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OUTER INVARIANCE ENTROPY FOR DISCRETE-TIME LINEAR SYSTEMS ON LIE GROUPS

Fritz Colonius[1,*], João A.N. Cossich[2] and Alexandre J. Santana[2]

Abstract. We introduce discrete-time linear control systems on connected Lie groups and present an upper bound for the outer invariance entropy of admissible pairs (K, Q). If the stable subgroup of the uncontrolled system is closed and K has positive measure for a left invariant Haar measure, the upper bound coincides with the outer invariance entropy.

1. Introduction

In 2009, Colonius and Kawan [6] introduced the theory of invariance entropy for control systems. This concept is closely related to feedback entropy introduced by Nair, Evans, Mareels and Moran [14] in an engineering context. Specifically, for invariance entropy a pair (K, Q) of nonempty subsets of the state space is called admissible if K is a compact subset of Q and for each $x \in K$ there exists a control v such that the trajectory $\varphi(\mathbb{R}^+, x, v) \subset Q$. For $T > 0$ denote by $r_{\text{inv}}(T, K, Q)$ the minimal number of controls u such that for every initial point $x \in K$ there is u with trajectory $\varphi([0, T], x, u) \subset Q$. Then the invariance entropy is the exponential growth rate of these numbers as T tends to infinity,

$$h_{\text{inv}}(K, Q) := \limsup_{T \to \infty} \frac{1}{T} \log r_{\text{inv}}(T, K, Q).$$

From this paper the theory was developed culminating in Kawan's book [12] that compiled all the theory achieved until 2013. In particular, this book also developed the theory of invariance entropy for discrete time. Most of the results lead to more explicit expressions when studied in the case of linear systems. For linear discrete-time systems on Euclidean spaces the present authors in [5] described the invariance pressure and, in particular, the invariance entropy for subsets K of the control set D, which is unique and bounded under hyperbolicity assumptions. For the hyperbolic theory of invariance entropy for continuous-time nonlinear control systems, *cf.* Da Silva and Kawan [9] and also [10]. Invariance entropy for continuous-time linear systems on Lie groups has been analyzed by Ayala *et al.* [2], *cf.* also the references therein for the theory of continuous-time linear control systems on Lie groups.

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For discrete time, the present paper introduces linear control systems on connected Lie groups and starts the investigation of invariance entropy. We define discrete-time linear systems on connected Lie groups G as systems of the form

$$\Sigma \;:\; g_{k+1} = f(g_k, u_k), \quad u_k \in U,$$

where $f_u(\cdot) := f(\cdot, u) : G \to G$ is an automorphism for $u = 0$ and otherwise a diffeomorphism such that $f_u(g) = f_u(e) \cdot f_0(g)$. We refer to $g_{k+1} = f(g_k, 0) = f_0(g_k)$ as the uncontrolled system.

Outer invariance entropy weakens the requirements on the trajectories: the trajectories are allowed to go out to ε-neighborhoods $N_\varepsilon(Q)$ of Q, and

$$h_{\mathrm{inv,out}}(K, Q) := \lim_{\varepsilon \to 0} h_{\mathrm{inv}}(K, N_\varepsilon(Q)).$$

Our main result establishes that an upper bound for outer invariance entropy of an admissible pair (K, Q) of discrete-time linear systems is given by the sum of the logarithms of the modulus of the eigenvalues λ of the differential $(\mathrm{d}f_0)_e$ with $|\lambda| > 1$, where e is the identity of G. The upper bound coincides with the outer invariance entropy, if the stable subgroup of the uncontrolled system (*cf.* the definition after formula (2.2)) is closed and K has positive measure for a left invariant Haar measure μ on G. This bears some similarity to the characterization of outer invariance entropy in the continuous-time case, *cf.* Da Silva [7].

Section 2 presents the main concepts for discrete-time linear control system on Lie groups and some examples. In Section 3 we introduce discrete-time linear control system induced on homogeneous spaces and Section 4 proves the main result of the paper, the characterization of outer invariance entropy for discrete-time linear control systems on Lie groups. Finally, Section 5 derives a sufficient condition for closedness of the stable subgroup of the uncontrolled system.

2. Discrete-time linear control systems on Lie groups

In this section we present our definition of discrete-time linear control systems on Lie groups in analogy to the definition of the continuous-time linear control systems on Lie groups, derive some properties, and provide several examples.

Recall (*cf. e.g.* Sontag [16]) that a discrete-time control system on a topological space M is given by difference equations

$$x_{k+1} = f(x_k, u_k), \quad u_k \in U,$$

where $k \in \mathbb{N}_0$, the control range U is a nonempty set and $f : M \times U \to M$ is a map such that $f_u(\cdot) := f(\cdot, u) : M \to M$ is continuous for each $u \in U$. For initial value $x_0 := x \in M$ and control $u = (u_i)_{i \in \mathbb{N}_0} \in \mathcal{U} = U^{\mathbb{N}_0}$ the solution of this system is given by

$$\varphi(k, x, u) = \begin{cases} x & \text{for} \quad k = 0 \\ f_{u_{k-1}} \circ \cdots \circ f_{u_1} \circ f_{u_0}(x) & \text{for} \quad k \geq 1 \end{cases}.$$

Where convenient, we also write $\varphi_{k,u} := \varphi(k, \cdot, u)$. Now we present our definition of discrete-time linear control systems on Lie groups.

Definition 2.1. Let $U \subset \mathbb{R}^m$ with $0 \in U$. A discrete-time control system

$$\Sigma \;:\; g_{k+1} = f(g_k, u_k), \quad u_k \in U,$$

on a connected Lie group G is linear if $f_0 = f(\cdot, 0) : G \to G$ is an automorphism and for each $u \in U$, $f_u : G \to G$ satisfies

$$f_u(g) = f_u(e) \cdot f_0(g) = L_{f_u(e)}(f_0(g)) \text{ for all } g \in G.$$

Here " $\cdot$ " denotes the product of G, it will be omitted when it is clear by the context. Thus $f_u(g)$ is given by left translation $L_{f_u(e)}$ of $f_0(g)$ by $f_u(e)$, and $f_0(g)^{-1} = f_0(g^{-1})$ implies that for each $u \in U$ the map $f_u : G \to G$ is a diffeomorphism with inverse

$$(f_u)^{-1}(g) = (f_0)^{-1}\left((f_u(e))^{-1} \cdot g\right) = (f_0)^{-1} \circ L_{(f_u(e))^{-1}}(g), g \in G.$$

