A Universal Property of Symmetric L-theory and Quinn's Bordism Machine

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Abstract

In this thesis we demonstrate a universal property of symmetric L-theory as a space-valued functor from the category of Waldhausen categories with Spanier Whitehead products in the sense of [WW98]. Specifically, we characterise symmetric L-Theory as the target of the "universal bordism characteristic of symmetric Poincaré objects". Furthermore, we show that the construction of Quinn's bordism spaces of ad theories in the sense of [LM14] satisfies an analogous characterisation. The main novel ingredient of our work is the development of a simple abstract setting for universality that unifies both examples.

There are two parts to this thesis: Part I establishes the abstract foundations and describes applications. Part II is a technical extension of the first part, based on a further analysis of sufficient conditions for universal bordism characteristics and the problem of how to extend their targets to spectrum-valued functors in a natural way. We introduce a second more specialised framework for this investigation and illustrate the theory in two explicit examples; namely, Quinn's Bordism machine of ad theories and symmetric L-theory in the setting of additive categories with chain duality introduced by A. A. Ranicki.

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Contents

Introduction

Algebraic L-theory was developed by C. T. C. Wall, A. A. Ranicki and A. S. Mishchenko (see [Wal99, Ran92, Mis71]) as a receptacle for signature-type invariants of closed manifolds and, most notably, surgery obstructions. It is defined in terms of semi-simplicial spaces whose n-simplices are algebraic models of compact Δ^n-manifold-ads, i.e., manifolds with $n+1$ many boundary pieces intersecting transversely and with empty total intersection; for example, a Δ^0-manifold is just a manifold without boundary and a Δ^1-manifold is a manifold with boundary in the usual sense.

L-theory should be regarded as a cousin of algebraic K-theory, which geometrically serves as a receptacle for Euler-characteristic-type invariants. However, in contrast to the feature of additivity of Euler-characteristic-type invariants with respect to decompositions of the underlying space, the most important property of signature-type invariants is bordism invariance.

Recent research ([Bar16, BGT13]; see [Ste17] in particular, for an elementary and 1-categorical account) aimed at describing algebraic K-theory not only via its construction, but rather by characterising it via a universal property, briefly summarised as being "the universal additive characteristic".

One of the two motivations of this work was to find a similar description of algebraic L-theory via a universal property. The other motivation of this work is based on the idea, popularised by Frank Quinn in his work on the surgery exact sequence [Qui70, Qui95] and much further pursued by Gerd Laures and James E. McClure [LM14, LM13], that the construction of L-theory only relies on having a suitable notion of Δ^n-ads available, and so generalises to other contexts. In this generality the simplicial construction is often referred to as Quinn's bordism machine; in the case of manifolds, for instance, the machine gives rise to a geometric model for Thom spectra underlying bordism theory. One may then wonder to what extent Quinn's machine, in general, possesses a universal property.

In this thesis we show that indeed both algebraic L-theory and Quinn's bordism machine satisfy a universal property, briefly summarised as being "the universal bordism characteristic". We formalise an abstraction of both constructions, called *parametric realisation*, in order to unify the proofs, and then analyse two specific examples in detail, based on the availability of relevant results in the literature: First, the case of Waldhausen categories with duality in analogy to [Ste17] wherein parametric realisation corresponds to algebraic L-theory; and second, the case of ad theories and Quinn's bordism machine in the sense of Laures-McClure.

Let us now explain the results of this work in more detail: We start by placing ourselves into a minimalistic setting to describe parametric realisation and its universal property. A complete introduction is given in Chapter 1.

Let $\mathcal{C}$ be a category and denote by Δ^{op} the opposite of the category of non-empty finite ordered total sets $[n] = \{0 < \cdots < n\}$, where n is a natural number, and order-preserving injective maps. By a *parametrisation structure* on $\mathcal{C}$, we will mean a functor

$$\begin{aligned} p : \mathcal{C} \times \Delta^{op} &\to \mathcal{C} \\ (c, [n]) &\mapsto c[n] \end{aligned}$$

together with a natural transformation $\mu_c : c \cong c[0]$. We call the functor p the *parametrisation operator* on $\mathcal{C}$ and μ the *unit*.

Denote by $\mathbf{Top}_*$ the category of pointed spaces and pointed continous maps. In anticipation of our applications, one should regard a space-valued functor $Z : \mathcal{C} \to \mathbf{Top}_*$ on $\mathcal{C}$ as specifying the "closed objects" of $\mathcal{C}$, or the abstract analogues of closed manifolds, and the spaces $Z(c[n])$ as the space of n-parameter bordisms in $Z(c)$. For any functor $Z : \mathcal{C} \to \mathbf{Top}_*$ we then define its *parametric realisation* to be the functor $PZ : \mathcal{C} \to \mathbf{Top}_*$ determined by taking geometric realisation of the semi-simplicial spaces $[n] \mapsto Z(c[n])$, for all objects c in $\mathcal{C}$, i.e.,

$$PZ(c) := \|[n] \mapsto Z(c[n])\|.$$

Note that the construction PZ comes with a canonical natural transformation

$$\iota_Z(c) : Z(c) \hookrightarrow PZ(c),$$

given by the composition of the natural transformation $Z(\mu_c) : Z(c) \cong Z(c[0])$ and the inclusion of the 0-skeleton $Z(c[0]) \hookrightarrow PZ(c)$.

Definition 0.0.1. Let $Z : \mathcal{C} \to \mathbf{Top}_*$ be a functor. A *bordism characteristic* of Z is a pair (F, σ) consisting of a functor $F : \mathcal{C} \to \mathbf{Top}_*$ and a natural transformation $\sigma : Z \Rightarrow F$ such that the functor F satisfies the following condition called *bordism invariance*: all face maps of the semi-simplicial space $[n] \mapsto F(c[n])$ are weak equivalences, for all objects c in $\mathcal{C}$ and all natural numbers $n \geq 1$.

Bordism characteristics of any given $Z : \mathcal{C} \to \mathbf{Top}_*$ assemble into a category, which we denote by $\mathbf{Brd}(Z)$; a morphism $(F, \sigma) \to (F', \sigma')$ is a natural transformation $\eta : F \Rightarrow F'$ such that $\eta \circ \sigma = \sigma'$. We call a morphism η in $\mathbf{Brd}(Z)$ a *weak equivalence* if it is an objectwise weak equivalence and denote the corresponding category obtained by formally inverting weak equivalences by $\mathbf{hBrd}(Z)$.

It is evident from the definition that the operation of parametric realisation on a given functor Z defines a bordism characteristic (PZ, ι_Z) of Z precisely if PZ is bordism invariant. This latter assumption may not hold in general, though. We defer a counterexample to Example 1.3.2.

Nevertheless, under a mild symmetry assumption on the parametrisation operator, we show that whenever the pair (PZ, ι_Z) is a bordism characteristic of Z, then it is the universal one in the following homotopical sense:

Theorem 0.0.2. *Let $(\mathcal{C}, p, \mu)$ be a category with symmetric parametrisation structure. Furthermore, let $Z : \mathcal{C} \to \mathbf{Top}_*$ be a functor such that its parametric realisation PZ is bordism invariant. Then the pair (PZ, ι_Z) is an initial object in $\mathbf{hBrd}(Z)$.*

Theorem 0.0.2 will serve as a template for applications. Its proof depends on the theory of semi-simplicial spaces and reasoning analogous to that of [Ste17].

Let us next outline our two main applications of Theorem 0.0.2. Chapters 2 and 3 are dedicated to thorough discussions.

First, we consider **xWald**, the category of Waldhausen categories with a Spanier Whitehead product, alias Waldhausen categories with duality, as developed by Michael Weiss and Bruce Williams in [WW98]. The setting **xWald** was introduced in the work of Weiss-Williams on automorphisms of manifolds (see [WW88, WW89, WW14, WW01, WW98, WW00]) as a generalisation of the category of rings with involution, and should be viewed as a counterpart to the setting of Waldhausen categories for algebraic K-theory.

As closed objects of **xWald**, we consider the symmetric Poincaré objects, which satisfy a self-duality property in analogy to the Poincaré duality of manifolds. We will denote the corresponding functor by $\mathrm{sp} : \mathbf{xWald} \to \mathbf{Top}_*$. A parametrisation structure on the category **xWald** was essentially described in [WW98] and symmetric L-theory $L(\mathcal{C})$ of a Waldhausen category with duality $\mathcal{C}$ is defined as the parametric realisation of the functor sp applied to $\mathcal{C}$, i.e.,

$$L(\mathcal{C}) := P\,\mathrm{sp}(\mathcal{C}).$$

It was observed in [WW00] that symmetric L-theory satisfies the bordism invariance condition, or in other words, that the pair $(L, \iota_{\mathrm{sp}} : \mathrm{sp} \Rightarrow L)$ is a bordism characteristic of sp. We combine this observation with Theorem 0.0.2 to deduce that the pair (L, ι_{sp}) is in fact a *universal* bordism characteristic of sp, and, thus, satisfies a universal property:

Theorem 0.0.3. *The pair* (L, ι_{sp}) *is initial in* $\mathbf{hBrd}(\mathrm{sp})$.

To elaborate, Theorem 0.0.3 states that symmetric L-theory of Waldhausen categories with duality is the universal target of bordism characteristics of symmetric Poincaré objects, in the sense that every natural transformation $\mathrm{sp} \Rightarrow F$ to a bordism invariant functor F extends along $\iota_{\mathrm{sp}} : \mathrm{sp} \Rightarrow L$ to a natural transformation $L \Rightarrow F$, up to inverting weak equivalences. Furthermore, the extension is uniquely defined up to inverting weak equivalences.

As a demonstration of Theorem 0.0.3 we give the following application: One of the main achievements of [WW98] was to establish a relationship between algebraic L- and K-theory to explain the existence of Rothenburg sequences. The authors achieve this by constructing a natural transformation

$$\Xi : L \Rightarrow K^{th\mathbb{Z}_2}$$

from symmetric L-theory to the $\mathbb{Z}_2$-Tate cohomology of K-theory.

We prove that the natural transformation Ξ of Weiss-Williams can be characterised as a morphism of bordism characteristics of symmetric Poincaré objects, up to weak equivalence, as a consequence of the universal property of L-theory stated in Theorem 0.0.3.

In our second application of Theorem 0.0.2 we examine the category of ad theories introduced by Laures-McClure [LM14]. Ad theories are a modern reformulation of Quinn's "bordism-type theories" (cf. [Qui95]), serving as input for Quinn's bordism machine, and giving rise to highly structured models of bordism-type spectra, such as L-theory spectra and Thom spectra. They are defined as integer-graded (by dimension) categories $\mathcal{A}$ with involution together with an integer-graded set of diagrams in $\mathcal{A}$, called "ads", indexed by the face posets of ball complex pairs.

Let **Ad** denote the category of ad theories. A parametrisation operator on **Ad** is described in [LM14], yet the proof that the construction is well-defined was not presented there. We provide comprehensive details here to fill the gap, and, moreover, show that the parametrisation operator extends over the whole category of ball complex pairs.

We denote by $\mathbf{Ball}_2$ the category of ball complex pairs and let $\mathrm{cl}^0(\mathcal{A})$ (cl for closed) denote the set of Δ^0-ads of grading 0 of an ad theory $\mathcal{A}$. (The grading is chosen in order that the functor $\mathcal{Q}$ corresponds to the taking the zeroeth space of Quinn's bordism-spectra.) Quinn's bordism-space machine $\mathcal{Q}$ is defined as the parametric realisation of the closed objects functor cl^0, i.e.,

$$\mathcal{Q} := P\,\mathrm{cl}^0\,.$$

It turns out that results of [LM14] directly imply that the pair $(\mathcal{Q}, \iota_{\mathrm{cl}^0})$ is a bordism characteristic. We furthermore prove that this pair defines a universal bordism characteristic of the closed-object functor:

Theorem 0.0.4. *The pair* $(\mathcal{Q}, \iota_{\mathrm{cl}^0})$ *is initial in* $\mathbf{hBrd}(\mathrm{cl}^0)$.

The results described so far constitute the content of the first part of this work. Part II of this work serves as a technical extension and elaboration of Part I, based on a further investigation of parametric realisation and the conditions of Theorem 0.0.2. Our research is motivated by the following specific questions inspired by [Ste17]:

Let $(\mathcal{C}, p, \mu)$ be a category with parametrisation structure and $Z : \mathcal{C} \to \mathbf{Top}_*$ be a given space-valued functor on $\mathcal{C}$. Under which conditions on the functor Z and parametrisation operator p, can one deduce that:

1. The parametric realisation of Z is bordism invariant?

2. The parametric realisation of Z extends to an Ω-spectrum functor?

The motivating aim behind the first question is to understand better the role of the input functor Z in Theorem 0.0.2, and in particular, to find sufficient conditions on the functor Z itself, rather than on its parametric realisation, so that the pair (PZ, ι_Z) is universal. The second question is motivated by the fact that our main examples of parametric realisation are not just space-valued but in fact take values in the category of infinite loop spaces, i.e., they appear as zeroeth terms of Ω-spectrum-valued functors. It is then desirable to find a setting for parametric realisation in which the extension to spectra would be described naturally.

In this work, we do not examine Questions (1.) and (2.) in full generality but rather in the special situation that the parametrisation operator extends over the category of ball complex pairs, motivated by the example of ad theories and, furthermore, by the recent description of Tibor Macko and Spirou Adams-Florou in [AFM18] of additive categories with chain duality parametrised over ball complexes.

More specifically, we introduce an axiomatisation of the extended features of parametrisation in the category of ad theories called *extended parametrisation* *structures*: These are parametrisation structures (p, μ) such that the parametrisation operator p is given by a functor $p : \mathcal{C} \times \mathbf{Ball}_2^{op} \to \mathcal{C}$, together with the data of natural isomorphisms

$$\alpha_c^{(K,L),(M,N)} : (c[M,N])[K,L] \cong c[(K,L) \times (M,N)]$$

expressing associativity of the parametrisation operator with respect to the product '$\times$' of ball complexes (as CW complexes) and satisfying compatibility conditions with the unit $\mu_c : c \cong c[0]$. A formal introduction is presented in Chapter 4. We then establish the following specialisation of Theorem 0.0.2 in the setting of extended parametrisation structures, as a demonstrative answer to Questions (1.) and (2.):

Theorem 0.0.5. *Let $(\mathcal{C}, p, \mu, \alpha)$ be a category with extended parametrisation structure and $Z : \mathcal{C} \to \mathbf{Set}_*$ be a functor. Suppose that p is combinatorial and deloopable and that Z is local and surjective on expansions. Then the parametric realisation PZ is bordism invariant and extends to an Ω-spectrum-valued functor. In particular, the pair (PZ, ι_Z) is initial in $\mathbf{hBrd}(Z)$.*

The details and proof of Theorem 0.0.5 are described in Chapter 4.

For an illustration of the setting of extended parametrisation structures and Theorem 0.0.5, we discuss two examples in Chapters 5 and 6: First, we demonstrate that Theorem 0.0.5 applies in the case of the category of ad theories and the closed-objects functor cl^0, thereby obtaining an alternative and second proof of Theorem 0.0.4. This second proof highlights the special properties of the closed objects functor which imply universality of Quinn's bordism machine and, moreover, it recovers the extension of Quinn's bordism machine to spectra.

Second, we examine the setting of additive categories with chain duality for algebraic L-theory in the sense of Ranicki [Ran92]. This setting generalises that of rings with involution but is more specialised than that of Waldhausen categories with duality.

We prove that one may assemble additive categories with chain duality into a category with extended parametrisation structure such that symmetric L-theory is given as the parametric realisation of the symmetric-Poincaré-complexes functor. The definition of an extended parametrisation structure was partially described in [AFM18], in the sense that the authors construct additive categories with duality, for every pair $(\mathbb{A}, X)$ consisting of an additive category with duality $\mathbb{A}$ and ball complex X, but functoriality of the construction is not described there. Our work aims to provide the details, and, in particular extend the established functoriality, due to Ranicki [Ran92], over the category of simplicial complexes and simplicial inclusions to the category of ball complex pairs. As a consequence, we are able to formulate a universal property of L-theory in this setting.

In the course of editing the final drafts of this paper, the series of preprints [CDH$^+$20a, CDH$^+$20b, CDH$^+$20c] appeared in which a new variant of L-theory in an ∞-categorical setting is defined, and furthermore, shown to possess a universal property. The universal property described there is similar in spirit to the one demonstrated here but is not directly comparable, due to the difference of setups; in addition, we note that our setup also applies in non-algebraic situations.

Part I

Universality and Main Results

Chapter 1

An Abstract Universality Theorem

In this chapter we develop a minimalistic framework for the study of universality of certain space-valued functors, motivated by applications to L-theory and Quinn's bordism machine. The central feature introduced is a semi-simplicial construction on space-valued functors called *Parametric Realisation* (see Definition 1.1.7) that serves to abstract the definitions of both L-theory and Quinn's bordism machine. The aim of the framework is to establish a universal property for the parametric realisation of a given functor, as a blueprint for applications.

We introduce the notion of *categories with parametrisation structures* (Definition 1.1.3) to serve as minimal settings for parametric realisation. Roughly speaking, a parametrisation structure equips a category with an internal rule for determining "objects parametrised over an n-simplex", for every natural number $n \geq 0$. The exact definitions of parametrisation structures and parametric realisation are described in Subsection 1.1.1, assuming familiarity with the fundamentals of semi-simplicial theory. We refer the reader to [ERW19] for a concise reference on the latter.

The notions of *bordism invariant* functors and *bordism characteristics* are introduced in Subsection 1.1.2 in order to describe a universal property of parametric realisation. The precise statement of this universal property is recorded as the "Abstract Universality Theorem" (Theorem 1.2.1) and we have devoted Section 1.2 to its statement and proof.

The proof of the Abstract Universality Theorem is heavily inspired by the proof of a universal property of algebraic K-theory described in [Ste17]. In particular, we rely on two facts about the geometric realisation of semi-simplicial spaces that we record in Propositions 1.2.4 and 1.2.7. We expect these facts to be well known, although we did not find precise statements or accompanying proofs in the literature; we provide them here for completeness.

It turns out that the concept of Parametric Realisation has a broader scope than our main applications. In Section 1.3, we illustrate a few simple examples, including the singular construction of topological spaces and the classifying space construction for small categories.

1.1 Definitions

The following two subsections serve to collect preliminary definitions for the Abstract Universality Theorem.

1.1.1 Parametrisation Structures and Parametric Realisation

Definition 1.1.1. Let Δ denote the category with

- objects: the non-empty ordered sets $[n] := \{0 < 1 < \cdots < n\}$, where $n \geq 0$, and
- morphisms: the injective order-preserving maps $[n] \to [m]$.

Furthermore, denote its opposite category by Δ^{op}.

Remark 1.1.2. Note that our notation is nonstandard; the category Δ is usually denoted by Δ_{inj} in the literature (e.g., cf. [ERW19, §1.2]), and should not be confused by the usual simplex category consisting of the same objects as Δ but with all order-preserving maps as morphisms. We employ the simpler notation Δ as we will primarily deal with semi-simplicial objects without any specific degeneracy maps.

Definition 1.1.3. A *category with parametrisation structure* consists of a triple $(\mathcal{C}, p, \mu)$ where:

- $\mathcal{C}$ is a category,
- $p : \mathcal{C} \times \Delta^{op} \to \mathcal{C}$ is a functor, called the *parametrisation operator*, and
- $\mu : \mathrm{id}_{\mathcal{C}} \Rightarrow p(-, [0])$ is a natural isomorphism of functors $\mathcal{C} \to \mathcal{C}$, called the *unit*.

We will write $c[n]$ and $f[n]$ as shorthand for the notations $p(c, [n])$ and $p(f, \mathrm{id}_{[n]})$, respectively.

Remark 1.1.4. For intuition, it is helpful to regard the objects $c[1]$ and $c[n]$ as the objects of bordisms and n-parameter bordisms of elements in $c[0] \cong c$, respectively. This viewpoint is motivated by our main examples which arise from bordism theory and will be described in the upcoming chapters.

The main relevance of categories with parametrisation structures is that they serve as input for a simplicial construction on space-valued functors. Before coming to the definition, we first introduce preliminary notation.

Notation 1.1.5.

1. For any two given categories $\mathcal{A}, \mathcal{B}$, we denote by $\mathrm{Fun}(\mathcal{A}, \mathcal{B})$ the category of functors from $\mathcal{A}$ to $\mathcal{B}$ where morphisms are natural transformations. Furthermore, we employ the notation $ss\mathcal{B}$ in the case $\mathcal{A} = \Delta^{op}$, corresponding to the category of of semi-simplicial objects in $\mathcal{B}$.

2. We denote by $\mathbf{Top}_*$ the category of pointed compactly generated spaces and pointed continuous maps. Moreover, we let pt and $*$ denote a fixed terminal object of the categories $\mathbf{Top}_*$ and $ss\,\mathbf{Top}_*$, respectively.

3. We let $||-|| : ss\,\mathbf{Top}_* \to \mathbf{Top}_*$ denote geometric realisation (see Remark 1.1.6).

Remark 1.1.6. Geometric realisation $||-|| : ss\,\mathbf{Top}_* \to \mathbf{Top}_*$ is defined by assigning to a semi-simplicial pointed space $X : [n] \mapsto (X_n, x_n)$ the pointed space $(||X||, ||x_0||)$, where $||X||$ is the quotient space given by

$$||X|| := \left(\coprod_{n \geq 0} X_n \times |\Delta^n| \right) \Big/ \sim .$$

Here $|\Delta^n|$ denotes the standard topological n-simplex and $\sim$ is the equivalence relation generated by

$$(\phi^* x, t) \sim (x, \phi_* t), \text{ for all morphisms } \phi \in \mathrm{Mor}(\Delta).$$

For more comprehensive details about geometric realisation of semi-simplicial spaces, we refer the reader to [ERW19, §1.2].

Definition 1.1.7. Let $(\mathcal{C}, p, \mu)$ be a category with parametrisation structure.

1. Let $P_\bullet : \mathrm{Fun}(\mathcal{C}, \mathbf{Top}_*) \to \mathrm{Fun}(\mathcal{C}, \mathbf{Top}_*)$ be the functor given by precomposition with the parametrisation operator p and currying variables, i.e.,

$$P_\bullet : \mathrm{Fun}(\mathcal{C}, \mathbf{Top}_*) \xrightarrow{\circ p} \mathrm{Fun}(\mathcal{C} \times \Delta^{op}, \mathbf{Top}_*) \cong \mathrm{Fun}(\mathcal{C}, ss\,\mathbf{Top}_*).$$

We define an endofunctor $P : \mathrm{Fun}(\mathcal{C}, \mathbf{Top}_*) \to \mathrm{Fun}(\mathcal{C}, \mathbf{Top}_*)$, called *Parametric Realisation* on $\mathcal{C}$, as the following composition:

$$P : \mathrm{Fun}(\mathcal{C}, \mathbf{Top}_*) \xrightarrow{P_\bullet} \mathrm{Fun}(\mathcal{C}, ss\,\mathbf{Top}_*) \xrightarrow{||-||\circ} \mathrm{Fun}(\mathcal{C}, \mathbf{Top}_*).$$

2. We furthermore define a canonical natural transformation $\iota : \mathrm{id} \Rightarrow P$ as follows: For any given functor $F : \mathcal{C} \to \mathbf{Top}_*$, let $\iota_F : F \Rightarrow PF$ be the natural transformation whose component at an object c in $\mathcal{C}$ is given by the composition

$$\iota_F(c) : F(c) \overset{F(\mu_c)}{\cong} F(c[0])) \hookrightarrow PF(c),$$

where the hooked morphism $F(c[0]) \hookrightarrow PF(c)$ denotes the inclusion of the 0-skeleton.

For every natural number $n \geq 0$ and functor $F : \mathcal{C} \to \mathbf{Top}_*$, we will denote by

$$P_n F : \mathcal{C} \to \mathbf{Top}_*, c \mapsto F(c[n])$$

the composition of $P_\bullet F$ with the evaluation functor $ss\,\mathbf{Top}_* \to \mathbf{Top}_*, X_\bullet \mapsto X_n$ in degree n. Moreover we denote by $F(c[\bullet])$ the image of $P_\bullet F$ on an object c in $\mathcal{C}$.

1.1.2 Bordism Characteristics

We are interested in finding a characterisation of the pairs (PZ, ι_Z) for a given category with parametrisation structure $(\mathcal{C}, p, \mu)$ and space-valued functor Z depending upon conditions on the functor PZ and parametrisation operator p. In the subsequent definition, we introduce a special category of pairs (F, σ) consisting of a space-valued functor F from $\mathcal{C}$ and natural transformation $\sigma : Z \Rightarrow F$, called *the category of bordism characteristics*, as the most basic yet interesting setting for this characterisation.

