The complete cartwheel tiling

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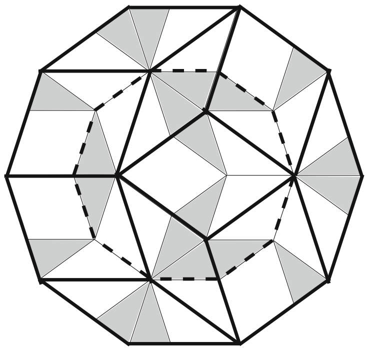
1. Introduction

In 1974, Penrose [10, 11] introduced a class of tilings of Euclidean plane which was based on the geometry of the regular pentagon. These tilings are aperiodic: They are not invariant under any translation. Yet there is another strong order, some self similarity with respect to a certain decomposition of the tiles ("inflation"). De Bruijn [2] gave a different description of Penrose tilings as orthogonal projections of subsets ("strips") of the regular 5-grid $\mathbb{Z}^5 \subset \mathbb{R}^5$ onto a certain 2-plane $E \subset \mathbb{R}^5$ ("cut and project method" or "projection method"). Such tilings will be called "projection type". De Bruijn has also used an equivalent construction, the "pentagrid method" (see Sect. 7).

The *complete cartwheel tiling* is a Penrose tiling of the entire plane introduced first by Gardner [5], see also [6]. It has no exact pentagonal symmetry but in the large it looks like a cartwheel with decagonal symmetry. It can be easily generated from a finite tiling which is self-similar under inflation: A decagon occurring frequently in every Penrose tiling.

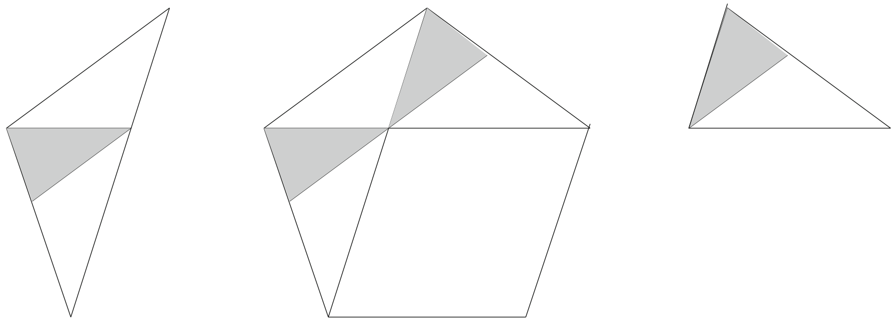


The dotted inner domain with its subdivision is (after a 180° turn) homothetic to the whole decagon with its subdivision by the fat lines. Subdividing again and again and enlarging the figure in every step by the golden ratio, we obtain a tiling of the whole plane which we call the *complete cartwheel tiling*. Our main theorem (Sect. 3) states that this is not a projection tiling.
We start with an elementary geometric definition of Penrose tilings which extend to the whole plane. It is not sufficient to prescribe tiles and matching rules stating which pairs of oriented edges may be adjacent to each other, such as in [10,14]. We show that there are traps: Finite tilings which obey all matching rules and yet cannot be extended over certain edges. In Sect. 3 we describe the complete cartwheel tiling and show in Sect. 6 that it cannot be obtained from the projection method. Before, in Sects. 4 and 5, we recall the geometry of projection tilings.

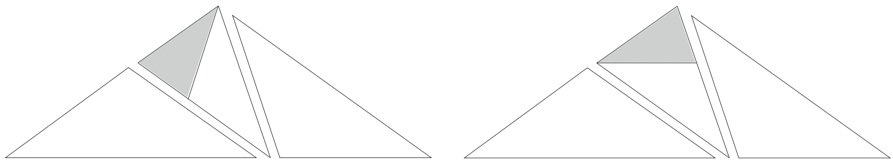
It is our pleasure to thank Rui Pacheco for hints and discussion.

2. Penrose tilings of the full plane

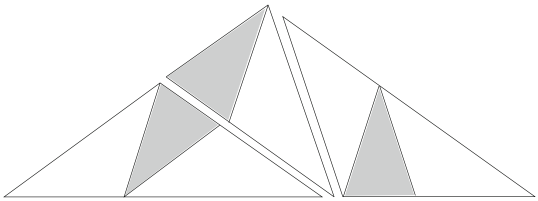
To construct all Penrose tilings which extend to the full plane, we may start with a regular pentagon with two diagonals enclosing one side. Draw a line segment parallel to the enclosed side through the intersection point of the diagonals. This bounds two narrow triangles, which are colored as in the subsequent figure. Now the two diagonals cut off two isosceles triangles, a broad one and a narrow one which come with a subdivision by similar triangles, scaled down by the inverse golden ratio $1/\tau$ with $\tau = \frac{1}{2}(1+\sqrt{5})$.



