic energy E_F (Fermi energy), the Anderson transition may be induced by changing E_F.

What does the transition actually look like? According to Mott [5] it originally was supposed to be <u>discontinuous</u>. This expectation was based on the following arguments. The dc-conductivity σ_0 (1) of a d-dimensional system can be written as

$$\sigma_0 = \left(\frac{e^2}{\hbar}\right) \frac{a^{2-d}}{\pi} \frac{\ell}{a} = \sigma_c \cdot \frac{\ell}{a} \qquad (8)$$

where we made use of $\ell = v_F \tau$, $v_F = \hbar k_F / m$ and where the density of a Fermi gas has been expressed as $n = a^{-d}$ (here k_F is the Fermi wave number which is connected with a , the average distance of the particles, by $k_F \approx \pi/a$) . The conductivity σ_c is an essentially universal quantity — it is independent of the disorder in the system. The disorder-dependence only enters via the mean free path ℓ , i.e. the ratio ℓ/a . When the disorder increases, ℓ and hence σ_0, decrease. On the other hand, (1) and (8) have been derived within the Boltzmann transport theory. So, for these equations to be valid at all, ℓ always has to be greater than the average particle distance a (Ioffe-Regel criterion); shorter ℓ makes no sense. Mott therefore postulated a "minimal metallic conductivity" σ_{min} , which is essentially given by σ_c (more precisely, $\sigma_{min} = c\,\sigma_c$, where $c \approx 0.08 - 0.3$ is a nonuniversal constant which is due to a reduction of the density of

states of the electrons at the Fermi surface compared to the Fermi gas). He argued that – as the metal-insulator transition was approached from the metallic side — the conductivity σ_0 would decrease proportional to ℓ and then, at $\ell = a$, would drop discontinuously from σ_{min} to zero (Fig. 6). However, already at this point

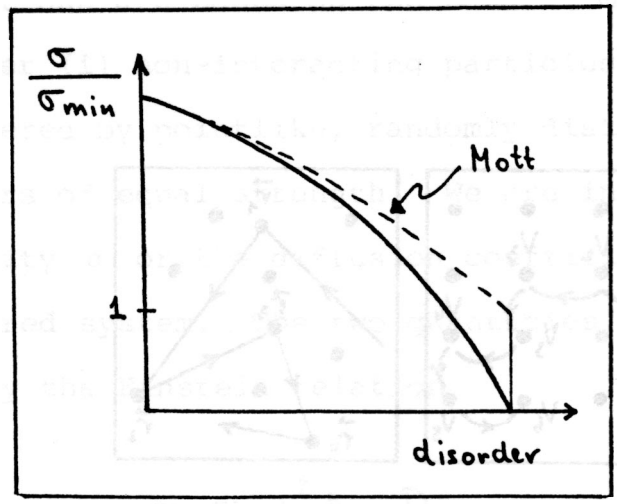


Fig. 6: Possible shape of the conductivity curve as a function of the disorder.

we should like to mention that this concept does not hold — in spite of many experimental results which appeared to support it for a long period of time. Indeed, in the last 3-4 years low temperature experiments have

measured metallic conductivities much below σ_{min} and have thereby falsified the concept of a minimal metallic conductivity [6]. We will later come back to that problem.

Theoretical Description of Disorder

In the model originally discussed by Anderson, a particle moves on a regular lattice, the energy of the lattice points being statistically distributed (Fig. 7a).

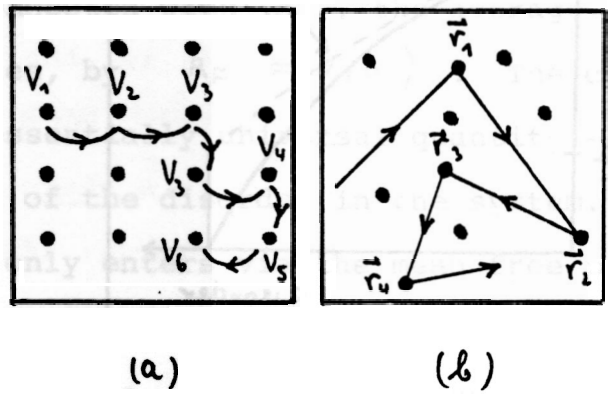


Fig. 7: Disorder and motion of a particle in the model of (a) Anderson, (b) Edwards.

The "disorder" is then exclusively due to the energy state of the lattice sites. An alternative model goes back to Edwards [7]. In his model particles of identical

energy are scattered off randomly distributed scattering centers of equal potential (Fig. 7b). The disorder is then due to the spatial distribution of the scatterers. While the first model starts from the localized regime, the starting point of the Edwards model is the regime of extended states (weak scattering). The latter model is particularly suitable for the formulation of a systematic perturbation theory which starts from the undisturbed, metallic regime and then includes a small impurity concentration, i.e. weak disorder. The following investigations are based on the Edwards model. For this we consider (i) non-interacting particles, which (ii) are scattered by pointlike, randomly distributed scattering centers of equal strength. We are interested in the conductivity σ or the diffusion coefficient D of such a disordered system. The two quantities are actually related by the Einstein relation

$$\sigma = e^2 N_F D \tag{9}$$

where N_F is the density of states at the Fermi surface.

We will measure the disorder by a dimensionless parameter λ with $\lambda \propto n_i V_0^2$, i.e. λ is essentially given by the impurity concentration n_i and the scattering strength V_0^2 of the scatterers. The parameter is often called "coupling constant".

In the case $d = 1$ we saw that even arbitrarily weak disorder $(\lambda \to 0)$ leads to localization. The "critical" value of the disorder, λ_c, above which one finds localization is therefore given by $\lambda_c = 0$. On the other hand, in the case $d = 3$ λ_c is different from zero $(\lambda_c > 0)$. For $\lambda < \lambda_c$ one finds metallic behavior, which is described by a finite dc-conductivity σ_0 (or diffusion constant D_0). For $\lambda > \lambda_c$ one deals with an insulator. Here $\sigma_0 = 0$; instead, an insulator has a finite polarizability, i.e. dielectric constant ϵ_0. At $\lambda = \lambda_c$ the Anderson transition is located.

Apparently, the case $d = 2$ (very thin films) marks a marginal dimension. The question is now, whether there is an Anderson transition or not in two dimensions, i.e. whether $\lambda_c = 0$ or $\lambda_c > 0$. In other words, one may ask an almost trivially sounding question: "Is there metallic conductivity in very thin films at $T = 0$?" Or is there, for example, a minimal metallic conductivity? The answer to this seemingly simple question has only been found in recent years. It led to unexpected insights into the physics of disordered systems and their transport properties.

The "Weakly" Localized Regime

We first consider the case of very weak disorder $(\lambda \ll 1)$. Therefore the starting point is the metallic regime. We want to understand how a small concentration

of impurities affects the metallic behavior. Since we are far away from the actual Anderson transition itself, these effects can be treated by means of a perturbation theory in the disorder parameter $\lambda \ll 1$. Weak disorder means that the mean free path ℓ is much greater than the average particle distance $a \sim k_F^{-1}$, i.e. $k_F \ell \gg 1$ or, equivalently, $E_F \tau \gg 1$. We will therefore choose

$$\lambda = \frac{1}{2\pi E_F \tau} \tag{10}$$

as our (small) perturbation parameter. Starting from the metallic regime we want to consider the precursor effects of localization, i.e. the corrections $\delta\sigma$ to the metallic conductivity

$$\sigma = \sigma_0 + \delta\sigma \quad , \quad |\delta\sigma| \ll \sigma_0 \tag{11}$$

These perturbational effects are commonly called "weak localization". Our aim is to calculate $\delta\sigma = f(L, \omega, T, H)$ as a function of several external parameters like the system's size L , the frequency ω , the temperature T or the magnetic field H .

Diffusion of Classical and Quantum Mechanical Particles

As mentioned before, the conductivity σ_0 ,(1)

$$\sigma_0 \propto \frac{1}{\lambda} \tag{12}$$

is a result of the Boltzmann transport theory. In this theory consecutive collisions of particles are assumed to be independent of each other, i.e. collisions are uncorrelated. This implies that multiple scattering of a particle at a particular scattering center is not taken into account. Consequently, if there is a finite probability for the repeated occurrence of such multiple scatterings, the basic assumption of the independence of scattering events breaks down and the validity of the result for σ_0 in (1) becomes, at least, questionable.

To investigate this fundamental point we consider the diffusive behavior of a particle in a d-dimensional disordered system. Let the particle be located at $\vec{r}_0$ at time $t=0$ (Fig. 8a). Due to its diffusive motion the particle moves away from $\vec{r}_0$. At some later time t we will only be able to make a probability statement about its position: it will be located within some smooth volume (Fig. 8b) whose size is determined by the probability distribution $P(\vec{r},t)$, which is the solution of the diffusion equation

$$\frac{\partial P}{\partial t} - D_0 \nabla^2 P = 0 \tag{13}$$

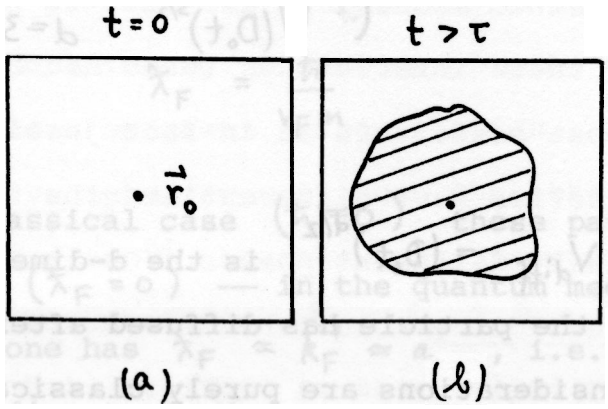


Fig. 8: Probability distribution of a diffusing particle: (a) $t=0$, (b) $t>\tau$.