For precision, note that by a *weak equivalence* in $\mathbf{Top}_*$, indicated by the notation '$\simeq$', we will mean the standard notion of a map $f : X \to Y$ of pointed spaces inducing an bijection on the set of path components and an isomorphism on all pointed homotopy groups.

Definition 1.1.8. Let $(\mathcal{C}, p, \mu)$ be a category with parametrisation structure.

1. A functor $F : \mathcal{C} \to \mathbf{Top}_*$ is said to be *bordism invariant* if, for every object c in $\mathcal{C}$, all face maps of the semi-simplicial space $F(c[\bullet])$ are weak equivalences, i.e.,
$$F(d_i) : F(c[n]) \xrightarrow{\simeq} F(c[n-1])$$
for all objects c in $\mathcal{C}$, $n \geq 0$ and $0 \leq i \leq n$.

2. Let $Z : \mathcal{C} \to \mathbf{Top}_*$ be a functor. A *bordism characteristic* of Z is a pair (F, σ) consisting of

   - a bordism invariant functor $F : \mathcal{C} \to \mathbf{Top}_*$, and
   - a natural transformation $\sigma : Z \Rightarrow F$ of functors $\mathcal{C} \to \mathbf{Top}_*$.

   The natural transformation σ is called a *characteristic*.

3. We assemble bordism characteristics of Z into a category $\mathbf{Brd}(Z)$, whose morphisms $\eta : (F, \sigma) \to (F', \sigma')$ are natural transformations $\eta : F \Rightarrow F'$ which commute with the bordism characteristics, i.e., such that $\eta \circ \sigma = \sigma'$.

4. A natural transformation of $\mathbf{Top}_*$-valued functors on $\mathcal{C}$, $\eta : F \Rightarrow F$, is called a *weak equivalence* if, each component $\eta_c : F(c) \to F'(c)$ is a weak equivalence of spaces, for all objects c of $\mathcal{C}$. A morphism $\eta : (F, \sigma) \to (F', \sigma')$ of bordism characteristics of Z is called a *weak equivalence* if $\eta : F \Rightarrow F$ is a weak equivalence of functors $\mathcal{C} \to \mathbf{Top}_*$.

5. We define *the homotopy category of bordism characteristics of* Z, denoted $\mathbf{hBrd}(Z)$, to be the category obtained by formally inverting weak equivalences (see [DHKS04, §26.5] for the construction).

6. An initial object in the category $\mathbf{hBrd}(Z)$ is called a *universal bordism characteristic of* Z.

Remark 1.1.9. The term *bordism invariant* issuggested by our main examples in which case the objects $c[n]$, where $n \geq 0$, describe the "n-parameter bordisms" in c. The condition that all face maps of the semi-simplicial functor $P_\bullet F$ are weak equivalences may then be interpreted to mean that the functor F is invariant under bordisms and all higher bordisms.

Remark 1.1.10. The definition of the category of bordism characteristics of Z is inspired by the definition of the category of global Euler characterstics given in [Ste17].

Finally, we introduce a symmetry condition on parametrisation structures that asserts that the operation of parametrisation is commutative up to natural isomorphisms:

Definition 1.1.11. Let $(\mathcal{C}, p, \mu)$ be a category with parametrisation structure. We say the parametrisation structure (p, μ) is *symmetric*, if there are natural isomorphisms of functors $\mathcal{C} \times \Delta^{op} \times \Delta^{op} \to \mathcal{C}$,

$$s_c^{n,m} : (c[n])[m] \cong (c[m])[n],$$

such that the following properties hold:

1. $s_c^{n,m} \circ s_c^{m,n} = \mathrm{id}_c$, for all $c \in \mathrm{ob}(\mathcal{C})$ and $m, n \geq 0$.
2. The symmetry morphisms $s_c^{n,m}$ are compatible with the unit μ in the sense that the following trianges commutes, for all $c \in \mathrm{ob}(\mathcal{C})$ and $k \geq 0$:

$$\begin{tikzcd} & c[k] \arrow[dl, "\mu_c[k]"'] \arrow[dr, "\mu_{c[k]}"] & \\ (c[0])[k] \arrow[rr, "s_c^{0,k}"] & & (c[k])[0] \end{tikzcd} \quad \begin{tikzcd} & c[k] \arrow[dl, "\mu_{c[k]}"'] \arrow[dr, "\mu_c[k]"] & \\ (c[k])[0] \arrow[rr, "s_c^{k,0}"] & & (c[0])[k]. \end{tikzcd}$$

We call the morphisms $s_c^{n,m}$ *symmetry morphisms.*

1.2 The Universality Theorem

In this section we will show that parametric realisation yields universal bordism characteristics under suitable conditions. Precisely, we will prove the following theorem:

Theorem 1.2.1. (Abstract Universality Theorem)
Let $(\mathcal{C}, p, \mu)$ be a category with symmetric parametrisation structure and let $Z : \mathcal{C} \to \mathbf{Top}_*$ be a functor such that its parametric realisation PZ is bordism invariant. Then the pair (PZ, ι_Z) is a universal bordism characteristic of Z

The proof of Theorem 1.2.1 relies on three preliminary lemmas, recorded as Lemmas 1.2.6, 1.2.10 and 1.2.11 below.

In preparation for the first preliminary lemma, we recall the notion of a homotopy cartesian square and a key result about the homotopy type of semi-simplicial spaces.

Definition 1.2.2. Let $C \times_D^h B$ denote the homotopy pullback alias homotopy limit of the diagram of spaces $(C \to D \leftarrow B)$. A strictly commutative diagram of spaces

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

is called a *homotopy cartesian square* or *homotopy pullback square* if the canonical map $A \to C \times_D^h B$ is a weak equivalence.

Remark 1.2.3. We refer the reader to the book [MV15] for a comprehensive reference on homotopy limits. The previous definition is based on Definitions 3.2.4 and 3.3.1 therein.

Proposition 1.2.4. *Let $X_\bullet$ be a semi-simplicial space such that all of its face maps are weak equivalences. Then the inclusion of the 0-skeleton*

$$X_0 \hookrightarrow \|X_\bullet\|$$

is a weak equivalence.

Proof. A map of semi-simplicial spaces $f_\bullet : X_\bullet \to Y_\bullet$ is called *homotopy cartesian* ([ERW19, Definition 2.9]) if the squares

$$\begin{tikzcd} X_n \arrow[r, "d_i"] \arrow[d, "f_n"] & X_{n-1} \arrow[d, "f_{i-1}"] \\ Y_n \arrow[r, "d_i"] & Y_{n-1} \end{tikzcd}$$

are homotopy cartesian for all $n \geq 1$ and $0 \leq i \leq n$.

The assumption that all face maps of X are weak equivalences may thus be translated to the statement that the unique map $X \to *$ to the terminal semi-simplicial space $*$ is homotopy cartesian. The result then follows from [ERW19, Theorem 2.12], which states that for a homotopy cartesian map of semisimpicial spaces $f_\bullet : X_\bullet \to Y_\bullet$, the following square is homotopy cartesian:

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d, "f_0"] & \|X_\bullet\| \arrow[d, "\|f\|"] \\ Y_0 \arrow[r] & \|Y_\bullet\|. \end{tikzcd}$$

□

Remark 1.2.5. The analogous statement with homotopy equivalences in place of weak equivalences goes back to Segal; see [Seg74, Proposition 1.6].

The first preliminary lemma is now immediate and highlights a crucial consequence of bordism invariance condition:

Lemma 1.2.6. *Let $(\mathcal{C}, p, \mu)$ be a category with parametrisation structure and $F : \mathcal{C} \to \mathbf{Top}_*$ be a bordism invariant functor. Then the natural transformation $\iota_F : F \Rightarrow PF$ is a weak equivalence.*

Proof. The result is obtained by applying Proposition 1.2.4 objectwise, i.e., to the semi-simplicial spaces $F(c[\bullet])$, for all $c \in \mathrm{ob}(\mathcal{C})$. □

By a pointed *bi-semi-simplicial space* we will mean a functor

$$X : (\Delta \times \Delta)^{op} \to \mathbf{Top}_* \,.$$

Our second preliminary lemma depends crucially on the fact that different ways of realising bi-semi-simplicial spaces are equivalent. We formulate this precisely in the following proposition:

Proposition 1.2.7. *Let $X_{\bullet,\bullet}$ be a bi-semi-simplicial space. Then there are canonical homeomorphisms*

$$\|X_{\bullet,\bullet}\| \cong \|[n] \mapsto \|[m] \mapsto X_{m,n}\|\| \cong \|[m] \mapsto \|[n] \mapsto X_{n,m}\|\|.$$

Proof. We demonstrate the first homeomorphism. The second one is analogous. Observe that

$$\begin{aligned}
\|X_{\bullet,\bullet}\| &:= \left(\bigsqcup_{m,n\geq 0} X_{m,n} \times |\Delta^m| \times |\Delta^n| \right) / \sim_{m,n} \\
&\cong \left(\bigsqcup_{n\geq 0} \left(\bigsqcup_{m\geq 0} X_{m,n} \times |\Delta^m| \times |\Delta^n| \right) / \sim'_m \right) / \sim'_n \\
&\cong \left(\bigsqcup_{n\geq 0} \left(\bigsqcup_{m\geq 0} X_{m,n} \times |\Delta^m| \right) / \sim_m \times |\Delta^n| \right) / \sim_n \\
&= \left(\bigsqcup_{n\geq 0} \|[m] \mapsto X_{m,n}\| \times |\Delta^n| \right) / \sim_n \\
&= \|[n] \mapsto \|[m] \mapsto X_{m,n}\|\|,
\end{aligned}$$

where:

- $\sim_{m,n}$ denotes the bi-semi-simplicial face relations generated by

  $$((\phi \times \psi)^* x, s, t) \sim (x, \phi_* s, \psi_* t),$$

  where $\phi \times \psi \in \mathrm{Mor}(\Delta \times \Delta)$.

- $\sim'_m, \sim'_n$ denote the equivalence relations generated by

  $$\begin{aligned}
  &(x, s, t) \sim ((\phi \times \mathrm{id})^* x, \phi_* s, t) \text{ and} \\
  &(x, s, t) \sim ((\mathrm{id} \times \psi)^* x, s, \psi_* t),
  \end{aligned}$$

  respectively, where $\phi, \psi \in \mathrm{Mor}(\Delta)$.

- $\sim_m, \sim_n$ denote the simplicial relations in the p, q coordinates, respectively.

The first isomorphism is obtained from formal manipulation of colimits. The second isomorphism follows from the fact that $|\Delta^m|$ is compact, for all $m \geq 0$, and hence the product and quotient commute here. □

Remark 1.2.8. The result of the previous proposition is stated in [ERW19, p. 5] for the setting of compactly generated spaces. The proof above serves to complete the details, and also to demonstrate that the assumption of the setting of compactly generated space is not in fact necessary.

Remark 1.2.9. The analogous result in the *simplicial* case is well known; e.g., see [Qui73, pp. 10-11].

We are ready for the second preliminary lemma:

Lemma 1.2.10. *Let $(\mathcal{C}, p, \mu)$ be a category with symmetric parametrisation structure. Then, for every functor $F : \mathcal{C} \to \mathbf{Top}_*$, the two canonical natural transformations*

$$\iota_{PF}, P\iota_F : PF \Rightarrow P^2F$$

differ by an automorphism of P^2F.

Proof. Fix an object c in $\mathcal{C}$. By the previous proposition, the space $P^2F(c)$ is homeomorphic to the realisation of the pointed bi-semi-simplicial space given by $([m], [n]) \mapsto F((c[m])[n])$, for all $m, n \geq 0$.

Let $\tau^{n,m} : |\Delta^n| \times |\Delta^m| \to |\Delta^m| \times |\Delta^n|$ denote the homeomorphism exchanging factors and $s_c^{n,m}$ denote the symmetry isomorphisms corresponding to c, for all $n, m \geq 0$. Then, the homeomorphisms

$$F((c[n])[m]) \times |\Delta^n| \times |\Delta^m| \xrightarrow{F(s_c^{n,m}) \times \tau^{n,m}} F((c[m])[n])) \times |\Delta^m| \times |\Delta^n|,$$

for all $n, m \geq 0$, assemble to a homeomorphism

$$\bigsqcup_{n,m \geq 0} F((c[n])[m]) \times |\Delta^n| \times |\Delta^m| \cong \bigsqcup_{n,m \geq 0} F((c[m])[n])) \times |\Delta^m| \times |\Delta^n|.$$

Naturality of the symmetry morphisms $s_c^{n,m}$ in the n and m variables implies that the homeomorphism above descends to a well-defined self-homemorphism

$$\Psi(c) : P^2F(c) \cong P^2F(c).$$

Furthermore, naturality of the morphisms $s_c^{n,m}$ in the variable c implies that the homeomorphisms $\Psi(c)$, where $c \in \mathrm{ob}(\mathcal{C})$, form the components of a natural isomorphisms $\Psi : P^2F \Rightarrow P^2F$.

Now, observe that the natural isomorphism Ψ restricts to a natural isomorphism $\psi : P(P_0F) \Rightarrow P_0(PF)$ of the 0-skeleta of P^2F, and consider the following diagram of space-valued functors:

$$\begin{tikzcd}
& PF \arrow[dl, Rightarrow, "P(F(\mu))"'] \arrow[dr, Rightarrow, "PF(\mu)"] & \\
P(P_0F) \arrow[rr, Rightarrow, "\cong", "\psi"'] \arrow[d, Rightarrow, "h"] & & P_0(PF) \arrow[d, Rightarrow, "v"] \\
P^2F \arrow[rr, Rightarrow, "\cong", "\Psi"'] & & P^2F.
\end{tikzcd}$$

The arrows h and v in the diagram label the inclusion of the horizontal and vertical 0-skeleton of P^2F, respectively. By definition of the maps Ψ and ψ, the lower square commutes. Furthermore, the upper triangle commutes by the assumption that the symmetry morphisms are compatible with the unit μ. The desired result of this lemma now follows from the observation that the natural transformations ι_{PF} and $P\iota_F$ correspond to the composites $v \circ PF(\mu)$ and $h \circ P(F(\mu))$, respectively. $\square$

The final preliminary lemma asserts that the category $\mathbf{hBrd}(Z)$ is closed under parametric realisation in the following sense:

Lemma 1.2.11. *Let $(\mathcal{C}, p, \mu)$ be a category with parametrisation structure. The endofunctor $P : \mathrm{Fun}(\mathcal{C}, \mathbf{Top}_*) \to \mathrm{Fun}(\mathcal{C}, \mathbf{Top}_*)$ and the natural transformation $\iota : \mathrm{id} \Rightarrow P : \mathrm{Fun}(\mathcal{C}, \mathbf{Top}_*) \to \mathrm{Fun}(\mathcal{C}, \mathbf{Top}_*)$ extend to $\mathbf{hBrd}(Z)$ for any given $Z : \mathcal{C} \to \mathbf{Top}_*$.*

Proof. For any bordism characteristic (F, σ) of Z, set $P(F, \sigma) := (PF, \iota_F \circ \sigma)$. In order for this to be well-defined, we must show that PF is bordism invariant. Let c be an object in $\mathcal{C}$, $d_i : c[n] \to c[n-1]$ be a face map, and consider the following commutative diagram:

$$\begin{tikzcd} F(c[n]) \arrow[r, "F(d_i)"] \arrow[d, "\iota_F(c[n])"'] & F(c[n-1]) \arrow[d, "\iota_F(c[n-1])"] \\ PF(c[n]) \arrow[r, "PF(d_i)"] & PF(c[n-1]). \end{tikzcd}$$

By assumption, the map $F(d_i)$ is a weak equivalence, and hence by Lemma 1.2.6 the vertical arrows are too. It follows immediately then that the face map $PF(d_i)$ is also weak equivalence. Thus, $P(F, \sigma)$ is a bordism characteristic.

Next, let $\eta : (F, \sigma) \to (F', \sigma')$ be a weak equivalence in $\mathbf{Brd}(Z)$ and consider the following commutative diagram:

$$\begin{tikzcd} F \arrow[r, Rightarrow, "\eta", "\simeq"'] \arrow[d, Rightarrow, "\iota_F"', "\simeq"] & F' \arrow[d, Rightarrow, "\simeq"', "\iota_{F'}"] \\ PF \arrow[r, Rightarrow, "P\eta"] & PF'. \end{tikzcd}$$

The map $P\eta : PF \Rightarrow PF'$ is a weak equivalence by the 2-out-of-3 property for weak equivalences. Hence, the functor P preserves weak equivalences, and therefore descends to the homotopy category $\mathbf{hBrd}(Z)$.

Lastly, observe that for any given bordism characteristic (F, σ), the natural transformation $\iota_F : F \Rightarrow PF$ determines a well-defined morphism of bordism characteristics $\iota(F, \sigma) : (F, \sigma) \to P(F, \sigma)$. $\square$

We finally come to the proof of the Abstract Universality Theorem 1.2.1.

Proof. Let $Z : \mathcal{C} \to \mathbf{Top}_*$ be given and (F, σ) a bordism characteristic of Z. Consider the following commutative diagram of functors $\mathcal{C} \to \mathbf{Top}_*$:

$$\begin{tikzcd} Z \arrow[r, "\iota_Z"'] \arrow[d, "\iota_Z"'] \arrow[rr, bend left, "\sigma"] & PZ \arrow[r, dashed, "\eta"] \arrow[d, "\iota_{PZ}"] & F \arrow[d, "\iota_F"] \\ PZ \arrow[r, "P\iota_Z"] \arrow[rr, bend right, "P\sigma"'] & P^2 Z \arrow[r, dashed, "P\eta"] & PF. \end{tikzcd} \tag{1.1}$$

We claim there exists a unique morphism $(PZ, \iota_Z) \to (F, \sigma)$ in $\mathbf{hBrd}(Z)$.

- Proof of Existence:
Since F is bordism invariant, it follows from Lemma 1.2.11 that the pair $P(F,\sigma)$ is a bordism characteristic. Thus, we may consider the zigzag

$$(PZ,\iota_Z) \overset{P\sigma}{\to} P(F,\sigma) \overset{\iota_F}{\leftarrow} (F,\sigma)$$

of bordism characteristics of Z. We claim that the previous zigzag represents a well-defined morphism $(PZ,\iota_Z) \to (F,\sigma)$ in $\mathbf{hBrd}(Z)$: Indeed, this is true since the map ι_F is a weak equivalence by Lemma 1.2.6, and moreover, the outer square of Diagram 1.1 commutes, i.e., $(\iota_F)^{-1} \circ P\sigma \circ \iota_Z = \sigma$, by Lemma 1.2.11 and functoriality of P.

- Proof of Uniqueness:
Note that, by assumption, the parametric realisation PZ of Z is bordism invariant. It follows from Lemma 1.2.6 that the natural transformation ι_{PZ} is a weak equivalence. Hence, by Lemma 1.2.10, the natural transformation $P\iota_Z$ is a weak equivalence too. Consider now the following zigzag:

$$\tilde{\sigma} : (PZ,\iota_Z) \overset{\iota_{PZ}}{\to} P(PZ,\iota_Z) \overset{P\iota_Z}{\leftarrow} (PZ,\iota_Z) \overset{P\sigma}{\to} P(F,\sigma) \overset{\iota_F}{\leftarrow} (F,\sigma).$$

Lemma 1.2.11, and the commutativity of the outer square and left hand square of Diagram 1.1 imply that $\tilde{\sigma}$ represents a well-defined morphism $(PZ,\iota_Z) \to (F,\sigma)$ in $\mathbf{hBrd}(Z)$. On the other hand, suppose we are given a morphism in $\mathbf{hBrd}(Z)$, $\eta : (PZ,\iota_Z) \to (F,\sigma)$. Then, by commutativity of the right hand square and lower triangle of Diagram 1.1, the morphism η must also be represented by $\tilde{\sigma}$. □

Remark 1.2.12. The Abstract Universality Theorem 1.2.1 and its proof were inspired by the universality theorem [Ste17, Theorem 0.2] for algebraic K-theory of Waldhausen categories.

Remark 1.2.13. The assumption in Theorem 1.2.1 that PZ is bordism invariant is clearly necessary but non-trivial; we present a counterexample in Example 1.3.2 in the next section.

Remark 1.2.14. The proof of Theorem 1.2.1 does not rely on any special features about pointed spaces. In fact, the evident analogue of our results hold for the category, **Top**, of unpointed spaces and continuous maps in place of $\mathbf{Top}_*$, and may be proven by completely analogously. Our emphasis on pointed spaces here is rather intended to prepare for our work in Part II and the discussion of spectrum-valued functors.

1.3 First Examples

This section is dedicated to illustrating some simple examples of categories with parametrisation structures.

We start with a trivial example to show that the bordism invariance condition may be empty.

Example 1.3.1. Every category $\mathcal{C}$ admits a trivial parametrisation operator $p : \mathcal{C} \times \Delta^{op} \to \mathcal{C}$ given by projection onto the first factor and unit given by the identity $\mu = \mathrm{id}_{\mathcal{C}}$ on $\mathcal{C}$. In this case, every space-valued functor from $\mathcal{C}$ is bordism invariant and weakly equivalent to its parametric realisation. In particular, the identity transformation $\mathrm{id} : F \Rightarrow F$ serves as a universal bordism characteristic of F for every space-valued functor from $\mathcal{C}$.

In our next example, we will demonstrate the opposite extreme case in which bordism invariant functors are those that are homotopically constant, i.e., weakly equivalent to a constant functor. This example also serves to highlight the fact that parametric realisation does not automatically yield bordism invariant functors and therefore the assumption of bordism invariance in Theorem 1.2.1 is non-trivial.

Example 1.3.2. Consider the category **Top** of compactly generated spaces. Every space X determines a semi-simplicial space $[n] \mapsto X[n]$ with empty higher simplices given by $X[0] = X$ and $X[n] = \emptyset$, for all $n \geq 1$. It is clear that bordism invariant functors $F : \mathbf{Top} \to \mathbf{Top}_*$ are precisely the homotopically constant functors and are determined by the value $F(\emptyset)$. However, the parametric realisation of a functor $F : \mathbf{Top} \to \mathbf{Top}_*$ is not homotopically constant in general.

As an example, consider the embedding of the category of spaces into pointed spaces given by adding a disjoint basepoint denoted by

$$\mathrm{id}_+ : \mathbf{Top} \to \mathbf{Top}_*, X \mapsto X_+.$$

The functor id_+ itself is evidently *not* homotopically constant, yet

$$\mathrm{id}_+(X) \simeq P\,\mathrm{id}_+(X)$$

is a natural weak equivalence since the geometric realisation of the terminal semi-simplicial set $*$ is contractible: Indeed, a nullhomotopy from the identity map $\mathrm{id}_{||*||}$ on $||*||$ to the constant map $c_{||*_0||}$ at the point $||*_0||$ is given by

$$\begin{aligned} H : ||*|| \times |\Delta^1| &\to ||*|| \\ (||*_n, t||, s) &\mapsto ||*_{n+1}, ((1-s)t, s)||. \end{aligned}$$

Nevertheless, note that a universal bordism characteristic exists for every functor $F : \mathbf{Top} \to \mathbf{Top}_*$, and is given by the unique natural transformation from F to the constant functor at a point.

Our next two examples will show that the singular construction of topological spaces and the construction of classifying space of categories yield examples of universal bordism characteristics. In these examples bordism invariance translates to a homotopy invariance condition.

Example 1.3.3. Consider the category $\mathbf{Top}_*$. Let $X^{|\Delta^n|}$ denote the pointed space of continuous maps from the topological n-simplex $|\Delta^n|$ to X, based at the constant map onto the baspoint in X. A symmetric parametrisation structure on $\mathbf{Top}_*$ is given by

$$X[n] := X^{|\Delta^n|},$$

together with the evident unit $\mu_X : X \cong X^{|\Delta^0|}$. In this case, bordism invariance of a functor $F : \mathbf{Top}_* \to \mathbf{Top}_*$ is equivalent to homotopy invariance, in the sense that the functor F, maps pointed homotopy equivalences to weak equivalences. Indeed, a functor with the latter property is clearly bordism invariant. For the converse, observe that the face maps

$$F(d_0), F(d_1) : F\big(X^{|\Delta^1|}\big) \to F\big(X^{|\Delta^0|}\big)$$

admit a common section $F(s_0) : F(X) \to F\big(X^{|\Delta^1|}\big)$ induced by the degeneracy map $s_0 : X \to X^{|\Delta^1|}$ that takes a point in X to the constant map at that point. Hence, if F is bordism invariant, the maps $F(d_0)$ and $F(d_1)$ are *equal* in the homotopy category $\mathbf{hTop}_*$ obtained from $\mathbf{Top}_*$ by inverting weak equivalences. It follows that for any homotopy equivalence $f : X \to Y$ with homotopy inverse $g : Y \to X$, the following diagram commutes in $\mathbf{hTop}_*$:

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"'] \arrow[rr, bend left, "\mathrm{id}"] & F(Y) \arrow[r, "F(g)"'] \arrow[rr, bend right, "\mathrm{id}"'] & F(X) \arrow[r, "F(f)"'] & F(Y) \end{tikzcd}$$

The 2-out-of-6 property for isomorphisms in $\mathbf{hTop}_*$ then implies that $F(f)$ is a weak equivalence, and hence F is homotopy invariant since f was chosen arbitrarily.

