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PSEUDO-HOLOMORPHIC CURVES AND PERIODIC ORBITS ON COTANGENT BUNDLES

By K. CIELIEBAK

ABSTRACT. — On the cotangent bundle over a compact manifold M consider a 1-periodic time-dependent Hamiltonian which is asymptotically quadratic in the fibres. Studying a boundary value problem for a nonlinear Cauchy-Riemann operator, the existence of infinitely many 1-periodic solutions of the corresponding Hamiltonian system is shown, provided that the fundamental group of M is finite.

1. Introduction

To a 1-periodic time-dependent smooth Hamiltonian $H \in C^{\infty}(S^1 \times V, \mathbb{R})$, $S^1 := \mathbb{R}/\mathbb{Z}$ on a symplectic manifold (V, ω) we associate a vector field X_H defined by

$$dH_t(x) = \omega(X_H(t, x), .)$$

for all (t, x) in $S^1 \times V$, where we have put $H_t := H(t, .)$. Let

$$\mathfrak{C} := \mathfrak{C}(V, H) := \{ x \in C^{\infty}_{\text{contr}}(S^1, V) \mid \dot{x} = X_H(t, x) \}$$

be the set of all contractible 1-periodic solutions of the Hamiltonian system $\dot{x} = X_H(t, x)$.

If V is compact and both ω and the first Chern class of V vanish over the second homotopy group $\pi_2(V)$, the Arnold conjecture as proved by A. Floer ([F12], Th. 1) estimates the number $\#\mathfrak{C}$ from below in terms of the topology of V (more precisely, the cuplength and the sum of the Betti numbers of V).

In this paper we will give estimates of $\#\mathfrak{C}$ for some natural class of noncompact symplectic manifolds, namely for cotangent bundles over compact base manifolds, provided that H satisfies a certain "asymptotic quadraticity" condition.

Therefore, let M be a compact manifold of dimension n (if we just say 'manifold' we will always mean a smooth manifold without boundary), $\tau^* : T^*M \to M$ its cotangent bundle and $T\tau^* : TT^*M \to TM$ the differential of τ^*. T^*M carries a natural 1-form θ defined as

$$\theta(\xi) := p(T\tau^* \xi) \qquad \text{for} \quad \xi \in T_p T^*M$$

and a symplectic form $\omega := -d\theta$. Furthermore, there is a canonical vector field η on T^*M satisfying

$$d\theta(\eta, .)=\theta.$$

We say that $H \in C^\infty(S^1 \times T^*M, \mathbb{R})$ satisfies *condition* (H) if

(H1)
$$dH(\eta)(t, q, p) - H(t, q, p) \geqq \kappa |p|^2 - d_1,$$

(H2)
$$\begin{cases} \left|\dfrac{\partial^2 H}{\partial p_i \partial p_j}(t, q, p)\right| \leqq d_2, \\ \left|\dfrac{\partial^2 H}{\partial p_i \partial q_j}(t, q, p)\right| \leqq d_2, \end{cases}$$

for all $(t, q, p) \in S^1 \times T^*M$, with respect to a suitable metric on the bundle $T^*M \to M$ and constants $\kappa > 0$, d_1 and d_2. Here q and p denote the base and fibre part of an $x=(q, p) \in T^*M$, and in (H2) $q_1, \ldots, q_n, p_1, \ldots, p_n$ are coordinates on T^*M induced by geodesic normal coordinates $q_1, \ldots, q_n$ on M.

Let $\Lambda := \Lambda M := H^{1,2}(S^1, M)$ be the loop space of M (*see* Lemma 4.1) and Λ^{contr} the component of the constant loops. For $x \in \Lambda$ we define the action

$$\Phi(x) := \Phi_H(x) := \int_0^1 x^* \theta - \int_0^1 H(t, x(t))\, dt.$$

Finally, for a metric space X and a ring R we set

$$\text{cuplength}_R(X) := \sup\{k \in \mathbb{N} \mid \text{There exist } \alpha_i \in H^{n_i}(X, R),\ n_i \geqq 1,$$
$$1 \leqq i \leqq k-1 \text{ such that } \alpha_1 \cup \ldots \cup \alpha_{k-1} \neq 0\}$$

with $\mathbb{N}$ denoting the natural numbers including 0 and H^* the singular cohomology. Our main theorem is the following:

1.1. Main Theorem. – *Let* M, H *and* $\mathfrak{C} = \mathfrak{C}(T^*M, H)$ *be as above.*

(i) *If* $\pi_1(M)$ *is finite then* $\# \mathfrak{C} = \infty$, *and* Φ_H *is not bounded from above on* $\mathfrak{C}$.

(ii) *In any case* $\# \mathfrak{C} \geqq \text{cuplength}_{\mathbb{Z}_2}(\Lambda^{\text{contr}})$.

Remarks. – (1) For M being a torus this result has already been proved in 1983 by C. Conley and E. Zehnder ([CZ], *see* also [Jo] for a generalisation). In the general case no such result is known to us. A variational problem in which the cohomology of the loop space occurs was used in 1988 by H. Hofer and C. Viterbo in their proof of the Weinstein conjecture on cotangent bundles ([HV1], especially compare their Prop. 1 to our Theorem 7.6).

(2) In 1983 V. Benci obtained the corresponding result for Lagrangian systems [Be]. However, since we have made no convexity assumption on H, there is no Legendre transform available, so here Benci's method doesn't work.

(3) If $\pi_1(M)$ is infinite we cannot in general expect $\mathfrak{C}$ to be infinite. For example, one can easily construct a Hamiltonian H on the cotangent bundle of the 2-torus T^2 satisfying the assumptions of 1.1 and such that $\#\mathfrak{C}=3=\text{cuplength}_{\mathbb{Z}_2}(T^2)$ (so the estimate given by 1.1 (ii) is optimal in this case).

(4) Note that condition (H1) is invariant under changes of the metric and under diffeomorphisms of M. It cannot be removed completely as simple examples show. In contrast, condition (H2) depends sensitively on the metric, and it can presumably be dropped. Condition (H) is satisfied by all physical Hamiltonians, *i.e.* those of the form $H(t, q, p)=(1/2)|p-A(t, q)|^2+V(t, q)$.

As immediate corollaries we have:

1.2 Corollary. — $\#\mathfrak{C}\geqq \text{cuplength}_{\mathbb{Z}_2}(M)$.

Proof. — From the Leray-Hirsch theorem ([Hu], Ch. 16, Th. 1.1 and Rem. 1.2) applied to the fibration of Λ^{contr} over M it follows that $\text{cuplength}_{\mathbb{Z}_2}(\Lambda^{\text{contr}})\geqq \text{cuplength}_{\mathbb{Z}_2}(M)$. □

1.3 Corollary (C. Conley, E. Zehnder, 1983). — *Let* T^n *be the n-dimensional torus and* $H\in C^\infty(S^1\times T^n\times \mathbb{R}^n, \mathbb{R})$ *satisfying*: $H(t, q, p)=(1/2)\langle p, Bp\rangle+\langle A, p\rangle$ *for* $|p|\geqq C>0$. *Here* $\langle\ ,\ \rangle$ *is the scalar product on* $\mathbb{R}^n$, $B\in\mathbb{R}^{n\times n}$ *a constant symmetric positive definite matrix and* $A\in\mathbb{R}^n$ *a constant.*

Then there are at least $n+1$ *contractible* 1-*periodic solutions of* $\dot{x}=X_H(t, x)$ *contained in* $\{(q, p)\in T^n\times\mathbb{R}^n \mid |p|<C\}$.

Proof (*cf.* [CZ], Theorem 3). — H satisfies the assumptions of 1.1, so we have $\#\mathfrak{C}\geqq\text{cuplength}_{\mathbb{Z}_2}(T^n)=n+1$. By integrating the Hamiltonian equation for $|p|\geqq C$ one easily sees that periodic solutions with $|p(t)|\geqq C$ for some $t\in S^1$ cannot be contractible (*see* [CZ], proof of Theorem 3). Hence all $x\in\mathfrak{C}$ stay in $\{(q, p)\in T^n\times\mathbb{R}^n \mid |p|<C\}$. □

Applying 1.1 to the Hamiltonian $H(t, q, p)=(1/2)|p|^2$ we obtain the classical result of L. Lusternik and A. Fet that every compact Riemannian manifold with finite fundamental group possesses a nonconstant contractible closed geodesic.

Finally we note that the method used to prove Theorem 1.1 also works to give the following generalization of a result by V. Benci, D. Fortunato and F. Giannoni:

1.4 Proposition. — *Let* $(M_1, \langle\ ,\ \rangle_1)$ *be a compact Riemannian manifold with* $\pi_1(M_1)$ *finite. Let* $M:=\mathbb{R}\times M_1$ *be a static space-time,* i.e. *the Lorentz metric on* M *is given by* $\langle\ ,\ \rangle=\langle\ ,\ \rangle_1-\beta(q_1)\,dq_0^2$ *for* $(q_0, q_1)\in\mathbb{R}\times M_1$ *and a strictly positive* $\beta\in C^\infty(M_1, \mathbb{R})$. *Let* $H\in C^\infty(S^1\times T^*M, \mathbb{R})$ *be of the form* $H(t, q, p)=H_1(t, q_1, p_1)-(1/2)\beta(q_1)p_0^2$ *with* $H_1\in C^\infty(S^1\times T^*M_1, \mathbb{R})$ *satisfying* (H1) *and* (H2). *Define* X_H *to be the Hamiltonian vector field with respect to the canonical symplectic form on* T^*M. *For* $T>0$ *set*

$$\mathfrak{C}_T:=\{x=(x_0, x_1)=(q_0, p_0, q_1, p_1)\in C^\infty([0, 1], T^*M) \mid p_0 \text{ and } x_1 \text{ are } 1\text{-periodic and contractible},\ q_0(0)=0,\ q_0(1)=T,\ \dot{x}=X_H(t, x)\}.$$

Then for every $T>0: \#\mathfrak{C}=\infty$.

Proof (*cf.* [BFG], Theorem 1.5). — For $x_1=(q_1, p_1)\in C^\infty(S^1, TM_1)$ and $T>0$ we define

$$\Phi_T(x_1):=\int_0^1 x_1^*\theta_1-\int_0^1 H_1(t, x_1)\,dt-\frac{1}{2}T^2\left[\int_0^1 \frac{1}{\beta(q_1)}\,dt\right]^{-1}.$$

A simple calculation analogous to [BFG], proof of Theorem 2.1, shows the equivalence of the following two statements:

(*a*) $x=(x_0, x_1)\in \mathfrak{C}_T$.

(*b*) $x_1\in C^\infty_{\text{contr}}(S^1, TM_1)$ with $\Phi'_T(x_1)=0$. Furthermore, $x_0=(q_0, p_0)\in C^\infty([0, 1], \mathbb{R}^2)$ with $\dot q_0(t)=T\left[\int_0^1 (1/\beta(q_1(\tau)))\,d\tau\right]^{-1}(1/\beta(q_1(t)))$, $q_0(0)=0$ and $p_0=\dot q_0$.

In addition, if (*a*) or (*b*) holds then $\Phi_H(x)=\Phi_T(x_1)$.

Now the functional Φ_T doesn't fit exactly in the framework of 1.1. But since $\Phi_T(x_1)=\Phi_{H_1}(x_1)+f(q_1)$ and $\Phi'_T(x_1)=\Phi'_{H_1}(x_1)+g(q_1)$, Φ_T behaves asymptotically like Φ_{H_1}, and the proof of 1.1 still works for Φ_T and gives the existence of infinitely many solutions of (*b*), hence of (*a*). □

Note added after revision. — A refinement of our proof also yields the following two results similar to those obtained with different methods by C. Gole in [Go]:

1. If $H(t, q, p)=|p|^2$ for $|p|\geqq c$ then there are at least $\text{cuplength}_{\mathbb{Z}_2}(M)$ elements of $\mathfrak{C}$ staying in $\{(q, p)\in T^*M \mid |p|\leqq c\}$.

2. In any free homotopy class of T^*M there are at least two 1-periodic orbits.

The proof of 1.1 will occupy the rest of this paper (see the next section for a sketch of the proof). We have omitted some arguments usually considered as "standard". Detailed proofs of all statements may be found in [Ci].

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2. Setup and sketch of the proof

By means of the metric-induced isomorphism $TM\cong T^*M$ we shall transfer everything from the cotangent to the tangent bundle, without changing our notation.

