Entropy for practical stabilization

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ENTROPY FOR PRACTICAL STABILIZATION*

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Abstract. For deterministic continuous time nonlinear control systems, ε-practical stabilization entropy, and practical stabilization entropy are introduced. Here the rate of attraction is specified by a $\mathcal{KL}$-function. Upper and lower bounds for the diverse entropies are proved, with special attention to exponential $\mathcal{KL}$-functions. The relation to feedbacks is discussed; the linear case and several nonlinear examples are analyzed in detail.

Key words. practical stabilization, practical stabilization entropy, invariance entropy

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1. Introduction. This paper analyzes entropy properties for practical stabilization of control systems described by ordinary differential equations. The constructions are similar to the theory of invariance entropy motivated by digitally connected control systems. A controller which in finite time intervals $[0, \tau]$ receives only finite data can generate only finitely many (open-loop) control functions on $[0, \tau]$. Invariance entropy abstracts from this situation by counting the number of control functions needed in order to achieve invariance on $[0, \tau]$ and then looks at the exponential growth rate of this number as τ tends to infinity. The specific relations to minimal data rates are worked out in the monograph Kawan [15], where also the relations to feedback entropy introduced in the pioneering work by Nair et al., [23] are clarified. Related work includes Kawan and Da Silva [16] using hyperbolicity conditions, Huang and Zhong [11] for a dimension-like characterization, and Wang, Huang, and Sun [27] for a measure-theoretic version. A similar approach is taken in Colonius [5] for entropy of exponential stabilization, and analogous constructions are also used for state estimation in Liberzon and Mitra [20] as well as in Matveev and Pogromsky [21, 22] and Kawan, Matveev and Pogromsky [17]. Related work is also due to Berger and Jungers [2], where finite data rates for linear systems with switching are analyzed.

Our motivation to consider practical stabilizability is twofold (throughout the paper *practical stabilizability* means *practical asymptotic stabilizability*): Firstly, there are systems where instead of stabilization only practical stabilization is possible. Some examples are provided in section 5. Secondly, and perhaps more relevant, is the fact that standard stabilization algorithms may only lead to practical stabilization, although, theoretically, stabilization is possible. This is the case for economic model predictive control schemes (cf. Zanon and Faulwassser [28, Theorem 1 and Theorem 3]), where practical stabilization is achieved, but stabilization does not hold. Furthermore, sampled feedback of stabilizable systems may only lead to practical

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stabilization; cf. Grüne [9, section 9.4] for a simple example. Similarly, the restriction to other classes of feedbacks may entail that only practical stabilization is possible.

The purpose of the present paper is to contribute to an understanding of entropy for practical stabilization. We specify not only the sets Γ of initial states and the "target set" Λ of final states, but also the convergence rate given by a $\mathcal{KL}$-function, similarly as in the definition of practical stability in Hamzi and Krener [10]. For this purpose we introduce new notions of entropy for ε-practical stabilization, practical stabilization, and also stabilization. We consider the minimal number of control functions needed in order to achieve the practical stabilization goal on a finite time interval. Then we let time tend to infinity and consider the exponential growth rate of these numbers. This is similar to the familiar definition of invariance entropy as exposed, in particular, in Kawan [15]. Note, however, that here in contrast to [15], the set of initial states is in general not a subset of the target set. The relation to feedbacks is briefly discussed based on a new notion of entropy for feedbacks.

In more specific terms the basic construction for entropy is the following. Consider a control system in $\mathbb{R}^d$ of the form $\dot{x}(t) = f(x(t), u(t))$ with a set $\mathcal{U}$ of admissible control functions u and trajectories denoted by $\varphi(t, x_0, u), t \geq 0$. Fix subsets $\Gamma, \Lambda \subset \mathbb{R}^d$ and a $\mathcal{KL}$-function ζ. For $\tau > 0$ a set $\mathcal{S} \subset \mathcal{U}$ of controls is said to be $(\tau, \zeta, \Gamma, \Lambda)$-spanning if for every initial value $x_0 \in \Gamma$ there exists $u \in \mathcal{S}$ with

$$(1.1) \qquad d(\varphi(t, x_0, u), \Lambda) \leq \zeta\left(d(x_0, \Lambda), t\right) \text{ for all } t \in [0, \tau].$$

Denoting by $r(\tau, \zeta, \Gamma, \Lambda)$ the minimal cardinality of a $(\tau, \zeta, \Gamma, \Lambda)$-spanning set we define the stabilization entropy by $\overline{\lim}_{\tau \to \infty} \frac{1}{\tau} \log r(\tau, \zeta, \Gamma, \Lambda)$. This number measures how fast the average number of required controls increases, when the system should approach the set Λ with the bound (1.1) as time τ tends to infinity. In order to guarantee the existence of finite spanning sets of controls, this notion has to be slightly modified; cf. Remark 2.4. Practical stabilizability properties, which are the focus of the present paper, are obtained if we require that the solutions approach Λ only approximately; cf. Definition 2.2. We remark that the construction of entropy via spanning sets follows the classical construction of entropy for dynamical systems in metric spaces due to Bowen and Dinaburg. The logarithm with base 2 is directly related to the number of bits needed to choose a control u; for continuous time systems, as considered here, the natural logarithm is more convenient.

The contents of this paper are as follows. Section 2 introduces ε-practical stabilization entropy, practical stabilization entropy, and stabilization entropy about a compact set Λ for compact sets Γ of initial states and compact control ranges. Also modifications for noncompact control ranges and noncompact sets of initial states are indicated. Section 3 proves upper bounds for the diverse entropies, and lower bounds based on volume growth arguments are established. Special attention is given to exponential $\mathcal{KL}$-functions of the form $\zeta(r, s) = e^{-\alpha s} M r, r, s \geq 0$, with $\alpha > 0$ and $M \geq 1$. Section 4 briefly discusses the relation to feedbacks. Section 5 first analyzes linear systems. Then two scalar nonlinear examples and a higher dimensional system are analyzed which are not locally C^1 stabilizable at the origin. Here practically stabilizing quadratic and piecewise linear feedbacks, respectively, can be constructed, and estimates for the entropies can be obtained. The analysis reveals some subtleties in the constructions. The proofs for the practical stabilizability of the two scalar example are provided in the appendix. Finally, section 6 draws some conclusions and presents open questions.

Notation. A $\mathcal{KL}$-function is a continuous function $\zeta : [0, \infty) \times [0, \infty) \to [0, \infty)$ such that $\zeta(r, s)$ is strictly increasing in r for fixed s with $\zeta(0, s) = 0$ and strictly

decreasing with respect to s for fixed r with $\lim_{s\to\infty} \zeta(r,s) = 0$. In a metric space, the distance of a point x to a nonvoid set A is $d(x,A) := \inf\{d(x,a)\,|a \in A\}$, and for a compact set A the ε-neighborhood is $N(A;\varepsilon) = \{x\,|d(x,A) < \varepsilon\}$. For a point a we write the ball with radius ε around a as $B(a,\varepsilon) = \{x\,|d(x,a) < \varepsilon\}$. The cardinality, viz. the number of elements, of a finite set A is denoted by $\#A$. The natural logarithm is denoted by $\log$, and the limit superior is denoted by $\overline{\lim}$.

2. Entropy notions. We consider control systems in $\mathbb{R}^d$ of the form

$$(2.1) \qquad \dot{x}(t) = f(x(t), u(t)), \quad u(t) \in U,$$

where $f : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$ is continuous. The control range U is a nonvoid subset of $\mathbb{R}^m$, and the set of admissible control functions is given by

$$\mathcal{U} = \{u \in L^\infty([0,\infty), \mathbb{R}^m) : u(t) \in U \text{ for almost all } t\}.$$

We assume standard conditions on f guaranteeing existence and uniqueness of solutions $\varphi(t, x_0, u), t \geq 0$, with $\varphi(0, x_0, u) = x_0$ for all $x_0 \in \mathbb{R}^d$ and $u \in \mathcal{U}$ as well as continuous dependence on initial values.

Consider the following stability properties for a differential equation $\dot{x} = g(x,t)$ with (unique) solutions $\psi(t, x_0), t \geq 0$, for initial conditions $\psi(0, x_0) = x_0$.

DEFINITION 2.1. *Let $\Gamma, \Lambda \subset \mathbb{R}^d$, and let ζ be a $\mathcal{KL}$-function. Then the system is (ζ, Γ, Λ)-stable if every $x_0 \in \Gamma$ satisfies*

$$d(\psi(t, x_0), \Lambda) \leq \zeta(d(x_0, \Lambda), t) \text{ for } t \geq 0.$$

For $\varepsilon > 0$ it is ε-practically (ζ, Γ, Λ)-stable if every $x_0 \in \Gamma$ satisfies

$$d(\psi(t, x_0), \Lambda) \leq \zeta(d(x_0, \Lambda), t) + \varepsilon \text{ for } t \geq 0.$$

Recall that stability of an equilibrium e is equivalent to the existence of a $\mathcal{KL}$-function for $\Lambda = \{e\}$; cf. Clarke, Ledyaev, and Stern [4, Lemma 2.6]. We are interested in the data rate needed to make the control system (2.1) stable or at least ε-practically stable for certain $\varepsilon > 0$ or for all $\varepsilon > 0$. Again, let subsets $\Gamma, \Lambda \subset \mathbb{R}^d$ and a $\mathcal{KL}$-function ζ be given. For $\tau, \varepsilon > 0$ we call a set $\mathcal{S} \subset \mathcal{U}$ of controls $(\tau, \varepsilon, \zeta, \Gamma, \Lambda)$-spanning if for every $x_0 \in \Gamma$ there exists $u \in \mathcal{S}$ with

$$(2.2) \qquad d(\varphi(t, x_0, u), \Lambda) \leq \zeta\left(d(x_0, \Lambda) + \varepsilon, t\right) \text{ for all } t \in [0, \tau].$$

(Here $\varepsilon > 0$ is introduced in order to guarantee that finite spanning sets exist; cf. Remark 2.4.) Furthermore, a set $\mathcal{S} \subset \mathcal{U}$ of controls is called practically $(\tau, \varepsilon, \zeta, \Gamma, \Lambda)$-spanning if for every $x_0 \in \Gamma$ there exists $u \in \mathcal{S}$ with

$$(2.3) \qquad d(\varphi(t, x_0, u), \Lambda) \leq \zeta\left(d(x_0, \Lambda) + \varepsilon, t\right) + \varepsilon \text{ for all } t \in [0, \tau].$$

The minimal cardinality of a $(\tau, \varepsilon, \zeta, \Gamma, \Lambda)$-spanning set and a practically $(\tau, \varepsilon, \zeta, \Gamma, \Lambda)$-spanning set are denoted by $r_s(\tau, \varepsilon, \zeta, \Gamma, \Lambda)$ and $r_{ps}(\tau, \varepsilon, \zeta, \Gamma, \Lambda)$, respectively. If there is no finite spanning set or no spanning set at all, we set these numbers equal to $+\infty$.

DEFINITION 2.2. *Let $\Gamma, \Lambda \subset \mathbb{R}^d$, and let ζ be a $\mathcal{KL}$-function.*
(i) For $\varepsilon > 0$ the ε-stabilization entropy and the stabilization entropy are

$$h_s(\varepsilon, \zeta, \Gamma, \Lambda) = \overline{\lim_{\tau\to\infty}} \frac{1}{\tau} \log r_s(\tau, \varepsilon, \zeta, \Gamma, \Lambda) \text{ and } h_s(\zeta, \Gamma, \Lambda) = \lim_{\varepsilon\to 0} h_s(\varepsilon, \zeta, \Gamma, \Lambda).$$

(ii) *For $\varepsilon > 0$ the ε-practical stabilization entropy $h_{\mathrm{ps}}(\varepsilon, \zeta, \Gamma, \Lambda)$ and the practical stabilization entropy $h_{\mathrm{ps}}(\zeta, \Gamma, \Lambda)$ are*

$$(2.4) \qquad h_{\mathrm{ps}}(\varepsilon, \zeta, \Gamma, \Lambda) = \varlimsup_{\tau \to \infty} \frac{1}{\tau} \log r_{\mathrm{ps}}(\tau, \varepsilon, \zeta, \Gamma, \Lambda) \text{ and } h_{\mathrm{ps}}(\zeta, \Gamma, \Lambda) = \lim_{\varepsilon \to 0} h_{\mathrm{ps}}(\varepsilon, \zeta, \Gamma, \Lambda).$$

For $\varepsilon_1 > \varepsilon_2 > 0$ the following inequalities are easily seen:

$$(2.5) \qquad h_{\mathrm{ps}}(\varepsilon_1, \zeta, \Gamma, \Lambda) \leq h_{\mathrm{ps}}(\varepsilon_2, \zeta, \Gamma, \Lambda) \leq h_{\mathrm{ps}}(\zeta, \Gamma, \Lambda) \leq h_{\mathrm{s}}(\zeta, \Gamma, \Lambda).$$

This shows, in particular, that in (2.4) the limit for $\varepsilon \to 0$ exists and coincides with the supremum over $\varepsilon > 0$ (it may equal $+\infty$). Our results will mainly concern compact control ranges U and compact sets Γ and Λ, but cf. Remarks 2.7 and 2.8 for generalizations. Definition 2.2 does not require that Γ is a neighborhood of Λ. Nevertheless, situations where Γ is a neighborhood of Λ or, at least, has nonvoid interior are certainly most interesting.

First we will ascertain that under weak assumptions finite spanning sets exist.