Consider now the forgetful functor $U : \mathbf{Top}_* \to \mathbf{Set}_*$ which takes a space to its underlying set. By definition, for any space X,

$$P_\bullet U(X) = U\big(X^{|\Delta^\bullet|}\big) = \mathrm{Sing}_\bullet(X)$$

where $\mathrm{Sing}_\bullet$ denotes the usual singular construction on X, and hence,

$$PU(X) = \|\mathrm{Sing}_\bullet(X)\|$$

for any space X. It is well known that there is a natural weak equivalence,

$$\|\mathrm{Sing}_\bullet(X)\| \simeq X,$$

given by evaluation (for the simplicial case, see [Mil57, Theorem 4]. Moreover, the semi-simplicial case is a consequence of [RS71, Proposition 2.1].) In fact the canonical characteristic $\iota_U : U(X) \hookrightarrow \|\mathrm{Sing}_\bullet(X)\|$ is weakly equivalent to the inclusion of points $inc : U(X) \to X$. Since the identity functor $\mathrm{id}_{\mathbf{Top}_*}$ is clearly bordism invariant, the Abstract Universality Theorem then implies that both ι_U and inc define universal bordism characteristics. In particular, every natural transformation $\sigma : U(X) \Rightarrow F(X)$ from U to a homotopy invariant functor $F : \mathbf{Top}_* \to \mathbf{Top}_*$ automatically extends to a natural transformation $\tilde{\sigma} : X \Rightarrow F(X)$, uniquely up to weak equivalence.

Example 1.3.4. Let **Cat** denote the category of small categories. A symmetric parametrisation structure on **Cat** is given by

$$\mathcal{C}[n] := \mathrm{Fun}([n], \mathcal{C}),$$

coupled with the obvious unit $\mu_{\mathcal{C}} : \mathcal{C} \cong \mathrm{Fun}([0], \mathcal{C})$.

We call a functor $F : \mathcal{C} \to \mathcal{C}'$ a *lax weak equivalence* if there exists a functor $G : \mathcal{C}' \to \mathcal{C}$ such that the two compositions, GF and FG, agree up to a zigzag of natural transformations with the identity functors $\mathrm{id}_{\mathcal{C}}$ and $\mathrm{id}_{\mathcal{C}'}$, respectively. It can be shown in analogy with the reasoning in Example 1.3.3 that the class of bordism invariant functors from **Cat** are those which take lax weak equivalences to weak equivalences.

Now consider the object functor $\mathrm{ob}_+ : \mathbf{Cat} \to \mathbf{Set}_*, \mathcal{C} \mapsto \mathrm{ob}_+(\mathcal{C})$ that assigns to a category $\mathcal{C}$ its set of objects together with an additional basepoint. The parametric realisation of the functor ob_+ is easily seen to be weakly equivalent to the usual classifying space construction on $\mathcal{C}$ equipped with a disjoint basepoint, denoted by $B_+\mathcal{C}$, i.e.,

$$P\,\mathrm{ob}_+(\mathcal{C}) \simeq B_+\mathcal{C}.$$

It is a standard exercise to prove that a natural transformation between any two functors induces a homotopy equivalence on classifying spaces (e.g., see [Seg74, Proposition 2.1]), whence it follows that the functor B_+ is bordism invariant. By Theorem 1.2.1 we deduce that the natural transformation $\mathrm{ob}_+ \Rightarrow B_+$ is universal among natural transformations from ob_+ whose target converts lax weak equivalences to weak equivalences.

Remark 1.3.5. For comparison, we would like to mention a number of other examples that we found in the literature, which closely resemble parametric realisation and the concept of bordism invariance, though do not fit into our framework immediately. These include the operation of homotopisation in the context of a homotopy invariant K-theory of rings discussed in [Wei13, Ch. IV, §§11-12]; an auxiliary simplicial construction used by Waldhausen in relating A-theory with the Whitehead space (see [Wal85, §3]; in particular p. 402); and the construction of “concordance invariant” sheaves on the site of smooth manifolds described in [EBBdBP19].

Chapter 2

Application I: A Universal Property of Symmetric L-Theory

The theory developed in the previous chapter was primarily motivated by an investigation into whether algebraic L-theory satisfies a universal property. In this chapter we will now consider L-theory in a setting for which the theory is directly applicable. Namely, we will examine universality of symmetric L-Theory of *Waldhausen categories with duality* as developed by Michael Weiss and Bruce Williams in [WW98].

Our work in this chapter should be regarded as a complement to [WW98]. The setting of Waldhausen categories with duality was introduced there to serve as a common input category for algebraic L- and K-theory of Waldhausen categories, and we recall the necessary background definitions in Section 2.1. Note that we assume familiarity with the fundamentals of Waldhausen's algebraic K-theory throughout this chapter and recommend the foundational article [Wal85] as a reference.

A construction of a parametrisation operator on the category of Waldhausen categories with duality has been already described in [WW98]; however, not all the details about its functorial properties were presented there. We review and clarify these details in Section 2.2. Furthermore, we prove that the parametrisation operator is part of a symmetric parametrisation structure.

One advantage of equipping Waldhausen categories with a notion of duality is that it allows for the notion of symmetric Poincaré objects. Symmetric L-theory can then be defined as the parametric realisation of the functor of symmetric Poincaré objects. It turns out that the bordism invariance of symmetric L-theory had already been considered, and proven in [WW98]. We recall the details in Section 2.3, and deduce from this observation a universal property of L-theory (see Theorem 2.3.6): It states that symmetric L-theory of Waldhausen categories with duality is the universal target for bordism characteristics of the functor of symmetric Poincaré objects.

Our last Section 2.4 is devoted to an application of this universal property which may be briefly summarised as follows: The main construction in [WW98] (see also [WW89] for the more classical case of rings with involution) describes a natural transformation, denoted by " Ξ ", between symmetric L-Theory and the $\mathbb{Z}_2$-Tate cohomology of K-Theory. It was observed in [WW00] that the $\mathbb{Z}_2$-Tate cohomology of K-Theory is bordism invariant. We apply this observation and the universal property of symmetric L-theory to obtain a characterisation of Ξ as a morphism of bordism characteristics.

2.1 Waldhausen Categories with Duality

We start by recalling the setting of [WW98]. The most important feature is the notion of a *SW product* defined as follows (cf. [WW98, Definition 1.1]):

Definition 2.1.1. Let $\mathcal{C}$ be a Waldhausen category and $\mathbf{0}$ denote its zero object. A *SW product* on $\mathcal{C}$ is a functor

$$\begin{aligned}\odot : \mathcal{C} \times \mathcal{C} &\to \mathbf{Top}_*,\\ (C, D) &\mapsto C \odot D,\end{aligned}$$

satisfying the following conditions:

- *w-invariance*: The functor $\odot$ takes pairs of weak equivalences to homotopy equivalences.
- *Symmetry*: The functor $\odot$ comes with a natural isomorphism $\tau : C \odot D \cong D \odot C$, whose square is the identity on $C \odot D$.
- *Bilinearity*: For fixed but arbitrary D, the functor $C \mapsto C \odot D$ takes any *cofibre square* in $\mathcal{C}$ to a homotopy pullback square of spaces. (A cofibre square is a commutative pushout square in which either the horizontal or the vertical arrows are cofibrations.) Bilinearity also means that $\mathbf{0} \odot D$ is contractible.

The isomorphisms $\tau : C \odot D \cong D \odot C$ are called *symmetry operators*.

Definition 2.1.2. We denote by **xWald** the category defined as follows:

- The objects of **xWald** are pairs $(\mathcal{C}, \odot)$ consisting of a Waldhausen category $\mathcal{C}$ and a Spanier Whitehead Product $\odot : \mathcal{C} \times \mathcal{C} \to \mathbf{Top}_*$, satisfying certain axioms (see [WW98, §2, Axioms 1-5]). We call such a pair a *Waldhausen category with duality* (WCD).
- A morphism of WCDs $(E, \psi) : (\mathcal{C}, \odot_{\mathcal{C}}) \to (\mathcal{D}, \odot_{\mathcal{D}})$ consists of
  - an exact functor $E : \mathcal{C} \to \mathcal{D}$, and
  - a natural transformation $H : A \odot_{\mathcal{C}} B \to E(A) \odot_{\mathcal{D}} E(B)$, which commutes with the symmetry operators of the products $\odot_{\mathcal{C}}$ and $\odot_{\mathcal{D}}$, and maps nondegenerate components (see [WW98, Definition 3.6]) to nondegenerate components.

Remark 2.1.3. The category **xWald** originally appears in [WW98] (see Definitions 7.1, 9.2 and §12 therein).

Remark 2.1.4. We have introduced the terminology "Waldhausen category with duality" to emphasise that the SW product prescribes a notion of duality on the underlying Waldhausen category, allowing for a description of L-theory and an involution on K-theory.

Moreover, the notation '**xWald**' is based on the ubiquitous x-prefix in the paper [WW98].

2.2 The Parametrisation Structure

The construction of a parametrisation structure on the category **xWald** has essentially been carried out in [WW98]. This section is dedicated to giving a precise description and defining corresponding unit and symmetry operators. We start by recalling the definition of a Waldhausen category with duality parametrised over a given finite simplicial complex X as described in [WW98, Definition 1.5].

Notation 2.2.1. Let $\mathrm{sub}(X)$ denote the poset of simplicial subcomplexes of X ordered by inclusion and let $\mathcal{F}(X)$ denote the subposet on non-empty faces of X.

Definition 2.2.2. Let $(\mathcal{C}, \odot)$ be a WCD and X a simplicial complex. We define a WCD

$$(\mathcal{C}[X], \odot^X)$$

as follows:

- $\mathcal{C}[X]$ is the category whose objects are the functors $\mathrm{sub}(X) \to \mathcal{C}$, which take all morphisms to cofibrations, take $\emptyset$ to the zero object $\mathbf{0}$ and unions to pushouts. The morphisms of $\mathcal{C}[X]$ are natural transformations of such functors.

- A *cofibration* in $\mathcal{C}[X]$ is a morphism $\eta : F \to F'$ such that $F(A) \to F'(A)$ is a cofibration, for all simplicial subcomplexes $A \subset X$, and such that for each pair of subcomplexes $A \subset B$ of X, the evident morphism

$$\operatorname{colim}\big(F'(A) \leftarrow F(A) \to F(B)\big) \to F'(B)$$

  is a cofibration.

- A *weak equivalence* in $\mathcal{C}[X]$ is a morphism $\eta : F \to F'$ such that the morphisms $F(A) \to F'(A)$ are weak equivalences, for all simplicial subcomplexes $A \subset X$.

- The SW product $\odot^X$ on $\mathcal{C}[X]$ is given by

$$F \odot^X F' := \operatorname{holim}_{\sigma \in \mathcal{F}(X)} F(\sigma) \odot F'(\sigma),$$

  where the homotopy limit is taken over the poset $\mathcal{F}(X)$.

Remark 2.2.3. Homotopy limit is understood here in the Bousfield-Kan sense (cf. [WW98, §0.1]). In our calculations, we use the formula given in [MV15, Definition 8.2.1].

Remark 2.2.4. Note that the symmetry operators

$$\tau^X : C \odot^X D \cong D \odot^X C$$

of the SW product $\odot^X$ are those induced from the symmetry operators $(\tau : C(\sigma) \odot D(\sigma) \cong D(\sigma) \odot C(\sigma))_{\sigma \in \mathcal{F}(X)}$ of the product $\odot$.

It is proven in [WW98, §8] that every such pair $(\mathcal{C}[X], \odot^X)$ is a WCD, and also, that the construction is functorial with respect to inclusions of simplicial complexes $A \hookrightarrow X$: Indeed, a simplicial inclusion $f : A \hookrightarrow X$ induces an exact functor $f^* : \mathcal{C}[X] \to \mathcal{C}[A]$ and a natural transformation

$$C \odot^X D \to f^*C \odot^A f^*D$$

by restriction of parameters.

The functoriality of the parametrisation construction with respect to morphism of WCDs is implicit from [WW98, §12], but the details are not explicitly described. Proposition 2.2.7 below serves to fill this gap. Its proof relies on the following characterisation of nondegenerate components of the product $\odot^X$ demonstrated in [WW98, Proposition 8.8]:

Proposition 2.2.5. *For any objects B, C in $\mathcal{C}[X]$ and any path component $[\eta] \in \pi_0(B \odot^X C)$, the following are equivalent:*

1. *$[\eta]$ is nondegenerate.*
2. *For each face $s \subset X$, the image of $[\eta]$ under the specialisation map*

$$\pi_0(B \odot^X C) \to \pi_{|s|}(B(s) \odot C(s/\partial s))$$

is nondegenerate.

Remark 2.2.6. The notation $C(s/\partial s)$ was introduced in [WW98, p. 565] and denotes the cofibre of the map $C(\partial s) \to C(s)$. Moreover, the notation $|s|$ denotes the dimension of a face $s \subset X$.

To elaborate, the specialisation map in Proposition 2.2.5 is induced from a map of spaces

$$C \odot^X D \to \Omega^{|s|}(C(s) \odot D(s/\partial s))$$

defined (cf. [WW98, p. 566]) as the composition of the map

$$C \odot^X D = \mathrm{holim}_{\sigma \in \mathcal{F}(X)} C(\sigma) \odot D(\sigma) \to \mathrm{holim}_{\tau \in \mathcal{F}(s)} C(\tau) \odot D(\tau/\tau \cap \partial s)$$

with a homotopy inverse of the inclusion

$$\Omega^{|s|}(C(s) \odot D(s/\partial s)) \cong \mathrm{holim}_{\tau \in \mathcal{F}(s)} G(s) \hookrightarrow \mathrm{holim}_{\tau \in \mathcal{F}(s)} C(\tau) \odot D(\tau/\tau \cap \partial s),$$

where G denotes the functor $G : \mathrm{sub}(s) \to \mathcal{C}$ given by: $G(\tau) = pt$, if $\tau \neq s$, and $G(s) = C(s) \odot D(s/\partial s)$.

Proposition 2.2.7. *A morphism $(E, \psi) : (\mathcal{C}, \odot) \to (\mathcal{C}', \odot')$ of WCDs induces a morphism $(E_*, \psi_*) : (\mathcal{C}[X], \odot^X) \to (\mathcal{C}'[X], (\odot')^X)$ of WCDs.*

Proof. We define the pair (E_*, ψ_*) by applying the functor E and the natural transformation ψ pointwise. It is clear from definitions that the functor E_* is exact and that ψ_* respects the symmetry operators. What is perhaps not immediate is that the natural transformation ψ_* maps nondegenerate components to nondegenerate components. To see this, consider the following diagram:

$$\begin{array}{ccc} \pi_0(B \odot^X C) & \longrightarrow & \pi_{|s|}(B(s) \odot C(s/\partial s)) \\ \downarrow & & \downarrow \\ \pi_0(E_*(B)(\odot')^X E_*(C)) & \longrightarrow & \pi_{|s|}(E(B(s)) \odot' E(C(s/\partial s))). \end{array}$$

The horizontal rows are specialisation maps and the left and right vertical map are induced by the natural transformations ψ_* and ψ, respectively. The diagram can be seen to commute, by the naturality of ψ, the definition of the specialisation maps, and the assumption that E is exact. It then follows from Proposition 2.2.5 and the assumption that ψ preserves nondegenerate components that the natural transformation ψ^X preserves nondegenerate components. $\square$

It is straightforward to check that the previous constructions of induced morphism commute, in the sense that they define an unambiguous morphism of WCDs

$$(\mathcal{C}[X], \odot^X) \to (\mathcal{C}'[A], \odot^A),$$

for all pairs (f, E) consisting of a simplicial inclusion $f : A \hookrightarrow X$ and morphism $E : (\mathcal{C}, \odot) \to (\mathcal{C}', \odot')$ of WCDs. In summary, we obtain a functor

$$\hat{p} : \mathbf{xWald} \times \mathbf{Simp}^{op} \to \mathbf{xWald}$$
$$(X, (\mathcal{C}, \odot)) \mapsto (\mathcal{C}[X], \odot^X),$$

where **Simp** denotes the category of finite simplicial complexes and simplicial inclusions. Since the category Δ embeds into **Simp** (cf. [Lan78, VII.5]) by associating the set $[n]$ to the standard topological n-simplex, for all natural numbers $n \geq 0$, we may define a parametrisation operator $p : \mathbf{xWald} \times \Delta^{op} \to \mathbf{xWald}$ on **xWald** to be the corresponding restriction of the functor $\hat{p}$ to the subcategory $\mathbf{xWald} \times \Delta^{op} \subset \mathbf{xWald} \times \mathbf{Simp}^{op}$.

Proposition 2.2.8. *The parametrisation operator p admits a unit, i.e., there is a natural isomorphism*

$$\mu_{\mathcal{C}} : \mathcal{C} \cong \mathcal{C}[0]$$

of WCDs.

Proof. Observe that an object $F : \mathrm{sub}(\Delta^0) \to \mathcal{C}$ in $\mathcal{C}[0]$ and natural transformation of functors $F_1 \Rightarrow F_2$ in $\mathcal{C}[0]$ are uniquely defined by their value on the non-empty cell σ. A natural transformation $\mu = (\mu, \phi)$ may now be defined as follows:

- Let $\mu_{\mathcal{C}} : \mathcal{C} \cong \mathcal{C}[0]$ be the isomorphism of Waldhausen categories which takes an object c of $\mathcal{C}$ to the object $\mu_{\mathcal{C}}(c)$ in $\mathcal{C}[0]$ given by $\mu_{\mathcal{C}}(c)(\sigma) = c$, and a morphism $a \xrightarrow{f} b$ in $\mathcal{C}$ to the natural transformation $\mu_{\mathcal{C}}(f) : \mu_{\mathcal{C}}(a) \Rightarrow \mu_{\mathcal{C}}(b)$ given by $\mu_{\mathcal{C}}(f)(\sigma) := \mu_{\mathcal{C}}(a)(\sigma) \xrightarrow{f} \mu_{\mathcal{C}}(b)(\sigma)$.

- Let $\mathrm{Map}(pt, C \odot D)$ denotes the space of maps from the point to the space $C \odot D$ with compact open topology. Then, clearly

$$\mu_{\mathcal{C}}(C) \odot^{\Delta^0} \mu_{\mathcal{C}}(D) = \mathrm{holim}\, \mu_{\mathcal{C}}(C)(\sigma) \odot \mu_{\mathcal{C}}(D)(\sigma) = \mathrm{Map}(pt, C \odot D)$$

for all $C, D \in \mathrm{ob}(\mathcal{C})$. We now define $\phi_{\mathcal{C}} : C \odot D \cong \mathrm{Map}(pt, C \odot D)$ to be the natural homeomorphism given by $x \mapsto (f_x : pt \mapsto x)$.

The inverse to $(\mu_{\mathcal{C}}, \phi_{\mathcal{C}})$ is given by $(\mu_{\mathcal{C}}^{-1}, \phi_{\mathcal{C}}^{-1})$ where $\mu_{\mathcal{C}}^{-1} : \mathcal{C}[0] \to \mathcal{C}$ evaluates at the cell σ. □

Proposition 2.2.9. *The parametrisation structure (p, μ) on* **xWald** *is symmetric.*

Proof. Let $(\mathcal{C}, \odot)$ be a WCD and X and Y be simplicial complexes. We define a natural morphism of WCDs

$$(s_{\mathcal{C}}^{X,Y}, \psi_{\mathcal{C}}^{X,Y}) : (\mathcal{C}[X])[Y] \to (\mathcal{C}[Y])[X]$$

as follows: The morphism $s_{\mathcal{C}}^{X,Y} : (\mathcal{C}[X])[Y] \to (\mathcal{C}[Y])[X]$ exchanges the order of evaluation on subcomplexes of X and Y. More precisely, $s_{\mathcal{C}}^{X,Y}$ sends an object F of $(\mathcal{C}[X])[Y]$ to the object $s(F)$ in $(\mathcal{C}[Y])[X]$ that satisfies

$$\begin{aligned} s(F)(A)(B) &= F(B)(A), \\ s(F)(A)(B \subset B') &= F(B \subset B')(A), \text{ and} \\ s(F)(A \subset A')(B) &= F(B)(A \subset A'), \end{aligned}$$

for all $A, A' \in \mathrm{sub}(X)$ and $B, B' \in \mathrm{sub}(Y)$.

Furthermore, $s_{\mathcal{C}}^{X,Y}$ is defined to take a natural transformation $\eta : F_1 \Rightarrow F_2$ to the natural transformation $s(\eta) : s(F_1) \Rightarrow s(F_2)$ given by

$$s(\eta)(A)(B) = \eta(B)(A),$$

for all $A \in \mathrm{sub}(X)$ and $B \in \mathrm{sub}(Y)$.

It is routine to check that the functor $s_{\mathcal{C}}^{X,Y}$ is exact and has inverse $s_{\mathcal{C}}^{Y,X}$.

Next, observe that for every pair C, D of objects in $\mathcal{C}$, we have equivalences of pointed spaces:

$$\begin{aligned} C(\odot^X)^Y D &:= \mathrm{holim}_{\sigma \in \mathcal{F}(Y)} C(\sigma) \odot^X D(\sigma) \\ &= \mathrm{holim}_{\sigma \in \mathcal{F}(Y)} \mathrm{holim}_{\tau \in \mathcal{F}(X)} C(\sigma)(\tau) \odot D(\sigma)(\tau) \\ &\cong \mathrm{holim}_{\tau \in \mathcal{F}(X)} \mathrm{holim}_{\sigma \in \mathcal{F}(Y)} C(\sigma)(\tau) \odot D(\sigma)(\tau) \\ &= \mathrm{holim}_{\tau \in \mathcal{F}(X)} \mathrm{holim}_{\sigma \in \mathcal{F}(Y)} s(C)(\tau)(\sigma) \odot s(D)(\tau)(\sigma) \\ &= \mathrm{holim}_{\tau \in F(X)} s(C)(\tau) \odot^Y s(D)(\tau) \\ &=: s(C)(\odot^Y)^X s(D), \end{aligned}$$

where the isomorphism comes from the natural isomorphism commuting homotopy limits (cf. [MV15, Proposition 8.5.5]). We therefore obtain a natural isomorphism

$$\psi_{\mathcal{C}}^{X,Y} : C(\odot^X)^Y D \cong s(C)(\odot^Y)^X s(D).$$

A straightforward inspection shows that the morphisms $\psi_{\mathcal{C}}^{X,Y}$ are compatible with the symmetry operators. Moreover, the fact that $\psi_{\mathcal{C}}^{X,Y}$ maps nondegenerate components to nondegenerate components can be seen using the characterisation of nondegenerate components in Proposition 2.2.5. More precisely, it may be verified that, for all $\sigma \in \mathcal{F}(Y)$ and $\tau \in \mathcal{F}(X)$, the following diagram commutes:

$$\begin{tikzcd}
\pi_0(C(\odot^X)^Y D) \arrow[r] \arrow[d, "\pi_0(\psi_{\mathcal{C}}^{X,Y})"'] & \pi_{|\sigma|+|\tau|}(C(\sigma)(\tau) \odot D(\sigma/\partial\sigma)(\tau/\partial\tau)) \arrow[d, "="] \\
\pi_0(s(C)(\odot^Y)^X s(D)) \arrow[r] & \pi_{|\sigma|+|\tau|}(s(C)(\tau)(\sigma) \odot s(D)(\tau/\partial\tau)(\sigma/\partial\sigma)),
\end{tikzcd}$$

where the horizontal arrows are specialisation maps.