To find 1-periodic solutions the variational principle for the action functional $\Phi=\Phi_H$ will be used: If $x\in C^\infty(S^1, TM)$ and $\xi\in C^\infty(x^*TTM)$ is a smooth section in the pullback bundle $x^*TTM\to S^1$ we have the derivative

$$d\Phi_H(x)\xi:=\left.\frac{d}{ds}\right|_{s=0}\Phi_H(\exp_x s\xi)\in\mathbb{R}.$$

We introduce an "L^2 gradient" $\Phi'_H(x) \in L^2(x^* TTM)$ by requiring

$$d\Phi_H(x)\xi = \int_0^1 \langle \Phi'_H(x), \xi \rangle \, dt \qquad \text{for all} \quad \xi \in C^\infty(x^* TTM),$$

where $\langle \cdot, \cdot \rangle = \langle T\tau \cdot, T\tau \cdot \rangle + \langle K \cdot, K \cdot \rangle$ is the product on $TTM \to TM$ induced by the metric $\langle \; , \; \rangle$ on M and its Levi-Civita connection $K : TTM \to TM$. One easily sees:

$$x \text{ is a 1-periodic solution} \Leftrightarrow d\Phi(x) = 0 \Leftrightarrow \Phi'(x) = 0.$$

The reason that Φ' is – in contrast for example to the $H^{1,2}$-gradient of Φ – accessible to variational methods lies in its connection with pseudo-holomorphic curves, which were shown by M. Gromov [Gr] to have very nice analytical properties. To see this, let $J : TTM \to TTM$ be the almost complex structure on TM defined by

$$T\tau \circ J = K \qquad \text{and} \qquad K \circ J = -T\tau.$$

Then

$$\Phi'(x) = -J(x)\dot{x} - H'(t, x),$$

and the equation of the "flow" of Φ' becomes $\partial_s u = -J(u)\partial_t u - H'(t, u)$, *i.e.* a perturbed Cauchy-Riemann equation.

Let

$$X := \left\{ u \in C^\infty(\mathbb{R} \times S^1, TM) \,\middle|\, \partial_s u + J(u)\partial_t u + H'(t, u) = 0, \int_{\mathbb{R} \times S^1} |\partial_s u|^2 \, ds \, dt < \infty \right\}$$

be the space of "connecting trajectories", turned into a metric space with the C^∞_{loc}-topology (7.4). Although Φ is neither bounded from above nor from below on $C^\infty(S^1, TM)$, the map $\hat{\Phi}(u) := \Phi(u(0, .))$ turns out to be bounded from below on X. There is a natural flow on X given by translation in the negative $\mathbb{R}$-direction. $\hat{\Phi}$ is strictly decreasing along the flow lines, and the rest points of the flow are exactly the 1-periodic solutions (7.5).

So we have successfully transformed the original problem into a more obscure one, now having to find rest points of a flow on some metric space X. But in fact, the space X is not that bad! Namely, it has compact flow-invariant subspaces which carry a lot of nontrivial cohomology:

(A) The sets $\Phi^c_+ := \{ u \in X \mid \Phi(u(s, .)) \leq c \text{ for all } s \in \mathbb{R} \}$ are compact for each $c \in \mathbb{R}$ (7.5).

(B) If $\pi : X \to \Lambda$ denotes the projection $u \mapsto u(0, .)$, then for each nontrivial cohomology class $\alpha \in \bar{H}^*(\Lambda^{contr}, \mathbb{Z}_2)$ there exists a $c \in \mathbb{R}$ such that $(\pi|_{\Phi^c_+})^* \alpha \neq 0$ in $\bar{H}^*(\Phi^c_+, \mathbb{Z}_2)$ (7.6).

1.1 (ii) now follows directly by Lusternik-Schnirelman theory. To prove (i) we associate to each $0 \neq \alpha \in \bar{H}^*(\Lambda^{contr}, \mathbb{Z}_2)$ a minimax value c^0_α such that there exists a fixed point u of the flow with $\hat{\Phi}(u) = c^0_\alpha$ (8.3). Using a result by D. Sullivan on the homology of the loop space of a simply connected manifold (8.1), we deduce that the family (c_α^0) cannot be bounded from above (8.4).

Of course, the hard part is the proof of (A) and (B).

The main difficulty in (A) is the lack of an *a priori* C^0-bound. This is overcome as follows: We cut the cylinder $\mathbb{R} \times S^1$ into small cylinders on the boundaries of which we obtain uniform C^0-bounds (5.3). Then we use a refined maximum principle (5.1) to extend the C^0-estimate to their interior (5.4).

Once we have a C^0-bound the boundedness in C^∞ (6.4) follows by standard bubbling-off analysis. As we also need a compactness result for finite cylinders with boundary, we must investigate the bubbling-off not only of spheres but also of holomorphic disks with boundary in a Lagrangian submanifold (6.3).

For (B) we consider the nonlinear Cauchy-Riemann operator on finite cylinders with certain Lagrangian boundary conditions. Namely, given a $q \in C^\infty_{\text{contr}}(S^1, M)$, regard $u \mapsto \partial_s u + J(u)\partial_t u + H'(u)$ as a smooth operator D from the Hilbert manifold

$$\mathscr{E}_q := \{ u \in H^{2,2}([0, R] \times S^1, TM) \mid u(0, t) \in M,\ u(R, t) \in T_{q(t)} M \},$$

into a suitable Hilbert space F (4.1). Some restriction of D turns out to be a proper Fredholm operator of index 0 and $\mathbb{Z}_2$-degree 1. Thus the same is true for the operator $f_1 : \mathscr{E}^{\text{contr}} := \{ (u, q) \mid q \in \Lambda^{\text{contr}},\ u \in \mathscr{E}_{\mathscr{I}_q} \} \to F \times \Lambda^{\text{contr}}$ given by $(u, q) \mapsto (Du, q)$ (4.3). (Here $\mathscr{I} : \Lambda \to \Lambda$ is a smoothing operator.) Approximating Λ^{contr} by finite dimensional manifolds (7.2) one shows that f_1, and therefore also the projection $\{ (u, q) \in \mathscr{E}^{\text{contr}} \mid Du = 0 \} \to \Lambda^{\text{contr}}$, induces an injection in $\mathbb{Z}_2$-cohomology (7.3). Finally we stretch the cylinder and use a continuity property of the Alexander-Spanier cohomology to obtain (B).

The sections of this paper can in principle be read independently (except Sections 7 and 8).

In Section 3 we calculate the Fredholm index of the linear Cauchy-Riemann operator on finite cylinders with Lagrangian boundary conditions (3.3). For the proof of 1.1 only 3.3 (ii) with $l = 0$ will actually be needed.

Section 4 is rather technical. Here the Fredholm index and $\mathbb{Z}_2$-degree of the nonlinear Cauchy-Riemann operator f_1 is determined.

Section 5, which might be most interesting for the reader familiar with the subject, is concerned with the C^0-bound.

The bubbling-off analysis is contained in section 6.

Section 7 deals with the injection in $\mathbb{Z}_2$-cohomology. Together with Section 8 it constitutes the skeleton the proof.

3. A linear boundary value problem

This section closely follows [Sa], Section 5.4.

Let $H^{m,p}$, $m \in \mathbb{N}$, $1 \leqq p < \infty$ be the Sobolev classes and $\| \ \|_{m,p}$ the corresponding norms. In this section we will denote by $(\ ,\ )$ the real scalar product on $\mathbb{R}^n$ or $\mathbb{C}^n$ and by $\langle f, g \rangle := \int (f, g)\, dx$ the canonical pairing between L^p and L^q with $1 \leqq p, q \leqq \infty$, $(1/p)+(1/q)=1$.

Define

$$\begin{aligned} B_R &:= \{ z = s + it \in \mathbb{C} \mid |z| < R \}, \\ B_R^+ &:= \{ z \in B_R \mid \operatorname{Im}(z) \geqq 0 \}, \\ D_R &:= C_0^\infty (B_R, \mathbb{C}^n), \\ D_R^+ &:= \{ u \in C_0^\infty (B_R^+, \mathbb{C}^n) \mid u(s, 0) \in \mathbb{R}^n \text{ for all } s \in \mathbb{R} \}. \end{aligned}$$

Let i denote the canonical complex structure in $\mathbb{C}^n$, $\bar{\partial} := \partial_s + i \partial_t$ the Cauchy-Riemann operator and $\partial := \partial_s - i \partial_t$ its formal adjoint.

We are not going to prove the following standard result (*cf.* [Fl1], Lemma 2.2):

3.1 Lemma. – *For $m \in \mathbb{N}$, $1 < p < \infty$ and $R > 0$ there exists a constant $c = c(m, p, R)$ so that*

$$\begin{aligned} \| u \|_{H^{m+1,p}(B_R)} &\leqq c \| \bar{\partial} u \|_{H^{m,p}(B_R)} \quad \textit{for all} \quad u \in D_R, \\ \| u \|_{H^{m+1,p}(B_R^+)} &\leqq c \| \bar{\partial} u \|_{H^{m,p}(B_R^+)} \quad \textit{for all} \quad u \in D_R^+. \quad \square \end{aligned}$$

For a fixed $R > 0$ we set $Z_R := [0, R] \times S^1$. Let $\Lambda_0, \Lambda_R : S^1 \to \mathcal{L}(n)$ be smooth loops of Lagrangian subspaces of $\mathbb{C}^n$ (equipped with the canonical symplectic structure). For $m \in \mathbb{N}$, $1 < p < \infty$ and $\Lambda = (\Lambda_0, \Lambda_R)$ we define

$$H_\Lambda^{m,p} := \{ u \in H^{m,p}(Z_R, \mathbb{C}^n) \mid u(0, t) \in \Lambda_0(t),\ u(R, t) \in \Lambda_R(t) \text{ for almost all } t \in S^1 \}.$$

Here "for almost all" t means: For all t outside a set of measure zero. This makes sense because there exist continuous trace operators $H^{1,p}(Z_R) \to L^p(\partial Z_R)$ ([A1], A 5.7).

Covering Z_R by disks and half-disks, trivializing Λ_0, Λ_R and applying 3.1 one shows: There exist constants $c = c(m, p, R)$ such that, if $u \in H_\Lambda^{1,p}$ and $\bar{\partial} u \in H^{m,p}(Z_R, \mathbb{C}^n)$, then $u \in H_\Lambda^{m+1,p}$ and

$$\| u \|_{m+1,p} \leqq c (\| \bar{\partial} u \|_{m,p} + \| u \|_{0,p}).$$

Now generally, if E, F, G are Banach spaces, $T \in L(E, F)$ a bounded linear operator and $K \in K(E, G)$ a compact linear operator such that

$$\|x\|_E \leq c(\|Tx\|_F + \|Kx\|_G) \qquad \text{for all} \quad x \in E$$

then T is a semi-Fredholm operator.

It follows that $\bar{\partial}: H_\Lambda^{m+1,p} \to H^{m,p}(Z_R, \mathbb{C}^n)$ is a semi-Fredholm operator.

3.2 Proposition. — *Regard* $\bar{\partial}: L^p \supset H_\Lambda^{1,p} \to L^p$ *as an unbounded operator on* $L^p(Z_R, \mathbb{C}^n)$ *with dense domain* $H_\Lambda^{1,p}$ *and let* $(1/p)+(1/q)=1$. *Then its adjoint operator is given by*

$$-\partial: L^q \supset H_{i\Lambda}^{1,q} \to L^q.$$

Proof. — For $v \in H_\Lambda^{1,p}$ and $w \in H_{i\Lambda}^{1,q}$ we have

$$\langle \bar{\partial} v, w \rangle + \langle v, \partial w \rangle = \int_{\partial Z_R} (v, w)\, dt = 0$$

because $(v, w)=0$ on almost all of ∂Z_R. The harder part is to show that, if $w \in L^q(Z_R, \mathbb{C}^n)$ satisfies

$$|\langle w, \bar{\partial} v \rangle| \leq c\|v\|_{0,p} \qquad \text{for all} \quad v \in H_\Lambda^{1,p},$$

then $w \in H_{i\Lambda}^{1,q}$. By localization this statement can be reduced to the following: If $w \in L^q(B_r^+, \mathbb{C}^n)$ vanishes outside some compact $K \subset B_r^+$ and satisfies

$$|\langle w, \bar{\partial} v \rangle| \leq c\|v\|_{0,p} \qquad \text{for all} \quad v \in D_r^+,$$

then $w \in H^{1,q}$ and $w(s, 0) \in \mathbb{R}^n$ for almost all $s \in (-r, r)$. Note that $w(s, 0) \in \mathbb{R}^n$ rather than $i\mathbb{R}^n$ because the roles of s and t have been interchanged.

We shall make use of the following smoothing operator: Choose a $\rho \in C^\infty(\mathbb{R}^k, [0, 1])$ having its support in $B_1(\mathbb{R}^k) := \{x \in \mathbb{R}^k \mid |x| < 1\}$ and satisfying $\rho(x) = \rho(|x|)$, $\int_{\mathbb{R}^k} \rho(x)\, dx = 1$. For $f \in L_{loc}^1(\mathbb{R}^k, \mathbb{R}^l)$ and $\varepsilon > 0$ define

$$\mathscr{I}_\varepsilon f(x) := \varepsilon^{-k} \int_{\mathbb{R}^k} \rho\left(\frac{x-y}{\varepsilon}\right) f(y)\, dy,$$

(*see* [Fr], Chapter 6 for some elementary properties of $\mathscr{I}_\varepsilon$).