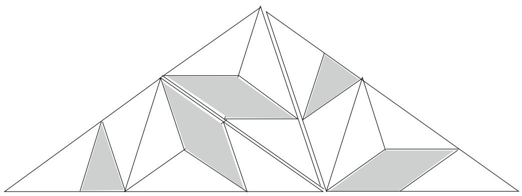
Since the narrow triangle is a subset of the broad one, we only consider the broad triangle with its subdivision. This is our initial (finite) Penrose tiling T_o, where the tiles are the small triangles.
We may subdivide the small triangles in the same way as before. There are always two mirror symmetric ways to do this. But insisting that along each edge the subdivisions from both sides agree, we get uniqueness. In fact, the narrow tile in the middle may have one of the following two subdivisions:



However, only the left subdivision can be extended over the right edge of the narrow tile since both acute angles of the broad tile are uncolored. Hence the subdivisions of the neighboring broad tiles are already fixed by the subdivision of the edge adjacent to the narrow tile:

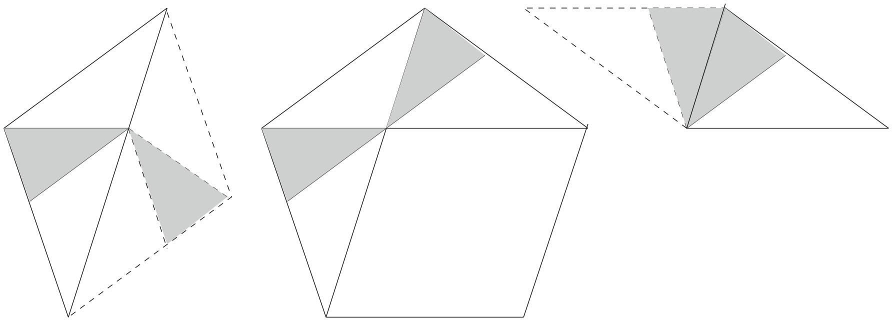


Call T_{n+1} the tiling which arises from T_n by subdividing all tiles and enlarging by the factor τ; then the tiles in T_{n+1} have the same size as the tiles in T_n. The tiling T_1 is in the figure above, and T_2 looks as follows.

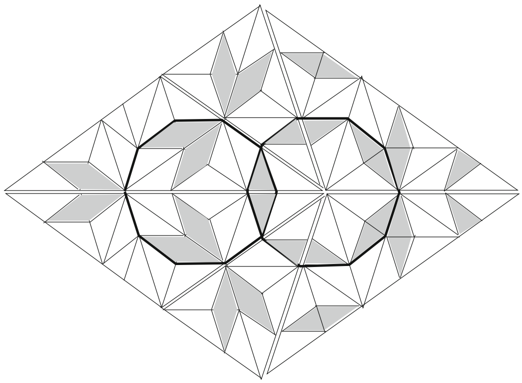


Definition. An *extensible Penrose tiling* is a tiling composed of the two isosceles triangles (broad and narrow) in the regular pentagon, in such a way that any finite subset of tiles is isometric to a subset of some T_n.

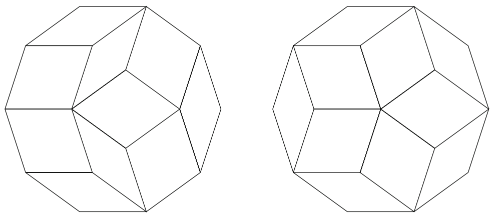
Since the third edge of any of the two isosceles triangles can be adjacent only to the third edge of a mirror symmetric copy of itself, the triangles in the interior of a Penrose tiling always compose to rhombs.



Using this doubling, our third Penrose tiling T_3 looks as follows.

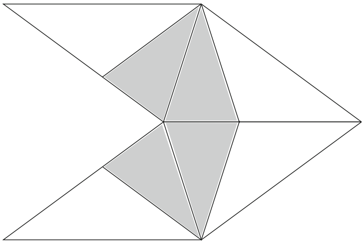


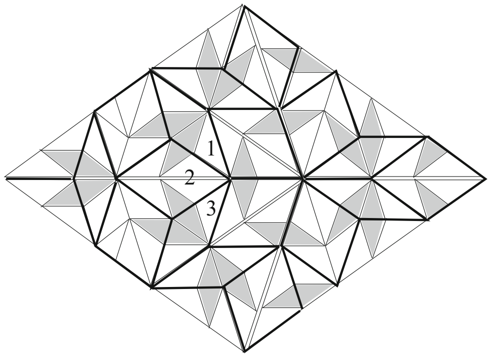
We have marked two decagons in the tiling where the right one is pentagon-symmetric while the left one is only reflection symmetric.



Repeatedly subdividing the right decagon which we call *symmetric decagon*, we construct the two Penrose tilings with full decagonal symmetry. The other decagon, on the left, will be called *cartwheel decagon*. It turns out that the interior part of its first subdivision is similar to the original tiling, see Fig. 1 below. Thus considering the union of all iterated subdivisions, each one scaled by τ and reflected at the vertical axis, we obtain a tiling of the full plane as explained in the next section. We will show that this tiling cannot be obtained by the projection method.

We still need to explain why matching rules never give sufficient conditions for unlimited extensibility. The following figure shows the simplest situation which is build legally in terms of matching rules but cannot be extended.





The right figure which is the subdivision of T_2 (fat lines) shows why the left figure is legal with respect to any matching rules: The broad tile 2 is symmetric and bounded by the narrow tiles 1 and 3, hence 1 and 3 could be also adjacent to each other as shown in the left figure. But the left figure cannot be extended over the left boundary since the colored vertices of the two tiles do not fit into the gap. Of course, the same problem may arise after an arbitrary number of subdivisions of the two tiles.

