Metric invariance entropy, quasi-stationary measures and control sets

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The topological notion of invariance entropy has been studied since some time and a number of results are available (cf. Kawan [6]). In particular, under hyperbolicity assumptions da Silva and Kawan [5] could show that the invariance entropy of control sets is determined by the exponential growth rate of the unstable determinant.

In view of the fruitful interplay between topological and measure theoretic versions of entropy of dynamical systems it seems desirable to develop also a measure-theoretic version of invariance entropy. The research reported undertakes some steps in that direction. Since here the behavior of trajectories within a non-invariant subset of the state space is of interest, I use a generalization of invariant measures given by quasi-stationary measures with respect to a given probability measure on the control range. Quasi–stationary measures are frequently employed in the theory of absorbed Markov processes, where they occur as Yaglom limits and describe the behavior under the condition that the trajectory remains in the considered subset. General references to quasi-stationary measures are the monograph [2] and the survey [1].

Compared to entropy for dynamical systems, in the construction of invariance entropy one replaces partitions and open covers by invariant partitions and invariant open covers, respectively, which use feedbacks keeping the system in the given subset of the state space up to a finite time. Due to quasi-stationarity, the relevant probability measure for the associated Shannon-entropy has to be weighted according to the considered time. Since the minimal required bit rate is of relevance for control theoretic purposes, the infimum of the associated bit rates over all invariant partitions is taken.

The main results show that this entropy, which is always bounded above by the topological invariance entropy, is invariant under measurable transformations and that it is already determined by control sets, which are certain subsets of the state space which are characterized by controllability properties.

Open problems in this field include the question if measure theoretic invariance entropy can be arbitrarily close to the topological version. Furthermore, there is a lack of explicit examples where the measure theoretic invariance entropy can be computed.

References

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