For w as above we define $\hat{w} \in L^q(B_r, \mathbb{C}^n)$ as

$$\hat{w}(s, t) := \begin{cases} w(s, t) & \text{if} \quad t \geq 0, \\ \overline{w(s, -t)} & \text{if} \quad t < 0. \end{cases}$$

Let $\hat{\phi}_m := \mathscr{I}_{1/m} \hat{w} \in C_0^\infty(B_r)$ and $\phi_m := \hat{\phi}_m|_{B_r^+} \in D_r^+$. By the Riesz representation theorem there exists a $u \in L^q(B_r^+)$ such that $\langle w, \bar{\partial} v \rangle = \langle u, v \rangle$ for all $v \in D_r^+$.

Given sufficiently large $l, m \in \mathbb{N}$ and any $v \in D_r^+$ with support in the interior of K we calculate:

$$\begin{aligned} 2|\langle \partial\phi_m - \partial\phi_l, v\rangle_{B_r^+}| &= 2|\langle \phi_l - \phi_m, \bar\partial v\rangle_{B_r^+}| \\ &= |\langle \hat\phi_l - \hat\phi_m, \bar\partial \hat v\rangle_{B_r}| \\ &= |\langle \hat w, \bar\partial(\mathcal{J}_{1/l} - \mathcal{J}_{1/m})\hat v\rangle_{B_r}| \\ &= 2|\langle w, \bar\partial(\mathcal{J}_{1/l} - \mathcal{J}_{1/m})\hat v\rangle_{B_r^+}| \\ &= 2|\langle u, (\mathcal{J}_{1/l} - \mathcal{J}_{1/m})\hat v\rangle_{B_r^+}| \\ &= |\langle (\mathcal{J}_{1/l} - \mathcal{J}_{1/m})\hat u, \hat v\rangle_{B_r}| \\ &\leq c\|(\mathcal{J}_{1/l} - \mathcal{J}_{1/m})\hat u\|_{0,q}\|v\|_{0,p} \end{aligned}$$

with a constant c independent of l, m and v. Note that we used the hypothesis on w in the fifth equality since the restriction of $(\mathcal{J}_{1/l} - \mathcal{J}_{1/m})\hat v$ to B_r^+ is in D^+ but need no longer have support in the interior of K.

It follows that $(\partial\phi_m)$ is a Cauchy sequence in $L^q(K, \mathbb{C}^n)$. If we replace $\bar\partial$ by ∂ in 3.1 we obtain that (ϕ_m) is a Cauchy sequence in $H^{1,q}(K)$. Since $(\phi_m) \subset D^+$ and obviously $\phi_m \to w$ in L^q, the conclusion for w follows. □

As an immediate corollary of 3.2 we have: $\bar\partial : H_\Lambda^{0,p} \to L^p$ is a Fredholm operator of index

$$\dim\ker(\bar\partial : H_\Lambda^{0,p} \to L^p) - \dim\ker(\partial : H_{i\Lambda}^{0,q} \to L^q).$$

To describe the Fredholm index we will use the Maslov index $\mu : C^0(S^1, \mathcal{L}(n)) \to \mathbb{Z}$ for loops of Lagrangian subspaces as defined in [Sa], Section 5.4. It has the following nice property ([Sa], Lemma 5.4.1):

Two loops $\Lambda_1, \Lambda_2 \in C^0(S^1, \mathcal{L}(n))$ are homotopic if and only if $\mu(\Lambda_1) = \mu(\Lambda_2)$.

The main result of this section is

3.3 Theorem. – *Let $m \in \mathbb{N}$, $1 < p < \infty$ and $\Lambda_0, \Lambda_R \in C^\infty(S^1, \mathcal{L}(n))$. Then*
(i) *$\bar\partial : H_\Lambda^{m+1,p} \to H^{m,p}(Z_R, \mathbb{C}^n)$ is a Fredholm operator of index*

$$\mu(\Lambda_R) - \mu(\Lambda_0).$$

(ii) *If $\Lambda_0(t) = e^{i\pi lt}\mathbb{R} \times \mathbb{R}^{n-1}$ and $\Lambda_R(t) = i(e^{i\pi lt}\mathbb{R} \times \mathbb{R}^{n-1})$ for some $l \in \mathbb{Z}$, then $\bar\partial$ is an isomorphism.*

Proof. – From regularity theory it follows that the kernel and cokernel of $\bar\partial : H_\Lambda^{m+1,p} \to H^{m,p}(Z_R, \mathbb{C}^n)$ do not depend on m and p. So we may take $m = 0$.

In view of the homotopy invariance of the Fredholm index and the above property of the Maslov index it suffices to calculate the Fredholm index for some particular choice of loops Λ_0, Λ_R with $\mu(\Lambda_0) = k$, $\mu(\Lambda_R) = l$, for given $k, l \in \mathbb{Z}$. According to [Sa], 5.4.3, we choose $\Lambda_0(t) := e^{i\pi kt}\mathbb{R} \times \mathbb{R}^{n-1}$, $\Lambda_R(t) = i(e^{i\pi lt}\mathbb{R} \times \mathbb{R}^{n-1})$.

If $\bar\partial u=0$, u can be expanded in a Laurent series. The boundary conditions then give relations between the coefficients, and one easily obtains:

$$\dim\ker(\bar\partial)=\begin{cases} l-k, & \text{if } l>k;\\ 0, & \text{if } l\leqq k.\end{cases}$$

Similarly, for the adjoint operator $-\partial$ of 3.3 we find:

$$\dim\ker(\partial)=\begin{cases} k-l, & \text{if } l\leqq k;\\ 0, & \text{if } l\geqq k.\end{cases}$$

The theorem now follows. □

4. A nonlinear boundary value problem

If A is a compact l-dimensional manifold-with-boundary, $m\in\mathbb{N}$, $1<p<\infty$ such that $mp>l$, we denote by $H^{m,p}(A, M)$ the Banach manifold of $H^{m,p}$ maps from A to M (*see* [El], Theorem 5.1, and [Pa2], Corollary 9.7). For $k\in\mathbb{N}$, $k<m$ and $1<q<\infty$ or $k=m$ and $1<q\leqq p$ we have Banach bundles $H^{k,q}(H^{m,p}(A, M)^* TM)$ consisting of $H^{k,q}$ sections in TM over base curves of class $H^{m,p}$ ([El], Theorem 6.1 and [Pa2], Theorem 9.6). We imbed M isometrically into $\mathbb{R}^N$ for some $N\in\mathbb{N}$ [Na] and choose a tubular neighborhood U of M in $\mathbb{R}^N$ with smooth projection $p:U\to M$.

Let $0<\delta<1/3\, i(M)$ such that $\{x\in\mathbb{R}^N \mid \operatorname{dist}(x, M)<\delta\}\subset U$, where $i(M)$ denotes the injectivity radius of M. Choose $a\geqq 1$ big enough so that, if $x\in M$ and $y\in\mathbb{R}^N$ with $\|x-y\|<\delta/a$, then $d(x, p(y))<\delta$ (d denoting the metric on M).

We define $\varepsilon:\Lambda\to\mathbb{R}^+$ by $\varepsilon(q):=(1/a^2)\min((\delta/\|q\|_{1,2})^2, \delta)$ and $\mathcal{J}:\Lambda\to\Lambda$ by

$$\mathcal{J}q:=p\circ\mathcal{J}_{\varepsilon(q)}q$$

with $\mathcal{J}_\varepsilon$ as in Section 3. The following properties follow immediately from the construction and the properties of $\mathcal{J}_\varepsilon$:

(i) $\mathcal{J}:\Lambda\to C^\infty(S^1, M)$ is continuous.

(ii) $d(\mathcal{J}q(t), q(t))<\delta$ for all $t\in S^1$.

(iii) $\mathcal{J}\sim\mathrm{id}:\Lambda\to\Lambda$.

(iv) For $mp>1$, $\mathcal{J}:\{q\in\Lambda \mid \|q\|_{1,2}<\delta\}\to H^{m,p}(S^1, M)$ is a smooth map between Banach manifolds.

We shall identify M with the zero section in $\tau:TM\to M$

4.1. Lemma and Definition. — (i) *There exists a (smooth) Hilbert manifold*

$$\mathcal{E}:=\{(u, q)\in H^{2,2}(Z_R, TM)\times\Lambda M \mid u(0, t)\in M,\ u(R, t)\in T_{\mathcal{J}q(t)}M \textit{ for all } t\in S^1\}.$$

(ii) *There exists a (smooth) Hilbert bundle* $\mathcal{F}\to\mathcal{E}$ *with fibres* $\mathcal{F}_{(u,q)}=H^{1,2}(u^* TTM)$.

Proof. — (i) We shall construct charts for $\mathscr{E}$ as a submanifold of $\mathscr{E}' := H^{2,2}(Z_{\mathbf{R}}, TM) \times \Lambda$. Therefore, let K be the Levi-Civita connection on $\tau : TM \to M$ and $\exp : TM \to M$ the exponential map. K induces a symmetric connection K_T on $\tau_1 : TTM \to TM$ ([El], Theorem 3.1) with the corresponding exponential map given by

$$\exp_T = T\exp \circ (\tau_1, T\tau, K)^{-1} \circ (T\tau, \tau_1, K) : TTM \to TM.$$

Take $\mathscr{D} := \{\xi \in TM \mid |\xi| < \delta\}$ and $\mathscr{D}^T$ any neighborhood of the zero section in TTM such that $(\tau_1, \exp_T) : \mathscr{D}^T \to TM \times TM$ is injective. Then by [El], Theorem 5.1 charts for $\mathscr{E}'$ are given by

$$\exp_T \times \exp : H^{2,2}(u^* \mathscr{D}^T) \times H^{1,2}(q^* \mathscr{D}) \to U' \subset \mathscr{E}'$$

with $u \in C^\infty(Z_{\mathbf{R}}, TM)$, $q \in C^\infty(S^1, M)$. If in addition

$$(u, q) \in \mathscr{E}, \qquad (\eta, \xi) \in H^{2,2}(u^* \mathscr{D}^T) \times H^{1,2}(q^* \mathscr{D}) \qquad \text{and} \qquad t \in S^1$$

we calculate:

$$\begin{aligned} d(\mathscr{I} q(t), \mathscr{I} \exp \xi(t)) &\leqq d(\mathscr{I} q(t), q(t)) + d(q(t), \exp \xi(t)) + d(\exp \xi(t), \mathscr{I} \exp \xi(t)) \\ &\leqq \delta + |\xi(t)| + \delta \\ &\leqq 3\delta < i(M). \end{aligned}$$

So there exist a unique section $\bar{\xi} \in C^\infty((\mathscr{I} q)^* TM)$ such that $\exp \bar{\xi}(t) = \mathscr{I} \exp \xi(t)$ for all $t \in S^1$. Using $T\tau \circ \eta(R, t) \in T_{\mathscr{I} q(t)} M$ and $\tau \exp_T = \exp \circ T\tau$ we see:

$$\begin{aligned} \tau \exp_T \eta(R, t) = \mathscr{I} \exp \xi(t) &\Leftrightarrow \exp \circ T\tau \circ \eta^h(R, t) = \exp \bar{\xi}(t) \\ &\Leftrightarrow T\tau \circ \eta^h(R, t) = \bar{\xi}(t). \end{aligned}$$

Here η^h and η^v denote the horizontal and vertical part of η according to the splitting $TTM = T^h TM \oplus T^v TM$ induced by K. By means of a cutoff function we extend $(T\tau|_{T^h TM})^{-1} \circ \bar{\xi}$ to a section $\xi' \in C^\infty(u^* T^h TM)$. It follows from property (iv) of $\mathscr{I}$ that $\xi \mapsto \xi'$ defines a smooth map $H^{1,2}(q^* \mathscr{D}) \to H^{2,2}(u^* TTM)$. Thus we have shown:

$$\exp_T \eta(R, t) \in T_{\mathscr{I} \exp \xi(t)} M \Leftrightarrow \eta^h(R, t) - \xi'(R, t) = 0.$$

Similarly,

$$\exp_T \eta(0, t) \in M \Leftrightarrow \eta^v(0, t) = 0.$$

Changing notation from η to $\eta - \xi'$ we define

$$\begin{aligned} E &:= \{(\eta, \xi) \in H^{2,2}(u^* TTM) \times H^{1,2}(q^* TM) \mid \eta^v(0, t) = 0 = \eta^h(R, t) \text{ for all } t \in S^1\}, \\ V &:= \{(\eta, \xi) \in E \mid \|\xi\|_{1,2} < \delta,\ \eta + \xi' \in H^{2,2}(u^* \mathscr{D}^T)\}. \end{aligned}$$

Then E is a closed subspace of $H^{2,2}(u^* TTM) \times H^{1,2}(q^* TM)$. $V \subset E$ is open, and a chart for $\mathscr{E}$ is given by $\phi^{-1} : V \to U \subset \mathscr{E}$,

$$\phi^{-1}(\xi, \eta) = (\exp_T(\eta + \xi'), \exp \xi).$$

(ii) follows immediately from [El], Theorem 6.1. With the notation as in (i) a local trivialization is given by $\Phi^{-1}: E \times F \supset V \times F \to \mathscr{F}|_U$,

$$\Phi^{-1}((\eta, \xi), \zeta) = \nabla_2 (\exp_T)_{(\eta+\xi')} \zeta$$

with $F := H^{1,2}(u^* TTM)$ and

$$\nabla_2 (\exp_T)_y := T_y \exp \circ (K_T|_{T_y^v TTM})^{-1} : T_x TM \to T_{\exp_T(y)} TM, \quad y \in T_x TM. \quad \square$$

By [El], Theorem 6.2, $\bar{\partial}: (u, q) \mapsto \bar{\partial} u := \partial_s u + J(u) \partial_t u$ is a smooth section in the bundle $\mathscr{F} \to \mathscr{E}$ (with J as in Section 2), and so is

$$g_\lambda(u, q) := \begin{cases} \bar{\partial} u + 2\lambda H_0'(u) & \text{for } \lambda \in [0, 1/2], \\ \bar{\partial} u + 2(1-\lambda) H_0'(u) + (2\lambda - 1) H'(t, u) & \text{for } \lambda \in [1/2, 1], \end{cases}$$

where H, $H_0 \in C^\infty(S^1 \times TM, \mathbb{R})$, $H_0(q, p) = (1/2)|p|^2$.