Example 2.2. Consider on the additive Lie group $G = \mathbb{R}^d$ the control system given by

$$x_{k+1} = Ax_k + Bu_k, \quad u_k \in U,$$

where $A \in GL(d, \mathbb{R}), B \in \mathbb{R}^{d \times m}$, and $0 \in U \subset \mathbb{R}^m$. In this case, $f : \mathbb{R}^d \times U \to \mathbb{R}^d$ is given by $f(x, u) = Ax + Bu$. Note that $f_0(x) = Ax$ is an automorphism of $\mathbb{R}^d$ and $f_u(e) = f_u(0) = Bu$, hence $f_u(x) = f_0(x) + f_u(0) = f_u(0) + f_0(x)$. In this case, the solutions are given by

$$\varphi(k, x, u) = A^k x + \sum_{j=0}^{k-1} A^{k-1-j} Bu_j.$$

The next result shows that for linear systems the solution starting in a point g is a translation of the solution starting in the identity e.

Proposition 2.3. *Consider a discrete-time linear control system $g_{k+1} = f(g_k, u_k)$, $u_k \in U$, on a Lie group G. Then it follows for all $g \in G$ and $u = (u_i) \in \mathcal{U}$ that*

$$\varphi(k, g, u) = \varphi(k, e, u) f_0^k(g) \text{ for all } k \in \mathbb{N}. \tag{2.1}$$

Proof. The proof will follow by induction over $k \in \mathbb{N}$. Note initially that for $g \in G$ and $u \in \mathcal{U}$ we have

$$\varphi(1, g, u) = f_{u_0}(g) = f_{u_0}(e) f_0(g) = \varphi(1, e, u) f_0(g).$$

Now, suppose that the equality holds for $k \in \mathbb{N}$, that is, $\varphi(k, g, u) = \varphi(k, e, u) f_0^k(g)$. With the k-shift to the right given by $\Theta_k(u_i)_{i \in \mathbb{N}} = (u_{i+k})_{i \in \mathbb{N}}$, this implies

$$\begin{aligned}
\varphi(k+1, g, u) &= \varphi(1, \varphi(k, g, u), \Theta_k u) = \varphi(1, \varphi(k, e, u) f_0^k(g), \Theta_k u) \\
&= f_{u_k}(\varphi(k, e, u) f_0^k(g)) = f_{u_k}(e) f_0(\varphi(k, e, u) f_0^k(g)) \\
&= f_{u_k}(e) f_0(\varphi(k, e, u)) f_0^{k+1}(g) = f_{u_k}(\varphi(k, e, u)) f_0^{k+1}(g) \\
&= \varphi(k+1, e, u) f_0^{k+1}(g).
\end{aligned}$$

$\square$

Remark 2.4. For a continuous-time linear control system on a connected Lie group the solution is given by

$$\varphi(t, g, u) = \varphi(t, e, u) \varphi(t, g, 0),$$

cf. Ayala *et al.* ([3], formula (7)). The solution formula (2.1) for discrete-time linear systems is analogous.

Example 2.5. Consider the connected component of the identity for the affine group

$$Aff(2, \mathbb{R})_0 = \left\{ \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}; \ x > 0 \ \text{and} \ y \in \mathbb{R} \right\}.$$

This group can be seen as $\mathbb{R}_+ \times \mathbb{R}$ provided with the product

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2, x_1 y_2 + y_1).$$

Define $f : Aff(2, \mathbb{R})_0 \times U \to Aff(2, \mathbb{R})_0$ as $f((x, y), u) = (xe^u, ye^{2+u} + u)$, where $U \subset \mathbb{R}$ with $0 \in U$. Elementary calculations show that $f_0 : Aff(2, \mathbb{R})_0 \to Aff(2, \mathbb{R})_0$ is an automorphism and $f_u(x, y) = f_u(1, 0) \cdot f_0(x, y)$, for all $(x, y) \in Aff(2, \mathbb{R})_0$. Hence,

$$(x_{k+1}, y_{k+1}) = f((x_k, y_k), u_k), \ u_k \in U,$$

is a discrete-time linear control system on $Aff(2, \mathbb{R})_0$.

For $u = (u_i)_{i \in \mathbb{N}_0} \in \mathcal{U}$, denote by $S_k(u)$ the sum $\sum_{i=0}^{k} u_i$. Proposition 2.3 and induction over k imply that for all $k \in \mathbb{N}_0$, $(x, y) \in Aff(2, \mathbb{R})_0$ and $u \in \mathcal{U}$,

$$
\begin{aligned}
\varphi(k, (x, y), u) &= \varphi(k, (1, 0), u) f_0^k(x, y) \\
&= \left(e^{S_{k-1}(u)}, \sum_{j=0}^{k-1} u_j e^{2(k-1-j) + S_{k-1}(u) - S_j(u)} \right) (x, ye^{2k}) \\
&= \left(xe^{S_{k-1}(u)}, ye^{2k + S_{k-1}(u)} + \sum_{j=0}^{k-1} u_j e^{2(k-1-j) + S_{k-1}(u) - S_j(u)} \right).
\end{aligned}
$$

Example 2.6. Let

$$G = \left\{ \begin{bmatrix} 1 & x_2 & x_1 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{bmatrix}; \ (x_1, x_2, x_3) \in \mathbb{R}^3 \right\}$$

be the Heisenberg group. Note that G is diffeomorphic to $\mathbb{R}^3$ with the product

$$(x_1, x_2, x_3) \cdot (y_1, y_2, y_3) = (x_1 + y_1 + x_2 y_3, x_2 + y_2, x_3 + y_3).$$

Consider the automorphism $f_0 : G \to G$ given by

$$f_0(x_1, x_2, x_3) = \left(x_1 + x_2 + \frac{x_2^2}{2}, x_2, x_2 + x_3 \right)$$

and for each $u \in U$, define the diffeomorphism $f_u : G \to G$ by

$$f_u(x_1, x_2, x_3) = \left(x_1 + x_2 + \frac{x_2^2}{2} + ux_2 + ux_3 - \frac{u}{2} - \frac{u^2}{3}, x_2 + u, x_2 + x_3 - \frac{u}{2} \right).$$

It is not difficult to see that for each $g \in G$, $f_u(g) = f_u(e) f_0(g)$. Hence, the system

$$\Sigma : \ g_{k+1} = f(g_k, u_k)$$

on G is linear. By Proposition 2.3, the solutions are given by

$$\varphi(k, g, u) = \varphi(k, e, u) f_0^k(g) = \varphi(k, e, u) \left(x_1 + kx_2 + \frac{k}{2} x_2^2, x_2, kx_2 + x_3 \right).$$