LEMMA 2.3. *Consider for a control system of the form (2.1) subsets $\Gamma, \Lambda \subset \mathbb{R}^d$ and a $\mathcal{KL}$-function ζ. Assume further that Γ is compact, and fix $\varepsilon > 0$.*

(i) *Suppose that for every $x_0 \in \Gamma$ there is a control $u \in \mathcal{U}$ with*

$$d(\varphi(t, x_0, u), \Lambda) < \zeta(d(x_0, \Lambda) + \varepsilon, t) \text{ for all } t \geq 0.$$

Then for every $\tau > 0$ there is a finite set $\mathcal{S} = \{u_1, \ldots, u_n\} \subset \mathcal{U}$ such that for every $x_0 \in \Gamma$ there is $u_j \in \mathcal{S}$ with

$$d(\varphi(t, x_0, u_j), \Lambda) < \zeta(d(x_0, \Lambda) + 2\varepsilon, t) \text{ for all } t \in [0, \tau].$$

(ii) *Suppose that for every $x_0 \in \Gamma$ there is a control $u \in \mathcal{U}$ with*

$$d(\varphi(t, x_0, u), \Lambda) < \zeta(d(x_0, \Lambda) + \varepsilon, t) + \varepsilon \text{ for all } t \geq 0.$$

Then for every $\tau > 0$ there is a finite set $\mathcal{S} = \{u_1, \ldots, u_n\} \subset \mathcal{U}$ such that for every $x_0 \in \Gamma$ there is $u_j \in \mathcal{S}$ with

$$d(\varphi(t, x_0, u_j), \Lambda) < \zeta(d(x_0, \Lambda) + 2\varepsilon, t) + 2\varepsilon \text{ for all } t \in [0, \tau].$$

Proof. (i) For every $x_0 \in \Gamma$ choose a control $u \in \mathcal{U}$ with

$$d(\varphi(t, x_0, u), \Lambda) < \zeta(d(x_0, \Lambda) + \varepsilon, t) \text{ for all } t \in [0, \tau].$$

By continuous dependence on initial values (as assumed for (2.1)) there is δ with $0 < \delta < \varepsilon$ such that, for all $x_1 \in \mathbb{R}^d$ with $\|x_0 - x_1\| < \delta$ and for all $t \in [0, \tau]$,

$$d(\varphi(t, x_1, u), \Lambda) < \zeta(d(x_0, \Lambda) + \varepsilon, t) \leq \zeta(\|x_0 - x_1\| + d(x_1, \Lambda) + \varepsilon, t)$$
$$< \zeta(\delta + d(x_1, \Lambda) + \varepsilon, t) < \zeta(d(x_1, \Lambda) + 2\varepsilon, t).$$

Here we have used that for all $x, y \in \mathbb{R}^d$ one has $d(x, \Lambda) \leq \|x - y\| + d(y, \Lambda)$ together with the monotonicity properties of the $\mathcal{KL}$-function ζ. Now compactness of Γ shows that there is a finite set $\mathcal{S} = \{u_1, \ldots, u_n\} \subset \mathcal{U}$ such that for each $x_1 \in \Gamma$ there is $u_j \in \mathcal{S}$ satisfying, for all $t \in [0, \tau]$,

$$d(\varphi(t, x_1, u_j), \Lambda) < \zeta(d(x_1, \Lambda) + 2\varepsilon, t).$$

(ii) This is proved analogously. $\qquad \qquad \square$

We observe that the assumption in Lemma 2.3(i) holds, in particular, if there exists a stabilizing feedback. See also section 4 for more on the relations to feedbacks.

The following remarks refer to related results in the literature and elaborate on some variants of the entropy notions introduced in Definition 2.2.

Remark 2.4. In Colonius [5] the following notion of entropy for exponential stabilization about an equilibrium in the origin is introduced. Consider a compact set $\Gamma \subset \mathbb{R}^d$ of initial states, and let $\alpha > 0, M > 1$, and $\varepsilon > 0$. For a time $\tau > 0$ a subset $\mathcal{S} \subset \mathcal{U}$ is called $(\tau, \varepsilon, \alpha, M, \Gamma)$-spanning if for all $x_0 \in \Gamma$ there is $u \in \mathcal{S}$ with

(2.6) $$\|\varphi(t, x_0, u)\| < e^{-\alpha t}(\varepsilon + M\|x_0\|) \text{ for all } t \in [0, \tau].$$

The minimal cardinality of such a set is denoted by $s_{\text{stab}}(\tau, \varepsilon, \alpha, M, \Gamma)$, and the stabilization entropy is defined by

$$h_{\text{stab}}(\alpha, M, \Gamma) = \lim_{\varepsilon \to 0} \overline{\lim_{\tau \to \infty}} \frac{1}{\tau} \log s_{\text{stab}}(\tau, \varepsilon, \alpha, M, \Gamma).$$

The spanning condition (2.6) can be rewritten in the following way: With $\Lambda = \{0\}$ let a $\mathcal{KL}$-function ζ be defined by $\zeta(r, s) := e^{-\alpha s} M r$. With $M\varepsilon$ instead of ε, condition (2.6) is

$$d(\varphi(t, x_0, u), \{0\}) < e^{-\alpha t} M(\varepsilon + \|x_0\|) = \zeta(d(x_0, \{0\}) + \varepsilon, t) \text{ for all } t \in [0, \tau].$$

Thus $h_{\text{stab}}(\alpha, M, \Gamma)$ is a special case of stabilization entropy as specified in Definition 2.2(i). In [5, Proposition 2.2] it is shown that finite spanning sets can only be expected for positive $\varepsilon > 0$ in (2.6). This is the reason why we also consider positive ε in Definition 2.2(i). Observe that one could similarly relax the condition for ε-practical stability in Definition 2.1 by requiring

$$d(\psi(t, x_0), \Lambda) \leq \zeta(d(x_0, \Lambda) + \varepsilon, t) + \varepsilon \text{ for } t \geq 0.$$

Relations of entropy to minimal bit rates for (nonexponential) stabilization are given in [5, Lemma 5.2 and Theorem 5.3].

Remark 2.5. Hamzi and Krener [10] call a control system locally practically stabilizable around an equilibrium in the origin if for every $\varepsilon > 0$ there exists an open set D containing the closed ball $\mathbf{B}(0, \varepsilon)$, a $\mathcal{KL}$-function ζ_ε, a positive constant $\delta = \delta(\varepsilon)$, and a control law $u = k_\varepsilon(x)$ such that for any initial value $x(0)$ with $\|x(0)\| < \delta$, the solution $x(t)$ of the feedback system $\dot{x} = f(x, k_\varepsilon(x))$ exists and satisfies

(2.7) $$d(x(t), \mathbf{B}(0, \varepsilon)) \leq \zeta_\varepsilon(d(x(0), \mathbf{B}(0, \varepsilon)), t) \text{ for all } t \geq 0.$$

Note that here $\delta(\varepsilon) < \varepsilon$ is admitted; hence attractivity is not required. The trajectories may leave $\mathbf{B}(0, \varepsilon)$, but the bound (2.7) ensures that they converge to it for $t \to \infty$. In view of the fact that here the $\mathcal{KL}$-function ζ_ε depends on ε the following variant of practical stabilization entropy might be considered: Define for $\varepsilon > 0$ the ε-practical stabilization entropy by

$$\hat{h}_{\text{ps}}(\varepsilon, \Gamma, \Lambda) = \inf_{\zeta_\varepsilon} \overline{\lim_{\tau \to \infty}} \frac{1}{\tau} \log r_{\text{ps}}(\tau, \varepsilon, \zeta_\varepsilon, \Gamma, \Lambda),$$

where the infimum is taken over all $\mathcal{KL}$-functions ζ_ε, and let a practical stabilization entropy be $\hat{h}_{\text{ps}}(\Gamma, \Lambda) = \lim_{\varepsilon \to 0} \hat{h}_{\text{ps}}(\varepsilon, \Gamma, \Lambda)$. Also a corresponding local version might

be introduced by requiring the practical $(\tau, \varepsilon, \zeta, \Gamma, \Lambda)$-spanning condition (2.3) only for all $x_0 \in \Gamma_\varepsilon$, where Γ_ε is compact neighborhood of Λ. If one requires this spanning condition for all initial values x_0 in a compact neighborhood $\Gamma_{\delta(\varepsilon)}$ of Λ containing a $\delta(\varepsilon)$-neighborhood of Λ, $\delta(\varepsilon) > 0$, one obtains a local notion without attractivity requirement.

Remark 2.6. For control-affine systems, Da Silva and Kawan define in [7] a version of invariance entropy (for "practical stabilization") in the special situation, where (in our notation) $\Gamma = \Lambda$ is a compact subset of a control set D with nonvoid interior (i.e., a maximal set with approximate controllability). Then they consider the maximum of the corresponding entropies taken over all $\Gamma = \Lambda$ contained in D. Under a uniform hyperbolicity condition for cl$D$, [7, Theorem 9] shows that the corresponding entropy varies continuously with respect to system parameters.

Remark 2.7. In the examples in subsections 5.2 and 5.3 (cf. Theorems 5.4 and 5.7) also unbounded closed control ranges U occur, where it will be appropriate to employ a reduction to compact control ranges by using the following modified notion: For compact sets $K \subset \mathbb{R}^m$ a subset $\mathcal{S} \subset \mathcal{U}$ of controls with values in $U \cap K$ is practically $(\tau, \varepsilon, \zeta, \Gamma, \Lambda, U \cap K)$-spanning if for every $x_0 \in \Gamma$ there exists $u \in \mathcal{S}$ with

$$d(\varphi(t, x_0, u), \Lambda) \le \zeta\,(d(x_0, \Lambda) + \varepsilon, t) + \varepsilon \text{ for all } t \in [0, \tau].$$

Denoting the minimal cardinality of such a spanning set by $r_{\mathrm{ps}}(\tau, \varepsilon, \zeta, \Gamma, \Lambda, U \cap K)$ we define the ε-practical stabilization entropy by

$$h_{\mathrm{ps}}(\varepsilon, \zeta, \Gamma, \Lambda, U) = \inf_K \varlimsup_{\tau \to \infty} \frac{1}{\tau} \log r_{\mathrm{ps}}(\tau, \varepsilon, \zeta, \Gamma, \Lambda, U \cap K),$$

where the infimum is taken over all compact subsets $K \subset \mathbb{R}^m$. Then the practical stabilization entropy again is obtained by letting $\varepsilon \to 0$. In the case of an exponential $\mathcal{KL}$-function $\zeta(r, s) = e^{-\alpha s} M r$, the relevant quantity is the exponential rate α. In the examples in subsections 5.2 and 5.3 constants M which depend on ε occur while α does not.

Remark 2.8. In the theory developed below, compactness of the set Γ of initial states plays a crucial role. For general closed sets Γ a reasonable notion of practical stabilization entropy might be introduced as $h_{\mathrm{ps}}(\zeta, \Gamma, \Lambda) := \sup_K h_{\mathrm{ps}}(\zeta, \Gamma \cap K, \Lambda)$, where the supremum is taken over all compact sets $K \subset \mathbb{R}^d$. This is in the same spirit as the definition of topological entropy for uniformly continuous maps on metric spaces; cf. Walters [26, Definition 7.10].

3. Bounds for practical stabilization entropy. In this section we derive upper and lower bounds for the practical stabilization entropy and the stabilization entropy.

First we present an upper bound for the ε-practical stabilization entropy. The proof is based on a cut-off function and is a modification of the proofs in Katok and Hasselblatt [12, Theorem 3.3.9] (for topological entropy) as well as Colonius and Kawan [6, Theorem 4.2] (for invariance entropy).

For compact sets $\Gamma, \Lambda \subset \mathbb{R}^d$, a $\mathcal{KL}$-function ζ, and $\varepsilon \ge 0$ define the compact set

$$P_\varepsilon := \left\{ x \in \mathbb{R}^d \,\middle|\, d(x, \Lambda) \le \zeta\left(\max_{y \in \Gamma} d(y, \Lambda) + \varepsilon, 0 \right) + \varepsilon \right\},$$

and define $L_\varepsilon := \sup_{(x,u) \in P_\varepsilon \times U} \|f_x(x, u)\|$ where $f_x(x, u) = \frac{\partial f}{\partial x}(x, u)$. Observe that L_ε depends on ζ and Γ and, naturally, on Λ. We suppose that $\zeta(r, 0) \ge r$ for $r \in [0, \infty)$, implying $\Gamma \subset P_\varepsilon$.

THEOREM 3.1. *Consider for control system* (2.1) *compact sets* $\Gamma, \Lambda \subset \mathbb{R}^d$, *and let* ζ *be* $\mathcal{KL}$*-function. Suppose that the control range* U *is compact, that* f *is differentiable with respect to* x, *and that the partial derivative* $f_x(x, u)$ *is continuous in* (x, u).

(i) *Fix* $\varepsilon > 0$. *If for every* $x_0 \in \Gamma$ *there is a control* $u \in \mathcal{U}$ *with*

$$(3.1) \qquad d(\varphi(t, x_0, u), \Lambda) < \zeta(d(x_0, \Lambda) + \varepsilon, t) + \varepsilon \text{ for all } t \geq 0,$$

then the 2ε*-practical stabilization entropy satisfies* $h_{\mathrm{ps}}(2\varepsilon, \zeta, \Gamma, \Lambda) \leq L_\varepsilon d$.

(ii) *If the assumption in* (i) *holds for all* $\varepsilon > 0$, *the practical stabilization entropy satisfies* $h_{\mathrm{ps}}(\zeta, \Gamma, \Lambda) \leq L_0 d$.

Proof. Define for $\varepsilon \geq 0$

$$R_\varepsilon := \{(x_0, u) \in \Gamma \times \mathcal{U} \mid d(\varphi(t, x_0, u), \Lambda) \leq \zeta(d(x_0, \Lambda) + \varepsilon, t) + \varepsilon \text{ for all } t \geq 0\}.$$

Note that every $(x_0, u) \in R_\varepsilon$ satisfies $\varphi(t, x_0, u) \in P_\varepsilon$ for $t \geq 0$, since

$$d(\varphi(t, x_0, u), \Lambda) \leq \zeta(d(x_0, \Lambda) + \varepsilon, t) + \varepsilon \leq \zeta(\max_{y \in \Gamma} d(y, \Lambda) + \varepsilon, 0) + \varepsilon,$$

using that ζ is increasing in the first argument and decreasing in the second argument.

(i) Fix $\varepsilon > 0$, and let $\tilde{\varepsilon}, \tau > 0$ be given. Since the compact set $P_{\varepsilon+2\tilde{\varepsilon}}$ is contained in the interior of $P_{\varepsilon+3\tilde{\varepsilon}}$ one can choose a C^1-function $\theta : \mathbb{R}^d \to [0, 1]$ with $\theta(x) = 1$ for all $x \in P_{\varepsilon+2\tilde{\varepsilon}}$ and support contained in $P_{\varepsilon+3\tilde{\varepsilon}}$ (cf. Abraham, Marsden, and Ratiu [1, Proposition 5.5.8, page 380]). We define $\tilde{f} : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$ by $\tilde{f}(x, u) := \theta(x)f(x, u)$ (note that $\tilde{f}$ depends on $\tilde{\varepsilon}$). Then $\tilde{f}$ is continuous, and the derivative with respect to the first argument is continuous in (x, u). Consider the control system

$$(3.2) \qquad \dot{x}(t) = \tilde{f}(x(t), u(t)), \quad u(t) \in U.$$

The right-hand side of this system is globally bounded, and thus solutions exist globally (see, e.g., Sontag [24, Proposition C.3.7]). We denote the solution map associated with (3.2) by ψ and observe that

$$(\psi([0, \tau], x_0, u) \subset P_{\varepsilon+2\tilde{\varepsilon}} \text{ or } \varphi([0, \tau], x_0, u) \subset P_{\varepsilon+2\tilde{\varepsilon}})$$
$$\Rightarrow \psi(t, x_0, u) = \varphi(t, x_0, u) \text{ for all } t \in [0, \tau].$$

A global Lipschitz constant for $\tilde{f}$ on $\mathbb{R}^d \times U$ with respect to the first variable is given by

$$(3.3) \qquad \tilde{L} := \max\left\{\| \tilde{f}_x(x, u) \| \,\big|\, (x, u) \in \mathbb{R}^d \times U\right\},$$

which satisfies

$$\tilde{L} = \tilde{L}_{\varepsilon+3\tilde{\varepsilon}} := \max\left\{\| \tilde{f}_x(x, u) \| \,\big|\, (x, u) \in P_{\varepsilon+3\tilde{\varepsilon}} \times U\right\}.$$

Using continuity of $\tilde{f}_x$ and ζ and compactness of $P_{\varepsilon+3\tilde{\varepsilon}} \times U$ one finds

$$(3.4) \qquad \tilde{L}_{\varepsilon+3\tilde{\varepsilon}} \to L_\varepsilon \text{ for } \tilde{\varepsilon} \to 0.$$

Every (y, u) in $R_{\varepsilon+\tilde{\varepsilon}}$ satisfies $\varphi(t, y, u) \in P_{\varepsilon+\tilde{\varepsilon}}$, and hence $\varphi(t, y, u) = \psi(t, y, u), t \geq 0$.