The diffusion constant D_0 is given by $D_0 = v_F^2\tau/d$, where v_F is the characteristic velocity of the particles. The explicit solution of (13) is given by

$$P(\vec{r},t) = \frac{e^{-\frac{|\vec{r}-\vec{r}_0|^2}{4Dt}}}{(4\pi D_0 t)^{d/2}} \tag{14}$$

At times $t \gg \tau$ the exponential in (14) is unimportant, so that

$$P(\vec{r},t) \propto \frac{1}{V_{diff}} = \begin{cases} 1/(D_0 t)^{1/2}, & d=1 \\ 1/D_0 t, & d=2 \\ 1/(D_0 t)^{3/2} & d=3 \end{cases} \tag{15}$$

We observe that $V_{diff} = (D_0 t)^{d/2}$ is the d-dimensional volume into which the particle has diffused after time t.

These considerations are purely classical. To understand the differences in the diffusive behavior of classical and quantum mechanical particles, we take a look at the path of a particle diffusing from point A to point B (Fig. 9) [8].

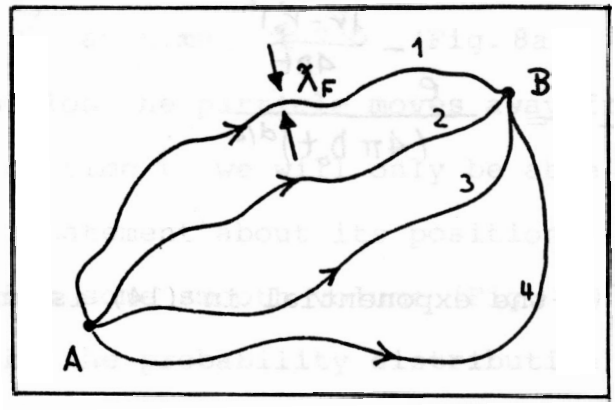


Fig. 9: Possible paths of a particle diffusing from A to B.

This transport can take place via different trajectories (in Fig. 9 four examples are shown). The trajectories, or "tubes", have a typical width given by the Fermi wavelength

$$\lambda\!\!\!^{-}_F = \frac{\hbar}{v_F m} \tag{16}$$

In the classical case $(\hbar = 0)$ these paths are arbitrarily sharp $(\lambda\!\!\!^{-}_F = 0)$ — in the quantum mechanical case, however, one has $\lambda\!\!\!^{-}_F \simeq k_F^{-1} \approx a$, i.e. the tubes have a finite diameter. We now assume that (i) the disorder λ is very small $(\lambda_F/\ell \propto \lambda \ll 1)$ and that (ii) the temperature is low enough such that inelastic processes, characterized by an inelastic scattering time τ_{in} , occur only very rarely $(\tau_{in} \gg \tau)$.

Since the transport from A to B may take place along different trajectories, there is a probability amplitude A_i connected to every path i . The total probability W to reach point B from A is then given by the square of the magnitude of the sum of all amplitudes:

$$W = \left|\sum_i A_i\right|^2 \tag{17a}$$

$$= \sum_i{}' |A_i|^2 + \sum_{i \neq j}{}' A_i A_j^* \tag{17b}$$

The first term in (17b) describes separate, i.e. non-

interfering paths — this is the classical case, in which the tubes are infinitely sharp. On the other hand, the second term represents the contribution due to interference of the path-amplitudes,which is therefore an exclusively quantum mechanical effect. In the Boltzmann theory these interference terms had been neglected. In most cases this is in fact justified: since the trajectories have different lengths the amplitudes A_i carry different phases. On the average this leads to destructive interference. Hence the quantum mechanical interferences in Fig. 9 are generally unimportant.

There is, however, one particular exception to this conclusion, namely if point A and B coincide (Fig. 10)!

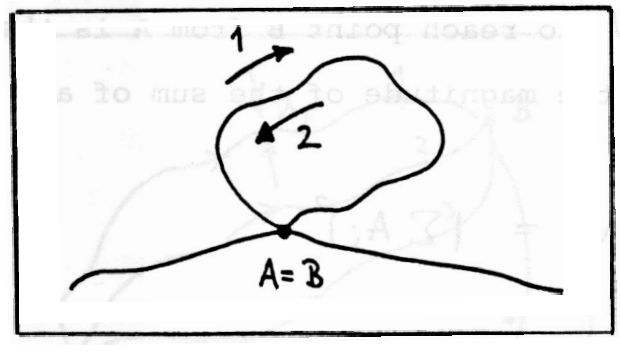


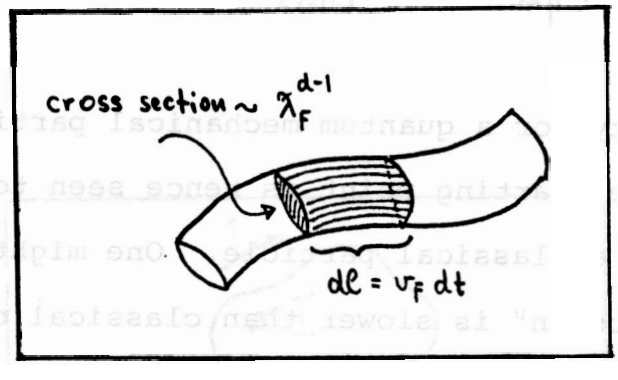
Fig. 10: Return of a particle to its starting point.

In this case starting-point and end-point are identical, such that the path in between can be traversed in two opposite directions: forward and backward. The probability to go from A to B is then nothing but the <u>return</u>-probability to the starting-point. Since paths 1 and 2 in Fig. 10 are equal, the Amplitudes A_1 and A_2 have a coherent phase relation. This leads to <u>constructive</u> interference, such that the quantum mechanical contribution to W becomes very important. Eq.(17b) then tells us that for $A_1 = A_2 \equiv A$ the classical return probability (due to the neglect of the interference terms) is given by $W_{class} = 2|A|^2$, while the quantum mechanical case yields $W_{qm} = 2|A|^2 + 2A_1A_2^* = 4|A|^2$. Hence one obtains

$$W_{qm} = 2W_{class} \tag{18}$$

The probability for a quantum mechanical particle to return to some starting point is hence seen to be <u>twice</u> that of a classical particle. One might say "quantum-diffusion" is slower than classical diffusion because in the first case there exists a more effective back scattering effect. In other words: quantum mechanical particles in a disordered medium are (at low temperatures) less mobile than classical particles. This in turn leads to a correspondingly lower conductivity σ.

To estimate this effect on σ we consider the change $\delta\sigma$ relative to the metallic conductivity σ_0, i.e. $\delta\sigma/\sigma_0$. Because of the expected <u>lowering</u> of σ, the sign of $\delta\sigma/\sigma_0$ will be negative. Furthermore, the change will be proportional to the probability that during diffusion a closed path as in Fig. 10 occurs at all. (This is the probability to find a particle in a closed tube, i.e. the probability for the trajectory to intersect itself during the diffusion.) Let us therefore have a look at a d-dimensional tube (Fig. 11) with diameter λ_F, i.e. cross-section λ_F^{d-1}. During



<u>Fig. 11</u>: Enlarged section of a quantum mechanical trajectory.

the time interval dt the particle moves a distance $dl = v_F\, dt$, such that the corresponding volume-

element of the tube is given by $dV = v_F\, dt\, \lambda^{d-1}$. On the other hand, the maximally attainable volume for the diffusing particle is given by (15), i.e. by $= (D_0 t)^{d/2}$. The above mentioned probability for a particle to be in a closed tube is therefore given by the ratio of these two volumes. We find

$$W = \int_{\tau}^{\tau_{in}} \frac{dV}{V_{diff}} = v_F \lambda_F^{d-1} \int_{\tau}^{\tau_{in}} \frac{dt}{(D_0 t)^{d/2}} \qquad (19)$$

where we have integrated over all times $\tau \leq t \leq \tau_{in}$: τ is the microscopic time for a single elastic collision (shorter times are not sensible), while τ_{in} is the shortest inelastic relaxation time in the system. It determines the maximal time during which coherent interference of the path-amplitudes is possible. Because of $D_0 \propto 1/\lambda$ and $\lambda_F \sim \hbar$ we obtain

$$\frac{\delta\sigma}{\sigma_0} \propto -\lambda \times \begin{cases} (\tau_{in}/\tau)^{1/2}, & d=1 \\ \hbar \ln (\tau_{in}/\tau), & d=2 \\ \hbar^2 (\tau_{in}/\tau)^{-1/2}, & d=3 \end{cases} \qquad (20)$$

If we assume, that for $T \to 0$ the inelastic relaxation rate vanishes with some power of T, i.e. $1/\tau_{in} \propto T^p$, where p is a constant, (20) is given by

$$\frac{\delta\sigma}{\sigma_0} \propto -\lambda \begin{cases} T^{-p/2}, & d=1 \\ \hbar \frac{p}{2} \ln\left(\frac{\hbar/\tau}{k_B T}\right), & d=2 \\ \hbar^2 T^{p/2}, & d=3 \end{cases} \tag{21}$$

We observe the following: (i) the conductivity decreases for decreasing temperature, (ii) the relative correction $\delta\sigma/\sigma_0$ is linear in the disorder parameter $\lambda \ll 1$ (lowest order in λ), (iii) except for $d=1$, these corrections are of quantum mechanical origin, i.e. they disappear for $\hbar \to 0$. (In the case $d=1$ the "tube" in Fig. 11 has no finite diameter — just as in the classical situation; furthermore, since in $d=1$ there is only forward and backward scattering all paths are trivially "closed".)