3. The cartwheel tiling C

Subdivision of the cartwheel decagon gives the pattern shown in Fig. 1 below. It is remarkable that a copy of the original cartwheel decagon also occurs in the center of the subdivided pattern. Therefore, by successively subdividing and rescaling we get a tiling of the entire plane that is invariant under subdivision. In fact, we start with the cartwheel decagon C_1, subdivide, reflect at the vertical axis through the center and rescale by the golden ratio τ; then the

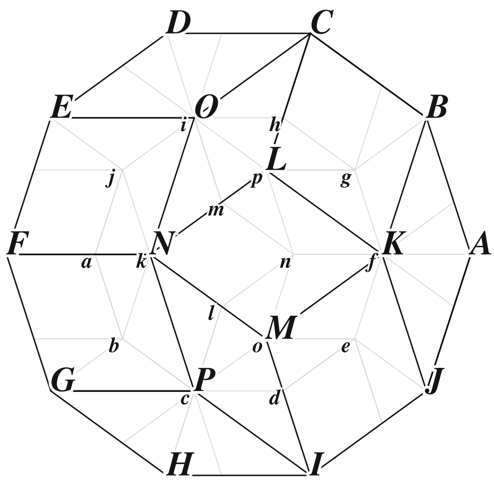


FIGURE 1 Subdivision of the cartwheel decagon

points k, l, m, n in Fig. 1 are mapped onto K, L, M, N. Thus we obtain a new tiling C_2 of a τ-times larger decagon which contains C_1. It is clear that we can repeat this process arbitrarily: Given C_k, we obtain C_{k+1} by subdividing, reflecting and enlarging by τ, and $C_{k+1} \supset C_k$. The *complete cartwheel tiling* is the union of all C_k,

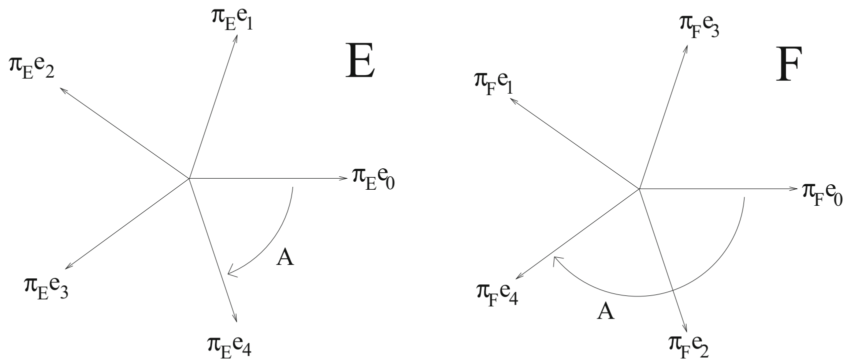
$$C = \bigcup_{k \in \mathbb{N}} C_k,$$

which certainly satisfies our definition since any finite subset lies in some C_k which in turn is a subset of some T_n; recall $C_1 \subset T_3$.

Theorem. *The complete cartwheel tiling C is not of projection type.*
The rest of this article is devoted to the proof of this theorem.

4. Projection tilings

The tilings produced by the projection method [2] arise as follows. We consider the cyclic permutation $A = (12345)$ as an orthogonal matrix permuting the 5 coordinates of $\mathbb{R}^5$. It decomposes $\mathbb{R}^5$ orthogonally as $\mathbb{R}^5 = \mathbb{R}\mathbf{d} + E + F$ where $\mathbf{d} = (1, 1, 1, 1, 1)^T$ is a fixed vector of A and E, F are two invariant planes on which A acts by rotations of 72 and 144 degrees, respectively.



Let $\mathbf{a} \in \mathbb{R}^5$ such that $a := \langle \mathbf{a}, \mathbf{d} \rangle$ is an integer and such that no point of $E + \mathbf{a}$ has more than 2 integer coordinates; this property of $\mathbf{a}$ is called *general position*. Then the E-projection of the set

$$\Sigma_{\mathbf{a}} = \big((0, 1)^5 + E + \mathbf{a}\big) \cap \mathbb{Z}^5$$

is the vertex set of a tiling $T_{\mathbf{a}}$ on E, and this is a Penrose tiling in the sense of our definition [2,3,13]. The elements of $\Sigma_{\mathbf{a}} \subset \mathbb{Z}^5$ are called *admissible* for the tiling $T_{\mathbf{a}}$.

Remark. By using an open hypercube, our definition seems to differ from for example Baake and Grimm who use a closed hypercube, see [1] page 278. However, that difference is irrelevant since the cut and project method require that the $E + \mathbf{a}$ must be in general position or what [1] calls generic.

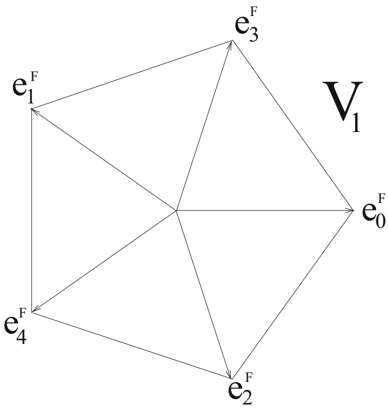
This condition can be transformed into planar geometry. Note first that any grid point $\mathbf{x} \in \mathbb{Z}^5$ lies on one of the hyperplanes

$$H_k = \{\mathbf{x} \in \mathbb{R}^5 : \langle \mathbf{x}, \mathbf{d} \rangle = k\}$$

for some $k \in \mathbb{Z}$. The admissibility of a point $\mathbf{x} \in \mathbb{Z}^5 \cap H_k$ is decided by the so called *window* [2]

$$V_k = \pi_F \left((0,1)^5 \cap H_k \right), \tag{1}$$

which is nonempty only for $k \in \{1, 2, 3, 4\}$; the subsequent figure shows V_1.



