Infinite dimensional symmetric spaces

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Introduction

The main goal of this work is to extend to infinite dimensional settings well known results in differential geometry, more precisely in the theory of symmetric spaces. The idea of studying various aspects of local differential geometry (like covariant derivative and geodesics) in infinite dimensions, more precisely on open subsets of Banach spaces, goes back to the 1930s. In the 1950s, due mainly to J. Eells, formal infinite dimensional manifold structures have been given to different spaces of functions. The study of infinite dimensional manifolds has been particularly intense in the 1960s and 1970s. One should mention here in connection to our work the important paper ([3]) of J. Eells.

A central example of an infinite dimensional manifold is provided by the set of maps from a compact manifold to another finite dimensional manifold. In our work we will deal extensively with the well known particular case of loop groups, for which the domain space is the circle S^1 and the target space a Lie group. More generally, the set of cross sections of a differentiable fibre bundle over a compact manifold admits a manifold structure. This type of manifolds lies at the heart of the branch of mathematics called global analysis - see [21]. Another example of infinite dimensional manifolds (and Lie groups) we will consider is that of the Kac-Moody Lie groups. The Kac-Moody Lie algebras have been introduced by V. Kac and R. Moody in the mid-1960s. They are infinite dimensional and a special class of them, the so called affine Kac-Moody Lie algebras, come from a class of infinite dimensional Lie groups, the affine Kac-Moody Lie groups. The Kac-Moody Lie groups of type 1 can be obtained as a torus bundle over the loop group LG of a compact Lie group G.

The extension of the basic theory concerning the finite dimensional differentiable manifolds to manifolds modeled on Banach spaces can be found in detail in [2] and [13]. The classical theory of Lie groups has also been extended successfully to the case of Lie groups modeled on Banach spaces - see [1]. This will be helpful to us, especially in the first chapter.

On manifolds modeled on Hilbert spaces one can consider (strong) Riemannian metrics, which induce on the tangent space at any point a scalar product generating the initial Hilbert space structure (determined by the coordinate charts from that of the modeling space). There is a well developed theory concerning the Riemannian manifolds modeled on Hilbert spaces, which generalizes most of the classical results about finite dimensional Riemannian manifolds - see [13], [6]; there are nevertheless a few exceptions, mostly involving local compactness (like the Hopf-Rinow theorem). Weak metrics can be considered even on manifolds modeled by more general topological vector spaces, like Banach or Fréchet. Weak means here that the tangent spaces are only pre-Hilbert with respect to the induced scalar products. Unfortunately, many of the results about Riemannian manifolds are no more valid in the case of weak metrics. All the metrics we will consider in this work are actually weak metrics.

A *Riemannian symmetric space* is a Riemannian manifold admitting a symmetry at each point, i.e. an isometry which fixes the point and reverses all the geodesics passing through it. Finite dimensional Riemannian symmetric spaces have many nice properties and have been studied intensively over the years, culminating with their classification, accomplished by E. Cartan - see for example [10]. One can similarly define the notion of pseudo-Riemannian symmetric space, by replacing the Riemannian metric with a pseudo-Riemannian one; most of the properties of the Riemannian symmetric spaces are common to the pseudo-Riemannian symmetric spaces as well - see [19].

Infinite dimensional symmetric spaces have also been considered. In [9], P. de la Harpe gives a (possibly complete) list of Hilbert symmetric spaces (with strong metrics). His examples are obtained from the canonical infinite dimensional extensions of the classical Lie groups. Nevertheless, a comprehensive theory, analogue to that in the finite dimensional case, does not exist in infinite dimension.

In this thesis we study two different classes of infinite dimensional symmetric spaces. Both classes are derived from loop groups of compact Lie groups and one object from each class corresponds essentially to any simply connected symmetric space of compact type. In both cases some work is needed to find manifold structures.

In the first chapter we consider a class of Hilbert manifolds (modeled by the separable Hilbert space l^2). An object in this class is a quotient space, obtained from a loop group of Sobolev H^1 loops by dividing the fixed point subgroup of some involution. We put on it a weak Riemannian metric, derived from the L^2 scalar product. The existence of symmetries at each point is easy to check. A first difficulty due to the weakness of the metric is the failure of the theorem stating the existence of the Levi-Civita connection. For this reason we adopt a rather unusual approach for studying symmetric spaces: we determine first on the loop group a bi-invariant metric admitting as Levi-Civita connection a well known pointwise connection, make in this way the canonical submersion into a Riemannian submersion, and use afterwards the standard relations for Riemannian submersions to determine a Levi-Civita connection on the quotient space. We investigate then facts which are characteristic for finite dimensional symmetric spaces and find several analogies: The geodesic exponential is determined from the group exponential (we make use here of the O'Neill tensor A), the curvature tensor is parallel and it can be expressed in terms of the Lie bracket, and in particular the sectional curvature is nonnegative, which indicates a compact type behavior. Motivated by this we construct a kind of dual symmetric space -based also on loop groups- and study its similar properties. Even though the Hadamard-Cartan theorem is not available in this setting (again because of the weakness of the metric), we prove that the dual space is diffeomorphic to a Hilbert space. Another important result which we prove is the correspondence between totally geodesic submanifolds and Lie triple systems, and between flat submanifolds and abelian subalgebras.

In the second chapter, motivated mainly by the relation found by C.-L. Terng in [24] with polar actions on Hilbert spaces, we consider the extension of a loop group to an affine Kac-Moody Lie group of type 1. In order to obtain a Lie group structure we must restrict to smooth loops and work with the Fréchet C^∞ topology. This deprives us of the use of very important tools, like the inverse function theorem, the existence and uniqueness theorems for ordinary differential equations and the Frobenius theorem. One section of this chapter is therefore devoted to proving that the Kac-Moody groups are tame Fréchet manifolds, for which a weaker version of the inverse function theorem holds. In Section 2 we show how to define a unique torsionfree left invariant linear connection on an arbitrary Fréchet Lie group. We also pay special attention to the exponential of the Kac-Moody groups, giving an explicit description of it. A pseudo-Riemannian symmetric space of index 1 is then obtained from the Kac-Moody group by dividing the fixed point subgroup of some involution (involution which extends the one used in the previous chapter). We treat similar problems to those in Chapter I, trying to overcome the lack of a well developed theory concerning the Fréchet manifolds and Lie groups. By methods analogue to those used in the first chapter we obtain similar results concerning the geodesics (whose existence and uniqueness are not negatively influenced by the lack of the usual existence and uniqueness theorems for ordinary differential equations) and the curvature, but we are unable to construct a dual symmetric space in this case. Unlike the case studied in the first chapter, we find here a conjugacy class of finite dimensional maximal totally geodesic and flat submanifolds.

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Chapter 1

A weak Hilbert Riemannian symmetric space

To a simply connected symmetric space of compact type G/G^ρ we associate the quotient space LG/LG^ρ, where LG is the loop group of H^1 loops and the involution ρ on LG is obtained from the involution (denoted by the same ρ) on G. It admits a canonical Hilbert differentiable structure (allowing smooth partitions of unity). With a certain weak Riemannian metric it becomes a symmetric space. We associate a Levi-Civita connection to this metric. In infinite dimensions we have to work with a slightly stronger definition of a linear connection. We find several analogies with the theory of finite dimensional symmetric spaces, including the duality compact-noncompact.

1.1 Prerequisites

A *Hilbert manifold* is a Hausdorff topological space with an atlas of coordinate charts taking values in Hilbert spaces, such that the coordinate transition functions are all smooth maps between Hilbert spaces. *Banach manifolds* can be defined in a similar way. In this chapter we will deal exclusively with manifolds modeled by separable Hilbert spaces, which are all isomorphic with l^2. Let $\tilde{M}$ be such a manifold.

For a vector field $X \in \mathfrak{X}(M)$ and a chart $M \supset U \xrightarrow{\varphi} \varphi(U) \subset \mathbb{M}$, with $\mathbb{M}$ modeling Hilbert space for M, we consider the principal part $X_\varphi : \varphi(U) \to \mathbb{M}$ of X defined by $X_\varphi(\varphi(x)) = pr_2 \circ T\varphi(X_x)$. More generally, let E be a vector bundle over M. For any trivialization

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\Phi} & \varphi(U) \times \mathbb{E} \\ \pi \downarrow & & \downarrow pr_1 \\ U & \xrightarrow{\bar{\varphi}} & \varphi(U), \end{array}$$

where (φ, U) is a chart for M and $\mathbb{E}$ is another Banach space, we denote the principal part of a section X by X_φ.

We explain now the notion of *linear connection* in the case of infinite dimensional (Hilbert or Banach) manifolds. A connection on the vector bundle E can be given in several ways. A first one is to express it as a mapping $\nabla : \mathcal{X}(M) \times \Gamma(E) \to \Gamma(E)$ (write $\nabla_X Y$ instead of $\nabla(X,Y)$) such that for any trivialization (Φ, φ, U) as before, there is a smooth mapping

$$\Gamma_\varphi : \varphi(U) \to L(\mathbb{M}, \mathbb{E}; \mathbb{E})$$

with $(\nabla_X Y)_\varphi(x) = DY_\varphi(x) \cdot X_\varphi + \Gamma_\varphi(x)(X_\varphi Y_\varphi)$, $\forall x \in \varphi(U)$. By $L(\mathbb{M}, \mathbb{E}; \mathbb{E})$ we denote the space of bilinear continuous maps. It is a Banach space with the norm given by $\| A \| = \sup_{\|u\| \leq 1, \|v\| \leq 1} A(u,v)$ ($\| A \|$ is finite if and only if A is continuous).

∇ defined in this way still has the properties of linearity known from the finite dimensional case, but conversely they do not imply that the Christoffel symbol is a mapping $\Gamma_\varphi : \varphi(U) \to L(\mathbb{M}, \mathbb{M}; \mathbb{M})$ anymore. This is because in infinite dimension the $\mathcal{F}(\varphi(U))$-bilinearity of $\Gamma_\varphi = (\nabla_X Y)_\varphi - DY_\varphi \cdot X_\varphi$ does not imply that Γ_φ is a tensor field over $\varphi(U)$. For the same reason, this stronger definition is essential for proving that the curvature and the torsion are really tensors. The Christoffel symbols Γ_φ are uniquely determined by ∇ only if M admits partitions of unity. Otherwise it is possible not to have enough global vector fields to determine them.

In the following we restrict ourselves to linear connections on the tangent bundle. A second way to define a connection (possible also for general vector bundles) is via a connection mapping $K : TTM \to TM$ such that for any chart (φ, U) of M, there is again a smooth map $\Gamma_\varphi : \varphi(U) \to L(\mathbb{M}, \mathbb{M}; \mathbb{M})$ which determines K locally:

$$K_\varphi(x, y, z, w) = (x, w + \Gamma_\varphi(x)(z, y)),$$

where $K_\varphi := T\varphi \circ K \circ TT\varphi^{-1} : \varphi(U) \times \mathbb{M} \times \mathbb{M} \times \mathbb{M} \to \varphi(U) \times \mathbb{M}$.

Given K, the covariant derivative ∇ is obtained from the formula $\nabla_X Y = K \circ TY(X)$. The same formula defines the covariant derivative along mappings $f : N \to M$, in case $X \in \mathcal{X}(N)$, $Y \in \mathcal{X}(f) = \{X : N \to TM \text{ smooth with } \tau \circ X = f\}$.

The set of connection mappings K is in bijection with the set of collections of Christoffel symbols (which satisfy a certain transformation rule). When partitions of unity exist (we will show in Section 3 that the manifolds which we are studying satisfy this condition), then this gives a bijection with the set of covariant derivatives ∇. Otherwise there may exist several connection maps K determining the same ∇. In the case partitions of unity exist, it is enough to produce the Christoffel symbols for a set of trivializations which cover M. All the linear connections which we will use in this chapter will be introduced through formulas giving the operator ∇ in terms of previously defined connection operators, with the pointwise connection explained below serving as starting point. We will then determine the Christoffel symbols for trivializations of type $(\varphi, U, \Phi = T\varphi)$. Even though we will show that the spaces we consider admit partitions of unity, we could completely avoid using them. For this we should either check the transformation rule for our partial set of Christoffel symbols, or equivalently to determine an associated connection map K - having the Christoffel symbols satisfy the transformation rule is essential for many things, including the good definition of the torsion and curvature tensors.

Let now V be an euclidian vector space with scalar product $\langle\ ,\ \rangle$. We are interested in the spaces LV of loops of V, i.e. maps from S^1 to V. We always regard loops as maps on the interval $I = [0, 2\pi]$ with identical end values. There are several possible choices, depending on the regularity the loops should satisfy. The most natural ones turn out to be the spaces $C^k(I,V)$ of loops possessing continuous derivatives of order $\leq k$ for some integer K with $1 \leq k \leq \infty$, the spaces $L^p(I,V)$ of (Lebesgue) measurable maps u satisfying $\| u \|_p = (\int |u(t)|^p dt)^{\frac{1}{p}} < \infty$ (one can also define the space $L^\infty(I,V)$ of measurable and almost everywhere bounded maps) and the Sobolev spaces $H^p(I,V)$. $C^\infty(I,V)$ is a Fréchet space with the seminorms $\| u \|_n = \sup\{|u^{(n)}(t)| \,|\, t \in I\}$, $n \in \mathbb{N} \cup 0$. For $k < \infty$, $C^k(I,V)$ has a Banach structure, with the norm $\| u \| = \sum_{j=o}^k \sup\{|u^{(j)}(t)| \,|\, t \in I\}$. For each positive real number $p \geq 1$, $L^p(I,V)$ is a Banach space with norm $\| \ \|_p$ (once we identify maps which are equal except for a set of measure zero). For $p < q$, $L^q(I,V)$ is a dense subspace of $L^p(I,V)$.

For every $p \in \mathbb{N}$, $H^p(I,V)$ is the space of all L^2 loops u whose distribution derivatives $u^{(k)}$ are L^2 maps for $k \leq p$. One can more generally define $H^s(I,V)$ for each $s \in \mathbb{R}$ as the space of distributions (viewed as generalized functions) f satisfying $\Lambda_s f \in L^2$, where Λ_s is an operator on the space of distributions, defined by means of the Fourier transform. The elements of $H^s(I,V)$ need not to be functions for $s \leq 0$. For $s < t$ $H^t(I,V)$ is a dense subspace of $H^s(I,V)$. $H^1(I,V)$ can be described alternatively as the space of all absolute continuous maps whose first derivative belongs to $L^2(I,V)$ - see [15]. All $H^s(I,V)$ are Hilbert spaces. For $s = p \in \mathbb{N}$ the scalar product is given by $\langle u, v\rangle = \sum_{k=0}^p \int u^{(k)}(t) v^{(k)}(t) dt$. The Sobolev embedding theorem provides in our case the inclusion $H^s(I,V) \subset C^k(I,V)$ for $s > k + \frac{1}{2}$. If we restrict to integers we obtain a chain of inclusions, written shortly as follows: $L^1 \supset L^1 \supset L^2 = H^0 \supset L^3 \supset ... \supset L^\infty \supset C^0 \supset H^1 \supset C^2 \supset H^2 \supset ... \supset C^\infty$. All the inclusions are dense. For more details about this approach to Sobolev spaces, see [7].

Given a finite dimensional manifold, one can define C^k and H^k loops on it. This contrasts with the L^p case, because the L^p property is not preserved by the composition with diffeomorphisms (even analytic) between open subsets of $\mathbb{R}^n$ ($\int |f| < \infty$ does not imply $\int |\varphi(f)| < \infty$). For simplicity, in this chapter we will work with the H^1 loops. The advantage of the Hilbert space structure seems not to be essential, since the Riemannian metric we will work with is a weak metric.

Let M be a finite dimensional connected C^∞ differential manifold, g a Riemannian metric on it and let ∇ be its Levi-Civita connection. Denote by $\exp_x$ the exponential mapping at $x \in M$ and by d the induced distance on M. Let $\tau : TM \to M$ the canonical projection. We define a loop $c : I \to M$ to be of type H^1 if for any chart (φ, U) of M, $\varphi \circ c$ is of type H^1 on any compact subinterval $I' \subset I$ with $c(I') \subset U$.

We denote by LM the set of all H^1 loops of M. A H^1 vector field along $c \in LM$ is a H^1 loop X on TM such that $\tau \circ X = c$. The space of vector fields along c is isomorph with the space of sections $\Gamma(c^*TM)$ of the pullback c^*TM (we regard them as identified in the following) and is a separable Hilbert space with the scalar product $\langle u, v\rangle = \int_0^{2\pi} g(u(t), v(t))dt + \int_0^{2\pi} g(\frac{D}{dt}u(t), \frac{D}{dt}v(t))dt$.

Let now 0_c be the 0-section and define:

$$\begin{aligned} B_\epsilon(c) &:= \{\gamma \in LM \mid d(c(t), \gamma(t)) < \epsilon, \forall t \in I\}, \\ B_\epsilon(0_c) &:= \{X \in \Gamma(c^*TM) \mid \ \| X(t) \| < \epsilon, \forall t \in I\}, \\ \exp_c &: B_\epsilon(0_c) \to B_\epsilon(c),\ \exp_c(X)(t) = \exp_{c(t)} X(t). \end{aligned}$$

Remark: $\| X \|_\infty = \max_{t\in I} \| X(t) \|$ defines a norm on $\Gamma(c^*TM)$. In [6] there is a nice proof, due to H. Karcher, of the inequality $\| X \|_\infty \leq k \| X \|$ for some $k > 0$ which depends on I (he obtained $k = 11/9$ for $I = [0, 1]$; for $I = [0, 2\pi]$ his argument gives $k = 1 + 16\pi^3/9$). This shows that the topology induced by $\| \ \|_\infty$ is weaker than that induced by $\langle\, ,\rangle$, and so $B_\epsilon(0_c)$ is open in the second topology. The two topologies are not equivalent, as it is shown by the sequence $X_n \in LT_xM$, $X_n(t) = 1/n \sin ntX$, for any $x \in M$ and $X \in T_xM$.

LM has a Hilbert manifold structure such that $\{(\exp_c^{-1}, B_\epsilon(c)) \mid c \in LM\}$ is a C^∞ atlas. The tangent space T_cLM is the space $\Gamma(c * TM)$ of vector fields along c.

Remark: The differentiable structure on LM does not depend on the chosen Riemannian structure on M. A functor L is thus obtained:

$$\begin{array}{rcl} M & \mapsto & LM, \\ f : M \to N & \mapsto & Lf : LM \to LN, Lf(g)(t) = f(g(t)) \end{array}$$

from the category of finite dimensional manifolds and smooth maps to the category of Hilbert manifolds and smooth maps. L commutes with the functor T which gives the tangent space.

For any connection on M, there is a well known *pointwise connection* on LM. If K is the connection mapping corresponding to ∇ on M, then the connection on LM is given by $LK : LTTM = TTLM \to LTM = TLM$. The induced covariant derivative ∇^L satisfies $\nabla^L_{LX} LY = L\nabla_X Y$ for any $X, Y \in TM$.

Let $c \in LM$ and consider as before the chart $(\exp_c^{-1}, B_\epsilon(c)) := (\tilde\varphi, \tilde U)$. Consider also the family of charts of M $(\varphi_t, U_t) := ((\exp|_{B_\epsilon(0_{c(t)})})^{-1}, B_\epsilon(c(t)))$. Then the Christoffel symbols $\Gamma_{\tilde\varphi}$ of the pointwise connection on LM (for the given chart) are deduced from those (denoted by $\Gamma_t := \Gamma_{\varphi_t}$) of the connection on M by the following formula:

$$\Gamma_{\tilde\varphi}(\tilde\varphi(\eta))(X_{\tilde\varphi}, Y_{\tilde\varphi})(t) = \Gamma_t(\varphi(\eta(t)))(X_{\tilde\varphi}(t), Y_{\tilde\varphi}(t))$$

for any $\eta \in \tilde U$ and $X, Y \in T_\eta LM$. Notice that $X_{\tilde\varphi}, Y_{\tilde\varphi} \in T_cLM$.

Any Riemannian metric g on M induces the L^2 Riemannian metric $\langle\, ,\rangle$ on LM: $\langle X, Y\rangle := \int_0^{2\pi} g(X(t), Y(t))dt,\ \forall c \in LM,\ X; Y \in T_cLM$. It is a *weak metric* in the sense that the tangent space at any point in LM is only pre-Hilbert with respect to it. In general, there may be no Levi-Civita connection for a weak metric on a Hilbert manifold. If there is one, then it is unique (the relation used in the finite dimensional case for defining the Levi-Civita connection still holds, and the uniqueness follows from it).

Proposition 1.1.1. *If ∇ is the Levi-Civita connection for the metric g on M, then ∇^L is the Levi-Civita connection for the L^2 metric.*

Proof. A connection is torsionfree if and only if all its Christoffel symbols are symmetric. From the above formula giving the Christoffel symbols of the pointwise connection it follows that the pointwise connection induced by a torsionfree connection is torsionfree.

To show that ∇^L is metric, we have to check the relation

$$\frac{d}{ds}\langle X_s, Y_s\rangle_{c(s)} = \left\langle \frac{D^L X_s}{ds}, Y_s \right\rangle_{c(s)} + \left\langle X_s, \frac{D^L Y_s}{ds} \right\rangle_{c(s)}$$

for any smooth curve $(s \mapsto c_s)$ on LM and any two vector fields $s \mapsto X_s$ and $s \mapsto Y_s$ along c - see [6], I.3.8 for the equivalence of the various ways to state the metric condition. The symbol D^L is used for the covariant differentiation induced by the pointwise connection ∇^L. But $\frac{d}{ds}\langle X_s, Y_s\rangle_{c_s} = \frac{d}{ds}\int_I g_{c_s(t)}(X_s(t), Y_s(t))dt = \int_I \frac{d}{ds} g_{c_s(t)}(X_s(t), Y_s(t))dt$ $= \int_I g_{c_s(t)}\left(\frac{DX_s(t)}{ds}, Y_s(t)\right) dt + \int_I g_{c_s(t)}\left(X_s(t), \frac{DY_s(t)}{ds}\right) dt = \langle \frac{D^L X_s}{ds}, Y_s\rangle_{c(s)} + \langle X_s, \frac{D^L Y_s}{ds}\rangle_{c(s)}$. We used the fact that ∇ is a metric connection, as well as the relation $\frac{D^L X_s}{ds}(t) = \frac{DX_s(t)}{ds}$, characteristic for the pointwise connection. □

Details about many of the facts presented in this section can be found (in a slightly different setting) in [6].

1.2 Constructions

We start with a compact, connected, simply connected and semisimple Lie group G and denote by $\mathfrak{g}$ its Lie algebra. Consider the loop group LG of all H^1 loops $\gamma : S^1 \to G$ (with pointwise multiplication $(\gamma \cdot \eta)(t) = \gamma(t) \cdot \eta(t)$). This is a well known object and it has a Hilbert Lie group structure. Its Lie algebra is $L\mathfrak{g}$, the vector space of all H^1 loops of $\mathfrak{g}$ with the pointwise Poisson bracket: [u,v](t)=[u(t),v(t)]. The exponential mapping is just $L\exp : L\mathfrak{g} \to LG$ and we call it by abuse of notation simply exp. An atlas of charts for LG can be constructed easier than in the general case of loop manifolds LM, using the group exponential and the left translations. Moreover, it is easy to see that this gives, like in the finite dimensional case, an analytic structure for LG. For the generalization of the notion of analytic functions to infinite dimensions see N. Bourbaki - [2].

An involution $\rho : G \to G$ determines an involution (of the second kind) of LG. We make another abuse of notation, denoting it by the same ρ. It is given by $\rho(\gamma)(t) = \rho(\gamma(2\pi - t))$, $\forall \gamma \in LG$. Let τ be the automorphism of LG which reverses all the loops, i.e. $\tau(\gamma)(t) = \gamma(2\pi - t)$. It is obviously a smooth involution. It holds $\rho = L\rho \circ \tau$, so in particular ρ is smooth.

Let $LG^\rho := \{\gamma \in LG \mid \rho(\gamma) = \gamma\}$. We can construct now the quotient space LG/LG^ρ. We will study this space extensively and look for analogies with the finite dimensional symmetric spaces (of compact type).

Remark: One could also consider the involution of first kind, determined by $\rho'(\gamma)(t) = \rho(\gamma(t))$, $\forall \gamma \in LG$. This case is nevertheless trivial in the following sense: Obviously, $L(G^\rho) = LG^{\rho'}$. Moreover, when G/G^ρ is simply connected, then the map $LG/LG^{\rho'} \ni \gamma LG^{\rho'} \xrightarrow{\Phi} (t \mapsto \gamma(t)G^\rho) \in L(G/G^\rho)$ is bijective (G is the total space of a $G^\rho-$principal bundle over G/G^ρ; if G/G^ρ is simply connected, then its pullback under any loop $\bar{\gamma} : S^1 \to G/G^\rho$ is trivial, and thus $\bar{\gamma}$ admits a lift $\gamma : S^1 \to G$, hence Φ is surjective; the injectivity and the well-definedness are easy to check). Even if G and G/G^ρ are not simply connected, Φ determines a bijection between the connected component of $eLG^{\rho'}$ in $LG/LG^{\rho'}$ and the connected component of eG^ρ in $L(G/G^\rho)$.

The involution $\rho : G \to G$ induces a Lie algebra involutive automorphism ρ_* of $\mathfrak{g}$. We denote by the same symbol the involution on $L\mathfrak{g}$ given by $\rho_*(u)(t) = \rho_*(u(2\pi - t))$. It is the differential of the involution ρ on LG and it induces a splitting $L\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with $\mathfrak{k} = \{u \mid \rho_*(u) = u\}$ and $\mathfrak{p} = \{u \mid \rho_*(u) = -u\}$. In addition, $\mathfrak{k} = Lie(LG^\rho)$. We get thus a kind of orthogonal symmetric Lie algebra $(L\mathfrak{g}, \rho_*)$ corresponding to the symmetric space LG/LG^ρ. The fixed point set $\mathfrak{k}$ of ρ_* is of course not a compact subalgebra in this case.