Now let $\mathcal{S}^+ = \{(y_1, u_1), \ldots, (y_n, u_n)\} \subset R_{\varepsilon+\tilde{\varepsilon}}$ be a subset with the property that for every $x_0 \in \Gamma$ there exists $(y_i, u_i) \in \mathcal{S}^+$ with

$$\max_{t \in [0, \tau]} d(\psi(t, x_0, u_i), \psi(t, y_i, u_i)) < \tilde{\varepsilon}.$$

Thus also $\psi(t, x_0, u_i) = \varphi(t, x_0, u_i)$, since $\varphi(t, x_0, u_i) \in P_{e+2\tilde{\varepsilon}}, t \in [0, \tau]$. By continuity and compactness, we may in fact assume that $\mathcal{S}^+$ has finite cardinality, and we take $\mathcal{S}^+$ with minimal cardinality denoted by $r^+(\tau, \tilde{\varepsilon})$. We claim that for $2\tilde{\varepsilon} < \varepsilon$,

$$(3.5) \qquad r_{\mathrm{ps}}(\tau, 2\varepsilon, \zeta, \Gamma, \Lambda) \leq r_{\mathrm{ps}}(\tau, \varepsilon + 2\tilde{\varepsilon}, \zeta, \Gamma, \Lambda) \leq r^+(\tau, \tilde{\varepsilon}).$$

The first inequality follows by monotonicity in the second argument. The second inequality follows, since for a minimal set $\mathcal{S}^+$ as above and $x_0 \in \Gamma$ there are $(y_i, u_i) \in \mathcal{S}^+$ such that for all $t \in [0, \tau]$,

$$\begin{aligned}
d(\psi(t, x_0, u_i), \Lambda) &< d(\psi(t, x_0, u_i), \psi(t, y_i, u_i)) + d(\psi(t, y_i, u_i), \Lambda) \\
&< \tilde{\varepsilon} + \zeta(d(y_i, \Lambda) + \varepsilon + \tilde{\varepsilon}, t) + \varepsilon + \tilde{\varepsilon} \\
&\leq \zeta(\|y_i - x_0\| + d(x_0, \Lambda) + \varepsilon + \tilde{\varepsilon}, t) + \varepsilon + 2\tilde{\varepsilon} \\
&< \zeta(d(x_0, \Lambda) + \varepsilon + 2\tilde{\varepsilon}, t) + \varepsilon + 2\tilde{\varepsilon}.
\end{aligned}$$

Since $\psi(t, x_0, u_i) = \varphi(t, x_0, u_i), t \in [0, \tau]$, it follows that $\mathcal{S}^+$ is practically $(\tau, \varepsilon + 2\tilde{\varepsilon}, \zeta, \Gamma, \Lambda)$-spanning, and (3.5) is proved.

Next define the sets

$$\Gamma_i := \left\{ x_0 \in \Gamma \mid \max_{t \in [0, \tau]} d(\psi(t, x_0, u_i), \psi(t, y_i, u_i)) < \tilde{\varepsilon} \right\}, \quad i = 1, \ldots, n = r^+(\tau, \tilde{\varepsilon}).$$

By the definitions, $\Gamma = \bigcup_{i=1}^n \Gamma_i$. Let $x_0 \in \mathbb{R}^d$ be a point with $\|x_0 - y_i\| < e^{-\tilde{L}\tau}\tilde{\varepsilon}$ for some $i \in \{1, \ldots, r^+(\tau, \tilde{\varepsilon})\}$. By (3.3) it follows that

$$(3.6) \quad \|\psi(t, x_0, u_i) - \psi(t, y_i, u_i)\| \leq \|x_0 - y_i\| + \tilde{L} \int_0^t \|\psi(\sigma, x_0, u_i) - \psi(\sigma, y_i, u_i)\| d\sigma$$

for all $t \geq 0$. By Gronwall's lemma this implies for all $t \in [0, \tau]$,

$$(3.7) \qquad \|\psi(t, x_0, u_i) - \psi(t, y_i, u_i)\| \leq \|x_0 - y_i\| e^{\tilde{L}t} < \tilde{\varepsilon}.$$

It follows that $x_0 \in \Gamma_i$, and thus Γ contains the union of the balls $\mathbf{B}(y_i, e^{-\tilde{L}\tau}\tilde{\varepsilon})$.

Now assume that there exists a cover $\mathcal{V}$ of Γ consisting of balls $\mathbf{B}(x_i, e^{-\tilde{L}\tau}\tilde{\varepsilon}), x_i \in \Gamma$ for $i = 1, \ldots, N$, such that $N = \#\mathcal{V} < \#\mathcal{S}^+ = r^+(\tau, \tilde{\varepsilon})$. By assumption (3.1) we can assign to each point x_i a control function v_i with $(x_i, v_i) \in R_{e+\tilde{\varepsilon}}$. Then, by the arguments above, the ball $\mathbf{B}(x_i, e^{-\tilde{L}\tau}\tilde{\varepsilon})$ is contained in the set

$$\left\{ x_0 \in \mathbb{R}^d \mid \max_{t \in [0, \tau]} d(\psi(t, x_0, v_i), \psi(t, x_i, v_i)) < \tilde{\varepsilon} \right\}.$$

This contradicts the minimality of $\mathcal{S}^+$. Let $c(\delta, Z)$ is the minimal cardinality of a cover of a bounded subset $Z \subset \mathbb{R}^d$ by δ-balls. We have shown that $r^+(\tau, \tilde{\varepsilon}) \leq c(\delta, \Gamma)$ with $\delta := e^{-\tilde{L}\tau}\tilde{\varepsilon}$.

Recall that for a bounded subset $Z \subset \mathbb{R}^d$ the upper box or fractal dimension satisfies

$$\dim_F(Z) := \overline{\lim_{\delta \searrow 0}} \frac{\log c(\delta, Z)}{\log(1/\delta)} \leq d;$$

cf., e.g., Boichenko, Leonov, and Reitmann [3, Proposition 2.2.2 in Chapter III]. Since

$$\tilde{L}\tau = \log(e^{\tilde{L}\tau}\tilde{\varepsilon}^{-1}) + \log\tilde{\varepsilon} = \log(e^{\tilde{L}\tau}\tilde{\varepsilon}^{-1})\left(1 + \frac{\log\tilde{\varepsilon}}{\log(e^{\tilde{L}\tau}\tilde{\varepsilon}^{-1})}\right),$$

it follows that

$$\overline{\lim_{\tau\to\infty}} \frac{1}{\tau}\log r^+(\tau,\tilde{\varepsilon}) \le \overline{\lim_{\tau\to\infty}} \frac{1}{\tau}\log c(e^{-\tilde{L}\tau}\tilde{\varepsilon},\Gamma) = \tilde{L}\overline{\lim_{\tau\to\infty}} \frac{\log c(e^{-\tilde{L}\tau}\tilde{\varepsilon},\Gamma)}{\tilde{L}\tau}$$

$$(3.8) \qquad = \tilde{L}\overline{\lim_{\tau\to\infty}} \frac{\log c(e^{-\tilde{L}\tau}\tilde{\varepsilon},\Gamma)}{\log(e^{\tilde{L}\tau}\tilde{\varepsilon}^{-1})\left(1 + \frac{\log\tilde{\varepsilon}}{\log(e^{\tilde{L}\tau}\tilde{\varepsilon}^{-1})}\right)} = \tilde{L}\overline{\lim_{\tau\to\infty}} \frac{\log c(e^{-\tilde{L}\tau}\tilde{\varepsilon},\Gamma)}{\log(e^{\tilde{L}\tau}\tilde{\varepsilon}^{-1})}$$

$$= \tilde{L}\dim_F(\Gamma).$$

As $\tilde{\varepsilon}$ tends to zero, the Lipschitz constants $\tilde{L} = \tilde{L}_{\varepsilon+3\tilde{\varepsilon}}$ tend to L_ε by (3.4). Taking into account also (3.5), this implies

$$h_{\text{ps}}(2\varepsilon,\zeta,\Gamma,\Lambda) \le \left(\lim_{\tilde{\varepsilon}\to 0}\hat{L}_{\varepsilon+3\tilde{\varepsilon}}\right)\dim_F(\Gamma) = L_\varepsilon\dim_F(\Gamma) \le L_\varepsilon d,$$

which proves assertion (i).

(ii) If the assumptions in (i) hold for all $\varepsilon > 0$, then the Lipschitz constants L_ε converge for $\varepsilon \to 0$ to L_0, and the assertion follows from

$$h_{\text{ps}}(\zeta,\Gamma,\Lambda) = \lim_{\varepsilon\to 0}h_{\text{ps}}(\varepsilon,\zeta,\Gamma,\Lambda) \le \lim_{\varepsilon\to 0}L_\varepsilon d = L_0 d. \qquad \square$$

The following theorem gives a similar estimate for the stabilization entropy with exponential $\mathcal{KL}$-function. For compact control range U and $\varepsilon \ge 0, M \ge 0$, define the compact set

$$P_\varepsilon^s := \left\{x \in \mathbb{R}^d \,\Big|\, d(x,\Lambda) \le M\left(\max_{y\in\Gamma}d(y,\Lambda) + \varepsilon\right)\right\}$$

and the constant $L_0^s := \max_{(x,u)\in P_0^s\times U}\|f_x(x,u)\|$.

THEOREM 3.2. *Consider an exponential $\mathcal{KL}$-function $\zeta(r,s) = e^{-\alpha s}Mr, r,s \ge 0$, with constants $\alpha > 0, M \ge 1$, and suppose that the assumptions of Theorem 3.1 are satisfied for control system (2.1). Assume that for every $\varepsilon > 0$ and for every $x_0 \in \Gamma$ there is a control $u \in \mathcal{U}$ with*

$$(3.9) \qquad d(\varphi(t,x_0,u),\Lambda) < e^{-\alpha t}M(d(x_0,\Lambda) + \varepsilon) \text{ for all } t \ge 0.$$

Then the stabilization entropy satisfies $h_s(\zeta,\Gamma,\Lambda) \le (L_0^s + \alpha)d$.

Proof. This proof follows similar steps as the proof of Theorem 3.1 but is somewhat simpler. Define, for $\varepsilon \ge 0$,

$$R_\varepsilon^s := \{(x_0,u) \in \Gamma \times \mathcal{U} \,|\, d(\varphi(t,x_0,u),\Lambda) < e^{-\alpha t}M(d(x_0,\Lambda) + \varepsilon) \text{ for all } t \ge 0\}.$$

Fix $\varepsilon > 0$, and choose a C^1-function $\theta : \mathbb{R}^d \to [0,1]$ with $\theta(x) = 1$ for all $x \in P_{3\varepsilon}^s$ and support contained in $P_{4\varepsilon}^s$. We define $\tilde{f} : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$ by $\tilde{f}(x,u) := \theta(x)f(x,u)$ and consider the control system

$$\dot{x}(t) = \tilde{f}(x(t),u(t)), \quad u(t) \in U.$$

The solution map ψ associated with this system satisfies for $\tau > 0$,

$$(\psi([0,\tau],x_0,u) \subset P_{3\varepsilon}^s \text{ or } \varphi([0,\tau],x_0,u) \subset P_{3\varepsilon}^s)$$
$$\Rightarrow \psi(t,x_0,u) = \varphi(t,x_0,u) \text{ for all } t \in [0,\tau].$$

Now let $\mathcal{S}^* = \{(y_1, u_1), \ldots, (y_n, u_n)\} \subset R^s_\varepsilon, n = r^*(\tau, \varepsilon)$, be a minimal subset with the property that for every $x_0 \in \Gamma$ there exists $(y_i, u_i) \in \mathcal{S}^*$ with

$$\max_{t \in [0,\tau]} d(\psi(t, x_0, u_i), \psi(t, y_i, u_i)) < \varepsilon e^{-\alpha\tau} \text{ for } t \in [0, \tau].$$

Define

$$\Gamma_i := \left\{ x_0 \in \Gamma \mid \max_{t \in [0,\tau]} d(\psi(t, x_0, u_i), \psi(t, y_i, u_i)) < \varepsilon e^{-\alpha\tau} \right\}, \quad i = 1, \ldots, n = r^*(\tau, \tilde\varepsilon),$$

$$L^s_{4\varepsilon} := \max\left\{ \| \tilde f_x(x, u) \| \,|\, (x, u) \in P_{4\varepsilon} \times U \right\} = \max\left\{ \| \tilde f_x(x, u) \| \,|\, (x, u) \in \mathbb{R}^d \times U \right\}.$$

Consider $x_0 \in \mathbb{R}^d$ with $\|x_0 - y_i\| < e^{-(L^s_{4\varepsilon} + \alpha)\tau}\varepsilon$ for some $i \in \{1, \ldots, n\}$. Then Gronwall's lemma implies instead of (3.7), for $t \in [0, \tau]$,

$$\|\psi(t, x_0, u_i) - \psi(t, y_i, u_i)\| \le \|x_0 - y_i\|e^{L^s_{4\varepsilon}\tau} < e^{-(L^s_{4\varepsilon} + \alpha)\tau}\varepsilon e^{L^s_{4\varepsilon}\tau} = \varepsilon e^{-\alpha\tau}.$$

It follows that $x_0 \in \Gamma_i$, and thus Γ contains the union of the balls $\mathbf{B}(y_i, e^{-(L^s_{4\varepsilon} + \alpha)\tau}\varepsilon)$. Instead of (3.5) we obtain $r_s(\tau, 3\varepsilon, \zeta, \Gamma, \Lambda) \le r^*(\tau, \varepsilon)$, since for a minimal set $\mathcal{S}^*$ as above and $x_0 \in \Gamma$ there are $(y_i, u_i) \in \mathcal{S}^*$ such that for all $t \in [0, \tau]$,

$$\begin{aligned}
d(\psi(t, x_0, u_i), \Lambda) &< d(\psi(t, x_0, u_i), \psi(t, y_i, u_i)) + d(\psi(t, y_i, u_i), \Lambda) \\
&< e^{-\alpha t}\varepsilon + e^{-\alpha t}M(d(y_i, \Lambda) + \varepsilon) \\
&\le e^{-\alpha t}M(\|y_i - x_0\| + d(x_0, \Lambda) + 2\varepsilon) \\
&\le e^{-\alpha t}M(d(x_0, \Lambda) + 3\varepsilon).
\end{aligned}$$

Furthermore $\psi(t, x_0, u_i) \in P^s_{3\varepsilon}$; hence $\psi(t, x_0, u_i) = \varphi(t, x_0, u_i), t \in [0, \tau]$, and the set of controls $\{u_1, \ldots, u_n\}, n = r^*(\tau, \varepsilon)$, is $(\tau, 3\varepsilon, \Gamma, \Lambda)$-spanning. One finds that

$$r_s(\tau, 3\varepsilon, \zeta, \Gamma, \Lambda) \le r^*(\tau, \varepsilon) \le c(\delta, \Gamma) \text{ with } \delta := e^{-(L^s_{4\varepsilon} + \alpha)\tau}\varepsilon$$

and

$$\overline{\lim_{\tau \to \infty}} \frac{1}{\tau} \log r^*(\tau, \varepsilon) \le (L^s_{4\varepsilon} + \alpha) \dim_F(\Gamma) \le (L^s_{4\varepsilon} + \alpha)d.$$

For $\varepsilon \to 0$ the Lipschitz constants $L^s_{4\varepsilon}$ converge to L^s_0, and the assertion follows. $\quad\square$

Remark 3.3. Theorem 3.2 generalizes and improves Colonius [5, Theorem 3.3], where an upper bound for stabilization entropy about an equilibrium is given using a global Lipschitz constant L.