In $d=2$ one therefore obtains a <u>logarithmic</u> temperature dependence of the conductivity correction $\delta\sigma$. We note that the elastic scattering due to the disorder in principle leads to a <u>divergent</u> temperature behavior of $\delta\sigma$ in $d \leq 2$. For the initial assumption $|\delta\sigma| \ll \sigma_0$ to remain valid, the results in (21) for $d \leq 2$ may therefore not be used at too low temperatures. In particular, (21) does not allow to draw conclusions about $\delta\sigma$ at exactly $T=0$!

Finally, it should be pointed out that the results in (21) have been derived with a tacit assumption:

namely that the path-amplitudes A_1 and A_2 in Fig. 9 were perfectly coherent $(A_1 = A_2)$. So we actually took for granted that particle states with momentum $\vec{k}$ and $-\vec{k}$ are equivalent. Such a "time reversal invariance" is given if there are no external magnetic fields and if the impurities are non-magnetic. Otherwise a new situation arises which we will discuss later.

We should like to stress once more, that (20) and (21) are based on the explicit consideration of backscattering effects, i.e. multiple scattering and the correlation of consecutive collisions. Thus they cannot be obtained within the framework of the Boltzmann transport theory. Also CPA ("coherent potential approximation" [9]), an almost classical approximation method in problems involving disorder, is not able to obtain these results because it makes similar assumptions as the Boltzmann theory ("single site approximation", etc.).

Systematic Calculation of Corrections to the Conductivity

The arguments leading to (21) already contain the essential physics. They enabled us to understand the temperature dependence of $\delta\sigma$. On the other hand, we cannot deduce more than proportionality relations from them. For example, the precise prefactors in (20) can thereby not be determined. Furthermore, the perturbation theory cannot be extended beyond first order in λ.

For this we need a systematic approach. This involves calculational methods which can only be formulated within a certain mathematical framework which we cannot discuss here. To obtain a precise calculation of $\delta\sigma$ we will therefore "use" methods without really explaining their theoretical background. The necessary concepts and terms will thus only be mentioned by appealing to their plausibility.

In spite of all quantum mechanical effects the functional form of the results expressed in (21) are due to the diffusive behavior of the particles. Their probability distribution $P(\vec{r},t)$, (14), is determined by the diffusion equation (13). Fourier-transforming (13), or $P(\vec{r},t)$, leads to $P(\vec{q},\omega)$. Using (13) plus proper boundary conditions one can easily convince oneself that one finds

$$P(\vec{q},\omega) \propto \frac{1}{-i\omega + D_0 q^2} \tag{22}$$

This is called a "diffusion pole", because $P(\vec{q},\omega)$ diverges for $q, \omega \to 0$. Its origin is exclusively due to particle conservation during the diffusion.

The knowledge of $P(\vec{r},t)$, which is a local quantity describing a density distribution, or of $P(\vec{q},\omega)$ is not sufficient, however, if we want to know a dynamical quantity like the conductivity

or the diffusion coefficient $D(\vec{q},\omega)$. For this one needs a more general function, e.g. the so-called "density-density correlation function" $\chi(\vec{r},t)$. It describes the dependence of a density distribution at one point in space and time on that at some other point in space and time. Its Fourier transform has the general structure [10]

$$\chi(\vec{q},\omega) = \frac{D(\vec{q},\omega)\,q^2}{-i\omega + D(q,\omega)\,q^2}\,\chi(\vec{q},0) \qquad (23)$$

i.e. it also has a diffusion pole, where $D(\vec{q},\omega)$ is now a q and ω-dependent diffusion coefficient. Once we know $\chi(\vec{q},\omega)$ we of course also know $D(\vec{q},\omega)$. This quantity is therefore of fundamental importance for a systematic calculation of corrections $\delta\sigma$, δD to σ_0, D_0 due to the impurity scattering. It can be derived within a perturbation theory using a diagrammatic "sign language" (Feynman diagrams) [11]. In this way the microscopic scattering processes are described graphically: (i) the motion of a particle is characterized by a line with an arrow ("propagator"), while (ii) the scattering at an impurity is symbolized by a dashed line with a cross. An example is shown in Fig. 12. The upper line, pointing to the right, describes a "particle" with energy $E_F+\omega$ and momentum $\vec{p}+\vec{q}/2$, while the lower line, pointing to the left, describes

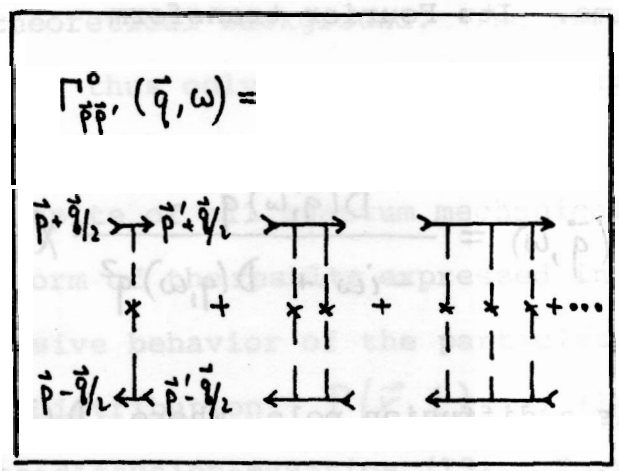


Fig. 12: Particle-hole ladder diagrams describing diffusion.

a "hole" with energy E_F and momentum $\vec{p} - \vec{q}/2$. In this way the density-density correlation of a particle-hole pair due to the usual diffusion (namely a sequence of independent collisions) is described by means of intuitively appealing diagrams. It should be pointed out that the dashed line describes only a correlation and not an interaction (which so far has not been considered at all) between particle and hole. The arrows carry the initial and final momenta of the particle and the hole.

The diagrams in Fig. 12 can easily be summed up. In the case of small energy - and momentum differences between particles and holes (i.e. $\omega \ll E_F$, $|\vec{q}| \ll p_F = \hbar k_F$) one finds

$$\Gamma^{0}_{\vec{p}\vec{p}'}(\vec{q},\omega) = \frac{V_0 \tau}{-i\omega + D_0 q^2} \tag{24}$$

where V_0 is the potential energy of the scatterers. So, as expected, we have obtained a diffusion pole.

To understand the properties of the correlation leading to (24) we note that a particle-hole pair with small energy - and momentum difference can be graphically represented in momentum space as shown in Fig. 13.

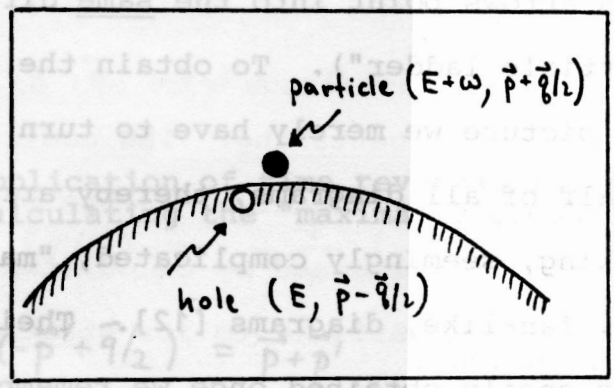


Fig. 13: Particle-hole excitation at the Fermi surface

The partners of such a pair always propagate in the same direction, experience the same collisions and are hence correlated for all time. This is a consequence of particle number conservation and this is the reason why the diffusion pole (24) always exists — even in the case of inelastic processes.

In the presence of time reversal invariance (which, by the way, does not refer to a global invariance of the whole system but to that of single-particle states!), particle states with momenta $\vec{k}$ and $-\vec{k}$ are equivalent. It means that we may invert the momentum of one of the arrows ("propagators"), i.e. replace $\vec{k}$ by $-\vec{k}$, without changing anything. To understand its consequences we now consider the particle-hole ladder in Fig. 11: we first turn around all the lower propagators and at the same time invert all the momenta as mentioned $(\vec{p} - \vec{q}/2 \rightarrow -\vec{p} + \frac{\vec{q}}{2}$, etc). In this way (Fig. 14) we again arrive at a diagrammatic ladder but this time both arrows point into the same direction ("particle-particle ladder"). To obtain the usual particle-hole picture we merely have to turn around the total lower half of all diagrams, thereby arriving at strangely looking, seemingly complicated, "maximally crossed", i.e. fan-like, diagrams [12]. Their contribution [13] is easily obtained once we remember that they originated from a particle-hole ladder (24) in which, however, the momentum transfer is now given by

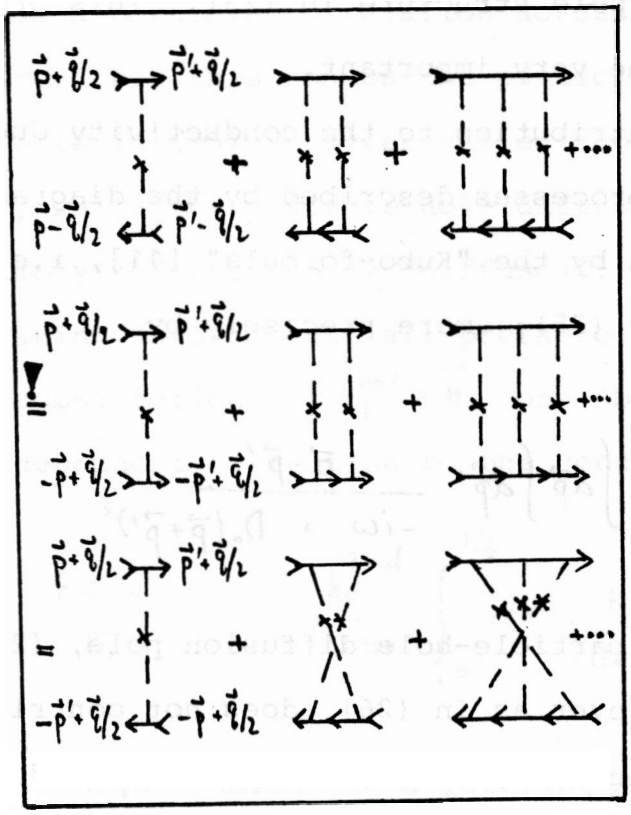


Fig. 14: Application of time reversal invariance for calculating the "maximally crossed" diagram.