We want to find a dual to LG/LG^ρ like in the finite dimensional case. It should be induced by the pair $(L\mathfrak{g}', \rho_*)$, where $L\mathfrak{g}' = \mathfrak{k} \oplus i\mathfrak{p}$ and ρ_* is the involution on $L\mathfrak{g}'$ with +1 eigenspace $\mathfrak{k}$ and -1 eigenspace $i\mathfrak{p}$.

We construct the dual rigorously in the following way (we adapt to our case and develop to the group level an idea of C.-L. Terng, presented in [25]): Consider the complexification $\mathfrak{g}_\mathbb{C} = \mathfrak{g} + i\mathfrak{g}$ and let $G_\mathbb{C}$ be the unique connected simply connected Lie group with Lie algebra $\mathfrak{g}_\mathbb{C}$. Define two involutive automorphisms

$$\rho_* : \mathfrak{g}_\mathbb{C} \to \mathfrak{g}_\mathbb{C}, \quad \rho_0 : \mathfrak{g}_\mathbb{C} \to \mathfrak{g}_\mathbb{C},$$

$$x + iy \xrightarrow{\rho_*} \rho_*(x) - i\rho_*(y), \quad x + iy \xrightarrow{\rho_0} x - iy.$$

Both ρ_0 and ρ_* are conjugate linear and $\rho_0\rho_* = \rho_*\rho_0$.

Let $\mathfrak{g} = \mathfrak{k}_\mathfrak{g} \oplus \mathfrak{p}_\mathfrak{g}$ the splitting into the eigenvalues of ρ_*. Then $(\mathfrak{g}, \rho_*)$ and $(\mathfrak{k}_\mathfrak{g} \oplus i\mathfrak{p}_\mathfrak{g}, \rho_0)$ are dual symmetric pairs with $\mathfrak{g} = \mathfrak{g}_\mathbb{C}^{\rho_0}$, $\mathfrak{k}_\mathfrak{g} \oplus i\mathfrak{p}_\mathfrak{g} = \mathfrak{g}_\mathbb{C}^{\rho_*}$. We extend this picture to a similar one involving infinite dimensional Lie algebras:

Construct now $L(\mathfrak{g}_\mathbb{C}) = (L\mathfrak{g})_\mathbb{C} = \mathbb{C} \otimes L\mathfrak{g}$ and define the involutions

$$\rho_* : L\mathfrak{g}_\mathbb{C} \to L\mathfrak{g}_\mathbb{C}, \qquad \rho_0 : L\mathfrak{g}_\mathbb{C} \to L\mathfrak{g}_\mathbb{C},$$

$$\rho_*(u)(t) = \rho_*(u(-t)), \qquad \rho_0(u)(t) = \rho_0(u(t)).$$

One can easily check that they are Lie algebra automorphisms, they are conjugate linear and $\rho_*\rho_0 = \rho_\circ\rho_*$. The fixed point subalgebras are Fix $\rho_0 = L\mathfrak{g}$ and Fix $\rho_* = \mathfrak{k} \oplus i\mathfrak{p}$. These are real forms of $L\mathfrak{g}_{\mathbb{C}}$ and ρ_0 restricts to $\mathfrak{k} \oplus i\mathfrak{p}$ with Fix $\rho_0|_{\mathfrak{k}\oplus i\mathfrak{p}} = \mathfrak{k}$, ρ_* restricts to $L\mathfrak{g}$ with Fix $\rho_*|_{L\mathfrak{g}} = \mathfrak{k}$. Thus $(L\mathfrak{g}, \rho_*)$and $(\mathfrak{k} \oplus i\mathfrak{p}, \rho_0)$ are dual symmetric pairs.

Remark: If one defines as Killing form on $L\mathfrak{g}$ the average of the Killing form on $\mathfrak{g}$: $B(u, v) = \int_0^{2\pi} B(u(t), v(t))dt$, then B is negative definite on $L\mathfrak{g}$, giving a first reason to consider $(L\mathfrak{g}, \rho_*)$ a symmetric pair of compact type and $L\mathfrak{g}$ a compact real form of $L\mathfrak{g}_{\mathbb{C}}$.

Since G_C is simply-connected, there exist unique Lie group automorphisms ρ_0 and ρ of $G_{\mathbb{C}}$ such that $d{\rho_0}_e = \rho_0$, respectively $d\rho_e = \rho_*$ for $\rho_0, \rho_* : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ defined above. Take next the loop group $LG_{\mathbb{C}}$, corresponding to the Lie algebra $L\mathfrak{g}_{\mathbb{C}}$ and consider the involutions ρ and ρ_0 on $LG_{\mathbb{C}}$ constructed from the corresponding involutions on $G_{\mathbb{C}}$ by $\rho(\gamma)(t) = \rho(\gamma(2\pi - t))$, $\rho_0(\gamma)(t) = \rho_0(\gamma_0(t))$. They commute.

Because G is simply connected, it lies as a compact real form in $G_{\mathbb{C}}$ (it is the unique connected subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g} \subset \mathfrak{g}_{\mathbb{C}}$). Hence $LG \subset LG_{\mathbb{C}}$ and $LG^\rho \subset LG^\rho_{\mathbb{C}}$ - see for example [26]. Furthermore, the involution ρ on $LG_{\mathbb{C}}$ extends the previously defined involution ρ on LG.

Consider $LG^\rho_{\mathbb{C}0}$, the connected component of the identity in $LG^\rho_{\mathbb{C}}$. It is the unique connected subgroup of $LG_{\mathbb{C}}$ with Lie algebra $\mathfrak{k} \oplus i\mathfrak{p}$. $LG^\rho \cap LG^\rho_{\mathbb{C}0} \subset LG^\rho_{\mathbb{C}0}$ is the subgroup with Lie algebra $\mathfrak{k}$ (the fixed point set of the involution $\rho_0|_{LG^\rho_{\mathbb{C}0}}$).
$LG^\rho_{\mathbb{C}0}/LG^\rho \cap LG^\rho_{\mathbb{C}0}$ will be then a symmetric space associated to the orthonormal symmetric pair $(\mathfrak{k} \oplus i\mathfrak{p})$, and thus dual to LG/LG^ρ.

Remark: Loops fixed by ρ are completely determined by only one half of them, so if G^ρ and $G^\rho_{\mathbb{C}}$ are the fixed point subgroups of the involutions $\rho : G \to G$ respectively $\rho : G_{\mathbb{C}} \to G_{\mathbb{C}}$, then:

$$\begin{aligned} LG^\rho &\cong \{\gamma : [0, \pi] \to G \mid \gamma(0), \gamma(\pi) \in G^\rho\}, \\ LG^\rho_{\mathbb{C}} &\cong \{\gamma : [0, \pi] \to G_{\mathbb{C}} \mid \gamma(0), \gamma(\pi) \in G^\rho_{\mathbb{C}}\}. \end{aligned}$$

Because $G_{\mathbb{C}}$ is simply connected, it follows $LG^\rho_{\mathbb{C}0} \cong \{\gamma : [0, \pi] \to G_{\mathbb{C}} | \gamma(0), \gamma(\pi) \in G^\rho_{\mathbb{C}0}\}$, where $G^\rho_{\mathbb{C}0}$ is the connected component of the identity in $G^\rho_{\mathbb{C}}$. But we can now use the following result of Rashevsky (see [20], pp. 108): If G is a simply connected Lie group and $s \in Aut\mathfrak{g} = AutG$ is a semisimple automorphism all of whose (complex) eigenvalues are equal to unity in absolute value, then the subgroup G^s is connected. This result holds in particular for involutions, and thus G^ρ and $G^\rho_{\mathbb{C}}$ are connected subgroups. Because $G^\rho \subset G^\rho_{\mathbb{C}}$ and both are simply connected groups, we finally obtain $LG^\rho_0 = LG^\rho \subset LG^\rho_{\mathbb{C}0} = LG^\rho_{\mathbb{C}}$. The dual symmetric space is thus simply $LG^\rho_{\mathbb{C}}/LG^\rho$.

1.3 Manifold structures

In this section we are looking for manifold structures for the quotient spaces LG/LG^ρ and $LG^\rho_{\mathbb{C}}/LG^\rho$. We will show first that $LG^\rho_{\mathbb{C}}$ is an embedded Lie subgroup of $LG_{\mathbb{C}}$ and that LG^ρ is an embedded Lie subgroup of both LG and $LG^\rho_{\mathbb{C}}$.

In the finite dimensional case, it is well known that a closed subgroup of a Lie group is a Lie subgroup, a Lie subgroup is closed if and only if it is embedded (it has the relative topology), and the quotient of a Lie group by a closed subgroup has a unique manifold structure such that the canonical projection is smooth. In the infinite dimensional case (Banach), the latter fact is still true. More precisely, given an embedded Lie subgroup K of a Lie group H, there is a unique analytic manifold structure on H/K such that the canonical projection is a submersion (see N. Bourbaki - [1], III.6, Prop. 11).

On the other hand, a closed subgroup A of a Lie group G need not be an (embedded) Lie subgroup anymore. A simple example, which can be found in N. Bourbaki - [1], is the following: Let G be the the Hilbert space l_2 considered as a Lie group. Let G_n be the set of $(x_1, x_2, ...) \in G$ such that $x_m \in \frac{1}{m}\mathbb{Z}$ for $1 \leq m \leq n$. The G_n are closed subgroups of G, so $H = \cap_n G_n$ is also a closed subgroup of G, but not an embedded Lie subgroup.

We explain the reason in the following: The standard way to prove the result in finite dimensions (see for example F. Warner,[27]) is to show that the set $\mathfrak{a} = \{X \in Lie(G) \mid \exp tX \in A \text{ for all } t \in \mathbb{R}\}$ is a subspace of $Lie(G)$, and then to find neighborhoods U of 0 in $Lie(G)$ and V of e in G diffeomorphic under exp such that $\exp(U \cap \mathfrak{a}) = V \cap A$. For this, one takes a complementary subspace $\mathfrak{b}$ to $\mathfrak{a}$ and supposes that there is a sequence (Y_i) contained in $\mathfrak{b}$ such that $Y_i \overset{i\to\infty}{\longrightarrow} 0$, and $\exp Y_i \in A$ for all i. Because of the local compactness, it follows that the lines generated by a subsequence of Y_i converge to a line contained in $\mathfrak{a}$, which is a contradiction. In infinite dimensions the above argument fails because local compactness is no more valid. It is now clear that the trouble is made by the possibility of arbitrarily small vectors Y with $\exp Y \in A$ and $\exp tY \nsubseteq A$. B. Maissen proves in [14] that if the infimum of the norms of all such vectors is strictly positive, than the connected component of the identity A_0 is a Lie subgroup of G.

Lemma 1.3.1. *Let G a Hilbert Lie group. Let $\rho \in Aut(G)$ and G^ρ the fixed point subgroup of G. Then G^ρ is an embedded Lie subgroup of G.*

Proof. Cf. [1], III 1.3, it is enough to find a neighborhood of e in G such that the restriction of G^ρ to this neighborhood is an embedded submanifold of G. Notice that Bourbaki uses the terms submanifold and Lie subgroup in the strong sense of embedded submanifold, respectively embedded Lie subgroup.

Let V be an open neighborhood of 0 in $\mathfrak{g} = Lie(G)$ and W an open neighborhood of e in G such that $\exp|_V : V \to W$ is a diffeomorphism (this is always possible for

Banach Lie groups, as a consequence of the inverse function theorem). Denote by i the inclusion of G^ρ into G. Take $\tilde{V} := V \cap \rho_*^{-1}(V)$ and $\tilde{W} := \exp(\tilde{V})$. Then $\tilde{V}$ is an open neighborhood of 0 such that $\rho_*(\tilde{V}) \subset V$. Moreover,

$$x \in \tilde{W} \Rightarrow x = \exp X,\ X \in \tilde{V} \Rightarrow \rho(x) = \rho(\exp X) = \exp \rho_*(X) \in \exp(V) = W,$$

hence $\rho(\tilde{W}) \subset W$. Then it is easy to check that $\exp|_{\tilde{V}}$ maps $\tilde{V} \cap \mathfrak{g}^\rho$ onto $\tilde{W} \cap G^\rho$:

$$x \in \tilde{W} \cap G^\rho \Rightarrow x = \exp X,\ \rho \exp X = \exp X \Rightarrow \exp \rho_* X = \exp X;$$

but X and $\rho_* X$ belong to V, hence $\rho_* X = X$, i.e. $X \in \mathfrak{g}^\rho$.

Since $\mathfrak{g}^\rho$ is a closed linear subspace of $\mathfrak{g}$ (and any closed linear subspace of a Hilbert space admits a direct complement), we obtained that $\tilde{W} \cap G^\rho \subset G$ satisfies the conditions from the definition of a submanifold - $\exp|_{\tilde{V}}^{-1} \circ i|_{\tilde{W} \cap G^\rho}$ provides a global chart. (see [2],5.8.). □

Remark: The lemma holds even in Banach case, provided $\mathfrak{g}^\rho$ admits a direct complement.

The lemma and the aforementioned result about quotient spaces give now the following:

Proposition 1.3.2. *$LG^\rho_{\mathbb{C}}$ is an embedded Lie subgroup of $LG_{\mathbb{C}}$. LG^ρ is an embedded Lie subgroup of both LG and $LG^\rho_{\mathbb{C}}$. LG/LG^ρ and $LG^\rho_{\mathbb{C}}/LG^\rho$ have unique analytic manifold structures such that the canonical projections are submersions. The actions of LG respectively $LG^\rho_{\mathbb{C}}$ by left translations are analytic.*

Remark: It also holds $Ker\pi_{*x} = l_{x*}T_e LG^\rho, \forall x \in LG$. As previously remarked, the exponential $\exp : L\mathfrak{g} \to LG$ is obtained by applying the functor L to $\exp : \mathfrak{g} \to G$, hence $\exp_{*0} = id_{L\mathfrak{g}}$. It follows that $(\pi \circ \exp|_{\mathfrak{p}})_{*0}$ is an isomorphism of the Hilbert spaces $\mathfrak{p}$ and $T_{\hat{e}} LG/LG^\rho$. Using the inverse function theorem, we obtain that $(\varphi_g, \varphi_g^{-1}(U))$ is a differentiable atlas for LG/LG^ρ, where U is some open neighborhood of 0 in $\mathfrak{p}$ and $\varphi_g = (l_g \circ \pi \circ \exp|_U)^{-1}$.

We end this section by showing that the manifolds we are concerned with admit partitions of unity. We can put (strong) Riemannian metrics on LG, $LG_{\mathbb{C}}$ and $LG^\rho_{\mathbb{C}}$ in an obvious way: start with scalar products which make $L\mathfrak{g}$, $L\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{k} \oplus i\mathfrak{p}$ Hilbert spaces (the standard example was given in the introduction) and extend them by left translations to a left invariant metric on LG, $LG_{\mathbb{C}}$ respectively $LG^\rho_{\mathbb{C}}$. This done, LG, $LG_{\mathbb{C}}$ admit $LG^\rho_{\mathbb{C}}$ partitions of unity as follows: A strong Riemannian metric defines like in the finite dimensional a distance which induces the initial topology, so Riemannian manifolds are metrizable. But metric spaces are paracompact and any paracompact manifold of class C^p modeled on a separable Hilbert space admits partitions of unity (of class C^p). Details can be found in [13]. On the other hand, the considered Lie groups are modeled on (closed subspaces of) Hilbert spaces of H^1 loops and the Sobolev space of H^1 functions is known to be separable (and thus isomorphic with l^2). Partitions of unity can now be obtained on LG/LG^ρ and $LG^\rho_{\mathbb{C}}/LG^\rho$ with the help of the Cartan embeddings $LG/LG^\rho \hookrightarrow LG$ and $LG^\rho_{\mathbb{C}}/LG^\rho \hookrightarrow LG^\rho_{\mathbb{C}}$ - see section 1.7.

It is also not hard to prove that $LG, LG/LG^\rho, LG_{\mathbb{C}}$ and $LG^\rho_{\mathbb{C}}/LG^\rho$ are separable topological spaces.

1.4 The geometry of LG/LG^ρ

We start with a bi-invariant metric $\langle , \rangle$ on G (remember that G is compact and semisimple, so we choose it such that $\langle \, , \rangle_e$ is a negative constant times the Killing form). We are looking for a Riemannian metric on LG/LG^ρ which makes it a symmetric space. We want the translations

$$l_g : LG/LG^\rho \to LG/LG^\rho, l_g(hLG^\rho) = ghLG^\rho$$

to be isometries for any $g \in LG$, so we need as usual an LG^ρ-invariant scalar product on $T_{\hat{e}}LG/LG^\rho$, which we then translate to a metric on LG/LG^ρ. $\pi_* : L\mathfrak{g} \to T_{\hat{e}}LG/LG^\rho$ provides a natural identification $T_{\hat{e}}LG/LG^\rho \cong \mathfrak{p}$. Because $\pi_* \circ Ad(g) = l_{g_*} \circ \pi$ for any $g \in LG^\rho$, we actually need an $\mathrm{Ad}(LG^\rho)$-invariant scalar product on $\mathfrak{p} \subset L\mathfrak{g}$ (notice that since $\mathrm{Ad}(\rho(g)) \circ \rho_* = \rho_* \circ \mathrm{Ad}(g)$, $\mathfrak{p}$ is $\mathrm{Ad}(LG^\rho)$ - invariant).

We remark now that the scalar product that makes $\mathfrak{p}$ (and $L\mathfrak{g}$) a Hilbert space:

$$\langle u, v \rangle = \int_0^{2\pi} \langle u(t), v(t) \rangle dt + \int_0^{2\pi} \langle u'(t), v'(t) \rangle dt$$

is not Ad-invariant: just take $u = v$ constant in $\mathfrak{p}$ and $\gamma \in LG^\rho$ such that $Ad(\gamma)$ is not constant. Therefore we consider the Ad-invariant L^2 scalar product on $L\mathfrak{g}$:

$$\langle u, v \rangle = \int_0^{2\pi} \langle u(t), v(t) \rangle dt$$

Remark: Because the scalar product $\langle \, , \rangle$ on $\mathfrak{g}$ comes from the Killing form, any automorphism of $\mathfrak{g}$, in particular ρ_*, is an isometry of $\mathfrak{g}$. It can be seen directly that the involution ρ_* on $L\mathfrak{g}$ is then an isometry with respect to the L^2 scalar product. This implies that the decomposition of $L\mathfrak{g}$ into its eigenspaces is orthogonal, i.e. $\mathfrak{k} \perp \mathfrak{p}$.

We restrict $\langle , \rangle$ to $\mathfrak{p}$ and then extend it with the help of the left translations l_g to a metric on LG/LG^ρ. The disadvantage of this metric is that $L\mathfrak{g}$, $\mathfrak{p}$ and accordingly the tangent space of LG/LG^ρ at any point are only pre-Hilbert (not complete) with respect to it. For this reason we can not use the theory of Hilbert Riemannian manifolds (studied extensively in [13], [6]) to get results like for example the existence of the Levi-Civita connection. We will construct in the following a Levi-Civita connection. Its uniqueness still follows from the general theory.

Let ∇^G be the Levi-Civita connection on G and let ∇ be the pointwise connection induced by it on LG. Since the L^2 scalar product on $L\mathfrak{g}$ is Ad-invariant, we can extend it to a bi-invariant metric on LG and then ∇ is the Levi-Civita connection with respect to this metric. This is a particular case of the one mentioned in the introduction.

Proposition 1.4.1. *With the given metrics $\pi : LG \to LG/LG^\rho$ becomes a Riemannian submersion.*

Proof. Since $\mathfrak{p}$ is perpendicular to $\mathfrak{k} = Ker\pi_{*e}$, it generates the left invariant horizontal distribution. The result follows now from the relation $l_g\pi = \pi l_g$, $\forall g \in LG$ between the left translations (both denoted by l_g) on LG and LG/LG^ρ. □

Proposition 1.4.2. *LG/LG^ρ admits a (unique) Levi-Civita connection $\bar\nabla$ associated to the previously constructed weak metric.*

Proof. Define $\bar\nabla_{\bar X}\bar Y := \pi_*\nabla_X Y$, where $\bar X, \bar Y$ are vector fields on LG/LG^ρ and X, Y are their horizontal lifts. We check first that $\bar\nabla$ is well defined: because the right translations r_g on LG with elements g of LG^ρ are fiber preserving isometries ($\pi \circ r_g = \pi$), we have $r_{g*}X = X, r_{g*}Y = Y$, so $r_{g*}(\nabla_X Y)_x = (\nabla_{r_{g*}X} r_{g*}Y)_{xg} = (\nabla_X Y)_{xg}$ and thus $\pi_*(\nabla_X Y)_{xg} = \pi_* r_{g*}(\nabla_X Y)_x = \pi_*(\nabla_X Y)_x$

One has to check thereafter that $\bar\nabla$ is a connection in the sense made clear in the beginning: given a chart $LG/LG^\rho \supset \bar U \xrightarrow{\bar\varphi} \bar\varphi(\bar U) \subset \mathfrak{p}$, there is a C^∞-mapping $\bar\Gamma_{\bar\varphi} : \bar\varphi(\bar U) \to L(\mathfrak{p}, \mathfrak{p}; \mathfrak{p})$ such that $(\bar\nabla_{\bar X}\bar Y)_{\bar\varphi} = D\bar Y_{\bar\varphi} \cdot \bar X_{\bar\varphi} + \bar\Gamma_{\bar\varphi}(\bar X_{\bar\varphi}, \bar Y_{\bar\varphi})$.

Since π is a submersion, we can cover both LG and LG/LG^ρ with pairs $(\varphi, \bar\varphi)$ of charts for which the following diagram is commutative:

$$\begin{array}{ccc} LG \supset U & \xrightarrow{\varphi} & \varphi(U) \subset L\mathfrak{g} \\ \pi\downarrow & & \downarrow pr_2 \\ LG/LG^\rho \supset \bar U & \xrightarrow{\bar\varphi} & \bar\varphi(\bar U) \subset \mathfrak{p} \end{array}$$

Take $\bar X, \bar Y$ vector fields on $\bar U$ and X, Y their horizontal lifts restricted to U. We know $(\nabla_X Y)_\varphi = DY_\varphi \cdot X_\varphi + \Gamma_\varphi(X_\varphi, Y_\varphi)$: We also know $\bar\nabla_{\bar X}\bar Y = \pi_*\nabla_X Y$, which means locally

$$(*)(\bar\nabla_{\bar X}\bar Y)_{\bar\varphi} \circ pr_2 = \mathrm{Dpr}_2(\nabla_X Y)_\varphi$$

We prove now that $DY_\varphi \cdot X_\varphi$ is pr_2-correlated with $D\bar Y_{\bar\varphi} \cdot \bar X_{\bar\varphi}$ (think of them as vector fields on $\varphi(U)$, respectively $\bar\varphi(\bar U)$).

Write $X_\varphi = X^1_\varphi + X^2_\varphi$, $X^1_\varphi \in \mathfrak{k}, X^2_\varphi \in \mathfrak{p}$. Decompose Y_φ in the same way. X_φ and Y_φ are lifts of $\bar X_{\bar\varphi}$ and $\bar Y_{\bar\varphi}$ - not necessarily horizontal, since φ (which preserves the vertical distributions) is not necessarily an isometry. It follows $X^2_\varphi = \bar X_{\bar\varphi} \circ \mathrm{pr}_2$ and $Y^2_\varphi = \bar Y_{\bar\varphi} \circ \mathrm{pr}_2$.

Next $DY_\varphi \cdot X_\varphi = D_1 Y_\varphi \cdot X^1_\varphi + D_2 Y_\varphi \cdot X^2_\varphi$ and $\mathrm{Dpr}_2 DY_\varphi \cdot X_\varphi = \mathrm{pr}_2 D_1 Y_\varphi \cdot X^1_\varphi + \mathrm{pr}_2 D_2 Y_\varphi \cdot X^2_\varphi = D_1 Y^2_\varphi \cdot X^1_\varphi + D_2 Y^2_\varphi \cdot X^2_\varphi$.

But $D_1 Y^2_\varphi = 0$ because $Y^2_\varphi = \bar{Y}_{\bar\varphi} \circ \mathrm{pr}_2$ is constant along the fibers, so

$$(**) \mathrm{Dpr}_2 DY_\varphi \cdot X_\varphi = D_2 Y^2_\varphi \cdot X^2_\varphi = D\bar{Y}_{\bar\varphi} \cdot \bar{X}_{\bar\varphi} \circ \mathrm{pr}_2$$

Define now $\bar\Gamma_{\bar\varphi}$ by $\bar\Gamma_{\bar\varphi}(\bar{X}_{\bar\varphi}, \bar{Y}_{\bar\varphi}) \circ pr_2 = \mathrm{Dpr}_2 \Gamma_\varphi(X_\varphi, Y_\varphi)$. From the relations $(*)$, $(**)$ and $\mathrm{Dpr}_2 \Gamma_\varphi(X_\varphi, Y_\varphi) = \mathrm{Dpr}_2(\nabla_X Y)_\varphi - \mathrm{Dpr}_2(DY_\varphi \cdot X_\varphi)$ we obtain both the well-definedness and the formula $(\bar\nabla_{\bar X} \bar Y)_{\bar\varphi} = D\bar{Y}_{\bar\varphi} \cdot \bar{X}_{\bar\varphi} + \bar\Gamma_{\bar\varphi}(\bar{X}_{\bar\varphi}, \bar{Y}_{\bar\varphi})$.

The dependence of $\bar\Gamma_{\bar\varphi}$ only on the pointwise values of $\bar{X}_{\bar\varphi}$ and $\bar{X}_{\bar\varphi}$ is obvious from its definition. This completes the proof that $\bar\nabla$ is a well defined connection on LG/LG^ρ.