Next we prove lower bounds for the ε-practical stabilization entropy based on a volume growth argument. For general $\mathcal{KL}$-functions the lower bound is given by the divergence $\text{div}_x f(x, u) = \text{tr} f_x(x, u)$ of f with respect to x, while for exponential $\mathcal{KL}$-functions a stronger result involving also the exponential bound holds.

THEOREM 3.4. *Consider for control system* (2.1) *compact sets* $\Gamma, \Lambda \subset \mathbb{R}^d$, *where* Γ *has positive Lebesgue measure, and let* ζ *be a* $\mathcal{KL}$-*function. Suppose that* f *is differentiable with respect to* x *and the partial derivative* $f_x(x, u)$ *is continuous in* (x, u) *with* $\inf_{(x,u) \in A \times U} f_x(x, u) > -\infty$ *for bounded sets* $A \subset \mathbb{R}^d$.

(i) *For $\varepsilon > 0$ the ε-practical stabilization entropy and the practical stabilization entropy satisfy*

$$\infty \geq h_{\mathrm{ps}}(\varepsilon, \zeta, \Gamma, \Lambda) \geq \min_{(x,u) \in \overline{N(\Lambda; \varepsilon)} \times U} \operatorname{div}_x f(x, u),$$

$$\infty \geq h_{\mathrm{ps}}(\zeta, \Gamma, \Lambda) \geq \min_{(x,u) \in \Lambda \times U} \operatorname{div}_x f(x, u).$$

(ii) *Let $\Lambda = \{0\}$ and $\zeta(r, s) := e^{-\alpha s} M r, r, s \geq 0$, with constants $\alpha > 0, M \geq 1$. Then the ε-practical stabilization entropy and the practical stabilization entropy satisfy*

$$\infty \geq h_{\mathrm{ps}}(\varepsilon, \zeta, \Gamma, \{0\}) \geq \alpha d + \min_{\|x\| \leq \varepsilon, u \in U} \operatorname{div}_x f(x, u),$$

$$\infty \geq h_{\mathrm{ps}}(\zeta, \Gamma, \{0\}) \geq \alpha d + \min_{u \in U} \operatorname{div}_x f(0, u).$$

Proof. (i) If $h_{\mathrm{ps}}(\varepsilon, \zeta, \Gamma, \Lambda) = \infty$, the inequalities in (i) are trivially satisfied. Hence we may assume that for $\tau > 0$ there is a finite practically $(\tau, \varepsilon, \zeta, \Gamma, \Lambda)$-spanning set $\mathcal{S} = \{u_1, \ldots, u_n\}$ of controls, and we pick $\mathcal{S}$ with minimal cardinality; hence $n = r_{\mathrm{ps}}(\tau, \varepsilon, \zeta, \Gamma, \Lambda)$. Define, for $i = 1, \ldots, n$,

$$\Gamma_i := \{x_0 \in \Gamma \mid d(\varphi(t, x_0, u_i), \Lambda) < \zeta(d(x_0, \Lambda) + \varepsilon, t) + \varepsilon \text{ for all } t \in [0, \tau]\}.$$

Denote $\kappa := \max_{x \in \Gamma} d(x, \Lambda)$ and $\delta(t) := \zeta(\kappa + \varepsilon, t), t \in [0, \tau]$. Then, for $i = 1, \ldots, n$,

$$(3.10) \qquad \varphi(t, \Gamma_i, u_i) \subset N(\Lambda, \zeta(\kappa + \varepsilon, t) + \varepsilon) = N(\Lambda; \delta(t) + \varepsilon), t \in [0, \tau],$$

implying for the Lebesgue measures

$$(3.11) \qquad \lambda(\varphi(t, \Gamma_i, u_i)) \leq \lambda(N(\Lambda; \delta(t) + \varepsilon)).$$

On the other hand, by the transformation theorem for diffeomorphisms and Liouville's trace formula (cf. Teschl [25, Lemma 3.11]) we get, for $i = 1, \ldots, n$,

$$\lambda(\varphi(\tau, \Gamma_i, u_i)) = \int_{\Gamma_i} \left| \det \frac{\partial \varphi}{\partial x_0}(\tau, x_0, u_i) \right| dx_0 \geq \lambda(\Gamma_i) \cdot \inf_{(x_0, u)} \left| \det \frac{\partial \varphi}{\partial x_0}(\tau, x_0, u) \right|$$

$$(3.12) \qquad = \lambda(\Gamma_i) \cdot \inf_{(x_0, u)} \exp \left(\int_0^\tau \operatorname{div}_x f(\varphi(s, x_0, u), u(s)) ds \right).$$

Here, and in the rest of this proof, $\inf_{(x_0, u)}$ denotes the infimum over all $(x_0, u) \in \Gamma \times \mathcal{U}$ with $\varphi(t, x_0, u) \subset N(\Lambda; \delta(t) + \varepsilon)$ for all $t \in [0, \tau]$. Fix $\tau_0 \in [0, \tau]$. Then (x_0, u) as above satisfies the estimate

$$\int_0^\tau \operatorname{div}_x f(\varphi(s, x_0, u), u(s)) ds$$

$$(3.13) \qquad \geq \int_0^{\tau_0} \operatorname{div}_x f(\varphi(s, x_0, u), u(s)) ds + (\tau - \tau_0) \min_{(y,v)} \operatorname{div}_x f(y, v),$$

where the minimum is taken over all $(y, v) \in \mathbb{R}^d \times U$ with $d(y, \Lambda) \leq \zeta(\kappa + \varepsilon, \tau_0) + \varepsilon = \delta(\tau_0) + \varepsilon$. This holds since the function ζ is decreasing in the second argument, and for all $s \in [\tau_0, \tau]$,

$$d(\varphi(s, x_0, u), \Lambda) < \zeta(\max_{x_0 \in \Gamma} d(x_0, \Lambda) + \varepsilon, s) + \varepsilon \leq \zeta(\kappa + \varepsilon, \tau_0) + \varepsilon.$$

We may assume that $\lambda(\Gamma_1) = \max_{i=1,\dots,n} \lambda(\Gamma_i)$. Inequalities (3.12) and (3.11) imply

$$0 < \lambda(\Gamma) \le \sum_{i=1}^{n} \lambda(\Gamma_1) \le n \cdot \lambda(\Gamma_1) \le n \cdot \frac{\lambda(\varphi(\tau, \Gamma_1, u_1))}{\inf_{(x_0, u)} \exp\left(\int_0^\tau \operatorname{div}_x f(\varphi(s, x_0, u), u(s)) ds\right)}$$

$$\le n \cdot \frac{\lambda(N(\Lambda; \delta(\tau) + \varepsilon))}{\inf_{(x_0, u)} \exp\left(\int_0^\tau \operatorname{div}_x f(\varphi(s, x_0, u), u(s)) ds\right)};$$

hence

$$n = r_{\mathrm{ps}}(\tau, \varepsilon, \zeta, \Gamma, \Lambda) \ge \frac{\lambda(\dot{\Gamma})}{\lambda(N(\Lambda; \delta(\tau) + \varepsilon))} \inf_{(x_0, u)} \exp\left(\int_0^\tau \operatorname{div}_x f(\varphi(s, x_0, u), u(s)) ds\right).$$

Using (3.13) and taking the logarithm on both sides one finds

$$\log r_{\mathrm{ps}}(\tau, \varepsilon, \zeta, \Gamma, \Lambda) \ge \log \lambda(\Gamma) - \log \lambda(N(\Lambda; \delta(\tau) + \varepsilon))$$

$$+ \inf_{(x_0, u)} \int_0^{\tau_0} \operatorname{div}_x f(\varphi(s, x_0, u), u(s)) ds + (\tau - \tau_0) \min_{(y,v)} \operatorname{div}_x f(y, v).$$

This yields the inequality

(3.14)
$$\varlimsup_{\tau \to \infty} \frac{1}{\tau} \log r_{\mathrm{ps}}(\tau, \varepsilon, \zeta, \Gamma, \Lambda)$$

$$\ge \varlimsup_{\tau \to \infty} \left[-\frac{1}{\tau} \log \lambda(N(\Lambda; \delta(\tau) + \varepsilon)) + \frac{\tau - \tau_0}{\tau} \min_{(y,v)} \operatorname{div}_x f(y, v) \right].$$

Since $\delta(\tau) \le \delta(0)$ and $\lambda(N(\Lambda; \varepsilon)) > 0$, we find

$$\varlimsup_{\tau \to \infty} -\frac{1}{\tau} \log \lambda(N(\Lambda; \delta(\tau) + \varepsilon)) = 0.$$

It follows that

$$\varlimsup_{\tau \to \infty} \frac{1}{\tau} \log r_{\mathrm{ps}}(\tau, \varepsilon, \zeta, \Gamma, \Lambda) \ge \min_{(y,v)} \operatorname{div}_x f(y, v).$$

Recall that the minimum is on the set $\{(y, v) \in \mathbb{R}^d \times U \,|\, d(y, \Lambda) \le \delta(\tau_0) + \varepsilon\}$. In the Hausdorff metric, these compact sets converge to $\overline{N(\Lambda; \varepsilon)} \times U$ for $\tau_0 \to \infty$. This proves the first assertion in (i); the second follows by taking the limit for $\varepsilon \to 0$.

(ii) For $\varepsilon > 0$ inequality (3.14) holds. If we employ the maximum-norm in $\mathbb{R}^d$, we obtain for the Lebesgue measure

$$\lambda(N(\{0\}, \delta(\tau) + \varepsilon)) = \lambda(\mathbf{B}(0, \delta(\tau) + \varepsilon)) \le (2\delta(\tau) + 2\varepsilon)^d,$$

and, by the choice of ζ,

$$\lim_{\tau \to \infty} \frac{1}{\tau} \log(\delta(\tau) + \varepsilon) = \lim_{\tau \to \infty} \frac{1}{\tau} \log \left[e^{-\alpha\tau} M(\kappa + \varepsilon) + \varepsilon \right] = -\alpha.$$

Hence (3.14) implies

$$\varlimsup_{\tau \to \infty} \frac{1}{\tau} \log r_{\mathrm{ps}}(\tau, \varepsilon, \zeta, \Gamma, \Lambda) \ge \alpha d + \varlimsup_{\tau \to \infty} \left(\frac{\tau - \tau_0}{\tau} \min_{(y,v)} \operatorname{div}_x f(y, v) \right)$$

$$= \alpha d + \min_{(y,v)} \operatorname{div}_x f(y, v),$$

and the inequalities in assertion (ii) follow as in (i).   □

Remark 3.5. Note that in Theorem 3.4 the assumption $\inf_{(x,u)\in A\times U} f_x(x,u) > -\infty$ for bounded sets A is of interest only if the control range U is unbounded. The lower bounds provided by this theorem are independent of Γ and ζ. They may be negative and then yield no information. In the linear case, this can be improved; cf. Theorem 5.1.

Remark 3.6. Theorem 3.4(ii) improves Colonius [5, Theorem 3.2], where a similar lower bound for the stabilization entropy (which may be greater than the practical stabilization entropy) is proved.

4. Relations to feedbacks. We will prove an upper bound of the ε-stabilization entropy under the assumption that a feedback exists such that the system satisfies an appropriate stability property. This is illustrated in the linear case.

Consider system (2.1), and fix $\varepsilon > 0$ and a $\mathcal{KL}$-function ζ. Let $k_\varepsilon : \mathbb{R}^d \to U$ be a Lipschitz continuous feedback such that the solutions $\psi(t, x_0; k_\varepsilon(\cdot)), t \geq 0$, of

$$(4.1) \qquad x(0) = x_0, \quad \dot{x}(t) = f(x(t), k_\varepsilon(x(t)))$$

are well defined and depend continuously on the initial value. Define the ε-entropy of $k_\varepsilon(\cdot)$ in the following way. Let $\Gamma \subset \mathbb{R}^d$ be compact, and define for every $x_0 \in \Gamma$ a control by

$$(4.2) \qquad u_{x_0}(t) = k_\varepsilon(\psi(t, x_0; k_\varepsilon(\cdot))), \quad t \geq 0.$$

For $\tau > 0$ a set $E = \{y_1, \ldots, y_n\} \subset \Gamma$ is called $(\tau, \varepsilon, \zeta, \Gamma)$-spanning for $k_\varepsilon(\cdot)$ if for all $x_0 \in \Gamma$ there is $j \in \{1, \ldots, n\}$ with

$$\|x_0 - y_j\| < \varepsilon \text{ and } \left\|\varphi(t, x_0, u_{y_j}) - \varphi(t, y_j, u_{y_j})\right\| \leq \zeta(\|x_0 - y_j\| + \varepsilon, t) \text{ for } t \in [0, \tau].$$

DEFINITION 4.1. *For system* (2.1), *a set* $\Gamma \subset \mathbb{R}^d$ *of initial states, and a* $\mathcal{KL}$-*function* ζ *the* ε-*entropy of the feedback* $k_\varepsilon(\cdot)$ *is*

$$h_{\mathrm{fb}}(\varepsilon, \zeta, k_\varepsilon(\cdot), \Gamma) = \varlimsup_{\tau \to \infty} \frac{1}{\tau} \log \min\{\#E \,|\, E \text{ is } (\tau, \varepsilon, \zeta, \Gamma)\text{-spanning for } k_\varepsilon(\cdot)\}.$$

If the feedback $k_\varepsilon(\cdot)$ *is independent of* ε, *we define the entropy of the feedback* $k(\cdot) := k_\varepsilon(\cdot)$ *as*

$$h_{\mathrm{fb}}(\zeta, k(\cdot), \Gamma) = \lim_{\varepsilon \to 0} h_{\mathrm{fb}}(\varepsilon, \zeta, k(\cdot), \Gamma).$$

These notions of entropy are based on the concept that only in the beginning, at time $t = 0$, an estimate of the initial point is used. The control is not corrected at any later time.