Remark 2.7. For a discrete-time linear control system $g_{k+1} = f(g_k, u_k)$, $u_k \in U$, on a Lie group G, we have that $(\mathrm{d}f_0)_e : \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra isomorphism, because f_0 is an automorphism of G. Also, we can see that $(\mathrm{d}f_0^n)_e = [(\mathrm{d}f_0)_e]^n$ for each $n \in \mathbb{Z}$ and hence for each $X \in \mathfrak{g}$

$$f_0^n(\exp X) = \exp([(\mathrm{d}f_0)_e]^n X).$$

Given the automorphism f_0, we can define several Lie subalgebras that are intrinsically associated with its dynamics. Consider an eigenvalue α of $(\mathrm{d}f_0)_e$ and its generalized eigenspace

$$\mathfrak{g}_\alpha = \{ X \in \mathfrak{g}; \ ((\mathrm{d}f_0)_e - \alpha)^n X = 0, \ \text{for some } n \geq 1 \}.$$

The following proposition, whose proof can be found in Ayala and Da Silva ([1], Prop. 2.1), will be useful in order to define dynamical subalgebras of $\mathfrak{g}$ related to the differential of f_0 at $e \in G$.

Proposition 2.8. *If α and β are eigenvalues of $(\mathrm{d}f_0)_e$, then*

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha\beta},$$

where $\mathfrak{g}_{\alpha\beta} = \{0\}$ if $\alpha\beta$ is not an eigenvalue of $(\mathrm{d}f_0)_e$.

Due to Proposition 2.8, we define the unstable, center, and unstable Lie subalgebras (*cf.* [1]) by

$$\mathfrak{g}^+ = \bigoplus_{|\alpha|>1} \mathfrak{g}_\alpha, \quad \mathfrak{g}^0 = \bigoplus_{|\alpha|=1} \mathfrak{g}_\alpha, \quad \mathfrak{g}^- = \bigoplus_{0<|\alpha|<1} \mathfrak{g}_\alpha. \tag{2.2}$$

Proposition 2.8 implies that $\mathfrak{g}^+$ and $\mathfrak{g}^-$ are nilpotent Lie subalgebras of $\mathfrak{g}$. Since $(\mathrm{d}f_0)_e$ is a Lie algebra isomorphism, the decomposition $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^-$ holds. The center-unstable and center-stable Lie subalgebras are

$$\mathfrak{g}^{+,0} = \mathfrak{g}^+ \oplus \mathfrak{g}^0 \quad \text{and} \quad \mathfrak{g}^{-,0} = \mathfrak{g}^- \oplus \mathfrak{g}^0.$$

We will also need the connected Lie groups $G^{-,0}$, $G^{+,0}$, G^- and G^+ corresponding to $\mathfrak{g}^{-,0}$, $\mathfrak{g}^{+,0}$, $\mathfrak{g}^-$ and $\mathfrak{g}^+$, respectively. In particular, we refer to G^- and G^+ as the stable and the unstable subgroup of the uncontrolled system, resp.

Remark 2.9. The restrictions of $(\mathrm{d}f_0)_e$ to the Lie subalgebras $\mathfrak{g}^+$, $\mathfrak{g}^0$, and $\mathfrak{g}^-$ satisfy:

$$|[(\mathrm{d}f_0)_e]^n X| \geq c\sigma^{-n} |X|, \quad \text{for any } X \in \mathfrak{g}^+, \ n \in \mathbb{N},$$

and

$$|[(\mathrm{d}f_0)_e]^n Y| \leq c^{-1}\sigma^n |Y|, \quad \text{for any } Y \in \mathfrak{g}^-, \ n \in \mathbb{N},$$

for some $c \geq 1$ and $\sigma \in (0, 1)$ and, for all $a > 0$ and $Z \in \mathfrak{g}^0$ it holds that

$$|[(\mathrm{d}f_0)_e]^n Z| \sigma^{a|n|} \to 0 \ \text{ as } \ n \to \pm\infty.$$

3. Linear systems induced on homogeneous spaces

In this section, we define a induced (discrete-time) linear system on a homogeneous space G/H from a discrete-time linear system on a Lie group G. This construction will be important to get a formula for the outer invariance entropy presented in the Theorem 4.4.

Consider the discrete-time linear control system

$$g_{k+1} = f(g_k, u_k), \quad u = (u_i)_{i \in \mathbb{N}_0} \in \mathcal{U}, \tag{3.1}$$

on a Lie group G and let H a Lie subgroup of G which is f_0-invariant. If H is closed, this induces a control system on the homogeneous space G/H in the following way: define $\bar{f} : G/H \times U \to G/H$ as $\bar{f}(gH, u) = f(g, u)H$. The f_0-invariance of H implies that $\bar{f}$ is well defined. Note also that for each $u \in U$, the map $\bar{f}_u : G/H \to G/H$ is a diffeomorphism with inverse $\bar{f}_u^{-1}(gH) = f_u^{-1}(g)H$, because $\bar{f}_u \circ \pi = \pi \circ f_u$ and $\bar{f}_u^{-1} \circ \pi = \pi \circ f_u^{-1}$ are differentiable maps, where $\pi : G \to G/H$ is the natural projection. This induces a discrete-time control system

$$x_{k+1} = \bar{f}(x_k, u_k), \ x_k \in G/H, \ u = (u_i)_{i \in \mathbb{N}_0} \in \mathcal{U}, \tag{3.2}$$

on G/H with solutions denoted by $\bar{\varphi}$. Since for each $k \in \mathbb{N}$, $\bar{f}_0^k \circ \pi = \pi \circ f_0^k$, we have that

$$\pi(\varphi(k, g, u)) = \bar{\varphi}(k, \pi(g), u),$$

for all $k \in \mathbb{N}_0$, $g \in G$ and $u \in \mathcal{U}$. In other words, $(\pi, \mathrm{id}_{\mathcal{U}})$ is a semi-conjugacy between the systems (3.1) and (3.2) (*cf.* Kawan [12], Def. 2.4). For each $g \in G$ we denote by $\mathcal{L}_g$ the left translation $G/H \ni xH \mapsto gxH \in G/H$. Then we find that

$$\begin{aligned}
\bar{\varphi}(k, gH, u) = \pi(\varphi(k, g, u)) &= \pi(\varphi_{k,u}(e)f_0^k(g)) \\
&= \varphi_{k,u}(e)f_0^k(g)H = \varphi_{k,u}(e)\bar{f}_0^k(gH) \\
&= \mathcal{L}_{\varphi_{k,u}(e)}(\bar{f}_0^k(gH)).
\end{aligned} \tag{3.3}$$

This shows that the solutions of (3.2) satisfy properties similar to equality (2.1) for a linear system on G, hence we call it the **induced linear system** on the homogeneous space G/H.