Checking that $\bar\nabla$ is metric and torsionfree follows easily from the fact that ∇ is metric respectively symmetric:

$\bar\nabla$ symmetric: $\bar\Gamma_{\bar\varphi}(\bar{X}_{\bar\varphi}, \bar{Y}_{\bar\varphi}) = Dpr_2 \Gamma_\varphi(X_\varphi, Y_\varphi) = Dpr_2 \Gamma_\varphi(Y_\varphi, X_\varphi) = \bar\Gamma_{\bar\varphi}(\bar{Y}_{\bar\varphi}, \bar{X}_{\bar\varphi})$.

$\bar\nabla$ metric: $\bar{X}\langle \bar{Y}, \bar{Z}\rangle \circ \pi = \pi_* X \langle \bar{Y}, \bar{Z}\rangle = X(\langle \bar{Y}, \bar{Z}\rangle \circ \pi) = X\langle Y, Z\rangle$ and $\langle \bar\nabla_{\bar X} \bar{Y}, \bar{Z}\rangle \circ \pi = \langle \pi_* \nabla_X Y, \bar{Z} \circ \pi\rangle = \langle h\nabla_X Y, Z\rangle = \langle \nabla_X Y, Z\rangle$. □

We want now to obtain in this setting as many of the properties of finite dimensional symmetric spaces (of compact type) as possible. Before proceeding, we change for the sake of simplicity the notation of the Levi-Civita connection on LG/LG^ρ, using for it the symbol ∇.

We start with the existence of symmetries at each $x \in LG/LG^\rho$. This works as in the case of a finite dimensional space G/H with (G, H) Riemanian symmetric pair: the symmetry at $\hat{e} = eLG^\rho$ is

$$\sigma_{\hat e}(hLG^\rho) = \bar\rho(hLG^\rho) = \rho(h)LG^\rho$$

It satisfies $\bar\rho_{*\hat e} = -id$ (because $\rho_*|_{\mathfrak p} = -id$, $T_{\hat e} LG/LG^\rho = \mathfrak{p}$) and it is isometric everywhere because $\bar\rho = l_{\rho(g)} \circ \bar\rho \circ l_{g^{-1}}$. The symmetry at an arbitrary point $\hat g$ is $\sigma_{\hat g} = l_g \circ \bar\rho \circ l_{g^{-1}}$. It is easy to check $\sigma_{\hat g} = \sigma_{\widehat{gh}}$ for $h \in LG^\rho$.

It is easy to see that the geodesics of LG are made up pointwisely of geodesics of G, hence the geodesics passing through the origin are precisely the 1-parameter subgroups of LG. We determine next the geodesics of LG/LG^ρ. For this we introduce the O'Neill tensor A on LG. It is defined in general for a Riemannian submersion by the formula $A_X Y = \mathcal{H}\nabla_{\mathcal{H}X}\mathcal{V}Y + \mathcal{V}\nabla_{\mathcal{H}X}\mathcal{H}Y$ (see [18]). For our purposes we need it applied only to horizontal vector fields, for which $A_X Y = \mathcal{V}\nabla_X Y$. Since the tensoriality does not follow from the simple argument used in the finite dimensional case, we need to prove:

Lemma 1.4.3. *A is a tensor.*

Proof. As we just mentioned, we restrict A to horizontal vector fields. Let X and Y be two such fields. The tensoriality in the first argument is clear from the similar property of ∇. We choose therefore X to be left invariant. Take a (countable) orthonormal basis $(e_n)_{n\in\mathbb{N}}$ for the separable Hilbert space $\mathfrak{p}$ (with the H^1 scalar product $\langle u, v\rangle_1 = \sum_{k=0}^1 \int \langle u^{(k)}(t), v^{(k)}(t)\rangle dt$). Each e_n can be extended to a horizontal left invariant vector field E_n on LG. Then Y can be written as an infinite sum $Y = \sum_{n\in\mathbb{N}} y_n E_n$, with y_n smooth functions on LG given by $y_n = \langle Y, E_n\rangle$. Since ∇, X and E_n are all left invariant, each $\nabla_X E_n$ is a left invariant vector field. Since $\nabla_X E_n = \frac{1}{2}[X, E_n]$, $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k} = \mathcal{V}_e$ and X_e, $e_n \in \mathfrak{p} = \mathcal{H}_e$, it follows that $\nabla_X E_n \subset \mathcal{V}$. We want to prove that $A_X Y = \sum_{n=1}^{\infty} y_n \nabla_X E_n$. This would show that $A_X Y(g)$ depends only on the value of Y at g.

The problem can be reduced to showing that ∇_X restricts to a continuous operator on the Hilbert space of left invariant vector fields on LG (identified with $L\mathfrak{g}$) - just consider the sequence of partial sums $Y^k = \sum_{n=1}^{k} y_n E_n$, for which the relation $\mathcal{V}\nabla_X Y^k = \sum_{n=1}^{k} y_n \nabla_X E_n$ holds. But this restriction is just the mapping $Y \mapsto \frac{1}{2}[X, Y]$ and the continuity of the bracket on $L\mathfrak{g}$ follows from the general theory and can also be checked directly in a trivial way. □

Remark: One can prove more generally for the pointwise connection induced by any left invariant connection ∇^G on G that the operator defined previously is bounded, and thus continuous:

The relation $(l_{g*}Z)(t) = l_{g(t)*}Z(t)$ for $g \in LG$ and $Z \in L\mathfrak{g}$ (i.e. the commutativity of the functors T and L mentioned in the introduction) gives a correspondence between left invariant vector fields on LG and families of left invariant vector fields on G. The connection ∇^G on G generates a bi-linear mapping $B : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$, $B(Z, W) = \nabla^G_Z W$. Writing $B_t := B(X(t), \cdot)$, we can describe the restriction $\nabla_X : L\mathfrak{g} \to L\mathfrak{g}$ as follows: $(\nabla_X Z)(t) = \nabla^G_{X(t)} Z(t) = B_t Z(t)$. We use this description to show $\| \nabla_X Z \| \leq C \| Z \|$ for some constant $C > 0$ and for all $Z \in L\mathfrak{g}$. Assuming for simplicity $\| Z \| = 1$, we are looking for an estimate for

$$\int_0^{2\pi} \langle (B_t Z)(t), (B_t Z)(t)\rangle dt + \int_0^{2\pi} \langle (B_t Z)'(t), (B_t Z)'(t)\rangle dt.$$

The first term is smaller than $M := (\max_{t\in I}\{\| B_t \|\})^2$. For the second term we have to interpret B_t as a H^1 loop on the space $\mathfrak{gl}(n,\mathbb{R})$ of $n \times n$ matrices. This is easy to do if we decompose $X(t)$, and accordingly B_t with respect to a basis on $\mathfrak{g}$ (B_t will be then a finite sum $\sum_i X^i(t) B_i$ where $B_i \in \mathfrak{gl}(n,\mathbb{R})$ and the components X^i of X are H^1 periodic functions). We get thus

$$\begin{aligned}\int_0^{2\pi} \langle (B_t Z)'(t), (B_t Z)'(t)\rangle dt &= \int_0^{2\pi} \langle B_t Z'(t), B_t Z'(t)\rangle dt + \int_0^{2\pi} \langle (B_t)' Z(t), (B_t)' Z(t)\rangle dt \\ &\quad + 2\int_0^{2\pi} \langle B_t Z'(t), (B_t)' Z(t)\rangle dt.\end{aligned}$$

We have $\int_0^{2\pi} \langle B_t Z'(t), B_t Z'(t)\rangle dt \leq M \int_0^{2\pi} \langle Z'(t), Z'(t)\rangle dt \leq M \| Z \| = M$. Remember now from the introduction that $\| Z \|^2_\infty = (\max_{t\in I}\{\| Z(t) \|\})^2 \leq k \| Z \|^2$ for some $k > 0$. Thus $\int_0^{2\pi} \langle (B_t)' Z(t), (B_t)' Z(t)\rangle dt$ is smaller than $k \int_0^{2\pi} \| B_t' \|^2 dt$. The latter integral is smaller than the square root of the product of the former two integrals, as can be seen by applying the Cauchy - Schwartz inequality for the $L^2 = H^0$ scalar product (notice that both $(B_t)'Z(t)$ and $B_tZ'(t)$ are L^2 loops). This finishes the proof.

We obtain for A the same relations as in the finite dimensional case, i.e. $A_XX = 0$ and $A_XY = \mathcal{V}[X,Y]$ for X and Y horizontal. The geodesics of LG/LG^ρ can be now completely described (their existence and uniqueness is a basic fact on any Hilbert manifold):

Proposition 1.4.4. *In the Riemannian submersion $\pi : LG \to LG/LG^\rho$, the horizontal lifts of geodesics are geodesics. The geodesical exponential Exp of LG/LG^ρ at $\hat{e} = e \cdot LG^\rho$ is given therefore by $Exp = \pi \circ \exp|_{\mathfrak{p}}$, where* exp *is as before the group and geodesical exponential of LG. The geodesics passing through other points can be obtained by left translating the geodesics through $\hat{e}$.*

Proof. Lift horizontally any geodesic on LG/LG^ρ and use the decomposition $\nabla_XX = A_XX + \mathcal{H}\nabla_XX = \mathcal{H}\nabla_XX$, as well as the formula defining the connection on the base space. For the second part use the fact that the geodesics on LG through the origin are the 1-parameter subgroups. Since ∇ commutes with any isometry (see section 1.7), the last assertion follows. ☐

The horizontal lift of a geodesic passing through $\hat{e}$ is thus a 1-parameter subgroup of LG, $t \mapsto \exp(tX)$ with $X \in \mathfrak{p}$. But this can be also interpreted as the group of transvections along the geodesic $t \mapsto \mathrm{Exp}(tX)$, such that $\mathfrak{p}$ is the space of infinitesimal transvections. The transvections can be obtained as usual by composing two symmetries: $\sigma_{\widehat{\exp tX}} \circ \sigma_{\hat{e}} = l_{\exp tX} \circ \bar{\rho} \circ l_{\exp tX^{-1}} \circ \bar{\rho} = l_{\exp tX} \circ l_{\rho(\exp tX)^{-1}} = l_{\exp 2tX}$ for $X \in \mathfrak{p}$. The fact that transvections translate vectors along γ parallely reduces to the fact that ∇ commutes with the isometries.

As in the finite dimensional case, a vector $u \in L\mathfrak{g}$ determines a Killing vector field X_u on LG/LG^ρ. The two identifications $\mathfrak{p} \cong T_{\hat{e}}LG/LG^\rho$ given by $\pi_*|_{\mathfrak{p}}$ and by the map $\mathfrak{p} \ni u \to X_u(\hat{e})$ are obviously the same.

The curvature tensor R is defined just as in the finite dimensional case. Proving that it is indeed a tensor is an easy computation in local coordinates. We prove now the following result:

Theorem 1.4.5. *The curvature tensor on LG/LG^ρ with the above connection is given by $R(X,Y)Z_{\hat{e}} = [Z,[X,Y]]_{\hat{e}}$. Here X, Y, Z are should be regarded either as vectors in $\mathfrak{p} \subset L\mathfrak{g}$ or as the corresponding Killing vector fields on LG/LG^ρ (restricted to $\hat{e}$ on the left-hand side).*

Remark: There is a minus sign difference between performing $[,]$ in the Lie algebra $L\mathfrak{g}$ and performing it on the induced Killing vector field in the Lie algebra $\mathcal{X}(LG/LG^\rho)$. Because there are two brackets in the expression of R the two minus signs will cancel each other.

Proof. We follow the proof given in [5] for the finite dimensional case. We sketch it first, showing then that it applies to our case.

- prove that $(\nabla X_n)_{\hat{e}} = 0$, when X_n is the Killing vector field corresponding to $u \in \mathfrak{p} \subset L\mathfrak{g}$;
- prove that $\nabla^2_{A,B}X = \nabla_A\nabla_B X - \nabla_{\nabla_A B}X$ is tensorial in A and B;
- choose X Killing vector field and define $L(A,B) = \nabla^2_{A,B}X + R(X,A)B$;
- check $L(A,B) = L(B,A)$ by using the Bianchi identity;
- see that Killing fields restricted to geodesics satisfy the Jacoby equation; use this equation to prove $L(A,A) = 0$;
- conclude that $\nabla^2_{A,B}X + R(X,A)B = 0$; use this and the vanishing of the torsion tensor to prove $\nabla_X[Y,Z] = R(Y,Z)X$ for $X, Y, Z \in \mathfrak{p}$ (regarded as vector fields on LG/LG^ρ)) -use also $(\nabla X_n)_{\hat{e}} = 0$;
- use Bianchi identity and again the torsion free property of ∇ and $(\nabla X_n)_{\hat{e}} = 0$ to finish the proof.

What is left is to check the ingredients on which the above facts are based. We have already seen that ∇ is torsionfree.

Bianchi's identity holds for any symmetric connection. This can be shown as usual by getting rid of the terms involving Lie brackets - extend any given three vectors to vector fields whose principal parts with respect to some trivialization are constant. The fact that $\nabla^2_{A,B}X$ is tensorial can be proved in a similar way to the tensoriality of R - here is essential that ∇ satisfies the definition of a linear connection by means of the local Christoffel symbols, definition presented in the introduction.

One computes

$$\begin{aligned}(\nabla_A\nabla_B X)_\varphi &= D(\nabla_B X)_\varphi \cdot A_\varphi + \Gamma_\varphi(A_\varphi, (\nabla_B X)_\varphi)\\ &= D\Gamma_\varphi(B_\varphi, X_\varphi)\cdot A_\varphi + \Gamma_\varphi(A_\varphi, \Gamma_\varphi(B_\varphi X_\varphi))\\ &\quad +\Gamma_\varphi(A_\varphi, DX_\varphi \cdot B_\varphi) + D(DX_\varphi\cdot B_\varphi)\cdot A_\varphi\end{aligned}$$

$$\begin{aligned}(\nabla_{\nabla_A B}X)_\varphi &= DX_\varphi(\nabla_A B)_\varphi + \Gamma_\varphi((\nabla_A B)_\varphi, X_\varphi)\\ &= DX_\varphi \dot{D}B_\varphi \cdot A_\varphi + DX_\varphi\cdot\Gamma_\varphi(A_\varphi, B_\varphi)\\ &\quad +\Gamma_\varphi(DB_\varphi\cdot A_\varphi, X_\varphi) + \Gamma_\varphi(\Gamma_\varphi(A_\varphi, B_\varphi), X_\varphi)\end{aligned}$$

Thus, because $D(DX_\varphi\cdot B_\varphi)\cdot A_\varphi = D^2X_\varphi(A_\varphi, B_\varphi) + DX_\varphi\cdot DB_\varphi\cdot A_\varphi$ and moreover $D\Gamma_\varphi(B_\varphi, X_\varphi)\cdot A_\varphi = D\Gamma_\varphi\cdot A_\varphi(B_\varphi, X_\varphi) + \Gamma_\varphi(DB_\varphi\cdot A_\varphi, X_\varphi) + \Gamma_\varphi(B_\varphi, DX_\varphi\cdot A_\varphi)$, we get $(\nabla_A\nabla_B X - \nabla_{\nabla_A B})_\varphi = D\Gamma_\varphi\cdot A_\varphi(B_\varphi, X_\varphi) + \Gamma_\varphi(B_\varphi, DX_\varphi\cdot A_\varphi) + \Gamma_\varphi(A_\varphi, \Gamma_\varphi(B_\varphi, X_\varphi)) + \Gamma_\varphi(A_\varphi, DX_\varphi\cdot B_\varphi) + D^2X_\varphi(A_\varphi, B_\varphi) - DX_\varphi\cdot\Gamma_\varphi(A_\varphi, B_\varphi) - \Gamma_\varphi(\Gamma_\varphi(A_\varphi, B_\varphi), X_\varphi)$.

From this expression it is clear that $\nabla_A\nabla_B X - \nabla_{\nabla_A B}X$ at some point depends only on the values of A and B at that point (all terms involving differentials of A or B have been canceled), so it is tensorial in A and B. One does not have to check the transformation rule for tensors, because we already have $\nabla_A\nabla_B X - \nabla_{\nabla_A B}$ defined globally.

The fact that $(\nabla X)_{\hat{e}} = 0$ for X Killing vector field corresponding to some $u \in \mathfrak{p}$ follows straightforward (as in Eschenburg-[5]) from the relation $\frac{D}{\partial s}\frac{\partial}{\partial t}\alpha(s,t) = \frac{D}{\partial t}\frac{\partial}{\partial s}\alpha(s,t)$, while the fact that Killing vector fields restricted to geodesics satisfy the Jacoby identity follows (as usual) from the relation $\frac{D}{\partial s}\frac{D}{\partial t}X(s,t) - \frac{D}{\partial t}\frac{D}{\partial s}X(s,t) = R(\frac{\partial\alpha}{\partial t}, \frac{\partial\alpha}{\partial s})X(s,t)$. For both relations $\alpha : \mathbb{R} \to LG/LG^\rho$ is a parametrised surface, and in the second one $X(s,t)$ is a vector field along α. The first relation is a rewriting of $T \equiv 0$, while the second comes from the definition of R - see [6]. ☐

Corollary 1.4.6. *LG/LG^ρ has non-negative sectional curvature.*

Proof. $\langle R(X,Y)Y, X\rangle = \langle [Y,[X,Y]], X\rangle = \langle [X,Y],[X,Y]\rangle \geq 0$. ☐

1.5 The dual symmetric space

In the first part of this section we define on $LG^\rho_\mathbb{C}/LG^\rho$ a Riemannian metric and a Levi-Civita connection which generate the usual properties of (dual) symmetric spaces. In the second part we prove that $LG^\rho_\mathbb{C}/LG^\rho$ is actually diffeomorphic with a Hilbert space. Let $\langle\, ,\rangle_\mathbb{C}$ be the hermitian scalar product on $\mathfrak{g}_\mathbb{C} = Lie(G_\mathbb{C})$ defined by:

$$\begin{cases} \langle x,y\rangle_\mathbb{C} = \langle x,y\rangle, \\ \langle x,iy\rangle_\mathbb{C} = 0, \qquad\qquad \forall\, x,y \in \mathfrak{g} \\ \langle ix,iy\rangle_\mathbb{C} = \langle x,y\rangle, \end{cases}$$

We integrate it to obtain again an L^2 scalar product, this time on $L\mathfrak{g}_\mathbb{C}$. It induces a left invariant metric on $LG_\mathbb{C}$ (which is not right invariant). On the other side, the scalar product on $\mathfrak{g}_\mathbb{C}$ can be extended to a left invariant metric on $G_\mathbb{C}$. On $LG_\mathbb{C}$ there is a connection ∇, pointwise with respect to the left invariant Levi-Civita connection on $G_\mathbb{C}$. This is the Levi-Civita connection for the metric just constructed. It is also a left invariant connection.

On the Lie subgroup $LG^\rho_\mathbb{C}$ of $LG_\mathbb{C}$ constructed before we put the induced metric, such that $i : LG^\rho_\mathbb{C} \hookrightarrow LG_\mathbb{C}$ becomes an isometric immersion. We use the same notation $\langle\, ,\rangle$ for the metrics on $LG_\mathbb{C}$ and $LG^\rho_\mathbb{C}$. As in the finite dimensional case, the Gauss equation gives the corresponding Levi-Civita connection on $LG^\rho_\mathbb{C}$:

$L\mathfrak{g}_\mathbb{C}$ splits into the ± 1 eigenspaces with respect to ρ_*, $L\mathfrak{g}_\mathbb{C} = L\mathfrak{g}^{tg}_\mathbb{C} \oplus L\mathfrak{g}^\perp_\mathbb{C}$, so that

$$L\mathfrak{g}^{tg}_\mathbb{C} = \mathrm{Lie}(\mathrm{LG}^\rho_\mathbb{C}) = \mathfrak{k} \oplus i\mathfrak{p} \text{ and } \mathrm{L}\mathfrak{g}^\perp_\mathbb{C} = i\mathfrak{k} \oplus \mathfrak{p}.$$

Remember that $\rho_*|_{L\mathfrak{g}}$ is an isometry. It follows easily that the extension $\rho_* : L\mathfrak{g}_{\mathbb{C}} \to L\mathfrak{g}_{\mathbb{C}}$ is also isometry, and therefore $L\mathfrak{g}_{\mathbb{C}}^{tg} \perp L\mathfrak{g}_{\mathbb{C}}^{\perp}$.

Let now X and Y be two vector fields on $LG_{\mathbb{C}}^{\rho}$ and decompose ∇ with respect to the two perpendicular left-invariant distributions determined by $L\mathfrak{g}_{\mathbb{C}}^{tg}$ and $L\mathfrak{g}_{\mathbb{C}}^{\perp}$: $\nabla_X Y = \nabla_X Y^{tg} + \nabla_X Y^{\perp}$. We define a linear connection $\tilde{\nabla}$ on $LG_{\mathbb{C}}^{\rho}$ by $\tilde{\nabla}_X Y := \nabla_X Y^{tg}$.

It is easy to see that $\tilde{\nabla}$ is well defined as an infinite dimensional connection: $\tilde{\Gamma}_{\tilde{\varphi}}$ (corresponding to $\tilde{\nabla}$) is just the $L\mathfrak{g}_{\mathbb{C}}^{tg}$-component of the Christoffel symbol Γ_{φ} corresponding to ∇. Here we take φ such that $\tilde{\varphi}$ - defined as the restriction of φ to $LG_{\mathbb{C}}^{\rho}$ - is a chart for $LG_{\mathbb{C}}^{\rho}$ with values in $L\mathfrak{g}_{\mathbb{C}}^{tg}$. The fact that $\tilde{\nabla}$ is torsionfree and Riemannian follows directly from the similar properties of the pointwise connection ∇ on $LG_{\mathbb{C}}$. We obtain thus:

Proposition 1.5.1. *The weak Riemannian manifold $(LG_{\mathbb{C}}^{\rho}, \langle\, , \rangle)$ admits a Levi-Civita connection.*

The next thing we do is to project the metric on $LG_{\mathbb{C}}^{\rho}$ to a metric on $LG_{\mathbb{C}}^{\rho}/LG^{\rho}$. Remark first that the tangent bundle $TLG_{\mathbb{C}}^{\rho}/LG^{\rho}$ splits itself in two perpendicular left invariant distributions: a horizontal one determined by $i\mathfrak{p}$ and a vertical one determined by $\mathfrak{k}$. By identifying G with a subgroup of $GL(n,\mathbb{R})$ (always possible since G is compact), $G_{\mathbb{C}}$ can be seen as the connected Lie subgroup of $GL(n,\mathbb{C})$ with Lie algebra $\mathfrak{g}_{\mathbb{C}}$. The identity $\mathrm{Ad}(g)iX = i\mathrm{Ad}(g)X$ is thereby trivial on $G_{\mathbb{C}}$. From it we get $\mathrm{Ad}(\gamma)iX = i\mathrm{Ad}(\gamma)X$ (in particular) for any $\gamma \in LG^{\rho}$ and $X \in \mathfrak{p}$. It follows that the scalar product defined on $L\mathfrak{g}_{\mathbb{C}}$ restricts to an $\mathrm{Ad}(LG^{\rho})$-invariant scalar product on $i\mathfrak{p} \subset L\mathfrak{g}_{\mathbb{C}}$. This corresponds at the algebraic level to the identity $\langle \xi, [\mu, \delta]\rangle = \langle [\xi, \mu], \delta\rangle$ for $\mu \in \mathfrak{k}, \xi, \delta \in i\mathfrak{p}$ (which fails for example for $\xi \in \mathfrak{k}, \mu, \delta \in i\mathfrak{p}$).

With the same argument used in the case of LG/LG^{ρ}, the $\mathrm{Ad}(LG^{\rho})$-invariance makes possible to define a (weak) Riemannian metric on $LG_{\mathbb{C}}^{\rho}/LG^{\rho}$ such that the restriction of $\pi_{*x} : T_x LG_{\mathbb{C}}^{\rho} \to T_{\pi(x)} LG_{\mathbb{C}}^{\rho}/LG^{\rho}$ to the horizontal distribution is an isometry (the canonical projection π becomes thus a Riemannian submersion). Moreover, the action of $LG_{\mathbb{C}}^{\rho}$ on $LG_{\mathbb{C}}^{\rho}/LG^{\rho}$ (by left translations) is also isometric. Symmetries at each point are obtained exactly as in the case of LG/LG^{ρ}.

We put a connection on $LG_{\mathbb{C}}^{\rho}/LG^{\rho}$ in the same way as for LG/LG^{ρ}. Remember that when we checked that ∇ is well-defined we needed only the $\mathrm{Ad}(LG^{\rho})$-invariance of the scalar product on $\mathfrak{p}$. By the same argument we get a well-defined connection on $LG_{\mathbb{C}}^{\rho}/LG^{\rho}$.

Then analogous ∇ on $LG_{\mathbb{C}}^{\rho}/LG^{\rho}$ is metric, torsionfree and satisfies all the properties needed to prove that $R(X,Y)Z = [Z, [X,Y]]$.

Similar to the finite dimensional case, if we identify the tangent spaces of LG/LG^{ρ} and $LG_{\mathbb{C}}^{\rho}/LG^{\rho}$ (via the identification of $\mathrm{i}\mathfrak{p}$ with $\mathfrak{p}$), the curvature tensors of the two spaces differ only by a sign. Accordingly, $LG_{\mathbb{C}}^{\rho}/LG^{\rho}$ has non-positive sectional curvature.

Determining the geodesics is also an easy task. Since the Levi-Civita connection on $LG_{\mathbb{C}}^{\rho}$ is the projection on $TLG_{\mathbb{C}}^{\rho}$ of a pointwise connection on $LG_{\mathbb{C}}$, the argument used to show that the O'Neill tensor A is really a tensor is valid in this case (the connection on $LG^\rho_{\mathbb{C}}$ is left invariant and it restricts to a bounded operator on $\mathfrak{k} \oplus i\mathfrak{p}$ - an upper bound is provided by the similar operator determined by the connection ∇ of $LG_{\mathbb{C}}$ on $L\mathfrak{g}_{\mathbb{C}}$) - see the Remark following Lemma 1.4.3.