As a consequence of Kuiper's theorem every Hilbert bundle with infinite fibre dimension over a paracompact Hilbert manifold is trivial ([Pa1], Chapter 9A, Corollary C6). Now the Banach manifolds $H^{m,p}(A, M)$ are imbedded in the Banach spaces $H^{m,p}(A, \mathbb{R}^N)$, hence metrizable and paracompact, and the same is true for $\mathscr{E}$. So we get a smooth bundle isomorphism $\Psi: \mathscr{F} \to \mathscr{E} \times F$. Denoting by π_F and π_Λ the projections from $\mathscr{E} \times F$ to F and from $\mathscr{E}$ to Λ we define

$$f_\lambda := (\pi_F \circ \Psi \circ g_\lambda, \pi_\Lambda) : \mathscr{E} \to F \times \Lambda.$$

As we will have to talk about the Fredholm property and the $\mathbb{Z}_2$ degree of f_λ it is a considerable simplification that we can deal with a map between manifolds instead of a section in a bundle. This is the reason why we chose the Hilbert bundle setting. Let $\Delta_\lambda := f_\lambda^{-1}(\{0\} \times M)$, where $M \subset \Lambda$ as constant loops, and $\Delta_I := \{(\lambda, x) \mid \lambda \in I, x \in \Delta_\lambda\}$ for $I \subset [0, 1]$.

4.2. PROPOSITION. — (i) *If* $g_\lambda(x) = 0$ *then* $Df_\lambda(x): T_x\mathscr{E} \to F \times \Lambda$ *is a semi-Fredholm operator.*

(ii) *For* $\lambda \in [0, 1/2]$, Δ_λ *consists of constant maps.*

(iii) *For* $x \in \Delta_0$, $Df_0(x)$ *is an isomorphism.*

Proof. — (i) Take a local trivialization (U, E, ϕ, F, Φ) as in the proof of 4.1. The representation of $\bar{\partial}$ in this trivialization is

$$\begin{aligned} \bar{\partial}_\Phi(y, x) &:= \Phi \circ \bar{\partial} \circ \phi^{-1}(y, x) \\ &= (\nabla_2 (\exp_T)_{(y+x')})^{-1} \circ \bar{\partial} \circ \exp_T(y+x') \\ &= (\nabla_2 (\exp_T)_{(y+x')})^{-1} \circ [T \exp_T(y+x') \circ T(y+x') \partial_s \\ &\quad + J(\exp_T(y+x')) \circ T \exp_T(y+x') \circ T(y+x') \partial_t] \\ &= K_T \circ T(y+x') \partial_s + \tilde{J}(y+x') \circ K_T \circ T(y+x') \partial_t \\ &= \nabla_s(y+x') + \tilde{J}(y+x') \nabla_t(y+x') \end{aligned}$$

for $(y, x) \in V \subset E$, where ∇ denotes covariant derivatives with respect to K_T, and $\tilde{J}(z) = (\nabla_2 (\exp_T)_z)^{-1} \circ J(\exp_T(z)) \circ \nabla_2 (\exp_T)_z$. Taking the derivative, it follows from the smoothing property of $x \mapsto x'$ that

$$\begin{aligned} D\bar{\partial}_\Phi (y, x)(\eta, \xi) &= \nabla_s \eta + \tilde{J}(y+x') \nabla_t \eta + A_1(y, x)(\eta, \xi) \\ &= \partial_s \eta + \tilde{J}(y+x') \partial_t \eta + A_2(y, x)(\eta, \xi) \end{aligned}$$

for $(\eta, \xi) \in E$, with compact operators $A_i(y, x) \in K(E, F)$. Now $\partial_s + \tilde{J}(y+x') \partial_t$ is a linear Cauchy-Riemann operator, and it follows as in Section 3:

$$\| \eta \|_{2,2} \leqq c (\| \partial_s \eta + \tilde{J}(y+x') \partial_t \eta \|_{1,2} + \| \eta \|_{0,2}).$$

Since the linearization of g_λ differs from $\partial_s \eta + \tilde{J}(y+x') \partial_t \eta$ only by a compact operator we obtain

$$\| \eta \|_{2,2} \leqq c (\| D(g_\lambda)_\Phi (y, x) \|_{1,2} + \| A_3(y, x)(\eta, \xi) \|_{1,2})$$

for some $A_3(y, x) \in K(E, F)$.

If $g_\lambda(\phi^{-1}(y, x)) = 0$ then $D(g_\lambda)_\Phi(y, x)$ and $D(\pi_F \times \Psi \times g_\lambda)(\phi^{-1}(y, x))$ are related through composition with an isomorphism. Adding the second component of f_λ we get

$$\| (\eta, \xi) \| \leqq c [\| Df_\lambda(\phi^{-1}(y, x))(\eta, \xi) \| + \| A(y, x)(\eta, \xi) \|_F$$

for $(\eta, \xi) \in T_{\phi^{-1}(y, x)} \mathscr{E}$, with some compact operator A, and (i) follows.

(ii) Let $(u, q) \in \Delta_\lambda$, $\lambda \in [0, 1/2]$. Since $q(t) = q_0$ is constant, from the boundary conditions on u we get $\int_0^1 u(0, .)^* \theta = \int_0^1 u(R, .)^* \theta = 0$, $H_0(0, t) = 0$, $H_0(R, t) \geqq 0$, and therefore

$$\begin{aligned} 0 &\geqq \Phi_{2\lambda H_0}(u(R)) - \Phi_{2\lambda H_0}(u(0)) \\ &= \int_0^R \langle \Phi'_{2\lambda H_0}(u(s)), \partial_s u(s) \rangle_{0,2} \, ds \\ &= \int_{Z_R} | \partial_s u(s, t) |^2 \, ds \, dt, \end{aligned}$$

hence $\partial_s u = 0$ and so $u(s, t) = q_0$ for all $(s, t) \in Z_R$.

(iii) If $(u, q) \in \Delta_0$ then from (ii) $u = q$ is constant. Using the representation $\bar{\partial}_\Phi$ of $\bar{\partial}$ in a local trivialization from (i) and taking the derivative at $(y, x) = (0, 0)$ we obtain

$$\begin{aligned} D\bar{\partial}_\Phi(0, 0)(\eta, \xi) &= \nabla_s \eta + J(u) \nabla_t \eta + L_1(\xi) \\ &= \partial_s \eta + J(u) \partial_t \eta + L_1(\xi) \end{aligned}$$

with $L_1 \in L(H^{1,2}(S^1, T_q M), H^{1,2}(Z_R, T_u TM))$. Identifying $(T_u TM, J(u), T_u^v TM, T_u^h TM)$ with $(\mathbb{C}^n, i, \mathbb{R}^n, i\mathbb{R}^n)$ and arguing as in (i) we see that $Df_0(u, q)$ is equivalent to the operator

$$T: H^{2,2}_{\Lambda} \times H^{1,2}(S^1, \mathbb{R}^n) \to H^{1,2}(Z_{\mathbb{R}}, \mathbb{C}^n) \times H^{1,2}(S^1, \mathbb{R}^n),$$
$$(\eta, \xi) \mapsto (\bar{\partial}\eta + L_2(\xi), \xi)$$

with $\Lambda = (\mathbb{R}^n, i\,\mathbb{R}^n)$ and some $L_2 \in L(H^{1,2}(S^1, \mathbb{R}^n), H^{1,2}(Z_{\mathbb{R}}, \mathbb{R}^n))$. Since by 3.3 $\bar{\partial}: H^{2,2}_{\Lambda} \to H^{1,2}(Z_{\mathbb{R}}, \mathbb{C}^n)$ is an isomorphism, so is T. □

A C^1 map $F: X \to Y$ between Banach manifolds is called (semi-)Fredholm if every linearization is a linear (semi-)Fredholm operator. For a triple (F, U, y) where F is a C^2 Fredholm operator, $U \subset X$ open, $F|_{\bar{U}}$ proper and $y \in Y \setminus F(\partial U)$ let deg (F, U, y) be the $\mathbb{Z}_2$ degree (as defined in [Sm]).

4.3. Theorem. — *Let* H *satisfy condition* (H) *and* f_λ *be defined as above. Then for each compact* $K \subset \Lambda^{\text{contr}}$ *there exists an open neighborhood* U *of* $f_1^{-1}(\{0\} \times K)$ *in* $\mathscr{E}$ *such that*:

(i) $f_1|_{\bar{U}}$ *is a proper Fredholm operator of index* 0.

(ii) $\deg(f_1, U, (0, q)) = 1$ *for all* $q \in K$.

Proof. — (1) By 4.2(i), all $Df_\lambda(x)$ are semi-Fredholm operators for $x \in \Delta_0$. Since the set of semi-Fredholm operators is open, there exists a connected open neighborhood U_0 of Δ_0 in $\mathscr{E}$ such that all $f_\lambda|_{U_0}$ are semi-Fredholm operators. By 4.2 (ii), for $x \in \Delta_0$ $Df_0(x)$ is a Fredholm operator of index 0. Since the Fredholm operators are open and closed in the set of semi-Fredholm operators and the Fredholm index is continuous, all $f_\lambda|_{U_0}$ are Fredholm operators of index 0. Then $f|_{[0, 1/2] \times U_0}: (\lambda, x) \mapsto f_\lambda(x)$ is a smooth Fredholm operator of index 1. By 4.2 (iii), $\Delta_{[0, 1/2]} \cong [0, 1/2] \times M$ is compact. Since Fredholm operators are locally proper ([Sm], Theorem 1.6) there exists a neighborhood U of $\Delta_{[0, 1/2]}$ such that $f|_{\bar{U}}$ is proper. For each $\lambda_0 \in [0, 1/2]$ we find a neighborhood I of λ_0 and an open $V \subset \mathscr{E}$ such that $\Delta_I \subset I \times V \subset U$. Because of the invariance of the $\mathbb{Z}_2$-degree under proper homotopies, for every $q \in M$, $\deg(f_\lambda, V, (0, q))$ is independent of $\lambda \in I$. Using the excision property of the degree we conclude:

$$\begin{aligned}\deg(f_{1/2}, U_0, (0, q)) &= \deg(f_0, U_0, (0, q)) \\ &= 1 \quad \text{by 4.2 (ii) and (iii).}\end{aligned}$$

(2) In Theorem 6.4 (i) we shall prove: $\Delta_{[1/2, 1]}$ is compact. Arguing as in (1) one shows that for each $q \in M$

$$\begin{aligned}\deg(f_1, U_1 (0, q)) &= \deg(f_{1/2}, U_0, (0, q)) \\ &= 1\end{aligned}$$

for some appropriate neighborhood U_1 of Δ_1.

(3) Now let $K \subset \Lambda^{\text{contr}}$ be compact. By enlarging K we may assume that it is connected and contains a constant loop $q_0 \in M$. Again from Theorem 6.4 (i) it will follow that $f_1^{-1}(\{0\} \times K)$ is compact. As in (1) we find a neighborhood U_K of $f_1^{-1}(\{0\} \times K)$ such that $f_1|_{\bar{U}_K}$ is a proper Fredholm operator of index 0. Since $\{0\} \times K$

is contained in a connected component of $F \times \Lambda \setminus f_1(\partial U_K)$, for each $q \in K$ it follows:

$$\begin{aligned} \deg(f_1, U_K, (0, q)) &= \deg(f_1, U_K, (0, q_0)) \\ &= \deg(f_1, U_1, (0, q_0)) \\ &= 1. \quad (\square) \end{aligned}$$

Remark. — We have not taken the more natural homotopy with $g_\lambda := \bar{\partial} + \lambda H'$ because in that case we wouldn't have got the compactness of $\Delta_{[0,1]}$ so easily from Theorem 6.4.

5. Boundedness

Let N be a compact manifold with smooth boundary ∂N and $\dot{N} := N \setminus \partial N$ its interior.