More precisely, for any point $\mathbf{x} \in \mathbb{Z}^n \cap H_k$ and for $a = \langle \mathbf{a}, \mathbf{d} \rangle$ we have

$$\mathbf{x} \in \Sigma_{\mathbf{a}} \iff \pi_F(\mathbf{x}) \in \pi_F(\mathbf{a}) + V_{k-a}\,. \tag{2}$$

In fact, note that $\mathbf{x} \in \Sigma_{\mathbf{a}} \iff \mathbf{x} \in \left((0,1)^5 + E + \mathbf{a}\right) \cap H_k \iff$

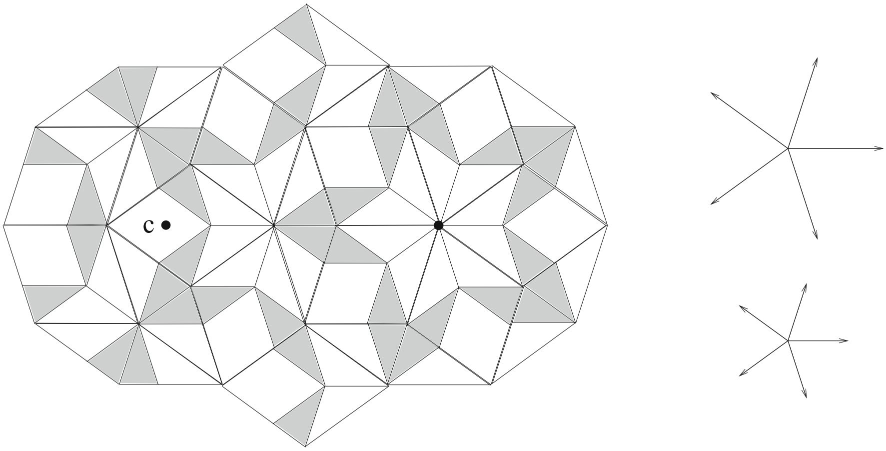
$$\pi_F(\mathbf{x}) \in \pi_F\left(\left((0,1)^5 + \mathbf{a}\right) \cap H_k\right) = \pi_F(\mathbf{a}) + V_{k-a},$$

since $\mathbf{x} = \mathbf{x}^o + \mathbf{a} \in H_k \iff \mathbf{x}^o \in H_k - \mathbf{a} = H_{k-a}$. For any $\mathbf{x} \in \Sigma_{\mathbf{a}}$ we denote

$$\operatorname{ind} \mathbf{x} := \langle \mathbf{x} - \mathbf{a}, \mathbf{d} \rangle = \langle \mathbf{x}, \mathbf{d} \rangle - a$$

the *index* of $\mathbf{x}$ and $\pi_E(\mathbf{x})$. In particular, when $\mathbf{a} = \frac{a}{5}\mathbf{d}$ for $a \in \{1, 2, 3, 4\}$, the tiling has the full pentagonal symmetry. Then $\mathbf{d} \in \Sigma_{\frac{a}{5}\mathbf{d}}$, and the symmetry center $\pi_E(\mathbf{d}) \in T_{\frac{a}{5}\mathbf{d}}$ has index $5 - a$. Note that $-I$ maps the cases $a = 1, 3$ isometric onto $a = 4, 2$, respectively. Figure 2 below shows the case $\mathbf{a} = \frac{1}{5}\mathbf{d}$ where the symmetry center has index 4.

We have marked a cartwheel decagon in this tiling. Its center is located at $\pi_E(\mathbf{c})$ with $\mathbf{c} = \mathbf{e}_2 + \mathbf{e}_3 - \mathbf{e}_0$; its mirror image $\pi_E(-\mathbf{e}_2 - \mathbf{e}_3 + \mathbf{e}_0)$ is a vertex of the tiling. The next figure shows the subdivision which forms the other tiling with full pentagonal symmetry.



On the right margin we see the vectors $\pi_E(\mathbf{e}_i)$. Advancing by each of these vectors increases the index by one. Since the index can move only between 1 and 4, the symmetry center in the right decagon must have index 4 for the coarse tiling (long arrows) and index 2 for the fine tiling (short arrows). We see again the small cartwheel decagon in the fine tiling inside the large one in the coarse tiling. However, after blowing up the subdivision by the factor τ, the center of the cartwheel decagon is moved to the right. Therefore we change the origin to $\pi_E(\mathbf{c})$ by translating $T_{\mathbf{d}/5}$ to $T_{-\mathbf{c}+\mathbf{d}/5} = T_{\mathbf{d}/5} - \pi_E(\mathbf{c})$.

5. Subdivision of projection tilings

The subdivision of a projection tiling can be obtained from a linear map S on $\mathbb{R}^5$ with eigenspaces E, F and $\mathbb{R}\mathbf{d}$ which contracts on E and expands on F,

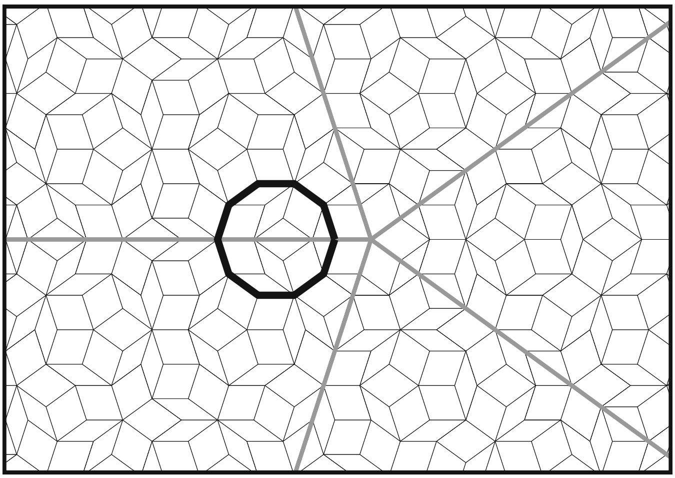
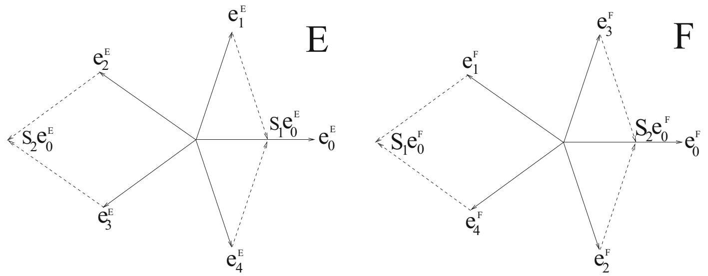