Finally, a direct computation shows that the triangle of Waldhausen categories and exact functors

$$\begin{tikzcd}
 & \mathcal{C}[n] \arrow[dl, "\mu_{\mathcal{C}[n]}"'] \arrow[dr, "\mu_{\mathcal{C}}[n]"] & \\
(\mathcal{C}[n])[0] \arrow[rr, "s_{\mathcal{C}}^{n,0}"] & & (\mathcal{C}[0])[n]
\end{tikzcd}$$

commutes for all $n \geq 0$ and Waldhausen categories $\mathcal{C}$, and that the triangle of spaces

$$\begin{tikzcd}
 & C \odot^{\Delta^n} D \arrow[dl, "\phi_{\mathcal{C}[n]}"'] \arrow[dr, "\phi_{\mathcal{C}}[n]"] & \\
C(\odot^{\Delta^n})^{\Delta^0} D \arrow[rr, "\psi_{\mathcal{C}}^{n,0}"] & & C(\odot^{\Delta^0})^{\Delta^n} D
\end{tikzcd}$$

also commutes for all pairs of objects C, D of any given WCD $(\mathcal{C}, \odot)$. Thus, the symmetry morphisms are compatible with the unit μ. $\square$

2.3 Symmetric Poincaré objects and L-Theory

We next turn to the description of the symmetric Poincaré objects and symmetric L-Theory of WCDs following [WW98, §9].

Notation 2.3.1.

1. Let $\mathbb{Z}_2$ denote the cyclic group of order 2 and $E\mathbb{Z}_2$ be a contractible space with a free $\mathbb{Z}_2$-action.
2. For an object C of a WCD $(\mathcal{C}, \odot)$, let $(C \odot C)^{h\mathbb{Z}_2}$ denote the homotopy fixed point space of $C \odot C$ with respect to the $\mathbb{Z}_2$-action coming from the symmetry morphisms, i.e., the space of all $\mathbb{Z}_2$-maps $\phi : E\mathbb{Z}_2 \to C \odot C$. Furthermore, for any homotopy fixed point $\phi : E\mathbb{Z}_2 \to C \odot C$, let $\phi_0 \in C \odot C$ denote the value of ϕ on the basepoint.

Definition 2.3.2. Let $(\mathcal{C}, \odot)$ be a WCD. A *symmetric Poincaré object* in $\mathcal{C}$ is an object C in $\mathcal{C}$ together with a point $\phi \in (C \odot C)^{h\mathbb{Z}_2}$ whose image ϕ_0 in $C \odot C$ is in a nondegenerate component. The set of symmetric Poincaré objects in $\mathcal{C}$ is denoted by $\mathrm{sp}(\mathcal{C})$ and regarded as a pointed space equipped with the discrete topology, where the basepoint is given by the zero object $\mathbf{0}$ of $\mathcal{C}$ and the homotopy fixed point $\phi_{\mathbf{0}} \in (\mathbf{0} \odot \mathbf{0})^{h\mathbb{Z}_2}$ determined by the basepoint of $\mathbf{0} \odot \mathbf{0}$.

Remark 2.3.3. Note, we have dropped the usual prefix "0-dimensional" in the definition of a symmetric Poincaré object from [WW98], for simplicity.

Functoriality of sp is defined as follows: For every morphism of WCDs $(E, \psi) : (\mathcal{C}, \odot_{\mathcal{C}}) \to (\mathcal{D}, \odot_{\mathcal{D}})$, the induced map of sets $\mathrm{sp}(\mathcal{C}) \to \mathrm{sp}(\mathcal{D})$ is given by the assignment $(C, \phi) \to (E(C), \psi(\phi))$. We denote the corresponding functor by

$$\mathrm{sp} : \mathbf{xWald} \to \mathbf{Top}_* \,.$$

Definition 2.3.4. *Symmetric L-theory $L : \mathbf{xWald} \to \mathbf{Top}_*$* is defined as the parametric realisation of the symmetric-Poincaré-objects functor sp, i.e.,

$$L := P\,\mathrm{sp}\,.$$

Proposition 2.3.5. *The functor $L : \mathbf{xWald} \to \mathbf{Top}_*$ is bordism invariant.*

Proof. By definition, bordism invariance of the functor L means that all face maps of the semi-simplicial spaces $L(\mathcal{C}[\bullet])$ are weak equivalences for all Waldhausen categories with duality $(\mathcal{C}, \odot)$. This fact was stated in [WW00, p. 695] as part of the proof that the spaces $L(\mathcal{C})$ admit a bi-semi-simplicial model given by the rule $([m], [n]) \mapsto L((\mathcal{C}[m])[n])$. We note that the proof strategy for bordism invariance is to study the induced maps on homotopy groups directly, using the fact that the semi-simplicial sets $\mathrm{sp}(\mathcal{C}[\bullet])$ are Kan for all $\mathcal{C}$ (see [WW98, p. 570]). $\square$

Propositions 2.2.9 and 2.3.5 imply that the conditions of Theorem 1.2.1 are satisfied. Thus, we deduce that symmetric L-theory of Waldhausen categories with duality is the universal target for bordism characteristics of symmetric Poincaré-objects:

Theorem 2.3.6. *The pair (L, ι_{sp}) is a universal bordism characteristic of* sp.

Remark 2.3.7. One may view Theorem 2.3.6 as a formalisation of the idea presented in [WW98, p. 536, Example 2] of symmetric L-theory as *the* bordism theory of symmetric Poincaré objects.

2.4 Characterisation of the Weiss-Williams map

The purpose of this section is to demonstrate how the universal property of symmetric L-theory can be used to characterise the Weiss-Williams Ξ-map from symmetric L-Theory to $\mathbb{Z}_2$-Tate cohomology of K-theory constructed in [WW98]. We start with a recollection of the description of Tate K-theory given in [WW98].

Notation 2.4.1.

1. Let **Spectra** denote the category of CW-spectra and maps in the sense of Adams (see [Ada74, §III.2]; and also [Swi02, Ch. 8] for another account). Moreover, denote by $\mathbb{Z}_2$- **Spectra** the category whose objects are CW-spectra $\mathbf{X}$ equipped with a celluar automorphism $\mathbf{X} \to \mathbf{X}$ and morphisms are maps of CW-spectra compatible with the involutions.

2. Let $\Omega^\infty : \mathbf{Spectra} \to \mathbf{Top}_*$ denote the functor taking a spectrum to its infinite loop space given by $\Omega^\infty\,\mathbf{X} = \mathrm{hocolim}_n\,\Omega^n \mathbf{X}_n$, where $n \in \mathbb{N}$.

3. Let $(-)^{th\mathbb{Z}_2} : \mathbb{Z}_2\text{-}\mathbf{Spectra} \to \mathbf{Spectra}$ denote the Tate construction (see [WW98, §9.10]).
4. Let $\mathbf{K} : \mathbf{xWald} \to \mathbb{Z}_2\text{-}\mathbf{Spectra}$ denote the K-theory functor of Weiss-Williams (see [WW98, §7]) and $K : \mathbf{xWald} \to \mathbf{Top}_*$ its composition with Ω^∞.

Definition 2.4.2. The $\mathbb{Z}_2$-Tate cohomology of $\mathbf{K}$ is defined as the composition $\Omega^\infty \circ (-)^{th\mathbb{Z}_2} \circ \mathbf{K} : \mathbf{xWald} \to \mathbf{Top}_*$ and will be denoted by $K^{th\mathbb{Z}_2}$.

Remark 2.4.3. The functor $\mathbf{K}$ is defined in a similar way to Waldhausen's K-theory functor (see [Wal85]). In fact, it is shown in [WW98, §7] that there is a natural equivalence between them, obtained by forgetting the $\mathbb{Z}_2$-action.

Remark 2.4.4. The Tate construction on a $\mathbb{Z}_2$-spectrum $\mathbf{X}^{th\mathbb{Z}_2}$ can be described (see [WW98, Properties 9.11]) as the cofibre of a certain norm map

$$N : \mathbf{X}_{h\mathbb{Z}_2} \to \mathbf{X}^{h\mathbb{Z}_2}$$

from the homotopy orbit spectrum $\mathbf{X}_{h\mathbb{Z}_2}$ of X to the homotopy fixed point spectrum $\mathbf{X}^{h\mathbb{Z}_2}$ of $\mathbf{X}$. See also [WW89, §2] for another description.

Our interest in the $\mathbb{Z}_2$-Tate cohomology of K-theory is its bordism invariance property. The following proposition records this result:

Proposition 2.4.5. *The functor $K^{th\mathbb{Z}_2} : \mathbf{xWald} \to \mathbf{Top}_*$ is bordism invariant.*

Proof. The property that the face maps of the semi-simplicial space $K^{th\mathbb{Z}_2}(\mathcal{C}[\bullet])$ are homotopy equivalences, for arbitrary $\mathcal{C}$, is proven in [WW00, p. 696] and obtained from the analogous statement on the level of spectra given in [WW98, Theorem 9.12]. For the convenience of the reader we sketch the argument for the proof of the latter here:

Central to the proof is the notion of an *induced spectrum* which is defined as follows: a spectrum $\mathbf{X}$ with an action of a discrete group G is said to be *induced* if there exists a spectrum $\mathbf{Y}$ and an equivariant map $\mathbf{X} \wedge G_+ \to \mathbf{Y}$ from the smash product $\mathbf{X} \wedge G_+$ to $\mathbf{Y}$ which is an ordinary homotopy equivalence (cf. [WW98, p. 572]). It turns out to be a formal property of the Tate construction that it vanishes on induced spectra (see [WW98, Properties 9.11])

Consider now the semi-simplicial spectrum $[m] \mapsto \mathbf{K}(\mathcal{C}[m])$, for any given Waldhausen category with duality $\mathcal{C}$. It is shown in [WW98, Lemma 9.4]) that the homotopy fibres of the last vertex maps $\mathbf{K}(\mathcal{C}[m]) \to \mathbf{K}(\mathcal{C}[0])$ are induced as $\mathbb{Z}_2$-spectra, for arbitrary m. The result then follows by the vanishing property of the Tate construction. □

We next recall the construction of the Weiss-Williams map $\Xi : L \Rightarrow K^{th\mathbb{Z}_2}$ relating L-theory to $\mathbb{Z}_2$-Tate cohomology of K-theory. The map Ξ is induced from a certain natural transformation $\xi : \mathrm{sp} \Rightarrow K^{th\mathbb{Z}_2}$ defined on [WW00, p. 571]. Before coming to the definition, we introduce more notation following [WW98, Remark 4.4].

Notation 2.4.6. For any WCD $(\mathcal{C}, \odot)$, let $xw\mathcal{C}$ denote the topological category whose objects are triples (B, D, z) where B, D are objects in the category of weak equivalences $w\mathcal{C}$ of $\mathcal{C}$ and $z \in B \odot D$ is nondegenerate. A morphism from (B_1, D_1, z_1) to (B_2, D_2, z_2) is a pair of weak equivalences $B_1 \to B_2, D_1 \to D_2$ such that the induced map $B_1 \odot D_1 \to B_2 \odot D_2$ takes z_1 to z_2. Furthermore, let $|xw\mathcal{C}|$ denote the classifying space of $xw\mathcal{C}$.

It is a part of their construction (see [WW98, §7] for details) that the spaces $K(\mathcal{C})$, where $\mathcal{C}$ denotes an arbitrary WCD, come with a natural transformation

$$\chi : |xw\mathcal{C}| \Rightarrow K(\mathcal{C})$$

that is compatible with the $\mathbb{Z}_2$-actions. Hence, there is an induced map on homotopy fixed points,

$$\chi^{h\mathbb{Z}_2} : |xw\mathcal{C}|^{h\mathbb{Z}_2} \Rightarrow K(\mathcal{C})^{h\mathbb{Z}_2}.$$

Furthermore, observe that there is a natural tranformation

$$\nu : \mathrm{sp}(\mathcal{C}) \Rightarrow |xw\mathcal{C}|^{h\mathbb{Z}_2}$$

defined by assigning to a symmetric Poincaré object (C, ϕ) of the given Waldhausen category with duality $(\mathcal{C}, \odot)$ the point

$$|(C, C, \phi_0)| \in |xw\mathcal{C}|^{h\mathbb{Z}_2}.$$

Finally, let $\epsilon : K^{h\mathbb{Z}_2} \Rightarrow K^{th\mathbb{Z}_2}$ be the canonical inclusion. The natural transformation $\xi : \mathrm{sp} \Rightarrow K^{th\mathbb{Z}_2}$ is then defined by the following composition:

$$\xi : \mathrm{sp}(\mathcal{C}) \overset{\nu}{\Rightarrow} |xw\mathcal{C}|^{h\mathbb{Z}_2} \overset{\chi^{h\mathbb{Z}_2}}{\Longrightarrow} K^{h\mathbb{Z}_2}(\mathcal{C}) \overset{\epsilon}{\Rightarrow} K^{th\mathbb{Z}_2}(\mathcal{C}).$$

By Proposition 2.4.5, we know that the functor $K^{th\mathbb{Z}_2}$ is bordism invariant. It follows that $(K^{th\mathbb{Z}_2}, \xi)$ is a bordism characteristic of the symmetric-Poincaré-objects functor, i.e., $(K^{th\mathbb{Z}_2}, \xi) \in \mathbf{Brd}(\mathrm{sp})$.

Remark 2.4.7. In contrast, note that the functor $K^{h\mathbb{Z}_2}$ does *not* yield a bordism characteristic of sp, since it is not bordism invariant. In fact, the parametric realisation of $K^{h\mathbb{Z}_2}$ is related to $K^{th\mathbb{Z}_2}$ by a chain of weak equivalences. This fact is proven as [WW98, Theorem 9.14] and is the main reason that the functor $K^{th\mathbb{Z}_2}$ was considered there (cf. [WW89, §0]).

By the universality property of symmetric L-theory (Theorem 2.3.6), we deduce immediately that the natural transformation $\xi : \mathrm{sp} \Rightarrow K^{th\mathbb{Z}_2}$ extends to a natural transformation $\Xi : L \Rightarrow K^{th\mathbb{Z}_2}$ along $\iota_{\mathrm{sp}} : \mathrm{sp} \Rightarrow L$, which is unique up to homotopy:

Corollary 2.4.8. *There exists a unique morphism*

$$\Xi : (L, \iota_{\mathrm{sp}}) \to (K^{th\mathbb{Z}_2}, \xi)$$

in $\mathbf{hBrd}(\mathrm{sp})$*, the homotopy category of bordism characteristics of* sp*.*

Corollary 2.4.8 thus gives a simple characterisation of the morphism Ξ between symmetric L-theory and $\mathbb{Z}_2$-Tate cohomology of K-theory as a morphism of bordism characteristics of the functor sp.

Remark 2.4.9. Note that the original natural transformation $L \Rightarrow K^{th\mathbb{Z}_2}$ defined in [WW98, Theorem 9.14] has in fact the same definition as the natural transformation constructed in Theorem 1.2.1 as part of the existence proof; namely, Ξ is represented by the composite:

$$\Xi : L \xRightarrow{P\xi} PK^{th\mathbb{Z}_2} \underset{\simeq}{\xLeftarrow{\iota_{K^{th\mathbb{Z}_2}}}} K^{th\mathbb{Z}_2}.$$

Chapter 3

Application II: A Universal Property of Quinn's Bordism Machine

This chapter is devoted to our second main application of the Abstract Universality Theorem 1.2.1. We examine Quinn's bordism machine in the setting of ad theories as developed by Laures-McClure in [LM14], and show it is characterised as the universal target for bordism characteristics of a suitable functor of "closed objects". In fact, we regard Quinn's bordism machine as yielding the *prototypical* example of a universal bordism characteristic because of the generality of the setting of ad theories.

Our results are dependent and supplementary to those of [LM14]. Therein (see [LM14, p. 1170]), it was stated that one can define ad theories, for any ball complex and any ad theory. However, the details about the functorial nature of the construction were not given. Our aim is to provide those details here and, in addition, show that the construction yields a symmetric parametrisation structure on the category of ad theories. Section 3.2 is dedicated to these tasks and constitutes the main technical contribution of this chapter.

We will not define the parametrisation structure on the category of ad theories directly but rather present an extended definition to allow ball complex *pairs* as input, in preparation for our work in Part II. The extension of the parametrisation structure to ball complex pairs requires a slight restriction of the axioms of an ad theory. For accuracy, we will make a thorough review of the foundations and axioms in Section 3.1.

In subsection 3.3.1, we recall the definition of Quinn's bordism-space machine from [LM14] and identify it as the parametric realisation of the closed-objects functor. Thereafter in subsection 3.3.2, we show how the bordism invariance property of Quinn's bordism machine follows from results of [LM14] and thereby admits a universal property by Theorem 3.3.9.

We note that the definitions related to ad theories are based on the foundational paper [LM14] and the more recent articles [BL17] and [BLM19]. We refer the reader to these sources for comparison and examples.

3.1 The Setting of Ad Theories

This section is devoted to the description of the category of ad theories. Our exposition is based on [LM14, §§2-3], and also influenced by [BL17, §2] and [BLM19, §3].

A description of the category of ad theories requires introducing certain diagram categories indexed over certain posets of *ball complex pairs*. The main relevance of ball complex pairs is that they form a convenient category of combinatorial spaces closed under forming products.

We have organised our exposition as follows: Section 3.1.1 records the basic definitions pertaining to ball complexes and their associated posets. Section 3.1.2 introduces certain diagram categories indexed over such posets. Finally, Section 3.1.3 presents the definition of the category of ad theories.

Note, we assume familiarity with basic notions from Piecewise-Linear (PL) topology and the incidence theory of regular CW complexes; we refer the reader to [RS72] for an introduction to the former, and to [Whi78, Ch. II., §6] or [Mas91, Ch. 4., §5-7] for details about the latter.

3.1.1 Ball Complexes and Associated Cell Posets

We start by collecting the necessary background about ball complexes from [LM14, §2] and [BRS76, pp. 4-5].

The category of ball complexes

Definition 3.1.1. 1. Let K be a finite collection of PL balls in some Euclidean space, and write $|K|$ for the union $\cup_{\sigma \in K}\sigma$. We say that K is a *ball complex* if the interiors of the balls of K are disjoint and the boundary of each ball of K is a union of balls of K. The balls of K will be called *(closed) cells* of K.

2. An *isomorphism of ball complexes* $K \to L$ is a PL homeomorphism $|K| \to |L|$ which takes closed cells of K to closed cells of L.

3. A *subcomplex* of a ball complex K is a subset of K which is itself a ball complex with the inherited cell structure.

4. A *morphism of ball complexes* $K \to L$ is the composite of an isomorphism with an inclusion of a subcomplex.

We denote the category of ball complexes by **Ball**.

Definition 3.1.2. 1. A *ball complex pair* (K, L) is a pair of ball complexes such that $L \subset K$ is a subcomplex.

2. A morphism of ball complex pairs $(K, L) \to (K', L')$ is a morphism of ball complexes $f : K \to K'$ such that $f(L) \subseteq L'$.

We denote the category of ball complex pairs by $\mathbf{Ball}_2$. Furthermore, we let K denote the ball complex pair $(K, \emptyset)$.

A technically useful feature of the category of ball complexes, in contrast to the category of simplicial complexes, is that it is closed under the formation of *products*. These are defined as follows:

Definition 3.1.3. 1. Let K and M be ball complexes. The *product of K and M* is the ball complex denoted by $K \times M$ whose closed cells are the products of closed cells in K and M.

2. Let (K, L) and (M, N) be ball complex pairs. The *product of (K, L) and (M, N)* is the ball complex pair $(K \times M, L \times N \cup K \times N)$.

Another important feature of ball complexes is that the cells of a ball complex K induce a regular CW structure on its underlying space. Thus, we may speak about the *incidence number* (see [Whi78, p. 82]), denoted by $[(\sigma, o), (\sigma', o')] \in \{0, \pm 1\}$, of a pair of oriented cells $((\sigma, o), (\sigma', o'))$ of K, where $|\sigma| = |\sigma'| + 1$. For reference in later calculations, we will record here the canonical orientations on products of ball complexes following [Whi78, p. 88]:

Given two oriented ball complexes (X, o_X) and (Y, o_Y), we denote by $o_X \times o_Y$ the orientation on $X \times Y$ defined by the following incidence relations, where $\tau \subseteq \tau'$ and $\sigma \subseteq \sigma'$ are oriented cells of X and Y, respectively:

$$\begin{aligned} [\sigma' \times \tau, \sigma \times \tau] &= [\sigma', \sigma] \\ [\sigma \times \tau', \sigma \times \tau] &= (-1)^{|\sigma|}[\tau', \tau]. \end{aligned} \tag{3.1}$$

Next, we define the notion of subdivision and residual complex in accordance with [LM14, Definition 2.2].

Definition 3.1.4. A *subdivision* of a ball complex K is a ball complex K' with the following two properties:

- $|K| = |K'|$, and
- each closed cell of K' is contained in a closed cell of K.

A subdivision of a ball complex pair (K, L) as a ball complex pair (K', L') such that K' and L' are subdivisions of K and L, respectively.

We call a subcomplex of K which is also a subcomplex of K' a *residual subcomplex.*

Finally, we record that the category of ball complexes receives a faithful functor $\Delta \hookrightarrow \mathbf{Ball}_2$ taking the set $[n]$ to the standard n-simplex with its standard PL-structure, for all $n \geq 0$. We denote the image of $[n]$ under this embedding by Δ^n.

Cell Posets

A fundamental property about ball complexes is that they naturally yield posets via cell inclusion. We make this precise in the following definition:

Definition 3.1.5. For any ball complex pair (X, Y), we define its face poset, denoted $\mathcal{F}(X, Y)$, to be the poset consisting of the closed cells σ in X not in Y under the relation of face inclusion i.e., $\sigma \leq \sigma'$ in $\mathcal{F}(X, Y)$ if and only if $\sigma \subseteq \sigma'$.

Remark 3.1.6. Our notation for face posets is based on that of [BVS⁺93, Appendix 4.7].

We will also need to work with an inflated version of these posets as defined in [LM14, Example 3.6]. Preliminary to the definition is the notion of a $\mathbb{Z}$-*graded category* defined as follows:

Definition 3.1.7.

1. An *involution* i on a category $\mathcal{A}$ is an endofunctor $i : \mathcal{C} \to \mathcal{C}$ such that its square is strictly equal to the identity functor on $\mathcal{C}$.

2. A $\mathbb{Z}$-*graded category* is a quadruple $(\mathcal{A}, i, \dim, \emptyset)$, consisting of a small category $\mathcal{A}$ with involution i, an involution preserving functor $\dim : \mathcal{A} \to \mathbb{Z}$ into the poset of integers $\mathbb{Z}$ (equipped with trivial involution), and an involution preserving section $\emptyset : \mathbb{Z} \to \mathcal{A}$ to $\dim : \mathcal{A} \to \mathbb{Z}$, such that the objects $\emptyset(n)_{n\in\mathbb{Z}}$ (called *basepoints*) are initial in the following sense:

   For all $n \in \mathbb{Z}$, there is a unique morphism $\emptyset(n) \to a$, whenever $\dim(a) \geq n$.

   We call dim the *dimension function* of $\mathcal{A}$ and set $\emptyset_n := \emptyset(n)$, for all $n \in \mathbb{Z}$.

3. A morphism between $\mathbb{Z}$-graded categories $(\mathcal{A}, i, \dim, \emptyset)$ and $(\mathcal{A}', i', \dim', \emptyset)$ is a functor $F : \mathcal{A} \to \mathcal{A}'$ of the underlying categories which decreases the dimension of objects by an integer k, preserves the basepoints and strictly commutes with the involutions, i.e.,

   - $\dim(F(a)) = \dim(a) - k$, for all $a \in \operatorname{ob} \mathcal{A}$,
   - $F(\emptyset_n) = \emptyset_{n-k}$, for all $n \in \mathbb{Z}$, and
   - $F \circ i = i' \circ F : \mathcal{A} \to \mathcal{A}'$.

   The integer k is called the *degree* of the morphism and a morphism $F : \mathcal{A} \to \mathcal{A}'$ of degree k is called a k-*morphism*. We denote the corresponding category of $\mathbb{Z}$-graded categories by $\mathbf{Cat}_{\mathbb{Z}}$.

Remark 3.1.8. Definition 3.1.7 is based on [LM14, Definition 3.3]. However, note that we additionally demand the existence of unique morphisms from basepoints to any other object in $\mathcal{A}$. This assumption is innocuous though necessary for our formulation of the axioms (see Remark 3.1.15). We note that it is trivially satisfied in the standard examples of ad theories (see [LM14, §§6,7,9 and 11]).

Remark 3.1.9. Note in particular that 0-morphisms of $\mathbb{Z}$-graded categories are dimension preserving.