Let L be a strongly positive elliptic differential operator of second order, *i.e.* in local charts L can be written as

$$L = -\sum_{i,j=1}^{n} a_{ij}(x)\,\partial_{x_i}\partial_{x_j} + \sum_{i=1}^{n} a_i(x)\,\partial_{x_i} + a_0(x),$$

where a_{ij}, a_i, $a_0 \in C^\infty(U)$, $a_0(x) \geqq 0$, and $(a_{ij}(x))$ is positive definite for all x.

For $\alpha \in (0, 1)$ and $k \in \mathbb{N}$ let $C^{k+\alpha}(N)$ denote the set of C^k functions on N whose k-th derivatives are Hölder continuous with exponent α.

The results of this section are based on the following

5.1. Proposition. — *The Dirichlet problem:*

$$\left\{\begin{array}{ll} Lu = \lambda u & \text{in } \dot{N} \\ u = 0 & \text{on } \partial N \end{array}\right\}$$

has a unique smallest eigenvalue λ_1 (i.e. $|\lambda| > |\lambda_1|$ *for all other eigenvalues*) *which is real and positive. If* $\lambda \in \mathbb{R}$, $\lambda < \lambda_1$ *and* $u \in C^{2+\alpha}(N)$ *such that*

$$\left\{\begin{array}{ll} (L-\lambda)u \geqq 0 & \text{in } \dot{N}, \\ u \geqq 0 & \text{on } \partial N \end{array}\right\}$$

then $u \geqq 0$ *on* N.

Proof. — The proof combines the maximum principle with the Krein-Rutman theorem. It can be found in [Am], Theorem 4.3 and Theorem 4.4 for domains in $\mathbb{R}^n$ but remains valid also for manifolds. □

5.2. Lemma. — *For* $\varepsilon, \delta > 0$ *let* $Z_{\varepsilon,\delta} := [-\varepsilon, \varepsilon] \times (\mathbb{R}/\delta\mathbb{Z})$. *Then the smallest eigenvalue of*

$$\left\{\begin{array}{ll} -\Delta u = \lambda u & \text{in } \dot{Z}_{\varepsilon,\delta}, \\ u = 0 & \text{on } \partial Z_{\varepsilon,\delta} \end{array}\right\}$$

is $\lambda_1 = \pi^2/4\varepsilon^2$.

Proof. — Using Fourier series one easily calculates that the spectrum consists of the $(\pi^2 n^2/4\varepsilon^2)+(4\pi^2 k^2/\delta^2)$ with $n \in \mathbb{N}\setminus\{0\}$ and $k \in \mathbb{N}$. □

Now let $H \in C^\infty(S^1 \times TM, \mathbb{R})$ satisfy condition (H). By integration along the fibres we obtain from (H2):

(H3) $$|H'(t, q, p)| \leqq 2d_2|p| + d_3$$

for some constant d_3.

5.3. Lemma. — $\{x \in H^{1,2}(S^1, TM) \mid \Phi_H(x) \leqq a, \|\Phi'_H(x)\|_{0,2}^2 \leqq b\}$ *is bounded in the* $H^{1,2}$ *norm by a constant* $c(a, b, \kappa, d_1, d_2, d_3)$.

Proof. — Using $\dot{x} = X_H(t, x) + J(x)\Phi'(x)$ we calculate for $x=(q, p)$:

$$\begin{aligned} a &\geqq \Phi(x) \\ &= \int_0^1 (\theta(\dot{x}) - H(t, x))\,dt \\ &= \int_0^1 (d\theta(\eta(x), \dot{x}) - H(t, x))\,dt \\ &= \int_0^1 (\omega(X_H(x), \eta(x)) - H(t, x) + \omega(J(x)\Phi'(x), \eta(x)))\,dt \\ &= \int_0^1 (dH_t(\eta(x)) - H(t, x) + \langle \Phi'(x), \eta(x)\rangle)\,dt \\ &\geqq \kappa\|p\|_{0,2}^2 - d_1 - \|\Phi'(x)\|_{0,2}\|p\|_{0,2}, \end{aligned}$$

where in the last step we have used (H2). This implies a bound for $\|p\|_{0,2}$ and thus for $\|x\|_{0,2}$. Inserting this in the equation for $\dot{x}$ and using (H3) we derive a bound for $\|\dot{x}\|_{0,2}$. □

Remark. — 5.3 also implies the following "Palais-Smale condition" for Φ:

If $(x_n) \subset H^{1,2}(S^1, TM)$ is a sequence with $\Phi(x_n)$ bounded and $\|\Phi'(x_n)\|_{0,2} \to 0$ then (x_n) possesses a subsequence converging in $H^{1,2}$ to some x with $\Phi'(x)=0$.

Now we are able to prove the main result of this section. Let $Z := \mathbb{R} \times S^1$.

5.4. Theorem. — *Let* $H \in C^\infty(S^1 \times TM, \mathbb{R})$ *satisfy condition* (H). *Then*

(i) $X_a^b := \{u \in C^\infty(Z, TM) \mid \partial_s u = \Phi'_H(u), a \leqq \Phi_H(u(s)) \leqq b$ *for all* $s \in \mathbb{R}\}$ *is bounded in the* C^0*-norm by a constant* $c(a, b, \kappa, d_1, d_2, d_3)$.

(ii) $X_{R, q_0} := \{u \in C^\infty(Z_R, TM) \mid \partial_s u = \Phi'_H(u), u(0, t) \in M, u(R, t) \in T_{q_0(t)}M\}$ *is for each* $q_0 \in C^\infty(S^1, M)$ *bounded in the* C^0*-norm by a constant* $c(\|q_0\|_{C^1}, \kappa, d_1, d_2, d_3)$.

Proof. — For $u=(q, p)$, $\partial_s u = \Phi'_H(u)$ writes explicitly as

$$\begin{aligned} \nabla_s p - \partial_t q + KH'(u) &= 0, \\ \nabla_t p + \partial_s q + T\tau H'(u) &= 0. \end{aligned}$$

For $(s, t) \in Z$ or Z_R we calculate:

$$\begin{aligned}
\frac{1}{2}\Delta|p|^2 &= \partial_s\langle p, \nabla_s p\rangle + \partial_t\langle p, \nabla_t p\rangle \\
&= |\nabla_s p|^2 + |\nabla_t p|^2 + \langle p, \nabla_s^2 p + \nabla_t^2 p\rangle \\
&= |\nabla_s p|^2 + |\nabla_t p|^2 + \langle p, \nabla_s(\partial_t q - \mathrm{KH}'(u)) - \nabla_t(\partial_s q + \mathrm{T}\tau\mathrm{H}'(u))\rangle \\
&= |\nabla_s p|^2 + |\nabla_t p|^2 - \langle p, \nabla_s \mathrm{KH}'(u) + \nabla_t \mathrm{T}\tau\mathrm{H}'(u)\rangle \\
&= |\nabla_s p|^2 + |\nabla_t p|^2 - \langle p, \mathrm{KT}(\mathrm{KH}')(u)\,\partial_s u + \mathrm{KT}(\mathrm{T}\tau\mathrm{H}')(u)\,\partial_t u\rangle \\
&\geqq |\nabla_s p|^2 + |\nabla_t p|^2 - |p|(d_2|\partial_s u| + d_2|\partial_t u|) \\
&\geqq |\nabla_s p|^2 + |\nabla_t p|^2 - d_2|p|(|\nabla_s p| + |\nabla_t p| + |\partial_s q| + |\partial_t q|) \\
&\geqq |\nabla_s p|^2 + |\nabla_t p|^2 - d_2|p|(2|\nabla_s p| + 2|\nabla_t p| + |\mathrm{KH}'(u)| + |\mathrm{T}\tau\mathrm{H}'(u)|) \\
&\geqq |\nabla_s p|^2 + |\nabla_t p|^2 - d_2|p|(2|\nabla_s p| + 2|\nabla_t p| + 4d_2|p| + 2d_3) \\
&\geqq -\frac{1}{2}\mu(|p|^2 + 1)
\end{aligned}$$

with a constant $\mu(d_2, d_3)$. Putting $\phi := -|p|^2$ we thus have: $(-\Delta - \mu)\phi \geqq -\mu$. Choose an $\varepsilon > 0$ such that $\mu\varepsilon^2 \leqq 1$, $\mu\varepsilon \leqq \kappa$.

(i) By assumption,

$$\begin{aligned}
\int_{s_1}^{s_2} \|\Phi_H'(u(s))\|_{0,2}^2\, ds &= \int_{s_1}^{s_2} \partial_s[\Phi_H(u(s))]\, ds \\
&= \Phi_H(u(s_2)) - \Phi_H(u(s_1)) \\
&\leqq b - a.
\end{aligned}$$

Therefore to any given $s_0 \in \mathbb{R}$ we find a $s_1 \in [s_0 - \varepsilon, s_0]$ and a $s_2 \in [s_0, s_0 + \varepsilon]$ with $\|\Phi_H'(u(s_i))\|_{0,2}^2 \leqq (b-a)/\varepsilon$, $i = 1, 2$.

Let $\delta := (s_2 - s_1)/2 \leqq \varepsilon$ and $Z_0 := [s_1, s_2] \times S^1$.

By 5.3, $u|_{\partial Z_0}$ is bounded in $H^{1,2}$ by a constant $c(a, b, \kappa, d_1, d_2, d_3)$. So $\phi \geqq -\nu$ on ∂Z_0 for some $\nu(a, b, \kappa, d_1, d_2, d_3) \geqq 0$.

We define $v(s, t) := (1/2)\mu(1+\nu)(\delta^2 - (s - s_1 - \delta)^2) + \nu$. An easy calculation shows: $(-\Delta - \mu)v \geqq \mu$ in $\dot{Z}_0$ and $v = \nu$ on ∂Z_0. Thus

$$\left\{\begin{aligned} (-\Delta - \mu)(\phi + v) &\geqq 0 \quad \text{in } \dot{Z}_0, \\ (\phi + v) &\geqq 0 \quad \text{on } \partial Z_0. \end{aligned}\right\}$$

By 5.2 the Dirichlet problem of $-\Delta$ on Z_0 has the smallest eigenvalue $\lambda_1 = \pi^2/4\delta^2 > 1/\varepsilon^2 \geqq \mu$. So 5.1 is applicable and gives: $(\phi + v) \geqq 0$ and therefore $|p|^2 \leqq v \leqq (1/2)(1 + 3\nu)$ on Z_0.

Since ν was independent of s_0, we have shown: $|p|^2 \leqq 2\nu + 1$ on all of Z.

(ii) From (H1) we get $H(t, q, p) \geqq (\kappa/2)|p|^2 - d_4$ with some constant d_4. Now the boundary conditions imply:

$$\Phi_H(u(0)) = -\int_0^1 H(u(0, t))\,dt$$
$$\geqq -d_1,$$
$$\Phi(u(R)) \leqq \int_0^1 \left(\langle \dot{q}_0(t), p(R, t)\rangle - \frac{\kappa}{2}|p(R, t)|^2 + d_4\right) dt$$
$$\leqq \frac{1}{2\kappa}\|\dot{q}_0\|_{0,2}^2 + d_4,$$

where we have used that $x^*\theta = \langle p, \dot{q}\rangle dt$ for $x = (q, p) \in C^\infty(S^1, TM)$.

Again we find $s_0 \in [0, \varepsilon]$ and $s_1 \in [R - \varepsilon, R]$ with $\phi(s_i, t) \geqq -\nu_0(\|q_0\|_{C^1}, \kappa, d_1, d_2, d_3)$, $i = 0, 1$, and it follows: $|p|^2 \leqq c(\|q_0\|_{C^1}, \kappa, d_1, d_2, d_3)$ on $[s_0, s_1] \times S^1$. Because of $|p(0, t)|^2 = 0$ we also get the C^0 bound over $[0, s_0] \times S^1$. So we only have to deal with $[s_1, R] \times S^1$.

Let $-\nu := \min_{t \in S^1} \phi(R, t)$. If $\nu \leqq \nu_0$ we are done. Otherwise let $s_2 := R$ and define δ, Z_0, v as in (i). Again it follows that $(\phi + v) \geqq 0$ in Z_0, and the minimum 0 is attained in some $z_2 = (R, t_2)$. Then $|p(z_2)|^2 = \nu$ and

$$0 \geqq \partial_s(\phi + v)(z_2)$$
$$= -\partial_s|p|^2(z_2) - \mu(1 + \nu)(s_2 - s_1 - \delta)$$
$$= 2\langle p(z_2), -\nabla_s p(z_2)\rangle - \mu\delta(1 + |p(z_2)|^2)$$
$$\geqq 2\langle p(z_2), -\dot{q}_0(t_2) + KH'(u(z_2))\rangle - \kappa(|p(z_2)|^2 + 1)$$
$$\geqq 2(-\langle p(z_2), \dot{q}_0(t_2)\rangle + H(t_2, u(z_2)) + \kappa|p(z_2)|^2 - d_1) - \kappa(|p(z_2)|^2 + 1)$$
$$\geqq \frac{\kappa}{2}(|p(z_2)|^2 - c),$$

with a constant $c(\|q_0\|_{C^1}, d_1, d_4)$. Therefore $|p(R, t)|^2 \leqq |p(z_2)|^2 \leqq c$ for all $t \in S^1$. The C^0-bound over Z_0 now follows as in (i). □

6. Compactness

Throughout this section, (Σ, i) will denote a compact Riemannian surface with boundary $\partial\Sigma$ and (Ω, i) a second countable Riemannian surface (without boundary).