$(\vec{p}+\vec{q}/2) - (-\vec{p}'+\vec{q}/2) = \vec{p}+\vec{p}'$ instead of $\vec{q}$ [14].

Their sum $\Lambda^{o}_{\vec{p}\vec{p}'}(\vec{q},\omega)$ is then found to be

$$\Lambda^{o}_{\vec{p}\vec{p}'}(\vec{q},\omega) = \frac{V_o\tau}{-i\omega + D_o(\vec{p}+\vec{p}')^2} \tag{25}$$

The essential difference to the particle-hole diffusion pole in (24) is, that the diffusion pole in (25) is no longer due to particle number conservation. This implies that, for example, inelastic processes (during which particles change their energy state) will destroy ("cut off") the pole structure in (25). This property will later become very important.

The contribution to the conductivity due to the scattering processes described by the diagrams in Fig. 14 is given by the "Kubo-formula" [11], i.e. by an integral over (25); more precisely by

$$\delta\sigma \propto \int d\vec{p} \int d\vec{p}\,' \; \frac{\vec{p}\cdot\vec{p}\,'}{-i\omega + D_0(\vec{p}+\vec{p}\,')^2} \tag{26}$$

(Note, that the particle-hole diffusion pole, (24), when integrated over as in (26), does not contribute because the integrand is then odd in $\vec{p}$ and $\vec{p}\,'$! The main contribution of the integrand in (26) comes from the region $\vec{p}+\vec{p}\,' \approx 0$, i.e. $\vec{p}\,' = -\vec{p}$ and is hence due to <u>backscattering</u>. The contribution considered here is in fact identical to one previously discussed, which we had obtained by calculating the return probability of a particle to its starting point. Clearly, a pronounced back scattering will favor localization. Since all scattering processes take place very close to the Fermi-surface $(|\vec{p}| \approx \hbar k_F)$

backscattering means that one deals with a momentum change of $2\hbar k_F$. One therefore often speaks of "$2k_F$ - scattering". Here we encounter an essential similarity with Cooper pairing in superconductors, where particles with momenta $\vec{k}$ and $-\vec{k}$ couple, i.e. where one also has a particle correlation across the whole Fermi sphere. For this reason the particle-particle diffusion pole in (25) is often called "Cooper-pole" or "Cooperon", while the particle-hole diffusion pole in (24) is simply referred to as "diffuson".

Taking into account all prefactors in (26) and using the substitution $\vec{p}-\vec{p}\,' = \vec{k}$ one obtains a frequency dependent correction to the conductivity [15]

$$\frac{\delta\sigma(\omega)}{\sigma_0} = -\frac{k_F^{2-d}}{\pi m}\int_0^{1/\ell} dk\, \frac{k^{d-1}}{-i\omega + D_0 k^2} \tag{27}$$

The negative sign, indicating a lowering of the conductivity, is due to the minus sign in $\vec{p}\,' = -p + k$, i.e. is due to the back scattering. In $d=2$ one finds

$$\delta\sigma(\omega) = -\frac{1}{2\pi^2}\left(\frac{e^2}{\hbar}\right)\ln\frac{1}{\omega\tau} \tag{28}$$

For $\omega \to 0$ the conductivity indeed decreases.

$$\sigma = \sigma_0 \left[1 - 2\lambda \ln\left(\frac{L}{\ell}\right) \right] \tag{30b}$$

i.e. the correction $\delta\sigma/\sigma_0$ is linear in $\lambda \ll 1$ and has a negative sign. The conductivity thus decreases when the system's size is increased — this is the precursor effect of localization.

How can one check these results experimentally? Well, we first have to bear in mind that all experiments are performed at finite temperatures. So, besides the elastic impurity scattering, one will also always have inelastic processes, which are described by some inelastic scattering time τ_{in} and for which we again assume that $\tau_{in}^{-1} \propto T^p$. In this way a new energy scale τ_{in}^{-1} enters the problem. As a consequence the energy $-i\omega$ in (27) is replaced by $-i\omega + \tau_{in}^{-1}$. The particle-particle diffusion pole is then cut off, such that $\omega \to 0$ does not lead to a divergence anymore. In turn, the frequency dependence in (28), $\ln(1/\omega\tau)$ is replaced by $\ln(\tau_{in}/\tau) \propto \ln(\hbar/k_B\tau T)$, i.e. by a temperature dependence just as in (20), (21). Alternatively one may say, that the inelastic processes introduce a new length scale, an "inelastic diffusion length", $L_{in} = (D_0\tau_{in})^{1/2}$, the so-called "Thouless-length". It provides the length scale on which a particle suffers an inelastic process and is scattered

out of its energy state. For $L_{in} < L$ the size of the system is quite irrelevant: a particle will then only experience L_{in} as the relevant length. In this way L has to be replaced by L_{in} in (29), (30) which leads to exactly the same temperature dependence of $\delta\sigma$ as in (20),(21).

A decrease of the conductivity $(\delta\sigma < 0)$ is equivalent to an increase of the resistance $(\delta R > 0)$. The corresponding logarithmic temperature dependence of the resistance of very thin films $(d=2)$, as shown in Fig. 15, has been found in numerous experiments (for a

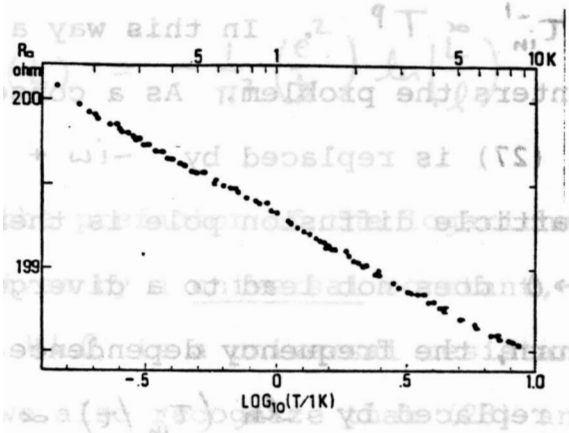


Fig. 15: Logarithmic temperature dependence of the resistivity of thin Au-Pd films [16].

review see [17]).

The preceding discussion was limited to the so-called "normal" scattering, i.e. scattering by non-magnetic impurities. Therefore the spin of the particles was unimportant. However, in the case that the impurities carry a magnetic moment, spin scattering will occur, causing the spin of the particles to flip. Therefore the particles experience something similar to a fluctuating magnetic field. The time reversal invariance is then lifted so that the Cooperon (25) does not diverge any longer (the pole is cut off by a constant term $1/\tau_s$, where τ_s is a characteristic collision time for spin scattering [18]). Field theoretical investigations [19,20], which we cannot discuss here in spite of their fundamental importance [21], have shown, that nevertheless even in this new situation one finds a logarithmic correction in $d=2$ just as in (30b). However, now the prefactor goes like λ^2 instead of λ, i.e. the correction is even smaller than in the case of normal scattering. Since Cooperons no longer yield a divergent contribution, these logarithmic corrections must be due to diffusons only, i.e. due to the usual diffusion process. It has not yet been possible to understand this result by means of the simple probability arguments used before in the case of normal scattering.

Impurities with a heavy nucleus lead to yet another type of scattering, namely to spin-orbit scattering of the particles. Theoretical investigations [20,22]

have again predicted a logarithmic correction for as in (30b) – but this time with a positive sign. The conductivity therefore increases with decreasing temperature! A simple quantum mechanical explanation, and experimental results fully supporting these findings, can be found in [17].