Definition 3.1.10. For any ball complex pair (K, L) let $\operatorname{cell}(K, L)$ be the $\mathbb{Z}$-graded category whose underlying category has object set:

$$\operatorname{ob}(\operatorname{cell}(K, L)) := \{(\sigma, o) \,|\, \sigma \in \mathcal{F}(K, L), o \text{ an orientation of } \sigma\} \sqcup \{\emptyset_n\}_{n\in\mathbb{Z}}.$$

The objects of $\operatorname{cell}(K, L)$ are called *oriented cells* and the empty cells $\{\emptyset_n\}_{n\in\mathbb{Z}}$ serve as basepoints. In addition to the identity morphisms, there are unique morphisms:

- $(\sigma, o) \to (\sigma, o')$, if $\sigma \lneq \sigma'$, and o, o' are arbitrary,

- $\emptyset_n \to (\sigma, o)$, if $n \leq |\sigma'|$, and
- $\emptyset_n \to \emptyset_m$, if $n \leq m$.

The involution swaps orientations of cells and acts trivially on the basepoints. The dimension function is defined by taking the dimension of cells and the index of basepoints.

$$
\begin{tikzcd}
\tau^+ & \tau_0^+ \arrow[r] \arrow[dr] & \sigma^+ & \tau_0^+ \arrow[r] \arrow[dr] & \sigma^+ & \tau_1^+ \arrow[l] \arrow[dl] \\
\tau^- & \tau_0^- \arrow[r] \arrow[ur] & \sigma^- & \tau_0^- \arrow[r] \arrow[ur] & \sigma^- & \tau_1^- \arrow[l] \arrow[ul]
\end{tikzcd}
$$

Figure 3.1: A schematic of the posets $\mathrm{cell}(\Delta^0)$, $\mathrm{cell}(\Delta^1, \{1\})$ and $\mathrm{cell}(\Delta^1)$ (from left to right) with basepoints suppressed. The cells labelled by τ^ϵ, and τ_i^ϵ, for $\epsilon = \pm$ and $i = 1, 2$, denote the 0-dimensional cells of Δ^0, and Δ^1, respectively. The cells labelled by σ^ϵ denote the 1-dimensional cells of Δ^1. The superscripts $\epsilon = \pm$ refer to the two possible orientations and the arrows indicate the relations between the cells.

Some examples of cell posets are illustrated in Figure 3.1. It was remarked in [BL17, p. 2] that the assignment

$$(K, L) \mapsto \mathrm{cell}(K, L)$$

is functorial. We describe the induced maps explicitly here, for completeness:

A morphism of ball complexes $f : (K, L) \to (M, N)$ induces a 0-morphism of $\mathbb{Z}$-graded categories $\mathrm{cell}(K, L) \to \mathrm{cell}(M, N)$ determined by the following rules:

$$
\begin{aligned}
(\sigma, o) &\mapsto (f(\sigma), f_* o), && \text{if } \sigma \notin L, o \text{ arbitrary} \\
(\sigma, o) &\mapsto \emptyset_{|\sigma|}, && \text{if } \sigma \in L, o \text{ arbitrary,}
\end{aligned}
$$

where the notation $f_* o$ denotes the induced orientation of the cell $f(\sigma)$, given by pushforward of the orientation class o. We will not distinguish between a morphism of ball complex and its induced map on posets.

It will be important to deal with morphisms of cell posets *not* necessarily induced by a map of ball complex pairs. We introduce the following category as a setting for such abstract morphisms between ball complexes:

Definition 3.1.11. The category $\mathbf{CBall}_2$ has object and morphism sets defined by

$$
\begin{gathered}
\mathrm{ob}\,\mathbf{CBall}_2 := \mathrm{ob}\,\mathbf{Ball}_2 \\
\mathrm{Mor}_{\mathbf{CBall}_2}((K, L), (M, N)) := \mathrm{Mor}_{\mathbf{Cat}_\mathbb{Z}}(\mathrm{cell}(K, L), \mathrm{cell}(M, N)).
\end{gathered}
$$

For all $k \in \mathbb{Z}$, we call a k-morphism (resp. k-isomorphism) $\theta : (K, L) \to (M, N)$ in $\mathbf{CBall}_2$ a *k-morphism (resp. k-isomorphism) of ball complexes.*

The most important class of abstract morphisms of ball complexes are those which are *incidence compatible*. We define this condition in agreeance with [LM14, Definition 3.7(i)]:

Definition 3.1.12. Let $\theta : (K, L) \to (K', L')$ be a k-morphism, where $k \in \mathbb{Z}$. We say θ is *incidence compatible* if

$$[(\sigma, o), (\sigma', o')] = (-1)^k [\theta(\sigma, o), \theta(\sigma', o')],$$

for all pairs of oriented cells $((\sigma, o), (\sigma', o'))$ in $\mathrm{cell}(K, L)$ such that $|\sigma| = |\sigma'| + 1$.

3.1.2 Categories Parametrised over Ball Complexes

For any $\mathbb{Z}$-graded category $\mathcal{A}$ and ball complex pair (K, L), we introduce functor categories denoted $\mathcal{A}[K, L]$ as a generalisation of the construction of the categories $\mathcal{A}[K]$ from [LM14, p. 1170] to pairs.

Definition 3.1.13. We define a functor

$$\begin{aligned} \mathbf{Cat}_{\mathbb{Z}} \times \mathbf{Ball}_2^{op} &\to \mathbf{Cat}_{\mathbb{Z}}, \\ (\mathcal{A}, (K, L)) &\mapsto \mathcal{A}[K, L], \end{aligned}$$

as follows: Let (K, L) be a ball complex pair and $(\mathcal{A}, \dim, i, \emptyset)$ be a $\mathbb{Z}$-graded category. The underlying category of $\mathcal{A}[K, L]$ is defined to have object set

$$\mathrm{pre}_{\mathcal{A}}(K, L) := \mathrm{Mor}_{\mathbf{Cat}_{\mathbb{Z}}}(\mathrm{cell}(K, L), \mathcal{A}),$$

and morphisms given by natural transformations. An element of $\mathrm{pre}_{\mathcal{A}}(K, L)$ is called a (K, L)-*pread* in $\mathcal{A}$. We say a pread *has dimension* k if it is a $-k$-morphism of $\mathbb{Z}$-graded categories. The set of k-dimensional preads is denoted by $\mathrm{pre}_{\mathcal{A}}^{-k}(K, L)$.

The unique (K, L)-pread of dimension k which takes values in the basepoints of $\mathcal{A}$ is defined to be the k-dimensional basepoint of $\mathcal{A}[K, L]$ and will be called the *trivial k-dimensional (K, L)-pread in* $\mathcal{A}$. The involution on $\mathcal{A}[K, L]$ postcomposes a diagram X with the involution i of $\mathcal{A}$.

We declare a morphism $f : (K, L) \to (K', L')$ of ball complexes to induce a morphism of $\mathbb{Z}$-graded categories

$$f^* : \mathcal{A}[K', L'] \to \mathcal{A}[K, L],$$

for every $\mathbb{Z}$-graded category $\mathcal{A}$, by precomposition. Furthermore, a k-morphism $F : \mathcal{A} \to \mathcal{A}'$ of $\mathbb{Z}$-graded categories induces a k-morphism

$$F_* : \mathcal{A}[K, L] \to \mathcal{A}'[K, L],$$

for all ball complex pairs (K, L), by postcomposition. More generally, a pair (f, F) with f and F as above, induces a morphism of $\mathbb{Z}$-graded categories

$$\begin{aligned} (f^*, F_*) : \mathcal{A}[K', L'] &\to \mathcal{A}'[K, L] \\ X &\mapsto F \circ X \circ f. \end{aligned}$$

Remark 3.1.14. The terminology "pread" and the notation "$\mathrm{pre}_{\mathcal{A}}^k(K, L)$" are based on [LM14, Definition 3.8].

Remark 3.1.15. The sets $\mathrm{pre}_{\mathcal{A}}(K, L)$ were originally defined as certain *subsets* of the sets $\mathrm{pre}_{\mathcal{A}}(K)$ (cf. [LM14, Definition 3.8(iii)]). In the defintion above we have followed the conventions of [BL17, BLM19]. The equivalence of both definitions was remarked in [LM14, Remark 3.9]. For the convenience of the reader, we demonstrate the comparison here: For any ball complex pair (K, L), let $\varepsilon^*_{(K,L)} : \mathrm{pre}_{\mathcal{A}}(K, L) \to \mathrm{pre}_{\mathcal{A}}(K)$ be the map of sets induced by the inclusion $\varepsilon_{(K,L)} : K \hookrightarrow (K, L)$, i.e., the map $\varepsilon^*_{(K,L)}$ is given by extending a (K, L)-ad to a K-ad by evaluating to the basepoint over the subcomplex L.

Then for any $\mathbb{Z}$-graded category $\mathcal{A}$, the maps $\varepsilon^*_{(K,L)} : \mathrm{pre}_{\mathcal{A}}(K, L) \to \mathrm{pre}_{\mathcal{A}}(K)$ identify $\mathrm{pre}_{\mathcal{A}}(K, L)$ with the subset of $\mathrm{pre}_{\mathcal{A}}(K)$ whose elements are those K-preads in $\mathcal{A}$ that restrict to the trivial L-pread in $\mathcal{A}$ via the inclusion $L \hookrightarrow K$. Indeed, an inverse to ε^* is given by the map $\eta^*_{(K,L)} : \mathrm{pre}_{\mathcal{A}}(K) \to \mathrm{pre}_{\mathcal{A}}(K, L)$ induced from the inclusion of posets $\eta_{(K,L)} : \mathrm{cell}(K, L) \hookrightarrow \mathrm{cell}(K)$. Note, in particular, that in order for the composite $\varepsilon^*_{(K,L)} \circ \eta^*_{(K,L)}$ to be the identity, we require the additional assumption that the basepoints in $\mathcal{A}$ are initial (see Definition 3.1.7) to exclude the existence of non-trivial maps from basepoints.

More generally, abstract morphisms of ball complexes induce morphisms on categories of preads. The following definition serves to explain this and generalises [LM14, Definition 3.7(ii)]:

Definition 3.1.16. Let $\mathcal{A}$ be a $\mathbb{Z}$-graded category with involution i. For any k-morphism

$$\theta : \mathrm{cell}(K, L) \to \mathrm{cell}(K', L'),$$

in $\mathbf{CBall}_2$, we define a k-morphism of $\mathbb{Z}$-graded categories

$$\theta^* : \mathcal{A}[K', L'] \to \mathcal{A}[K, L],$$

by assigning to each l-dimensional (K', L')-pread X in $\mathcal{A}$, the $(l-k)$-dimensional (K', L')-pread in $\mathcal{A}$,

$$\theta^* X := i^{kl} \circ X \circ \theta.$$

Moreover, each natural transformation $\eta : X \Rightarrow X'$ of l-dimensional (K', L')-pread X, X' in $\mathcal{A}$ is assigned the natural transformation $\theta^*\eta : \theta^* X \Rightarrow \theta^* X'$ defined by

$$\theta^*\eta(\sigma) = i^{kl}(\eta(\theta(\sigma))),$$

for all $(\sigma, o) \in \mathrm{cell}(K, L)$.

Remark 3.1.17. Definition 3.1.16 should be regarded as a partial extension of Definition 3.1.13. Indeed, the definition of the induced morphisms $\theta^* : \mathcal{A}[K', L'] \to \mathcal{A}[K, L]$ of the former agree with the latter in the case that $\theta : (K, L) \to (K', L')$ comes from a map of ball complexes, as those morphisms are 0-morphisms. Furthermore, the morphisms θ^* are natural in the variable $\mathcal{A}$.

The reader should be aware, though, that the assignment $\theta \mapsto \theta^*$ is not strictly functorial, but rather satisfies $(\theta \circ \psi)^* = i^{kl}\psi^* \circ \theta^*$ for all k-morphisms θ and l-morphisms ψ in $\mathbf{CBall}_2$.

Remark 3.1.18. The involution terms i^{kl} do not play an essential role in this chapter and the reader may choose to ignore them for now. However, they will be needed later in Chapter 5 in order to get signs correct.

3.1.3 Ad Theories

Let $\mathbf{Set}_{\mathbb{Z}}$ denote the category of $\mathbb{Z}$-graded sets.

Definition 3.1.19. An *ad theory* consists of a quintuple

$$(\mathcal{A}, i, \emptyset, \dim, \mathrm{ad}_{\mathcal{A}} : \mathbf{Ball}_2^{op} \to \mathbf{Set}_{\mathbb{Z}})$$

where the quadruple $(\mathcal{A}, i, \emptyset, \dim)$ is a $\mathbb{Z}$-graded category and $\mathrm{ad}_{\mathcal{A}}$ is an i-invariant subfunctor of $\mathrm{pre}_{\mathcal{A}}$ satisfying the axioms below. The elements of $\mathrm{ad}_{\mathcal{A}}(K, L)$ are called (K, L)-*ads* in $\mathcal{A}$.

- absolute: A (K, L)-pread is a (K, L)-ad if and only if it extends to a K-ad (see Remark 3.1.15).
- pointed: The trivial K-preads in $\mathcal{A}$ are K-ads.
- local: A K-pread is an K-ad if it restricts to a σ-ad for each closed cell σ of K.
- cylinder: There is a natural transformation $J_{\mathcal{A}} : \mathrm{ad}_{\mathcal{A}}(-) \Rightarrow \mathrm{ad}_{\mathcal{A}}(- \times \Delta^1)$ of functors $\mathbf{Ball} \to \mathbf{Set}_{\mathbb{Z}}$ with the following properties:
  - J maps trivial ads to trivial ads.
  - For every K-ad X, the restrictions of $J_{\mathcal{A}}(X)$ to $K \times \{0\}$ and $K \times \{1\}$ coincide with X, i.e.,

    $$j_0^* X = X = j_1^* X,$$

    where j_k denotes the composition of the evident isomorphism of ball complexes $K \cong K \times \{k\}$ and inclusion $K \times \{k\} \hookrightarrow K \times \Delta^1$, for $k = 0, 1$.
- gluing: For each subdivision (K', L') of (K, L), and each (K', L')-ad X, there is a (K, L)-ad which agrees with X on each residual subcomplex.
- reindexing: For every incidence compatible k-isomorphism

  $$\theta : (K, L) \to (M, N)$$

  in $\mathbf{CBall}_2$, the induced k-isomorphism of sets,

  $$\theta^* : \mathrm{pre}_{\mathcal{A}}(M, N) \to \mathrm{pre}_{\mathcal{A}}(K, L)$$

  maps ads isomorphically to ads, i.e, restricts to a k-isomorphism

  $$\theta^* : \mathrm{ad}_{\mathcal{A}}(M, N) \xrightarrow{\cong} \mathrm{ad}_{\mathcal{A}}(K, L).$$

The functor $\mathrm{ad}_{\mathcal{A}}$ is called *the ad structure of* $\mathcal{A}$. We will denote the quintuples $(\mathcal{A}, i, \emptyset, \dim, \mathrm{ad}_{\mathcal{A}} : \mathbf{Ball}_2^{op} \to \mathbf{Set}_{\mathbb{Z}})$ more simply by $(\mathcal{A}, \mathrm{ad}_{\mathcal{A}})$, or by $\mathcal{A}$, when the ad structure is clear from context.

A *morphism of ad theories* $(\mathcal{A}, \mathrm{ad}_A) \to (\mathcal{A}', \mathrm{ad}_{\mathcal{A}'})$ is a 0-morphism $F : \mathcal{A} \to \mathcal{A}'$ of $\mathbb{Z}$-graded categories such that, for all ball complex pairs (K, L), the induced map of sets

$$\mathrm{pre}_F(K, L) : \mathrm{pre}_{\mathcal{A}}(K, L) \to \mathrm{pre}_{\mathcal{A}'}(K, L)$$

maps ads to ads.

We denote the corresponding category of ad theories by $\mathbf{Ad}$.

Remark 3.1.20. The definition of the axioms above is a partially restricted version of [LM14, Definition 3.10] and also influenced by [BL17, Definition 2.1]. We have imposed the following restrictions:

1. Our gluing axiom is a relative version of that given in [LM14]. We need the relative version so that the functor category $\mathcal{A}[M, N]$ inherits an ad structure from the ad theory $\mathcal{A}$ for any ball complex *pair* (see Proposition 3.2.3 below). The point being that we need to ensure that trivial ads are glued to trivial ads.

2. A morphisms of ad theories must be dimension preserving. The reason is that we want certain shift functors (see Section 3.3) to define *non-trivial* automorphisms on **Ad**.

3.2 Parametrisation in the Category of Ad theories

The aim of this section is to demonstrate that the category of ad theories admits a symmetric parametrisation structure.

3.2.1 The Ad Structure on Categories of Preads

We begin by showing how the $\mathbb{Z}$-graded categories $\mathcal{A}[M, N]$ can be equipped with an ad structure, for all ball complex pairs (M, N) and ad theories $(\mathcal{A}, \mathrm{ad}_{\mathcal{A}})$. The absolute case, $(M, N) = (M, \emptyset)$, was described in [LM14, p. 1170]. The idea of the construction is to transfer the ad structure from $\mathcal{A}$ to the functor categories $\mathcal{A}[M]$ using the product-hom adjunction

$$(\mathcal{A}[M])[K] \cong \mathcal{A}[K \times M],$$

natural in all variables (see [BL17, Lemma 3.3]). Explicitly, one identifies every arbitrary k-dimensional pread in $(\mathcal{A}[M])[K]$,

$$\begin{aligned} X : \mathrm{cell}(K) &\to \mathcal{A}[M] \\ (\sigma, o) &\mapsto X[\sigma, o], \end{aligned}$$

with the corresponding k-dimensional pread in $\mathcal{A}[K \times M]$ given by

$$\begin{aligned} \alpha(X) : \mathrm{cell}(K \times M) &\to \mathcal{A} \\ (\sigma \times \tau, o \times o') &\mapsto X[\sigma, o](\tau, o'). \end{aligned}$$

The same idea applies in the case of ball complex pairs, yielding the following proposition:

Proposition 3.2.1. *There are natural isomorphisms*

$$\alpha_{\mathcal{A}}^{(M,N),(K,L)} : (\mathcal{A}[M, N])[K, L] \cong \mathcal{A}[(K, L) \times (M, N)]$$

of functors $\mathbf{Cat}_{\mathbb{Z}} \times \mathbf{Ball}_2^{op} \times \mathbf{Ball}_2^{op} \to \mathbf{Cat}_{\mathbb{Z}}$.

The morphisms $\alpha_{\mathcal{A}}^{(M,N),(K,L)}$ from Proposition 3.2.1 will be called *associativity morphisms.* We defer to Section 4.1 for an explanation of the terminology.

Definition 3.2.2. Fix an ad theory $(\mathcal{A}, \mathrm{ad}_{\mathcal{A}})$ and ball complex pair (M, N). We define an ad structure $\mathrm{ad}_{\mathcal{A}[M,N]}$ on $\mathcal{A}[M, N]$ as follows:

Let $n \in \mathbb{Z}$. The n-dimensional (K, L)-ads in the category $\mathcal{A}[M, N]$ are the n-dimensional (K, L)-preads in $\mathcal{A}[M, N]$ which are adjoint to n-dimensional $(K \times M, K \times N \cup L \times M)$-ads in $\mathcal{A}$ via the adjunction α given in Proposition 3.2.1, i.e.,

$$X \in \mathrm{ad}^n_{\mathcal{A}[M,N]}(K, L) \text{ if and only if } \alpha(X) \in \mathrm{ad}^n_{\mathcal{A}}((K, L) \times (M, N)).$$

Proposition 3.2.3. *The pair* $(\mathcal{A}[M.N], \mathrm{ad}_{\mathcal{A}[M.N]})$ *is an ad theory, for all ad theories* $\mathcal{A}$ *and ball complex pairs* (M, N).

Proof. Checking the axioms is straightforward from the definitions. The main idea is to use the adjunction α to transfer the problem to $\mathcal{A}$ and work with the axioms there. We record the details here for completeness. Fix a ball complex pair (K, L).

- absolute:
  Let X be a (K, L)-pread in $\mathcal{A}[M, N]$. We must prove that X is a (K, L)-ad in $\mathcal{A}[M, N]$ if and only if $\varepsilon^*_{(K,L)}(X)$ is a K-ad in $\mathcal{A}[M, N]$.

  Observe that by the definition of the ad structure on $\mathcal{A}[M, N]$, the previous statement is equivalent to proving that $\alpha(X)$ is a $(K, L) \times (M, N)$-ad in $\mathcal{A}$ if and only if $\alpha(\varepsilon^*_{(K,L)}(X))$ is a $K \times (M, N)$-ad in $\mathcal{A}$. By the absolute axiom for the ad theory $\mathcal{A}$, this corresponds to showing that $\varepsilon^*_{(K,L)\times(M,N)}\alpha(X)$ is a $K \times M$-ad in $\mathcal{A}$ if and only if $\varepsilon^*_{K\times(M,N)}(\alpha(\varepsilon^*_{(K,L)}X))$ is a $K \times M$-ad in in $\mathcal{A}$. The latter statement follows from the fact that there is an equality of $K \times M$-ads in $\mathcal{A}$:

  $$\varepsilon^*_{K\times(M,N)}(\alpha(\varepsilon^*_{(K,L)}X)) = \varepsilon^*_{(K,L)\times(M,N)}\alpha(X)$$

  Indeed, the equality can be seen considering the following diagram, whose top square commutes by naturality of α and lower triangle commutes by functoriality of $\mathrm{pre}_{\mathcal{A}}$:

  $$\begin{tikzcd}
  \mathrm{pre}_{\mathcal{A}[M,N]}(K,L) \arrow[r, "\varepsilon^*_{(K,L)}"] \arrow[d, "\alpha"] & \mathrm{pre}_{\mathcal{A}[M,N]}(K) \arrow[d, "\alpha"] \\
  \mathrm{pre}_{\mathcal{A}}((K,L)\times(M,N)) \arrow[r, "(\varepsilon_{(K,L)}\times\mathrm{id})^*"] \arrow[dr, "\varepsilon^*_{(K,L)\times(M,N)}"'] & \mathrm{pre}_{\mathcal{A}}(K\times(M,N)) \arrow[d, "\varepsilon^*_{K\times(M,N)}"] \\
  & \mathrm{pre}_{\mathcal{A}}(K\times M).
  \end{tikzcd}$$

- pointed:
  The trivial K-pread in $\mathcal{A}[M, N]$ corresponds under the bijection α to the trivial $(K \times M, K \times N)$-pread in $\mathcal{A}$. The latter is an ad in $\mathcal{A}$ by the absolute axiom since it extends to the trivial $K \times M$-ad in $\mathcal{A}$.

- local:
  Let X be a K-pread in $\mathcal{A}[M, N]$, which restricts to a σ-ad, for each closed cell σ of K. We need to show that X is an ad, i.e., that $\alpha(X)$ is an ad

in $\mathcal{A}$, or equivalently that its extension $\varepsilon^*_{K\times(M,N)}\alpha(X)$ is a $K \times M$-ad in $\mathcal{A}$. By the locality axiom for the ad theory $\mathcal{A}$, it suffices to show that $\varepsilon^*_{K\times(M,N)}\alpha(X)$ restricts to a $\sigma \times \sigma'$-ad, for every pair of closed cells σ, σ' of K and of M, respectively.

Let σ, σ' be given. By naturality of α and the definition of the ad structure $\mathrm{ad}_{\mathcal{A}[M,N]}$, we know that the restriction of $\alpha(X)$ to $\sigma \times (M, N)$ is a $\sigma \times (M, N)$-ad in $\mathcal{A}$. Now, consider the commutative diagram

$$\begin{tikzcd}
\mathrm{pre}_{\mathcal{A}}(K \times (M,N)) \arrow[r] \arrow[d] & \mathrm{pre}_{\mathcal{A}}(K \times M) \arrow[d] \arrow[dr] & \\
\mathrm{pre}_{\mathcal{A}}(\sigma \times (M,N)) \arrow[r] & \mathrm{pre}_{\mathcal{A}}(\sigma \times M) \arrow[r] & \mathrm{pre}_{\mathcal{A}}(\sigma \times \sigma'),
\end{tikzcd}$$

induced from the diagram of inclusions of ball complexes

$$\begin{tikzcd}
K \times (M,N) & K \times M \arrow[l] & \\
\sigma \times (M,N) \arrow[u] & \sigma \times M \arrow[l] \arrow[u] & \sigma \times \sigma'. \arrow[l] \arrow[ul]
\end{tikzcd}$$

The fact that $\varepsilon^*_{K\times(M,N)}\alpha(X)$ restricts to a $\sigma \times \sigma'$-ad then follows from an easy diagram chase.

