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CONFORMAL MAPPINGS OF SURFACES AND CAUCHY-RIEMANN INEQUALITIES *

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ABSTRACT

H. Hopf has introduced the use of holomorphic quadratic forms to study surfaces with constant mean curvature in R^3. The same techniques have been used to study surfaces immersed in spaces forms, complex projective spaces and also to study harmonic maps.

Following Hopf's ideas we consider holomorphic type forms,that is, forms whose zeroes behave like zeroes of holomorphic forms, to study surfaces such that the norm of the mean curvature vector or of its covariant derivative satisfies some inequality. We also apply the same techniques to study maps whose tension field satisfies similar property.

0. Introduction

A famous theorem of H. Hopf states that the round sphere is the only compact surface of genus zero which is immersed with constant mean curvature in euclidean 3-space [H]. The idea of the proof was to construct a certain holomorphic form on the surface, whose zeros are precisely the umbilic points. Such forms must vanish identically if the surface has genus zero, and so the immersed surface must be a totally umbilic sphere. The holomorphicity of this form is precisely equivalent to the constant mean curvature condition. Since then, this idea with some improvement has been used repeatedly by several authors to study constant mean curvature and minimal surfaces and harmonic maps (cf. [Ch],[ChG],[Y],[EW],[ChW],[EGT],[ET],[B2]).

However, holomorphicity is not really used in this argument. The form vanishes since otherwise its zeros would have negative index which is impossible if the surface has genus 0. In fact this was Hopf's original argument. So we only need a "holomorphic type" behaviour near the zeros. This is satisfied already under much weaker geometric assumptions and leads to new results about mappings of surfaces under very general assumptions. The key point in the proofs is to establish a Cauchy-Riemann type differential inequality which implies the desired behaviour at the zeros.

1. Statement of the results

Let M denote a Riemann surface with a compatible metric ds^2 of Gauß curvature K. Let $(P,\langle\ ,\ \rangle)$ be a Riemannian manifold and $f : M \to P$ a smooth mapping. Let γ always denote a local L^p-function on M, where $p > 2$. Throughout the paper, pointwise norms will be denoted by $|\ |$ in order to avoid confusion with function norms. The first theorem is local and extends a result of Gulliver et al. [GOR]:

THEOREM 1 *Let* $f : M \to P$ *be weakly conformal with conformal factor* μ *and tension field* τ. *If*

(1) $$|\tau| \le \gamma\cdot\mu$$

then f *is a branched immersion.*

From now on, we suppose that M is compact. Let $I = f^*(\langle\ ,\ \rangle)$. This is a quadratic form on M, called 1st fundamental form. Let $I^{(2,0)}$ denote its (2,0)-part, i.e. the dz^2-part of I for any holomorphic chart z of M. The following result generalizes a theorem of Chern and Goldberg [CG]:

THEOREM 2 *If* M *has genus* 0 *and* $f : M \to P$ *is a map with*

(2) $$|\tau| \le \gamma\cdot|I^{(2,0)}|$$

then f *is conformal and harmonic, hence a branched minimal immersion.*

The next theorem generalizes the classical theorem of Hopf [H] mentioned in the introduction.

THEOREM 3 *Let M be of genus 0 and P a 3-space of constant curvature c, and let $f : M \to P$ be an isometric immersion such that the mean curvature H satisfies*

(3) $$|dH| \le \varepsilon \cdot (H^2 - K + c)^{1/2} .$$

Then M is isometric to a round sphere and f is totally umbilic.

This result remains true also for generalized surfaces of mean curvature H (cf. [HH], [ET]). Note that the condition (3) is valid for a large number of immersions, e.g. for all those without umbilic points.

The next result was proved by S.T.Yau [Y] in the case of parallel mean curvature vector. Denote by $D^\perp$ the connection in the normal bundle Nf and by $A : TM \otimes TM \to Nf$ the 2^{nd} fundamental form. Then $\langle A,A\rangle$ is a symmetric 4-form whose (4,0)-part we denote by $\langle A,A\rangle^{(4,0)}$.

THEOREM 4 *Let M be of genus 0 and P be an n-manifold of constant sectional curvature c. Let $f : M \to P$ be an isometric immersion with 2^{nd} fundamental form A and mean curvature vector field H satisfying*

(4) $$|D^\perp H| \le \varepsilon \cdot |\langle A,A\rangle^{(4,0)}| .$$

Then either M is isometric to a round sphere and $f(M)$ is totally umbilic in some 3-dimensional totally geodesic submanifold, or f is a superminimal immersion in some totally umbilic hypersurface.

Here, superminimality means that the ellipse of curvature

$$E_p := \{A(x,x);\ x \in T_pM\ ,\ |x| = 1\}$$

is a circle centered at 0 for any $p \in M$ (cf. [B1]).