Hence the geodesics on $LG^\rho_{\mathbb{C}}/LG^\rho$ are projections of horizontal geodesics on $LG^\rho_{\mathbb{C}}$. For X and Z horizontal left invariant vector fields on $LG^\rho_{\mathbb{C}}$ ($X, Z \in i\mathfrak{p}$) we have:

$$\begin{aligned} \langle \nabla_X X, Z\rangle &= -\langle X, \nabla_X Z\rangle = -\langle X, \nabla_Z X\rangle - \langle X, [X,Z]\rangle \\ &= -\frac{1}{2} Z\langle X, X\rangle - \langle X, [X,Z]\rangle = 0 \ ([X,Z] \in \mathfrak{k} \perp i\mathfrak{p}). \end{aligned}$$

Therefore the horizontal geodesics on $LG^\rho_{\mathbb{C}}$ are again the 1-parameter subgroups ($t \mapsto \exp tX$) for $X \in i\mathfrak{p}$, and thus the formula $\mathrm{Exp} = \pi \circ \exp|_{i\mathfrak{p}}$ describes the geodesic exponential of the dual symmetric space.

Remark: The 1-parameter subgroups ($t \mapsto \exp tX$), with $X \in i\mathfrak{p}$, are geodesics even in the ambient space $LG_{\mathbb{C}}$, since it is easy to see that $\langle X, [X,Z]\rangle = \langle [X,X], Z\rangle = 0$ holds for any $Z \in L\mathfrak{g}_{\mathbb{C}}$.

Summarizing, we get:

Proposition 1.5.2. *$LG^\rho_{\mathbb{C}}/LG^\rho$ admits a weak Riemannian metric and a corresponding Levi-Civita connection such that:*

i)It is a symmetric space (it admits symmetries at each point);

ii)the exponential is given by $Exp = \pi \circ \exp|_{i\mathfrak{p}}$;

iii)The curvature tensor can be expressed at $\hat{e}$ by $R(X,Y)Z=-[[X,Y],Z]$;

iv)It is dual to LG/LG^ρ (by identifying $\mathfrak{p}$ with $i\mathfrak{p}$ the isotropy action of LG^ρ is the same and the curvature is just the opposite), in particular it has non-positive sectional curvature.

In the finite dimensional case, the symmetric spaces of noncompact type are diffeomorphic (via the exponential) with the modeling space $\mathbb{R}^n$. This can be proved by using the Hadamard-Cartan theorem. In his Ph.D. thesis (at Columbia University, 1965), J. McAlpin proved the Hadamard-Cartan theorem in the Hilbert setting. The two obstacles he had to overcome were the absence of the Hopf-Rinow theorem, which fails to be true for Hilbert manifolds, and the fact that injective linear endomorphisms may not be surjective. It turns out that the use of the Hopf-Rinow theorem can be avoided with some extra work. The completeness of the tangent spaces with respect to the scalar product induced by the Riemannian metric is crucial on both points in McAlpin's proof. Since this is not the case for our weak metric, we had to find another approach to show that $LG^\rho_{\mathbb{C}}/LG^\rho$ is (diffeomorphic to) a Hilbert space. Fortunately, one only has to look closer at the way it was constructed.

We prove:

Theorem 1.5.3. *$Exp : i\mathfrak{p} \to LG^\rho_{\mathbb{C}}/LG^\rho$ is a diffeomorphism.*

Proof. Since $G_{\mathbb{C}}/G$ is the noncompact dual symmetric space of G, $\mathrm{Exp} : i\mathfrak{g} \to G_C/G$ is a diffeomorphism. The geodesical exponential Exp equals $pr \circ \exp$, where exp is the group exponential and pr the projection of $G_{\mathbb{C}}$ onto G. It induces a diffeomorphism $L\mathrm{Exp} : iL\mathfrak{g} = Li\mathfrak{g} \to L(G_{\mathbb{C}}/G)$.

Define $f : LG^{\rho}_{\mathbb{C}}/LG^{\rho} \to L(G_{\mathbb{C}}/G)$, $f(\hat{g}) = \bar{g}$, where $\bar{g}(t) = g(t)G \in G_{\mathbb{C}}/G$. It is easy to see that f is well-defined.

Since G is the fixed point set of ρ_0 in $G_{\mathbb{C}}$, there is a well known embedding of $G_{\mathbb{C}}/G$ in $G_{\mathbb{C}}$, which we denote it by Φ. It is defined by $\Phi(xG) = x\rho_o(x)^{-1}$ and it induces an embedding $L\Phi : L(G_{\mathbb{C}}/G) \to LG_{\mathbb{C}}$. If $\bar{\gamma} \in L(G_{\mathbb{C}}/G)$ and $\gamma \in LG_{\mathbb{C}}$ such that $\gamma(t)G = \bar{\gamma}(t)$, then obviously $L\Phi(\bar{\gamma}) = \gamma\rho_0(\gamma)^{-1}$.

We consider now the following diagram:

$$\begin{array}{ccccc} Li\mathfrak{g} & \xrightarrow{L\mathrm{Exp}} & L(G_{\mathbb{C}}/G) & \xrightarrow{L\Phi} & LG_{\mathbb{C}} \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle f} & & \\ i\mathfrak{p} & \xrightarrow{\mathrm{Exp}} & LG^{\rho}_{\mathbb{C}}/LG^{\rho} & & \end{array}$$

Since $\mathrm{Exp} : i\mathfrak{p} \to LG^{\rho}_{\mathbb{C}}/LG^{\rho}$ can also be expressed as $\pi \circ \exp$ (and this exp is actually Lexp), the commutativity follows straightforward.

Let $\hat{g}_1, \hat{g}_2 \in LG^{\rho}_{\mathbb{C}}/LG^{\rho}$ such that $f(\hat{g}_1) = f(\hat{g}_2)$. Then $g_2^{-1}g_1(t) \in G$, $\forall t \in I$. But $g_1, g_2 \in LG^{\rho}_{\mathbb{C}}$, hence $g_2^{-1}g_1 \in LG^{\rho}$ and thus $\hat{g}_1 = \hat{g}_2$, i.e. f is injective. To prove that $\mathrm{Exp} : i\mathfrak{p} \to LG^{\rho}_{\mathbb{C}}/LG^{\rho}$ is bijective is thus enough to show that $L\mathrm{Exp}|_{i\mathfrak{p}} : i\mathfrak{p} \to L(G_{\mathbb{C}}/G)$ maps $i\mathfrak{p}$ onto $\mathrm{Im} f$. Let $\hat{g} = gLG^{\rho} \in LG^{\rho}_{\mathbb{C}}/LG^{\rho}$. Then there is a unique $u \in Li\mathfrak{g}$ such that $L\mathrm{Exp}(u) = f(\hat{g})$.

We want $u \in i\mathfrak{p}$, that is $\rho_*(u) = u$. We know $L\Phi \circ f(\hat{g}) = g\rho_0(g)^{-1}$ and therefore $L\Phi \circ L\mathrm{Exp}(u) = g\rho_0(g)^{-1}$. Furthermore, for $X \in i\mathfrak{g}$ it holds $\Phi \circ \mathrm{Exp}(X) = \Phi \circ pr \circ \exp(X) = \exp(X)(\rho_0 \exp X)^{-1} = \exp X(\exp \rho_0 X)^{-1} = \exp X \exp(-X)^{-1} = \exp 2X$, hence $\Phi \circ \mathrm{Exp} = \mathrm{Exp} \circ \tau$, where τ is the multiplication with 2.

It follows $L\Phi \circ L\mathrm{Exp}(u) = L(\Phi \circ \mathrm{Exp})(u) = L(\exp \circ \tau)(u) = L\exp(2u) = \exp(2u)$. We obtained $g\rho_0(g)^{-1} = \exp(2u)$. From the relations $\rho_*\rho_0 = \rho_0\rho_*$ and $\rho(g) = g$ we get $\rho(g\rho_0(g)^{-1}) = g\rho_0(g)^{-1}$. It follows $\rho_*(\exp(2u)) = \exp(2u)$, whence $\exp(\rho_*(2u)) = \exp(2u)$. Since $u \in Li\mathfrak{g}$, $\rho_*(Li\mathfrak{g}) \subseteq Li\mathfrak{g}$ and $\exp|_{Li\mathfrak{g}}$ is injective, we finally obtain $\rho_*(u) = u$, and thus $u \in i\mathfrak{p}$.

We proved that $\mathrm{Exp} : i\mathfrak{p} \to \mathrm{LG}^{\rho}_{\mathbb{C}}/\mathrm{LG}^{\rho}$ is bijective. The smoothness of $\mathrm{Exp} = \pi \circ \exp$ is obvious. To get the smoothness of its inverse, we only need the smoothness of f. If we denote by j the inclusion of $LG^{\rho}_{\mathbb{C}}$ in $LG_{\mathbb{C}}$, then $f \circ \pi = Lpr \circ j : LG^{\rho}_{\mathbb{C}} \to L(G_{\mathbb{C}}/G)$, which means that $f \circ \pi$ is smooth. Since π is a surjective submersion, we conclude that Exp: $i\mathfrak{p} \to LG^{\rho}_{\mathbb{C}}/LG^{\rho}$ is a diffeomorphism. □

1.6 Totally geodesic submanifolds and maximal flats

We begin with some considerations about the submanifolds of LG/LG^ρ. Given a submanifold S, we can restrict the L^2 metric to a metric on S. In order to get an associated Levi-Civita connection one needs a normal bundle for S, i.e. an orthogonal complement for the tangent space of S at each point. This is in not possible in general, because the tangent spaces of LG/LG^ρ are only pre-Hilbert with the given metric. Nevertheless, when a normal bundle for S exists, then the orthogonal decomposition of $\nabla_X Y$, for X and Y vector fields on S, gives as usual the Gauss equation $\nabla_X Y = \nabla^S_X Y + \alpha(X,Y)$, where ∇^S is the Levi-Civita connection corresponding to the induced metric on S and α is the second fundamental form. $\nabla_X Y$ makes sense as the covariant derivative along the inclusion $i : S \to LG/LG^\rho$. One can easily see that α is a tensor: Taking charts (φ, U) for S and (Φ, V) for LG/LG^ρ with respect to which the inclusion of S in LG/LG^ρ is just the inclusion of a closed vector space in $\mathfrak{p}$, the local expression for α is

$$\alpha_\varphi(u,v) = \Gamma_\Phi(u,v) - \Gamma^S_\varphi(u,v).$$

Γ^S_φ and Γ^S_φ are as before the Christoffel symbols for ∇ and ∇^S in the given charts.

Because of the problem mentioned above, we will not talk about geodesics of S. One can still say that S is *totally geodesic at a point* $p \in S$ provided any geodesic of LG/LG^ρ tangent to S at p is contained in S. S is said to be *totally geodesic* if it is totally geodesic at each point. In this case S is said to be *flat* if the restriction of the curvature tensor to the tangent spaces of S is identically zero.

In analogy with the finite dimensional case, totally geodesic submanifolds correspond to Lie triples (which we require to be closed subspaces of $L\mathfrak{g}$). As previously remarked, closed (Hilbert) subgroups of Lie groups need not be Lie subgroups. Therefore, we need the following basic fact:

Lemma 1.6.1. *Let G be a Hilbert Lie group, H a Lie subgroup of G and K an embedded Lie subgroup of G. Then $H \cap K$ is an embedded Lie subgroup of H and $Lie(H \cap K) = Lie(H) \cap Lie(K)$.*

Proof. Cf. [1], III-1.3.6, it suffices to find a point $x \in H \cap K$ and a neighborhood U of x in H such that $K \cap U$ is an embedded submanifold of H. We can choose $x = e$, take a normal neighborhood of it $\tilde{U} = \exp \tilde{V}$ for some neighborhood $\tilde{V}$ of 0 in $Lie(G)$ and let $U := \exp(Lie(H) \cap \tilde{V})$. Thus $U \cap K$ is the image under exp of the embedded submanifold $\tilde{U} \cap Lie(K)$ of $\tilde{U} := Lie(H) \cap \tilde{V}$.

□

We can prove:

Lemma 1.6.2. *Given G, H, and K as in lemma 1, $H/H \cap K$ is a submanifold of G/K.*

Proof. The proof is identical to that given by Helgason in [10], Prop. II.4.4.(a) and is based essentially on the inverse function theorem.

□

We also need:

Lemma 1.6.3. *Let M be a weak Hilbert Riemannian manifold admitting a Levi-Civita connection and let S be a totally geodesic submanifold of M. Then the parallel translation along curves in S transports tangents to S into tangents to S.*

Proof. Choose an arbitrary point p in S. Let V be an open neighborhood of p in S and (Φ, U) a chart for M around p with $\Phi : U \to \Phi(U) \subset E$ such that $V \subset U$ and $\Phi(V)$ is the intersection between $\Phi(U)$ and a closed subspace F of the Hilbert space E. Let γ be a geodesic tangent to S at some $q \in V$. Then γ is contained in S, hence $\Phi \circ \gamma \subset F$. In the given chart the geodesic equation takes the form $(\Phi \circ \gamma)''(t) + \Gamma_\Phi(\Phi \circ \gamma(t))((\Phi \circ \gamma)'(t), (\Phi \circ \gamma)'(t)) = 0$.

Let H be a complementary subspace for F in E. Decomposing the Christoffel symbol accordingly into $\Gamma_\Phi = \Gamma_\Phi^F + \Gamma_\Phi^H$, we obtain $\Gamma_\Phi^H(\Phi \circ \gamma(t))((\Phi \circ \gamma)'(t), (\Phi \circ \gamma)'(t)) = 0$. In particular, for t=0, it follows $\Gamma_\Phi^H(q)(X, X) = 0$, $\forall X \in F$. But $\Gamma_\Phi(q)$ is symmetric, so $\Gamma_\Phi^H(q)|_{F \times F} = 0$ for any point q in V. Let now $c : I \to V$ be an arbitrary curve and let $Y(t)$ be a parallel vector field along c such that $Y(0)$ is tangent to S. We have $Y_\Phi'(t) + \Gamma_\Phi(\Phi \circ c(t))(Y_\Phi(t), (\Phi \circ c)'(t)) = 0$ and $Y(0)_\Phi \in F$. Take the H-part $(Y_\Phi'(t))^H + \Gamma_\Phi^H(\Phi \circ c(t))(Y_\Phi(t)(\Phi \circ c)'(t)) = (Y_\Phi^H)'(t) + \Gamma_\Phi^H(\Phi \circ c(t))(Y_\Phi^H(t), (\Phi \circ c)'(t)) = 0$. This linear differential equation with initial condition $Y_\Phi^H(0) = 0$ has a unique solution $Y_\Phi^H \equiv 0$, which shows that Y is tangent to S everywhere. For vector fields along arbitrary curves (not contained in a chart) the set of all t for which $Y(t)$ is tangent to S is both open and closed, so it must be the whole interval. □

Theorem 1.6.4. *Let $\mathfrak{s}$ be a Lie triple contained in $\mathfrak{p}$. Then it exists a totally geodesic submanifold S of LG/LG^ρ such that $T_{\hat{e}}S = \mathfrak{s}$ and $Exp\, \mathfrak{s} \subset S$. Conversely, the tangent space of a totally geodesic submanifold passing through $\hat{e} = eLG^\rho$ is a Lie triple. Furthermore, the totally geodesic submanifold S is flat if and only if $\mathfrak{s}$ is abelian. A similar correspondence exists between totally geodesic submanifolds of $LG_{\mathbb{C}}^\rho/LG^\rho$ and Lie triples in $i\mathfrak{p}$. In this case the equality $S = Exp\, \mathfrak{s}$ holds.*

Remark: We will show in the next section that the loop group LG can be considered as a particular case of a coset space LG'/LG'^ρ. On the other side, unlike the group exponential of G, that of LG need not be surjective. One example is given by $G = SU(2)$ - see [23]. This shows that the inclusion Exp $\mathfrak{s} \subset S$ can be strict. It provides also a counterexample to the Hopf-Rinow theorem. It should be pointed out here that although the Hopf-Rinow theorem fails in general in infinite dimensions, it holds for the loop manifold LM provided with the natural strong H^1 Riemannian metric - see [4], where the result is proved for a more general class of Hilbert manifolds of Sobolev sections of some fibre bundle.

Proof. For simplicity we consider the proof for LG/LG^ρ, mentioning the differences which may appear in the case of the dual space. Let first S be a totally geodesic submanifold containing $\hat{e}$. Since S may lack induced geometrical structure, we can not argue that $\mathfrak{s} := T_{\hat{e}}S$ is invariant under the curvature tensor. Therefore we follow the more involved proof given in [10]. For X, $Y \in \mathfrak{s}$ and $t \in \mathbb{R}$, the vector $A = d\mathrm{Exp}_{tY}(X)$ is tangent to S, while the family $dl_{\exp(-tY)}A$ is its parallel extension to $t \mapsto \exp tY$. By Lemma 1.6.3, $dl_{\exp(-tY)}A \in \mathfrak{s}$.

The differential of the exponential $\exp : \mathfrak{h} \to H$ (for any Lie group H) is expressed by the formula $d\exp_Y = dl_{\exp Y} \circ \sum_{m=0}^{\infty} \frac{(-adY)^m}{(m+1)!}$. Because the exponential exp of LH is actually $L\exp$ and because the functors L and T (the tangent functor) commute, we get an identical formula for the differential of the loop group exponential. Taking $H = G_\mathbb{C}$ and restricting to $G_\mathbb{C}^\rho$ we get the same result for $G_\mathbb{C}^\rho$. Using now the relation $\mathrm{Exp} = \pi \circ \exp|_\mathfrak{p}$ and denoting the restriction of $(\mathrm{ad}Y)^2$ to $\mathfrak{p}$ by T_Y, the following is obtained: $d\mathrm{Exp}_Y = dl_{\exp Y} \circ d\pi \sum_{m=0}^{\infty} \frac{(-adY)^m}{(m+1)!} = dl_{\exp Y} \circ \sum_{n=0}^{\infty} \frac{(T_Y)^n}{(2n+1)!}$ for any $Y \in \mathfrak{p}$.

Putting all together, it follows $\sum_{n=0}^{\infty} \frac{(T_{tY})^n}{(2n+1)!}(X)$. Because T_X is bounded, taking t small enough shows that $T_Y(X) \in \mathfrak{s}$. From this point we can just repeat the easy computation (involving the Jacobi identity) from [10], which gives $[X,[Y,Z]] \in \mathfrak{s}$ for $X,Y,Z \in \mathfrak{s}$. Thus $\mathfrak{s}$ is a Lie triple.

Conversely, suppose $\mathfrak{s}$ is a Lie triple and let $\overline{[\mathfrak{s},\mathfrak{s}]}$ be the closure of the subspace $[\mathfrak{s},\mathfrak{s}]$ of $\mathfrak{k}$. It is easy to check that $\mathfrak{s} + [\mathfrak{s},\mathfrak{s}]$ is a Lie subalgebra of $L\mathfrak{g}$. Its closure is $\mathfrak{g}' = \mathfrak{s} + \overline{[\mathfrak{s},\mathfrak{s}]}$ and is a closed Lie subalgebra of $L\mathfrak{g}$ ($[\,,]$ is continuous). It exists therefore a Lie subgroup G' of LG with Lie algebra $\mathfrak{g}'$. Since LG^ρ is an embedded Lie subgroup of LG, Lemma 1.6.1 shows that $G' \cap LG^\rho$ is an embedded subgroup of G'. From Lemma 1.6.2 now $G'/G' \cap LG^\rho$ is a submanifold of LG/LG^ρ. From the formula $\mathrm{Exp} = \pi \circ \exp|_\mathfrak{p}$ one sees that $\mathrm{Exp}\,\mathfrak{s} \subset G'/G' \cap LG^\rho$ and that the geodesics tangent to $G'/G' \cap LG^\rho$ are contained in $G'/G' \cap LG^\rho$. Since G' acts transitively by isometries on $G'/G' \cap LG^\rho$, it follows that $G'/G' \cap LG^\rho$ is totally geodesic. In the case of the dual the geodesic exponential $\mathrm{Exp} : i\mathfrak{p} \to LG_\mathbb{C}^\rho/LG^\rho$ is a diffeomorphism, and therefore the equality $\mathrm{Exp}\,\mathfrak{s} = G'/G' \cap LG^\rho$ holds.

Finally, if $\mathfrak{s}$ is abelian, then it is clear from the formula for R that $G'/G' \cap LG^\rho$ is flat. Conversely, for X and Y in $\mathfrak{s}$, $R(X,Y) = 0$ implies $0 = \langle X, R(X,Y)Y\rangle = \langle X,[Y,[X,Y]]\rangle = \pm\langle [X,Y],[X,Y]\rangle$, whence $[X,Y] = 0$ and thus $\mathfrak{s}$ is abelian (the minus sign appears in the case of the dual space). □

In the finite dimensional case the maximal flat totally geodesic submanifolds of a symmetric space M have two nice properties: they are closed topological subspaces of M and are all conjugate under isometries of M. The second property definitely fails in our case, while the first one may fail either (the usual proof is not working at least). One can easily find several conjugacy classes of maximal subalgebras contained in $\mathfrak{p}$. Let for example $\mathfrak{a}_0, \mathfrak{a}_1$ be maximal abelian subalgebras contained in $\mathfrak{g}$. Then for each $t_0 \in [o,\pi]$, the loops $\gamma \in \mathfrak{p}$ (in particular $\gamma(0), \gamma(\pi) \in \mathfrak{p}_\mathfrak{g}$) with $\gamma(t) \in \mathfrak{a}_0$ for $t \le t_0$ and $\gamma(t) \in \mathfrak{a}_1$

for $t_0 \leq t$ form a maximal abelian subalgebra in $\mathfrak{p}$. On the other side, any smooth map $S^1 \xrightarrow{\varphi} \{\mathfrak{a} \subset \mathfrak{g}$ maximal abelian $\} \subset G(k,n)$ (the Grassmanian of K -planes into $\mathfrak{g}$) gives rise to another maximal abelian subalgebra $\mathfrak{a}_\varphi = \{\gamma \in L\mathfrak{g} \mid \gamma(t) \in \mathfrak{a}_t = \varphi(t)\}$ of $L\mathfrak{g}$. Here $K = \text{rank}(G)$, $n = \dim G$. The two examples belong obviously to different conjugacy classes.

The space $\widehat{LG}/\widehat{LG}^\rho$ which we study in the next chapter is closely related to LG/LG^ρ. Topologically it is a torus bundle over LG/LG^ρ. Motivated by the results obtained in the next chapter about a class of maximal tori of $\widehat{LG}$, and correspondingly about a class of maximal flat totally geodesic submanifolds of $\widehat{LG}/\widehat{LG}^\rho$, we obtain the following (by smooth we can understand here either C^∞ or H^1):

Proposition 1.6.5. *Let $\mathfrak{a} \subset L\mathfrak{g}$ a maximal abelian subalgebra such that for each $v \in \mathfrak{a}$, $v(t) \in Ad(G)(v(0))$. Then it exists a smooth map $\varphi_t : I \to G$ such that $\mathfrak{a} = \{Ad(\varphi_t)v_0 \mid v_0 \in \mathfrak{a}_0\}$ and $\mathfrak{a}_0 = \{v_0 = v(0) \mid v \in \mathfrak{a}\}$ is maximal abelian in $\mathfrak{g}$.*

Proof. For each $t \in I$, $\mathfrak{a}$ determines an abelian subalgebra $\mathfrak{a}_t = \{v(t) \mid v \in \mathfrak{a}\}$. Let $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{g}$ such that $\mathfrak{a}_0 \subset \mathfrak{t} \subset \mathfrak{g}$. Because $\mathfrak{g}$ is compact it decomposes as $\mathfrak{g} = \mathfrak{t} \oplus \sum_{\alpha \in \Delta_+} \mathfrak{g}^\alpha$, where Δ_+ is the set of positive real roots $\mathfrak{g}$ and $\mathfrak{g}^\alpha$ are 2-dimensional real subspaces which allow complex structures (multiplications) such that

$$[H, X] = 2\pi i \alpha(H) X \text{ for } H \in t,\ x \in \mathfrak{g}^\alpha,\ \alpha \in \Delta_+.$$

The centralizer of any $H \in \mathfrak{t}$ is therefore $Z(H) = \mathfrak{t} \oplus \sum_{\substack{H \in Ker\alpha \\ \alpha \in \Delta_+}} \mathfrak{g}^\alpha$, with $\alpha : \mathfrak{t} \to \mathbb{R}$.

We index the roots α from 1 to n: $\Delta_+ = \{\alpha_1, ..., \alpha_n\}$ such that

$$\mathfrak{a}_0 \subset \ker \alpha_1, ..., \mathfrak{a}_0 \subset \ker \alpha_k,\ \mathfrak{a}_0 \not\subseteq \ker \alpha_{k+1}, ..., \mathfrak{a}_0 \not\subseteq \ker \alpha_n.$$

$\mathrm{Ker}\alpha_{\mathrm{k+1}} \cap \mathfrak{a}_0, ..., \ker \alpha_{\mathrm{n}} \cap \mathfrak{a}_0$ are hyperplanes of $\mathfrak{a}_0$. Choose $v_0 \in \mathfrak{a}_0$ outside all these hyperplanes; then $Z(v_0) = \mathfrak{t} \oplus \sum_{i=1}^{k} \mathfrak{g}^{\alpha_i}$. Now

$$\begin{aligned} v(t) &= \mathrm{Ad}(\varphi_t)v_0, \\ w(t) &= \mathrm{Ad}(\psi_t)w_0 \end{aligned}$$

for some φ_t, $\psi_t : I \to$ G smooth with $\varphi_0 = \psi_0 = e \in G$.

From $[v, w] = 0$ it follows $[\mathrm{Ad}(\varphi_t)v_0, \mathrm{Ad}(\psi_t)w_0] = 0$ for any $t \in I$, so $(Ad)(\psi_t)w_0 \in Z(\mathrm{Ad}(\varphi_t)v_0) = \mathrm{Ad}(\varphi_t)Z(v_0)$.

But $\mathrm{Ad}(G)w_0 \perp Z(w_0)$ at w_0 with respect to the Ad-invariant metric on $\mathfrak{g}$, hence $\mathrm{Ad}(G)w_0 \perp Z(v_0)$ at w_0 (because of the way we have chosen v_0, it holds $w_0 \in Z(v_0) \subseteq Z(w_0)$).