- <u>cylinder</u>:
We define the required natural transformation, $J_{\mathcal{A}[M,N]}$, by the formula

$$J_{\mathcal{A}[M,N]}(X) := \alpha^{-1}\big(\varepsilon^*_{(K\times\Delta^1\times(M,N))}\big)^{-1} T^* J_{\mathcal{A}} \varepsilon^*_{K\times(M,N)}\alpha(X),$$

where X is a K-ad in $\mathcal{A}[M,N]$, and

$$T^* : \mathrm{ad}_{\mathcal{A}}(K \times M \times \Delta^1) \to \mathrm{ad}_{\mathcal{A}}(K \times \Delta^1 \times M)$$

is induced from the isomorphism of ball complexes,

$$T : K \times \Delta^1 \times M \cong K \times M \times \Delta^1,$$

given by interchanging factors. In other words, $J_{\mathcal{A}[M,N]}$ fits into the following commutative diagram:

$$\begin{tikzcd}
\mathrm{ad}_{\mathcal{A}[M,N]}(K) \arrow[r, "J_{\mathcal{A}[M,N]}"'] \arrow[d, "\alpha", "\cong"'] & \mathrm{ad}_{\mathcal{A}[M,N]}(K \times \Delta^1) \arrow[d, "\alpha", "\cong"'] \\
\mathrm{ad}_{\mathcal{A}}(K \times (M,N)) \arrow[dd, "\varepsilon^*_{K\times(M,N)}"] & \mathrm{ad}_{\mathcal{A}}(K \times \Delta^1 \times (M,N)) \arrow[d, "\varepsilon^*_{K\times\Delta^1\times(M,N)}"] \\
& \mathrm{ad}_{\mathcal{A}}(K \times \Delta^1 \times M) \\
\mathrm{ad}_{\mathcal{A}}(K \times M) \arrow[r, "J_{\mathcal{A}}"'] & \mathrm{ad}_{\mathcal{A}}(K \times M \times \Delta^1). \arrow[u, "T^*", "\cong"']
\end{tikzcd}$$

We claim that $J_{\mathcal{A}[M,N]}$ is well-defined. First, we will show that it is well-defined as a map of sets: Let X be a K-ad in $\mathcal{A}[M,N]$. Then, by definition,

$\alpha(X)$ is a $K \times (M, N)$-ad in $\mathcal{A}$, and hence, the extension $\varepsilon^*_{K\times(M,N)}\alpha(X)$ restricts to the trivial $K \times N$-ad in $\mathcal{A}$. Now, since $J_{\mathcal{A}}$ preserves trivial ads, the ad $J_{\mathcal{A}}\varepsilon^*_{K\times(M,N)}\alpha(X)$ restricts to the trivial $K \times N \times \Delta^1$-ad, and thus the composition $\left(\varepsilon^*_{K\times\Delta^1\times(M,N)}\right)^{-1} \circ (T^* J_{\mathcal{A}}\varepsilon^*_{K\times(M,N)}\alpha)$ is well-defined.

Next, note that naturality of $J_{\mathcal{A}[M,N]}$ follows directly from the naturality of α and $J_{\mathcal{A}}$, and functoriality of $\mathrm{ad}_{\mathcal{A}}$. Furthermore $J_{\mathcal{A}[M,N]}$ maps trivial ads to trivial ads since it is a composition of maps with this property.

Finally, let X be a given K-ad and $k \in \{0, 1\}$. Then, the following computation is obtained by naturality of α, functoriality of $\mathrm{ad}_{\mathcal{A}}$ and the definitions of T^* and $J_{\mathcal{A}}$:

$$\begin{aligned}
j_k^* J_{\mathcal{A}[M,N]}(X) &= j_k^*\alpha^{-1}\left(\varepsilon^*_{(K\times\Delta^1\times(M,N))}\right)^{-1} T^* J_{\mathcal{A}}\varepsilon^*_{K\times(M,N)}\alpha(X) \\
&= \alpha^{-1} j_k^*\left(\varepsilon^*_{K\times\Delta^1\times(M,N)}\right)^{-1} T^* J_{\mathcal{A}}\varepsilon^*_{K\times(M,N)}\alpha(X) \\
&= \alpha^{-1}\left(\varepsilon^*_{K\times(M,N)}\right)^{-1} j_k^* T^* J_{\mathcal{A}}\varepsilon^*_{K\times(M,N)}\alpha(X) \\
&= \alpha^{-1}\left(\varepsilon^*_{K\times(M,N)}\right)^{-1} j_k^* J_{\mathcal{A}}\varepsilon^*_{K\times(M,N)}\alpha(X) \\
&= \alpha^{-1}\left(\varepsilon^*_{K\times(M,N)}\right)^{-1} \varepsilon^*_{K\times(M,N)}\alpha(X) \\
&= X.
\end{aligned}$$

- gluing
  Let (K', L') be a subdivision of (K, L) and let X' be a given (K', L')-ad in $\mathcal{A}[M, N]$. Our aim is to glue X' to a (K, L)-ad X in $\mathcal{A}[M, N]$, which agrees with X' on every residual subcomplex.

  Observe that the ball complex pair $(K', L') \times (M, N)$ is a subdivision of $(K, L) \times (M, N)$. Hence, by the gluing axiom for $\mathcal{A}$, we find a $(K, L) \times (M, N)$-ad in $\mathcal{A}$ which agrees with $\alpha(X')$ on every residual subcomplex. Denote this ad by Y and set $X := \alpha^{-1}(Y)$.

- reindexing:
  Let $\theta : (K, L) \to (K', L')$ be an incidence compatible k-morphism in $\mathbf{CBall}_2$. Then the k-morphism of cell posets

  $$\theta \times \mathrm{id} : (K, L) \times (M, N) \to (K', L') \times (M, N)$$

  given by

  $$\theta \times \mathrm{id}((\sigma, o) \times (\sigma', o')) := (\theta(\sigma, o) \times (\sigma, o')),$$

  where $(\sigma, o) \times (\sigma', o') \in \mathrm{cell}((K, L) \times (M, N))$, is also incidence compatible. Furthermore, the following diagram can be seen to commute by inspection:

  $$\begin{array}{ccc}
  \mathrm{pre}_{\mathcal{A}[M,N]}(K', L') & \xrightarrow{\theta^*} & \mathrm{pre}_{\mathcal{A}[M,N]}(K, L) \\
  \big\downarrow{\scriptstyle\alpha} & & \big\downarrow{\scriptstyle\alpha} \\
  \mathrm{pre}_{\mathcal{A}}((K', L') \times (M, N)) & \xrightarrow{(\theta\times\mathrm{id})^*} & \mathrm{pre}_{\mathcal{A}}((K, L) \times (M, N)).
  \end{array}$$

  The result now follows from the reindexing axiom for $\mathcal{A}$.

This completes the proof that $(\mathcal{A}[M, N], \mathrm{ad}_{\mathcal{A}[M,N]})$ is an ad theory. □

Remark 3.2.4. We mention that analogous arguments using the adjunction α had also been given in the proof of [BL17, Proposition 3.4].

3.2.2 Functoriality of Parametrisation

We turn to proving functoriality of the assignment

$$(\mathcal{A}, (M, N)) \mapsto (\mathcal{A}[M, N], \mathrm{ad}_{\mathcal{A}[M,N]}).$$

We will write $\mathcal{A}[M, N]$ for $(\mathcal{A}[M, N], \mathrm{ad}_{\mathcal{A}[M,N]})$ from now on, if it does not lead to confusion.

Proposition 3.2.5. *Let $\theta : (K, L) \to (K', L')$ be a morphism of ball complex pairs and $F : \mathcal{A} \to \mathcal{A}'$ be a morphism of ad theories. Then, the induced morphism of $\mathbb{Z}$-graded categories*

$$(\theta^*, F_*) : \mathcal{A}[K', L'] \to \mathcal{A}'[K, L]$$

defines a morphism of ad theories.

Proof. To ease readability, we only illustrate the proof of the case $L = L' = \emptyset$; the general case being analogous. We need to show that, for all ball complex pairs (M, N), the induced map on (M, N)-preads

$$\mathrm{pre}_{(\theta^*,F_*)}(M, N) : \mathrm{pre}_{\mathcal{A}[K']}(M, N) \to \mathrm{pre}_{\mathcal{A}'[K]}(M, N)$$

maps ads to ads. Let X be a (M, N)-ad in $\mathcal{A}[K']$. Observe that $\mathrm{pre}_{(\theta^*,F_*)}(M, N)$ is equal to the composition $\mathrm{pre}_{\theta^*}(M, N) \circ \mathrm{pre}_{F_*}(M, N)$ and consider the following diagram:

$$\begin{array}{ccccc}
\mathrm{pre}_{\mathcal{A}[K']}(M, N) & \xrightarrow{\mathrm{pre}_{F_*}} & \mathrm{pre}_{\mathcal{A}'[K']}(M, N) & \xrightarrow{\mathrm{pre}_{\theta^*}} & \mathrm{pre}_{\mathcal{A}'[K,L]}(M, N) \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} \\
\mathrm{pre}_{\mathcal{A}}((M, N) \times K') & \xrightarrow{\mathrm{pre}_F} & \mathrm{pre}_{\mathcal{A}'}((M, N) \times K') & \xrightarrow{(\mathrm{id} \times \theta)^*} & \mathrm{pre}_{\mathcal{A}'}((M, N) \times K').
\end{array}$$

The diagram commutes, by naturality of α, so it suffices to show that the pread $(\mathrm{id} \times\theta)^*(\mathrm{pre}_F(\alpha(X)))$ is an ad in $\mathcal{A}'$. By assumption, pre_F maps ads to ads. Thus, $\mathrm{pre}_F(\alpha(X))$ is an ad in $\mathcal{A}'$. Functoriality of the ad structure $\mathrm{ad}_{\mathcal{A}}$ then implies that $(\mathrm{id} \times\theta)^*(\mathrm{pre}_F(\alpha(X)))$ is also an ad in $\mathcal{A}'$. □

Propositions 3.2.3 and 3.2.5 taken together show that there is a well-defined functor:

$$\begin{aligned}\hat{p} : \mathbf{Ad} \times \mathbf{Ball}_2^{op} &\to \mathbf{Ad}\,. \\ (\mathcal{A}, (M, N)) &\mapsto \mathcal{A}[M, N].\end{aligned}$$

We let p denote the restriction of $\hat{p}$ to the subcategory $\mathbf{Ad} \times\Delta^{op} \subset \mathbf{Ad} \times\mathbf{Ball}_2^{op}$.

3.2.3 Properties of Parametrisation

Our goal in this subsection is to show that the parametrisation operator p on $\mathbf{Ad}$ extends to a symmetric parametrisation structure. We begin by constructing a unit:

For an ad theory $\mathcal{A}$, let $\mu_{\mathcal{A}} : \mathcal{A}[0] \to \mathcal{A}$ denote the functor given by evaluation at the cell $(\{0\}, +1)$ in $\mathrm{cell}(\Delta^0)$, where $+1$ denotes the canonical orientation on $\{0\}$.

Proposition 3.2.6. *The evaluation maps $\mu_{\mathcal{A}}$ define a natural isomorphism*

$$\mathcal{A}[0] \overset{\cong}{\Rightarrow} \mathcal{A}$$

of functors **Ad** $\to$ **Ad***, i.e., the parametrisation operator p on* **Ad** *admits a unit.*

Proof. Clearly the collection $\mu_{\mathcal{A}}$ define a natural isomorphism of the underlying $\mathbb{Z}$-graded categories. We must show that $\mathrm{pre}_{\mu_{\mathcal{A}}}$ maps ads isomorphically to ads for all ad theories $\mathcal{A}$. Fix an ad theory $\mathcal{A}$ and arbitrary ball complex pair (K, L). Now, let u denote the isomorphism of ball complexes

$$\begin{aligned} u : (K, L) &\overset{\cong}{\to} (K, L) \times \Delta^0 \\ x &\mapsto (x, \{0\}). \end{aligned}$$

and consider the following triangle:

$$\begin{tikzcd} & \mathrm{pre}_{\mathcal{A}[0]}(K, L) \arrow[dl, "\alpha"'] \arrow[dr, "\mathrm{pre}_{\mu_{\mathcal{A}}}"] & \\ \mathrm{pre}_{\mathcal{A}}((K, L) \times \Delta^0) \arrow[rr, "u^*"] & & \mathrm{pre}_{\mathcal{A}}(K, L). \end{tikzcd}$$

It is straightforward to check from definitions that the triangle commutes. Since both α and u^* map ads isomorphically to ads, it follows from commutativity that the map $\mathrm{pre}_{\mu_{\mathcal{A}}}$ must preserve ads too. □

Next, we construct symmetry morphisms for the parametrisation structure (p, μ) using the associativity isomorphisms α. The first observation is that the associativity morphisms α define morphisms of ad theories, not just of $\mathbb{Z}$-graded categories:

Proposition 3.2.7. *For all ball complex pairs (K, L) and (M, N), and each ad theory $\mathcal{A}$, the natural isomorphism*

$$\alpha_{\mathcal{A}}^{(M,N),(K,L)} : (\mathcal{A}[M, N])[K, L] \cong \mathcal{A}[(K, L) \times (M, N)]$$

maps ads isomorphically to ads.

Proof. For readability, we only present the proof for the absolute case; the relative case is completely analogous. Let $\mathcal{A}$ be an ad theory and K and M be ball complexes. Furthermore, let (U, V) be a ball complex pair, and set $\mathcal{B} = \mathcal{A}[M]$, $\mathcal{C} = (\mathcal{A}[M])[K]$ and $\mathcal{D} = \mathcal{A}[K \times M]$. Now consider the following diagram of sets:

$$\begin{tikzcd} \mathrm{pre}_{\mathcal{C}}(U, V) \arrow[r, "\mathrm{pre}_{\alpha_{\mathcal{A}}^{K,M}}(U,V)"] \arrow[d, "\alpha_{\mathcal{B}}^{K,(U,V)}"] & \mathrm{pre}_{\mathcal{D}}(U, V) \arrow[d, "\alpha_{\mathcal{A}}^{K \times M,(U,V)}"] \\ \mathrm{pre}_{\mathcal{B}}((U, V) \times K) \arrow[r, "\alpha_{\mathcal{A}}^{M,(U,V)\times K}"'] & \mathrm{pre}_{\mathcal{A}}((U, V) \times K \times M). \end{tikzcd}$$

The diagram commutes by associativity of the product of ball complexes. Furthermore, the definition of the ad structure on categories of preads implies that the maps, $\alpha_{\mathcal{A}}^{K\times M,(U,V)}$, $\alpha_{\mathcal{A}}^{M,(U,V)\times K}$ and $\alpha_{\mathcal{D}}^{K,(U,V)}$ preserve ads. It immediately follows that the map $\mathrm{pre}_{\alpha_{\mathcal{A}}^{K,M}}(U, V)$ preserves ads. □

Proposition 3.2.8. *The parametrisation structure (p, μ) on* **Ad** *is symmetric.*

Proof. Let $\mathcal{A}$ be an ad theory. For any ball complexes K and M, denote by $\tau_{M,K} : M \times K \cong K \times M$ the isomorphism of ball complexes given by interchanging factors, and define symmetry isomorphism $s_{\mathcal{A}}^{M,K}$ by the following composition:

$$s_{\mathcal{A}}^{M,K} : (\mathcal{A}[M])[K] \overset{\alpha^{K,M}}{\cong} \mathcal{A}[K \times M] \overset{\tau^*_{M,K}}{\cong} \mathcal{A}[M \times K] \overset{(\alpha^{M,K})^{-1}}{\cong} (\mathcal{A}[K])[M].$$

Notice that the relation $\tau^2_{M,K} = 1$ implies that

$$s_{\mathcal{A}}^{M,K} \circ s_{\mathcal{A}}^{K,M} = \mathrm{id},$$

for all ball complexes K and M.

Next, for a given $n \geq 0$, let $u_1 : \Delta^n \cong \Delta^0 \times \Delta^n$ and $u_2 : \Delta^n \cong \Delta^n \times \Delta^0$ denote the canonical isomorphisms, and consider the following diagram of ad theories:

$$\begin{tikzcd}
(\mathcal{A}[n])[0] \arrow[r, "\alpha^{0,n}"] \arrow[drr, "\mu_{\mathcal{A}[n]}"'] & \mathcal{A}[\Delta^0 \times \Delta^n] \arrow[rr, "\tau^*_{n,0}"] \arrow[dr, "u_1^*"] & & \mathcal{A}[\Delta^n \times \Delta^0] \arrow[dl, "u_2^*"'] & (\mathcal{A}[0])[n] \arrow[l, "\alpha^{n,0}"'] \arrow[dll, "\mu_{\mathcal{A}}[n]"] \\
& & \mathcal{A}[n]. & &
\end{tikzcd}$$

The outer triangles commute by inspection, and the inner triangle commutes by functoriality of the parametrisation operator. Therefore, the symmetry morphisms $s_{\mathcal{A}}^{n,0}$, where $n \geq 0$, are compatible with μ. ☐

3.3 Closed Objects and Quinn's bordism machine

In this section we introduce Quinn's bordism machine and show it defines a universal bordism characteristic.

3.3.1 Definitions

Definition 3.3.1. Let $(A, \mathrm{ad}_{\mathcal{A}})$ be an ad theory. We define the pointed graded set $\mathrm{cl}(\mathcal{A})$ of *closed objects* of the ad theory $\mathcal{A}$ to be the set of Δ^0-ads in $\mathcal{A}$, i.e.,

$$\mathrm{cl}(\mathcal{A}) := \mathrm{ad}_{\mathcal{A}}(\Delta^0),$$

where the trivial Δ^0-ads in $\mathcal{A}$ serve as basepoints. The pointed set of closed objects of $\mathcal{A}$ of dimension $-k$ will be denoted by $\mathrm{cl}^k(\mathcal{A})$.

Remark 3.3.2. The terminology is motivated by the ad theory of (oriented topological) manifolds (see [LM14, §6]), in which case closed objects correspond to closed manifolds.

Since morphisms of ad theories preserve ads by definition, taking closed objects of fixed grading k defines a functor $\mathrm{cl}^k : \mathbf{Ad} \to \mathbf{Set}_*$ on the category of ad theories. We introduce automorphisms $\Sigma^k : \mathbf{Ad} \to \mathbf{Ad}$ that shift the dimension of objects by an arbitrary integer k to relate the different gradings:

Definition 3.3.3. Let $(\mathcal{A}, i, \dim, \emptyset, \mathrm{ad}_{\mathcal{A}})$ be an ad theory and $k \in \mathbb{Z}$. We define $\Sigma^k\mathcal{A}$, the k^{th} shift of $\mathcal{A}$, to be the ad theory $(\mathcal{A}, i, \dim_{\Sigma^k\mathcal{A}}, \emptyset_{\Sigma^k\mathcal{A}}, \mathrm{ad}_{\Sigma^k\mathcal{A}})$, where

$$\begin{aligned} \dim_{\Sigma^k\mathcal{A}}(a) &:= \dim(a) - k, \\ \emptyset_{\Sigma^k\mathcal{A}}(n) &:= \emptyset(n-k), \\ \mathrm{ad}^n_{\Sigma^k\mathcal{A}}(K, L) &:= \mathrm{ad}^{n+k}_{\mathcal{A}}(K, L), \end{aligned}$$

for all objects a of $\mathcal{A}$, ball complexes (K, L) and integers $n \in \mathbb{Z}$.

The construction is evidently functorial. Moreover, it is clear that we have an equality of functors, $\mathrm{cl}^k = \mathrm{cl}^0 \circ \Sigma^{-k}$ for all $k \in \mathbb{Z}$. For simplicity, we distinguish the functor cl^0 and call it *the closed-objects functor*.

Definition 3.3.4. *Quinn's bordism-space machine* $\mathcal{Q} : \mathbf{Ad} \to \mathbf{Top}_*$ is defined to be the parametric realisation of the closed-objects functor cl^0, i.e.,

$$\mathcal{Q} := P\,\mathrm{cl}^0 .$$

Our definition of Quinn's bordism machine is different but equivalent to the one given in [LM14, §15]. There, Quinn's bordism machine is described in terms of the individual spaces of the Quinn spectrum associated to an ad theory. We recall how these were defined, following [LM14, Definition 15.4].

Definition 3.3.5. Let $\mathcal{A}$ be an ad theory.

1. For $k \geq 0$, let $P_k(\mathcal{A})$ denote the pointed semi-simplicial set with
   - n-simplices given by
   $$P_k(\mathcal{A})_n := \mathrm{ad}^k_{\mathcal{A}}(\Delta^n),$$
   - face maps induced by functoriality of the ad structure, and
   - n-dimensional basepoint defined to be the trivial Δ^n-ad in $\mathcal{A}$, for all $n \geq 0$.
2. Let $Q_k(\mathcal{A})$ denote the geometric realisation of the semi-simplicial set $P_k(\mathcal{A})$.

The comparison of $\mathcal{Q}$ with the assignments $\mathcal{A} \mapsto Q_k(\mathcal{A})$ may be now stated as follows:

Proposition 3.3.6. *There are natural isomorphisms*

$$\mathcal{Q}(\Sigma^k\mathcal{A}) \cong Q_k(\mathcal{A})$$

of functors $\mathbf{Ad} \to \mathbf{Top}_*$ *for every* $k \geq 0$.

Proof. Let $k \geq 0$ be given and $\mathcal{A}$ be an arbitrary ad theory. We define a natural isomorphism of pointed semi-simplicial sets

$$\mathrm{cl}^0((\Sigma^k\mathcal{A})[\bullet]) \cong P_k(\mathcal{A}) \tag{3.2}$$

with components $\mathrm{cl}^0((\Sigma^k\mathcal{A})[n]) \cong P_k(\mathcal{A})_n$, for $n \geq 0$, given by

$$\mathrm{cl}^0((\Sigma^k\mathcal{A})[n]) = \mathrm{ad}^0_{(\Sigma^k\mathcal{A})[n]}(\Delta^0) \overset{\alpha}{\cong} \mathrm{ad}^0_{\Sigma^k\mathcal{A}}(\Delta^0 \times \Delta^n) \overset{u^*}{\cong} \mathrm{ad}^0_{\Sigma^k\mathcal{A}}(\Delta^n) = \mathrm{ad}^k_{\mathcal{A}}(\Delta^n),$$

where u_2^* is induced from the evident isomorphism $u_2 : \Delta^n \cong \Delta^0 \times \Delta^n$. Composing the isomorphisms (3.2) with geometric realisation then yields the required natural isomorphisms. □

Remark 3.3.7. The shift functors Σ^k in fact commute with the parametrisation operator on **Ad**. We will prove this later in Lemma 5.1.5 of Chapter 5.

3.3.2 Universality of Quinn's machine

The final ingredient in the proof of universality of Quinn's bordism machine is the verification of bordism invariance:

Proposition 3.3.8. *Quinn's bordism machine $\mathcal{Q}$: **Ad** $\to$ **Top**$_*$ is bordism invariant.*

Proof. This is essentially a special case of [LM14, Lemma 17.10(i)], though obscured by notation. We clarify this here for the benefit of the reader:

Fix an ad theory $\mathcal{A}$. By definition, the semi-simplicial space $|R^2_2[\bullet]|$ in the statement of that lemma has p^{th} space $R^2_2[p]$ (where $p \geq 0$) given by

$$R^2_2[p] := \|[q] \mapsto \mathrm{ad}^2_{\mathcal{A}}(\Delta^p \times \Delta^q)\|.$$

Now, observe that for all $p, q \geq 0$ the composition of natural isomorphisms

$$\mathrm{ad}^2_{\mathcal{A}}(\Delta^p \times \Delta^q) \cong \mathrm{ad}^2_{\mathcal{A}}(\Delta^q \times \Delta^p) \cong \mathrm{ad}^2_{\mathcal{A}[p]}(\Delta^q)$$

given by functoriality of $\mathrm{ad}_{\mathcal{A}}$ and the definition of the ad structure on $\mathcal{A}[p]$ yield an isomorphism of topological spaces

$$R^2_2[p] \cong Q_2(\mathcal{A}[p]).$$

Part (i) of [LM14, Lemma 17.10] may then be translated as the claim that the face maps of the semi-simplicial space given by

$$[p] \mapsto Q_2(\mathcal{A}[p])$$

are all homotopy equivalences. Its proof, as described there, is obtained by direct comparison of homotopy groups (which is sufficient since all spaces involved are CW complexes). It follows from the natural identification $\mathcal{Q}(\Sigma^2\mathcal{A}) \cong Q_2(\mathcal{A})$ described in Proposition 3.3.6 and invertibility of the shift functor Σ^2 that the functor $\mathcal{Q}$ must be bordism invariant, as required. □

In summary, the combination of Propositions 3.2.8, 3.3.8 and Theorem 1.2.1 yields a characterisation of Quinn's bordism machine $\mathcal{Q}$ as the universal target for bordism characteristics of the closed-object functor:

Theorem 3.3.9. *The pair $(\mathcal{Q}, \iota_{\mathrm{cl}^0})$ is a universal bordism characteristic of* cl^0.