REMARK. If $\dim P = 4$, it follows from Gauß and Ricci equation that

$$|\langle A,A\rangle^{(4,0)}| = ((|H|^2 - K + c - K_N)(|H|^2 - K + c + K_N))^{1/2}$$

where K_N is the curvature of the normal connection (cf. [TG]). The formula remains valid in higher codimension if $A' := A - H\cdot ds^2$ spans a 2-dimensional subbundle of the normal bundle. Then K_N is the normal curvature in this plane.

Next, we generalize a result of Webster [W] for immersions into a Kähler manifold. There we have another invariant: the so called Kähler angle given by the pull-back of the Kähler form (see section 3).

THEOREM 5 *Let M have Euler number χ. Let P be a Kähler manifold and $f : M \to P$ a conformal smoothly branched immersion with mean curvature vector H and Kähler angle α such that*

(5) $$|H| \le \varepsilon \cdot \sin \alpha .$$

Then either f is holomorphic or anti-holomorphic, or the complex tangent planes are isolated. If P has real dimension 4, their number j (counted with multiplicities) is given by

$$j = \chi + \chi_N + b$$

where χ_N is the Euler number of the normal bundle and b the number of branch points (counted with multiplicities).

Here, the multiplicity of a complex tangent plane is the index as defined by Webster [W], see section 3.

Our last theorems generalize results of Eells - Wood [EW] and Chern - Wolfson [ChW] on conformal maps $f : M \to P$ where P is a Kähler manifold of constant holomorphic sectional curvature. On P, we have the hermitean form (,) given by

$$(v,w) = \langle v,w\rangle + i\,\langle v,Jw\rangle .$$

Using this, we consider the complex valued cubic form (Ddf,df) and its (3,0)-part $(Ddf,df)^{(3,0)}$.

<u>THEOREM 6</u> *Let P be a Kähler manifold of constant holomorphic sectional curvature and $f : M \to P$ a conformal map such that*

$$(6) \qquad |H|,\ |D^{\perp}H| \ \le\ \gamma\cdot|(Ddf,df)^{(3,0)}| .$$

If M has genus 0 then f is a totally isotropic branched minimal immersion.

<u>REMARK.</u> If P has real dimension 4 (Kähler *surface*) and f is an isometric immersion, then it follows from Gauß and Ricci equation that

$$|(Ddf,df)^{(3,0)}| = (\sin\alpha)\cdot(|H|^2 - K + K_N + 2\sigma)^{1/2}$$

where 4σ is the holomorphic sectional curvature of P. A similar formula does not hold in higher codimension since in general, the plane spanned by $A' = A - H\cdot ds^2$ does not ly in the complex closure of $df(TM)$.

<u>THEOREM 7</u> *Let P be a Kähler surface of constant holomorphic sectional curvature 4σ and $f : M \to P$ an isometric immersion of degree d.*

(a) If (5) holds then

$$\chi + \chi_N \le -|3d|$$

unless f is holomorphic of anti-holomorphic.

(b) If f satisfies

$$(6') \qquad |H|,\ |D^{\perp}H| \le \gamma\cdot(\sin\alpha)\cdot(|H|^2 - K + K_N + 2\sigma)^{1/2} ,$$

then either f is an isotropic minimal immersion or

$$\chi \le -|d| .$$

For the notion of degree, see section 4. The theorem generalizes to conformal smoothly branched immersions if χ is repaced with $\chi + b$ where b denotes the number of branch points (counted with multiplicities.)

In section 3, we prove the theorems 1,2,5 while the theorems 3,4,6,7 are proved in section 4.

2. Cauchy-Riemann inequalities

Let M be a Riemann surface and $E \to M$ a complex vector bundle with fibre $\mathbb{C}^n$. A smooth section s of E is called <u>of holomorphic type</u> if near any zero p of s we have

$$s = s_o(z-z(p))^k$$

for some positive integer k and some continuous section s_o with $s_o(p) \neq 0$, where z is any holomorphic chart in a neighborhood of p. Anti-holomorphic type is defined analogously. A section s is of anti-holomorphic type if it is of holomorphic type in $\overline{E}$, where $\overline{E}$ is the real vector bundle E with reversed multiplication by $i = \sqrt{(-1)}$.

If E is a line bundle, this number $k = k(s,p)$ also gives the index of s at the zero p. Thus by the Poincaré-Hopf index theorem (e.g. cf. [GHV]) we have

PROPOSITION 2.1 *Let* M *be a compact Riemann surface and* L *a complex line bundle over* M. *If* s *is a holomorphic type section of* L, *then the Euler number (1st Chern number) of* L *is*

$$\chi(L) = N(s) := \sum_{s(p)=0} k(s,p) .$$

In particular, $\chi(L) \geq 0$.