The following proposition shows that the ε-stabilization entropy can be bounded above by the ε-entropy of feedbacks for which the system satisfies an ε-stability property.

PROPOSITION 4.2. *Let* $\Gamma, \Lambda \subset \mathbb{R}^d$ *be compact and* $\varepsilon > 0$. *Suppose that there is a feedback* $k_\varepsilon(\cdot)$ *such that for every* $x_0 \in \Gamma$ *the solution* $\psi(t, x_0; k_\varepsilon(\cdot)), t \geq 0$, *of the feedback system* (4.1) *satisfies*

$$(4.3) \qquad d(\psi(t, x_0; k_\varepsilon(\cdot)), \Lambda) \leq \zeta(d(x_0, \Lambda) + \varepsilon, t) \text{ for } t \geq 0.$$

Then the ε-*stabilization entropy is bounded above by the* ε-*entropy of the feedback* $k_\varepsilon(\cdot)$,

$$h_{\mathrm{ps}}(2\varepsilon, 2\zeta, \Gamma, \Lambda) \leq h_{\mathrm{s}}(2\varepsilon, 2\zeta, \Gamma, \Lambda) \leq h_{\mathrm{fb}}(\varepsilon, \zeta, k_\varepsilon(\cdot), \Gamma).$$

Proof. Note that the first inequality trivially holds. Fix $\tau > 0$, and consider a minimal $(\tau, \varepsilon, \zeta, \Gamma)$-spanning set $E = \{y_1, \ldots, y_n\}$ for $k_\varepsilon(\cdot)$. Using (4.2) we associate to every y_i a control function u_{y_i} and claim that the set of control functions defined by

$$\mathcal{S}(E) = \{u_{y_1}, \ldots, u_{y_n}\}$$

is $(\tau, 2\varepsilon, 2\zeta, \Gamma, \Lambda)$-spanning. By assumption (4.3) for the feedback system we know

$$d(\varphi(t, y_j, u_{y_j}), \Lambda) = d(\psi(t, y_j; k_\varepsilon(\cdot)), \Lambda) \leq \zeta\left(d(y_j, \Lambda) + \varepsilon, t\right) \text{ for all } t \geq 0.$$

By the spanning property of E, for all $x_0 \in \Gamma$ there is j such that for all $t \in [0, \tau]$,

$$
\begin{aligned}
(4.4) \qquad d\left(\varphi(t, x_0, u_{y_j}), \Lambda\right) &\leq \left\|\varphi(t, x_0, u_{y_j}) - \varphi(t, y_j, u_{y_j})\right\| + d(\varphi(t, y_j, u_{y_j}), \Lambda) \\
&\leq \zeta(\|x_0 - y_j\| + \varepsilon, t) + d(\psi(t, y_j; k_\varepsilon(\cdot)), \Lambda) \\
&\leq \zeta(2\varepsilon, t) + \zeta\left(d(y_j, \Lambda) + \varepsilon, t\right) \\
&\leq \zeta(2\varepsilon, t) + \zeta\left(\|y_j - x_0\| + d(x_0, \Lambda) + \varepsilon, t\right) \\
&\leq 2\zeta(d(x_0, \Lambda) + 2\varepsilon, t).
\end{aligned}
$$

This shows the claim for $\mathcal{S}(E)$, and it follows that

$$
\begin{aligned}
&\min\{\#\mathcal{S} \,|\, \mathcal{S} \text{ is } (\tau, 2\varepsilon, 2\zeta, \Gamma, \Lambda))\text{-spanning}\} \\
&\leq \min\{\#E \,|\, E \text{ is } (\tau, \varepsilon, \zeta, \Gamma)\text{-spanning for } k_\varepsilon(\cdot)\}.
\end{aligned}
$$

Taking logarithms and the limit for $\tau \to \infty$, one obtains the assertion

$$
\begin{aligned}
h_s(2\varepsilon, 2\zeta, \Gamma, \Lambda) &= \varlimsup_{\tau \to \infty} \frac{1}{\tau} \log \min\{\#\mathcal{S} \,|\, \mathcal{S} \text{ is } (\tau, 2\varepsilon, 2\zeta, \Gamma, \Lambda)\text{-spanning}\} \\
&\leq \varlimsup_{\tau \to \infty} \frac{1}{\tau} \log \min\{\#E \,|\, E \text{ is } (\tau, \varepsilon, \Gamma)\text{-spanning for } k_\varepsilon(\cdot)\} = h_{\text{fb}}(\varepsilon, \zeta, k_\varepsilon(\cdot), \Gamma). \qquad \square
\end{aligned}
$$

Remark 4.3. If one replaces (4.3) by the weaker condition

$$d(\psi(t, x_0; k_\varepsilon(\cdot)), \Lambda) \leq \zeta(d(x_0, \Lambda) + \varepsilon, t) + \varepsilon \text{ for } t \geq 0,$$

one can prove analogously a bound for the ε-practical stabilization entropy,

$$h_{\text{ps}}(2\varepsilon, 2\zeta, \Gamma, \Lambda) \leq h_{\text{fb}}(\varepsilon, \zeta, k_\varepsilon(\cdot), \Gamma).$$

Next we illustrate Proposition 4.2 by considering linear systems of the form

$$(4.5) \qquad\qquad \dot{x}(t) = Ax(t) + Bu(t), \ u \in \mathcal{U},$$

with matrices $A \in \mathbb{R}^{d \times d}$, $B \in \mathbb{R}^{d \times m}$, control range $U = \mathbb{R}^m$, and $\Lambda = \{0\}$. For a linear feedback K the feedback system has the form

$$\dot{x}(t) = (A + BK)x(t), \ u \in \mathcal{U},$$

with solutions $\psi(t, x_0; K) = e^{(A + BK)t}x_0$. Suppose that K is stabilizing such that all eigenvalues λ_j of $A + BK$ satisfy $\operatorname{Re} \lambda_j < -\alpha$ for some $\alpha > 0$. Hence the solutions of the feedback system satisfy, for every initial value $x_0 \in \mathbb{R}^d$ and some constant $M \geq 1$,

$$(4.6) \qquad\qquad \|\psi(t, x_0; K)\| \leq e^{-\alpha t} M \|x_0\| = \zeta(\|x_0\|, t),$$

where $\zeta(r,s) := e^{-\alpha s}Mr, r, s \geq 0$. Thus assumption (4.3) in Proposition 4.2 holds; hence $h_s(2\varepsilon, 2\zeta, \Gamma, \{0\}) \leq h_{fb}(\varepsilon, \zeta, K, \Gamma)$. For a compact subset $\Gamma \subset \mathbb{R}^d$ the ε-entropy of K is determined by the following. Let, for $y_0 \in \mathbb{R}^d$,

$$u_{y_0}(t) = K\psi(t, y_0; K) = Ke^{(A+BK)t}y_0, \quad t \geq 0.$$

Note that, for $x_0, y_0 \in \mathbb{R}^d$,

$$\varphi(t, x_0, u_{y_0}) - \varphi(t, y_0, u_{y_0})$$
$$= e^{At}x_0 + \int_0^t e^{A(t-s)}Bu_{y_0}(s)ds - e^{At}y_0 - \int_0^t e^{A(t-s)}Bu_{y_0}(s)ds$$
$$= e^{At}(x_0 - y_0).$$

For $\varepsilon, \tau > 0$ a set $E = \{y_1, \ldots, y_n\}$ is $(\tau, \varepsilon, \zeta, \Gamma)$-spanning for the feedback K, if for all $x_0 \in \Gamma$ there exists j such that $\|x_0 - y_j\| < \varepsilon$ and, for $t \in [0, \tau]$,

$$(4.7) \qquad \left\|\varphi(t, x_0, u_{y_j}) - \varphi(t, y_j, u_{y_j})\right\| = \left\|e^{At}(x_0 - y_j)\right\| \leq e^{-\alpha t}M(\|x_0 - y_j\| + \varepsilon).$$

A classical result shows that the topological entropy $h_{top}(\Phi_t)$ of a linear flow $\Phi_t = e^{At}, t \in \mathbb{R}$, is given by $h_{top}(\Phi_t) = \sum_{i:\ \text{Re}\,\lambda_i > 0} \text{Re}\,\lambda_i$, where $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of A; cf. Walters [26, Theorem 8.14]. It follows that the topological entropy of the flow $e^{(A+\alpha I)t}, t \in \mathbb{R}$, is $\sum_{i:\ \text{Re}\,\lambda_i > -\alpha} \text{Re}\,\lambda_i$. By the definition of topological entropy of flows, for any compact set $\Gamma \subset \mathbb{R}^d$ a set $F = \{z_1, \ldots z_k\}$ is $(\tau, \varepsilon, \Gamma)$-spanning if for every $x_0 \in \Gamma$ there is z_j such that, for $t \in [0, \tau]$,

$$\left\|e^{(A+\alpha I)t}(x_0 - z_j)\right\| < \varepsilon, \quad \text{hence} \quad \left\|e^{At}(x_0 - z_j)\right\| < e^{-\alpha t}\varepsilon.$$

This shows that F is also $(\tau, \varepsilon, \zeta, \Gamma)$-spanning for the feedback K; cf. (4.7). Using Proposition 4.2 one finds for $\tau \to \infty$ that the ε-stabilization entropy and the ε-entropy of the feedback K satisfy

$$h_s(2\varepsilon, 2\zeta, \Gamma, \{0\}) \leq h_{fb}(\varepsilon, \zeta, K, \Gamma) \leq h_{top}\left(e^{(A+\alpha I)\cdot}\right) = \sum_{i:\ \text{Re}(\lambda_i) > -\alpha} \text{Re}\,\lambda_i.$$

Note that here $2\zeta(r, s) = e^{-\alpha s}2Mr, r, s \geq 0$. Since the right-hand side is independent of ε, it actually follows for $\varepsilon \to 0$ that the practical stabilization entropy and the stabilization entropy satisfy

$$(4.8) \qquad h_{ps}(2\zeta, K, \Gamma, \{0\}) \leq h_s(2\zeta, K, \Gamma, \{0\}) \leq h_{fb}(\zeta, K, \Gamma) \leq \sum_{i:\ \text{Re}(\lambda_i) > -\alpha} \text{Re}\,\lambda_i.$$

In Theorem 5.1 we will show that here equalities hold.

For general nonlinear systems, it seems very hard or impossible to derive explicit formulas for the ε-entropy of a feedback.

Remark 4.4. The papers by Liberzon and Hespanha [19] and De Persis [8] use input-to-state stability properties in order to derive stabilizing encoder/decoder controllers. This condition (Assumption 2 in [19]) requires (in our notation) that there exists a Lipschitz feedback law $u = k(x)$ which satisfies $k(0) = 0$ and renders the closed-loop system input-to-state stable with respect to measurement errors. This means that there exist $\mu \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ (i.e., $\gamma : [0, \infty) \to [0, \infty)$ is continuous, strictly increasing, and unbounded with $\gamma(0) = 0$) such that for every initial

state $x(t_0)$ and every piecewise continuous signal e the corresponding solution of the system $\dot{x} = f(x, k(x + e))$ satisfies

$$\|x(t)\| \leq \mu(\|x(t_0)\|, t - t_0) + \gamma(\sup_{s \in [t_0, t]} \|e(s)\|) \text{ for all } t \geq t_0.$$

The input-to-state stability property is used in order to estimate the effect of perturbations on feedbacks. In Proposition 4.2, we have used instead the entropy property of the feedback $k_\varepsilon(\cdot)$.

5. Applications. In this section we present several examples illustrating practical stabilization properties and estimates for the corresponding entropies. For linear control systems we show that the practical stabilization entropy and the stabilization entropy coincide and they are characterized by a spectral property. This uses inequality (4.8). Then two scalar examples are discussed, where quadratic feedbacks and piecewise linear feedbacks, respectively, only lead to practical stabilization properties. For these examples and a similar higher dimensional system we estimate the ε-practical stabilization entropy using the results from section 3. The three nonlinear systems analyzed here are not locally C^1 stabilizable.

5.1. Linear systems. In this subsection, the practical stabilization entropy is determined for linear control systems in $\mathbb{R}^d$ of the form

$$(5.1) \qquad \dot{x}(t) = Ax(t) + Bu(t), \quad u(t) \in U = \mathbb{R}^m$$

with matrices $A \in \mathbb{R}^{d \times d}$ and $B \in \mathbb{R}^{d \times m}$. The next theorem characterizes the practical stabilization entropy about the equilibrium $x = 0$.

THEOREM 5.1. *Consider a linear control system of the form (5.1). Assume that there are $\alpha > 0, M \geq 1$ such that for all initial values $0 \neq x_0 \in \mathbb{R}^d$ there is a control $u \in \mathcal{U}$ with*

$$(5.2) \qquad \|\varphi(t, x_0, u)\| < e^{-\alpha t} M/2 \|x_0\| \text{ for all } t \geq 0.$$

Then for the exponential $\mathcal{KL}$-function $\zeta(r, s) = e^{-\alpha t} Mr$ the ε-practical stabilization entropy, the practical stabilization entropy, and the stabilization entropy satisfy for every compact subset Γ with nonvoid interior

$$h_{\mathrm{ps}}(\varepsilon, \zeta, \Gamma, \{0\}) = h_{\mathrm{ps}}(\zeta, \Gamma, \{0\}) = h_{\mathrm{s}}(\zeta, \Gamma, \{0\}) = \sum_{\mathrm{Re}\,\lambda_i > -\alpha} (\alpha + \mathrm{Re}\,\lambda_i).$$

Here summation is over all eigenvalues λ_i of A, counted according to their algebraic multiplicity, with $\mathrm{Re}\,\lambda_i > -\alpha$.

Proof. One easily proves that assumption (5.2) holds if and only if there is a feedback K such that all eigenvalues of $A + BK$ satisfy $\mathrm{Re}\,\lambda_j < -\alpha$. Thus condition (4.6) holds, and it follows from (4.8) that

$$(5.3) \quad h_{\mathrm{ps}}(\zeta, K, \Gamma, \{0\}) \leq h_{\mathrm{s}}(\zeta, K, \Gamma, \{0\}) \leq h_{\mathrm{fb}}(\zeta/2, K, \Gamma) \leq \sum_{i:\ \mathrm{Re}(\lambda_i) > -\alpha} \mathrm{Re}\,\lambda_i.$$

For the proof of the converse inequalities note that $f(x, u) = Ax + Bu$ satisfies

$$\mathrm{div}_x f(x, u) = \mathrm{tr} f_x(x, u) = \mathrm{tr} A = \sum_{i=1}^{d} \lambda_i = \sum_{i=1}^{d} \mathrm{Re}\,\lambda_i.$$

Theorem 3.4(ii) can be applied to the system obtained by the projection π to the sum of the real generalized eigenspaces for all eigenvalues with real part larger than $-\alpha$ along the subspace corresponding to the sum of the other generalized eigenspaces. Then the ε-practical stabilization entropy of this projected system is bounded below by

$$\alpha \dim(\pi(\mathbb{R}^d)) + \sum_{\operatorname{Re}\lambda_i > -\alpha} \operatorname{Re}\lambda_i = \sum_{\operatorname{Re}\lambda_i > -\alpha} (\alpha + \operatorname{Re}\lambda_i).$$

The equality follows since the eigenvalues are counted according to their algebraic multiplicity. Since practically $(\tau, \varepsilon, \zeta, \Gamma, \{0\})$-spanning sets for the system in $\mathbb{R}^d$ yield practically $(\tau, \varepsilon, \zeta, \pi(\Gamma), \{0\})$-spanning sets for the projected system and $\pi(\Gamma)$ has nonvoid interior, it follows that $h_{\mathrm{ps}}(\varepsilon, \zeta, \Gamma, \{0\}) \geq \sum_{i:\operatorname{Re}\lambda_i > -\alpha}(\alpha + \operatorname{Re}\lambda_i)$. Together with (5.3) the assertion follows. $\qquad \square$

Remark 5.2. The characterization of stabilization entropy in Theorem 5.1 has already been proved in Colonius [5, Lemma 4.1 and Theorem 4.2]. The proof above is a considerable simplification.