Part II

A Study of Extended Parametrisation

Chapter 4

The Abstract Universality Theorem Revisited

In Chapter 1 we introduced the setting of categories with parametrisation structures to study the universality of parametric realisation, motivated by our applications to L-theory and Quinn's bordism machine. Our main result established there, the Abstract Universality Theorem, describes minimal conditions such that parametric realisation yields universal bordism characteristics.

Although the Abstract Universality Theorem was entirely sufficient for our purposes in Part I, it has two conceptual shortcomings: Firstly, it does not give intuition for the circumstances under which the bordism invariance condition is satisfied. Secondly, the theorem has no apparent connection to the stable nature of our examples, in the sense that they, in fact, take values in infinite loop spaces. Our desire for this latter connection is inspired by analogy with [Ste17] on the properties of global Euler characteristics.

The intent of this chapter is to demonstrate an example of a more structured setting for parametric realisation which addresses these shortcomings. Specifically, our objectives are to establish a setting for parametric realisation wherein natural sufficient conditions for bordism invariance exist and to formulate a refined version of the Abstract Universality Theorem that gives conditions for both bordism invariance and stability of the parametric realisation of a given functor.

The new setting is based on special features of the example of ad theories. Recall that the parametrisation operator on the category of ad theories extended over the category of ball complex pairs. Such an extension turns out to be technically convenient due to several formal properties of the category of ball complex pairs: namely, the combinatorial nature of its objects and the existence of a symmetric monoidal product '$\times$' and of relative objects (i.e. pairs). We abstract the properties of the parametrisation structure on ad theories with the notion of *extended parametrisation structures.* The details are described in Section 4.1.

A significant benefit of working in the setting of extended parametrisation structures is that there exist a rich variety of natural conditions on space-valued functors related to bordism invariance. We have devoted Sections 4.2, 4.3, 4.3 and 4.4 to introduce and compare these conditions.

In detail, section 4.2 describes two sufficient conditions for bordism invariance of a space-valued functor. The first condition is a formal strengthening of the bordism invariance condition, resembling a natural notion of homotopy invariance in this setting, inspired by the theory of homotopy invariant functors from the category of ball complex pairs to spaces from [BRS76]. The second condition is a combination of a reduction of the bordism invariance condition with a strong invariance property with respect to subdivisions (see Definition 4.2.7).

Section 4.3 introduces two important classes of space-valued functors for our reassessment of the Abstract Universality Theorem that we call *stable* and *linear*. Stable functors naturally extend to spectrum-valued functors under a certain invertibility assumption on the parametrisation operator (see Definition 4.3.1) and serve to conceptualise and abstract the delooping of Quinn's bordism machine $\mathcal{Q}$ to a spectrum-valued functor. It turns out that the condition of stability is related to bordism invariance under certain assumptions. We introduce the class of linear functors, inspired by the analogous notion from the theory of calculus of functors, as a convenient subclass of functors satisfying both of these conditions. In addition to being bordism invariant and stable, they also satisfy homotopy invariance and certain locality properties.

In Section 4.4 we show that the condition of linearity can be significantly reduced under a certain combinatorial assumption on an extended parametrisation structure (see Definition 4.4.1). The result is a key ingredient in the proof of a refinement of the Abstract Universality Theorem and is described precisely in Lemma 4.4.9.

Our final Section 4.5 is devoted to the establishment of the specialised Abstract Universality Theorem (Theorem 4.5.9) which gives conditions on an extended parametrisation structure and a discrete (i.e. set-valued) functor such that its parametric realisation is both bordism invariant and stable. As a byproduct we also formulate conditions such that the parametric realisation of a discrete functor is even linear (Theorem 4.5.7). We note that our analysis is restricted to discrete functors, motivated by the examples treated in subsequent chapters.

4.1 Extended Parametrisation Structures

Definition 4.1.1. *A category with extended parametrisation structure* consists of a quadruple $(\mathcal{C}, p, \mu, \alpha)$ where

- $\mathcal{C}$ is a category,
- $p : \mathcal{C} \times \mathbf{Ball}_2^{op} \to \mathcal{C}$ is a covariant functor,
- $\mu : \mathrm{id}_{\mathcal{C}} \Rightarrow p(-, \Delta^0)$ is a natural isomorphism of functors $\mathcal{C} \to \mathcal{C}$, and
- $\alpha = \left\{\alpha_c^{(K,L),(M,N)}\right\}$ is a natural isomorphism

$$\alpha_c^{(K,L),(M,N)} : (c[M,N])[K,L] \cong c[(K,L) \times (M,N)]$$

of functors $\mathcal{C} \times \mathbf{Ball}_2^{op} \times \mathbf{Ball}_2^{op} \to \mathcal{C}$ that is compatible with μ in the following sense: For every object X of $\mathbf{Ball}_2$ and every object $c \in \mathrm{ob}(\mathcal{C})$, the following two triangles commute:

$$\begin{tikzcd} & c[X] \arrow[dl, "\mu_c[X]"'] \arrow[dr, "\pi_1^*"] & \\ (c[\Delta^0])[X] \arrow[rr, "\alpha_c^{X,\Delta^0}"] & & c[X \times \Delta^0] \end{tikzcd} \qquad \begin{tikzcd} & c[X] \arrow[dl, "\mu_{c[X]}"'] \arrow[dr, "\pi_2^*"] & \\ (c[X])[\Delta^0] \arrow[rr, "\alpha_c^{\Delta^0,X}"] & & c[\Delta^0 \times (X)] \end{tikzcd}$$

where the morphisms π_1^* and π_2^* are induced from the projection isomorphisms $\pi_1 : X \times \Delta^0 \xrightarrow{\cong} X$ and $\pi_2 : \Delta^0 \times X \xrightarrow{\cong} X$, respectively. The morphisms of α are called *associativity morphisms.*

We will use the shorthand $c[K, L]$ and $f[K, L]$ for $p(c,(K, L))$ and $p(f, \mathrm{id}_{(K,L)})$, respectively. Moreover, we denote the morphims $p(\mathrm{id}_c, j)$ by j_c^*, or simply j^*, whenever $c \in \mathrm{ob}(\mathcal{C})$ is fixed.

Remark 4.1.2. The data μ and α of a category with extended parametrisation structure may be interpreted to express unit and associativity laws of the operator p in analogy with the notion of a category $\mathcal{C}$ with monoidal action (see [Lan78, p. 174]) by the monoidal category of ball complex pairs $(\mathbf{Ball}_2, \times, \Delta^0)$. However, note that we do not assume the usual coherence relations here so that a category with extended parametrisation structure need not necessarily come from an action.

Remark 4.1.3. Our notation j^* for the induced morphisms $p(\mathrm{id}_c, j)$ in $\mathcal{C}$, with c a fixed object, is chosen in order to be consistent with the notation used in the example of ad theories (see Chapter 3) and those relating to L-theory (see Chapters 2 and 6).

Example 4.1.4. Consider the category $\mathbf{Ball}_2^{op}$. The product $\times$ on $\mathbf{Ball}_2^{op}$ induces a parametrisation operator on $\mathbf{Ball}_2^{op}$ given by

$$(X, Y)[K, L] := (K, L) \times (X, Y),$$

for every pair of ball complex pairs (X, Y) and (K, L). The unit isomorphism $\mu_X : X \cong X \times \Delta^0$ is defined by inverse to the projection $\pi_1 : X \times \Delta^0 \cong X$; the associativity morphisms α are given by identity morphisms.

Example 4.1.5. Let $\mathcal{C} = \mathbf{Top}_*$. For any ball complex pair (K, L) and pointed space (X, x), let $p(X,(K, L)) := X^{(|K|,|L|)}$ be the space of continous maps of pairs $(|K|, |L|) \to (X, x)$, based at the constant map at the basepoint x of X. Furthermore, let $\mu : X \cong X^{|\Delta^0|}$ be the canonical natural homeomorphism. The exponential law for pairs of spaces induces natural homemomorphisms

$$\alpha_X^{(K,L),(M,N)} : \left(X^{(|M|,|N|)}\right)^{(|K|,|L|)} \cong X^{(|K \times M|, |K \times N \cup L \times M|)}.$$

It is not hard to show that the parametrisation structure of an extended parametrisation structure admits natural symmetry morphisms. We record this fact in the following proposition:

Proposition 4.1.6. *Let $(\mathcal{C}, p, \mu, \alpha)$ be a category with extended parametrisation structure and let $p_{|\Delta^{op}}$ denote the restriction of p to $\Delta^{op} \subset \mathbf{Ball}_2^{op}$. Then, there* *are natural isomophisms of functors $\mathcal{C} \times \mathbf{Ball}_2^{op} \times \mathbf{Ball}_2^{op} \to \mathcal{C}$,*

$$s_c^{(K,L),(M,N)} : (c[K,L])[M,N] \cong (c[M,N])[K,L],$$

such that $(p_{|\Delta^{op}}, \mu)$ defines a symmetric parametrisation structure on $\mathcal{C}$.

Proof. An analogous argument is presented in the proof of Proposition 3.2.8 and the details are omitted. The idea is to use the symmetry of the product $\times$ of $\mathbf{Ball}_2$ together with the additional data α, and its compatibility with the unit μ. □

The symmetry assumption for the parametrisation structure in the Abstract Universality Theorem 1.2.1 is thus automatically satisfied in this setting. We therefore obtain the following reformulation of the Abstract Universality Theorem in the context of extended parametrisations structures:

Theorem 4.1.7. *Let $(\mathcal{C}, p, \mu, \alpha)$ be a category with extended parametrisation structure and $Z : \mathcal{C} \to \mathbf{Top}_*$ be a functor. Suppose that parametric realisation of Z is bordism invariant. Then the pair (PZ, ι_Z) is universal bordism characteristic of Z.*

4.2 Two Sufficient Conditions for Bordism Invariance

In this section, we present two natural sufficient conditions for bordism invariance, under the assumption that the parametrisation structure is extended.

We first recall the notion of an elementary expansion of ball complexes (cf. [LM14, Definition 14.6]) in preparation for the first condition.

Definition 4.2.1. An inclusion of ball complex pairs $(K_0, L_0) \hookrightarrow (K, L)$ is called an *elementary expansion* if the following properties are satisfied:

1. $L_0 = L \cap K_0$.
2. K has exactly two cells, say A and a, that are not in K_0, such that a is a codimension-one face of A (cf. Figure 4.1).
3. A and a are either both in L or both not in L.

Furthermore, a morphism in $\mathbf{Ball}_2$ is called an *expansion*, if it is a composition of elementary expansions.

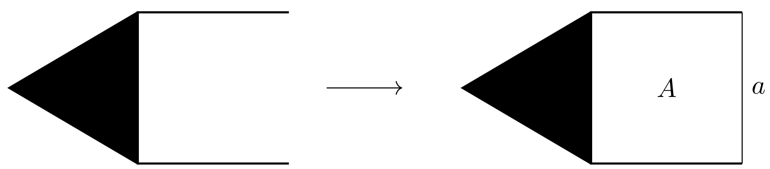

Figure 4.1: An elementary expansion with additional cells labelled A and a.

Definition 4.2.2. Let $(\mathcal{C}, p, \mu, \alpha)$ be a category with extended parametrisation structure and let $F : \mathcal{C} \to \mathbf{Top}_*$ be a functor. We say F is *homotopy invariant* if it takes expansions of ball complexes to weak equivalences, i.e., if for every object c in $\mathcal{C}$ and expansion of ball complex pairs $e : (K, L) \to (K', L')$, the induced map $F(e_c^*) : F(c[K', L]) \to F(c[K, L])$ is a weak equivalence.

Remark 4.2.3. The term "homotopy invariant" is inspired by the notion of homotopy invariant functors from the category of ball complex pairs to spaces from [BRS76] and based on the classical result (see [BRS76, Theorem 3.2]) that the homotopy category of ball complexes, i.e., the category whose objects are ball complexes and morphisms are homotopy classes of continuous maps of ball complex pairs, may be obtained by localising with respect to expansions.

Proposition 4.2.4. *Let $(\mathcal{C}, p, \mu, \alpha)$ be a category with extended parametrisation structure. If $F : \mathcal{C} \to \mathbf{Top}_*$ is homotopy invariant, then it is bordism invariant.*

Proof. This is immediate from the observation that the inclusion

$$\delta^i : \Delta_i^n \hookrightarrow \Delta^{n+1}$$

of the i^{th} face Δ_i^n of Δ^{n+1} is an expansion for all $n \geq 0$ and $i \in \{0, \dots, n+1\}$.

To prove this observation, let such n and i be given and choose $j \in \{0, \dots, n+1\} \setminus \{i\}$. We expand inductively over cells containing the vertices i and j of increasing dimension (cf. Figure 4.2).

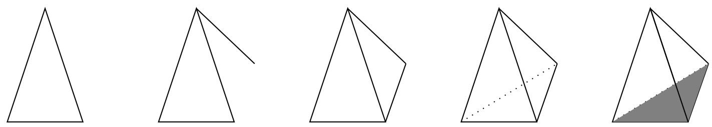

Figure 4.2: The inclusion $\delta^i : \Delta^p \hookrightarrow \Delta^{p+1}$ for $p = 2$ decomposed into elementary expansions; in the first step we adjoin a 1-cell. In the second and third step, a 2-cell. In the final step, we adjoin a 3-cell. The shading in the final figure is meant to indicate that the simplex and bottom face are filled.

Precisely, we first adjoin the pair of cells $(\langle i, j\rangle, \langle i\rangle)$ to the face Δ_i^n, then all pairs $(\langle i, j, k\rangle, \langle i, k\rangle)$, where $k \in \{0, \dots, n+1\} \setminus \{i, j\}$, to the union $\Delta_i^n \cup \langle i, j\rangle$, and so on until we have adjoined the pair $(\langle 0, \dots, n+1\rangle, \langle 0, \dots, \widehat{j}, \dots, n+1\rangle)$, consisting of the unique $n+1$ dimensional cell of Δ^{n+1} and its j^{th} face, to the horn $\Lambda_j^{n+1} \subset \Delta^{n+1}$. $\square$

Remark 4.2.5. Alternatively, one may deduce the result of the above Proposition from [BRS76, Theorem 3.2] and the observation that any face inclusion $\delta^i : \Delta_i^n \hookrightarrow \Delta^{n+1}$ is a homotopy equivalence.

We turn to our second sufficient condition for bordism invariance. It is based on the following condition that resembles a weaker form of the bordism invariance condition:

Definition 4.2.6. Let $(\mathcal{C}, p, \mu, \alpha)$ be a category with extended parametrisation structure and $F : \mathcal{C} \to \mathbf{Top}_*$ be a functor. We say F is *weakly bordism invariant* if the face maps

$$F(d_0), F(d_1) : F(c[\Delta^1]) \rightrightarrows F(c[\Delta^0])$$

are weak equivalences and homotopic, for all objects c in $\mathcal{C}$.

In our next lemma, we show that weak bordism invariance implies bordism invariance whenever the functor in question is *invariant under subdivisions* as described in the following definition:

Definition 4.2.7. We say F is *invariant under subdivisions* if, for every ball complex K and subdivision K' thereof, the inclusions of the ends

$$K \hookrightarrow (K \times \Delta^1) \underset{K\times\{1\}}{\cup} K' \hookleftarrow K'$$

induce weak equivalences

$$F(c[K]) \overset{\simeq}{\leftarrow} F(c[(K \times \Delta^1) \underset{K\times\{1\}}{\cup} K']) \overset{\simeq}{\rightarrow} F(c[K']).$$

Lemma 4.2.8. *Let $(\mathcal{C}, p, \mu, \alpha)$ be category with extended parametrisation structure and $F : \mathcal{C} \to \mathbf{Top}_*$ be a functor. If F is weakly bordism invariant and invariant under subdivisions, then F is bordism invariant.*

Proof. Let c be an object in $\mathcal{C}$ and let $p \geq 2$ and $0 \leq k \leq p$ be given. Observe that the ball complex $\Delta^1 \times \Delta^{p-1}$ is isomorphic to a subdivision of Δ^p, such that the k^{th} face Δ^p_k of Δ^p is identified with the subcomplex $\{0\} \times \Delta^{p-1} \subset \Delta^1 \times \Delta^{p-1}$ (cf. Figure 4.3).

Figure 4.3: Identifications of the prisms $(\Delta^1 \times \Delta^{p-1})$ with subdivisions of the $(p+1)$-simplex Δ^p for $p = 2$ and $p = 3$.

Now, denote by M^p the ball complex obtained by gluing $\Delta^1 \times \Delta^{p-1}$ to the face $\{1\} \times \Delta^p \subset \Delta^1 \times \Delta^p$, and let N^p denote the subcomplex $\Delta^1 \times \Delta^p_k \subset M^p$. Moreover, let $i_0 : \Delta^p \hookrightarrow M^p$ and $i_1 : \Delta^1 \times \Delta^{p-1} \hookrightarrow M^p$ denote the face inclusions and $i'_0 : \Delta^{p-1} \hookrightarrow N^p$ and $i'_1 : \{0\} \times \Delta^{p-1} \to N^p$ denote their restrictions to the subcomplexes $\Delta^p_k \subset \Delta^p$ and $\{0\} \times \Delta^{p-1} \subset \Delta^1 \times \Delta^{p-1}$, respectively (cf. Figure 4.4) .

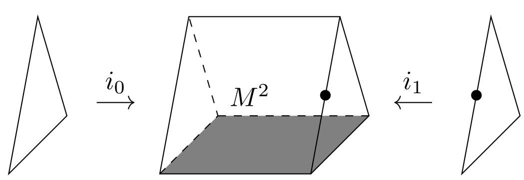

Figure 4.4: The ball complexes M^2 and N^2, and the inclusions i_0 and i_1; the subcomplex $N^2 \subset M^2$ is indicated by the shading.

By assumption, F is invariant under subdivisions, and hence the maps i_0 and i_1 induce weak equivalences

$$F(c[\Delta^p]) \xleftarrow[F(i_0^*)]{\simeq} F(c[M^p]) \xrightarrow[F(i_1^*)]{\simeq} F((c[\Delta^1 \times \Delta^{p-1}]).$$

Similarly, the face inclusion $j_0 : \Delta^{p-1} \cong \{0\} \times \Delta^{p-1} \hookrightarrow \Delta^1 \times \Delta^{p-1}$ induces a weak equivalence

$$F(j_0^*) : F(c[\Delta^1 \times \Delta^{p-1}]) \to F(c[\Delta^{p-1}]).$$

Now let $j_N : N^p \hookrightarrow M^p$ denote the inclusion and consider the following diagram of spaces:

$$\begin{tikzcd}
& F(c[M^p]) \arrow[dl, "F(i_0^*)"', "\simeq"] \arrow[d, "F(j_N^*)"] \arrow[dr, "F(i_1^*)", "\simeq"'] & \\
F(c[\Delta^p]) \arrow[dr, "F(d_k)"'] & F(c[N^p]) \arrow[d, shift right, "F(i_0'^*)"'] \arrow[d, shift left, "F(i_1'^*)"] & F(c[\Delta^1 \times \Delta^{p-1}]) \arrow[dl, "F(j_0^*)", "\simeq"'] \\
& F(c[\Delta^{p-1}]) &
\end{tikzcd} \tag{4.1}$$

In order to show that the face map $F(d_k)$ is a weak equivalence, it suffices to show that the outer square of Diagram 4.1 commutes up to homotopy. Functoriality of F implies that the inner squares of Diagram 4.1 commute, i.e., the composites,

$$F(d_k) \circ F(i_0^*), F(j_0^*) \circ F(i_1^*) : F(c[M^p]) \to F(c[\Delta^{p-1}])$$

are equal to the composition of the restriction map

$$F(j_N^*) : F(c[M^p]) \to F(c[N^p])$$

with the restriction maps

$$F(i_0'^*), F(i_1'^*) : F(c[N^p]) \to F(c[\Delta^{p-1}]),$$

respectively. The latter maps are in fact homotopic by the assumption that F is weakly bordism invariant. Indeed, this may be seen using associativity of the parametrisation operator. It thus follows that the composites $F(d_k) \circ F(i_0^*)$ and $F(j_0^*) \circ F(i_1^*)$ are also homotopic. $\square$

4.3 Stable and Linear functors

In this section we introduce two special classes of space-valued functors from a category with extended parametrisation structure. We begin by introducing a condition on parametrisation structures essential to the construction of deloopings of space-valued functors:

Definition 4.3.1. Let $(\mathcal{C}, p, \mu, \alpha)$ be a category with extended parametrisation structure and denote by $[\Omega]$ the functor $p(-, (\Delta^1, \partial\Delta^1)) : \mathcal{C} \to \mathcal{C}$. The parametrisation operator p is called *deloopable*, if the functor $[\Omega]$ is an equivalence of categories. We will write $c[\Omega]$ for $[\Omega]c$.

Remark 4.3.2. The terminology is based on the example of ad theories: For an ad theory $\mathcal{A}$, the object $\mathcal{A}[\Omega]$ describes the ad theory of $(\Delta^1, \partial\Delta^1)$-ads, or 'bordisms with empty boundary' of $\mathcal{A}$. It is suggestive to think of such bordisms as 'loops', or endomorphisms from and to the empty set, motivated by the theory of cobordism categories of manifolds wherein bordisms of manifolds are regarded as morphisms between manifolds. The assumption that the parametrisation operator p on a category $\mathcal{C}$ is deloopable may be then interpreted to mean that any object c in $\mathcal{C}$ *itself* describes the theory of loops of some other object, or admits a "delooping" in this sense.

We furthermore mention that the square bracket notation $[\Omega]$ is employed to emphasise that the operation of looping comes from the parametrisation operator itself.

Definition 4.3.3. Let $(\mathcal{C}, p, \mu, \alpha)$ be a category with extended parametrisation structure such that p is deloopable and let $F : \mathcal{C} \to \mathbf{Top}_*$ be a functor. Denote by $[\partial]$ and by $\overline{\{0\}}$ the functor $p(-, (\Delta^1, \{1\})) : \mathcal{C} \to \mathcal{C}$ and the ball complex pair $(\{0\}, \{0\})$, respectively. We say F is *stable* if the following conditions hold:

1. F is *reduced*, i.e., $F(c[K, K]) \simeq pt$, for every ball complex K.
2. F *vanishes on null bordisms*, i.e., for all objects c in $\mathcal{C}$,
$$F(c[\partial]) \simeq pt.$$
3. F is *additive*, i.e., for every object c in $\mathcal{C}$, the square
$$\begin{tikzcd} F(c[\Omega]) \arrow[r] \arrow[d] & F(c[\partial]) \arrow[d] \\ F(c[\overline{\{0\}}]) \arrow[r] & F(c) \end{tikzcd}$$
induced from the commutative square of ball complex pairs
$$\begin{tikzcd} (\Delta^1, \partial\Delta^1) & (\Delta^1, \{1\}) \arrow[l] \\ (\{0\}, \{0\}) \arrow[u] & \{0\} \arrow[l] \arrow[u] \end{tikzcd}$$
and the natural identification $\mu_c^{-1} : c[\Delta^0] \cong c$, is homotopy cartesian.

Remark 4.3.4. The term "vanishes on null bordisms" is also based on the example of ad theories, wherein the objects $c[\partial]$ describe the theory of null bordisms in c (cf. Remark 4.3.2). We will relate this vanishing condition to bordism invariance in the next section.

The additivity condition is inspired by the additivity theorem for cobordism categories of manifolds described in [Ste18].

The main importance of the class of stable functors is that they automatically extend to Ω-spectrum-valued-functors. By an "Ω-spectrum" we mean a slightly more general notion than the usual one (cf. [Swi02, Definition 8.41]) that admits more flexbility in defining structure maps.

Definition 4.3.5. 1. An Ω-*spectrum* consists of a pair (X, κ) where:

- $X = (X^i)_{i \in \mathbb{N}}$ is a sequence of homotopy cartesian squares of pointed spaces,

$$X^i = \left\{ \begin{tikzcd} X^i_{00} \arrow[r] \arrow[d] & X^i_{01} \arrow[d] \\ X^i_{10} \arrow[r] & X^i_{11} \end{tikzcd} \right\}$$

  such that the spaces X^i_{10} and X^i_{01} are contractible for all $i \in \mathbb{N}$.