FIGURE 2 A cartwheel decagon located in $T_{\frac{1}{5}\mathbf{d}}$

thus broadening the strip and making more integer grid points admissible, see [4] for details. We use the A-invariant linear maps

$$S_k : \mathbf{e}_j \mapsto \mathbf{e}_{j+k} + \mathbf{e}_{j-k}$$

for $k = 1, 2$ where the indices are computed modulo 5. We have $S_k\mathbf{d} = 2\mathbf{d}$, and E, F are eigenspaces for S_1, S_2 with eigenvalues $1/\tau, -\tau$ on E and $-\tau, 1/\tau$ on F, respectively, as can be read from the following figure:



The maps $S = -S_1$ and $R = S_2$ are inverse to each other modulo $\mathbf{d}$ since $SR(\mathbf{e}_0) = S(\mathbf{e}_3 + \mathbf{e}_2) = -(\mathbf{e}_2 + \mathbf{e}_4 + \mathbf{e}_1 + \mathbf{e}_3) = -\mathbf{d} + \mathbf{e}_0$, and similar $SR(\mathbf{e}_j) = -\mathbf{d}+\mathbf{e}_j$. Moreover, each integer grid point $\mathbf{x} \in \mathbb{Z}^5$ lies on one of the hyperplanes $H_k = \{\mathbf{x} \in \mathbb{R}^5 : \langle \mathbf{x}, \mathbf{d}\rangle = k\}$ for some $k \in \mathbb{Z}$, and S maps $H_k \cap \mathbb{Z}^5$ bijectively onto $H_{-2k} \cap \mathbb{Z}^5$. Since S expands on F by the factor τ, we have $S(V_k) \supset V_{k'}$ for $k' \equiv -2k$ mod 5 (see (1)) and hence $S(\Sigma_{\mathbf{a}} \cap H_k) \supset \Sigma_{S\mathbf{a}} \cap H_{k'}$ and

$$\pi_E S(\Sigma_{\mathbf{a}} \cap H_k) \supset \pi_E(\Sigma_{S\mathbf{a}} \cap H_{k'})\,. \tag{3}$$

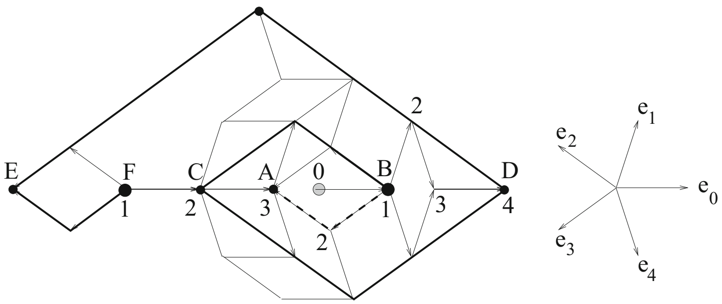
This shows that the vertex set of the tiling $S(T_{\mathbf{a}})$ (which is $T_{\mathbf{a}}$, scaled down by the factor $-1/\tau$) contains the vertex set of the tiling $T_{S\mathbf{a}}$; in fact $S(T_{\mathbf{a}})$ is the first subdivision of $T_{S\mathbf{a}}$.
In particular, $S(\frac{1}{5}\mathbf{d}) = -\frac{2}{5}\mathbf{d} \equiv \frac{3}{5}\mathbf{d}$ mod $\mathbf{d}$, and $S(\frac{3}{5}\mathbf{d}) = -\frac{6}{5}\mathbf{d} \equiv \frac{4}{5}\mathbf{d}$ mod $\mathbf{d}$, hence S maps each of the two symmetric tilings $T_{\mathbf{d}/5}$ and $T_{3\mathbf{d}/5}$ onto the first subdivision of the other, up to sign. The same is true for the translated tilings $P = T_{-\mathbf{c}+\mathbf{d}/5}$ and $Q = T_{-\mathbf{c}+3\mathbf{d}/3}$: If we let $P_0 = P$ and $Q_0 = Q$ and define recursively P_{k+1}, Q_{k+1} as the first subdivision of P_k, Q_k, respectively, then

$$P_{k+1} = S(Q_k), \quad Q_{k+1} = -S(P_k). \tag{4}$$

We will also consider the inverse map $R = S_2$, which expands on E by the factor $-\tau$ and maps P_{k+1} onto Q_k and Q_{k+1} onto $-P_k$. In particular, the cartwheel decagon $C_k \subset P_k$ (centered at the origin) is mapped onto the enlarged cartwheel decagon $C_{k+1} \supset C_k$.

6. The complete cartwheel tiling does not fit into the window

In Fig. 3, a path inside the third subdivision C_3 of the cartwheel ball $C_0 \subset P_0$ is marked. Both of its end point B, F have index 1, as the following figure shows.