5.2. A scalar example with quadratic feedback. In this subsection we discuss a scalar example, where only practical stabilization properties can be used. Our strategy is to construct quadratic feedbacks such that the closed-loop systems has, in addition to the unstable equilibrium at the origin, a stable equilibrium arbitrarily close to the origin. Thus for every $\varepsilon > 0$ the feedback systems are ε-practically stable in the sense of Definition 2.1, and we use the estimates for entropy from section 3.

Consider the scalar control system given by

$$(5.4) \qquad \dot{x} = f(x, u) = \lambda x + \alpha_0 x^2 + \beta_0 x u + \gamma_0 u^2,$$

where $\lambda > 0$ and $\alpha_0, \beta_0, \gamma_0$ with $\gamma_0 \neq 0$ are real parameters and the controls take values $u(t) \in U \subset \mathbb{R}$.

For system (5.4) the origin $x = 0$ is an equilibrium corresponding to $u = 0$ if $0 \in U$. For $x = 0$ the right-hand side of (5.4) is given by $f(0, u) = \gamma_0 u^2$. For $\gamma_0 > 0$ one has $f(0, u) > 0$ for all $0 \neq u \in \mathbb{R}$, and for $\gamma_0 < 0$ one has $f(0, u) < 0$ for all $0 \neq u \in \mathbb{R}$. Hence the system is not controllable around the origin. By Brockett's necessary condition (cf. Sontag [24, Theorem 22]) it is not locally C^1 stabilizable. Hence for $\gamma_0 > 0$ stabilization can only be expected for initial values in $(-\infty, 0)$ and for $\gamma_0 < 0$ for initial values in $(0, \infty)$.

For quadratic feedbacks of the form

$$(5.5) \qquad k_{\mathrm{quad}}(x) = kx + xqx \text{ with } k, q \in \mathbb{R},$$

the closed-loop system is

$$
\begin{aligned}
(5.6) \qquad \dot{x} &= \lambda x + \alpha_0 x^2 + \beta_0 x(kx + qx^2) + \gamma_0(kx + qx^2)^2 \\
&= \lambda x + (\alpha_0 + \beta_0 k + \gamma_0 k^2)x^2 + q(\beta_0 + 2\gamma_0 k)x^3 + \gamma_0 q^2 x^4.
\end{aligned}
$$

We denote the solutions of this equation by $\psi(t, x_0; k, q)$ on their existence intervals. The following theorem shows that with quadratic feedback (5.5) system (5.4) can be made ε-practically stable with exponential rate $\alpha \in (0, 3\lambda)$, where the constant M in $\zeta(r, s) = e^{-\alpha s} M r$ depends on ε (and Γ); cf. Definition 2.1.

THEOREM 5.3. *Consider system* (5.4) *with quadratic feedback* (5.5) *and* $\Lambda = \{0\}$. *Fix* $\varepsilon > 0$, *and let the control range be either* $U_\varepsilon^+ = [0, \rho(\varepsilon)]$ *or* $U_\varepsilon^- = [-\rho(\varepsilon), 0]$ *with*

$\rho(\varepsilon)$ *large enough. If* $\gamma_0 < 0$ *consider initial values in a compact set* $\Gamma = \Gamma^+ \subset (0, \infty)$; *if* $\gamma_0 > 0$ *consider a compact set* $\Gamma = \Gamma^- \subset (-\infty, 0)$.

Then for every $\alpha \in (0, 3\lambda)$ *there are* $k, q \in \mathbb{R}$ *such that for* $\zeta_\varepsilon(r, s) = e^{-\alpha s} M(\varepsilon) r$, $r, s \geq 0$, *with* $M(\varepsilon) \geq 1$, *the closed-loop system* (5.6) *is* ε-*practically* $(\zeta_\varepsilon, \Gamma, \{0\})$-*stable*.

The proof of Theorem 5.3 is given in the appendix.

Next we estimate the ε-practical stabilization entropy. The control ranges will vary; hence we add this argument in the notation for the entropy. In Theorem 5.4(ii) we employ the modified notion for noncompact control ranges in Remark 2.7.

THEOREM 5.4. *Consider system* (5.4) *with quadratic feedback* (5.5), *and let the assumptions of Theorem 5.3 be satisfied.*

(i) *For* $\varepsilon > 0$ *let the exponential* $\mathcal{KL}$-*function* ζ_ε *be given by Theorem 5.3. Then the* ε-*practical stabilization entropy satisfies*

$$h_{\mathrm{ps}}(2\varepsilon, \zeta_\varepsilon, \Gamma, \{0\}, U_\varepsilon^\pm) \leq \max\left\{|f_x(x, u)| \,\big|\, (x, u) \in P_\varepsilon \times U_\varepsilon^\pm\right\} < \infty,$$

where $\Gamma = \Gamma^+$ *for* $\gamma_0 < 0$ *and* $\Gamma = \Gamma^-$ *for* $\gamma_0 > 0$, *and*

$$f_x(x, u) = \lambda + 2\alpha_0 x + \beta_0 u \text{ and } P_\varepsilon = \left\{x \in \Gamma \,\big|\, \|x\| \leq M \max_{y \in \Gamma} |y| + \varepsilon(M + 1)\right\}.$$

(ii) *Suppose that the Lebesgue measure of* Γ *as in* (i) *is positive, and either* $\beta_0 > 0$ *and the control range is* $U = U^+ = [0, \infty)$, *or* $\beta_0 < 0$ *and the control range is* $U = U^- = (-\infty, 0]$. *Assume* $\mathrm{sign}(\gamma_0) = -\mathrm{sign}(\beta_0)$. *Then for every* $\alpha \in (0, 3\lambda)$ *and* $\zeta(r, s) = e^{-\alpha s} M r$ *with* $M \geq 1$, *the* ε-*practical stabilization entropy and the practical stabilization entropy satisfy*

$$\infty \geq h_{\mathrm{ps}}(\varepsilon, \zeta, \Gamma, \{0\}, U) \geq \alpha + \lambda - 3|\alpha_0|\varepsilon \text{ and } \infty \geq h_{\mathrm{ps}}(\zeta, \Gamma, \{0\}, U) \geq \alpha + \lambda.$$

For ζ_ε *given by Theorem 5.3, one has* $h_{\mathrm{ps}}(\varepsilon, \zeta_\varepsilon, \Gamma, \{0\}, U) < \infty$.

Proof. (i) Fix $\alpha \in (0, 3\lambda)$. Theorem 5.3 shows that for every $x_0 \in \Gamma$ there is a control $u(t) = k_{\mathrm{quad}}(\psi(t, x_0; k, q)), t \geq 0$, with values in $U_\varepsilon^\pm$ such that

$$d(\varphi(t, x_0, u), \{0\}) < \zeta_\varepsilon(d(x_0, \{0\}), t) + \varepsilon \text{ for all } t \geq 0.$$

Then Theorem 3.1(i) yields the upper bound

$$h_{\mathrm{ps}}(2\varepsilon, \zeta_\varepsilon, \Gamma, \{0\}, U_\varepsilon^\pm) \leq L_\varepsilon = \max\left\{|f_x(x, u)| \,\big|\, (x, u) \in P_\varepsilon \times U_\varepsilon^\pm\right\},$$

where $f_x(x, u)$ and P_ε are as stated in the assertion. This proves (i).

(ii) Recall the modified notion of ε-practical stabilization entropy for noncompact control ranges from Remark 2.7:

$$h_{\mathrm{ps}}(\varepsilon, \zeta, \Gamma, \{0\}, U) = \inf_K h_{\mathrm{ps}}(\varepsilon, \zeta, \Gamma^\pm, \{0\}, U \cap K),$$

where the infimum is taken over all compact subset $K \subset \mathbb{R}^m$. Let K_0 be compact with

$$\inf_K h_{\mathrm{ps}}(\varepsilon, \zeta, \Gamma, \{0\}, U \cap K) \geq h_{\mathrm{ps}}(\varepsilon, \zeta, \Gamma, \{0\}, U \cap K_0) - |\alpha_0|\varepsilon$$

$$\geq h_{\mathrm{ps}}(\varepsilon, \zeta, \Gamma, \{0\}, U_\varepsilon^\pm) - |\alpha_0|\varepsilon$$

if $\rho(\varepsilon) > 0$ is large enough such that $U \cap K_0 \subset U_\varepsilon^+$ if $\beta_0 > 0$ and $U \cap K_0 \subset U_\varepsilon^-$ if $\beta_0 < 0$. By Theorem 3.4(ii) we obtain

$$\begin{aligned} h_{\text{ps}}(\varepsilon, \zeta, \Gamma, \{0\}, U) &\geq \alpha + \min_{|x| \leq \varepsilon, u \in U_\varepsilon^\pm} f_x(x, u) - |\alpha_0| \varepsilon \\ &= \alpha + \lambda + \min_{|x| \leq \varepsilon, u \in U_\varepsilon^\pm} (2\alpha_0 x + \beta_0 u) - |\alpha_0| \varepsilon \\ &\geq \alpha + \lambda - 3|\alpha_0|\varepsilon + \min_{u \in U_\varepsilon^\pm} (\beta_0 u). \end{aligned}$$

For $\gamma_0 > 0, \beta_0 < 0$ the control range is $U_\varepsilon^- = [-\rho(\varepsilon), 0]$, and we get $\min_{u \in U_\varepsilon^-}(\beta_0 u) = 0$. For $\gamma_0 < 0, \beta_0 > 0$ the control range is $U_\varepsilon^+ = [0, \rho(\varepsilon)]$, and we get $\min_{u \in U_\varepsilon^+}(\beta_0 u) = \min_{u \in [0, \rho(\varepsilon)]}(\beta_0 u) = 0$. This proves the lower bound on the ε-practical stabilization entropy. The assertion for the practical stabilization entropy follows for $\varepsilon \to 0$. The final assertion follows from (i) by choosing $M(\varepsilon)$ from Theorem 5.3 and noting that $h_{\text{ps}}(\varepsilon, \zeta_\varepsilon, \Gamma, \{0\}, U^\pm) \leq h_{\text{ps}}(\varepsilon, \zeta_\varepsilon, \Gamma, \{0\}, U_\varepsilon^\pm)$. This completes the proof of assertion (ii). □

Remark 5.5. Observe that in Theorem 5.4(i) the upper bound L_ε converges to ∞ for $\varepsilon \to 0$ if $\rho(\varepsilon) \to \infty$ and $\beta_0 \neq 0$.

5.3. A scalar example with piecewise linear feedback. Consider the scalar system given by

$$(5.7) \quad \dot{x} = f(x, u) = \lambda x + \alpha_0 x^2 + \beta_0 x u + \gamma_0 u^2 + \alpha_1 x^3 + \beta_1 x^2 u + \gamma_1 x u^2 + \eta_1 u^3,$$

where $\lambda > 0, \alpha_0, \beta_0, \gamma_0, \alpha_1, \beta_1, \gamma_1$, and η_1 are real parameters with $\gamma_0, \eta_1 \neq 0$, and the controls take values $u(t) \in U \subset \mathbb{R}$.

We follow an approach in Hamzi and Krener [10] to construct piecewise linear feedbacks such that the closed-loop systems have, in addition to the unstable equilibrium at the origin, two stable equilibria arbitrarily close to the origin. Then we evaluate the bounds for entropy from section 3.

The origin $x = 0$ is an equilibrium corresponding to $u = 0$. For $x = 0$ the right-hand side of (5.7) is given by $f(0, u) = \gamma_0 u^2 + \eta_1 u^3$. If the control range is $U = [-\rho, \rho] \subset \mathbb{R}$ with $\rho > 0$ large enough, there are control values $u_1, u_2 \in U$ with $f(0, u_1) > 0$ and $f(0, u_2) < 0$; hence the system is controllable around the origin. It is not locally C^1 stabilizable, since the linearized system has the uncontrollable unstable eigenvalue $\lambda > 0$; cf. Sontag [24, Corollary 5.8.8].

First we will show that system (5.7) is practically stabilizable about the origin using a piecewise linear feedback with $k_1, k_2 \in \mathbb{R}$ of the form

$$(5.8) \qquad\qquad k(x) = \begin{cases} k_1 x & \text{for } x \geq 0, \\ k_2 x & \text{for } x \leq 0. \end{cases}$$

We denote the solutions of the feedback system by $\psi(t, x_0; k_1, k_2), t \geq 0$.

THEOREM 5.6. *Consider system (5.7) with piecewise linear feedback (5.8); let $\Gamma \subset \mathbb{R}$ be a compact set of initial values and $\Lambda = \{0\}$. For all $\varepsilon > 0$ and $\alpha > 0$ there is $M = M(\varepsilon, \alpha) \geq 1$ such that the $\mathcal{KL}$-function $\zeta_\varepsilon(r, s) = e^{-\alpha s} M r, r, s \geq 0$, satisfies the following property. If the control range is $U_\varepsilon = [-\rho(\varepsilon), \rho(\varepsilon)]$ with $\rho(\varepsilon)$ large enough, then there are $k_1, k_2 \in \mathbb{R}$ such that the feedback system is ε-practically $(\zeta, \Gamma, \{0\})$-stable.*

The proof of Theorem 5.6 is given in the appendix.

Next we estimate the ε-practical stabilization entropy. In Theorem 5.7(ii) we employ the modified notion for noncompact control ranges in Remark 2.7.