We obtain a path $\gamma : I \to \mathfrak{g}$, $\gamma(t) = \mathrm{Ad}(\varphi_t^{-1}\psi_t)w_0$ with $\gamma(0) = w_0$, $\gamma(I) \subset Z(v_0)$, and $\gamma'(0) \perp Z(v_0)$ hence $\gamma'(0) = 0$, i.e. $(t \mapsto \mathrm{Ad}(\varphi_t^{-1}\psi_t)w_0)'(0) = 0$.

But we can apply the same method for an arbitrary $t_0 \in I$: $\mathfrak{a}_{t_0}$ is an abelian subalgebra of $\mathfrak{g}$ and $Z(\mathrm{Ad}(g)v) = \mathrm{Ad}(g)Z(v)$ holds in general, therefore $Z(\mathrm{Ad}(\varphi_{t_0})v_0)$ has minimal dimension among the centralisers of all the elements $u(t_0) \in \mathfrak{a}_{t_0}$. Thus $Z(v(t_0)) \subset Z(u(t_0))$ for any u$\in \mathfrak{a}$ (with the same argument as before: let $\mathfrak{a}_{t_0} \subset \mathfrak{t}'$, where $\mathfrak{t}'$ is maximal abelian in $\mathfrak{g}$; it follows $Z(v(t_0)) = \mathfrak{t}' \oplus \sum\limits_{\substack{\alpha\in\Delta_+ \\ \mathfrak{a}_{t_0}\subset\ker\alpha}} \mathfrak{g}^\alpha$).

Now let $\tilde{\varphi}_s := \varphi_{t_0+s}\varphi_{t_0}^{-1}$, $\tilde{\psi}_s := \psi_{t_0+s}\psi_{t_0}^{-1}$. We get

$$[\mathrm{Ad}(\tilde{\varphi}_s)v_{t_0}, \mathrm{Ad}(\tilde{\psi}_s)w_{t_0}] = [\mathrm{Ad}(\varphi_{t_0+s})v_0, \mathrm{Ad}(\psi_{t_0+s})w_0] = 0,$$

hence $\mathrm{Ad}(\tilde{\psi}_s)w_{t_0} \in Z(\mathrm{Ad}(\tilde{\varphi}_s)v_{t_0}) = \mathrm{Ad}(\tilde{\varphi}_s)Z(v_{t_0})$, and thus $\mathrm{Ad}(\tilde{\varphi}_s^{-1}\tilde{\psi}_s)w_{t_0} \in Z(v_{t_0}) \subset Z(w_{t_0})$.

On the other side $\mathrm{Ad}(G)w_{t_0} \perp Z(w_{t_0})$ at w_{t_0}, so $\mathrm{Ad}(G)w_{t_0} \perp Z(v_{t_0})$. We obtained $\tilde{\gamma} : I \to \mathfrak{g}$, $\tilde{\gamma}(s) = \mathrm{Ad}(\tilde{\varphi}_s^{-1}\tilde{\psi}_s)w_{t_0}$, which satisfies $\tilde{\gamma}(0) = w_{t_0}$, $\tilde{\gamma}(I) \subset Z(v_{t_0})$ and $\tilde{\gamma}'(0) \perp Z(v_{t_0})$ hence $\tilde{\gamma}'(0) = 0$. But $\tilde{\varphi}_s^{-1}\tilde{\psi}_s = \varphi_{t_0}\varphi_{t_0+s}^{-1}\psi_{t_0+s}\psi_{t_0}^{-1}$, therefore $\tilde{\gamma}(s) = \mathrm{Ad}(\varphi_{t_0})\mathrm{Ad}(\varphi_{t_0+s}^{-1}\psi_{t_0+s})\mathrm{Ad}(\psi_{t_0}^{-1})(w_{t_0}) = \mathrm{Ad}(\varphi_{t_0})\mathrm{Ad}(\varphi_{t_0+s}^{-1}\psi_{t_0+s})(w_0)$ and $\tilde{\gamma}'(0) = 0$, i.e. $(t \mapsto \mathrm{Ad}(\varphi_t^{-1}\psi_t)(w_0))'(t_0) = 0$. Because t_0 was chosen arbitrary we get $\mathrm{Ad}(\varphi_t^{-1}\psi_t)w_0 = w_0$ for any t, which implies $\mathrm{Ad}(\varphi_t)w_0 = \mathrm{Ad}(\psi_t)w_0 = w(t)$. We can take thus a smooth curve $\varphi_t : I \to G$ such that any $u \in \mathfrak{a}$ can be written as $u(t) = \mathrm{Ad}(\varphi_t)u_0$, and because $\mathfrak{a}$ is maximal it follows that $\mathfrak{a}_0$ is maximal abelian in $\mathfrak{g}$. □

1.7 Miscellaneous

Any compact Lie group G is a symmetric space. It can be thought to be associated to an orthogonal symmetric Lie algebra if we represent it as the coset space $G \cong G \times G/\Delta(G \times G)$, where the diagonal $\Delta(G \times G)$ is the fixed point set of the involution $\rho : G \times G \to G \times G$, $\rho(g, h) = (h, g)$. We have a similar picture for LG:

For any $\eta \in LG$, let η^* be the reversed loop $\eta^*(t) = \eta(2\pi - t)$. Define the involution ρ of second kind on $L(G \times G) = LG \times LG$ by $\rho(\gamma, \eta) = (\eta^*, \gamma^*)$. The fixed point set $(LG \times LG)^\rho$ is $\{(\gamma, \gamma^*) \mid \gamma \in LG\}$. It is easy to check that the map $\Phi : LG \times LG/(LG \times LG)^\rho \to LG$, $\Phi(\widehat{(\gamma, \eta)}) = \gamma\eta^{*-1}$ is bijective and actually a diffeomorphism. If we start with a product bi-invariant metric on $G \times G$ and apply to this case the theory exposed in this chapter, we recover the L^2 metric, the pointwise Levi-Civita connection, the geodesics, the inverse mapping as geodesic symmetry and the expression of the curvature in terms of the Lie bracket. For example, if we consider the group exponential of $LG \times LG$ restricted to $\mathfrak{p} = \{(u, -u^*) \mid u \in L\mathfrak{g}\}$ and project it on $LG \times LG/(LG \times LG)^\rho$, then, via the identification provided by Φ, this is just the group exponential on LG.

The isotropy representation of LG/LG^ρ is equivalent to $\mathrm{Ad}_{LG}(LG^\rho)|_{\mathfrak{p}}$, the restricted adjoint action of LG^ρ on $\mathfrak{p}$. This (as well as the whole action of LG on $L\mathfrak{g}$) is not a Fredholm map: take a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$; then $L\mathfrak{a}$ is perpendicular to any orbit it meets, and so these have infinite codimension. In particular the isotropy representation is not polar.

A still open problem is to determine the isometry group of LG/LG^ρ. We expect the connected component $I_0(LG/LG^\rho)$ of it to equal LG, just like in the finite dimensional case. For this we have to assume nevertheless that G acts effectively on G/G^ρ, and thus $Z(G) \cap G^\rho = \emptyset$. Since G is simple $Z(G)$ is finite. Consequently, $Z(LG) = LZ(G) \cong Z(G)$ is finite and $Z(LG) \cap LG^\rho = \emptyset$. From this it is not hard to show that LG acts effectively on LG/LG^ρ.

One can also prove the curvature tensor R is parallel for both LG/L^ρ and $LG^\rho_{\mathbb{C}}/LG^\rho_0$. The proof is immediate and similar to that for finite dimensional (locally) symmetric spaces. All we need is the following result, used also on previous occasions:

Lemma 1.7.1. *Let M a (weak) Hilbert Riemannian manifold (for which the Levi-Civita connection ∇ exists). Then any isometry σ of M commutes with ∇: $\sigma_* \nabla_X Y = \nabla_{\sigma_* X} \sigma_* Y$.*

Proof. From the torsionfree and the metric property of ∇ we can deduce as usual the formula

$$2\langle Z, \nabla_X Y\rangle = X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle + \langle [X, Y], Z\rangle - \langle [Y, Z], X\rangle + \langle [Z, X], Y\rangle.$$

Notice that only the uniqueness of ∇ can be deduced from it (the existence does not follow anymore, since the linear map from T_xM to T_xM^* defined by $X \mapsto \langle \cdot, X\rangle$ is injective but not surjective for $(T_xX, \langle\ ,\ \rangle)$ pre-Hilbert space). Using this formula, it is easy to check that $\langle \sigma_* Z, \nabla_{\sigma_* X} \sigma_* Y\rangle \circ \sigma = \langle Z, \nabla_X Y\rangle$, from which we obtain $\langle \sigma_* Z, \nabla_{\sigma_* X} \sigma_* Y\rangle = \langle \sigma_* Z, \sigma_* \nabla_X Y\rangle$, and thus $\sigma_* \nabla_X Y = \nabla_{\sigma_* X} \sigma_* Y$. ☐

We end the study of LG/LG^ρ with the Cartan embedding into LG, i.e. the map $LG/LG^\rho \overset{j}{\hookrightarrow} LG$, $j(gLG^\rho) = g\rho(g)^{-1}$. It is easy to check that j is well defined and injective. The composition $j \circ l_g \circ \mathrm{Exp}(X) = j \circ l_g \circ \pi \circ \exp|_{\mathfrak{p}}(X) = j \circ \pi \circ l_g \circ \exp|_{\mathfrak{p}}(X) = g \exp 2X \rho(g)^{-1}$ shows that j is a smooth immersion. Let M be the image of LG/LG^ρ under j. To show that j is, like in the finite dimensional case, an embedding, it is now enough to show that for any $g \in M$ and for any sequence (g_n) in M with $g_n \overset{n\to\infty}{\longrightarrow} g$, $j^{-1}(g_n) \overset{n\to\infty}{\longrightarrow} j^{-1}(g)$ holds. Let $N \subset G$ be the image of the similar embedding of G/G^ρ into G, so $N = \{x\rho(x)^{-1} \mid x \in G\}$. From the compactness of G it follows easily that N is an embedded submanifold of G. From Terng,[24] we use the following description of M:

$$M = \{g \in LG \mid g(t)^{-1} = \rho(g(-t)),\ g(0), g(\pi) \in N\}.$$

The inclusion of M in the right hand side is clear. Let now $g \in LG$ with the properties above. Choose $x,\ y \in G$ such that $\rho(x)^{-1} = g(0),\ y\rho(y)^{-1} = g(\pi)$. Let $r : [0, \pi] \to G$ be any H^1 map with $r(0) = x,\ r(\pi) = y$. Define

$$h(t) = \begin{cases} r(t), & \text{if } t \in [0, 2\pi]; \\ g(t)\rho(r(2\pi - t)), & \text{if } t \in [0, 2\pi]. \end{cases}$$

Then $h \in LG$ and $h\rho(h)^{-1} = g$, i.e. $g \in M$. Back to the convergence problem, notice that $g_n \stackrel{n\to\infty}{\longrightarrow} g$ implies that $g_n(0) \stackrel{n\to\infty}{\longrightarrow} g(0)$ and $g_n(\pi) \stackrel{n\to\infty}{\longrightarrow} g(\pi)$. Using the embedding of G/G^ρ into G, we can therefore choose $x_n, y_n \in G$ satisfying $x_n\rho(x_n)^{-1} = g_n(0)$, respectively $y_n\rho(y_n)^{-1} = g_n(\pi)$ such that $x_n \stackrel{n\to\infty}{\longrightarrow} x$ and $y_n \stackrel{n\to\infty}{\longrightarrow} y$. Take finally H^1 paths r_n from x_n to y_n which converge to r in the H^1 topology. They generate as before loops $h_n \in LG$ such that $h_n \stackrel{n\to\infty}{\longrightarrow} h$ and $h_n\rho(h_n)^{-1} = g_n$, hence $h_nLG^\rho \stackrel{n\to\infty}{\longrightarrow} hLG^\rho$ and $j(h_nLG^\rho) = g_n$, q.e.d.

Chapter 2

A Fréchet pseudo-Riemannian symmetric space

In this chapter we treat another class of examples of infinite dimensional symmetric spaces. It is related in a remarkable way to the only known examples of polar actions on Hilbert spaces (more exactly, the isotropy action is, under some restrictions, polar - see [24]). The affine Kac-Moody group $\widehat{LG}$ is a torus bundle over LG. It can be made into a Fréchet Lie group. We extend the involution ρ of LG to $\widehat{LG}$ and consider the quotient space $\widehat{LG}/\widehat{LG}^{\rho}$. It has a Fréchet differentiable structure and it admits a pseudo-Riemannian metric of index 1 for which the basic properties of a symmetric space are verified.

2.1 Introduction

We start again with the loop group LG of a compact, connected, simply-connected, semisimple Lie group G and its Lie algebra of loops $L\mathfrak{g}$. We also consider an Ad-invariant scalar product $\langle\ ,\ \rangle$ on $\mathfrak{g}$. It determines a 2-form ω on $L\mathfrak{g}$ by:

$$\omega(u,v) = \frac{1}{2\pi}\int_0^{2\pi} \langle u(t), v'(t)\rangle dt.$$

The affine Kac-Moody algebra of type 1, denoted by $\widehat{L\mathfrak{g}}$, admits the following realization as an extension of $L\mathfrak{g}$:

$$\widehat{L\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{R}c \oplus \mathbb{R}d$$

with the Lie bracket defined by

$$\begin{aligned}&[u,v] = [u,v]_0 + \omega(u,v)c,\\ &[d,u] = u',\ \ [c,u] = [c,d] = 0,\end{aligned}$$

for $u, v \in L\mathfrak{g}$. $[\ ,\ ]_0$ denotes now the Lie bracket of $L\mathfrak{g}$. This object belongs to the more general class of Kac-Moody algebras, introduced by V. Kac and R. Moody in the mid-1960s, which can be obtained roughly as follows: A semisimple complex Lie algebra can be recovered from its Cartan matrix by a construction due to Serre. If one relaxes the conditions the Cartan matrix should satisfy (allowing arbitrary non-positive entries outside the diagonal, and in particular non-positive principal minors), then the Kac-Moody algebras are obtained. In particular the affine Kac -Moody algebras are obtained from singular Cartan matrices with positive proper principal minors. To the Kac-Moody Lie algebras can be associated groups called the Kac-Moody groups. In general they are not manifolds and thus not Lie groups. Their construction is presented in detail in [12].

If we would let as before $L\mathfrak{g}$ be the space of H^1 loops, then $\widehat{L\mathfrak{g}}$ would not be a Lie algebra (because $[d, u] = u'$ would not belong to $L\mathfrak{g}$). Similarly, the affine Kac-Moody group $\widehat{LG}$ which we will consider next would not be a Lie group. Therefore, in this chapter we will restrict to smooth (C^∞) loops. We use this time the same notations $L\mathfrak{g}$ and LG for the algebra, respectively the group of smooth loops.

The Lie algebra $\widehat{L\mathfrak{g}}$ corresponds to a Lie group $\widehat{LG}$ precisely when the scalar product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ which defines ω satisfies an integrality condition ($\langle h_\alpha, h_\alpha\rangle$ is an even integer for each coroot h_α of G). Scalar products satisfying the integrality condition can be easily obtained by scaling with proper constants. $\widehat{LG}$ is the semidirect product of S^1 with the central extension $\widetilde{LG}$ of LG, which is a non-trivial S^1-bundle over LG. For the construction of $\widetilde{LG}$ and $\widehat{LG}$ see [23] and [24]. $\widehat{LG}$ has a Fréchet Lie group structure with Lie algebra $\widehat{L\mathfrak{g}}$ (which is itself a Fréchet space).

There is a useful description of $\widetilde{LG}$ which we now give: Extend ω to a left invariant form on LG. The integrality condition means that $\omega/2\pi i$ belongs to an integral cohomology class of LG. For any loop $\gamma : S^1 \to LG$, let $C(\gamma) = \exp(i\int_S \omega)$ for some surface S in LG bounded by γ (C is independent of the choice of S). Consider the set of all triples (g, p, z) with $g \in LG$, p a path in LG joining e to g and $z \in S^1$. Define on it the relation of equivalence

$$(g_1, p_1, z_1) = (g_2, p_2, z_2) \Leftrightarrow g_1 = g_2 \text{ and } z_1 = z_2 C(p_2 * p_1^{-1}),$$

where "$*$" denotes the concatenation of paths. The set of equivalence classes forms a group, denoted $\widetilde{LG}$, with the multiplication

$$(g_1, p_1, z_1) \cdot (g_2, p_2, z_2) = (g_1 g_2, p_1 * (g_1 \cdot p_2), z_1 z_2).$$

$\widehat{LG}$ is then the semi-direct product $S^1 \ltimes \widetilde{LG}$, with S^1 acting on $\widetilde{LG}$ by $e^{is} \cdot (g, p, z) = (e^{is} \cdot g, e^{is} \cdot p, z)$ (where $e^{is} \cdot g(t) = g(s + t)$ and $e^{is} \cdot p$ is the path $r \mapsto e^{is} \cdot p(r)$).

Given an involutive automorphism ρ on G, we consider the involution of second kind $\rho : LG \to LG$ defined in the first chapter. One can extend this involution to an involution ρ of of $\widehat{LG}$, $\rho((e^i r, (g, p, z))) = (e^{-ir}, (\rho(g), \rho(p), z^{-1}))$. Here $\rho(p)(s) =$

$\rho(p(s))$. Its fixed point set is denoted as usual by $\widehat{LG}^{\rho}$. The differential $\rho_* : \widehat{L\mathfrak{g}} \to \widehat{L\mathfrak{g}}$ is an extension of the previous $\rho_* : L\mathfrak{g} \to L\mathfrak{g}$ with $\rho_*(c) = -c$, $\rho_*(d) = -d$. The intermediary involution ρ on $\widetilde{LG}$, with fixed point set $\widetilde{LG}^{\rho}$, will be also considered occasionally.

One obtains thus a quotient space $\widehat{LG}/\widehat{LG}^{\rho}$. As for LG/LG^{ρ}, we want to show in a rigorous way that this is a symmetric space, and to study its properties. For this we take a closer look at the group $\widehat{LG}$. We begin with facts concerning arbitrary Fréchet manifolds and Lie groups

2.2 Fréchet manifolds and Lie groups

A *Fréchet manifold* is a Hausdorff topological space with an atlas of coordinate charts taking values in Fréchet spaces, such that the coordinate transition functions are smooth maps between Fréchet spaces. A *Fréchet Lie group* is a Fréchet manifold with a group structure such that which the multiplication and the inverse function are smooth. Working in the Fréchet setting presents major disadvantages. Most of the theory about differential manifolds, Riemannian manifolds and Lie groups, successfully extended in the Hilbert and even Banach cases, is absent here. The main tool which is not available in this setting is the inverse function theorem. Related to it is the absence of a theory concerning existence and uniqueness of ordinary differential equations. A direct consequence is the possibility of vector fields without trajectories - see [11] for a nice example. Valuable information can be found in [8] and [11] about Fréchet manifolds and in [16] about Fréchet Lie groups.

A pseudo-Riemannian metric on a Fréchet manifold is, analogue to the Riemannian case on Hilbert manifolds, a smooth assignment of a scalar product (a continuous non-singular symmetric bilinear form) to the tangent space at every point of the manifold. The term "smooth" needs to be made precise carefully: The vector space of continuous linear maps from a Fréchet vector space to some other space is in general not a Fréchet space. Essentially because of this reason, one can not construct bundles of tensors over Fréchet manifolds (except for the tangent bundle). A (smooth) metric can not be regarded therefore as a (smooth) section of some vector bundle. We will say that a metric g on a Fréchet manifold M is smooth provided that g is expressed locally for any given chart $M \supset U \xrightarrow{\varphi} V \subset F$ by a smooth map $g_{\varphi} : U \times F \times F \to \mathbb{R}$, $g_{\varphi}(x,u,v) = g(T\varphi^{-1}(x,u), T\varphi^{-1}(x,v))$. One can associate to the metric g the quadratic form $q : TM \to \mathbb{R}$, $q(X) = g(X,X)$. Because of the formula $2g(X,Y) = q(X+Y) - q(X) - q(Y)$ and because TM is a manifold which has $T\varphi$ as chart, it is easy to see that the smoothness of g is equivalent to the smoothness of $q : TM \to \mathbb{R}$. Notice that a metric on a Fréchet manifold (which is not Hilbert) is necessarily weak in the sense made clear in the first chapter.

Let now G be a Fréchet Lie group. From a given scalar product on T_eG we construct a left invariant metric on G. Its smoothness can be seen as follows: As remarked in [16], TG is itself a Lie group with multiplication $p_* : TG \times TG \to TG$, where p is the multiplication map $p(g,h) = gh$ of G. TG can be expressed as a semi-direct product $TG \cong T_eG \circledS G$, with G acting on T_eG by the adjoint action. This identification comes from a short exact sequence of groups $0 \to T_eG \to TG \xrightarrow{\pi} G \to 0$ which splits. A right inverse for π is the map which assigns to each $g \in G$ the null vector 0_g. The identification $TG \xrightarrow{\alpha} G \times T_eG$ is given thus by $\alpha^{-1}(g,X) = p_*(0_g, X) = l_{g*}X$. Under this identification q is a function on $G \times T_eG$ which actually depends only on the second variable. The smoothness is now obvious.

As usual, the bi-invariant metrics are precisely the left invariant metrics which generate an Ad-invariant scalar product on $Lie(G)$.

We are now interested in the notion of a linear connection on a Fréchet manifold. Again because the space of linear maps between Fréchet is not a Fréchet space, we need to modify slightly the definition given in the previous chapter for the Hilbert setting. Namely, we require the Christoffel map Γ_φ to be smooth as a map

$$\varphi(U) \times \mathbb{M} \times \mathbb{M} \to \mathbb{M},\ (x,u,v) \mapsto \Gamma_\varphi(x) \cdot (u,v).$$

Proposition 2.2.1. *Any Fréchet Lie group G admits a unique left invariant connection such that $\nabla_X Y = \frac{1}{2}[X,Y]$ for any two left invariant vector fields X and Y. It is a torsionfree connection. If G admits a a bi-invariant (pseudo-)Riemannian metric, then ∇ is the corresponding Levi-Civita connection.*

Proof. Let $G \supset U \xrightarrow{\varphi} \varphi(U) \subset \mathbb{G}$ a chart for G, with $\mathbb{G}$ a modeling Fréchet space for G. Define $\Phi : \varphi(U) \times \varphi(U) \times \mathbb{G} \to \mathbb{G}$, $\Phi(\varphi(x), \varphi(y), u) = pr_2 \circ T\varphi \circ l_{yx^{-1}*} \circ T\varphi^{-1}(\varphi(x), u)$. We show first that Φ is smooth. We remark that the relation $l_{yx^{-1}*} \circ T\varphi^{-1}(\varphi(x), u) = p_*(0_{yx^{-1}}, T\varphi^{-1}(\varphi(x), u))$ holds. The smoothness is now evident. More precisely, if we take a chart (Ψ, V) around the identity with $UU^{-1} \subset V$ and denote the inverse map of G by i, then we can write $0_{yx^{-1}} = T\Psi^{-1}(\Psi \circ p(\varphi^{-1}(\varphi(x), i \circ \varphi^{-1}(\varphi(y)))), 0)$ and all the maps involved are smooth.

Define now the Christoffel symbol

$$\Gamma_\varphi(\varphi(x), u, v) = -\frac{1}{2}(D_2\Phi(\varphi(x), \varphi(x), v) \circ u + D_2\Phi(\varphi(x), \varphi(x), u) \circ v).$$

The reason for this is the fact that the map $\varphi(y) \mapsto \Phi(\varphi(x), \varphi(y), u)$ represents locally a left invariant vector field. Fix now an arbitrary point $x \in U$ and denote the latter map by $\tilde{u}^\varphi$.

For $X, Y \in \mathfrak{X}(U)$ left invariant it follows:
$(\nabla_X Y)_\varphi(\varphi(x)) = DY_\varphi(\varphi(x)) \cdot X_\varphi(\varphi(x)) - \frac{1}{2}(DY_\varphi(\varphi(x)) \cdot X_\varphi(\varphi(x)) + DX_\varphi(\varphi(x)) \cdot Y_\varphi(\varphi(x)) = \frac{1}{2}[X,Y]_\varphi(\varphi(x))$. Because the above defined Γ_φ are symmetric, the connection is torsionfree.