We obtain the broken line DE by applying three times the mapping R to the small broken line AB, and we reach the final point F by adding $-(\mathbf{e}_2+\mathbf{e}_3)$. Hence we may describe the transition from the initial point $B=\pi_E(\mathbf{e}_0)$ to the final point F by the affine map $T(\mathbf{x})=R^3\mathbf{x}+\mathbf{b}$ with $\mathbf{b}=-(\mathbf{e}_2+\mathbf{e}_3)-\mathbf{d}$ (The meaning of the additional term $-\mathbf{d}$ which projects to 0 on E will become clear below). Then we have $F=T(B)$. We iterate this process by putting $x_0=\mathbf{e}_0$ and $\mathbf{x}_{j+1}=T(\mathbf{x}_j)$, then $\pi_E(\mathbf{x}_0)=B$ and $\pi_E(\mathbf{x}_1)=F$. In every step the path is prolongated, see Fig. 4 for $\mathbf{x}_2=T^2\mathbf{x}_0=R^6\mathbf{e}_0+R^3\mathbf{b}+\mathbf{b}$. Note that all points $\pi_E\mathbf{x}_j$ lie on the horizontal line and have index 1.
Now we want to show that the F-projection of the path $\mathbf{x}_0,\mathbf{x}_1,\mathbf{x}_2,\ldots$ does not fit into the window V_1 since its length $\lim_{j\to\infty}|\pi_F(\mathbf{x}_j)-\pi_F(\mathbf{x}_0)|$ is precisely the diameter of V_1 along the horizontal axis, $L=|\frac{1}{2}(\mathbf{e}_1^F+\mathbf{e}_4^F)-\mathbf{e}_0^F|$ (which is $(1+\frac{1}{2}\tau)|\mathbf{e}_0^F|$), see the figure after (1). If C were a projection tiling, $\pi_F(\mathbf{x}_0)$ would have to lie somewhere in the open window V_1, thus for some $\epsilon>0$, every point $\pi_F(\mathbf{x}_j)$ must have distance $\leq L-\epsilon$ from $\pi_F(\mathbf{e}_0)$, a contradiction.

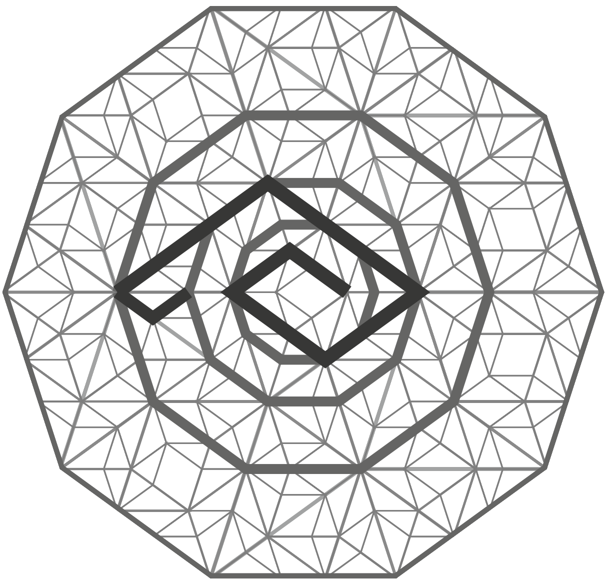


FIGURE 3 A path in C_3

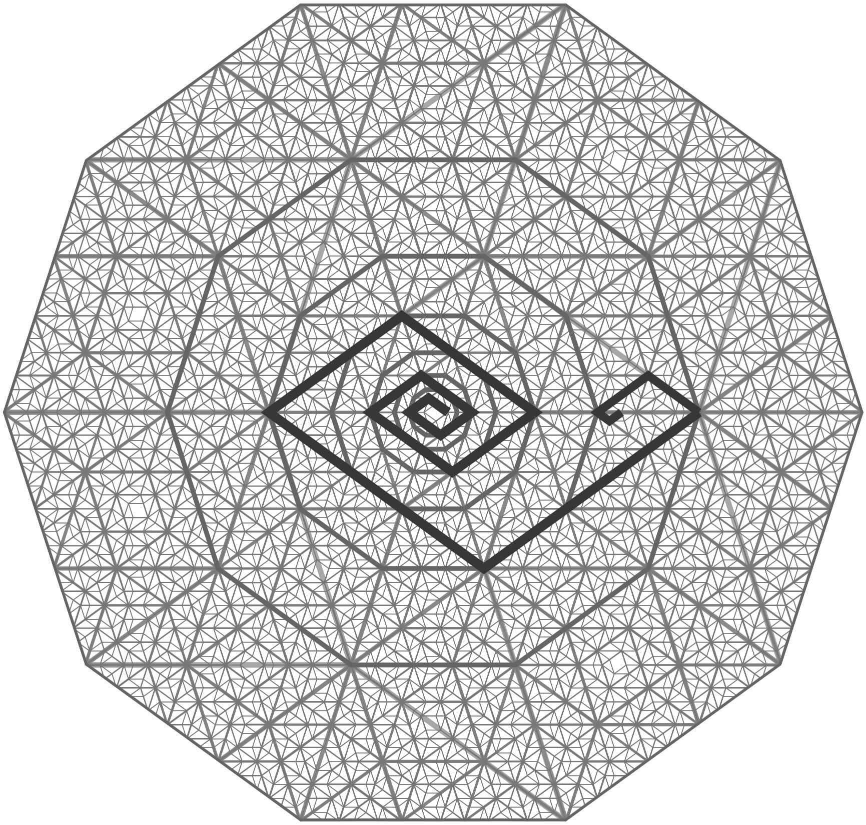


FIGURE 4 A path in C_6

To compute the length of the path on F we show first that

$$\mathbf{f} := \tfrac{1}{2}(\mathbf{e}_1 + \mathbf{e}_4) = \tfrac{1}{2}S_1\mathbf{e}_0$$