We will give a sufficient condition for a section to be of holomorphic type. We say that a section s of E satisfies a Cauchy-Riemann inequality if locally

$$|\partial s_U/\partial \bar{z}| \leq \gamma \cdot |s_U|$$

for some L^p-function γ with $p > 2$, where $z : U \to \mathbb{C}$ is a holomorphic chart and $s_U : U \to \mathbb{C}^n$ a local expression of s in bundle coordinates over an open subset U of M. Apparently, this condition is independent of the choice of the holomorphic chart z and the bundle chart. If D is any complex linear connection on E, the above inequality is equivalent to

$$|Ds/\partial \bar{z}| \leq \gamma \cdot |s|$$

for some other L^p-function γ, where we put (as usual)

$$D/\partial z = \tfrac{1}{2}(D/\partial x - iD/\partial y) , \quad D/\partial \bar{z} = \tfrac{1}{2}(D/\partial x + iD/\partial y) .$$

if $z = x + iy$. This is because we have $Ds/\partial \bar{z} = \partial s_U/\partial \bar{z} + A_U \cdot s_U$ for some matrix valued function A_U on U.

PROPOSITION 2.2 *If a smooth section* s *of* E *satisfies a Cauchy-Riemann inequality, it is either identically zero or of holomorphic type.*

This is an immediate consequence of the following lemma:

LEMMA 2.3 *Let* $U \subset \mathbb{C}$ *be an open domain containing* 0 *and* $f : U \to \mathbb{C}^n$ *a smooth function satisfying*

$$(*) \qquad |\partial f/\partial \bar{z}| \leq \gamma \cdot |f|$$

for some L^p*-function* γ *with* $p>2$. *Then near the origin, we have either* $f \equiv 0$ *or*

$$f(z) = z^k \cdot f_0(z)$$

for some nonnegative integer k *and a continuous function* f_0 *with* $f_0(0) \neq 0$.

The proof is an adaptation of an idea of Chern [Ch]. We need another lemma:

LEMMA 2.4 *Let* $g : U \setminus 0 \to \mathbb{C}^n$ *be a smooth function which is bounded near* 0 *and satisfies* $|\partial g/\partial \bar{z}| \leq \gamma \cdot |g|$ *for some* L^p*-function* γ *on* U *with* $p>2$. *Then* $\lim_{z\to 0} g(z)$ *exists, and for a suitably small closed disk* $D \subset U$ *of radius* R *centered at* 0, *the* L^q*-norms on* D *and its boundary* ∂D *are related by*

$$\|g\|_{q,D} / \|g\|_{q,\partial D} \leq C \cdot R^{1/p}$$

with a constant C *depending only on* $\|\gamma\|_p$, *where* $q^{-1}+p^{-1} = 1$.

PROOF OF LEMMA 2.4. Let $0 \neq \zeta \in \mathrm{Int}(D)$. Consider the 1-form $\eta = q(z)dz$ on $D_\varepsilon := D \setminus (B_\varepsilon(\zeta) \cup B_\varepsilon(0))$, where

$$q(z) = (g(z) - g(\zeta))/(z - \zeta) .$$

If we apply Stokes' theorem to η and let $\varepsilon \to 0$, we get the Cauchy formula

$$\text{(C)} \quad 2\pi i \cdot g(\zeta) = \int_{\partial D} g(z)(z-\zeta)^{-1}\, dz - \int_D g_{\bar z}(z)(z-\zeta)^{-1}\, d\bar z \wedge dz$$

for all $\zeta \in \mathrm{Int}(D)\setminus\{0\}$, where $g_{\bar z} := \partial g/\partial \bar z$ (cf. [FL]). Moreover, since $h := g_{\bar z}$ is L^p, the limit of the right hand side of (C) for $\zeta \to 0$ exists (cf. [A], p. 85/86). Thus by (C), $g(z)$ takes a limit as $z \to 0$.

Now let us estimate the L^q-norm. Since $q < 2$, the function $z \to g(z)(z-\zeta)^{-1}$ is L^q on D. Thus, by Hölder's inequality, the second term to the right of (C) is absolutely bounded by

$$\|g\|_p \cdot \left(\int_D |g(z)|^q\, |z-\zeta|^{-q}\, |d\bar z \wedge dz|\right)^{1/q} .$$

Taking the q^{th} power (which is a convex operation since $q > 1$), we get from (C)

$$(2\pi)^q\, |g(\zeta)|^q \le A \cdot R^{q-1} \cdot \int_{\partial D} |g(z)|^q\, |z-\zeta|^{-q}\, |dz|$$

$$+ \; B \cdot \int_D |g(z)|^q\, |z-\zeta|^{-q}\, |d\bar z \wedge dz|$$

where $A = (4\pi)^{q-1}$ and $B = 2^{q-1} \cdot (\|g\|_p)^q$. Integration over D with respect to ζ yields

$$(2\pi)^q (\|g\|_{q,D})^q \le A \cdot \alpha \cdot R^{q-1} (\|g\|_{q,\partial D})^q + B \cdot \alpha \cdot (\|g\|_{q,D})^q$$

where

$$\alpha = \sup_{z \in D} \left(\int_D |z-\zeta|^{-q}\, |d\bar\zeta \wedge d\zeta|\right) .$$

Note that $\alpha \to 0$ as $R \to 0$. Hence $(2\pi)^q - B \cdot \alpha$ is positive for small R, and we get the result.