THEOREM 5.7. *Consider system (5.7) with piecewise linear feedback (5.8), and let the assumptions of Theorem 5.6 be satisfied.*

(i) *For $\varepsilon > 0$ let the $\mathcal{KL}$-function ζ_ε be given by Theorem 5.6. Then the ε-practical stabilization entropy satisfies*

$$h_{\mathrm{ps}}(2\varepsilon, \zeta_\varepsilon, \Gamma, \{0\}, U_\varepsilon) \leq L_\varepsilon := \max\left\{ |f_x(x,u)| \, |(x,u) \in P_\varepsilon \times U_\varepsilon \right\} < \infty,$$

where

$$f_x(x,u) = \lambda + 2\alpha_0 x + \beta_0 u + 3\alpha_1 x^2 + 2\beta_1 x u + 2\gamma_1 u^2,$$

$$P_\varepsilon = \left\{ x \in \Gamma \left| |x| \leq M \max_{y \in \Gamma} |y| + \varepsilon(M+1) \right. \right\}.$$

(ii) *Suppose that the Lebesgue measure of Γ is positive and $\gamma_1 > 0, \beta_0 \neq 0$, and the control range is $U = \mathbb{R}$. Then for a $\mathcal{KL}$-function $\zeta = e^{-\alpha s} M r, r, s \geq 0$, with $\alpha > 0$ and $M \geq 1$ the ε-practical stabilization entropy and the practical stabilization entropy satisfy*

$$\infty \geq h_{\mathrm{ps}}(\varepsilon, \zeta, \Gamma, \{0\}, U) \geq \alpha + \lambda - 3|\alpha_0|\varepsilon - 3|\alpha_1|\varepsilon^2 - \frac{1}{4\gamma_1}(\beta_0 + 2\mathrm{sign}(\beta_0)|\beta_1|\varepsilon)^2,$$

$$\infty \geq h_{\mathrm{ps}}(\zeta, \Gamma, \{0\}, U) \geq \alpha + \lambda - \frac{\beta_0^2}{4\gamma_1}.$$

For ζ_ε given by Theorem 5.6 one has $h_{\mathrm{ps}}(\varepsilon, \zeta_\varepsilon, \Gamma, \{0\}, U) < \infty$.

Proof. (i) By Theorem 5.6 for every $x_0 \in \Gamma$ the control $u(t) = u(\psi(t, x_0; k_1, k_2))$, $t \geq 0$, with values in U_ε yields

$$d(\varphi(t, x_0, u), \Lambda) < \zeta(d(x_0, \Lambda) + \varepsilon, t) + \varepsilon \text{ for all } t \geq 0.$$

Then Theorem 3.1(i) yields the upper bound

$$h_{\mathrm{ps}}(2\varepsilon, \zeta, \Gamma, \{0\}, U_\varepsilon) \leq L_\varepsilon := \max\left\{ |f_x(x,u)| \, |(x,u) \in P_\varepsilon \times U_\varepsilon \right\},$$

where $f_x(x,u)$ and P_ε are as stated in the assertion. This proves (i).

(ii) As in the proof of Theorem 5.4 we use that for $\rho(\varepsilon)$ large enough

$$h_{\mathrm{ps}}(\varepsilon, \zeta, \Gamma, \Lambda, U) = \inf_K h_{\mathrm{ps}}(\varepsilon, \zeta, \Gamma, \Lambda, U \cap K) \geq h_{\mathrm{ps}}(\varepsilon, \zeta, \Gamma, \Lambda, U_\varepsilon) - |\alpha_0|\varepsilon.$$

Using the lower estimate provided by Theorem 3.4(ii) we obtain

$$h_{\mathrm{ps}}(\varepsilon, \zeta, \Gamma, \{0\}, U)$$
$$\geq \alpha + \min_{|x| \leq \varepsilon, u \in U_\varepsilon} f_x(x,u) - |\alpha_0|\varepsilon$$
$$= \alpha + \lambda + \min_{|x| \leq \varepsilon, u \in U_\varepsilon} \left\{ 2\alpha_0 x + 3\alpha_1 x^2 + (\beta_0 + 2\beta_1 x)u + 2\gamma_1 u^2 \right\} - |\alpha_0|\varepsilon$$
$$\geq \alpha + \lambda + \min_{|x| \leq \varepsilon}\{2\alpha_0 x + 3\alpha_1 x^2\} + \min_{|x| \leq \varepsilon}\min_{u \in U_\varepsilon}\left\{ (\beta_0 + 2\beta_1 x)u + 2\gamma_1 u^2 \right\} - |\alpha_0|\varepsilon.$$

Clearly, $\min_{|x| \leq \varepsilon}\{2\alpha_0 x + 3\alpha_1 x^2\} - |\alpha_0|\varepsilon \geq -3|\alpha_0|\varepsilon - 3|\alpha_1|\varepsilon^2$, and for the parabola $(\beta_0 + 2\beta_1 x)u + \gamma_1 u^2, u \in \mathbb{R}$, with $\gamma_1 > 0$, the minimum is attained in $u = -\frac{\beta_0 + 2\beta_1 x}{2\gamma_1}$. Hence for $\varepsilon > 0$,

$$\min_{|x|\le\varepsilon}\min_{u\in U_\varepsilon}\left\{(\beta_0+2\beta_1 x)u+2\gamma_1 u^2\right\}=\min_{|x|\le\varepsilon}\left(-\frac{(\beta_0+2\beta_1 x)^2}{2\gamma_1}+\frac{(\beta_0+2\beta_1 x)^2}{4\gamma_1}\right)$$

$$=-\frac{1}{4\gamma_1}\max_{|x|\le\varepsilon}(\beta_0+2\beta_1 x)^2$$

$$\ge-\frac{1}{4\gamma_1}(\beta_0+2\mathrm{sign}(\beta_0)\,|\beta_1|\,\varepsilon)^2.$$

Together this yields the lower estimate for $h_{\mathrm{ps}}(\varepsilon,\zeta,\Gamma,\{0\},U)$. The estimate for $h_{\mathrm{ps}}(\zeta,\Gamma,\{0\},U)$ follows by taking the limit for $\varepsilon\to 0$. The final assertion is a consequence of (i). This completes the proof of assertion (ii). □

5.4. A higher dimensional example. The following system is a generalization of the system in subsection 5.2 by connecting it with a chain of integrators. This system occurs in a quadratic normal form; cf. Krener, Kang, and Chang [18, Theorem 2.1]. We will rely on a practical stabilization result due to Hamzi and Krener [10]. Consider the control system in $\mathbb{R}^d$ given by

$$\dot{x}_1=\lambda x_1+\alpha_0 x_1^2+\beta_0 x_1 x_2+\sum_{j=2}^{d}\gamma_j x_j^2,\quad \dot{x}_2=x_3,\dots,\dot{x}_d=u,$$

where $\lambda>0$ and $\alpha_0,\beta_0,\gamma_2$ with $\gamma_2\ne 0$ are real parameters and the controls take values $u(t)\in U\subset\mathbb{R}$. Let

$$A_2=\begin{bmatrix}0&1&0&\cdots&0\\0&0&1&\cdots&0\\\vdots&&&\ddots&\vdots\\0&0&0&\cdots&1\\0&0&0&\cdots&0\end{bmatrix},\quad B_2=\begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix},$$

and abbreviate $z=(x_2,\dots,x_d)^\mathsf{T}$. Then we may write the system as

(5.9)
$$\dot{x}_1=\lambda x_1+\alpha_0 x_1^2+\beta_0 x_1 x_2+\sum_{j=2}^{d}\gamma_j x_j^2,$$
$$\dot{z}=A_2 z+B_2 u.$$

The system is not locally C^1 stabilizable since the linearized system has the uncontrollable unstable eigenvalue $\lambda>0$; cf. Sontag [24, Corollary 5.8.8]. We use linear feedback of the form

(5.10)
$$k(x_1,z)=k_1 x_1+K_2 z$$

with $k_1\in\mathbb{R}$ and choose $K_2\in\mathbb{R}^{1\times(d-1)}$ such that $A_2+B_2 K_2$ is stable. The feedback system becomes

(5.11)
$$\dot{x}_1=\lambda x_1+\alpha_0 x_1^2+\beta_0 x_1 x_2+\sum_{j=2}^{d}\gamma_j x_j^2,$$
$$\dot{z}=(A_2+B_2 K_2)z+B_2 k_1 x_1.$$

The following theorem shows a practical stabilizability result.

THEOREM 5.8. *Consider system (5.9) with linear feedback (5.10) and $\Lambda=\{0\}$. Suppose that $\lambda>0$ is sufficiently small. Fix $\varepsilon>0$, and let $\rho(\varepsilon)>0$ be large enough.*

If $\gamma_2 < 0$, let the control range be $U_\varepsilon^+ = [0, \rho(\varepsilon)]$, and consider initial values in a compact set $\Gamma = \Gamma^+ \subset (0, \infty)$. If $\gamma_2 > 0$, let the control range be $U_\varepsilon^- = [-\rho(\varepsilon), 0]$, and consider initial values in a compact set $\Gamma = \Gamma^- \subset (-\infty, 0)$.

Then there is a $\mathcal{KL}$-function ζ_ε such that the closed-loop system (5.6) is ε-practically $(\zeta_\varepsilon, \Gamma, \{0\})$-stable.

Proof. This follows from Hamzi and Krener [10, proof of Theorem 3, pages 44–47]. Here a center manifold reduction is used to show the following. If $\gamma_2 < 0$ there is $k_1 < 0$ such that the feedback system (5.11) has an equilibrium e^+ with positive first component and domain of attraction including Γ^+. If $\gamma_2 > 0$ there is $k_1 > 0$ such that the feedback system (5.11) has an equilibrium e^- with negative first component and domain of attraction including Γ^- and corresponding $\mathcal{KL}$-function ζ_ε. Choosing $|k_1|$ large enough, these equilibria are arbitrarily close to the origin This implies the assertion.   □

THEOREM 5.9. *Consider system (5.9) with linear feedback (5.10), and let the assumptions of Theorem 5.8 be satisfied. For every $\varepsilon > 0$ define*

$$L_\varepsilon := \max\left(1, \max_{x \in P_\varepsilon}\left\{\lambda + 2|\alpha_0|\, x_1 + |\beta_0|\, x_2 + 2\sum_{j=2}^{d}|\gamma_j|\, x_j\right\}\right),$$

where $P_\varepsilon := \left\{x \in \mathbb{R}^d\, \big|\, \|x\| \le \zeta_\varepsilon(\max_{y \in \Gamma}\|y\| + \varepsilon, 0) + \varepsilon\right\}$. *Then for every $\varepsilon > 0$ the ε-practical stabilization entropy satisfies*

$$\lambda - (2|\alpha_0| + |\beta_0|)\varepsilon \le h_{\mathrm{ps}}(\varepsilon, \zeta_\varepsilon, \Gamma, \{0\}) \le L_{\varepsilon/2}\, d.$$

Proof. Denoting the right-hand side of (5.9) by $f(x, u)$ one finds

$$f_x(x, u) = \begin{bmatrix} \lambda + 2\alpha_0 x_1 + \beta_0 x_2 & \beta_0 x_1 + 2\gamma_2 x_2 & \cdots & 2\gamma_d x_d \\ 0 & & & A_2 \end{bmatrix}.$$

Using the max-norm in $\mathbb{R}^d$ and $\mathrm{tr}A_2 = 0$, Theorem 3.4(i) yields the lower bound

$$\begin{aligned} h_{\mathrm{ps}}(\varepsilon, \zeta_\varepsilon, \Gamma, \{0\}) &\ge \min\{\mathrm{tr}f_x(x, u)\,|\,(x, u) \in \mathbf{B}(0, \varepsilon) \times U_\varepsilon\} \\ &= \min\{\lambda + 2\alpha_0 x_1 + \beta_0 x_2 + \mathrm{tr}A_2\,|\,x \in \mathbf{B}(0, \varepsilon)\} \\ &= \lambda - 2|\alpha_0|\,\varepsilon - |\beta_0|\,\varepsilon. \end{aligned}$$

Theorem 3.1(i) yields the upper bound $L_{\varepsilon/2}\, d$ with $L_{\varepsilon/2} := \max_{(x,u) \in P_{\varepsilon/2} \times U}\|f_x(x, u)\| < \infty$. Using the matrix norm induced by the max-norm in $\mathbb{R}^d$ we get

$$\|f_x(x, u)\| = \max\left\{|\lambda + 2\alpha_0 x_1 + \beta_0 x_2| + \sum_{j=2}^{d}|2\gamma_j x_j|, 1\right\};$$

hence

$$\begin{aligned} \max\left\{\|f_x(x, u)\|\,\big|\,(x, u) \in P_{\frac{\varepsilon}{2}} \times U_{\frac{\varepsilon}{2}}\right\} \\ = \max_{x \in P_{\frac{\varepsilon}{2}}}\left\{\lambda + 2|\alpha_0|\, x_1 + |\beta_0|\, x_2 + 2\sum_{j=2}^{d}|\gamma_j|\, x_j\right\}. \end{aligned}$$ □

6. Conclusions and open questions. In section 3, we have derived upper and lower bounds for ε-practical stabilization entropy and practical stabilization entropy (i.e., in the limit for $\varepsilon \to 0$) based on general $\mathcal{KL}$-functions ζ, with special attention to exponential $\mathcal{KL}$-functions. Section 4 presents an upper bound for ε-practical stabilization entropy based on an ε-entropy notion for feedbacks. In section 5 this is used for linear control systems in order to prove that the practical stabilization entropy and the stabilization entropy coincide provided that the system is stabilizable and to characterize them by a spectral condition. Furthermore, two scalar examples are analyzed where quadratic feedbacks and piecewise linear feedbacks, respectively, only lead to ε-practical stabilization for every $\varepsilon > 0$. Here and for a similar higher dimensional system the employed exponential $\mathcal{KL}$-functions depend on ε, and the upper bounds diverge for $\varepsilon \to 0$.

Major research problems include the following: Suppose that the considered control system is ε-practically stabilizable for every $\varepsilon > 0$ but not stabilizable (either by appropriate feedbacks or in the sense of (3.1), where open-loop controls are considered). Will the corresponding ε-practical stabilization entropies diverge for $\varepsilon \to 0$? It is also not clear to us when there exist $\mathcal{KL}$-functions which work for every $\varepsilon > 0$. Furthermore, suppose that the system is stabilizable. Is there a gap between the practical stabilization entropy and the stabilization entropy? In the linear case, Theorem 5.1 shows that both entropy notions coincide. The answer will be of interest for control devices which only lead to practical stability but not to stability. Furthermore, the relations of practical stabilization entropy to minimal data rates for digital communication channels merit exploration.

Our results do not yield formulas for practical stabilization entropy. In the well studied case of invariance entropy, only for hyperbolic control systems such strong results are available; cf. Kawan and Da Silva [16]. In this context Kawan [13] shows a lower bound for stabilization in terms of topological pressure under a uniform hyperbolicity assumption. See also Kawan [14] for a general discussion of hyperbolicity in the context of control systems. However, hyperbolicity conditions are not directly applicable in our framework, since it is not local (with respect to Λ).