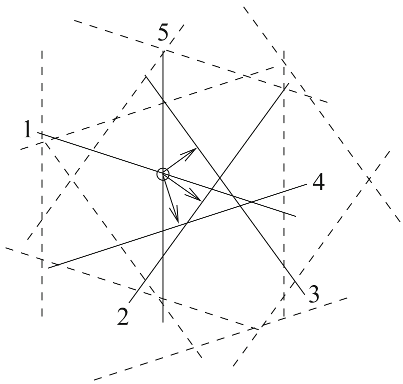
is a fixed vector for the affine map $T = S_2^3 + \mathbf{b}$ on $\mathbb{R}^5$. In fact,

$$\begin{aligned} S_2^3\mathbf{f} + \mathbf{b} &= \tfrac{1}{2}S_2^3S_1\mathbf{e}_0 + \mathbf{b} \\ &= \tfrac{1}{2}S_2^2(\mathbf{d} - \mathbf{e}_0) - S_2\mathbf{e}_0 - \mathbf{d} \\ &= \tfrac{1}{2}(2I + S_1)(\mathbf{d} - \mathbf{e}_0) - (S_2 + S_1)\mathbf{e}_0 + S_1\mathbf{e}_0 - \mathbf{d} \\ &= \mathbf{d} - \mathbf{e}_0 + \tfrac{1}{2}S_1(\mathbf{d} - \mathbf{e}_0) - \mathbf{d} + \mathbf{e}_0 + S_1\mathbf{e}_0 - \mathbf{d} \\ &= \tfrac{1}{2}S_1\mathbf{e}_0 \;=\; \mathbf{f} \end{aligned}$$

using $S_2S_1\mathbf{e}_0 = (S_2 + S_1)\mathbf{e}_0 = \mathbf{d} - \mathbf{e}_0$ and $S_2^2 = 2I + S_1$ and $S_1\mathbf{d} = 2\mathbf{d}$. Hence the affine map $T_F := \pi_F \circ T|_F$ has the fixed point $\pi_F(\mathbf{f})$. The linear part S_2^3 of T has eigenvalue $1/\tau^3 < 1$ on F, hence T_F is a strong contraction. Thus the sequence $\pi_F(\mathbf{x}_j) = \pi_F(T^j(\mathbf{x}_0)) = T_F^j(\pi_F(\mathbf{x}_0)$ converges to the fixed point of T_F, which is $\pi_F(\mathbf{f}) = \frac{1}{2}(\mathbf{e}_1^F + \mathbf{e}_4^F)$. The initial point of the sequence is $\pi_F(\mathbf{x}_0) = \mathbf{e}_0^F$. Thus the difference vectors $\pi_F(\mathbf{x}_j) - \pi_F(\mathbf{x}_0)$ converge for $j \to \infty$ to the diameter vector $\frac{1}{2}(\mathbf{e}_1^F + \mathbf{e}_4^F) - \mathbf{e}_0^F$ of the window V_1, and therefore $|\pi_F(\mathbf{x}_j) - \pi_F(\mathbf{x}_0)| \to L$. This finishes the proof that the decagonal tiling C is not of projection type.

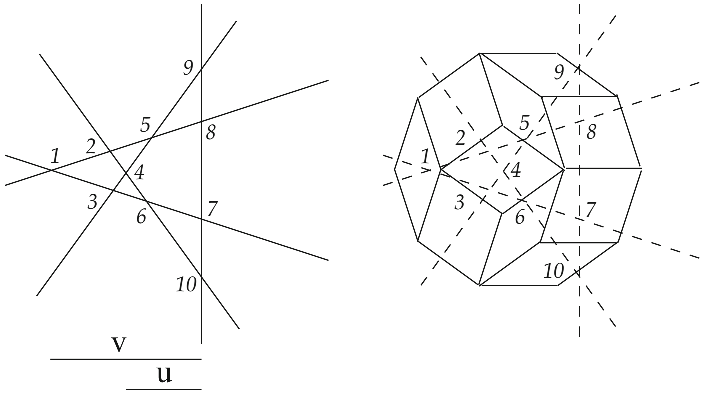
7. The pentagrid

Pentagrids introduced by de Bruijn [2] consist of five families of equidistant parallel lines in the plane. The lines point into the five pentagonal directions and only two of them intersect in a common point.



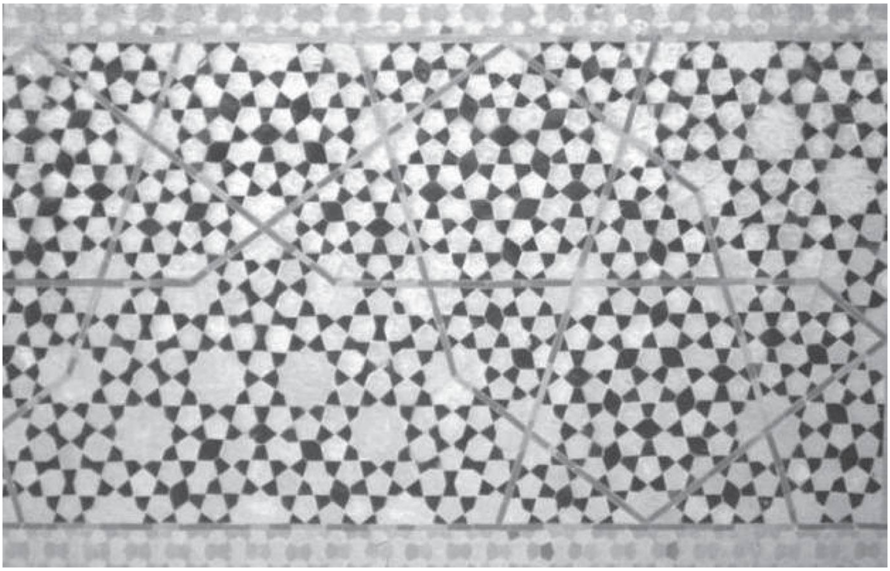
Pentagrids are in one-to-one correspondence to Penrose tilings of projection type (see Sect. 4). The vertex set of the projection tiling corresponding to an affine plane $E_{\mathbf{a}} = E + \mathbf{a} \subset \mathbb{R}^5$ arises by projecting $\Sigma_{\mathbf{a}} = \big((0,1)^5 + E_{\mathbf{a}}\big) \cap \mathbb{Z}^5$ onto E. The related pentagrid on the plane $E_{\mathbf{a}}$ is the intersection of $E_{\mathbf{a}}$ with the coordinate hyperplanes $H_{ik} = \{\mathbf{x} \in \mathbb{R}^{\mathbf{5}} : \mathbf{x_i} = \mathbf{k}\}$ for $i = 1, \dots, 5$ and $k \in \mathbb{Z}$. Vice versa, from a pentagrid and the intersection point of two of its lines (say, taken from the last two families), we obtain three numbers $a_i \in (0,1)$, $i = 1, 2, 3$, which are the signed distances to the nearest hyperplane H_{ik} (the lengths of the arrows in the figure above). The point $\mathbf{a} = (a_1, a_2, a_3, 0, 0) \in \mathbb{R}^5$ determines an affine plane $E_{\mathbf{a}} = E + \mathbf{a} \subset \mathbb{R}^5$ whose intersection with all hyperplanes H_{ik} is a pentagrid isometric to the given one.