We check now that the transformation rule:

$$\begin{aligned}\Gamma_\psi(\psi(x)) &= D(\psi\varphi^{-1})(\varphi(x)) \circ (D^2(\varphi\psi^{-1})(\psi(x)) \\ &\quad +\Gamma_\varphi(\varphi(x)) \circ (D(\varphi\psi^{-1})(\psi(x)) \times D(\varphi\psi^{-1})(\psi(x)))\end{aligned}$$

is satisfied for any (φ, U) and (ψ, V) charts and $x \in U \cap V$. Indeed, $\tilde{u}^\psi \circ (\psi\varphi^{-1}) = D(\psi\varphi^{-1})\tilde{u}^\varphi$, and thus

$$\begin{aligned}\Gamma_\psi(\psi(x))(u,v) &= -\frac{1}{2}(D(D(\psi\varphi^{-1})\tilde{v}^\varphi \circ (\varphi\psi^{-1}))(\psi(x)) \cdot D(\psi\varphi^{-1})(\varphi(x))u \\ &\quad +D(D(\psi\varphi^{-1})\tilde{u}^\varphi \circ (\varphi\psi^{-1}))(\psi(x)) \cdot D(\psi\varphi^{-1})(\varphi(x))v) \\ &= -D^2(\psi\varphi^{-1})(\varphi(x))(u,v) - \frac{1}{2}(D(\psi\varphi^{-1})(\varphi(x))D\tilde{v}^\psi(\psi(x)) \cdot u \\ &\quad +D(\psi\varphi^{-1})(\varphi(x))D\tilde{u}^\psi(\psi(x)) \cdot v)\end{aligned}$$

(because $D(D(\psi\varphi^{-1})\tilde{v}^\varphi \circ (\varphi\psi^{-1}))(\psi(x)) \cdot D(\psi\varphi^{-1})(\varphi(x))u = D^2(\psi\varphi^{-1})(\varphi(x))(u,v) + D(\psi\varphi^{-1})(\varphi(x))D\tilde{v}^\varphi(\varphi(x))u$). The transformation rule follows now from the equality

$$\begin{aligned}&D^2(\psi\varphi^{-1})(\varphi(x))(D(\varphi\psi^{-1})(\psi(x))u, D(\varphi\psi^{-1})(\psi(x))v) \\ =\; &-D(\psi\varphi^{-1})(\varphi(x))D^2(\varphi\psi^{-1})(\psi(x))(u,v).\end{aligned}$$

Finally, for any three left invariant vector fields X, Y and Z it holds $Xg(Y,Z) = 0 = \frac{1}{2}(g([X,Y],Z) + g(Y,[X,Z])) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$. Translating both terms for a given chart φ, we obtain at each $y = \varphi(x)$ the relations $Xg(Y,Z)(x) = D_1 g_\varphi(y, Y_\varphi(y), Z_\varphi(y)) \cdot X_\varphi(y) + g_\varphi(y, DY_\varphi \cdot X_\varphi(y), Z_\varphi(y)) + g_\varphi(y, Y_\varphi(y), DZ_\varphi \cdot X_\varphi(y))$ and $g(\nabla_X Y, Z)(x) = g_\varphi(y, DY_\varphi \cdot X_\varphi(y) + \Gamma_\varphi(y, X_\varphi(y), Y_\varphi(y)), Z_\varphi(y))$. Similar formulas are obtained for $g(Y, \nabla_X Z)(x)$. Comparing these expressions we get

$$\begin{aligned}D_1 g_\varphi(y, Y_\varphi(y), Z_\varphi(y)) \cdot X_\varphi(y) &= g_\varphi(y, \Gamma_\varphi(y, X_\varphi(y), Y_\varphi(y)), Z_\varphi(y)) \\ &\quad +g_\varphi(y, Y_\varphi(y), \Gamma_\varphi(y, X_\varphi(y), Z_\varphi(y))).\end{aligned}$$

This equality depends only on the values of X, Y and Z at x, and therefore it gives back $Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$ for arbitrary vector fields X, Y and Z. □

2.3 Tameness of Fréchet manifolds

Our goal now is to prove that $\widehat{LG}$ is a tame Fréchet manifold. In the category of tame Fréchet manifolds, a weaker version of the inverse function theorem holds, theorem which is otherwise untrue in the Fréchet setting. We first give the definitions and the basic facts concerning the tame Fréchet spaces. All these, plus a proof of the inverse function theorem for tame Fréchet manifolds and some applications of this theorem, can be found in detail in Hamilton's paper [8].

Definition 2.3.1. *A grading on a Fréchet space is a collection* $\{\| \ \|_n \mid n \in J\}$ *of seminorms indexed by the integers* $J = \{0, 1, ...,\}$ *which are increasing in strength, i.e.* $\| f \|_0 \leq \| f \|_1 \leq ...$, *for all* f, *and which define the topology. A graded Fréchet space is one with a choice of a grading.*

Remark: For any Fréchet space, a grading can be obtained from a countable collection of seminorms $\| \ \|_n$ which defines the topology by constructing a new collection $\| \ \|'_n = \| \ \|_0 + \| \ \|_1 + ... + \| \ \|_n$

Definition 2.3.2. *Two gradings* $\{\| \ \|_n\}$ *and* $\{\| \ \|'_n\}$ *are tamely equivalent of degree* r *and base* b *if*

$$\| f \|_n \leq C \| f \|'_{n+r} \quad \textit{and} \quad \| f \|'_n \leq C \| f \|_{n+r}$$

for any f, *all* $n \geq b$ *and a constant* C *which may depend on* n.

- **Tame linear maps:**

Definition 2.3.3. *A linear map* $L : F \to G$, *with* F, G *graded spaces, satisfies a tame estimate of degree* r *and base* b *if*

$$\| Lf \|_n \leq C \| f \|_{n+r}$$

for each $n \geq b$ *(the constant* C *may depend on* n*).* L *is tame if it satisfies a tame estimate for some* r *and* b.

Remark: A tame linear map is continuous.

Definition 2.3.4. L *is a tame isomorphism if* L *is a linear isomorphism and both* L *and* L^{-1} *are tame.*

Remark: Two gradings are tamely equivalent if and only if the identity map is a tame isomorphism from the space with one grading to the space with the other.

Remark: The composition of tame linear maps is tame. Example: Let $\Sigma(B)$ denote the space of sequences $\{f_k\}$ of elements in a Banach space B such that

$$\| \{f_k\} \|_n = \sum_{k=0}^{\infty} e^{nk} \| f_k \|_B < \infty \quad \text{for all} \ \ n$$

where $\| \ \|_B$ is the norm of B. Then $\Sigma(B)$ is a graded space with these norms.

We consider the definitions given in [8] for a Fréchet vector bundle, for the action of a Fréchet Lie groups on a Fréchet manifold, and the principal Fréchet G-bundle with structural group the Fréchet Lie group G.

- **Tame Fréchet spaces:**

Definition 2.3.5. *Let F and G be graded spaces. We say that F is a tame direct summand of G if there are tame linear maps $L : F \to G$ and $M : G \to F$ such that $M \circ L = id_F$*

Definition 2.3.6. *A graded space is tame if it is a tame direct summand of a space $\Sigma(B)$ defined as above.*

Lemma 2.3.7. *A tame direct summand of a tame space is tame.*

Lemma 2.3.8. *A cartesian product of two tame spaces is tame.*

Theorem 2.3.9. *If X is a compact manifold than $C^\infty(X)$ (the space of smooth functions on X) is tame.*

Corollary 2.3.10. *If X is a compact manifold and V is a vector bundle over X, then the space $C^\infty(X, V)$ of sections of V over X is tame.*

- **Tame maps**

Definition 2.3.11. *Let F and G be graded spaces and $P : U \subseteq F \to G$ a nonlinear map. We say that P satisfies a tame estimate of degree r and base b if $\| P(f) \|_n \leq C(1+ \| f \|_{n+r})$, $\forall f \in U$ and $n \geq b$ (C depends on n). P is called a tame map if it is defined on an open set, it is continuous, and satisfies a tame estimate in a neighborhood of each point . (r, b and C depend on the neighborhood).*

Theorem 2.3.12. *A map is a tame linear map if and only if it is linear and tame.*

Theorem 2.3.13. *A composition of tame maps is tame.*

Definition 2.3.14. *A map P is a smooth tame map if P is smooth and all the derivatives $D^k P$ are tame.*

Let X be a compact manifold, V and W vector bundles over X, $U \subset V$ open, $p : U \subseteq V \to W$ a fibre preserving smooth map. Let $C^\infty(X, V)$ be the space of sections. Then the set $\tilde{U}$ of sections contained in U is open in $C^\infty(X, V)$.

Define $P : \tilde{U} \subseteq C^\infty(X, V) \to C^\infty(X, W)$, $Pf(x) = p(f(x))$, p is called nonlinear vector bundle map and P nonlinear vector bundle operator.

Theorem 2.3.15. *A nonlinear vector bundle operator is tame.*

- **Tame Fréchet manifolds**

Definition 2.3.16. *A tame Fréchet manifold is one with coordinate charts in tame spaces whose coordinate transition functions are smooth tame maps.*

Theorem 2.3.17. *Let X be a compact manifold and B a fibre bundle over X. Then the manifold $C^\infty(B)$ of smooth sections of B is a tame Fréchet manifold.*

Corollary 2.3.18. *If X and Y are finite dimensional manifolds with X compact, then the space $\mathcal{M}(X,Y)$ of smooth maps from X to Y is a tame manifold.*

In particular, in our case, LG is a tame manifold. We will prove in the following that $\widehat{LG}$ is a tame manifold as well.

We first make clear the setting for our result:

Definition 2.3.19. *We say that E is a Fréchet fibre bundle over M with fibre F if E, M and F are Fréchet manifolds and there exists a projection map $\Pi : E \to M$ such for any point in M there is an open neighborhood U of it and a diffeomorfism $\Phi : \Pi^{-1}(U) \to U \times F$ with $p_1 \circ \Phi = \Pi|_{\Pi^{-1}(U)}$, where $p_1 : U \times F \to U$ is the projection onto the first factor.*

Remark: For our purpose we do not need the notion of fibre bundle with structural group.

We now prove the following:

Theorem 2.3.20. *Let (E, Π, M) be a Fréchet fibre bundle with fibre a Banach manifold F. If M is a tame Fréchet manifold then E is also a tame Fréchet manifold.*

Proof. We need to find an atlas for E with coordinate transition functions tame maps between Fréchet vector spaces. For this we construct coordinate charts using the local trivialisation maps of the fibre bundle E. We take $U \subset M$ with $\Phi : \Pi^{-1}(U) \to U \times F$ diffeomorphism, $p_1 \circ \Phi = \Pi$, and $\varphi : U \to \varphi(U) \subset \mathbb{M}$ coordinate chart for M ($\mathbb{M}$ is a Fréchet vector space). Moreover, we take $\psi : V \subset F \to \psi(V) \subset \mathbb{F}$ a coordinate chart for the Banach manifold F ($\mathbb{F}$ is a Banach space).

We obtain thus a coordintate chart for E:

$$\tilde{\Phi} : \tilde{U} = \Phi^{-1}(U \times V) \to \tilde{\Phi}(\tilde{U}) = \varphi(U) \times \psi(V) \subset \mathbb{M} \times \mathbb{F},$$

$$\tilde{\Phi} = (\varphi, \psi) \circ \Phi|_{\tilde{U}}$$

Obviously, we can cover E with charts of this type.

For two such charts

$$\tilde{\Phi}_i : \tilde{U}_i \to \tilde{\Phi}_i(\tilde{U}_i) \subset \mathbb{M} \times \mathbb{F}, \quad i = 1, 2, \text{with} \quad \tilde{U}_1 \cap \tilde{U}_2 = \tilde{U} \neq \Phi,$$

we must prove that

$$\tilde{\Phi}_{12} := \tilde{\Phi}_2 \circ \tilde{\Phi}_1^{-1} : \tilde{\Phi}_1(\tilde{U}) \to \tilde{\Phi}_2(\tilde{U})$$

is tame map between tame spaces.

First, it is easy to see that any Banach space B is tame Fréchet space as a tame direct summand of the above defined $\sum(B)$ (we consider B endowed with the seminorms $\| \|_n = \| \|_B$ for any n, where $\| \|_B$ is the Banach norm, the required tame linear maps being just the canonical inclusion of B into $\sum(B)$, respectively the projection onto the first component).

Second, a cartesian product $X = X_1 \times X_2$ of tame Fréchet spaces is tame as a Fréchet space with the seminorms $\| (x_1, x_2) \|_n^X = \| x_1 \|_n^{X_1} + \| x_2 \|_n^{X_2}$.

It follows that the product $\mathbb{M} \times \mathbb{F}$ is a tame Fréchet space.

So all we have to check now is that $\tilde{\Phi}_{12}$ satisfies a tame estimate in a neighborhood of each point, i.e. there are $b, r \in \mathbb{N}$ such that:

$$\| \tilde{\Phi}_{12}(x, y) \|_n \leq C(1 + \| (x, y) \|)_{n+r}$$

for all $n \geq b$, where $(x, y) \in \tilde{\Phi}_1(\tilde{U}) \subset \mathbb{M} \times \mathbb{F}$ lies in a neighborhood of some fixed point (x_0, y_0), and C is a constant which may depend on n.

But $\tilde{\Phi}_{12}$ can be expressed in terms of Φ_{12}, the transition functions given by the trivialization of the fibre bundle,

$$\Phi_{12} = \Phi_2 \circ \Phi_1^{-1} : U \times F \to U \times F, \quad U = U_1 \cap U_2 \subset M.$$

Because $p_1 \circ \Phi_i = \Pi, \quad i = 1, 2, \quad \Phi_{12}(x, y) = (x, f(x, y))$ for some smooth function $f : U \times F \to F$. (In the case of a fibre bundle with structural group G, we have the more exact form $\Phi_{12} = (x, f(x) \cdot y)$ for some $f : U \to G$, where $f(x) \cdot y$ denotes the action of G on F).

Thus,

$$\begin{aligned}\tilde{\Phi}_{12}(x, y) = \tilde{\Phi}_2 \circ \tilde{\Phi}_1^{-1}(x, y) &= (\varphi_2, \psi_2) \circ \Phi_2 \circ \tilde{\Phi}_1^{-1} \circ (\varphi_1, \psi_1)^{-1}(x, y) \\ &= (\varphi_2, \psi_2) \circ \tilde{\Phi}_{12}(\varphi_1^{-1}(x), \psi_1^{-1}(y)) \\ &= (\varphi_2, \psi_2)(\varphi_1^{-1}(x), f(\varphi_1^{-1}(x), \psi_1^{-1}(y)) \\ &= (\varphi_2 \circ \varphi_1^{-1}(x), \psi_2(f(\varphi_1^{-1}(x), \psi_1^{-1}(y))))\end{aligned}$$

so

$$\|\tilde{\Phi}_{12}(x, y)\|_n = \|\varphi_2 \circ \varphi_1^{-1}(x)\|_n + \|\psi_2(f(\varphi_1^{-1}(x), \psi_1^{-1}(y)))\|_n.$$

Since M is tame manifold and $\varphi_2 \circ \varphi_1^{-1}$ coordinate transition function, we can assume $\|\varphi_2 \circ \varphi_1^{-1}(x)\|_n \leq C(1 + \|x\|_{n+r})$. Let us denote $\psi_2(f(\varphi_1^{-1}(x), \psi_1^{-1}(y)))$ by $\tilde{f}(x, y)$. Then $\|\tilde{\Phi}_{12}(x, y)\|_n \leq C(1 + \|x\|_{n+r}) + \|\tilde{f}(x, y\|_n$; it is enough to prove

$$\|\tilde{f}(x, y\|_n \leq C'(1 + \|y\|_{n+r}),$$

because then

$$\begin{aligned}\|\tilde{\Phi}_{12}(x, y)\|_n &\leq C(1 + \|x\|_{n+r}) + C'(1 + \|y\|_{n+r}) \\ &\leq (C + C')(1 + \|x\|_{n+r} + \|y\|_{n+r}) = C''(1 + \|x + y\|_{n+r}).\end{aligned}$$

But since $\tilde{f}$ is defined from the Fréchet space $\mathbb{M} \times \mathbb{F}$ to the Banach space $\mathbb{F}$, we can even obtain a stronger estimate, namely $\|\tilde{f}(x, y\|_n \leq C'$ (following the argument used by Hamilton to show that any continuous map from a graded Fréchet space to a Banach space is tame):

Take $C' > \|\tilde{f}(x_0, y_0)\|$, $\|\cdot\|$ being the Banach norm on $\mathbb{F}$, and $D = \{(x,y) \in \mathbb{M} \times \mathbb{F} \mid \|\tilde{f}(x,y)\| < C'\} \cap \tilde{\Phi}_1(\tilde{U})$. Then D is an open (because $\tilde{f}$ is continuous) neighborhood of (x_0, y_0). For any $n \in \mathbb{N}$ and $(x,y) \in D$ it holds $\|\tilde{f}(x,y)\|_n = \|\tilde{f}(x,y)\| \leq C'$, which finishes the proof. □

As a corollary, the result holds for principal G bundles (with G a Banach Lie group) and vector bundles, in particular for the Kac-Moody group $\widehat{LG}$.

2.4 The exponential of the affine Kac-Moody group

Remark: Let G and H be Lie groups such that G acts on H on the left. One can consider then the semi-direct product $G \ltimes H$. A curve $t \mapsto (\gamma_t, \delta_t)$ is one-parameter subgroup of $G \ltimes H$ if and only if γ_t is a one-parameter subgroup of G and δ_t is solution of the differential equation $l^{-1}_{\delta_t *}\delta'_t = \gamma_{t*}\delta'_0$.

To show the existence of the exponential of $\widehat{LG}$ we need thus to prove that the above differential solution admits solutions on $\widetilde{LG}$. Even more generally, $\widetilde{LG}$ is a regular Lie group (cf. def. from [16]). This is true because of the regularity of LG (which is a direct consequence of the regularity of G) and because extensions of regular groups with regular groups are regular (see [11]).

For $X = u + r_1 c + r_2 d \in \widehat{L\mathfrak{g}}$, the 1-parameter subgroup $(t \mapsto \exp tX)$ is a lift of a curve $(t \mapsto c_t \in LG)$ determined as follows: For any $s \in I$, $(t \mapsto c_t(s) \in G)$ is the unique curve in G starting at e such that its tangent at t belongs to the left invariant vector field determined by $u(s+t)$.

Using now the description of the elements of $\widetilde{LG}$ in terms of triples (g, p, z), with $g \in LG$, p a path in LG from e to g and $z \in S^1$, we get a concrete description of the exponential mapping for $\widetilde{LG}$:

Proposition 2.4.1. *The group exponential* $\exp : \widetilde{L\mathfrak{g}} \to \widetilde{LG}$ *can be obtained from* $\exp : L\mathfrak{g} \to LG$ *by* $\exp(u + rc) = (\exp u, (s \mapsto \exp su)|_0^1, e^{ir})$*, for any* $u \in L\mathfrak{g}$ *and* $r \in \mathbb{R}$*.*

Proof. We notice first that $t \mapsto (\exp tu, (s \mapsto \exp stu)|_0^1, e^{irt})$ is 1-parameter subgroup of $\widetilde{LG}$. We only have to show therefore that its derivative at 0 is $u + rc$. In other words, we need to prove that the map $\Psi : \widetilde{L\mathfrak{g}} \to Lie(\widetilde{LG})$ defined by $\Psi(u + rc) = (t \mapsto (\exp tu, (s \mapsto \exp stu)|_0^1, e^{irt}))'(0)$ is a Lie algebra isomorphism. It is not difficult to see that Ψ is a vector space isomorphism (which maps $L\mathfrak{g}$ onto the subspace $\mathcal{H} = \{(t \mapsto (\exp tu, (s \mapsto \exp stu)|_0^1, 1))'(0) \mid u \in L\mathfrak{g}\}$). Since both $L\mathfrak{g}$ and $\widetilde{LG}$ are central extensions, it follows $\Psi([\tilde{u}, rc]) = [\Psi(\tilde{u}), \Psi(rc)] = 0$.

It only remained to prove $\Psi([u,v]) = [\Psi(u), \Psi(v)]$ for $u, v \in L\mathfrak{g} \subset \widetilde{L\mathfrak{g}}$. $\widetilde{LG}$ can be described alternatively as follows (for the equivalence of the two descriptions see [23]): There is a circle bundle P over LG and a connection form α on P whose curvature is just ω such that $\widetilde{LG}$ is the group of all bundle isomorphisms of P which preserve ω and cover the left translation with some $g \in LG$ ($\widetilde{LG}$ is actually an S^1-bundle isomorph with P). Each element of $Lie(\widetilde{LG})$ induces thus a vector field on P. The map $\Phi : Lie(\widetilde{LG}) \to \mathfrak{X}(P)$ thus obtained is a Lie algebra antihomomorphism. It is also injective. Even stronger, if $\Phi_x(Z) := \Phi(Z)_x$ for $x \in P$, then Φ_x is injective (because $\widetilde{LG}$ acts on P without fixed points). It is easy to see that Φ_x is actually a linear isomorphism.

We will prove that $(\Phi \circ \Psi([u,v]))_m = -[\Phi \circ \Psi(u), \Phi \circ \Psi(v)]_m$ for some $m \in P$. To simplify the computation, we chose $m \in \pi^{-1}(e)$, where $e \in LG$ is the identity element.

We make the notations: $g_t = (\exp tu, (s \mapsto \exp stu)|_0^1, 1)$, $X = \Phi \circ \Psi(u)$, $h_t = (\exp tv, (s \mapsto \exp stv)|_0^1, 1)$, $Y = \Phi \circ \Psi(v)$. Let γ, η be the 1-parameter subgroups of LG determined by $u, v \in L\mathfrak{g}$, let $s \mapsto c_t(n)(s)$ be the curve obtained by translating n parallelly along $I(\gamma(t))(\eta)$ and Hol_σ the holonomy around some loop σ. Then:

$$\begin{aligned}[\Phi \circ \Psi(u), \Phi \circ \Psi(v)]_m &= [X,Y]_m = \lim_{t\to 0} \frac{Y_m - g_{t*}Y_{g_{-t}(m)}}{t} \\ &= \lim_{t\to 0} \frac{Y_m - (s \mapsto g_t \circ h_s(g_{-t}(m)))'(0)}{t}.\end{aligned}$$

We remark that $g_t \circ h_s(g_{-t}(m))$ can be obtained through parallel translation along the loop $\sigma_{st} = \gamma|_0^t \star \gamma(t)\eta|_0^s \star \gamma(t)\eta(s)\gamma|_0^{-t} \star I(\gamma(t))\eta|_s^0$ followed by parallel translation along $I(\gamma(t))\eta|_0^s$: $g_t \circ h_s(g_{-t}(m)) = c_t(\mathrm{Hol}_{\sigma_{st}}(m))(s)$. Thus $(s \mapsto g_t \circ h_s(g_{-t}(m)))'(0) = c_t(m)'(0) + \frac{d}{ds}\big|_0 \mathrm{Hol}_{\sigma_{st}}(m)$. Since $Y_m = c_0(m)'(0)$, we obtain

$$[\Phi \circ \Psi(X), \Phi \circ \Psi(Y)]_m = \lim_{t\to 0} \frac{c_0(m)'(0) - c_t(m)'(0)}{t} - \frac{d}{dt}\bigg|_0 \frac{d}{ds}\bigg|_0 \mathrm{Hol}_{\sigma_{st}}(m).$$

The first limit is just minus the horizontal lift at m of $[u,v]_0 \in T_eLG$. It is not hard to check that the second term is just the curvature, more precisely $\frac{d}{dt}\big|_0 \frac{d}{ds}\big|_0 \mathrm{Hol}_{\sigma_{st}}(m) = \Phi \circ \Psi(\omega(u,v)c)$.

On the other hand, $\Phi \circ \Psi[u,v] = \Phi \circ \Psi([u,v]_0 + \omega(u,v)c)$. As $\Phi \circ \Psi([u,v]_0)_m$ is also the horizontal lift of $[u,v]_0$, we obtained the desired isomorphism. □

Remark: For any $u \in L\mathfrak{g}$, $\Phi \circ \Psi(u)$ is not horizontal vector field. It is horizontal nevertheless at the points in the fiber $\pi^{-1}(e)$.

2.5 A Fréchet symmetric space

We try first to give a description for $\widehat{LG}^\rho$. Using the description of the elements of $\widetilde{LG}$ in terms of triples as before, we get:

Proposition 2.5.1. $\widehat{LG}^\rho \cong \{(\pm 1, (g, p, \pm 1)) | \rho(g) = g, \rho(p(r)) = p(r)\}$. *It is a covering of* LG^ρ *with four leaves.*

Proof. It is clear that any element from the right hand side set belongs to $\widehat{LG}^{\rho}$. Conversely, let $\gamma = (e^{is}, (g, p, z)) \in \widehat{LG}^{\rho}$. Then $e^{is} = \pm 1$ and $(g, p, z) \sim (\rho(g), \rho(p), z^{-1})$, so $\rho(g) = g$ and $z^2 = C(\rho(p) * p^{-1}) = \exp(i \int_S \omega)$ for any surface $S \subset LG$ bounded by $\rho(p) * p^{-1}$. Take any path q in LG^{ρ} connecting g and e - remember that LG^{ρ} is connected. Take a surface S_1 bounded by $p * q^{-1}$ and let $S_2 = \rho(S_1), S = S_1 \cup S_2$. Since ρ reverses the loops, it is easy to check that $\omega(\rho(u), \rho(v)) = -\omega(u, v)$ for all $u, v \in \mathfrak{g}$, and therefore $\rho^* \omega = -\omega$. Choose now an orientation on S. ρ restricts to an orientation-changing homeomorphism of S (take two linearly independent vectors tangent to S at some point on the path q, one of them along q, and apply ρ_* to them). We obtain $\int_{S_1} \omega = \int_{-S_2} \rho * \omega = \int_{S_2} \omega$ and thus $\int_S \omega = 2 \int_{S_1} \omega$, so $\pm z = C(p * q^{-1})$, which means $(g, p, z) \sim (g, q, 1)$ or $(g, p, z) \sim (g, q, -1)$. In particular, if $(g, p, 1)$ and $(g, q, 1)$ are elements with $\rho(g) = g$, $\rho(p(r)) = p(r)$ and $\rho(q(r)) = q(r)$ for all $r \in [0, 1]$, then $(g, p, 1) \sim (g, q, 1)$ or $(g, p, 1) \sim (g, q, -1)$, which finishes the proof. $\square$

The identity $\widehat{LG}^{\rho} = \mathbb{Z}_2 \times \widetilde{LG}^{\rho}$ is obvious. When G^{ρ} is simply connected then $\widehat{LG}^{\rho}$ is a trivial covering.

Our interest is directed now toward the coset space $\widehat{LG}/\widehat{LG}^{\rho}$. From the previous proposition it follows easily that this is a torus bundle over LG/LG^{ρ}, with projection map $\tau : \widehat{LG}/\widehat{LG}^{\rho} \stackrel{\tau}{\longrightarrow} LG/LG^{\rho}$, $\tau((e^{is}, (g, p, z)) \cdot \widehat{LG}^{\rho}) = e^{-is}(g) \cdot LG^{\rho}$ (observe that the map $(e^{is}, (g, p, z)) \cdot \widehat{LG}^{\rho} \mapsto (e^{2is}, e^{-is}((g, p, z)) \cdot \widetilde{LG}^{\rho})$ is an isomorphism between $\widehat{LG}/\widehat{LG}^{\rho}$ and $S^1 \times \widetilde{LG}/\widetilde{LG}^{\rho}$).