PROOF OF LEMMA 2.3. We show first that f has a nonzero Taylor expansion around 0 unless $f \equiv 0$ near 0. In fact, suppose that $f(z) = o(|z|^k)$ for all $k \ge 0$, but that there exists z_0 with $f(z_0) \neq 0$ and $r := |z_0| < R$ for sufficiently small R. Let D be the disk of radius R centered at 0. Put $g_k = f/z^k$. Since $|f(z_0)| > 0$, we have

$$\|g_k\|_{q,D} \ge a \cdot r^{-k} ,$$

but on the other hand

$$\|g_k\|_{q,\partial D} \le b \cdot R^{-k}$$

for suitable constants a, b independent of k. Thus

$$\|g_k\|_{q,D} / \|g_k\|_{q,\partial D} \ge (a/b)(R/r)^k ,$$

and this goes to ∞ as $k \to \infty$, a contradiction to Lemma 2.4.

Now let k be the degree of the first nonzero Taylor polynomial of f at 0. Put $g = f/z^k$. Then g has a limit $a \neq 0$ as $z \to 0$, by the first part of Lemma 2.4. Thus

$$f(z) = a \cdot z^k + O(|z|^{k+1})$$

which finishes the proof.

REMARK. In general, the function f_0 in Lemma 2.3 is not smooth, e.g. take $f(z) = z^k + \bar z^{k+1}$. However, if the linear inequality (*) in Lemma 2.3 is replaced with a linear equality

$$\partial f/\partial \bar z = A \cdot f$$

for some smooth matrix value function A on U, then f_0 is smooth (cf. Lemma 2.1 and 2.2 in [ET]).

3. The differential of a mapping of a Riemann surface

Let M be a Riemann surface. Let ds^2 be an admissible metric on M, i.e. for any holomorphic chart $z = x + iy : U \to \mathbb{C}$ on some open subset $U \subset M$ we have

$$ds^2 = \lambda^2 \, dz \, d\bar{z}$$

for some positive function λ called <u>conformal factor</u> of z. let us put

$$\partial_z = ½(X - iY) , \quad \partial_{\bar{z}} = ½(X + iY) ,$$

where $X = \partial/\partial x$, $Y = \partial/\partial y$, and for any complex vector bundle E over M with complex linear connection D we let

$$D_z = ½(D_X - iD_Y) , \quad D_{\bar{z}} = ½(D_X + iD_Y) .$$

Let T^*M denote the bundle of real valued 1-forms and $T'M \subset T^*M \otimes \mathbb{C}$ the bundle of (1,0)-forms; those are $\mathbb{C}$-linear with respect to the almost complex structure (90°-rotation) j on M.

Let $(P, \langle \; , \; \rangle)$ be a Riemannian manifold and $f : M \to P$ a smooth mapping. The canonical connections on TM and f^*TP and all related bundles will be denoted by D . The differential df is a 1-form with values in the pull back bundle f^*TP , i.e. a section of $T^*M \otimes f^*TP$. If $(P, \langle \; , \; \rangle)$ carries a Kähler structure J, we consider $E := f^*TP$ as a complex vector bundle. Otherwise, we put $E = f^*TP \otimes \mathbb{C}$. In both cases, the canonical connection on E is complex linear. We have

$$df = \partial_z f \cdot dz + \partial_{\bar{z}} f \cdot d\bar{z}$$

and $\partial_z f$, $\partial_{\bar{z}} f$ are local sections of E .

The 2^{nd} derivative Ddf is a symmetric 2-form (hessean form) with values in f^*TP . Let τ be its trace with respect to the metric ds^2. This is called the <u>tension field</u> of f . By conformality of z we have $D_z\partial_{\bar{z}} = D_{\bar{z}}\partial_z = 0$ for the Levi-Civita connection on M and therefore we get:

<u>LEMMA 3.1</u> *Let $f : M \to P$ be a smooth mapping with tension field τ and z a holomorphic chart. Then*

$$D_z\partial_{\bar{z}} f = D_{\bar{z}}\partial_z f = ¼ \lambda^2 \tau .$$

Let us suppose first that we have no Kähler structure on $(P, \langle \; , \; \rangle)$. We consider the <u>1st fundamental form</u>

$$I = f^*(\langle \; , \; \rangle) = \langle df, df \rangle$$

on M . Its (2,0)-part (the dz^2-part) is a section of the complex line bundle $L = T'M \otimes_{\mathbb{C}} T'M$ which is locally given by

$$I^{(2,0)} = \langle \partial_z f, \partial_z f \rangle \, dz^2 ,$$

where the metric $\langle \; , \; \rangle$ is complex bilinearly extended to $E = f^*TP \otimes \mathbb{C}$. Let g always denote an L^p-function for some $p>2$.