The Influence of a Magnetic Field on Localization

In the case of normal impurity scattering the localization effects originate from the quantum coherence of paths 1 and 2 in Fig. 10, i.e. from their ability to interfere. Therefore they are very sensitive to any kind of disturbance of time reversal invariance of the momentum states $\vec{k}$ and $-\vec{k}$. Such a perturbation is, for example, caused by a magnetic field. In its presence a state is no longer characterized by a momentum $\vec{k}$, but rather by the electromagnetic momentum $\vec{k} - 2e\vec{A}$. Here, A is the vector potential and the factor $2e$ (instead of simply e) is due to the correlation of two particles just as in superconductivity. If we now let $\vec{k}$ go into $-\vec{k}$, the momentum states, i.e. the paths 1 and 2 in Fig. 10, are no longer equivalent. Mathematically speaking this is a consequence of the fact, that now the amplitudes A_1 and A_2 carry field dependent phase factors [8]:

$$A_1 \rightarrow A \exp\left[\frac{ie}{\hbar c}\oint d\vec{\ell}\cdot\vec{A}\right]$$
$$= A \exp\left[ieHS/c\right] \qquad (31)$$
$$A_2 = A_1^*$$

The phases are given by the line integral over the vector potential $\vec{A}$, i.e. by the magnetic flux HS, where H is the magnetic field and S is the area of the closed path in Fig. 10 (c = velocity of light). Since the motion of the particles is diffusive, S is given by $S = D_0 t$. The return probability W_H of a particle to its starting point in the presence of a magnetic field is again given by (17). One therefore obtains

$$W_H = 2|A|^2\left[1 + \cos\left(2eHD_0 t/c\right)\right] \qquad (32)$$

If $H=0$, we find the old result $W_{H=0} = 4|A|^2$. The conductivity correction in the presence of a magnetic field, $\delta\sigma(H)$, is again determined by the return probability W_H. The <u>total</u> change of the conductivity due to a magnetic field, $\Delta\sigma(H) = \delta\sigma(H) - \delta\sigma(0)$, therefore depends on the probability <u>difference</u> $\Delta W = W_H - W_{H=0}$, such that

$$\Delta\sigma(H) = -\int_{\tau}^{\tau_{in}} v_F \lambda_F^{d-1} \frac{dt}{(D_0 t)^{d/2}} \left[\cos(2eHD_0 t/c) - 1\right] \quad (33)$$

In $d = 2$ (33) can be written as $\Delta\sigma(H) = e^2 F(x)$, where $x = 2eHD_0\tau_{in}/c$. The function $F(x)$ has the limits

$$F(x) = x^2, \quad x \ll 1$$
$$F(x) = \ln x, \quad x \gg 1 \quad (34)$$

For weak magnetic fields $x \ll 1$ one therefore finds

$$\Delta\sigma(H) \propto H^2 \tau_{in}^2 \quad (35)$$

while stronger fields $x \gg 1$ give rise to a logarithmic field dependence

$$\Delta\sigma(H) \propto \ln(H\tau_{in}) \quad (36)$$

In any case, $\Delta\sigma$ is always <u>positive</u> $(\Delta R < 0)$, so the resistance <u>decreases</u> with increasing magnetic field ("anomalous magnetoresistance")[20]. The reason lies in the disturbance of the phase coherence by the magnetic field, leading to a weakening of the localization

effects. The "critical" field H_c, determined by $x=1$, at which the change from the H^2 to the $\ln H$ behavior occurs, depends on τ_{in} and thus on temperature. At temperatures commonly used in experiments, H_c is of the order $H_c \approx 100-500$ Gauss ($\approx 10-50$ mT). This should be contrasted with the classical result $(\Delta\sigma(H)/\sigma_0) \simeq -(\omega_L\tau)^2$, which is not only many orders of magnitude smaller but also has a different sign (ω_L is the Larmor frequency)! So we see that even very small magnetic fields have a drastic influence on localization.

The Anderson transition

We have so far calculated small corrections to the metallic conductivity σ_0 due to the elastic impurity scattering. These corrections are the result of a perturbation theory and are lowest order in the disorder parameter $\lambda \ll 1$. On the other hand, for $d \leq 2$ we found that, in spite of the smallness of λ, the corrections diverge in the limits ω or $T \to 0$ or $L \to \infty$. In this situation the perturbation theory breaks down, i.e. fails because the condition $|\delta\sigma| \ll \sigma_0$ is no longer fulfilled. What now? One could try to calculate higher order corrections in λ, but that doesn't get us very far and is also very complicated. Besides that, a phase transition and

the critical behavior can never be obtained by a finite order perturbation theory. Since an exact solution of the problem does not a priori appear to be possible, one has to use other methods. Two such possibilities will be discussed here: (i) a scaling theory, which is based on renormalization group ideas and which determines and (ii) a self-consistent approach which calculates $\sigma(\omega)$.

Scaling Theory

A very successful scaling theory of the Anderson localization problem was introduced in 1979 by Abrahams et al. [13]. These authors developed a one-parameter scaling theory for the so-called "conductance" g of a d-dimensional system and connected it to a perturbation theory, in which the previously discussed length dependence of the correction $\delta\sigma$ was calculated for the first time.

The conductance g of a d-dimensional hypercube of side length L is given by

$$g = \left(\frac{e^2}{\hbar}\right)^{-1} \cdot \frac{1}{R} = \left(\frac{e^2}{\hbar}\right)^{-1} \sigma L^{d-2} \tag{37}$$

where R is the resistance and σ is the conductivity. The conductance g is a <u>number</u>, i.e. a dimensionless parameter, whose relevance for the localization problem

(particularly that of its length dependence) had first been recognized and discussed by Thouless [23] in several very important papers. In the metallic regime where Ohm's law holds, the conductivity is by definition length independent, so that the conductance is given by $g \propto L^{d-2}$. On the other hand in the insulating regime the wave functions fall off exponentially; one would therefore also expect an exponential length dependence of σ and thus of g, i.e. $g \propto \sigma \propto \exp(-L/\xi)$, where ξ is an unknown "localization length". What now is the length dependence of g in between these limiting cases? To answer this question, we take hypercubes of side length L and build up a larger hypercube with side length b·L. Now we ask how the conductance of the larger system, $g(bL)$, **depends on the conductance** of the initial cube, i.e. $g(L)$. In principle $g(bL)$ could be a function of $g(L)$, b, L and also all kinds of non-universal properties of the material (l, k_F, m etc.). The heart of the argument proposed by Abrahams et al. now is, to assume that the old g was in fact the <u>only</u> relevant parameter of the system that would determine the <u>new</u> g, **i.e. to have a relation**

$$g(bL) = f[b, g(L)] \tag{38a}$$

or, equivalently,

$$g(L) = f[b, g(L/b)] \tag{38b}$$

where f is some, yet unknown, function. In this way microscopic details are assumed to be unimportant. Eq. (38b) may be called a "scaling equation". Since the relation is supposed to hold for any value of b, we differentiate (38b) by b and then set $b = 1$. Then we obtain

$$\frac{d\,g(L)}{d\ln L} = \tilde{f}[g(L)] \tag{39}$$

where $\tilde{f}$ is a function involving f. The logarithmic derivative of g with respect to L therefore turns out to be a function of only g itself. To take a logarithmic instead of the usual derivative is indeed a convenient trick: one thereby achieves that length <u>scales</u> of L (e.g. ℓ or ξ in L/ℓ, L/ξ) which we do not know anyhow, drop out! Abrahams et al. [13] now defined a so-called "β-function"

$$\beta = \frac{1}{g}\frac{dg}{d\ln L} = \frac{d\ln g}{d\ln L} = F[g(L)] \tag{40}$$

which is essentially given by (39) and which is also only a function of $g(L)$. Its behavior under a change

of $g(L)$ determines the conductivity behavior of the system.

Starting from what we already know, we can easily calculate the limiting cases of β. In the localized regime $g \ll 1$, with $g \propto e^{-L/\xi}$, we find

$$\beta \simeq \ln g \tag{41}$$

while in the metallic regime $g \gg 1$ we have $\sigma = \sigma_0 + \delta\sigma$, where $\delta\sigma$ is given by (29a). Combining (37) and (40) one arrives at

$$\beta = d-2 - \frac{c_d}{g} + O\left(\frac{1}{g^2}\right) \tag{42}$$

The first term in (42), i.e. $d-2$, represents Ohm's law (purely metallic conductivity) while the $1/g$ correction term follows from the perturbation theory for the conductivity.

By means of (41),(42) which describe β in the limits $g \ll 1$ and $g \gg 1$ we may try to draw conclusions about the shape of β for <u>all</u> g. This is shown in Fig. 16, where $\beta(g)$ has been plotted versus $\ln g$. The calculated limits of β for the insulating and the "metallic" regime are shown by full curves. Note the important fact, that for $g \gg 1$ all β-curves are smoothly bending upward — a consequence of the negative

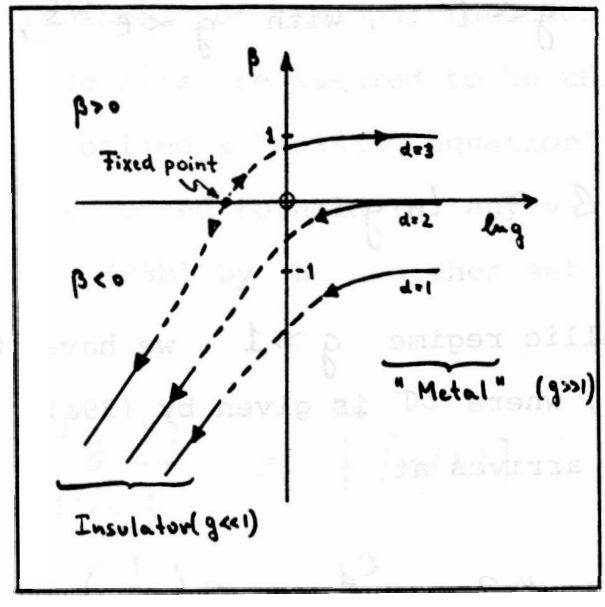


Fig. 16: The β-function for d = 1,2,3 dimensions

sign of the 1/g correction in (42). So, as g becomes smaller, the β-curves move away from the limiting value $d-2$. One may therefore make a sensible assumption about the unknown region in between: namely, that also there the β-curves have a smooth and monotonous shape as shown by the dashed curves in Fig. 16. **This figure** corresponds to a "flow diagram", which reflects the behavior of the system under changes of the system's size. For $d \leq 2$ this behavior is characteristically