(S, ω) will be a symplectic manifold of dimension $2n$, $A \subset S$ a compact subset, J an almost complex structure on S such that $g(\cdot, \cdot) := \omega(J\cdot, \cdot)$ is a Riemannian metric. Let $(S, g) \subset \mathbb{R}^N$ be an isometric imbedding according to [Na].

We define Banach spaces $H^{m,p}(\Sigma, \mathbb{R}^N)$, $C^k(\Sigma, \mathbb{R}^N)$ and Frechet spaces $C^\infty(\Sigma, \mathbb{R}^N)$, $H^{m,p}_{loc}(\Omega, \mathbb{R}^N)$, $C^k_{loc}(\Omega, \mathbb{R}^N)$, $C^\infty_{loc}(\Omega, \mathbb{R}^N)$ in the usual way.

Let $\Pi \to \Sigma \times S$ be the bundle with fibres

$$\Pi_{(z,s)} = \{ P \in L(T_z\Sigma, T_s S) \mid J(s) \circ P \circ i = P \}$$

and f a given smooth section in Π.

A $u \in H^{1,q}(\Sigma, \mathbb{R}^N) \cap C^0(\Sigma, S)$ defines an L^q section $\bar\partial u + f(z, u)$ in $(\mathrm{id} \times u)^* \Pi \to \Sigma$ by

$$z \mapsto T u(z) + J(u(z)) \circ T u(z) \circ i + f(z, u(z)).$$

Let $\Lambda : \partial\Sigma \to \{$Lagrangian submanifolds of S$\}$ be smooth in the following sense: For each $z_0 \in \partial\Sigma$ there exists a neighborhood U of z_0 in $\partial\Sigma$ and a map $\phi \in C^\infty(U \times \Lambda(z_0), S)$ such that for all $z \in U$ the restriction $\phi|_{\{z\} \times \Lambda(z_0)} : \{z\} \times \Lambda(z_0) \to \Lambda(z)$ is a diffeomorphism.

$$C^0_\Lambda(\Sigma, A) := \{ u \in C^0(\Sigma, \mathbb{R}^N) \mid u(\Sigma) \subset A,\ u(z) \in \Lambda(z) \text{ for all } z \in \partial\Sigma \},$$
$$H^{m,p}_\Lambda(\Sigma, A) := H^{m,p}(\Sigma, \mathbb{R}^N) \cap C^0_\Lambda(\Sigma, A) \text{ for } mp > 2.$$

6.1. Lemma. — *For $p > 2$ we have the following compact imbeddings*:

(i) $Y_{\Lambda,p} := \{ u \in H^{1,p}_\Lambda(\Sigma, A) \mid \bar\partial u + f(z, u) = 0 \} \hookrightarrow C^\infty(\Sigma, \mathbb{R}^N)$.

(ii) $Y_p := \{ u \in H^{1,p}_{\mathrm{loc}}(\Omega, A) \mid \bar\partial u + f(z, u) = 0 \} \hookrightarrow C^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$.

Proof. — This result is well-known (*cf.* [Ho], Proposition 1 or [Fl1], Lemma 2.3), so we are not going to prove it here. Roughly, the proof goes as follows: By elliptic regularity theory $Y_{\Lambda,p} \hookrightarrow C^\infty(\Sigma, \mathbb{R}^N)$. By Sobolev embedding a sequence $(u_n) \subset Y_{\Lambda,p}$ that is bounded in $H^{1,p}(\Sigma, \mathbb{R}^N)$ has a subsequence converging in C^0 to some $u \in C^0_\Lambda(\Sigma, A)$. Then localize around u and use 3.1 to show that (u_n) is a Cauchy sequence in C^∞. □

So we get compactness in C^∞ once we have boundedness in $H^{1,p}$ for some $p > 2$. Unfortunately, we will *a priori* only have a bound in $H^{1,2}$. The lack of a $H^{1,p}$-bound is described by the "bubbling off" of holomorphic spheres or disks which we shall now investigate. The key is the following observation (with B_1, B_1^+ as in Section 3):

6.2. Proposition (removal of singularities). — *Let* L *be a Lagrangian submanifold of* S.

(i) *If* $u \in C^\infty(B_1^+ \setminus \{0\}, A)$ *satisfies*:

$$u(s, 0) \in L \quad \textit{for} \quad 0 \neq s \in (-1, 1), \quad \bar\partial u = 0, \quad \int_{B_1^+ \setminus \{0\}} -u^* \omega < \infty,$$

then u can be extended smoothly to B_1^+.

(ii) *If* $u \in C^\infty(B_1 \setminus \{0\}, A)$ *satisfies*: $\bar\partial u = 0$, $\int_{B_1 \setminus \{0\}} -u^* \omega < \infty$, *then u can be extended smoothly to* B_1.

Proof. — *See* [Oh], but note that the assumption (which is missing in [Oh]) that the image of *u* is contained in a compact set A is necessary. □

In the following theorem let $D := \bar{B}_1$ denote the closed unit disk in $\mathbb{C}$ and S^2 the 2-sphere, both equipped with the standard complex structures.

6.3. Theorem. — (i) *Let* $Y_\Lambda := \{ u \in C^\infty_\Lambda(\Sigma, A) \mid \bar\partial u + f(z, u) = 0 \}$ *with the* C^∞*-topology. Then at least one of the following statements is true*:

(*a*) *Every sequence* $(u_n) \subset Y_\Lambda$ *on which* $\int_\Sigma - u_n^* \omega$ *remains bounded has a subsequence converging in* Y_Λ.

(*b*) *There exists a* $u \in C^\infty(S^2, A)$ *with* $\bar\partial u = 0$ *and* $\int_{S^2} u^* \omega < 0$.

(*c*) *There exist a* $z_0 \in \partial\Sigma$ *and a* $u \in C^\infty(D, A)$ *with* $u(\partial D) \subset \Lambda(z_0)$, $\bar\partial u = 0$ *and* $\int_D u^* \omega < 0$.

(ii) *Let* $Y := \{ u \in C^\infty_{\mathrm{loc}}(\Omega, A) \mid \bar\partial u + f(z, u) = 0 \}$ *with the* C^∞_{loc}*-topology. Then at least one of the following statements is true:*

(*a*) *Every sequence* $(u_n) \subset Y$ *on which* $\int_K - u_n^* \omega$ *remains bounded for each compact* $K \subset \Omega$ *has a subsequence converging in* Y.

(*b*) *There exists a* $u \in C^\infty(S^2, A)$ *with* $\bar\partial u = 0$ *and* $\int_{S^2} u^* \omega < 0$.

Proof. — (i) Assume that there exists a sequence $(u_n) \subset Y_\Lambda$ with $\int_\Sigma - u_n^* \omega \leqq c_0$ bounded and $\max_{z \in \Sigma} |\nabla u_n(z)| \to \infty$. Here $\nabla u_n(z)$ denotes the element in $T_z^* \Sigma \otimes T_{u_n(z)} S$ given by the differential of u and $|\ |$ the norm on $T_z^* \Sigma \otimes T_{u_n(z)} S$ induced by the metrics on Σ and S. We choose $z_n \in \Sigma$ such that $|\nabla u_n(z_n)| = \max_{z \in \Sigma} |\nabla u_n(z)|$. Without loss of generality we may assume that (z_n) converges to some $z_0 \in \Sigma$.

Case 1. — $z_0 \in \partial\Sigma$.

Let $(U, \phi, \mathbb{C}^+)$ be a holomorphic chart of Σ with $\phi(U) = B_1^+$, $\phi(z_0) = 0$ and $\phi(U \cap \partial\Sigma) = (-1, 1) \times \{0\}$. For n large enough $z_n \in U$, and we define

$$x_n := (s_n, t_n) := \phi(z_n), \qquad v_n := u_n \circ \phi^{-1}|_{B_1^+}, \qquad g_n := |\nabla v_n(x_n)|.$$

We may assume $|\nabla v_n(x)| \leqq 2 |\nabla v_n(x_n)|$ for all $x \in B_1^+$.

Case 1.1. — $|t_n| g_n \leqq a < \infty$ for all n.

Let $w_n(x) := v_n((s_n, 0) + x/g_n)$ for $x \in B^+_{g_n/2}$.

Since $g_n \to \infty$, there exists to any $R > 0$ a $n(R) \in \mathbb{N}$ such that for $n \geqq n(R)$ we have: $w_n \in C^\infty(B_R^+, A)$, $w((-R, R) \times \{0\}) \subset L$, $\bar\partial w_n + (1/g_n) f(\phi^{-1}(x), w_n) = 0$ and $|\nabla w_n(x)| \leqq 2$ for all $x \in B_R^+$. Thus $(w_n|_{B_R^+})$ is bounded in $H^{1,p}$ for any $p > 2$, so from 6.1 [which clearly remains true if $\bar{\partial} w_n + f(x, w_n) = 0$ is replaced by $\bar{\partial} w_n + (1/g_n) f(\phi^{-1}(x), w_n) = 0]$ we get a subsequence converging in C^∞ to some $w_R \in C^\infty(B^+_{R-1}, A)$.

Taking a diagonal sequence we obtain a subsequence of (w_n) converging in C^∞_{loc} to some $w \in C^\infty_{loc}(\mathbb{C}^+, A)$. It immediately follows that $\bar{\partial} w = 0$, $w(s, 0) \in \Lambda(z_0)$ for all $s \in \mathbb{R}$ and $\int_{\mathbb{C}^+} - w^* \omega < \infty$. Because of $\sup_{B_a^+} |\nabla w_n| \geqq |\nabla w_n((0, g_n t_n))| = 1$ we have $\sup_{B_a^+} |\nabla w| \geqq 1$, so w is not constant.

By 6.2 $z \mapsto w(z^{-1})$ can be smoothly extended over $z = 0$. Thus w can be smoothly extended to $\mathbb{C}^+ \cup \{\infty\}$ which is biholomorphic equivalent to D, and we obtain a $u \in C^\infty(D, A)$ with $u(\partial D) \subset \Lambda(z_0)$ and $\bar{\partial} u = 0$. Since w is nonconstant so is u. Therefore

$$\int_D u^* \omega = -\frac{1}{2} \int_D (|\partial_s u|^2 + |\partial_t u|^2)\, ds\, dt < 0,$$

and we have the situation of (*c*).

Case 1.2. — $|t_n| g_n \to \infty$.

Define $w_n(x) := v_n(x_n + (x/g_n))$ for $x \in B_{|t_n| g_n}$. Proceeding as in case 1.1 we find a nonconstant $w \in C^\infty(\mathbb{C}, A)$ with $\bar{\partial} w = 0$ and $\int_{\mathbb{C}} - w^* \omega < \infty$. By 6.2 w can be extended smoothly to S^2, and we have (*b*).

Case 2. — $z_0 \notin \partial \Sigma$.

Define $w_n(x) := v_n(x_n + (x/g_n))$ for $x \in B_{cg_n}$ with some properly chosen constant c and proceed as in case 1.2. Again we find (*b*).

If the assumption made at the beginning of the proof is wrong, every sequence $(u_n) \subset Y_\Lambda$ with $\int_\Sigma - u_n^* \omega \leqq c_0$ is bounded in $H^{1,p}$, $p > 2$. By 6.1 (u_n) then has a subsequence converging in Y_Λ, so (*a*) holds true.

(ii) follows in the same way, using the topological Lemma 3.3 of [HV2]. □

Now we will exclude the possibilities (*b*) and (*c*) in our special situation and thereby establish the following compactness result (where the notation is again as in the previous chapters):

6.4. Theorem. — *Let* $H \in C^\infty(S^1 \times TM, \mathbb{R})$ *satisfy condition* (H).

(i) *If* $K \subset \Lambda M$ *is compact*, $R > 0$, $m \in \mathbb{N}$ *and* $p > 1$ *with* $mp > 2$ *then*

$$\Gamma(R, K) := \{(\lambda, u, q) \in [0, 1] \times H^{m,p}(Z_R, TM) \times K \mid u(0, t) \in M,$$
$$u(R, t) \in T_{q(t)} M,\ \bar{\partial} u + (1-\lambda) H_0'(u) + \lambda H'(u) = 0\}$$

is compact in $[0, 1] \times C^\infty(Z_R, TM) \times K$.

(ii) *For* $a < b \in \mathbb{R}$

$$X_a^b := \{u \in C^\infty(Z, TM) \mid \bar{\partial} u + H'(u) = 0,\ a \leqq \Phi_H(u(s)) \leqq b \text{ for all } s \in \mathbb{R}\}$$

is compact in $C^\infty_{loc}(Z, TM)$.

Furthermore, there exist constants c_l, $l \in \mathbb{N}$, such that $|D^\alpha u(z)| \leqq c_l$ for all $u \in X^b_a$, $z \in Z$, $|\alpha| \leqq l$.