In [2, p. 9, figure 14], a part of a pentagrid is shown, which corresponds to a cartwheel decagon. The intersections in the pentagrid correspond to tiles whose edges meet the pentagrid lines perpendicularly. In the figure below, we have marked the intersection points of the pentagrid and the corresponding tiles by the same number. The figure still depends on two parameters u, v, which reduce to one if we obey the condition $\sum_i a_i \in \mathbb{Z}$. But as a consequence of our Theorem 3, none of these pentagrids represent the complete cartwheel tiling.

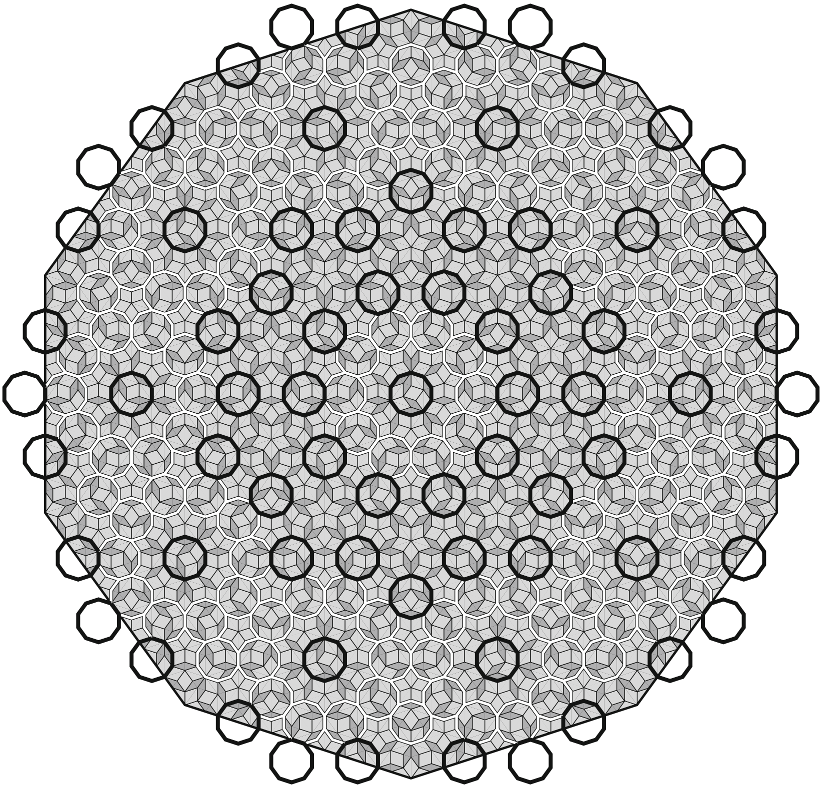


8. A link to traditional Islamic art

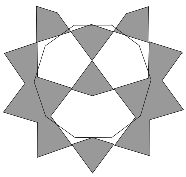
In 2005, Li and Steinhard [7] (see also [8,12]) observed that Penrose tilings are closely related to certain seventeenth century patterns in Islamic art. The subsequent figure shows an example from one of the entrance gates (called "The Master") into the courtyard of the Friday Mosque at Isfahan, Iran.



There is self similarity, and there are regular pentagons, like in Penrose tilings. However, the Isfahan pattern shows an exact decagonal symmetry which is impossible for Penrose tilings. Therefore the link between the two patterns is not completely obvious. It is revealed when we consider the pattern formed by the set of cartwheel decagons in a Penrose tiling, and when we look for a Penrose tiling with a quasi-decagonal symmetry. In our complete cartwheel tiling C there is such quasi-symmetry which becomes the more exact the farther we move away from the center. Thus we think that the Penrose pattern C is the closest one to the Isfahan pattern, see figure below which shows the eightfold subdivided cartwheel decagon C_8. The similarity between the Isfahan pattern and the cartwheel tiling has also been pointed out by Makovicky [9, p. 183].



The decagons which correspond to the white circles in the Isfahan pattern have been marked dark. Other decagons which are marked white correspond to the following figure composed of black and white stones.



The whole complicated arrangement is part of the geometry of the complete cartwheel tiling. Centuries ago it has been discovered by Iranian artists.

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