<u>PROPOSITION 3.2</u> *Let $f : M \to P$ be a smooth mapping with*

(2) $$|\tau| \le g \cdot |I^{(2,0)}| .$$

Then $I^{(2,0)}$ is of holomorphic type.

<u>PROOF.</u> Using Lemma 3.1 we have for $b = \partial_z f$

$$\begin{aligned} |\partial_{\bar{z}}\langle b,b\rangle| &= 2|\langle D_{\bar{z}} b, b\rangle| \\ &\le 2 \, |D_{\bar{z}} b||b| = ½ \lambda^2 |\tau||b| \\ &\le g|b||\langle b,b\rangle| \end{aligned}$$

since $|I^{(2,0)}| = 2|\langle b,b\rangle|/\lambda^2$. This finishes the proof by 2.2.

COROLLARY 3.3 *If M is compact under the assumptions of 3.2, then*

$$\chi(M) \le -2\cdot N(I^{(2,0)})$$

unless f is a comformal harmonic map.

PROOF. Since $\chi(L) = -2\,\chi(M)$, this follows from 3.2 using 2.1. If $I^{(2,0)} \equiv 0$, then f is conformal, and the harmonicity follows from (2) in 3.2.

Now Theorem 2 is also proved since it is a special case of 3.3.

Next suppose that $f : M \to P$ is a (weakly) conformal smooth map, i.e. $I = \mu^2\, ds^2$ for some function $\mu \ge 0$, the conformal factor of f with respect to ds^2, and $\langle \partial_z f, \partial_z f\rangle = 0$. Outside the zero set of df, the mapping f is a conformal immersion, and

$$\tau/\mu^2 = \mathrm{trace}(A) = 2H$$

where A denotes the 2nd fundamental form and H the mean curvature vector. Since $|\partial_z f| = \mu\lambda/\sqrt{2}$, we get from 2.2 for the (1,0)-part of df:

LEMMA 3.4 *Let $f : M \to P$ be a conformal map with conformal factor μ and tension field τ. If*

(1) $$|\tau| \le \beta\cdot\mu$$

then $df^{(1,0)}$ is of holomorphic type.

From the latter property it follows easily that f is a branched immersion (cf. [GOR], [ET]). In fact, if $\partial_z f = k\cdot z^{k-1}\cdot g$, choose coordinates u on P such that $\partial/\partial u^1 = \mathrm{Re}\, g(0)$, $\partial/\partial u^2 = \mathrm{Im}\, g(0)$ (these are linearly independent because of the conformality

property $\langle g,g\rangle = 0$ and $g(0) \ne 0$) and put

$$h(z) = u(f(0)) + (2\,\mathrm{Re}\, z^k,\ 2\,\mathrm{Im}\, z^k,\ 0,\dots,0)\ .$$

Then $b := u\circ f - h$ is real with $b(0) = 0$, $\partial_z b = O(|z|^k)$, thus $b = O(|z|^{k+1})$. This proves Theorem 1.

If $f : M \to P$ is a branched immersion, the plane bundle df(TM) extends continuously to the critical points of f. If this extension (called Tf) is a *smooth* subbundle of f^*TP, then f is called a smoothly branched immersion (cf. [ET]).

Now we suppose that a Kähler structure J is given on $(P, \langle\ ,\ \rangle)$ and that $f : M \to P$ is a conformal immersion. Recall that now $b_+ := \partial f/\partial z$ and $b_- := \partial f/\partial \bar z$ are sections of $E = f^*TP$. Outside the zero sets of these sections we have invariantly defined *complex* line bundles $L_\pm := \mathbb{C}\cdot b_\pm \subset E$ with $L_+ \perp L_-$. Further, $|b_\pm|^2 = \frac{1}{4}\lambda^2\,(1 \pm \cos\alpha)$. Here, λ denotes the conformal factor with respect to the induced metric $ds^2 = I$, and $\alpha \in [0,\pi]$ the so called Kähler angle (cf. [ChW], [EGT]). The latter is invariantly defined by

$$f^*\Omega = (\cos\alpha)\cdot dv$$

where dv is the volume form of ds^2 and Ω the Kähler form on P given by $\Omega(v,w) = \langle Jv, w\rangle$. In fact we have

$$\cos\alpha = \langle Je_1, e_2\rangle$$

for any oriented orthonormal frame (e_1,e_2) of df(TM). Thus we get from Lemma 3.1:

PROPOSITION 3.5 *Let P be a Kähler manifold and $f : M \to P$ a conformal mapping with conformal factor μ, tension field τ and Kähler angle α. If*

$$(5) \qquad |\tau| \le \tfrac{1}{8}\cdot\mu^2\cdot \sin\alpha$$

then f is a conformal branched immersion and $b_+ = \partial f/\partial z$ is of holomorphic and $b_- = \partial f/\partial \bar z$ of anti-holomorphic type. In particular, the zeros of b_+ and b_- are isolated, and the invariant line bundles $L_\pm = \mathbb{C}\cdot b_\pm$ can be continuously extended to these points.