7. Appendix. In this appendix we prove Theorem 5.3 and Theorem 5.6.

Proof of Theorem 5.3. Equilibria different from the trivial equilibrium $x = 0$ satisfy

$$(7.1) \qquad 0 = \lambda + (\alpha_0 + \beta_0 k + \gamma_0 k^2)x + q(\beta_0 + 2\gamma_0 k)x^2 + \gamma_0 q^2 x^3.$$

We choose the constant k in order to eliminate the quadratic term,

$$(7.2) \qquad \beta_0 + 2\gamma_0 k = 0, \text{ i.e., } k = -\frac{\beta_0}{2\gamma_0}.$$

Then it is immediately clear that the properties of the feedback system (5.6) do not depend on the sign of q. Furthermore, one finds

$$(7.3) \qquad \alpha_0 + \beta_0 k + \gamma_0 k^2 = \alpha_0 - \frac{\beta_0^2}{2\gamma_0} + \frac{\beta_0^2}{4\gamma_0} = \frac{4\alpha_0\gamma_0 - \beta_0^2}{4\gamma_0};$$

hence the equilibria are determined by

$$(7.4) \qquad x^3 + \frac{4\alpha_0\gamma_0 - \beta_0^2}{4\gamma_0^2 q^2}x + \frac{\lambda}{\gamma_0 q^2} = 0.$$

The solutions of this cubic equation in reduced form $x^3 + 3ax + b = 0$ are given by the classical Cardano formula; cf., e.g., Zwillinger [29, subsection 2.3.2]. For (7.4) one has $a = \frac{4\alpha_0\gamma_0 - \beta_0^2}{12\gamma_0^2 q^2}$ and $b = \frac{\lambda}{\gamma_0 q^2}$. If the discriminant

$$D := 4a^3 + b^2 = \frac{4}{q^6}\left(\frac{4\alpha_0\gamma_0 - \beta_0^2}{12\gamma_0^2}\right)^3 + \frac{1}{q^4}\frac{\lambda^2}{\gamma_0^2} > 0,$$

there is a unique real real solution, hence a unique nontrivial equilibrium, given by

$$e(q) = \left(-\frac{b}{2} + \frac{1}{2}\sqrt{D}\right)^{\frac{1}{3}} + \left(-\frac{b}{2} - \frac{1}{2}\sqrt{D}\right)^{\frac{1}{3}} = \left(-\frac{\lambda}{2\gamma_0 q^2} + \frac{1}{2}\sqrt{D}\right)^{\frac{1}{3}} + \left(-\frac{\lambda}{2\gamma_0 q^2} - \frac{1}{2}\sqrt{D}\right)^{\frac{1}{3}}.$$

The condition $D > 0$ holds for $|q|$ large enough. The dominant term for $|q| \to \infty$ in D is $\frac{1}{q^4}\frac{\lambda^2}{\gamma_0^2}$, and hence the dominant term in $e(q)$ is

$$(7.5) \qquad \left(-\frac{\lambda}{2\gamma_0 q^2} + \frac{1}{2}\frac{\lambda}{q^2\gamma_0}\right)^{1/3} + \left(-\frac{\lambda}{2\gamma_0 q^2} - \frac{1}{2}\frac{\lambda}{q^2\gamma_0}\right)^{1/3} = -\frac{1}{q^{2/3}}\frac{\lambda^{1/3}}{\gamma_0^{1/3}}.$$

Thus for $|q| \to \infty$ one has $e(q) \to 0$ with $|q|^{-2/3}$. Let $|q|$ be large enough. Then for $\gamma_0 > 0$ the initial values are taken in $\Gamma \subset (-\infty, e(q))$ and $e(q) < 0$. For $\gamma_0 < 0$, the initial values are taken in $\Gamma \subset (e(q), \infty)$ and $e(q) > 0$.

Next we analyze stability of the equilibria. Since the equilibrium in the origin is unstable, general properties of scalar autonomous differential equations imply that $e(q)$ is asymptotically stable with domain of attraction given by $(-\infty, 0)$ if $e(q) < 0$ and $(0, \infty)$ if $e(q) > 0$. In order to prove exponential stability, we compute the Jacobian of (5.6) with (7.2) and (7.3):

$$J(x) = \frac{\partial}{\partial x}\left[\lambda x + (\alpha_0 + \beta_0 k + \gamma_0 k^2)x^2 + q(\beta_0 + 2\gamma_0 k)x^3 + \gamma_0 q^2 x^4\right]$$

$$= \lambda + \frac{4\alpha_0\gamma_0 - \beta_0^2}{2\gamma_0}x + 4\gamma_0 q^2 x^3.$$

For $x = e(q)$ we obtain by (7.5) that for $|q| \to \infty$ the dominant term in the Jacobian $J(e(q))$ is

$$\lambda + \frac{4\alpha_0\gamma_0 - \beta_0^2}{2\gamma_0}\left(-\frac{1}{q^{2/3}}\frac{\lambda^{1/3}}{\gamma_0^{1/3}}\right) - 4\gamma_0 q^2 \frac{1}{q^2}\frac{\lambda}{\gamma_0} = \lambda - \frac{1}{q^{2/3}}\frac{4\alpha_0\gamma_0 - \beta_0^2}{2\gamma_0}\frac{\lambda^{1/3}}{\gamma_0^{1/3}} - 4\lambda \to -3\lambda.$$

Hence for $|q|$ large enough, the equilibrium $e(q)$ is locally exponentially stable with $\zeta(r, s) = e^{-\alpha s}r$ for $0 < \alpha < 3\lambda$.

Claim. There is $M = M(\varepsilon) \geq 1$ such that for every $\alpha \in (0, 3\lambda)$ and every $x_0 \in \Gamma$ the following exponential estimate holds:

$$(7.6) \qquad |\psi(t, x_0; k, q) - e(q)| \leq e^{-\alpha t}M|x_0 - e(q)| \text{ for } t \geq 0.$$

For the proof we first consider the case $e(q) > 0$. Then one can choose $z = z(q) \in \mathbb{R}$ in the domain of exponential attraction such that $e(q) < z$; hence

$$\psi(t, z; k, q) - e(q) \leq e^{-\alpha t}(z - e(q)) \text{ for } t \geq 0.$$

For every $x_0 \in \Gamma$ there is $T_{x_0} > 0$ with $\psi(T_{x_0}, x_0; k, q) = z$; hence, for $t \geq 0$,

$$(7.7) \quad \psi(t + T_{x_0}, x_0; k, q) - e(q) = \psi(t, \psi(T_{x_0}, x_0; k, q); k, q) - e(q) \leq e^{-\alpha t} (z - e(q)).$$

By compactness of Γ it follows that $T := \max_{x_0 \in \Gamma} T_{x_0} < \infty$, and hence with $M = e^{3\lambda T}$ it follows, for $x_0 \in \Gamma$, that

$$\psi(t, x_0; k, q) - e(q) \leq z - e(q) \leq e^{-\alpha T} M \max_{y \in \Gamma} \{y - e(q)\} \text{ for } t \in [0, T],$$

and for $t > T_{x_0}$ this yields together with (7.7) and (7.6),

$$\begin{aligned} \psi(t, x_0; k, q) - e(q) &= \psi(t - T_{x_0}, \psi(T_{x_0}, x_0; k, q); k, q) - e(q) \leq e^{-\alpha(t - T_{x_0})} (z - e(q)) \\ &\leq e^{-\alpha(t - T_{x_0})} e^{-\alpha T_{x_0}} M (x_0 - e(q)) = e^{-\alpha t} M (x_0 - e(q)). \end{aligned}$$

Note that T_{x_0} and T depend on q, since the point $z = z(q)$ is taken in the domain of exponential attraction of $e(q)$ and hence depends on q. This entails that M depends on ε, since $|q|$ is taken large enough in dependence on ε.

Analogously, one argues for $e(q) < 0$. Thus the claim is proved.

Choosing $|q|$ large enough, it follows that $e(q) \in (-\frac{\varepsilon}{2M}, \frac{\varepsilon}{2M})$; hence

$$\begin{aligned} |\psi(t, x_0; k, q)| &\leq |\psi(t, x_0; k, q) - e(q)| + |e(q)| \leq e^{-\alpha t} M |x_0 - e(q)| + |e(q)| \\ &\leq e^{-\alpha t} M |x_0| + \varepsilon. \end{aligned} \qquad \square$$

Thus the system is ε-practically $(\zeta_\varepsilon, \Gamma, \{0\})$-stable with $\zeta_\varepsilon(r, s) = e^{-\alpha s} M(\varepsilon) r$. Our assumptions on the control range $U_\varepsilon^\pm$ guarantee that the values of the quadratic feedback (5.5) can be taken in $U_\varepsilon^\pm$ for $x_0 \in \Gamma$.

Proof of Theorem 5.6. We will show that for every $\varepsilon > 0$ there are $k_1, k_2 \in \mathbb{R}$ such that, besides the trivial equilibrium at the origin, the feedback system has two equilibria in $(-\varepsilon, \varepsilon)$ which are locally exponentially stable.

Let $i \in \{1, 2\}$. Any nontrivial equilibrium x must satisfy

$$(7.8) \qquad 0 = \lambda + [\alpha_0 + \beta_0 k_i + \gamma_0 k_i^2] x + [\alpha_1 + \beta_1 k_i + \gamma_1 k_i^2 + \eta_1 k_i^3] x^2.$$

Abbreviate

$$\Delta_{0,i}(k_i) = \alpha_0 + \beta_0 k_i + \gamma_0 k_i^2 \text{ and } \Delta_{1,i}(k_i) = \alpha_1 + \beta_1 k_i + \gamma_1 k_i^2 + \eta_1 k_i^3.$$

Then the solutions of (7.8) are

$$(7.9) \qquad x_i^\pm(k_i) = \frac{-\Delta_{0,i}(k_i) \pm \sqrt{\Delta_{0,i}^2(k_i) - 4\lambda \Delta_{1,i}(k_i)}}{2\Delta_{1,i}(k_i)}.$$

Claim. The feedback system with $|k_i|$ large enough and $\operatorname{sign}(k_i) = -\operatorname{sign}(\eta_1)$ has three equilibria given by

$$e_2(k_2) := x_2^+(k_2) < 0 < e_1(k_1) := x_1^-(k_1).$$

For the proof of the claim observe first that $x_i^\pm(k_i) \in \mathbb{R}$ if

$$(7.10) \qquad \Delta_{0,i}^2(k_i) - 4\lambda \Delta_{1,i}(k_i) > 0,$$

and $x_1^{\pm}(k_1)$ is an equilibrium if and only if it is positive, and $x_2^{\pm}(k_2)$ is an equilibrium if and only if it is negative. For k_i as in the claim it follows that $\Delta_{1,i}(k_i) < 0$. Hence (7.10) holds, and $x_i^{\pm}(k_i) \in \mathbb{R}$. For $|k_i| \to \infty$ it follows that $\Delta_{1,i}(k_i) \to -\infty$ with $|k_i|^3$.

We distinguish the following two cases.

- Let $\gamma_0 > 0$. For $|k_i| \to \infty$ it follows that $\Delta_{0,i}(k_i) \to \infty$ with $|k_i|^2$; hence $x_i^{\pm}(k_i) \to 0$ with $|k_i|^{-1}$, and $x_i^-(k_i) > 0$ and $x_i^+(k_i) < 0$ for $|k_i|$ large enough.

- Let $\gamma_0 < 0$. For $|k_i| \to \infty$ it follows that $\Delta_{0,i}(k_i) \to -\infty$ with $|k_i|^2$; hence $x_i^{\pm}(k_i) \to 0$ with $|k_i|^{-1}$. Again, $x_i^-(k_i) > 0$ and $x_i^+(k_i) < 0$ for $|k_i|$ large enough.

Thus the claim is proved. Note that for $|k_i|$ large enough, the equilibria $e_i(k_i)$ are arbitrarily close to 0.

Next we analyze the stability properties of the equilibria. Since the equilibrium in the origin is unstable, it follows from general properties of scalar autonomous differential equations that $e_1(k_1)$ and $e_2(k_2)$ are asymptotically stable with domains of attraction $(0, \infty)$ and $(-\infty, 0)$, respectively. In order to prove exponential stability, we compute the Jacobian in a nontrivial equilibrium of the feedback system using the product rule and (7.8):

$$
\begin{aligned}
J(x) &= \frac{\partial}{\partial x}\left[x(\lambda + \alpha_0 x + \beta_0 k_i x + \gamma_0 k_i^2 x + \alpha_1 x^2 + \beta_1 x^2 k_i + \gamma_1 x k_i^2 x + \eta_1 k_i^3 x^2\right] \\
&= x\left[\alpha_0 + \beta_0 k_i + \gamma_0 k_i^2 + 2(\alpha_1 + \beta_1 k_i + \gamma_1 k_i^2 + \eta_1 k_i^3)x\right] \\
&= x\left[\Delta_{0,i}(k_i) + 2\Delta_{1,i}(k_i)x\right].
\end{aligned}
$$

For $x = x_1^-(k_1) = e_1(k_1) > 0$ one finds, by (7.9),

$$
\begin{aligned}
J(e_1(k_1)) &= e_1(k_1)\left[\Delta_{0,1}(k_1) + 2\Delta_{1,1}(k_1)e_1(k_1)\right] \\
&= -e_1(k_1)\sqrt{\Delta_{0,1}(k_1)^2 - 4\lambda\Delta_{1,1}(k_1)} < 0,
\end{aligned}
$$

and for $x = x_2^+(k_2) = e_2(k_2) < 0$ one obtains

$$
J(e_2(k_2)) = e_2(k_2)\sqrt{\Delta_{0,2}(k_2)^2 - 4\lambda\Delta_{1,2}(k_2)} < 0.
$$

Concluding, we have found constants k_1, k_2 with $|k_{1,2}|$ large enough such that there are, besides the trivial equilibrium at the origin, the two equilibria $e_2(k_2) < 0 < e_1(k_1)$ which are locally exponentially stable with domain of asymptotic attraction $(-\infty, 0)$ and $(0, \infty)$, respectively. Note that for $|k_i| \to \infty$ one has that $J(e_i(k_i)) \to -\infty$ with $|k_i|$. This follows since $e_i(k_i) \to 0$ with $|k_i|^{-1}$ and $\sqrt{\Delta_{0,i}(k_i)^2 - 4\lambda\Delta_{1,2}(k_i)} \to \infty$ with $|k_i|^2$. Hence the exponential rate α can be chosen arbitrarily large for $|k_i|$ large enough.

Similarly as in the proof of Theorem 5.3 one finds for $\alpha > 0$ a constant $M = M(\varepsilon, \alpha) \geq 1$ such that the solutions $\psi(t, x_0; k_1, k_2)$ satisfy for $x_0 \in \Gamma \cap (0, \infty]$ with $i = 1$ and for $x_0 \in \Gamma \cap (-\infty, 0)$ with $i = 2$,

$$
|\psi(t, x_0; k_1, k_2) - e_i(k_i)| \leq e^{-\alpha t}M|x_0|, \quad t \geq 0.
$$

With $\zeta_\varepsilon(r, s) = e^{-\alpha s}M(\varepsilon, \alpha)r$ one shows as in the proof of Theorem 5.3 that the system is ε-practically $(\zeta_\varepsilon, \Gamma, \{0\})$-stable for $|k_{1,2}|$ large enough. The required control values $k_i\psi(t, x_0, k_1, k_2), x_0 \in \Gamma$, are in the control range if $\rho(\varepsilon)$ is large enough. □

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