different from the one for $d>2$: (i) $\underline{d \leq 2}$; in this case β is always negative. Since β describes the change of the conductance g with the system's size L, this implies $dg/dL<0$. An enlargement of the system therefore always leads to a decrease of g: the curves in Fig. 16 always "flow" to the left into the insulating regime. This tells us that for $d \leq 2$ and $L \to \infty$ all states are localized. The system is always an insulator. Of the curves that always have a negative β the case $d=2$ is obviously special ("marginal"): the curve approaches $\beta = 0$ for $g \gg L$ but never reaches $\beta = 0$ at finite g ; so even here the states of the infinite system are always localized. This provides the answer to our initial question about the possibility of metallic conductivity of very thin films $(d=2)$ at T = 0: we now find that a truely metallic conductivity in $d=2$ is never possible! (ii) $\underline{d > 2}$; here two possibilities, $\beta < 0$ and $\beta > 0$, exist since the β-function has a zero at $g = g_c$. For systems with an initial value of the conductance $g < g_c$ (i.e. with a disorder $\lambda > \lambda_c$, where $g(\lambda_c) = g_c$) we find $\beta < 0$. Therefore an enlargement of the system again leads to an insulating behavior (flow of the curve to the left). However, for an initial value $g > g_c$ (i.e. $\lambda < \lambda_c$) one has $\beta > 0$, such that increasing the size of the system drives it to the right, i.e. into the <u>metallic</u> regime. The point $\beta(g_c) = 0$ is called "fixed point", because g stays

fixed when L is changed. This point represents the Anderson transition. So we find that for d > 2 there exists a transition at a finite g_c, i.e. at a finite value of the disorder, λ_c. Depending on the disorder $(\lambda \lessgtr \lambda_c)$ states are either extended (metal) or localized (insulator).

In the vicinity of the transition, $\beta(g) \approx 0$, the β-function may be linearized:

$$\beta = \frac{1}{\nu} \frac{g - g_c}{g_c} \tag{43}$$

where $1/\nu$ is the slope of β at $g = g_c$ in Fig. 16. Eq.(43) can be integrated and yields

$$\frac{L}{L_0} = \left(\frac{g - g_c}{g_0 - g_c} \right)^{\nu} \tag{44}$$

where $g_0 = g(L_0)$.

In the localized regime $(g, g_0 < g_c)$ one finds

$$g = g_0 \left[1 - \left(\frac{L}{\xi} \right)^{1/\nu} \right] \tag{45}$$

where ξ has been introduced as a "localization length". If we define a parameter $t = (g_0 - g_c)/g_c \propto (\lambda - \lambda_c)/\lambda_c$, which measures the disorder of the system relative to the corresponding critical value, i.e. the distance from the critical

point, then (44), (45) imply

$$\xi \propto |t|^{-\nu} \tag{46}$$

Thus the number ν turns out to be the critical exponent of the localization length ξ . Eq.(46) says that ξ diverges as the disorder approaches the critical value λ_c and is then the only relevant length in the problem.

In contrast, the metallic side $(g, g_0 > g_c)$ is characterized by a finite dc-conductivity $\sigma(0)$. Using (37) and (44) we find

$$\sigma(0) \propto t^{\nu(d-2)} \tag{47}$$

The critical exponent of $\sigma(0)$, which is usually referred to as s , is therefore given by

$$s = (d-2)\nu \tag{48}$$

This relation connects two critical exponents, one of which describes the behavior on the metallic side (s), while the other one determines the critical behavior on the insulating side (ν) . This scaling relation was first derived by Wegner [24] already in 1976, who showed (in a paper of fundamental importance for the Anderson localization problem) that there exists a close connection to the problem of critical phenomena.

Integration of (42) leads to an explicit value for s, namely $s = 1$. Within the approximations used, this result is valid at least close to 2 dimensions $(\epsilon = d-2 \ll 1)$, but it appears to hold even in $d = 3$.

Field theoretical methods have also led to a scaling theory in the case of magnetic impurities [19-22]. The results are qualitatively very similar to the ones obtained in the previously discussed case. In particular, one again expects an Anderson transition for $d > 2$, while for $d \leq 2$ all states are localized. The fact that the perturbational corrections due to the spin scattering are proportional to λ^2 (instead of λ as for normal scattering) is also reflected in the critical properties: the conductivity exponent s is here found to be $s = 1/2$ instead of $s = 1$.

Eq. (47) implies that the dc-conductivity goes to zero <u>continuously</u> as the transition is approached. This finding contradicts Mott's concept of a "minimal metallic conductivity" and the related <u>discontinuous</u> transition. The experimental confirmation of the continuous behavior will be discussed later.

The Method of Selfconsistency

The theoretical approach [13] as described above is based on the assumption of a one-parameter

scaling relation (40), whose validity cannot be further proven within this very framework.

Quite a different theoretical concept, which may be used to describe the conductivity behavior as well as the transition and the critical properties and which does not rely on a scaling assumption, is that of "selfconsistency" [25]. Within this method one attempts to express the frequency dependent conductivity $\sigma(\omega)$ (or the diffusion coefficient $D(\omega)$) by means of a non-trivial, generally approximate, relation which <u>itself</u> involves this quantity. So one wants to find an equation of the form

$$D(\omega) = \mathcal{F}[D(\omega)] \tag{49}$$

whose "self-consistent" solution then yields $D(\omega)$ for all ω and all disorder-parameters λ. For this to be successful it is necessary to start from known limiting cases (e.g. (49) has to agree with perturbation theory for $\lambda \ll 1$) such that the theory can be anchored to an exact result [14]. The selfconsistency is then used to go beyond perturbation theory, i.e. to the transition itself (and even beyond). It is therefore used as a substitute for an (infeasible) perturbation theory to infinite order.

Since $D(\omega)$ vanishes at the transition, its inverse $D_0/D(\omega)$ correspondingly diverges at that

point. Within a diagrammatic perturbation theory it has been shown that a selfconsistent calculation of the latter quantity can be performed by summing up the largest (i.e. most divergent) contributions of perturbation theory [14,26]. In this way one can indeed derive a selfconsistent equation. It has the simple structure

$$\frac{D_0}{D(\omega)} = 1 + \frac{1}{\pi N_F} \int \frac{d\vec{k}}{(2\pi)^d} \left(\frac{1}{-i\omega + D(\omega) k^2} \right) \tag{50}$$

where $D_0/D(\omega)$ is given by the integral over a diffusion pole involving the diffusion coefficient itself rather than the diffusion constant D_0. This relation can also be obtained by other methods [27-30]. Its solution can easily be obtained: one finds that for $d \leq 2$ the dc-conductivity $\sigma(0)$ is always zero, irrespective of how small the disorder is (insulating behavior). However, in dimension $d=2$ the localization length ξ is exponentially large for $\lambda \ll 1$ [14]:

$$\xi \propto e^{1/2\lambda} \tag{51}$$

For $d>2$ there exists a critical value of the disorder below which $\sigma(0)$ is finite (metallic regime), while for larger values it vanishes (insulating regime). Since the limit $\omega \to 0$ can be explicitly performed within this theory one obtains results which go beyond

the range of applicability of the scaling theory described above ($\omega \to 0$ means, that one leaves the critical regime). Besides that one obtains complete agreement with the results of the scaling theory. In particular one also finds the scaling relation (48) and the value $s=1$ for dimensions $2 < d < 4$. In fact one can show that (50) itself possesses scaling properties and therefore naturally leads to a scaling theory [26,28,31,32]. Performing the integral in (50) and choosing appropriate units such that one obtains dimensionless quantities G and z instead of D and ω , (50) can be written as [26]

$$\frac{d-2}{r_d} G = \pm z^{\frac{2-d}{d}} + G^{\frac{2-d}{2}} \tag{52}$$

where the plus (minus)-sign corresponds to the metallic (insulating) regime and r_d is a dimension-dependent constant. In (52) no material dependent parameters (k_F , m , λ etc.) are explicitly involved. It is therefore indeed a "scaling equation" which establishes a <u>universal</u> dependence of the conductivity or frequency. Its solution determines a complete and smooth curve for the β-function (40) [32].

A field theoretical analysis [30] of the present problem shows that, if a perturbational treatment is valid at all, (52) is an <u>exact</u> relation — at least close to two dimensions [33].

In conclusion we may say, that in spite of a few still open questions, the problem of Anderson localization, i.e. the behavior of non-interacting particles in a disordered medium, is basically well understood.

III. Interacting, Quantum Mechanical Particles in Disordered Systems

The theory discussed so far only involved the elastic scattering of non-interacting, quantum mechanical particles due to impurities. If, on the other hand, one wants to check the results experimentally, one always deals with systems, in which the particles also mutually interact. The most important example is that of electrons in a disordered metal. Even if one was always allowed to assume the Coulomb interaction between the electrons to be strongly screened, one would still have to know about its actual qualitative and quantitative influence compared with the effects of localization [3, 34].

Pure electron-electron interaction is strictly momentum conserving and therefore does not contribute to the metallic resistance. So, as much as before, the momentum change of particles due to impurity scattering is essential. To understand the combined effects, we

take a look at the inelastic interaction among the electrons themselves [35]. It is characterized by a momentum transfer q (with $q\ell \ll 1$) and an energy transfer $\hbar\omega$ (with $\omega\tau \ll 1$), where ℓ and τ are the mean free path and the average collision time of the electrons due to the impurities. One may then define an "interaction time" $t_0 = 2\pi/\omega$; obviously $t_0 \gg \tau$. This means that during the time t_0 many scatterings between electrons and impurities occur, leading to a qualitative change of the interaction itself. The "bare" interaction therefore becomes an "effective" interaction whose character depends on the energy transfer $\hbar\omega$. These effects lead to corrections $\delta\sigma$ and δN of the (metallic) conductivity and of the density of states close to the Fermi surface, respectively.