REMARK 1 If f is an immersion, we may replace $b_\pm$ with

$$s_\pm = \tfrac{1}{2}(e_1 \mp J\, e_2)$$

for any oriented orthonormal frame (e_1, e_2) of $df(TM)$. We still have $\mathbb{C}\cdot s_\pm = \mathbb{C}\cdot b_\pm = L_\pm$. This is still possible for conformal smoothly branched immersions. Here, we must replace $df(TM)$ by Tf.

REMARK 2 The sections s_+ and s_- cannot vanish at the same point since $|s_+ + s_-| = 1$. Thus, if P has real dimension 4, the bundles $L_+ = (L_-)^\perp$ and $L_- = (L_+)^\perp$ are globally defined and smooth.

REMARK 3 If the dimension of P is arbitrary and f is harmonic, i.e. $\tau = 0$, then b_+ and b_- satisfy Cauchy-Riemann equalities, by 3.1. Hence L_+ and L_- are globally defined and smooth also in this case.

Now suppose that $f : M \to P$ is a conformal smoothly branched immersion where P is a Kähler *surface* (i.e. real dimension 4). We consider the bundle map $\Phi : Tf \to Nf$,

$$\Phi(x) = (Jx)^N$$

which was introduced by Webster [W]. Here, $Nf = (Tf)^\perp$ denotes the normal bundle and ^{N} the normal component. For any oriented orthonormal frame (e_1, e_2) of Tf we have $(Je_1)^T = (\cos\alpha)\cdot e_2$ where ^{T} denotes the component in Tf. Therefore,

$$|\Phi(x)| = (\sin\alpha)\cdot|x|$$

for any $x \in Tf$, and hence Φ preserves angles. Wherever $\Phi \ne 0$, the 4-vector

$$e_1 \wedge e_2 \wedge \Phi(e_1) \wedge \Phi(e_2) = e_1 \wedge e_2 \wedge Je_1 \wedge Je_2$$

defines the negative orientation on P. (Apply $\Omega \wedge \Omega$!) So Φ is a section of the line bundle

$$L = \mathrm{Hom}_{\mathbb{C}}(Tf, \overline{Nf}) = (Tf)^* \otimes_{\mathbb{C}} \overline{Nf}$$

where $\overline{Nf}$ denotes Nf with reversed orientation. Thus in the compact case we have $-\chi(L) = \chi(Tf) + \chi(Nf)$, and we get from 2.1 and [ET], Theorem 4:

LEMMA 3.6 *If M is compact and Φ of holomorphic type, then*

$$-N(\Phi) = \chi(M) + \chi(Nf) + b$$

where b is the number of branch points, counted with multiplicities.

To derive a Cauchy-Riemann inequality for Φ, let us choose local unit vector fields E_1 of L_+ and E_2 of L_- (cf. [EGT]). These constitute a unitary frame of f^*TP, i.e. (E_1, JE_1, E_2, JE_2) is an orthonormal frame. Then $s_+ = u\cdot E_1$, $s_- = \bar v\cdot E_2$ for suitable $\mathbb{C}$-valued functions u, v . Consequently, we have

$$e_1 = u\cdot E_1 + \bar{v}\cdot E_2 , \quad e_2 = i(u\cdot E_1 - \bar{v}\cdot E_2) ,$$

and

$$e_3 := i(\bar{v}\cdot E_1 + u\cdot E_2) , \quad e_4 := -\bar{v}\cdot E_1 + u\cdot E_2$$

is an oriented orthonormal frame of $\overline{Nf}$ (as one easily checks when u and v are real positive). If we identify tangent and normal plane with $\mathbb{C}$ using these bases, Φ becomes the multiplication with $2uv$. Thus Φ is of holomorphic type if s_+ is of holomorphic and s_- of antiholomorphic type. Thus we get from 3.5:

<u>PROPOSITION 3.7</u> *Let P be a Kähler surface and $f : M \to P$ a conformal smoothly branched immersion satisfying*

$$(5) \qquad |H| \le \gamma\cdot\sin\alpha .$$

Then Φ is of holomorphic type.

Now Theorem 5 follows from 3.6 and 3.7.