Proof. — (i) Let $Y(R, K) := \{u \mid (\lambda, u, q) \in \Gamma(R, K)$ for some $\lambda, q\}$. By 6.1, $Y(R, K) \subset C^\infty(Z_R, TM)$. $\mathscr{I}(K)$ is compact and thus bounded in $C^1(S^1, M)$. The $(1-\lambda)H_0 + \lambda H$, $\lambda \in [0, 1]$ fulfill (H) and (H3) with uniform constants κ, d_1, d_2, d_3. Therefore $Y(R, K)$ is C^0-bounded by 5.4, *i.e.* its image is contained in some compact $A \subset TM$.

In the proof of 5.4 we have shown that $\{\Phi_H(u(s)) \mid u \in Y(R, K), s \in [0, R]\}$ is bounded from above and below. Therefore $\int_{Z_R} |\partial_s u|^2 \, ds \, dt$ and thus $\int_{Z_R} -u^* \omega$ is bounded on $Y(R, K)$.

Now let (λ_n, u_n, q_n) be a sequence in $\Gamma(R, K)$. We may assume that $\lambda_n \to \lambda_0$ in $[0, 1]$ and $q_n \to q_0$ in K. Repeating the proof of 6.3 for the sequence (u_n) one finds again that at least one of the following statements must be true:

(*a*) (u_n) has a subsequence converging in C^∞.

(*b*) There exists a $u \in C^\infty(S^2, TM)$ with $\bar{\partial} u = 0$ and $\int_{S^2} u^* \omega < 0$.

(*c*) There exists a $u \in C^\infty(D, TM)$ with $u(\partial D) \subset M$ or $\subset T_x M$ for some $x \in M$, $\bar{\partial} u = 0$ and $\int_D u^* \omega < 0$.

Since $\omega = -d\theta$, by Stokes' theorem ω vanishes on $\pi_2(TM)$, $\pi_2(TM, T_x M)$ and $\pi_2(TM, M) = \{0\}$. This excludes (*a*) and (*b*), thus proving that $\Gamma(R, K)$ is pre-compact in $[0, 1] \times C^\infty(Z_R, TM) \times K$. Clearly $\Gamma(R, K)$ is also closed.

(ii) For $u \in X^b_a$ and $\sigma \in \mathbb{R}$ define $(u.\sigma)(s, t) := u(s-\sigma, t)$ and apply 6.3 (ii) to $Y(a, b) := \{u.\sigma|_{(-2, 2) \times S^1} \mid u \in X^b_a, \sigma \in \mathbb{R}\}$. Again (*b*) cannot occur, so $Y(a, b)$ is compact in $C^\infty_{loc}((-2, 2) \times S^1, TM)$. In particular, all C^l-norms are bounded on $\{u \cdot \sigma|_{[-1, 1] \times S^1} \mid u \in X^b_a, \sigma \in \mathbb{R}\}$. This proves (ii). □

6.4 (i) finally finishes the proof of 4.3. Part (ii) will play an important role in the final section.

7. The injection in cohomology

We begin with a result about finite dimensional manifolds with boundaries. Again we denote by H^* the singular cohomology.

7.1. LEMMA. — *Let* A, B *be compact n-dimensional* C^1-*manifolds with boundaries* ∂A, ∂B. *Let* $f: A \to B$ *be a continuous map with* $\deg(f, A, b) = 1$ *for all* $b \in \dot{B} = B \setminus \partial B$.

Then $f^*: H^k(B, \mathbb{Z}_2) \to H^k(A, \mathbb{Z}_2)$ *is injective for all* $0 \leqq i \leqq n$.

Proof. — As in the proof of [Ho], Theorem 5, using the topological definition of the $\mathbb{Z}_2$-degree and Poincaré-Lefschetz duality. □

Next we show how to approximate the loop space Λ by finite dimensional manifolds with boundaries.

7.2. LEMMA. — *There exists a sequence of smooth imbeddings $g_k : \mathrm{P}_k \mathrm{M} \subsetneq \Lambda$ of compact finite dimensional manifolds with boundaries with the following property: For any field* R *and cohomology class $0 \neq \alpha \in \mathrm{H}^*(\Lambda, \mathrm{R})$ there is a $k_\alpha \in \mathbb{N}$ such that $g_k^* \alpha \neq 0$ for all $k \geqq k_\alpha$.*

Proof. — An even stronger result is stated in [Bo], Lecture 1. So we will just recall the definition of the $\mathrm{P}_k \mathrm{M}$.

Let d be the metric on M, $i(\mathrm{M})$ the injectivity radius and $0 < \varepsilon < i(\mathrm{M})$. For $p, q \in \mathrm{M}$ and $d(p, q) < i(\mathrm{M})$ we denote by $\mathrm{Geod}[p, q] : [0, 1] \to \mathrm{M}$ the minimal geodesic from p to q parametrized proportionally to the arclength. We define

$$\mathrm{P}_k \mathrm{M} := \left\{ (p_0, \ldots, p_{k-1}) \in \mathrm{M}^k \,\middle|\, \sum_{i=0}^{k-1} d(p_i, p_{i+1})^2 \leqq \varepsilon^2 \right\},$$

for $k \geqq 1$ and $p_k := p_0$. $g_k : \mathrm{P}_k \mathrm{M} \to \Lambda$ is defined by

$$g_k(p_0, \ldots, p_{k-1}) \left(\frac{i+t}{k} \right) := \mathrm{Geod}[p_i, p_{i+1}](t)$$

for $0 \leqq i \leqq k-1$ and $t \in [0, 1]$. □

From now on let $\mathrm{H} \in \mathrm{C}^\infty(\mathrm{S}^1 \times \mathrm{TM}, \mathbb{R})$ satisfy condition (H).

Let $\mathrm{X_R} := \{ u \in \mathrm{C}^\infty(\mathrm{Z_R}, \mathrm{TM}) \mid u(0, t) \in \mathrm{M},\ \bar{\partial} u + \mathrm{H}'(u) = 0 \}$ with the C^∞ topology, $\pi_\mathrm{R} : \mathrm{X_R} \to \Lambda$, $\pi_\mathrm{R} u(t) := \tau u(\mathrm{R}, t)$, $\mathrm{P}_m^{\mathrm{contr}}$ the component of the diagonal in $\mathrm{P}_m \mathrm{M}$ and $\mathrm{K}_m := g_m(\mathrm{P}_m^{\mathrm{contr}}) \subset \Lambda^{\mathrm{contr}}$.

By $\bar{\mathrm{H}}^*$ we denote the Alexander-Spanier cohomology as defined in [Sp], Chapter 6, Section 4.

7.3. PROPOSITION. — *For every $0 \neq \alpha \in \bar{\mathrm{H}}^*(\Lambda^{\mathrm{contr}}, \mathbb{Z}_2)$ there exists an $m_0 \in \mathbb{N}$ such that*

$$0 \neq (\pi_\mathrm{R} |_{\pi_\mathrm{R}^{-1} [\mathscr{I}(\mathrm{K}_m)]})^* \alpha \in \bar{\mathrm{H}}^*(\pi_\mathrm{R}^{-1}[\mathscr{I}(\mathrm{K}_m)], \mathbb{Z}_2)$$

for all $m \geqq m_0$ and $\mathrm{R} > 0$.

Proof. — Let $0 \neq \alpha \in \bar{\mathrm{H}}^*(\Lambda^{\mathrm{contr}}, \mathbb{Z}_2)$. As mentioned before, Λ^{contr} is a metrizable Hilbert manifold. Therefore by [Sp], Corollary 6.9.5, H and $\bar{\mathrm{H}}$ are naturally isomorphic on Λ^{contr} and $\mathrm{P}_m^{\mathrm{contr}}$. 7.2 remains valid if we replace $\Lambda \mathrm{M}$ by Λ^{contr} and $\mathrm{P}_m \mathrm{M}$ by $\mathrm{P}_m^{\mathrm{contr}}$. So there exists a $m_0 \in \mathbb{N}$ such that $0 \neq g_m^* \alpha \in \bar{\mathrm{H}}^*(\mathrm{P}_m^{\mathrm{contr}}, \mathbb{Z}_2)$ for all $m \geqq m_0$.

Because of the compactness of $\mathrm{K}_m \subset \Lambda^{\mathrm{contr}}$ by 4.3 we find a neighborhood U of $f_1^{-1}(\{0\} \times \mathrm{K}_m)$ in $\mathscr{E}$ such that $f_1|_{\bar{\mathrm{U}}}$ is a proper smooth Fredholm operator of index 0 and $\deg(f_1, \mathrm{U}, (0, q)) = 1$ for all $q \in \mathrm{K}_m$.

With the help of [Sm], Theorem 3.1 we find a C^1-approximation h of $(0, g_m) : \mathrm{P}_m^{\mathrm{contr}} \to \mathrm{F} \times \Lambda^{\mathrm{contr}}$, $x \mapsto (0, g_m(x))$ with the following properties:

The interior $\dot{B}$ of $B:=h(P_m^{contr})$ is contained in the component of $\{0\}\times K_m$ in $F\times\Lambda^{contr}\setminus\{0\}\times\partial K_m$, h is transversal to $f_1|_{\bar{U}}$ and $h\sim(0, g_m): P_m^{contr}\to F\times\Lambda^{contr}$ (where $\sim$ means "homotopic").

By [Sm], Theorem 3.3, $A:=f_1|_{\bar{U}}^{-1}(B)$ is a finite dimensional C^1-manifold with boundary which is compact because $f_1|_{\bar{U}}$ is proper.

Let $b\in\dot{B}$ be a regular value of $f_1|_A: A\to B$. Since $f_1|_{\bar{U}}$ is transversal to B, b is also a regular value of $f_1|_{\bar{U}}:\bar{U}\to F\times\Lambda^{contr}$, and we have:

$$\begin{aligned}\deg(f_1|_A, A, b) &= \# f_1|_A^{-1}(b)\\ &= \# f_1|_{\bar{U}}^{-1}(b)\\ &= \deg(f_1, U, b)\\ &= 1.\end{aligned}$$

From the definition of the $\mathbb{Z}_2$-degree it follows: $\deg(f_1|_A, A, b)=1$ for all $b\in\dot{B}$. By 7.1, $f_1|_A^*: H^*(B, \mathbb{Z}_2)\to H^*(A, \mathbb{Z}_2)$ and therefore $f_1|_A^*: \bar{H}^*(B, \mathbb{Z}_2)\to\bar{H}^*(A, \mathbb{Z}_2)$ is injective.

If $i: B\hookrightarrow F\times\Lambda$ is the inclusion and $\pi_\Lambda: F\times\Lambda\to\Lambda$ the projection, from $h\sim(0, g_m)$ and $g_m^*\alpha\neq 0$ we obtain: $i^*\pi_\Lambda^*\alpha\neq 0$, therefore $f_1|_A^* i^*\pi_\Lambda^*\alpha\neq 0$, and because of $A\subset U: f_1|_{\bar{U}}^*\pi_\Lambda^*\alpha\neq 0$. Since this is true for any neighborhood U of $f_1^{-1}(\{0\}\times K_m)$, the tautness property of the Alexander-Spanier cohomology ([Sp], Theorem 6.6.2. Remember that $\mathscr{E}$ is metrizable) implies: $f_1|_{f_1^{-1}(\{0\}\times K_m)}^*\pi_\Lambda^*\alpha\neq 0$.

From the commutative diagram

$$\begin{array}{ccc} f_1^{-1}(\{0\}\times K_m) & \xrightarrow{\pi_\Lambda} & \pi_{\mathbb{R}}^{-1}[\mathscr{J}(K_m)] \\ \downarrow f_1 & & \downarrow \pi_{\mathbb{R}} \\ F\times\Lambda^{contr} & \xrightarrow{\mathscr{J}\circ\pi_\Lambda} & \Lambda^{contr} \end{array}$$

and $\mathscr{J}\sim \mathrm{id}$ we finally get: $(\pi_{\mathbb{R}}|_{\pi_{\mathbb{R}}^{-1}[\mathscr{J}(K_m)]})^*\alpha\neq 0$. □

7.4. Definition. — $X:=\{u\in C_{loc}^\infty(Z, TM)\mid\bar{\partial}u+H'(u)=0, \int_Z|\partial_s u|^2\,ds\,dt<\infty\}$ with the C_{loc}^∞ topology.

$X^{contr}:=\{u\in X\mid u(0,\cdot)\text{ contractible}\}$.

$\pi: X\to\Lambda M$, $\pi u(t):=\tau u(0, t)$.

$\Phi_s: X\to\mathbb{R}$, $u\mapsto\Phi(u(s))$ for $s\in\mathbb{R}$.

$\Phi_\pm: X\to\mathbb{R}$, $u\mapsto\sup_{s\in\mathbb{R}}\Phi(u(s))$ resp. $\inf_{s\in\mathbb{R}}\Phi(u(s))$.

$\Phi_s^c:=\{u\in X\mid\Phi_s(u)\leq c\}$ for $c\in\mathbb{R}$, $s\in\mathbb{R}\cup\{+,-\}$.