Proposition 3.1. *Let G a Lie group with Lie algebra $\mathfrak{g}$. Assume that $\mathfrak{g}$ decomposes as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{l}$, where $\mathfrak{h}$ and $\mathfrak{l}$ are $(\mathrm{d}f_0)_e$-invariant Lie subalgebras of $\mathfrak{g}$. Consider the connected Lie subgroup H of G with Lie algebra $\mathfrak{h}$. If H is closed, then $(\mathrm{d}\bar{f}_0^k)_{eH} = (\mathrm{d}f_0)_e^k|_{\mathfrak{l}}$, for all $k \in \mathbb{N}_0$.*

Proof. The $(\mathrm{d}f_0)_e$-invariance of $\mathfrak{h}$ and $\mathfrak{l}$ allows us to consider the well defined linear isomorphism $\overline{(\mathrm{d}f_0)_e} : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h}$ given by $\overline{(\mathrm{d}f_0)_e}(X + \mathfrak{h}) = (\mathrm{d}f_0)_e X + \mathfrak{h}$. If π is the natural projection of G onto G/H, we can see that $\overline{(\mathrm{d}f_0)_e}$ satisfies $\overline{(\mathrm{d}f_0)_e} \circ (\mathrm{d}\pi)_e = (\mathrm{d}\pi)_e \circ (\mathrm{d}f_0)_e$. Hence, for all $X \in \mathfrak{g}$ and $k \in \mathbb{N}_0$ we have

$$\begin{aligned}
(\mathrm{d}\bar{f}_0^k)_{eH}(X + \mathfrak{h}) = d(\bar{f}_0^k \circ \pi)_e X = d(\pi \circ f_0^k)_e X &= (\mathrm{d}\pi)_e \circ (\mathrm{d}f_0^k)_e X \\
&= (\mathrm{d}\pi)_e \circ (\mathrm{d}f_0)_e^k X = (\overline{(\mathrm{d}f_0)_e})^k \circ (\mathrm{d}\pi)_e X \\
&= (\overline{(\mathrm{d}f_0)_e})^k (X + \mathfrak{h}).
\end{aligned}$$

By invariance, we can identify $\mathfrak{g}/\mathfrak{h}$ with $\mathfrak{l}$ and, therefore, $\overline{(\mathrm{d}f_0)_e}$ with $(\mathrm{d}f_0)_e^k|_{\mathfrak{l}}$ and the desired equality holds. $\square$

Definition 3.2. A measure μ on a homogeneous space G/H is a G-**invariant** Borel measure if $\mu(\mathcal{L}_g(A)) = \mu(A)$, for all $g \in G$ and all Borel set $A \subset G/H$.

The following proposition is an adaptation of Da Silva ([7], Prop. 4.6) in our context of discrete-time linear systems.

Proposition 3.3. *If the connected subgroup G^- corresponding to $\mathfrak{g}^-$ defined in (2.2) is closed, the homogeneous space G/G^- admits a unique (up to a scalar) G-invariant Borel measure.*

Proof. Denote by Δ_G and Δ_{G^-} the modular functions of G and G^-, respectively, *cf.* Chapter VIII, Section 2 of Knapp [13]. The result will follow from Theorem 8.36 in [13], if we can show that $(\Delta_G)|_{G^-} = \Delta_{G^-}$.

Since G^- is nilpotent, $\Delta_{G^-}(h) = 1$ for all $h \in G^-$. On the other hand, for any two eigenvalues α, β of $(\mathrm{d}f_0)_e$ we have

$$\mathrm{ad}(\mathfrak{g}_\alpha)^n \mathfrak{g}_\beta \subset \mathfrak{g}_{\alpha^n \beta}, \quad \forall\, n \in \mathbb{N},$$

which implies that for each $X \in \mathfrak{g}^-$, $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ is a nilpotent linear map. Writing for $X \in \mathfrak{g}^-$

$$X = \sum_{|\alpha| < 1} a_\alpha X_\alpha, \quad a_\alpha \in \mathbb{R},$$

where the sum is taken over all eigenvalues α of $(\mathrm{d}f_0)_e$ with $|\alpha| < 1$, we get

$$\mathrm{tr}(\mathrm{ad}(X)) = \sum_{|\alpha| < 1} a_\alpha \mathrm{tr}(\mathrm{ad}(X_\alpha)) = 0.$$

Now, consider $g \in G^-$. Then g can be written as

$$g = \exp X_1 \cdots \exp X_r,$$

for some $X_1, \ldots, X_r \in \mathfrak{g}^-$, because G^- is connected. Therefore,

$$
\begin{aligned}
\Delta_G(g) &= \Delta_G(\exp X_1 \cdots \exp X_r) = \Delta_G(\exp X_1) \cdots \Delta_G(\exp X_r) \\
&= |\det(\mathrm{Ad}(\exp X_1))| \cdots |\det(\mathrm{Ad}(\exp X_r))| \\
&= \left|\det\left(e^{\mathrm{ad}(X_1)}\right)\right| \cdots \left|\det\left(e^{\mathrm{ad}(X_r)}\right)\right| \\
&= \left|e^{\mathrm{tr}(\mathrm{ad}(X_1))}\right| \cdots \left|e^{\mathrm{tr}(\mathrm{ad}(X_r))}\right| = 1.
\end{aligned}
$$

This shows that $(\Delta_G)|_{G^-} = \Delta_{G^-}$ and, by Theorem 8.36 in [13], the result follows. $\square$

Remark 3.4. According to Theorem 8.36 of [13], if ν_G and ν_H are the left invariant Haar measures on G and H, respectively, the G-invariant Borel measure μ on G/H can be normalized so that

$$\int_G \phi(g)\, \mathrm{d}\nu_G(g) = \int_{G/H} \int_H \phi(gh)\, \mathrm{d}\nu_H(h)\, \mathrm{d}\mu(gH),$$

for all continuous function $\phi : G \to \mathbb{R}$ with compact support. Note that if $K \subset G$ is compact, then $\nu_G(K) > 0$ implies $\mu(\pi(K)) > 0$. This follows from the fact that $\overline{\chi_K}(gH) = \int_H \chi_K(gh)\, \mathrm{d}\nu_H(h)$ is a bounded positive function and $\overline{\chi_K} > 0$ if and only if $\chi_{\pi(K)} > 0$.