Estimating the Corrections

Calculating the corrections to σ to lowest order in the disorder λ , Altshuler and Aronov [35], found the surprising result, that they had the same structure as those in the non-interacting case. In particular, there is again a logarithmic temperature dependence of $\delta\sigma$ in d = 2, although the quantum mechanical interference processes that led to (19),(21) had not been considered by these authors at all! If initially one perhaps believed in an obscure coincidence,

we now know that these results are solely due to the disorder in the system. It leads to a diffusive behavior of the particles and thereby to a change of the interaction due to density fluctuations. In this way the naked, long range interaction between the electrons becomes an effective interaction of short range.

To study the implications we now investigate the inelastic scattering of two electrons (more precisely: of a particle-hole pair) with energy ϵ (relative to the Fermi surface) in which an energy transfer $\hbar\omega \approx \epsilon$ is assumed to occur [36]. According to the uncertainty principle the particle and the hole remain in mutually coherent states up to times $t \lesssim \hbar/\epsilon$, during which interference is possible. However, in the case of a short range interaction it is necessary that the particles actually meet at the same position during that time. The probability for this to happen we call $W(\epsilon)$. The short range interaction Λ is thereby changed to

$$\Lambda_{eff} = \Lambda(1 + W(\epsilon)) \tag{53}$$

To estimate $W(\epsilon)$ we observe once more that the motion of the particles is diffusive, i.e.

$$W(\epsilon) \propto \int_{\tau}^{\hbar/\epsilon} dt \frac{1}{(D_0 t)^{d/2}} \tag{54}$$

The integrand is thus given by $1/V_{diff}$, (15), just as in the localization problem. Consequently the integral leads to the same dimension dependent expression as in (20). In $d=2$ one obtains

$$W(\epsilon) \propto \ln\left(\hbar/\epsilon\tau\right) \tag{55}$$

We now assume that the relative corrections of physical quantities to lowest order in λ due to these effects are given by the change in the interaction, i.e. by

$$\frac{\delta\sigma}{\sigma_0}, \quad \frac{\delta N(\epsilon)}{N_F} \propto \delta\Lambda_{eff} = \Lambda W(\epsilon) \tag{56}$$

For example, the correction to the density of states in $d=3$ is then given by

$$\delta N(\epsilon) \propto \Lambda\sqrt{|\epsilon|} \tag{57}$$

In the case of repulsive interaction $(\Lambda > 0)$ the density of states is hence found to <u>decrease</u> in the vicinity of the Fermi surface, having a downward, non-analytical cusp at ε_F (i.e. $\epsilon = 0$). This behavior has in fact been observed in tunnel experiments [37] where the tunnel conductance dI/dV , which is proportional to the density of states, is measured as a function of the voltage V (instead of the energy ϵ); see Fig. 17.

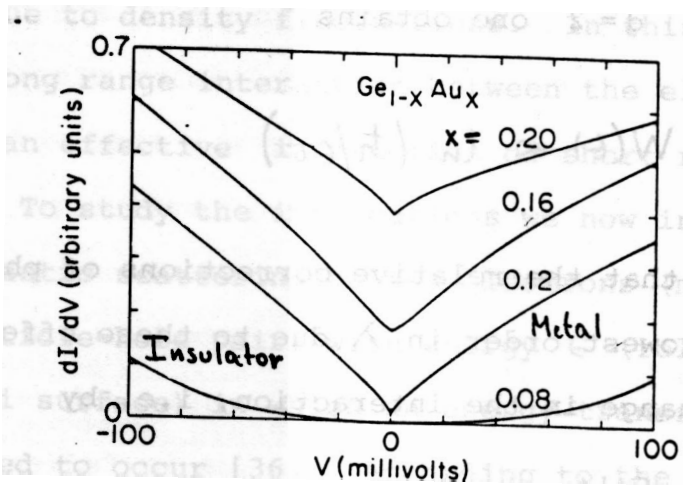


Fig. 17: Behavior of the density of states close to the Fermi energy in d = 3 [37].

To obtain the conductivity correction at finite temperatures the energy ϵ is replaced by T in (54):

$$\frac{\delta\sigma}{\sigma_0} \propto \Lambda \begin{cases} \ln\left(\frac{\hbar/\tau}{k_B T}\right), & d=2 \\ T^{\frac{d}{2}-1}, & d \neq 2 \end{cases} \qquad (58)$$

The temperature dependence of $\delta\sigma$ is therefore identical to the one previously found in the localization problem, although the quantum mechanical interference due to backscattering (58) is not involved at all. Only

the diffusive behavior of the electrons is of importance. In fact, inclusion of the "Cooperons" only leads to an insignificant change of the results in (58) [38].

The singularities in the density of states and other physical quantities show that, in fact, the disorder leads to essential changes of the properties of a Fermi liquid, thus disproving long held, opposite views.

In the meantime there even exists an exact solution of the interaction problem to lowest order in $\lambda \ll 1$ [39] which can also be understood within Fermi liquid theory [40]. Therefore, in the case of weak disorder, again a satisfactory situation has been reached.

The Metal-Insulator Transition

To go beyond perturbation theory and to be able to study the metal-insulator transition (including its critical properties), one needs a scaling theory or some other high-power mathematical method. At the present time field theoretical investigations as well as renormalization group treatments of the transition close to $d = 2$ dimensions are performed [41,42]. However, only in a few special cases (strong external magnetic field and magnetic spin scattering, respectively) have they led to definite conclusions. In these cases one finds that, just as in the localization problem, a

transition will only occur above $d = 2$ dimensions. Close to the transition the dc-conductivity is expected to vanish <u>linearly</u> in $\lambda_c - \lambda$ (i.e. with critical exponent $s = 1$, like in Anderson localization <u>without</u> magnetic field or spin scattering). However, the general behavior is still unknown. In fact, in the case of zero external magnetic field the renormalization-group analysis breaks down before the transition is reached [41,42]. This is due to the appearance of local magnetic moments (strong spin fluctuations) which lead to a divergence of the spin susceptibility for $T \to 0$.

Furthermore, the quantum interference effects due to backscattering which are responsible for the transition in the non-interacting system have not yet been fully considered. Therefore there remains a lot to be understood about the metal-insulator transition of the interacting system.

Experimental Results and Theoretical Conjectures

In an experiment the transition is usually not approached by increasing the disorder λ to λ_c but rather by <u>decreasing</u> the particle density n. In semi-conductors this can be achieved by a change of the doping concentration; in this way the energy of the particles

$$E = \frac{\hbar^2}{2m}\left(3\pi^2 n\right)^{2/3} \tag{59}$$

is lowered. If the critical energy at which the transition takes place is denoted by $E_c \equiv E(n_c)$, the system will be in the metallic regime for $n > n_c$ (i.e. $E > E_c$) and in the insulating regime for $n < n_c$ (i.e. $E < E_c$).

As the details of the transition have not yet been understood, the corresponding experiments have also not been explained conclusively [43]. In three-dimensional samples one finds for example that there is one group of systems (e.g. amorphous $Nb_x Si_{1-x}$ or compensated $Ge:Sb$) where $\sigma(0)$ vanishes <u>linearly</u> with $n-n_c$ (i.e. $s=1$; see Fig. 18a) while in another group (e.g. $Si:P$ or uncompensated $Ge:Sb$) one finds a <u>root</u>-behavior (i.e. $s = 1/2$; see Fig. 18b). (Note, that in both cases the conductivity goes to zero <u>continuously</u>; furthermore, in both cases conductivities significantly below the postulated minimal metallic conductivity σ_{min} (see the arrows in Fig. 18 a,b) have been measured.) The reason for this different critical behavior is still not really clear. However, the two groups of materials differ in <u>one</u> essential point and this has led to certain conjectures concerning the underlying physics [45,46]. The systems exhibiting