$\cdot: X\times\mathbb{R}\to X$, $(u.\sigma)(s, t):=u(s-\sigma, t)$.

7.5 LEMMA. — (i) X *is a metric space.* π, $\cdot$ *and* Φ_s, $s \in \mathbb{R}$ *are continuous.* $\Phi_\pm$ *is lower resp. upper semi-continuous.*

(ii) $u \mapsto u(0, \cdot)$ *gives a bijection between the fixed points of the flow in* X *and the* 1*-periodic solutions of* $\dot{x} = X_H(t, x)$.

(iii) *Let* $u \in X$. *Then every sequence* $s_n \to \pm\infty$ *possesses a subsequence such that* $u(s_n)$ *converges in* C^∞ *to an* $x \in C^\infty(S^1, TM)$ *with* $\Phi'(x) = 0$.

(iv) Φ_- *is bounded from below by a constant* $c^- \in \mathbb{R}$.

(v) Φ_+^c *is compact for every* $c \in \mathbb{R}$.

(vi) *For* $s \in \mathbb{R}$, Φ_s *is strictly decreasing along the flow trajectories,* i.e.: *If* $\Phi_s(u.\sigma) \geqq \Phi_s(u)$ *for a* $u \in X$ *and* $\sigma > 0$ *then u is a fixed point of the flow.*

Proof. — (i) and (ii) are obvious.

(iii) By assumption $u \in X_a^b$ for some $a \leqq b \in \mathbb{R}$, X_a^b as in 6.4. Hence all partial derivatives of u are bounded. In particular, $(u(s_n))$ is a bounded sequence in $C^\infty(S^1, TM)$ and therefore has a convergent subsequence $u(s_n) \to x \in C^\infty(S^1, TM)$. For $s \in \mathbb{R}$ let $b(s) := \int_0^1 |\partial_s u(s, t)|^2 \, dt$. b is a non-negative function with $\int_{-\infty}^{\infty} b(s) \, ds < \infty$ and $|b'(s)| \leqq \int_0^1 |\langle \partial_s u, \nabla_s \partial_s u \rangle| \, dt$ bounded. This implies $b(s) \to 0$ as $s \to \pm\infty$ and thus proves $\Phi'(x) = 0$.

(iv) The calculation in the proof of 5.3 shows that Φ_H is bounded from below on periodic orbits by $c^- = -d_1$. (iv) now follows from (iii).

(v) From (iv) we have $\Phi_+^c = X_{c^-}^c$ which is compact by 6.4.

(vi) If $0 \leqq \Phi(u(s_0 - \sigma)) - \Phi(u(s_0)) = -\int_{s_0-\sigma}^{s_0} \int_0^1 |\partial_s u|^2 \, ds \, dt$ then there is an $x \in C^\infty(S^1, TM)$ with $\Phi'(x) = 0$ such that $u(s, t) = x(t)$ for $s_0 - \sigma \leqq s \leqq s_0$. We define $v \in X$ by $v(s, t) := x(t)$ for $(s, t) \in Z$. Then u, v both satisfy a perturbed Cauchy-Riemann equation and agree on an open subset of Z. From Aronszajn's unique continuation theorem ([Ar], Remark 3 on p. 248) it follows that $u = v$ on all of Z (*cf.* [Ho], Lemma 4). □

7.6. THEOREM. — *To every* $0 \neq \alpha \in \bar{H}^*(\Lambda^{\mathrm{contr}} M, \mathbb{Z}_2)$ *there exists a* $b_\alpha \in \mathbb{R}$ *such that* $(\pi|_{\Phi_+^{b_\alpha}, \mathrm{contr}})^* \alpha \neq 0$.

Proof. — In the notation of 6.3 let $X_{R,m} := \pi_R^{-1}[\mathcal{I}(K_m)]$ and choose an $m \in \mathbb{N}$ such that $(\pi_{R|X_{R,m}})^* \alpha \neq 0$ for all $R > 0$.

We fix mappings $\phi_R \in C^\infty(\mathbb{R}, [0, 2R])$ with $\phi_R(s) = s + R$ for $|s| \leqq R - 1$ and define

$$P : X_{2R,m} \to C^\infty(Z, TM)$$

by

$$Pu(s, t) := u(\phi_R(s), t).$$

We have shown in the proof of 5.4 that there exist uniform constants a, b_α such that $a \leqq \Phi(u(s)) \leqq b_\alpha$ for all $u \in X_{2R,m}$, $0 \leqq s \leqq 2R$, $R > 0$.

We claim that for every neighborhood U of $\Phi_+^{b_\alpha,\,\mathrm{contr}}$ in $C^\infty_{\mathrm{loc}}(Z, TM)$ there exists a $R_0 \in \mathbb{R}$ such that $P(X_{2R,m}) \subset U$ for all $R \geqq R_0$.

Otherwise we would find sequences $R_k \to \infty$ and $u_k \in X_{2R_k,m}$ with $Pu_k \notin U$. For each fixed $R > 0$ and k sufficiently large, $(Pu_k|_{(-R-1,R+1)\times S^1})$ would be a sequence in $\{u \in C^\infty((-R-1, R+1) \times S^1, TM) \mid \bar{\partial} u + H'(u) = 0, a \leqq \Phi(u(s)) \leqq b_\alpha\}$. So by 6.4 a subsequence of $(Pu_k|_{[-R,R]\times S^1})$ would converge in C^∞ to a $v_R \in C^\infty([-R, R] \times S^1, TM)$ with $\bar{\partial} v_R + H'(v_R) = 0$ and $\Phi(v_R(R)) \leqq b_\alpha$. Taking a diagonal sequence we would obtain a subsequence of Pu_k converging in C^∞_{loc} to a $v \in C^\infty_{\mathrm{loc}}(Z, TM)$ with $\bar{\partial} v + H'(v) = 0$ and $\Phi_+(v) \leqq b_\alpha$. Thus $v \in \Phi_+^{b_\alpha,\,\mathrm{contr}}$, contradicting the assumption $Pu_k \notin U$ for all k.

So for every neighborhood U of $\Phi_+^{b_\alpha,\,\mathrm{contr}}$ in $C^\infty(Z, TM)$ and R sufficiently large we get the following commutative diagram:

$$\begin{array}{ccc} X_{2R,m} & \xrightarrow{P} & U \\ \downarrow{\scriptstyle \pi_R} & & \downarrow{\scriptstyle \pi} \\ \Lambda^{\mathrm{contr}} & \xrightarrow{\mathrm{id}} & \Lambda^{\mathrm{contr}} \end{array}$$

where $\pi_R u(t) = \tau u(R, t)$ and $\pi u(t) = \tau u(0, t)$. Since π_R is homotopic to π_{2R} and $(\pi_{2R}|_{X_{2R,m}})^* \alpha \neq 0$, it follows: $\pi|_U^* \alpha \neq 0$, hence the theorem by tautness of the Alexander-Spanier cohomology. □

8. Final arguments

Proof of 1.1 (i). — If M is compact and $\pi_1(M)$ finite the universal covering $\tilde{M}$ of M is compact. The Hamiltonian system on M is lifted under the projection $\tilde{\pi}: \tilde{M} \to M$ to a Hamiltonian system on $\tilde{M}$ still satisfying the assumptions of 1.1 (i). Conversely, 1-periodic solutions of the lifted Hamiltonian system project to contractible 1-periodic solutions of the original system having the same action. So without loss of generality we may (and will) assume M to be simply connected.

We need this assumption to apply the following result about the topology of the loop space:

8.1. Proposition (D. Sullivan). — *If* M *is a compact simply connected manifold then infinitely many rational Betti numbers of* ΛM *are nonzero.*

Proof. — [Su], p. 46, and the homotopy equivalence of ΛM and $C^0(S^1, M)$. □

8.2. Definition. — For every $0 \neq \alpha \in \bar{H}^*(\Lambda, \mathbb{Z}_2)$ we choose a b_α as in 7.6 such that $(\pi|_{\Phi^{b_\alpha}})^* \alpha \neq 0$ and then define

$$\mathscr{F}_\alpha := \{A \subset \Phi_+^{b_\alpha} \text{ compact} \mid \pi|_A^* \alpha \neq 0\},$$
$$c_\alpha^s := \inf_{A \in \mathscr{F}_\alpha} \sup_A \Phi_s \qquad \text{for} \quad s \in \mathbb{R}.$$

8.3. Lemma. — *For every* $0 \neq \alpha \in \bar{H}^*(\Lambda, \mathbb{Z}_2)$ *and* $s \in \mathbb{R}$:

(i) $\mathscr{F}_\alpha \neq \emptyset$, *and* $\mathscr{F}_\alpha$ *is invariant under the flow.*

(ii) $-\infty < c_\alpha^s = c_\alpha^0 < \infty$.

(iii) *There exists a fixed point u of the flow with* $\Phi_s(u) = c_\alpha^s$.

Proof. – (i) $\mathscr{F}_\alpha \neq \emptyset$ because $\Phi_+^{b_\alpha} \in \mathscr{F}_\alpha$ by 7.5 (v).

(ii) $c_\alpha^s \geqq c^- > -\infty$ by 7.5 (iv).

(iii) follows by the usual deformation argument, using 7.5 (vi) □

In view of 7.5 (ii) the following proposition will conclude the proof of 1.1 (i):

8.4. Proposition. – *The family* $(c_\alpha^0)_{0 \neq \alpha \in \bar{H}^*(\Lambda, \mathbb{Z}_2)}$ *is not bounded from above.*

Proof. – Assume that there exists a $c \in \mathbb{R}$ with $c_\alpha^0 \leqq c-1$ for all $0 \neq \alpha \in \bar{H}^*(\Lambda, \mathbb{Z}_2)$. For $0 \neq \alpha \in \bar{H}^*(\Lambda, \mathbb{Z}_2)$ let $K_{\alpha,s}^c := \Phi_+^{b_\alpha} \cap \Phi_s^c$. Then $(\pi|_{K_{\alpha,s}^c})^* \alpha \neq 0$ from the definition of c_α^s. By another continuity property of Alexander-Spanier cohomology ([Sp], Theorem 6.6.6),

$$(\pi|_{\bigcap_{s \in \mathbb{R}} K_{\alpha,s}^c})^* \alpha \neq 0.$$

But $\bigcap_{s \in \mathbb{R}} K_{\alpha,s}^c \subset \Phi_+^c$, hence $(\pi|_{\Phi_+^c})^* \alpha \neq 0$ for all $0 \neq \alpha \in \bar{H}^*(\Lambda, \mathbb{Z}_2)$. If $K := \pi(\Phi_+^c)$ and $i_K : K \subsetneq \Lambda$ is the inclusion then $i_K^* \alpha \neq 0$ for all $0 \neq \alpha \in \bar{H}^*(\Lambda, \mathbb{Z}_2)$.

Since K is compact and $E(q) := \int_0^1 |\dot{q}|^2 \, dt$ a continuous function on Λ, E is bounded on K. Using this one easily constructs a map $r : K \to P_m M$, m big enough, with the property: $g_m \circ r = i_K$ (*see* [Bo]). Thus

$$0 \neq g_m^* \alpha \in \bar{H}^*(P_m M, \mathbb{Z}_2)$$

for all $0 \neq \alpha \in \bar{H}^*(\Lambda, \mathbb{Z}_2)$.

From 8.1 and the universal coefficient theorem we obtain: $\bar{H}^k(\Lambda, \mathbb{Z}_2) \neq \{0\}$ and therefore $\bar{H}^k(P_m M, \mathbb{Z}_2) \neq \{0\}$ for infinitely many $k \in \mathbb{N}$. But $P_m M$ is a compact finite dimensional manifold-with-boundary, so we have a contradiction. □

Proof of 1.1 (ii). – Because the proof is completely analogous to [Ho], Proof of Theorem 3, we will only sketch it.

For a compact flow-invariant $Y \subset X$ we define

$$\mathcal{O} := \{ U \subset Y \text{ open} \mid \pi|_U^* : \bar{H}^k(\Lambda, \mathbb{Z}_2) \to \bar{H}^k(U, \mathbb{Z}_2) \text{ is the zero map for } k \geqq 1 \}$$

and for $A \subset Y$ the index

$$\operatorname{ind}(A) := \inf \left\{ k \in \mathbb{N} \mid \text{There are } U_1, \ldots, U_k \in \mathcal{O} \text{ with } A \subset \bigcup_{i=1}^k U_i \right\}.$$

It follows from classical Lusternik-Schnirelman theory that the number of fixed points of the flow in Y is at least ind(Y).

Now for given $\alpha_i \in \bar{H}^{n_i}(\Lambda^{\text{contr}}, \mathbb{Z}_2)$, $n_i \geqq 1$, $1 \leqq i \leqq k$ with $\alpha_1 \cup \ldots \cup \alpha_k \neq 0$ we find a $c \in \mathbb{R}$ such that $(\pi|_{\Phi_+^{c,\text{contr}}})^* (\alpha_1 \cup \ldots \cup \alpha_k) \neq 0$. An easy topological argument then shows $\operatorname{ind}(\Phi_+^{c,\text{contr}}) \geqq k+1$ and thus finishes the proof. □

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