4. Outer invariance entropy

In this section we prove our main result that provides an upper bound for outer invariance entropy of discrete-time linear control systems on Lie groups. We begin by recalling the definition of invariance entropy and outer invariance entropy as given in Kawan ([12], Defs. 2.2 and 2.3).

Consider a discrete-time control system

$$\Sigma \ : \ x_{k+1} = f(x_k, u_k), \ \ u_k \in U, \ x \in M,$$

with solutions $\varphi(k, x, u), k \in \mathbb{N}_0$. A pair (K, Q) of nonempty subsets of M is called **admissible**, if K is compact and for each $x \in K$, there exists $u \in \mathcal{U}$ such that $\varphi(k, x, u) \in Q$ for all $k \in \mathbb{N}_0$.

Given an admissible pair (K, Q) and $n \in \mathbb{N}$, we say that a set $\mathcal{S} \subset \mathcal{U}$ is (n, K, Q)-spanning if

$$\forall \ x \in K \ \exists \ u \in \mathcal{S} \ \forall \ j \in \{1, \dots, n\} : \ \varphi(j, x, u) \in Q.$$

Denote by $r_{\mathrm{inv}}(n, K, Q)$ the minimal number of elements such a set can have (if there is no finite set with this property we set $r_{\mathrm{inv}}(n, K, Q) = \infty$).

The existence of (n, K, Q)-spanning sets is guaranteed, since $\mathcal{U}$ is (n, K, Q)-spanning for every $n \in \mathbb{N}$.

Definition 4.1. Given an admissible pair (K, Q) for a discrete-time control system Σ, the invariance entropy of (K, Q) is defined by

$$h_{\mathrm{inv}}(K, Q) = h_{\mathrm{inv}}(K, Q; \Sigma) := \limsup_{n \to \infty} \frac{1}{n} \log r_{\mathrm{inv}}(n, K, Q).$$

Here, and throughout the paper, log denotes the logarithm with base 2.

The invariance entropy of (K, Q) measures the exponential growth rate of the minimal number of control functions sufficient to stay in Q when starting in K as time tends to infinity. Hence, invariance entropy is a nonnegative (possibly infinite) quantity which is assigned to an admissible pair (K, Q). For our main result we need the following related quantity. Note that for an admissible pair (K, Q) also every pair $(K, N_\varepsilon(Q)), \varepsilon > 0$, is admissible, where $N_\varepsilon(Q) = \{x \in M \,|\, d(x, Q) < \varepsilon\}$ denotes the ε-neighborhood of Q.

Definition 4.2. Given an admissible pair (K, Q) such that Q is closed in M and a metric d on M, we define the outer invariance entropy of (K, Q) by

$$h_{\mathrm{inv,out}}(K, Q) := h_{\mathrm{inv,out}}(K, Q, \Sigma, d) := \lim_{\varepsilon \to 0} h_{\mathrm{inv}}(K, N_\varepsilon(Q)) = \sup_{\varepsilon > 0} h_{\mathrm{inv}}(K, N_\varepsilon(Q)).$$

Obviously, the inequality $h_{\mathrm{inv}}(K, Q) \geq h_{\mathrm{inv,out}}(K, Q)$ holds. The next proposition (*cf.* Kawan [12], Prop. 2.13) describes the behavior of outer invariance entropy under semi-conjugacy. Recall that a semi-conjugacy of two discrete-time control systems on metric spaces M_1 and M_2, respectively, given by

$$x_{k+1} = f_1(x_k, u_k), u_k \in U, \ \text{and} \ \ y_{k+1} = f_2(y_k, v_k), V_k \in V,$$

with solutions φ_1 and φ_2, resp., is a pair (π, h) of continuous maps $\pi : M_1 \to M_2$ and $h : U^{\mathbb{N}_0} \to V^{\mathbb{N}_0}$ such that

$$\pi(\varphi_1(k, x, u)) = \varphi_2(k, \pi(x), h(u)) \ \text{for all} \ k \in \mathbb{N}_0, \ x \in M, u \in U^{\mathbb{N}_0}. \tag{4.1}$$

Proposition 4.3. *Consider two discrete-time control systems Σ_1 and Σ_2 and let (π, h) be a semi-conjugacy from Σ_1 to Σ_2. Then every admissible pair (K, Q) for Σ_1 with compact Q defines an admissible pair $(\pi(K), \pi(Q))$ for Σ_2 and*

$$h_{\mathrm{inv,out}}(K, Q; \Sigma_1) \geq h_{\mathrm{inv,out}}(\pi(K), \pi(Q); \Sigma_2).$$

Now, we will briefly recall the notion of topological entropy as defined by Bowen and Dinaburg (*cf. e.g.* Walters [17]). Let (M, d) be a metric space and $\phi : M \to M$ be a continuous map. Given a compact set $K \subset X$

and $n \in \mathbb{N}$ we say that a set $F \subset M$ is (n, ε)-spanning set for K with respect to ϕ if, for every $y \in K$, there exists $x \in F$ such that

$$d(\phi^j(x), \phi^j(y)) < \varepsilon, \quad \text{for all} \ \ j \in \{0, 1, \ldots, n\}.$$

If we denote by $r_n(\varepsilon, K)$ the minimal cardinality of an (n, ε)-spanning set for K with respect to ϕ, the topological entropy of ϕ over K is defined by

$$h_{\text{top}}(\phi, K) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log r_n(\varepsilon, K)$$

and the topological entropy of ϕ is

$$h_{\text{top}}(\phi) = \sup_{K \text{ compact}} h_{\text{top}}(\phi, K).$$

Alternatively, topological entropy can be defined using (n, ε)-separated sets $E \subset K$ with respect to ϕ which are defined as sets satisfying

$$\forall \ x, y \in E : x \neq y \Rightarrow \exists j \in \{0, \ldots, n\} : \ d(\phi^j(x), \phi^j(y)) > \varepsilon.$$

Let $s_n(\varepsilon, K)$ denote the largest cardinality of any (n, ε)-separated subset of K with respect to ϕ. If E is an (n, ε)-separated subset of K of maximal cardinality, then E is (n, ε)-spanning for K. The topological entropy of ϕ over K can also be characterized as the limit

$$h_{\text{top}}(\phi, K) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s_n(\varepsilon, K).$$

Bowen ([4], Cor. 16) shows that for an endomorphism ϕ of a Lie group G the topological entropy is

$$h_{\text{top}}(\phi) = \sum_{|\lambda| > 1} \log |\lambda|, \tag{4.2}$$

where the sum is taken over all eigenvalues λ of $d(\phi)_e$ with $|\lambda| > 1$.