<u>4. The second derivative</u>

From now on, let $f : M \to P$ be a weakly conformal map. As above, we consider df as a section of $T^*M \otimes E$ where $E = f^*TP$ if P is Kähler and $E = f^*TP \otimes \mathbb{C}$ otherwise. Let $z = x + iy$ be a holomorphic chart on M . Then

$$D_{\bar{z}}(D_z\partial_z f) = D_z(\tfrac14 \lambda^2 \tau) + [D_{\bar{z}}, D_z]\partial_z f .$$

On M we have $D_{\bar{z}}\partial_z = 0$ and $D_z\partial_z = \rho\cdot\partial_z$, where $\rho = \partial_z(\log\lambda^2)$.

So the left hand side gives

$$D_{\bar{z}}(D_z\partial_z f) \equiv D_{\bar{z}}(Ddf(\partial_z, \partial_z)) + \tfrac14 \lambda^2\cdot\rho\cdot\tau \quad \mathrm{mod}\ \partial_z f .$$

Moreover,

$$D_z(\lambda^2\cdot\tau) = \lambda^2(D_z\tau + \rho\cdot\tau) .$$

If P is a Riemannian manifold of constant sectional curvature or a Kähler manifold of constant holomorphic curvature, then

$$[D_{\bar{z}}, D_z]\partial_z f = (1/2i)\cdot R(\partial_x f, \partial_y f)\partial_z f \equiv 0 \quad \mathrm{mod}\ \partial_z f .$$

Thus we get:

<u>LEMMA 4.1</u> *Let $f : M \to P$ be weakly conformal where P is a Riemannian manifold of constant sectional curvature or a Kähler manifold of constant holomorphic curvature. Then*

$$D_{\bar{z}}(Ddf(\partial_z, \partial_z)) \equiv \tfrac14 \lambda^2\cdot D_z\tau \quad \mathrm{mod}\ \partial_z f$$

for any holomorphic chart z on M .

<u>COROLLARY 4.2</u> *Under the same assumptions, suppose that N is a parallel section of E with $N \perp \partial_z f$. Let Λ be the (2,0)-part of the quadratic form $\langle Ddf, N\rangle$. Suppose*

$$(3') \qquad |d\langle\tau, N\rangle| \le \gamma\cdot|\Lambda|$$

Then Λ is of holomorphic type.

<u>PROOF.</u> We have $\Lambda = \langle v, N\rangle dz^2$, where $v = Ddf(\partial_z, \partial_z)$. Then

$$\partial_{\bar{z}}\langle v, N\rangle = \langle D_{\bar{z}}v, N\rangle = \tfrac14 \lambda^2\, \partial_z\langle\tau, N\rangle ,$$

hence by (3'), $|\partial_{\bar{z}}\langle v, N\rangle| \le (\sqrt{2}/4)\lambda\cdot\gamma\cdot|\langle v, N\rangle|$, which proves the statement.

<u>REMARK.</u> The proof shows that the statement remains valid if v and N ly in a subbundle F of E with $DN \perp F$.

__PROPOSITION 4.3__ *Let P be a Riemannian manifold of constant sectional curvature and $f : M \to P$ a weakly conformal map. Let Λ be the (4,0)-part of the symmetric 4-form $\langle Ddf, Ddf \rangle$. Assume*

$$(4) \qquad |D\tau| \le \gamma \cdot |\Lambda| .$$

Then Λ is of holomorphic type.

__PROOF.__ We have $\Lambda = \langle v,v \rangle dz^4$, where $v = Ddf(\partial_z, \partial_z)$, and

$$\partial_{\bar z}\langle v,v\rangle = 2\,\langle D_{\bar z} v, v\rangle = \tfrac{1}{2}\,\lambda^2\,\langle D_z \tau, v\rangle$$

by 4.1; note that

$$\langle \partial_z f, v\rangle = \langle \partial_z f, D_z \partial_z f\rangle - \rho\cdot\langle \partial_z f, \partial_z f\rangle = 0 .$$

Since $|\Lambda| = 4\,\langle v,v\rangle/\lambda^4$, we get from (4) a Cauchy-Riemann inequality for $\langle v,v \rangle$ which proves the result by 2.2.

__REMARK.__ The argument shows: If v lies in a subbundle $F \subset f^*TP$ with orthogonal projection π_F, then it is sufficient to assume

$$(4') \qquad |\pi_F(D\tau)| \le \gamma \cdot |\Lambda| .$$

Now let P be a Kähler manifold. Recall that on P we have the hermitean form $(\, , \,)$ defined by

$$(v,w) = \langle v,w\rangle + i\langle v, Jw\rangle .$$

__PROPOSITION 4.4__ *Let P be a Kähler manifold of constant holomorphic sectional curvature and $f : M \to P$ a weakly conformal map. Let Λ be the (3,0)-part of the cubic form (Ddf, df). Assume*

$$(6) \qquad |\tau|,\ |D\tau| \le \gamma \cdot |\Lambda| .$$

Then Λ is of holomorphic type.