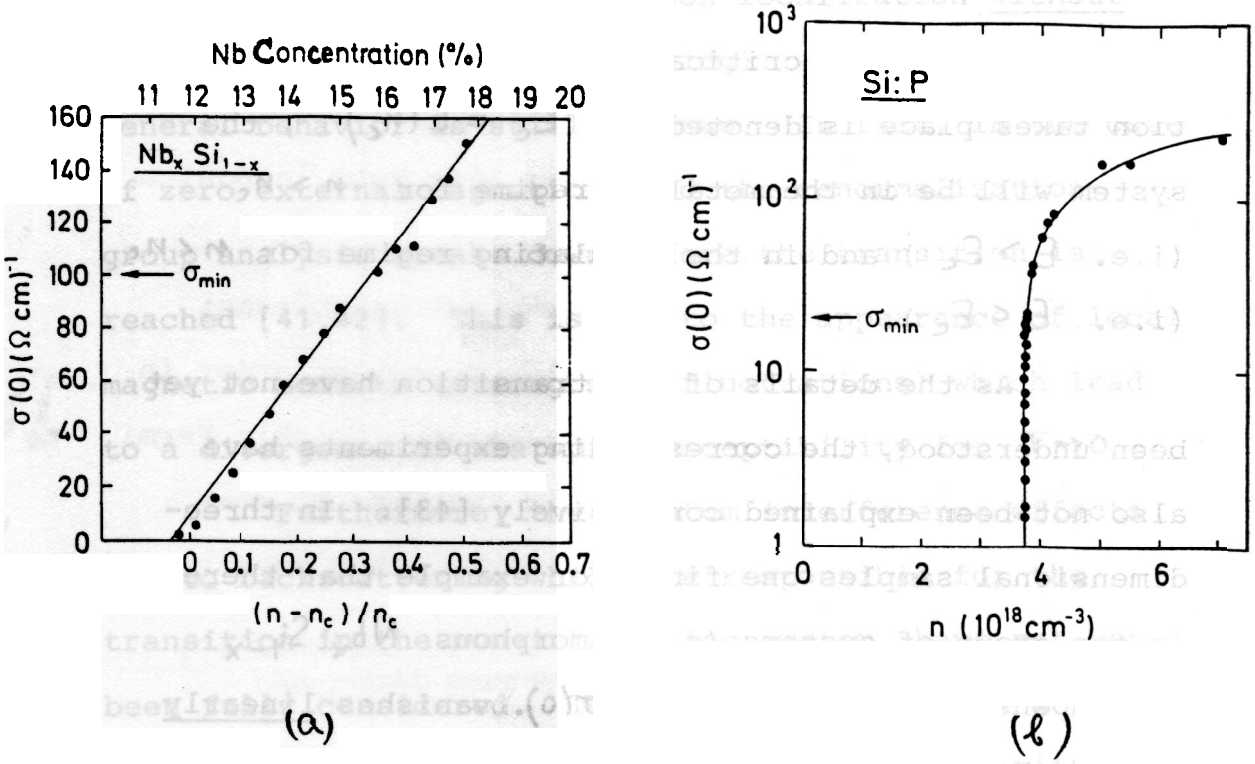


Fig. 18: Dc-conductivity of different systems close to the metal-insulator transition: (a) Exponent $s=1$ [44], (b) $s=1/2$ [43].

$s=1/2$ possess one valence electron per scattering center. (In the case of Si:P every P-atom contributes one additional valence electron to the electronic system, while the remaining P-ions represent the random potentials by which the electrons are scattered). In contrast to this, in the systems with $s=1$ there are several electrons per scattering center. (In the case of compensated Ge:Sb equal amounts of Sb and B are added, which then

exchange an electron; so they do not contribute additional charge carriers but only scattering centers.) One may now speculate that these additional scatterers lead to an increased scattering of the electrons and that therefore the conductivity behavior with $s = 1$ is essentially that of the Anderson transition. On the other hand $s=1$ is also compatible with the interaction theory for systems with strong spin scattering.

The situation in the group of materials with $s = 1/2$ is even more unclear. The behavior $s=1/2$ has so far only been found for non-interacting systems with strong spin scattering. However, in $Si:P$ only the electrons carry a spin. Now, it might just be so that the local magnetic moments (spin fluctuations), which the renormalization group calculations for the interacting system come across (and which lead to their ultimate breakdown) act as spin scattering centers themselves [46]. The interaction would then, through a collective behavior of the electrons, provide magnetic moments on which the electrons themselves scatter. Apart from that the electrons would "feel" a non-interacting system. Within this concept one would be able to understand $s = 1/2$. In fact this picture is supported by measurements of the specific heat and the spin susceptibility of these systems [43]. In the limit $T \rightarrow 0$ one finds that both quantities strongly increase – an effect that appears to be

explicable by means of a strongly enhanced effective mass. Here now seems to be a close connection to the theory of strongly correlated ("almost localized") Fermi systems, which already has found application in the theory of liquid ^{3}He [47]. In this way several remarkable connections between seemingly very different areas of condensed matter physics have recently emerged: connections between the problem of disordered systems, of Fermi liquid theory, of the theory of strongly correlated Fermi systems and also of superconductivity. We can be sure that a great deal of new and unexpected physics will evolve from the combination of these concepts.

References:

[1] O. Madelung; Introduction to Solid State Physics; Springer Series in Solid State Sciences, Vol. 2 ; Springer, Berlin (1978).

[2] Anderson Localization, Y. Nagaoka and H. Fukuyama, eds.; Springer Series in Solid State Sciences, Vol.39; Springer, Berlin (1982).

[3] P.A. Lee and T.V. Ramakrishnan; "Disordered Electronic Systems," Rev. Mod. Phys. 57 287 (A85).

[4] P.W. Anderson; Phys. Rev. 109 1492 (1958).

[5] N.F. Mott; Phil. Mag. 26 1015 (1972).

[6] A presentation of the historical development of this concept and of the localization problem from a rather more personal point of view can be found in N.F. Mott; Rep. Prog. Phys. 47 909 (1984).

[7] S.F. Edwards; Phil. Mag. 8 1020 (1958).

[8] Here we follow the discussion by D.E. Khmelnitskii in Proc. of the 17. Intern. Conf. on Low Temp. Phys. (LT - 17), part III, Physica 126 B+C, 235 (1984).

[9] See, for example: R.J. Elliot, J.A. Krumhansl and P.L. Lieb; Rev. Mod. Phys. 46 465 (1974).

[10] D. Forster, Hydrodynamic Fluctuations, Broken Symmetry and Correlation Functions; Benjamin, Reading (1975).

[11] See, for example: S. Doniach and E.H. Sondheimer; Green's Functions for Solid State Physicists; Benjamin, Reading (1974).

[12] J.S. Langer and T. Neal; Phys. Rev. Lett. 16 984 (1966).

[13] E. Abrahams, P.W. Anderson, D.C. Licciardello and T.V. Ramakrishnan; Phys. Rev. Lett. 42 673 (1979).

[14] D. Vollhardt and P. Wölfle; Phys. Rev. Lett. 45 482 (1980), Phys. Rev. B22, 4666 (1980).

[15] L.P. Gor'kov, A.J. Larkin and D.E. Khumelnitskii; JETP Lett. 30 248 (197

[16] G.J. Dolan and D.D. Osheroff; Phys. Rev. Lett. 43 721 (1979).

[17] G. Bergmann; Phys. Rep. 107, 1 (1984).

[18] P.A. Lee; J. Noncryst. Solids, 35 21 (1980).

[19] F. Wegner; Z. Phys. B35, 207 (1979).

[20] S. Hikami, A.I. Larkin and Y. Nagaoka; Prog. Theor. Phys. 63 707 (1980).

[21] See, for example: Ref. 3 as well as F. Wegner and S. Hikami in Ref. 2.

[22] R. Oppermann and K. Jungling; Phys. Lett. 76A 449 (1980).

[23] D.J. Thouless; Phys. Rep. 13C 93 (1974); Phys. Rev. Lett. 39 1167 (1977).

[24] F. Wegner; Z. Phys. B25 327 (1976).

[25] It was first introduced into the localization problem by W. Götze; Solid State Comm. 27 1393 (1978); Phil. Mag. 43 219 (1981).

[26] P. Wölfle and D. Vollhardt; in Ref. 2.

[27] P. Prelovsek; in Recent Developments in Condensed Matter Physics, ed. J.T. Devreese (Plenum Press, New York, 1981), Vol. II, p.191.

[28] D. Belitz, A Gold and W. Götze; Z. Phys. B44 273 (1981).

[29] C.S. Ting; Phys. Rev. B26 678 (1982).

[30] S. Hikami; in Ref. 2.

[31] B. Shapiro; Phys. Rev. B25 4266 (1982).

[32] D. Vollhardt and P. Wölfle; Phys. Rev. Lett. 48 699 (1982).

[33] I thank Prof. D.J. Amit for a discussion and a clarification of this point.

[34] For a comprehensive discussion of the interaction problem we refer the reader to the book

Electron-Electron Interactions in Disordered Systems, M. Pollak and B.L. Efros, eds; North Holland, Amsterdam (1984/1985) which is supposed to appear soon.

[35] B.L. Altshuler and A.G. Aronov; Zh. Eksp. Teor. Fiz. 77, 2028 (1979) [Sov. Phys. JETP 50, 968 (1980)].

[36] Here we follow the discussion by A.G. Aronov in Proceedings of LT-17 (see Ref. 8) and by B.L. Altshuler and A.G. Aronov in Ref. 33.

[37] W.L. McMillan and J. Mochel; Phys. Rev. Lett. 46 556 (1981).

[38] Y. Isawa and H. Fukuyama; J. Phys. Soc. of Japan; 53 1415 (1983).

[39] A.M. Finkelstein; Zh. Eksp. Teor. Fiz. 84 168 (1983) [Sov. Phys. JETP 57 97(1983)].

[40] B.L. Altshuler and A.G. Aronov; Solid State Comm. 46 429 (1983).

[41] A.M. Finkelstein; Z. Phys. B56 189 (1984).

[42] C. Castellani, C.Di Castro, P.A. Lee and M. Ma; Phys. Rev. B30 1596 (1984).

[43] An extensive discussion of experimental results can be found in T.F. Rosenbaum, R.F. Milligan, M.A. Paelanen, G.A. Thomas, R.N. Bhatt and W. Lin; Phys. Rev. B27 7509 (1983).

[44] B. Hertel, D.J. Bishop, E.G. Spencer, J.M. Powell and R.C. Dynes; Phys. Rev. Lett. 50 743 (1983).

[45] G.A. Thomas (unpublished); also see the Proceedings of the Conference on Doped Semiconductors etc. Santa Cruz, 1984, which will be published in "Solid State Electronics".

[46] E. Abrahams and T.M. Rice (private communication) and contributions in Ref. 45.

[47] D. Vollhardt; Rev. Mod. Phys. 56 99 (1984).