We consider the problem of finding a manifold structure for $\widehat{LG}/\widehat{LG}^{\rho}$. First, the loop group of smooth loops admits a manifold and Lie group structure in the same way as the loop group of H^1 loops. We show next how LG/LG^{ρ} becomes a Fréchet manifold, where by LG we understand this time the loop group consisting only of smooth loops. We use in this section the notations LG_1, LG_1^{ρ}, respectively LG_1/LG_1^{ρ} for the loop group of H^1 loops and the corresponding Lie subgroup, respectively homogeneous manifold. We also write $L\mathfrak{g}_1, \mathfrak{k}_1$ and $\mathfrak{p}_1$. There is thus a canonical inclusion $LG \subset LG_1$, which induces a well defined map $i : LG/LG^{\rho} \to LG_1/LG_1^{\rho}, gLG^{\rho} \mapsto gLG_1^{\rho}$. This map is injective, as can be checked easily. We consider also the natural inclusion $L\mathfrak{g} \subset L\mathfrak{g}_1$. Remember that points in LG (resp. LG_1) are loops in G which are mapped piecewise to smooth (resp. H^1) loops in $\mathbb{R}^n$ by charts of G. With respect to the above inclusions, $\exp : L\mathfrak{g}_1 \to LG_1$ restricts to $\exp : L\mathfrak{g} \to LG$.

As we saw in the first chapter (section I.3), an atlas for LG_1/LG_1^{ρ} is provided by the maps $\varphi_g = (l_g \circ \pi \circ \exp|_U)^{-1}$ for some open neighborhood U of e in $\mathfrak{p}_1$.

The elements $xLG_1^{\rho} \in LG_1/LG_1^{\rho}$ not contained in $i(LG/LG^{\rho})$ are precisely those for which the set xLG_1^{ρ} contains no smooth loop. The element $x = \exp X$ with $X \in U \subset \mathfrak{p}_1$ is not smooth precisely when the corresponding element in the Cartan embedding $x\rho(x)^{-1} = x^2 = \exp 2X$ is not smooth. From this it follows that $(\exp X)LG_1^{\rho} \in i(LG/LG^{\rho})$ if and only if X is a smooth loop in $\mathfrak{p}_1$. In conclusion, the chart φ_g of

LG_1/LG_1^ρ restricts for $g \in LG$ to a chart (denoted by the same φ_g) of LG/LG^ρ - U is open in the Fréchet topology as well.

For different $g, h \in LG$ we get:

$$\begin{aligned}
\varphi_h \varphi_g^{-1} &= \exp|_U^{-1} \circ \pi|_{\exp U}^{-1} \circ l_h^{-1} \circ l_g \circ \pi \circ \exp|_U \\
&= \exp|_U^{-1} \circ \pi|_{\exp U}^{-1} \circ l_{h^{-1}g} \circ \pi \circ \exp|_U \\
&= \exp|_U^{-1} \circ \pi|_{\exp U}^{-1} \circ \pi \circ l_{h^{-1}g} \circ \exp|_U = \exp|_U^{-1} \circ l_{h^{-1}g} \circ \exp|_U
\end{aligned}$$

whence the smoothness of the transition functions follows (because LG is a Fréchet Lie group).

Finding now manifold structures on $\widehat{LG}$ and $\widehat{LG}/\widehat{LG}^\rho$ is an easy task, since both $\widehat{LG}$ and $\widehat{LG}/\widehat{LG}^\rho$ are locally trivial fiber (torus) bundles - if we take a ball $B(0, r)$ centered at 0 in $\mathfrak{g}$ on which exp is a diffeomorphism, then $\widehat{LG}$ restricted to $L \exp B(0, r) = \exp LB(0, r)$ and $\widehat{LG}/\widehat{LG}^\rho$ restricted to $\pi \circ \exp(LB(0, r) \cap \mathfrak{p})$ are trivial. We obtained:

Proposition 2.5.2. *LG, LG/LG^ρ, $\widehat{LG}$ and $\widehat{LG}/\widehat{LG}^\rho$ admit Fréchet manifold structures (such that all the left translations are smooth and all the projections are submersions). LG and $\widehat{LG}$ are Fréchet Lie groups.*

Remark: If we consider the spaces of H^k Sobolev loops for all k, $L\mathfrak{g}$ is actually a projective limit of Hilbert spaces, and LG, LG/LG^ρ, $\widehat{LG}$ and $\widehat{LG}/\widehat{LG}^\rho$ become with the above structures inverse limits of Hilbert manifolds. Closely related to this is the concept of strong ILH Lie group structures, introduced by H. Omori in [17]. We believe that this concept may be useful for a further study of the objects considered here.

We are concerned next with determining a "proper" geometry (i.e. a pseudo-Riemannian metric and a corresponding Levi-Civita) for $\widehat{LG}/\widehat{LG}^\rho$. We follow the same procedure as in the first chapter, looking first for suitable structure on $\widehat{LG}$.

It is not hard to compute the general formula for the adjoint action of $\widehat{LG}$:

$$\begin{aligned}
Ad(x)u &= ge^{ir}(u)g^{-1} + \langle ge^{ir}(u)g^{-1}, g'g^{-1}\rangle c, \\
Ad(x)c &= c, \\
Ad(x)d &= g'g^{-1} + d + \frac{1}{2}\langle g'g^{-1}, g'g^{-1}\rangle c
\end{aligned}$$

for any $x = (e^{ir}, (g, p, z))$ and $u \in L\mathfrak{g}$.

We extend now the L^2 scalar product on $L\mathfrak{g}$ to a semi-definite (of index one) scalar product on $\widehat{L\mathfrak{g}}$:

$$\begin{aligned}
\langle u, v\rangle &= \int_0^{2\pi} \langle u(t), v(t)\rangle dt, \\
\langle c, d\rangle &= -1, \quad \langle u, c\rangle = \langle u, d\rangle = \langle c, c\rangle = \langle d, d\rangle = 0,
\end{aligned}$$

where $u, v \in L\mathfrak{g}$. The scalar product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ is that used in the introduction for defining ω, so it is Ad-invariant and satisfies the integrality condition (we can take for example the smallest one, called the normalized bi-invariant inner product, determined on each simple ideal by the condition $\langle h_\alpha, h_\alpha \rangle = 2$ for the coroot of a longest root α). A direct verification shows that $\langle\ ,\ \rangle$ on $\widehat{L\mathfrak{g}}$ is Ad-invariant. We translate it to a bi-invariant pseudo-Riemannian metric g. Using the results obtained in Section 2.2 about Fréchet Lie groups, we get:

Proposition 2.5.3. *$(\widehat{LG}, g)$ admits a unique Levi-Civita connection ∇. For X, Y left invariant vector fields on $\widehat{LG}$ it satisfies $\nabla_X Y = \frac{1}{2}[X, Y]$.*

One can check that the adjoint action restricted to $\widehat{LG}^\rho$ leaves $\hat{\mathfrak{p}}$ invariant. Since $\exp : \widehat{L\mathfrak{g}} \to \widehat{LG}$ is well defined, the invariance of $\hat{\mathfrak{p}}$ follows also from the usual general argument: $\rho(\exp \mathrm{Ad}(x)tX)) = x \exp(-tX)x^{-1}$, whence $\rho_* \mathrm{Ad}(x)X = -\mathrm{Ad}(x)X$. The relation $\mathfrak{k} \perp \mathfrak{p}$ obtained in Chapter I extends immediately for $\widehat{L\mathfrak{g}}$ to $\hat{\mathfrak{k}} \perp \hat{\mathfrak{p}}$. By identifying $T_e \widehat{LG}/\widehat{LG}^\rho$ with $\hat{\mathfrak{p}}$ via π_*, we obtain a $\widehat{LG}^\rho$-invariant scalar product on $T_e \widehat{LG}/\widehat{LG}^\rho$. A semi-Riemannian metric on $\widehat{LG}/\widehat{LG}^\rho$ is thus generated, such that the left translations with elements of $\widehat{LG}$ are isometries.

A symmetry at any point can be constructed as usual from ρ and the left translations. We have therefore:

Proposition 2.5.4. *$\widehat{LG}/\widehat{LG}^\rho$ admits a pseudo-Riemannian metric of index 1 which makes it a symmetric space and such that the submersion $\pi : \widehat{LG} \to \widehat{LG}/\widehat{LG}^\rho$ is pseudo-Riemannian. A corresponding Levi-Civita connection ∇ exists on $\widehat{LG}/\widehat{LG}^\rho$.*

Proof. The Levi-Civita connection ∇ on $\widehat{LG}/\widehat{LG}^\rho$ can be obtained in the same way as for LG/LG^ρ, by applying the Levi-Civita connection of $\widehat{LG}$ to the horizontal lifts of any two vector fields on $\widehat{LG}/\widehat{LG}^\rho$, and then projecting back the resulting vector field. We skip the details. □

The fact that the metric is not definite is not an impediment for obtaining similar properties with the positive definite case. However, a metric has to be given (an invariant semi-definite scalar product can not be obtained anymore by averaging an arbitrary one). A brief introduction to pseudo-Riemannian symmetric spaces can be found in [19].

One can now prove:

Proposition 2.5.5. *$\widehat{LG}/\widehat{LG}^\rho$ has a globally defined geodesical exponential Exp at each point. At $e\widehat{LG}^\rho$ it is defined by $Exp = \pi \circ \exp|_{\hat{\mathfrak{p}}}$.*

Proof. The proof is almost identical to that given in the first chapter for LG/LG^ρ. Only a little work is needed to adapt the proof of the tensoriality of the O'Neill operation

A, given for X and Y horizontal vector fields on $\widehat{LG}$ by $A_X Y = \mathcal{V}\nabla_X Y$. The problem is that a Fréchet space may not have a Schauder basis. But in our case a Schauder basis for the L^2 topology on $L\mathfrak{g}$, completed with the elements c and d, is a Schauder basis for Fréchet C^∞ topology. In other words, if we decompose some smooth loop $u \in L\mathfrak{g}$ into its Fourier components $u = \sum_{n\in\mathbb{Z}} u_n e_n$, then the sequence of partial sums $u^k = \sum_{|n|\leq k} u_n e_n$ converges to u not only in the L^2 topology, but also in the C^∞ topology. Here e_n denotes the function $t \mapsto \sin nt$ for $n < 0$ and the function $t \mapsto \cos nt$ in rest, and u_n are vectors in $\mathfrak{g}$. Remember that $u^k \overset{k\to\infty}{\longrightarrow} u$ in the C^∞ topology means that $\sup_{t\in I} |u^{(i)}(t) - u^{k^{(i)}}(t)| \overset{k\to\infty}{\longrightarrow} 0$ for all the i-derivatives. But $u^{k^{(i)}}$ form the sequence of partial sums corresponding to the Fourier series for $u^{(i)}$. All we need now is the fact that the Fourier series converges pointwise (which is true even for continuous loops admitting two-sided derivatives at each point - cf. a theorem of Dirichlet - see for example [22], Theorem 2.6.). $\square$

As a consequence, the 1-parameter subgroup $t \mapsto \exp tX$, with $X \in \hat{\mathfrak{p}}$, may be seen as the group of transvections along the geodesic $t \mapsto \operatorname{Exp} tX$.

We end this section by remarking that the proof of Theorem 1.4.3. holds entirely for $\widehat{LG}/\widehat{LG}^\rho$ with its Fréchet structure, allowing to express the curvature in terms of the Lie bracket of $\widehat{L\mathfrak{g}}$. From the formulas giving the Lie bracket of $\widehat{L\mathfrak{g}}$ (see the introduction) and the semidefinite scalar product (page 47), one sees that time-like vectors can not be obtained by multiplying two arbitrary vectors of $\widehat{L\mathfrak{g}}$. A short computation, analogue to Corollary 1.4.6., shows that $\widehat{LG}/\widehat{LG}^\rho$ has non-negative sectional curvature.

2.6 Maximal flats

We are motivated now to study the totally geodesic and the flat submanifolds of $\widehat{LG}/\widehat{LG}^\rho$. We encounter the same difficulties in defining them as in the case of LG/LG^ρ. Moreover, it is unclear if the Theorem 1.6.4 of chapter I is valid in the new context. This is because Fréchet spaces lack basic properties like the existence and uniqueness of solutions of ordinary differential equations. Nevertheless, there is a class of submanifolds which presents striking similarities with the finite dimensional case.

To understand it we must study first the maximal abelian subgroups of $\widehat{LG}$, and we actually begin with the maximal abelian subalgebras of $\widehat{L\mathfrak{g}}$. These fall into two classes: those which are contained into $\widetilde{L\mathfrak{g}}$ and those which are not. Since c belongs to the center of $L\mathfrak{g}$, the first ones have the form $\hat{\mathfrak{a}} = \mathfrak{a} + \mathbb{R}c$, with $\mathfrak{a}$ contained in $L\mathfrak{g}$. In this case $\mathfrak{a}$ sits in some maximal abelian subalgebra $\mathfrak{a}'$ of $L\mathfrak{g}$. Notice that because of the cocycle ω, $\mathfrak{a}'$ is not abelian in $\widehat{L\mathfrak{g}}$. If $\widehat{\mathfrak{a}}$ is finite dimensional, then it is not maximal: if $u_1, ..u_n$ is a basis for $\mathfrak{a}$, then $\phi : \mathfrak{a}' \to \mathbb{R}^n$, $\phi(v) = (\omega(v, u_1), ..., \omega(v, u_1))$ is linear and $\operatorname{codim} \ker \phi = n$, so $\ker\, \phi \subset \mathfrak{a}_\varphi$ is infinite dimensional, and we can take thus $u \in \ker\ \phi \setminus \operatorname{span}\{u_1, ..u_n\}$, which gives an abelian extension $\mathfrak{a} + \mathbb{R}u + \mathbb{R}c$ of $\mathfrak{a} + \mathbb{R}c$.

We obtain thus a large class of infinite dimensional maximal abelian subalgebras of $\widehat{L\mathfrak{g}}$, containing several conjugacy classes.

More interesting is the case of maximal abelian subalgebras of $\widehat{L\mathfrak{g}}$ which contain an element of the form $d+\tilde{u}$, where $\tilde{u}\in L\mathfrak{g}$. We call such subalgebras maximal abelian subalgebras of finite type. Again, such a subalgebra has the form $\hat{\mathfrak{a}}=\mathfrak{a}+\mathbb{R}c+\mathbb{R}(d+\tilde{u})$, with $\mathfrak{a}=\hat{\mathfrak{a}}\cap L\mathfrak{g}$ abelian subalgebra of $L\mathfrak{g}$. In addition,

$$v\in\mathfrak{a}\Rightarrow[v,d+\tilde{u}]=0\Leftrightarrow v'=[v,\tilde{u}]\Leftrightarrow\begin{cases}v'=[v,\tilde{u}]_0\\ \omega(v,\tilde{u})=0.\end{cases}$$

The first equation is a linear differential equation (called the Lax equation) with unique solution $v(t)=\mathrm{Ad}(\varphi_t)v_0$, where $\mathbb{R}\ni t\mapsto\varphi_t\in G$ is the solution of the differential equation $-\varphi_t'\varphi_t^{-1}=\tilde{u}(t)$ with initial condition $\varphi_0=e$. In order to obtain a loop we must impose the condition $\mathrm{Ad}(\varphi_{2\pi})(v_0)=v_0$ to the initial value $v(0)=v_0$, i.e. $v_0\in\mathfrak{g}^{\mathrm{Ad}(\varphi_{2\pi})}$. This condition is enough to obtain a smooth loop:

Let $\psi_s:=\varphi_s\cdot\varphi_{2\pi}$; then $\psi_s'\psi_s^{-1}=\varphi_s'\varphi_s^{-1}=-\tilde{u}(s)=-\tilde{u}(s+2\pi)=\varphi_{s+2\pi}'\varphi_{s+2\pi}^{-1}$, and so ψ_s and $\varphi_{s+2\pi}$ satisfy the same differential equation with the same initial value. Therefore $\varphi_s\cdot\varphi_{2\pi}=\psi_s=\varphi_{s+2\pi}$, hence $v(s+2\pi)=\mathrm{Ad}(\varphi_{s+2\pi})v_0=\mathrm{Ad}(\varphi_s)\mathrm{Ad}(\varphi_{2\pi})v_0=\mathrm{Ad}(\varphi_{s+2\pi})v_0=v(s)$.

The condition $\omega(v,\tilde{u})=0$ is also true for v thus obtained, because

$$\omega(\tilde{u},v)=\frac{1}{2\pi}\int_0^{2\pi}\langle\tilde{u},v'\rangle dt=\frac{1}{2\pi}\int_0^{2\pi}\langle\tilde{u},[v,\tilde{u}]\rangle dt=$$

$$-\frac{1}{2\pi}\int_0^{2\pi}\langle\tilde{u},[\tilde{u},v]\rangle dt=-\frac{1}{2\pi}\int_0^{2\pi}\langle[\tilde{u},\tilde{u}],v\rangle dt=0$$

for the Ad-invariant scalar product $\langle\ ,\ \rangle$ Ad-invariant scalar product on $\mathfrak{g}$.

For $\hat{\mathfrak{a}}$ to be abelian we further need $[v,w]=0$ for all $v,w\in\mathfrak{a}$, i.e.

$$\begin{cases}[v,w]_0=0\\ \omega(v,w)=0\end{cases}$$

But

$$[v,w]_0(t)=[v(t),w(t)]=[\mathrm{Ad}(\varphi_t)v_0,\mathrm{Ad}(\varphi_t)w_0]=\mathrm{Ad}(\varphi_t)[v_0,w_0]$$

and

$$\omega(v,w)=\frac{1}{2\pi}\int_0^{2\pi}\langle v,w'\rangle dt=\frac{1}{2\pi}\int_0^{2\pi}\langle v,[w,\tilde{u}]_0\rangle dt=\frac{1}{2\pi}\int_0^{2\pi}\langle[v,w]_0,\tilde{u}\rangle dt,$$

therefore it is enough to have $[v_0,w_0]=0$. Thus $\mathfrak{a}_0=\{v_0=v(0)\mid v\in\mathfrak{a}\}$ has to be a (maximal) abelian subalgebras of $\mathfrak{g}$ contained in $\mathfrak{g}^{\mathrm{Ad}(\varphi_{2\pi})}$.

Remark 1: Take $T \subset G$ maximal torus with $\varphi_{2\pi} \in T$. Then $\mathfrak{t} := Lie(T)$ is maximal abelian subalgebra of $\mathfrak{g}$ and $\mathfrak{t} \subset \mathfrak{g}^{\mathrm{Ad}(\varphi_{2\pi})}$. Thus, $\dim \mathfrak{a} = \dim \mathfrak{a}_0 = k = \mathrm{rank} G$. Moreover, if $\varphi_{2\pi}$ is regular, then $\mathfrak{t} = \mathfrak{g}^{\mathrm{Ad}(\varphi_{2\pi})}$ and there is only one choice for $\mathfrak{a}_0$. If $\varphi_{2\pi}$ is singular, then $\mathfrak{t} \subseteq \mathfrak{g}^{\mathrm{Ad}(\varphi_{2\pi})}$ and $\mathfrak{g}^{\mathrm{Ad}(\varphi_{2\pi})} = Lie(H) =$, where $H \subset G$ is the subgroup covered by all the maximal tori of G containing $\varphi_{2\pi}$.

Remark 2: If $\tilde{u} \in \mathfrak{a}$, then $d \in \hat{\mathfrak{a}}$, hence $[d, \mathfrak{a}] = 0$, which means that $\mathfrak{a} = \mathfrak{a}_0$ with the elements of $\mathfrak{a}_o$ regarded as constant loops. We get thus the canonical form $\hat{\mathfrak{a}} = \mathfrak{a}_0 + \mathbb{R}c + \mathbb{R}d$, where $\mathfrak{a}_0$ is a maximal abelian subalgebra of $\mathfrak{g}$. If $\tilde{u}$ is not constant, then $\tilde{u} \notin \mathfrak{a}$ (actually, there is no $u_0 \in \mathfrak{g}$ such that $\tilde{u}(t) = \mathrm{Ad}(\varphi_t)(u_0)$).

Remark 3: If $\tilde{u}(t) = \tilde{u} =$ constant, than $\varphi_t = \exp(-t\tilde{u})$ and $\tilde{u} \in \mathfrak{g}^{\mathrm{Ad}(\exp(-2\pi\tilde{u}))}$. In this case, if $\mathfrak{a}_0$ contains $\tilde{u}$, then $\hat{\mathfrak{a}}$ has the canonical form $\hat{\mathfrak{a}} = \mathfrak{a}_0 + \mathbb{R}c + \mathbb{R}d$.

In conclusion, for any $\tilde{u} \in L\mathfrak{g}$, we obtain one (if $\varphi_{2\pi}$ as above determined is regular) or more (for $\varphi_{2\pi}$ singular) finite dimensional maximal abelian subalgebras of $\widehat{L\mathfrak{g}}$, all of the same dimension. They have the form $\mathfrak{a} + \mathbb{R}c + \mathbb{R}(d + \tilde{u})$, with $\mathfrak{a} = \{v \mid v(t) = \mathrm{Ad}(\varphi_t)(v_0), v_0 \in \mathfrak{a}_0\}$, with $\mathfrak{a}_0$ maximal abelian subalgebra of $\mathfrak{g}$ contained in $\mathfrak{g}^{\mathrm{Ad}(\varphi_{2\pi})}$.

If $\tilde{u}$ is constant, at least one of the algebras corresponding to $\tilde{u}$ has canonical form.

Remark 4: The finite dimensional maximal abelian subalgebras of $\widehat{L\mathfrak{g}}$ do not cover $\widehat{L\mathfrak{g}}$, because the elements $v \in \mathfrak{a} = \widehat{\mathfrak{a}} \cap L\mathfrak{g}$ have the form $v(t) = \mathrm{Ad}(\varphi_t)v_0$, so $\mathrm{Im}\, v \subset S(0, \|v_0\|)$, the sphere of radius $\|v_0\|$ in $\mathfrak{g}$. The elements $u \in L\mathfrak{g}$ with $\|u(t)\|$ not constant (which are dense in $L\mathfrak{g}$) do not lie in any finite dimensional maximal abelian subalgebra.

We prove now the following result, due to E. Heintze:

Theorem 2.6.1. *Any two maximal abelian subalgebras of finite type are conjugate.*

Proof. Step I: Let $\hat{\mathfrak{a}} = \mathfrak{a} + \mathbb{R}c + \mathbb{R}(d + \tilde{u})$ be a maximal abelian subalgebra of finite type. We prove first that $\hat{\mathfrak{a}}$ is conjugate to some $\hat{\mathfrak{a}}' = \mathfrak{a}' + \mathbb{R}c + \mathbb{R}(d + \tilde{u}')$ with $\tilde{u}'$ constant. The gauge action $g \cdot u = gug^{-1} - g'g^{-1}$ of LG on $L\mathfrak{g}$ is polar, with a section Σ consisting of all constant loops in $\mathfrak{t}$, for $\mathfrak{t} \subset \mathfrak{g}$ maximal abelian. It follows that there is a constant loop x in $L\mathfrak{g}$ and a $g \in LG$ such that $g\tilde{u}g^{-1} - g'g^{-1} = x$. We know

$$
\begin{aligned}
\mathrm{Ad}(\hat{g})(c) &= c, \ \mathrm{Ad}(\hat{g})(u) = gug^{-1} + \langle gug^{-1}, g'g^{-1} \rangle c \\
\mathrm{Ad}(\hat{g})(d) &= -g'g^{-1} + d + \frac{1}{2}\|g'g^{-1}\|^2 c \quad \text{for } \hat{g} = (1, (g, p, z)) \in \widehat{LG}.
\end{aligned}
$$

Hence $\mathrm{Ad}(\hat{g})(d + \tilde{u}) = g\tilde{u}g^{-1} - g'g^{-1} + d + \lambda c = d + x + \lambda c$ for some $\lambda \in \mathbb{R}$. We get thus $\mathrm{Ad}(\hat{g})(\mathfrak{a} + \mathbb{R}c + \mathbb{R}(d + \tilde{u})) = g\, \mathfrak{a}\, g^{-1} + \mathbb{R}c + \mathbb{R}(d + x)$, with $x \in L\mathfrak{g}$ constant.

Step II: We prove now that any two maximal abelian subalgebras of finite type which contain the same element $d + x$ with $x \in L\mathfrak{g}$ constant are conjugate. The obvious fact that any two canonical subalgebras are conjugate (because maximal abelian subalgebras of $\mathfrak{g}$ are conjugate) then finishes the proof, because, as a consequence of Remark 3, at least one of the subalgebras containing $d + x$ is canonical.

Let thus $\widehat{\mathfrak{a}}_i = \mathfrak{a}_i + \mathbb{R}c + \mathbb{R}(d+x)$ be maximal abelian subalgebras of finite type for $i = 1, 2$. As already mentioned, $\varphi_t : I \to G$ determined by $\varphi_t' \varphi_t^{-1} = x$ takes the form $\varphi_t = \exp(-tx)$. Any $v \in \mathfrak{a}_i$ is given by $v(t) = \mathrm{Ad}(\exp(-tx))v_0 = e^{-t\mathrm{ad}x} v_0$, with $v_0 \in \mathfrak{a}_{i0} \subset \mathfrak{g}^{\mathrm{Ad}(\varphi_{2\pi})} = \mathfrak{g}^{\mathrm{Ad}(\exp(-2\pi x))} = \mathfrak{g}^{e^{-2\pi\mathrm{ad}x}} = \mathfrak{g}^{e^{2\pi\mathrm{ad}x}}$, for $i = 1, 2$.

Choose now $\varphi \in \mathrm{Int}\mathfrak{g}$ satisfying $\varphi(\mathfrak{g}^{2\pi\mathrm{ad}x}) = \mathfrak{g}^{2\pi\mathrm{ad}x}$ and $\varphi(\mathfrak{a}_{10}) = \mathfrak{a}_{20}$. Such an automorphism can be obtained explicitly as $\varphi = \mathrm{Ad}(\exp y) = \exp(\mathrm{ad}\, y)$ for some $y \in \mathfrak{g}^{e^{2\pi\mathrm{ad}x}}$.