__PROOF.__ We have $\Lambda = (v, \partial_{\bar z} f) dz^3$ where $v = Ddf(\partial_z, \partial_z)$, and

$$\partial_{\bar z}(v, \partial_{\bar z} f) = (D_{\bar z} v, \partial_{\bar z} f) + (v, D_z \partial_{\bar z} f) = \tfrac{1}{2}\,\lambda^2\cdot((D_z\tau, \partial_{\bar z} f) + (v,\tau))$$

by 4.1. Since $|\Lambda| = \sqrt{8}\cdot|(v,\partial_{\bar z} f)|/\lambda^3$, we get from (6) a Cauchy-Riemann inequality for $(v, \partial_{\bar z} f)$ and the result follows from 2.2.

Now suppose that $f : M \to P$ is an isometric immersion. Then $Ddf = A$ is the 2^{nd} fundamental form which has values in the normal bundle $Nf \subset f^*TP$. Putting $e_1 = \partial_x/\lambda$, $e_2 = \partial_y/\lambda$, and

$$a = \tfrac{1}{2}(A(e_1,e_1) - A(e_2,e_2)) , \quad b = A(e_1,e_2) ,$$

we have

$$v = A(\partial_z, \partial_z) = \tfrac{1}{2}\,\lambda^2(a - ib) .$$

First, let P be 3-dimensional and of constant sectional curvature c, and N a unit normal field along f. Then we get Theorem 3 from 4.2 and the following remark using 2.1 since for the 2-form $\Lambda = \langle v,N\rangle dz^2$ we have

$$|\Lambda|^2 = |\langle a,N\rangle - i\langle b,N\rangle|^2 = |a|^2 + |b|^2$$

$$= H^2 - \det\langle A,N\rangle = H^2 - K + c .$$

where $H = \tfrac{1}{2}$ trace $\langle A,N\rangle$ is the mean curvature.

In the case of arbitrary dimension, Theorem 4 and 6 follow from 4.3 and 4.4: By 2.1 we are reduced to the case $\Lambda \equiv 0$ which implies that f is harmonic (Thm.6) or has parallel mean curvature vector (Thm.4). Now we may apply the work of Yau [Y] and Eells-Wood [EW], Chern-Wolfson [CW] to get our results.

It remains to consider the case where P is a Kähler surface of constant holomorphic sectional curvature 4σ. We have the complex line bundles L_+, $L_- \subset E$ as introduced in section 3, and as in [EGT], p. 592, we get

$$\chi(L_+) = \tfrac{1}{2}\,(3d + \chi - \chi_N)\ ,$$

$$\chi(L_-) = \tfrac{1}{2}\,(3d - \chi + \chi_N)\ ,$$

where χ and χ_N denote the Euler numbers of M and its normal bundle, and

$$d = (\sigma/\pi) \int_M f^*\Omega = c_1(E)/3$$

the degree, where Ω denotes the Kähler form of P and c_1 the first Chern number. Let $df^{(1,0)} : TM \to L_+$, $df^{(0,1)} : TM \to L_-$ be the (1,0)- and (0,1)-part of df . We consider those as sections of the line bundles $T^*M \otimes_{\mathbb{C}} L_+$ and $T^*M \otimes_{\mathbb{C}} \overline{L_-}$. In 3.4 we saw that these sections are of holomorphic type if (5) holds. Then we get

$$N(df^{(1,0)}) = \chi(L_+) - \chi = 3d - (\chi + \chi_N)\ ,$$

$$N(df^{(0,1)}) = -\chi(L_-) - \chi = -3d - (\chi + \chi_N)\ ,$$

in particular

$$(*) \qquad \chi + \chi_N \le -|3d|$$

unless f is holomorphic or anti-holomorphic, which proves Theorem 7 (a).

To prove part (b), we consider the complex trilinear map w : $TM \otimes_{\mathbb{C}} L_+ \otimes_{\mathbb{C}} \overline{L_-} \to \mathbb{C}$,

$$w(x,a,b) = \tfrac{1}{2}((D_x a,b) - i(D_{jx} a,b))$$

where a, b are local sections of L_+ and L_- , and j denotes the almost complex structure (90°-rotation) on M. This is related to the cubic form Λ used above by

$$\Lambda(x) = w(x, df^{(1,0)}(x), df^{(0,1)}(x))\ .$$

It follows that w is of holomorphic type if $df^{(1,0)}$, $df^{(0,1)}$ and Λ are of holomorphic type. This is the case if (6') holds since it implies (5). So we get

$$N(w) = -(\chi + \chi(L_+) - \chi(L_-)) = -2\chi + \chi_N\ ,$$

unless $w \equiv 0$. Together with (*) we get $\chi \le -|d|$ unless $\Lambda \equiv 0$ which proves Theorem 7 (b). If we allow smooth branch points, we have to replace TM with Tf and therefore χ with $\chi + b$, by Theorem 4 of [ET].

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