Now we formulate the main result of this paper.

Theorem 4.4. *Let (K, Q) be an admissible pair for the discrete-time linear system $\Sigma : g_{k+1} = f(g_k, u_k)$, $u_k \in U$, on a connected Lie group G. Assume that Q is compact.*

i) Then the outer invariance entropy satisfies

$$h_{\text{inv,out}}(K, Q) \leq \sum_{|\lambda| > 1} \log |\lambda| = h_{\text{top}}(f_0),$$

where the sum is taken over all eigenvalues λ of $(\mathrm{d}f_0)_e$ with $|\lambda| > 1$.

ii) Moreover, suppose that the connected subgroup G^- corresponding to $\mathfrak{g}^-$ defined in (2.2) is closed and that $\nu_G(K) > 0$ for a left invariant Haar measure ν_G on G. Then

$$h_{\text{inv,out}}(K, Q) = \sum_{\lambda} \log |\lambda| = h_{\text{top}}(f_0),$$

where the sum is taken over all eigenvalues λ of $(\mathrm{d}f_0)_e$ with $|\lambda| > 1$.

Proof. (i) The characterization of the topological entropy of the automorphism f_0 follows from Bowen's result (4.2). Since Q is compact, Kawan ([12], Prop. 2.5) implies that the invariance entropy does not depend on the metric. Hence we may choose a left invariant Riemannian metric ϱ on G. This yields for $g \in G$,

$$\varrho(\varphi_{k,u}(g), \varphi_{k,u}(h)) = \varrho(\varphi_{k,u}(e)f_0^k(g), \varphi_{k,u}(e)f_0^k(h)) = \varrho(f_0^k(g), f_0^k(h)).$$

Now, fix $\varepsilon > 0$ and $n \in \mathbb{N}$. Let $E \subset K$ be an (n, ε)-separated subset of K with respect to f_0 of maximal cardinality $s_n(\varepsilon, K)$. Since (K, Q) is admissible, for each $h \in E$, there exists $u_h \in \mathcal{U}$ such that $\varphi(k, h, u_h) \in Q$ for $k = 1, \ldots, n$. We claim that $\mathcal{S} := \{u_h \in \mathcal{U}; \ h \in E\}$ is an $(n, K, N_\varepsilon(Q))$-spanning set. In fact, since the set E is also (n, ε)-spanning, one finds for all $g \in K$ an element $h \in E$ with

$$\varrho(\varphi_{k,u_h}(g), \varphi_{k,u_h}(h)) = \varrho(f_0^k(g), f_0^k(h)) < \varepsilon$$

for all $k \in \{0, \ldots, n\}$. This shows that $\varphi_{k,u_h}(g) \in N_\varepsilon(Q)$ and hence proves the claim. It follows that $r_{\mathrm{inv}}(n, K, N_\varepsilon(Q)) \leq s_n(\varepsilon, K)$ implying

$$h_{\mathrm{inv,out}}(K, N_\varepsilon(Q)) = \limsup_{n \to \infty} \frac{1}{n} \log r_{\mathrm{inv}}(n, K, N_\varepsilon(Q)) \leq \limsup_{n \to \infty} \frac{1}{n} \log s_n(\varepsilon, K).$$

Letting ε tend to 0, one finds by formula (4.2) for the topological entropy

$$h_{\mathrm{inv,out}}(K, Q) \leq h_{\mathrm{top}}(f_0) = \sum_{|\lambda| > 1} \log |\lambda|,$$

where the sum is taken over all eigenvalues λ of $(\mathrm{d}f_0)_e$ with $|\lambda| > 1$.

(ii) It remains to show the reverse inequality. Recall that $(\pi, \mathrm{id}_\mathcal{U})$ is a semi-conjugacy between system (3.1) and the induced system (3.2) with $H = G^-$. Hence Proposition 4.3 implies

$$h_{\mathrm{inv}}(K, N_\varepsilon(Q)) \geq h_{\mathrm{inv}}(\pi(K), N_\varepsilon(\pi(Q))),$$

and it suffices to estimate the right hand side. By Kawan ([12], Lem. A.3), there is $\varepsilon > 0$ small enough such that $\overline{N_\varepsilon(\pi(Q))}$ is compact. The G-invariant Borel measure μ on G/G^- whose existence is guaranteed by Proposition 3.3 satisfies $\mu(N_\varepsilon(\pi(Q))) \leq \mu(\overline{N_\varepsilon(\pi(Q))}) < \infty$, because this measure is finite on compact sets. Consider for $n \in \mathbb{N}$ an $(n, \pi(K), N_\varepsilon(\pi(Q)))$-spanning set $\mathcal{S} = \{u_1, \ldots, u_r\}$ with minimal cardinality $r = r_{inv}(n, \pi(K), N_\varepsilon(\pi(Q)))$. Define for $j \in \{1, \ldots, r\}$,

$$K_j := \{gG^- \in \pi(K); \ \bar{\varphi}(k, gG^-, u_j) \in N_\varepsilon(\pi(Q)), \ \forall \ k = 0, \ldots, n\}.$$

Each K_j is a Borel set and these sets cover $\pi(K)$ by the choice of $\mathcal{S}$. Since for each $u \in U$ the map $\bar{f}_u$ and hence also $\bar{\varphi}_{n,u_j}$ are diffeomorphisms on G/G^-, the set $\bar{\varphi}_{n,u_j}(K_j)$ is a Borel set. The inclusion $\bar{\varphi}_{n,u_j}(K_j) \subset N_\varepsilon(\pi(Q))$ implies

$$\mu(\bar{\varphi}_{n,u_j}(K_j)) \leq \mu(N_\varepsilon(\pi(Q))) < \infty.$$

Using the left invariance of μ equality (3.3) yields

$$\mu(\bar{\varphi}_{n,u_j}(K_j)) = \mu(\bar{\varphi}_{n,u_j}(e)\bar{f}_0^n(K_j)) = \mu(\bar{f}_0^n(K_j)).$$

Since $\bar{f}_0$ is an diffeomorphism, $\bar{f}_0 \circ \mathcal{L}_g = \mathcal{L}_{f_0(g)} \circ \bar{f}_0$ and $|\det d(\mathcal{L}_g)_{hG^-}| = 1$ for all $g, h \in G$, we obtain

$$
\begin{aligned}
|\det(d\bar{f}_0^n)_{gG^-}| &= |\det d(\mathcal{L}_{f_0^n(g)})_{eG^-}| |\det[(d\bar{f}_0^n)_{eG^-}]| |\det d(\mathcal{L}_{g^{-1}})_{gG^-}| \\
&= |\det(d\bar{f}_0^n)_{eG^-}| = \left|\det\left((df_0)_e^n|_{\mathfrak{g}^{+,0}}\right)\right|,
\end{aligned}
$$

where the last equality follows from Proposition 3.1.