It follows that $\mathrm{Ad}(\exp 2\pi x)y = y$, hence $I(\exp 2\pi x)(\exp sy) = (\exp sy)$, $\forall s \in \mathbb{R}$ (because both terms are 1-parameter subgroups with the same initial velocity). Thus $(\exp 2\pi x)(\exp y)(\exp -2\pi x) = \exp y$, which shows that $t \mapsto (\exp -tx)(\exp y)(\exp tx)$ is a smooth loop. Denote it by $g \in LG$.

For some $\hat{g} \in \widehat{LG}$ of the type $\hat{g} = (1, (g, p, z))$, we consider $\mathrm{Ad}(\hat{g}) : \widehat{L\mathfrak{g}} \to \widehat{L\mathfrak{g}}$ and check that it maps $\hat{\mathfrak{a}}_1$ into $\hat{\mathfrak{a}}_2$:

i)

$$v \in \mathfrak{a}_1 \Rightarrow v(t) = \mathrm{Ad}(\exp -tx)v_0,$$

$$\mathrm{Ad}(\hat{g})v = gvg^{-1} + \lambda(v)c = \mathrm{Ad}(g)v + \lambda(v)c,$$

$$\begin{aligned} gvg^{-1}(t) &= \mathrm{Ad}(\exp -tx)\mathrm{Ad}(\exp y)\mathrm{Ad}(\exp tx)\mathrm{Ad}(\exp -tx)v_0 \\ &= \mathrm{Ad}(\exp -tx)\mathrm{Ad}(\exp y)v_0 = \mathrm{Ad}(\exp -tx)\varphi(v_0), \end{aligned}$$

$$\varphi : \mathfrak{a}_{10} \to \mathfrak{a}_{20} \Rightarrow gvg^{-1} \in \mathfrak{a}_2;$$

ii) $\mathrm{Ad}(\hat{\mathfrak{g}})(c) = c$;

iii) $g(t) = \exp(-tx)\exp(y)\exp(tx) \;\Rightarrow g'(t) = -xg(t) + g(t)x$
$\Rightarrow g'g^{-1} = -x + gxg^{-1} \Rightarrow$

$$\mathrm{Ad}(\hat{\mathfrak{g}})(d+x) = -g'g^{-1} + d + \lambda_1 c + gxg^{-1} + \lambda_2 c$$

$$= x + d + (\lambda_1 + \lambda_2)c.$$

From i), ii) and iii) it follows $\mathrm{Ad}(\hat{\mathfrak{g}})(\hat{\mathfrak{a}}_1) = \hat{\mathfrak{a}}_2$, which finishes the proof. □

As a corollary, we obtain a similar result concerning the finite dimensional maximal tori of $\widehat{LG}$:

Proposition 2.6.2. *Each finite dimensional maximal abelian subalgebra of $\widehat{L\mathfrak{g}}$ is the Lie algebra of a maximal torus (a compact connected abelian subgroup) of $\widehat{LG}$. All these tori form a conjugacy class of subgroups of $\widehat{LG}$.*

Proof. Remark first that G is canonically embedded as a Lie subgroup of LG. The (non-trivial) torus bundle $\widehat{LG}$ over LG restricts to a trivial torus bundle over G (the 2-form ω restricted to G is identically null), i.e. $G \times S^1 \times S^1$ is embedded in $\widehat{LG}$. A subalgebra of canonical form $\mathfrak{a} = \mathfrak{a}_0 + \mathbb{R}c + \mathbb{R}d$ is obviously the Lie algebra of the torus $T = T_0 \times S^1 \times S^1 \subset G \times S^1 \times S^1 \subset \widehat{LG}$, where T_0 is the maximal torus of G with Lie algebra $\mathfrak{a}_0$. Any arbitrary finite dimensional maximal abelian subalgebra $\mathfrak{a}'$ of $\widehat{L\mathfrak{g}}$ equals $\mathrm{Ad}(\hat{g})\mathfrak{a}$ for some $\hat{g} \in \widehat{LG}$. The corresponding maximal torus is then $\mathrm{I}(\hat{g})T$. □

We pass now to the study of the maximal abelian subalgebras of $L\mathfrak{g}$ which are contained in $\hat{\mathfrak{p}}$. Suppose $\hat{\mathfrak{a}}$ is such a subalgebra. As before, if $\hat{\mathfrak{a}}$ is contained in $\mathfrak{p} + \mathbb{R}c$, then $\hat{\mathfrak{a}}$ is infinite dimensional (for example, if $u_0 \in \mathfrak{p}$, then $u_k \in \mathfrak{p}$, where $u_k(t) = \cos(kt)u_0(t)$).

Moreover, we do not have to take care about ω, because $\omega|_{\mathfrak{p}\times\mathfrak{p}} = 0$. This holds because

$$v \in \mathfrak{p} \ \Rightarrow \ v' \in \hat{\mathfrak{k}}, \ \omega(u, v) = \frac{1}{2\pi}\langle u, v'\rangle$$

and $\mathfrak{k} \perp \mathfrak{p}$, as shown in the previous chapter.

Let us take now $\mathfrak{a}$ maximal abelian of finite type, i.e.

$$\hat{\mathfrak{a}} = \mathfrak{a} + \mathbb{R}c + \mathbb{R}(d + \tilde{u}), \quad \mathfrak{a} \subset \mathfrak{p}, \quad \tilde{u} \in \mathfrak{p}.$$

We want to determine the loop part $\mathfrak{a}$ of $\hat{\mathfrak{a}}$. Consider again the solution $\varphi_t : I \to G$ of $-\varphi_t'\varphi_t^{-1} = \tilde{u}(t)$ with initial value $\varphi_0 = e$. The conditions

$$\begin{cases} [\mathfrak{a}, \tilde{u}] = 0 \\ \mathfrak{a} \text{ abelian} \end{cases}$$

determine in the same way as for the maximal abelian subalgebras of $\widehat{L\mathfrak{g}}$:

$$\mathfrak{a} = \{v \mid v(t) = \mathrm{Ad}(\varphi_t)v_0, \ v_0 \in \mathfrak{a}_0\},$$

with $\mathfrak{a}_0 \subset \mathfrak{g}^{\mathrm{Ad}(\varphi_{2\pi})}$ maximal abelian. In addition, $\mathfrak{a} \subset \mathfrak{p}$ implies $\mathfrak{a}_0 \subset \mathfrak{p}_{\mathfrak{g}}$.

It remains to check now that a loop v of this kind (satisfying $v(t) = \mathrm{Ad}(\varphi_t)v_0$ for some $v_0 \in \mathfrak{p}_{\mathfrak{g}}$) actually belongs to $\mathfrak{p}$:

From $\rho(\varphi_t)'\rho(\varphi_t)^{-1} = \rho_*\varphi_t'\rho(\varphi_t)^{-1} = \rho_*(\varphi_t'\varphi_t^{-1}) = \rho_*(-\tilde{u}(t)) = \tilde{u}(-t)$ and $\varphi_{-t}'\varphi_{-t}^{-1} = \tilde{u}(-t)$ it follows that $(t \mapsto \varphi_{-t})$ and $(t \mapsto \rho(\varphi_t))$ are solutions of the same differential equation $f_t'f_t^{-1} = \tilde{u}(-t)$, with initial value $f_0 = e$, hence $\rho(\varphi_t) = \varphi_{-t}$ (this equation takes locally the form $\frac{d}{dt}f_i(t) = x_i(t, f(t))$, where $x_i(t,\cdot)$, for $i = \overline{1,n}$, are the components of the right invariant vector determinated by $\tilde{u}(-t)$). We obtain thus

$$(\rho_*v)(t) = \rho_*(v(-t)) = \rho_*(\mathrm{Ad}(\varphi_{-t})v_0) = \mathrm{Ad}(\rho(\varphi_{-t}))\rho_*v_0 = -\mathrm{Ad}(\varphi_t)v_0 = -v(t)$$

$$\Rightarrow \rho_*v = -v \Rightarrow v \in \mathfrak{p}$$

($\rho_*\mathrm{Ad}(g)x = \mathrm{Ad}(\rho g)\rho_* x$ holds in general).

In conclusion

$$\mathfrak{a} = \{v \mid v(t) = \mathrm{Ad}(\varphi_t)v_0,\ v_0 \in \mathfrak{a}_0\}$$

with $\mathfrak{a}_0$ maximal abelian in $\mathfrak{g}^{\mathrm{Ad}(\varphi_{2\pi})} \cap \mathfrak{p}$.

Remark : We have obtained by similar arguments:

(i) $\rho(\varphi_t) = \varphi_{-t}$,

(ii) $\varphi_{t+2\pi} = \varphi_t\varphi_{2\pi}$, hence $\varphi_{2\pi}\varphi_{-2\pi} = e$.

Using these relations, we obtain $\rho(\varphi_{2\pi}) = \varphi_{2\pi}^{-1} = \varphi_{-2\pi}$. For any $x \in \mathfrak{g}^{\mathrm{Ad}(\varphi_{2\pi})}$, it holds now

$$\mathrm{Ad}(\varphi_{2\pi})\rho_* x = \mathrm{Ad}(\rho(\varphi_{-2\pi}))\rho_* x = \rho_*\mathrm{Ad}(\varphi_{-2\pi})x = \rho_*\mathrm{Ad}(\varphi_{2\pi}^{-1})x = \rho_* x.$$

Thus $\mathfrak{g}^{\mathrm{Ad}(\varphi_{2\pi})}$ is ρ_*-invariant, so it splits into $\mathfrak{g}^{\mathrm{Ad}(\varphi_{2\pi})} = \mathfrak{k}_\mathfrak{g} \cap \mathfrak{g}^{\mathrm{Ad}(\varphi_{2\pi})} \oplus \mathfrak{p}_\mathfrak{g} \cap \mathfrak{g}^{\mathrm{Ad}(\varphi_{2\pi})}$. All maximal abelian subalgebras of $\mathfrak{p} \cap \mathfrak{g}^{\mathrm{Ad}(\varphi_{2\pi})}$ are therefore conjugated (by the adjoint action of H^ρ restricted to $\mathfrak{p} \cap \mathfrak{g}^{\mathrm{Ad}(\varphi_{2\pi})}$, where H is the connected Lie subgroup of G with Lie algebra $\mathfrak{g}^{\mathrm{Ad}(\varphi_{2\pi})}$) - the conjugacy holds in general for Riemannian symmetric pairs, even though it appears in the literature in more restrictive settings.

We proceed now to prove the conjugacy of the finite type maximal abelian subalgebras of $\hat{\mathfrak{p}} \subset \widehat{L\mathfrak{g}}$.

Theorem 2.6.3. *All the finite type maximal abelian subalgebras of $\widehat{L\mathfrak{g}}$ contained in $\hat{\mathfrak{p}}$ are conjugated, (by the adjoint action of $\widehat{LG}^\rho$ restricted to $\hat{\mathfrak{p}}$). Their dimension equals the dimension of the maximal abelian subalgebras in $\mathfrak{p}_\mathfrak{g} \subset \mathfrak{g}$ plus two.*

Proof. The proof is similar to that of the preceding theorem:

For the step I we start with $\hat{\mathfrak{a}} = \mathfrak{a} + \mathbb{R}c + \mathbb{R}(d + \tilde{u})$, $\mathfrak{a} \subset \mathfrak{p}$, $\tilde{u} \in \mathfrak{p}$ and remark that LG^ρ acts polarly on $\mathfrak{p}$ (see [24]) by

$$g \cdot u = gug^{-1} - g'g^{-1}.$$

Elements $g \in LG^\rho$ and $x \in \mathfrak{p}$ constant loop (so $x \in \mathfrak{p}_\mathfrak{g}$) can thus be found such that $g\tilde{u}g^{-1} - g'g^{-1} = x$. Take $\hat{g} \in \widehat{LG}^\rho$, $\hat{g} = (1, (g, p, 1))$ with p arbitrary path connecting g to e.

Then $\mathrm{Ad}(\hat{g})(\mathfrak{a}) = \mathfrak{a}' = g\mathfrak{a}g^{-1} + \mathbb{R}c + \mathbb{R}(d + x)$ is contained in $\hat{\mathfrak{p}}$ and is again maximal abelian.

At step II we prove that any two maximal abelian subalgebras containing $d + x$ are conjugated.

Let $\hat{\mathfrak{a}}_i = \mathfrak{a}_i + \mathbb{R}c + \mathbb{R}(d + x)$, $\quad i = 1, 2$. Then

$$\mathfrak{a}_i = \{\mathrm{Ad}(-t\exp x)v_0 \mid v_0 \in \mathfrak{a}_{i0}\} \text{ with } \mathfrak{a}_{i0} \subset \mathfrak{g}^{\mathrm{Ad}(\exp(-2\pi x))} \cap \mathfrak{p} \text{ maximal abelian.}$$

We saw that $\mathfrak{g}^{\mathrm{Ad}(\exp(2\pi x))} = \mathfrak{g}^{\mathrm{Ad}(\exp(-2\pi x))}$ is ρ_*-invariant. H denotes as above the corresponding connected Lie group of G.

We find $\varphi = \mathrm{Ad}(\exp y) : \mathfrak{g} \to \mathfrak{g}$ with $y \in \mathfrak{g}^{\mathrm{Ad}(\exp(2\pi x))}$ and $\rho_* y = y$, so $\exp y \in H^\rho$. Then φ restricts to $\mathfrak{g}^{\mathrm{Ad}(\exp(2\pi x))}$and $\varphi(\mathfrak{a}_{10}) = \mathfrak{a}_{20}$ just as discussed before. Notice that g defined as before by $g(t) = \exp(-tx)\exp(y)\exp(tx)$ is now a loop in LG^ρ, as can be easily seen.

The rest follows as before. A canonical maximal abelian subalgebra of finite type contained in $\mathfrak{p}$ has the form

$$\hat{\mathfrak{a}} = \mathfrak{a} + \mathbb{R}c + \mathbb{R}d,$$

where $\mathfrak{a} = \mathfrak{a}_0 \subset \mathfrak{p} \subset \mathfrak{g}$ are constant loops in a maximal abelian in $\mathfrak{p} \subset \mathfrak{g}$, so $\dim \hat{\mathfrak{a}} = \dim \mathfrak{a} + 2$. □

Remark: It is easy to see directly that a maximal abelian subalgebra of $\mathfrak{g}^{\mathrm{Ad}(\exp(2\pi x))}$ contained in $\mathfrak{p}_{\mathfrak{g}}$ is maximal in $\mathfrak{p}_{\mathfrak{g}}$ for any $x \in \mathfrak{p}_{\mathfrak{g}}$. We did not have to check it, since the proof above showed indirectly this fact even in the more general case of x replaced with $\varphi_{2\pi}$, with φ determined as before.

Proposition 2.6.4. *A family of tori of $\widehat{LG}$ is obtained, associated to the subalgebras considered above. They are conjugated by elements of $\widehat{LG}^\rho$. Each of them can be written as* $\exp \hat{\mathfrak{a}}$ *for some subalgebra* $\hat{\mathfrak{a}}$ *as above.*

Proof. Let $\hat{T}_0 = \exp \hat{\mathfrak{a}}_0$, with $\hat{\mathfrak{a}}_0 = \mathfrak{a}_0 + \mathbb{R}c + \mathbb{R}d$, for an abelian subalgebra $\mathfrak{a}_0$ of $\mathfrak{g}$ contained in $\mathfrak{p}_{\mathfrak{g}}$, which is maximal with this properties. Then $\hat{T}_0 = T_0 \times (S^1)^2 \subset G \times (S^1)^2 \hookrightarrow \widehat{LG}$. The relation $I(g) \circ \exp = \exp \circ \mathrm{Ad}(g)$ and the previous result show that $\exp \mathfrak{a}$ is a torus for any $\mathfrak{a}$ as above, and that all the tori of this type are conjugated. □

Finally, we look at the images of these tori under the projection $\pi : \widehat{LG} \to \widehat{LG}/\widehat{LG}^\rho$. We start again with the canonical ones.

Remark: The embedding $G \hookrightarrow LG$ induces an embedding $G/G^\rho \hookrightarrow LG/LG^\rho$ of the finite dimensional symmetric space G/G^ρ. The embedding $G \times S^1 \times S^1 \hookrightarrow \widehat{LG}$ (see Prop. 2.6.2) induces an embedding $G \times S^1 \times S^1/(G \times S^1 \times S^1 \cap \widehat{LG}^\rho) = G \times S^1 \times S^1/G^\rho \times \mathbb{Z}_2 \times \mathbb{Z}_2 \cong G/G^\rho \times S^1 \times S^1 \hookrightarrow \widehat{LG}/\widehat{LG}^\rho$ of the Lorentzian product symmetric space $G/G^\rho \times (S^1)^2$.

Let $\hat{T}_0$ as before. The image $\pi(\hat{T}_0) = \pi \circ \exp(\hat{\mathfrak{a}}_0) = \mathrm{Exp}(\hat{\mathfrak{a}}_0)$ is, due to the previous remark, the embedded submanifold $M \times S^1 \times S^1 \subset G/G^\rho \times S^1 \times S^1 \hookrightarrow \widehat{LG}/\widehat{LG}^\rho$, where M is the maximal totally geodesic flat submanifold $\mathrm{Exp}\,\mathfrak{a}_0$ of the symmetric space G/G^ρ. The commutativity of the diagram

$$\begin{array}{ccc} \widehat{LG} & \xrightarrow{I(g)} & \widehat{LG} \\ \pi\downarrow & & \downarrow\pi \\ \widehat{LG}/\widehat{LG}^\rho & \xrightarrow{l_g} & \widehat{LG}/\widehat{LG}^\rho \end{array}$$

and the conjugacy of the tori involved show that each projection $\pi \circ \exp(\mathfrak{a})$ is a closed embedded submanifold of $\widehat{LG}/\widehat{LG}^{\rho}$ and that all are conjugate by left translations l_g with $g\widehat{LG}^{\rho}$.

We end with their geometric properties.

Remark: The closed subalgebra $\mathfrak{g} \subset L\mathfrak{g}$ of constant loops admits an orthogonal complement, the space of loops u satisfying $\int_I u = 0$. In terms of the Fourier decomposition they are the loops without constant component, i.e. $u(t) = \sum_{k>0} u_{-k}cos(kt) + \sum_{k>0} u_k sin(kt)$. More generally, each subspace $\mathfrak{s}$ of $\mathfrak{g}$ admits an orthogonal complement in $L\mathfrak{g}$, and therefore also in $\widehat{L\mathfrak{g}}$: Let $\mathfrak{s}^{\perp}$ be the orthogonal complement of $\mathfrak{s}$ in $\mathfrak{g}$. Then the direct sum of $L\mathfrak{s}^{\perp}$ with the space $\{u \in \mathfrak{s} \mid \int_I u = 0\}$ is the orthogonal complement of $\mathfrak{s}$ in $L\mathfrak{g}$.

In particular the maximal abelian subalgebras of finite type $\hat{\mathfrak{a}} \subset \hat{\mathfrak{p}}$ admit orthogonal complements. As a consequence, the submanifolds $\operatorname{Exp} \hat{\mathfrak{a}}$ admit induced Levi-Civita connections - see the beginning of Section 6, Chapter I. The relation giving the curvature shows that these submanifolds are flat. Their construction (as images under the geodesic exponential) shows that they are totally geodesic. In conclusion:

Theorem 2.6.5. *$\widehat{LG}/\widehat{LG}^{\rho}$ admits a family of maximal totally geodesic flat closed connected embedded submanifolds. They are all conjugate by left translations with elements $g \in \widehat{LG}^{\rho}$ and have finite dimension equal to m+2, where m is the rank of the symmetric space G/G^{ρ}.*

Proof. Only the maximality is left to be shown. Let $S = \operatorname{Exp} \hat{\mathfrak{a}}$ for some maximal abelian subalgebra $\hat{\mathfrak{a}} \subset \hat{\mathfrak{p}}$. Suppose $S \subset S'$, with S' connected totally geodesic flat submanifold of $\widehat{LG}/\widehat{LG}^{\rho}$. Since S is compact, there exists an $u \in \hat{\mathfrak{p}}$ not contained in $\hat{\mathfrak{a}}$ such that $\operatorname{Exp} tu \in S'$ for all $t \in \mathbb{R}$. Since S' is flat, we obtain $\langle [x,u],[x,u] \rangle = \langle R(x,u)u, x\rangle = 0$ for any $x \in \hat{\mathfrak{a}}$. We remark now that the multiplication of two elements of $\widehat{L\mathfrak{g}}$ never gives a light-like vector, and thus $[x,u] = 0$, which contradicts the maximality of $\hat{\mathfrak{a}}$. □

2.7 Duality and the isotropy representation

The dual symmetric space for $\widehat{LG}/\widehat{LG}^{\rho}$ should be a quotient $H/\widehat{LG}^{\rho}$, with $Lie(H) = \hat{\mathfrak{k}} \oplus i\hat{\mathfrak{p}}$, where the Lie algebra structure of $\hat{\mathfrak{k}} \oplus i\hat{\mathfrak{p}}$ is induced from that of $\widehat{L\mathfrak{g}}_{\mathbb{C}}$. We show that there can be no such dual space. More precisely, there is no Lie group H with $Lie(H) = \hat{\mathfrak{k}} \oplus i\hat{\mathfrak{p}}$.

Take $u \in \hat{\mathfrak{k}} \subset L\mathfrak{g}$ and decompose it into its Fourier series. Consider for simplicity $\operatorname{Im} u \subset \mathfrak{k}_{\mathfrak{g}}$, so $u(t) = \sum_{n\in\mathbb{N}} u_n \cos nt$ with all $u_n \in \mathfrak{k}_{\mathfrak{g}}$. For some $r > 0$ we get $[-ird, u] =$

$-iru'$, which allows us to deduce recursively:

$$\begin{aligned} ad(-ird)(u)(t) &= i\sum_{n\in\mathbb{N}} nru_n \sin nt, \\ ad^2(-ird)(u)(t) &= \sum_{n\in\mathbb{N}} n^2r^2u_n \cos nt, \text{ and for any odd k} \\ ad^k(-ird)(u)(t) &= i\sum_{n\in\mathbb{N}} n^kr^ku_n \sin nt. \end{aligned}$$

A similar expression is obtained for k even. If there would be a Lie group H as above then it would hold $\exp(-ird) \in H$ and $\mathrm{Ad}(\exp(-ird)) = e^{ad(-ird)}$, whence the relation

$$\mathrm{Ad}(\exp(-ird))(u)(t) = \sum_{n\in\mathbb{N}} \mathrm{cosh(nr)u_n \cos nt + i} \sum_{\mathrm{n\in\mathbb{N}}} \mathrm{sinh(nr)u_n \sin nt.}$$

But these two series are not converging in general (we only know that $\sum_{n\in\mathbb{N}} n^p|u_n| < \infty$ for any $p \in \mathbb{N}$ if u is C^∞ loop), and we come thus to a contradiction.

One could try to overcome this difficulty by considering the restriction to polynomial loops (see [23]). This approach has a major inconvenient: the subgroup of LG consisting of polynomial loops is not a manifold.

We end with a remark about the isotropy representation of our Kac-Moody symmetric space. The nice picture in this case, due to Terng, was one of the main motivations for studying the properties of $\widehat{LG}/\widehat{LG}^\rho$ as a symmetric space. We give a brief description, following [24].

The Lorentz scalar product on $\widehat{L\mathfrak{g}}$ is $\mathrm{Ad}(\widehat{LG})$-invariant, so the sphere of radius -1 is invariant under the adjoint action. The hypeplane $\{u + rc + d \mid u \in L\mathfrak{g}, r \in \mathbb{R}\}$ is also invariant and so is their intersection $R^\infty = \{u+rc+d \mid u \in L\mathfrak{g}, r = \frac{1}{2}(\langle u, u\rangle + 1)\}$. Looking at the general formula of the adjoint action on page 45, we see that the restricted action $\mathrm{Ad}_{\widetilde{LG}}(\widetilde{LG})$ to the subgroup $\widetilde{LG} \subset \widehat{LG}$ factors through LG. Moreover, the map $u+rc+d \mapsto u$ gives am isometry between R^∞ and $L\mathfrak{g}$. Under this identification the adjoint action of $\widetilde{LG}$ on the horosphere R^∞ is just the gauge action of LG on $L\mathfrak{g}$: $g \cdot u = gug^{-1} - g'g^{-1}$, which is polar.

If we consider now the eigenspace decomposition $\widehat{L\mathfrak{g}} = \hat{\mathfrak{k}} \oplus \hat{\mathfrak{p}}$ with respect to ρ_*, then $\hat{\mathfrak{k}}$ is the Lie algebra of $\widehat{LG}^\rho$ and $\mathrm{Ad}(\widehat{LG}^\rho)$ leaves $\hat{\mathfrak{p}}$ invariant, this being just the isotropy action at $e\widehat{LG}^\rho$ of the symmetric space $\widehat{LG}/\widehat{LG}^\rho$. This action leaves $R^\infty_{\mathfrak{p}} = R^\infty \cap \hat{\mathfrak{p}}$ invariant. As before, $R^\infty_{\mathfrak{p}} \cong \mathfrak{p}$, where $\mathfrak{p} = \{u \in L\mathfrak{g} \mid \rho(u(-t)) = -u(t)\}$, and the action of $\{+1\} \times \widetilde{LG}^\rho \subset \mathbb{Z}_2 \times \widetilde{LG}^\rho = \widehat{LG}^\rho$ on $R^\infty_{\mathfrak{p}}$ is equivalent to the polar gauge action of $P(G, K \times K)$ on $\{u \in H^0([0,1], \mathfrak{g}) \mid u(0), u(1) \in \mathfrak{p}_{\mathfrak{g}}\}$. We used here the equivalence $LG^\rho = \{g \in LG \mid \rho(g(-t)) = g(t)\} \equiv P(G, K \times K)$.

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