By the left invariance of μ we have

$$
\begin{aligned}
\mu(\bar{f}_0^n(K_j)) &= \int_{\bar{f}_0^n(K_j)} d\mu(gG^-) = \int_{K_j} |\det(d\bar{f}_0^n)_{gG^-}| d\mu(gG^-) \\
&= \left|\det\left((df_0)_e^n|_{\mathfrak{g}^{+,0}}\right)\right| \int_{K_j} d\mu(gG^-) = \left|\det\left((df_0)_e|_{\mathfrak{g}^{+,0}}\right)\right|^n \mu(K_j)
\end{aligned}
$$

Together, these relations yield

$$
\begin{aligned}
\mu(\pi(K)) &\leq \sum_{j=1}^r \mu(K_j) \leq r \max_{1 \leq j \leq r} \mu(K_j) = r \max_{1 \leq j \leq r} \frac{\mu(\bar{f}_0^n(K_j))}{\left|\det\left((df_0)_e|_{\mathfrak{g}^{+,0}}\right)\right|^n} \\
&= r \max_{1 \leq j \leq r} \frac{\mu(\bar{\varphi}_{n,u_j}(K_j))}{\left|\det\left((df_0)_e|_{\mathfrak{g}^{+,0}}\right)\right|^n} \leq r \frac{\mu(N_\varepsilon(\pi(Q)))}{\left|\det\left((df_0)_e|_{\mathfrak{g}^{+,0}}\right)\right|^n},
\end{aligned}
$$

which implies that

$$
r_{\mathrm{inv}}(n, \pi(K), N_\varepsilon(\pi(Q))) = r \geq \frac{\mu(\pi(K))}{\mu(N_\varepsilon(\pi(Q)))} \left|\det\left((df_0)_e|_{\mathfrak{g}^{+,0}}\right)\right|^n.
$$

Note that $\mu(\pi(K)) > 0$, because $\nu_G(K) > 0$ by hypothesis (see Rem. 3.4). Denoting $C := \dfrac{\mu(\pi(K))}{\mu(N_\varepsilon(Q))} > 0$ we get

$$
\begin{aligned}
h_{\mathrm{inv}}(\pi(K), N_\varepsilon(\pi(Q))) &\geq \limsup_{n \to \infty} \frac{1}{n} \log\left(C \left|\det\left((df_0)_e|_{\mathfrak{g}^{+,0}}\right)\right|^n\right) \\
&= \limsup_{n \to \infty} \frac{1}{n} \left(\log C + n \log\left|\det\left((df_0)_e|_{\mathfrak{g}^{+,0}}\right)\right|^n\right) \\
&= \log \prod_{|\lambda| \geq 1} |\lambda| = \sum_{|\lambda| > 1} \log|\lambda|,
\end{aligned}
$$

where the product is taken over all eigenvalues λ of $(df_0)_e$ with $|\lambda| \geq 1$ and the sum is taken over all eigenvalues λ of $(df_0)_e$ with $|\lambda| > 1$. Taking $\varepsilon \searrow 0$ the result follows. $\qquad \square$

Remark 4.5. The formula presented in Theorem 4.4 (ii) holds for the class of solvable, connected and simply connected Lie groups G. In fact, by San Martin ([15], Prop. 10.6) all connected subgroups of G are closed, hence G^- is closed. In Section 5 we show that G^- is still closed even when G is not solvable (see Thm. 5.3) and Theorem 4.4 (ii) can be applied for discrete-time linear systems on a broader class of Lie groups.

Example 4.6. Let (K, Q) be an admissible pair for the system defined in Example 2.5, where Q is compact and K has positive Haar measure. Note that the matrix of the differential of f_0 is given by

$$
(df_0)_{(1,0)} = \begin{bmatrix} 1 & 0 \\ 0 & e^2 \end{bmatrix}.
$$

In this case, $\mathfrak{g}^- = \{0\}$ and $G^- = \{(1,0)\}$, which is closed. By Theorem 4.4 (ii), $h_{\mathrm{inv,out}}(K,Q) = 2$.

Example 4.7. Again, consider an admissible pair (K,Q) of the system presented in Example 2.6. Assume that Q is compact and K has positive Haar measure. The matrix of $(\mathrm{d}f_0)_e$ in the canonical basis is

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

and we can see that the unique eigenvalue of $(\mathrm{d}f_0)_e$ is 1, hence $h_{\mathrm{inv,out}}(K,Q) = 0$ by Theorem 4.4.

Remark 4.8. For linear control systems in Euclidean space, *cf.* Example 2.2, natural candidates for admissible pairs (K,Q) such that Q has compact closure, as considered in Theorem 4.4, can be obtained by Colonius, Cossich and Santana ([5], Thm. 32) as follows: If A is hyperbolic, there exists a unique control set D with nonvoid interior (*i.e.*, a maximal set of approximate controllability with $\mathrm{int}\, D \neq \varnothing$) and it is bounded. Hence its closure $\overline{D}$ is compact and for any compact subset $K \subset Q := \overline{D}$ the pair (K,Q) is admissible with compact Q. Results on control sets for continuous-time linear systems on Lie groups are proved in Ayala *et al.* [2, 3]. For discrete-time linear systems on Lie groups, the control sets have not be studied.

5. CLOSEDNESS OF THE STABLE SUBGROUP

The formula for the outer invariance entropy in Theorem 4.4 has been derived under the assumption that the stable subgroup G^- is closed. In this section we provide a sufficient condition for this property. The arguments can also be applied to the unstable subgroup G^+. First we show that these subgroups are simply connected.

Proposition 5.1. *The stable subgroup G^- and the unstable subgroup G^+ are simply connected.*

Proof. We will just show this fact for G^-, because the proof for G^+ is analogous. Denote the exponential map $\exp : \mathfrak{g} \to G$ restricted to $\mathfrak{g}^-$ by $\exp^-$, which is the exponential map of G^-. Since G^- is nilpotent and connected, $\exp^- : \mathfrak{g}^- \to G^-$ is a covering map, hence it is surjective, continuous and open. Hence, if we show that $\exp^-$ is injective, it will be a homeomorphism. Therefore simple connectedness of $\mathfrak{g}^-$ implies simple connectedness of G^-.

In order to show that $\exp^-$ is injective, let $X, Y \in \mathfrak{g}^-$ such that $\exp^- X = \exp^- Y$. Consider open neighborhoods V and U of $0 \in \mathfrak{g}$ and $e \in G$, respectively, such that $\exp : V \to U$ is a diffeomorphism. Remark 2.9 implies that for all $Z \in \mathfrak{g}^-$ it holds

$$|(\mathrm{d}f_0)_e^n(Z)| \leq c\mu^n |Z| \text{ for all } n \in \mathbb{N},$$

for some $c \geq 1$ and $\mu \in (0,1)$. Consider n large enough such that $(\mathrm{d}f_0)_e^n(Y), (\mathrm{d}f_0)_e^n(Y) \in V$. Hence,

$$\exp^-((\mathrm{d}f_0)_e^n(X)) = f_0^n(\exp^- X) = f_0^n(\exp^- Y) = \exp^-((\mathrm{d}f_0)_e^n(Y)).$$

The injectivity of $\exp|_V$ implies that $\exp^-|_V = \exp|_{V \cap \mathfrak{g}^-}$ is injective, and hence $(\mathrm{d}f_0)_e^n(X) = (\mathrm{d}f_0)_e^n(Y)$. Since $(\mathrm{d}f_0)_e|_{\mathfrak{g}^-}$ is an isomorphism, we obtain $X = Y$. $\square$

Remark 5.2. The maps $\exp^\pm = \exp|_{G^\pm} : \mathfrak{g}^\pm \to G^\pm$ are diffeomorphisms, because $G^\pm$ are connected, simply connected and nilpotent (see Knapp [13], Thm. 1.127 or San Martin [15], Thm. 10.8).

The following theorem presents the announced sufficient condition for the closedness of G^- (and G^+).

Theorem 5.3. *If G is simply connected, the stable and the unstable subgroup G^- and G^+, resp., of the uncontrolled system are closed.*

Proof. We will just show this fact for G^-, because the proof for G^+ is analogous. Since G is simply connected and G^- is connected, G^- cannot be dense in G, hence the open set $G \setminus \overline{G^-}$ is non-empty. The continuity of $\exp : \mathfrak{g} \to G$ implies that $\exp^{-1}\left(G \setminus \overline{G^-}\right)$ is a non-empty open set of $\mathfrak{g}$. In order to show that $\mathfrak{g}^{+,0} \setminus \{0\} \subset \exp^{-1}\left(G \setminus \overline{G^-}\right)$, consider $Z \in \mathfrak{g}^{+,0} \setminus \{0\}$ and suppose that $\exp(Z) \in \overline{G^-}$. Then there is a sequence $(g_n) \subset G^-$ such that $g_n \to \exp(Z)$. Since $\exp^- := \exp|_{\mathfrak{g}^-} : \mathfrak{g}^- \to G^-$ is surjective, there is a sequence $(X_n) \subset \mathfrak{g}^-$ with $g_n = \exp(X_n)$. Hence $\exp(X_n) \to \exp(Z)$, so $X_n \to Z$. The closedness of $\mathfrak{g}^-$ implies that $Z \in \mathfrak{g}^- \cap \mathfrak{g}^{+,0}$, that is, $Z = 0$ which is a contradiction.

Assume that G^- is not closed and denote by $\overline{\mathfrak{g}^-}$ the Lie algebra of $\overline{G^-}$. Then $\mathfrak{g}^-$ is properly contained in $\overline{\mathfrak{g}^-}$. Consider $X \in \overline{\mathfrak{g}^-} \setminus \mathfrak{g}^-$. Then there are $Y \in \mathfrak{g}^+$, $Z \in \mathfrak{g}^0$ and $X^- \in \mathfrak{g}^-$ such that

$$X = Y + Z + X^-,$$

where Y or Z is non-zero. Hence $Y + Z = X - X^- \in \mathfrak{g}^{+,0} \cap \overline{\mathfrak{g}^-}$. But it implies that $\exp^{-1}\left(G \setminus \overline{G^-}\right) \cap \overline{\mathfrak{g}^-} \neq \emptyset$, which is a contradiction, because $\exp^{-1}\left(G \setminus \overline{G^-}\right)$ and $\overline{\mathfrak{g}^-}$ are disjoint. $\square$

The following example shows that the subgroups G^+ and G^- may not be closed, if the group G is not simply connected.

Example 5.4. Consider the following automorphism on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ which, naturally, is not simply connected:

$$f_0((x,y) + \mathbb{Z}^2) = (2x + y, x + y) + \mathbb{Z}^2.$$

Then $(\mathrm{d}f_0)_e : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$(\mathrm{d}f_0)_e = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Hence, the eigenvalues of $(\mathrm{d}f_0)_e$ are $\lambda_1 = \frac{1}{2}(3 + \sqrt{5}) > 1$ and $\lambda_2 = \frac{1}{2}(3 - \sqrt{5}) \in (0,1)$. Moreover, we have

$$\mathfrak{g}^+ = \mathrm{span}\left\{\left(\frac{1}{2}(3 + \sqrt{5}), 1\right)\right\}, \quad \mathfrak{g}^- = \mathrm{span}\left\{\left(\frac{1}{2}(3 - \sqrt{5}), 1\right)\right\}.$$

Therefore

$$G^+ = \left\{\exp\left(t\left(\frac{1}{2}(3 + \sqrt{5}), 1\right)\right); \ t \in \mathbb{R}\right\},$$

$$G^- = \left\{\exp\left(t\left(\frac{1}{2}(3 - \sqrt{5}), 1\right)\right); \ t \in \mathbb{R}\right\}.$$

Both G^+ and G^- are irrational flows on $\mathbb{T}^2$ which are dense in $\mathbb{T}^2$, hence they are not closed.

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