- $\kappa = (\kappa_i)_{i \in \mathbb{N}}$ is a sequence of homeomorphisms of pointed spaces,

$$\kappa_i : X^i_{11} \cong X^{i+1}_{00}.$$

We denote the spaces X^i_{11} and $X^i_{10} \times^h_{X^i_{11}} X^i_{01}$ by X_i and ΩX_i, respectively, for all $i \in \mathbb{N}$, and call the morphisms κ_i *connecting homomorphisms*.

2. A *map of* Ω-*spectra* $f : (X, \kappa) \to (Y, \eta)$ is a sequence $(f^i : X^i \to Y^i)_{i \in \mathbb{N}}$ of maps of homotopy cartesian squares compatible with the connecting homomorphisms, i.e., f consists of a collection of morphisms of pointed spaces $f^i_{kl} : X^i_{kl} \to Y_{kl}$, where $k, l \in \{0, 1\}$, such that the following cubes and squares are strictly commutative, for all $i \in \mathbb{N}$:

$$\begin{tikzcd} & X^i_{00} \arrow[rr] \arrow[dl, "f^i_{00}"'] \arrow[dd] & & X^i_{01} \arrow[dl, "f^i_{01}"] \arrow[dd] \\ Y^i_{00} \arrow[rr] \arrow[dd] & & Y^i_{01} \arrow[dd] & \\ & X^i_{10} \arrow[rr] \arrow[dl, "f^i_{10}"] & & X^i_{11} \arrow[dl, "f^i_{11}"] \\ Y^i_{10} \arrow[rr] & & Y^i_{11} & \end{tikzcd} \qquad \begin{tikzcd} X^i_{11} \arrow[r, "f^i_{11}"] \arrow[d, "\kappa_i"'] & Y^i_{11} \arrow[d, "\eta_i"] \\ X^i_{00} \arrow[r, "f^i_{00}"] & Y^i_{00}. \end{tikzcd}$$

We denote the category of Ω-spectra by $\Omega\mathbf{Spec}$.

Remark 4.3.6. The usual data of an Ω-spectrum, i.e., a sequence of pointed spaces X_i and weak equivalences $s_i : X_i \to \Omega X_{i+1}$ for $i \in \mathbb{N}$, can be seen to define an Ω-spectrum (X, κ) in the sense above, by considering the squares

$$X^i = \left\{ \begin{tikzcd} X_{i-1} \arrow[r, "s'_{i-1}"] \arrow[d] & \Lambda X_i \arrow[d] \\ pt \arrow[r] & X_i \end{tikzcd} \right\}$$

and maps $\kappa_i = \mathrm{id}_{X_i}$, where ΛX_i here denotes the based path space of X_i and the maps s'_{i-1} denote the composites $X_{i-1} \xrightarrow{s_{i-1}} \Omega X_i \subset \Lambda X_i$.

Proposition 4.3.7. *Let $(\mathcal{C}, p, \mu, \alpha)$ be a category with extended parametrisation structure such that the parametrisation operator is deloopable. Furthermore, let*

$F : \mathcal{C} \to \mathbf{Top}_*$ *be a stable functor. Then* F *extends to a functor* $\underline{F} : \mathcal{C} \to \Omega\mathbf{Spec}$ *such that the* 0^{th} *component of* $\underline{F}$ *is equal to* F*.*

Furthermore, every natural transformation $\eta : F \Rightarrow G$ *of stable functors on* $\mathcal{C}$ *extends to a natural transformation* $\underline{\eta} : \underline{F} \Rightarrow \underline{G}$ *of spectrum-valued functors.*

Proof. Choose an inverse Σ to $[\Omega]$ and a natural isomorphism $\nu : \mathrm{id}_{\mathcal{C}} \Rightarrow [\Omega] \circ \Sigma$. For $i \in \mathbb{N}$, denote the i^{th} iterate of Σ by Σ^i and moreover denote the objects $([\Omega] \circ \Sigma^i)c$ and $([\partial] \circ \Sigma^i)c$ by $\Sigma^i c[\Omega]$ and $\Sigma^i c[\partial]$, respectively. For a given object c in $\mathcal{C}$, we define an Ω-spectrum $i \mapsto (\underline{F}(c)_i, \kappa_i^c)$ by

$$\underline{F}(c)_i := \left\{ \begin{array}{ccc} F(\Sigma^i c[\Omega]) & \longrightarrow & F(\Sigma^i c[\partial]) \\ \big\downarrow & & \big\downarrow \\ F(\Sigma^i c[\overline{\{0\}}]) & \longrightarrow & F(\Sigma^i c) \end{array} \right\}$$

and

$$\kappa_i^c := F(\nu_{\Sigma^i c}) : F(\Sigma^i c) \cong F(\Sigma^{i+1} c[\Omega]),$$

for all $i \in \mathbb{N}$. A morphism $c \mapsto c'$ in $\mathcal{C}$ clearly induces a map of Ω-spectra $(\underline{F}(c), \kappa^c) \to (\underline{F}(c'), \kappa^{c'})$ by functoriality of the parametrisation operator on $\mathcal{C}$ and the functors Σ and F. The second statement of the proposition is immediate from definitions. $\square$

Remark 4.3.8. The notation $\Sigma^i c[\Omega]$ and $\Sigma^i c[\partial]$ in the definition above is justified since the functors Σ^i commute with the parametrisation operator p in the sense that there are natural isomorphisms of functors $\mathcal{C} \to \mathcal{C}$,

$$\kappa^{K,L} : \Sigma^i \circ p((K,L), -) \cong p((K,L), -) \circ \Sigma^i,$$

for all $i \in \mathbb{N}$. Indeed, the existence of the natural isomorphisms $\kappa^{K,L}$ follows formally from the existence of symmetry morphisms for the parametrisation operator p as in Proposition 4.1.6.

Remark 4.3.9. The choice of inverse Σ to $[\Omega]$ is not necessarily unique, although any two choices are naturally isomorphic. In the examples described in subsequent chapters, there is a natural choice for Σ given by "shifting the grading".

Definition 4.3.10. Let $(\mathcal{C}, p, \mu, \alpha)$ be a category with extended parametrisation structure. We call a functor $F : \mathcal{C} \to \mathbf{Top}_*$ *absolute*, if F is reduced and if, for every ball complex pair (K, L) and object c in $\mathcal{C}$, the following square induced by functoriality is homotopy cartesian:

$$\begin{array}{ccc} F(c[K,L]) & \longrightarrow & F(c[K]) \\ \big\downarrow & & \big\downarrow \\ F(c[L,L]) & \longrightarrow & F(c[L]). \end{array}$$

If, in addition, F takes every pushout diagram of ball complexes

$$\begin{array}{ccc} X \cap Y & \longrightarrow & X \\ \big\downarrow & & \big\downarrow \\ Y & \longrightarrow & X \cup Y \end{array}$$

to a homotopy pullback square

$$\begin{array}{ccc} F(c[X \cup Y]) & \longrightarrow & F(c[X]) \\ \downarrow & & \downarrow \\ F(c[Y]) & \longrightarrow & F(c[X \cap Y]), \end{array}$$

we say F is *local*.

Definition 4.3.11. Let $(\mathcal{C}, p, \mu, \alpha)$ be a category with extended parametrisation structure such that the parametrisation operator p is deloopable. We call a functor $F : \mathcal{C} \to \mathbf{Top}_*$ *linear* if it is homotopy invariant, stable and local.

Remark 4.3.12. The term linear is inspired by the notion of "linear" functors from the theory of functor calculus. We refer the reader to [MV15, §10.1] for further details about the latter theory.

We are primarily interested in linear functors since they form a convenient class of stable and bordism invariant functors. Observe that bordism invariance follows immediately from Proposition 4.2.4. We note this statement in the following proposition for future reference:

Proposition 4.3.13. *Let $(\mathcal{C}, p, \mu, \alpha)$ be a category with extended parametrisation structure such that the parametrisation operator p is deloopable, and let $F : \mathcal{C} \to \mathbf{Top}_*$ be a linear functor. Then F is bordism invariant.*

4.4 A Criterium For Linearity

Our goal in this section is to establish a reduced criterium for linearity of a space-valued functor from a category with extended parametrisation structure $\mathcal{C}$. Notice that linear functors are, by definition, necessarily absolute and vanish on null bordisms. The reduced criterium states that these latter conditions are in fact also sufficient under certain assumptions on the parametrisation operator. One of these assumptions is the following extended functoriality condition on the parametrisation operator that suggests that parametrisation only depends on the underlying posets of ball complexes:

Definition 4.4.1. Let $(\mathcal{C}, p, \mu, \alpha)$ be a category with extended parametrisation structure. We call the parametrisation operator p *combinatorial* if, for every incidence compatible 0-isomorphism of ball complexes in $\mathbf{CBall}_2$

$$\theta : (K, L) \cong (K', L'),$$

there is an isomorphism

$$\theta^* : c[K', L'] \cong c[K, L],$$

natural in the variable c and such that the following conditions are satisfied:

1. θ^* agrees with the induced maps given by the parametrisation operator whenever θ comes from a map of ball complexes.

2. $(\theta \circ \psi)^* = \psi^* \circ \theta^*$, for all incidence compatible 0-isomorphisms of ball complexes θ and ψ.

3. For any pair of morphisms of ball complexes (f, f') and any pair of incidence compatible 0-isomorphisms (θ, θ') such that the following diagram:

$$\begin{array}{ccc} (K, L) & \xrightarrow{\theta} & (K', L') \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ (M, N) & \xrightarrow{\theta'} & (M', L') \end{array}$$

commutes in $\mathbf{CBall}_2$, the induced square

$$\begin{array}{ccc} c[M', L'] & \xrightarrow{\theta'^*} & c[M, N] \\ \downarrow{\scriptstyle f'^*} & & \downarrow{\scriptstyle f^*} \\ c[K, L] & \xrightarrow{\theta^*} & c[K, L] \end{array}$$

commutes in $\mathcal{C}$.

Remark 4.4.2. Definition 4.4.1 is motivated by the reindexing axiom for ad theories and the extended functoriality of the construction of categories of preads (see Remark 3.1.17). Note that the axiom only stipulates the existence of additional morphisms for any dimension preserving maps into or from ball complex *pairs* since every incidence compatible 0-isomorphism of absolute ball complexes $K \to L$ already comes from a map of ball complexes. Indeed, this is easily seen from the fact that a morphism of ball complexes is determined by its induced map on face posets (see [BVS+93, Corollary 4.7.9]).

Remark 4.4.3. Condition (2.) of Definition 4.4.1 is not critical to our work, but is rather included because of its naturality.

We will build up to a criterium for linearity by establishing the conditions of stability, locality and homotopy invariance in turn. First, we record that under the combinatorial assumption on the parametrisation operator the absolute conditon described in Definition 4.3.10 can be strengthened to pairs:

Proposition 4.4.4. *Let $(\mathcal{C}, p, \mu, \alpha)$ be a category with extended parametrisation structure such that p is combinatorial. Let $F : \mathcal{C} \to \mathbf{Top}_*$ be an absolute functor. Then, for every ball complex K and pair of subcomplexes K_0 and L thereof, the square of spaces*

$$\begin{array}{ccc} F(c[K, K_0 \cup L]) & \longrightarrow & F(c[K, L]) \\ \downarrow & & \downarrow \\ F(c[K_0, K_0]) & \longrightarrow & F(c[K_0, K_0 \cap L]) \end{array}$$

induced from the commutative square of ball complex pairs

$$\begin{array}{ccc} (K_0, K_0 \cap L) & \longrightarrow & (K, L) \\ \downarrow & & \downarrow \\ (K_0, K_0) & \longrightarrow & (K, K_0 \cup L) \end{array}$$

is homotopy cartesian.

Proof. Observe that the inclusion maps

$$(K_0, K_0 \cap L) \hookrightarrow (K_0 \cup L, L) \text{ and } (K_0, K_0) \hookrightarrow (K_0 \cup L, K_0 \cup L)$$

induce 0-isomorphisms of posets

$$\mathrm{cell}(K_0, K_0 \cap L) \cong \mathrm{cell}(K_0 \cup L, L) \text{ and } \mathrm{cell}(K_0, K_0) \cong \mathrm{cell}(K_0 \cup L, K_0 \cup L).$$

Since p is combinatorial, it therefore suffices to show that the left hand square in the following commutative diagram is homotopy cartesian:

$$\begin{tikzcd}
F(c[K, K_0 \cup L]) \arrow[r] \arrow[d] & F(c[K, L]) \arrow[r] \arrow[d] & F(c[K]) \arrow[d] \\
F(c[K_0 \cup L, K_0 \cup L]) \arrow[r] & F(c[K_0 \cup L, L]) \arrow[r] \arrow[d] & F(c[K_0 \cup L]) \arrow[d] \\
 & F(c[L, L]) \arrow[r] & F(c[L]).
\end{tikzcd}$$

The result now follows immediately from the assumption that F is absolute and the pasting lemma for homotopy cartesian squares (see [MV15, Proposition 3.3.20]). $\square$

As a corollary of Proposition 4.4.4 we deduce that, in the presence of a deloopable and combinatorial parametrisation operator, absolute functors which vanish on null bordisms are automatically additive, and hence stable:

Corollary 4.4.5. *Let $(\mathcal{C}, p, \mu, \alpha)$ be a category with extended parametrisation structure such that p is deloopable and combinatorial. Let $F : \mathcal{C} \to \mathbf{Top}_*$ be an absolute functor which vanishes on null bordisms. Then F is stable.*

Proof. The functor F is reduced and vanishes on null bordism by assumption. Additivity of F follows from the previous proposition applied to the case $K = \Delta^1$, $K_0 = \{0\}$ and $L = \{1\}$. $\square$

Our next proposition states that, in the restricted setup of a combinatorial and deloopable parametrisation operator, an absolute and stable functor is in fact local.

Proposition 4.4.6. *Let $(\mathcal{C}, p, \mu, \alpha)$ be a category with extended parametrisation structure such that p is deloopable and combinatorial. Let $F : \mathcal{C} \to \mathbf{Top}_*$ be an absolute and stable functor. Then F is local.*

Proof. Let X, Y be ball complexes, and c be an object in $\mathcal{C}$. By the assumption that the parametrisation operator p is combinatorial, the inclusion of ball complex pairs

$$(X, X \cap Y) \hookrightarrow (X \cup Y, Y)$$

induces a homeomorphism

$$F(c[X \cup Y, Y]) \cong F(c[X, X \cap Y]),$$

for all objects c in $\mathcal{C}$. Hence, since F is absolute, the vertical homotopy fibres with respect to the basepoint components of the square

$$\begin{array}{ccc} F(c[X \cup Y]) & \longrightarrow & F(c[X]) \\ \downarrow & & \downarrow \\ F(c[Y]) & \longrightarrow & F(c[X \cap Y]) \end{array}$$

are weakly equivalent. By the assumption that F is stable, we furthermore deduce that the square is homotopy cartesian since all spaces and maps involved are loop spaces and maps thereof. To see the latter observation, let Σ denote an inverse to $[\Omega]$. Then, stability of F implies that there exists a zigzag of natural weak equivalences between the spaces $F(c[K])$ and the spaces $\Omega F(\Sigma c[K])$, for an arbitrary object c in $\mathcal{C}$ and ball complex K, where $\Omega F(\Sigma c[K])$ denotes the space of based loops of the pointed space $F(\Sigma c[K])$. $\square$

It remains to establish homotopy invariance. The following proposition is a critical component in the proof and shows the main technical advantage of working with stable functors:

Proposition 4.4.7. *Let $(\mathcal{C}, p, \mu, \alpha)$ be a category with extended parametrisation structure such that p is deloopable and combinatorial, and let $F : \mathcal{C} \to \mathbf{Top}_*$ be an absolute functor which vanishes on null bordisms. Moreover, let K be a ball complex, K_0 and L be two subcomplexes thereof, and $j : (K_0, K_0 \cap L) \hookrightarrow (K, L)$ denote the inclusion.*

Then, the induced map $F(j_c^*) : F(c[K, L]) \to F(c[K_0, K_0 \cap L])$ is a weak equivalence for all objects c in $\mathcal{C}$ if and only if the space $F(c[K, K_0 \cup L])$ is contractible for all objects c in $\mathcal{C}$.

Proof. Let $c \in \mathrm{ob}(\mathcal{C})$ be given. Since F is absolute, it follows from Propositon 4.4.4 that the space $F(c[K, K_0 \cup L])$ is a model for the homotopy fibre of $F(j_c^*)$ over the basepoint component. Hence, $F(c[K, K_0 \cup L])$ is contractible whenever $F(j_c^*)$ is a weak equivalence.

Conversely, assume that $F(c[K, L])$ is contractible, for all objects c in $\mathcal{C}$. By assumption, we also know that F is absolute and vanishes on null bordisms. It then follows from Corollary 4.4.5 that F is stable. Furthermore, Proposition 4.3.7 implies that the map $F(j_c^*)$ comes from a map of Ω-spectra. In particular, we may form the following commutative diagram:

$$\begin{array}{ccc} F(c[K, L]) & \xrightarrow{F(j_c^*)} & F(c[K_0, K_0 \cap L]) \\ \downarrow \cong & & \downarrow \cong \\ F(\Sigma c[K, L][\Omega]) & \xrightarrow{F(\Sigma(j_c^*[\Omega]))} & F(\Sigma c[K_0, K_0 \cap L][\Omega]) \\ \downarrow \simeq & & \downarrow \simeq \\ \Omega F(\Sigma c[K, L]) & \xrightarrow{\Omega F(j_{\Sigma c}^*)} & \Omega F(\Sigma c[K_0, K_0 \cap L]), \end{array}$$

where the vertical equivalences come from the structure maps of the Ω-spectra $\underline{F}(c[K, K_0 \cup L])$ and $\underline{F}(c[K_0, K_0 \cap L])$, and Σ denotes a fixed choice of inverse to $[\Omega]$.

From the diagram, it is clear that the map $F(j_c^*)$ is a weak equivalence if and only if the map $\Omega F(j_{\Sigma c}^*)$ is a weak equivalence. To prove the latter statement it is sufficient to show that the map

$$F(j_{\Sigma c}^*) : F(\Sigma c[K, L]) \to F(\Sigma c[K_0, K_0 \cap L])$$

induces an isomorphism on homotopy groups π_k over the basepoint component for all $k \geq 1$, yet this is a direct consequence of Proposition 4.4.4 since the homotopy fibre of $F(j_{\Sigma c}^*)$ over the basepoint is homotopic to the space $F(\Sigma c[K, K_0 \cup L])$, and is thus contractible by assumption. □

Proposition 4.4.8. *Let $(\mathcal{C}, p, \mu, \alpha)$ be a category with extended parametrisation structure such that p is combinatorial and deloopable. Furthermore, let $F : \mathcal{C} \to \mathbf{Top}_*$ be absolute and vanishing on null bordisms. Then F is homotopy invariant.*

Proof. It suffices to show that F takes elementary expansions to weak equivalences. Let c be an object of $\mathcal{C}$ and $e : (K_0, L_0) \to (K, L)$ be an elementary expansion. Moreover, denote by A, a the cells that do not belong to K_0.

If $A \in L$, then $F(e_c^*)$ is an isomorphism since e induces a 0-isomorphism of posets

$$\mathrm{cell}(K_0, L_0) \cong \mathrm{cell}(K, L).$$

Next, let n denote the dimension of K and X denote the ball complex $(\Delta^1, \partial\Delta^1)^{n-1}$. If $A \notin L$, then there is an incidence compatible 0-isomorphism of posets

$$\theta : \mathrm{cell}((\Delta^1, \{1\}) \times X) \cong \mathrm{cell}(K, K_0 \cup L),$$

which identifies the pair of cells $(\langle 0, 1\rangle^n, \{0\} \times \langle 0, 1\rangle^{n-1})$ with the pair (A, a). Furthermore, since p is combinatorial and associative we deduce that

$$F(c[K, K_0 \cup L]) \overset{F(\theta^*)}{\cong} F(c[(\Delta^1, \{1\}) \times X]) \overset{\alpha}{\cong} F((c[X])[\partial]).$$

By assumption, the latter space is contractible, and hence the former is too. It now follows from Proposition 4.4.7 that the map $F(e^*)$ is a weak equivalence. □

From the combination of Corollary 4.4.5 and Propositions 4.4.6 and 4.4.8, we immediately deduce the following criterium for linearity:

Lemma 4.4.9. *Let $(\mathcal{C}, p, \mu, \alpha)$ be a category with extended parametrisation structure such that the parametrisation operator p is deloopable and combinatorial. Furthermore, let $F : \mathcal{C} \to \mathbf{Top}_*$ be an absolute functor which vanishes on null bordisms. Then F is linear.*

Lemma 4.4.9 serves as a tool in proof of the specialised universality theorem, described in the next section. We also note the following corollary that gives a number of reformulations of the bordism invariance condition in the context of a combinatorial and deloopable parametrisation operator:

Corollary 4.4.10. *Let $(\mathcal{C}, p, \mu, \alpha)$ be a category with extended parametrisation such that p is combinatorial and deloopable. Let $F : \mathcal{C} \to \mathbf{Top}_*$ be an absolute functor. Then the following three conditions are equivalent:*

1. F is homotopy invariant.

2. F is bordism invariant

3. F vanishes on null bordisms

Proof. The implication of Conditions (1) to (2) and (3) to (1) follow directly from Propositions 4.2.4 and 4.4.8, respectively. Furthermore, the implication (2) to (3) can be easily seen by considering the following homotopy cartesian square:

$$\begin{tikzcd} F(c[\Delta^1, \{1\}]) \arrow[r] \arrow[d] & F(c[\Delta^1]) \arrow[d, "\simeq"', "F(d_0)"] \\ pt \simeq F(c[\{1\}, \{1\}]) \arrow[r] & F(c[\Delta^0]). \end{tikzcd}$$

□

Remark 4.4.11. The idea to deduce bordism invariance from the vanishing on null bordism condition was inspired by a similar idea used in the proof for the bordism invariance of $\mathbb{Z}_2$-Tate cohomology of K-theory described in [WW98, Theorem 9.12]).

4.5 A Specialised Universality Theorem

The aim of this section is to demonstrate a specialised version of the Abstract Universality Theorem in the case of a category with extended parametrisation structure $\mathcal{C}$. To simplify the analysis we will only consider functors $Z : \mathcal{C} \to \mathbf{Top}_*$ such that the spaces $Z(c)$ are equipped with the discrete topology. We call such functors *discrete* and employ the notation $Z : \mathcal{C} \to \mathbf{Set}_*$.

Recall that an ordered simplicial complex defines a semi-simplicial set with the same sets of simplices, and face maps induced by the ordering of the vertices (e.g., see [Ran92, Example 11.1]). Our first result serves as a key preliminary observation and states that, under the assumption of locality, the parametric realisation of Z at an object c of $\mathcal{C}$ represents the assignment $K \mapsto Z(c[K])$ over the category of finite ordered simplicial complexes:

Proposition 4.5.1. *Let $(\mathcal{C}, p, \mu, \alpha)$ be a category with extended parametrisation structure and K be a finite ordered simplicial complex. If $Z : \mathcal{C} \to \mathbf{Set}_*$ is local, then the set of semi-simplicial maps $K \to Z(c[\bullet])$ is isomorphic to the set $Z(c[K])$, for all objects c in $\mathcal{C}$. Furthermore, the isomorphism is natural with respect to inclusions $L \subset K$ of finite ordered simplicial complexes.*

Proof. For any finite simplicial complex K, we call a simplex $\sigma \in \mathcal{F}(K)$ *maximal* if for any $\tau \in \mathcal{F}(K)$, $\tau \geq \sigma$ implies that $\tau = \sigma$. We view K as the union of its maximal simplices $\{\sigma_i\}_{i=1,\dots,n}$, where $\sigma_j \neq \sigma_i$ for all $i \neq j$, and define the *rank of* K to be the number of maximal simplices. The proof will be obtained by induction on the rank of K. Let c denote a fixed object of $\mathcal{C}$.

Case $\mathrm{rk}(K) = 0$: This case corresponds to $K = \emptyset$ and is true by the assumption that Z is absolute since $Z(c[\emptyset]) = Z(c[\emptyset, \emptyset]) \cong pt$.

Case $\mathrm{rk}(K) = 1$: Without loss of generality, we may assume $K = \Delta^n$, for some $n \geq 0$. A semi-simplicial map $\Delta^n \to Z(c[\bullet])$ is determined uniquely by its value on the unique maximal simplex of Δ^n, i.e. by a choice of n-simplex in $Z(c[\bullet])$. Moreover, the n-simplices of $Z(c[\bullet])$ are just the elements of $Z(c[\Delta^n])$, by definition. The correspondence is thus clear.

To see that the correspondence is natural, let $\mathrm{Map}\big(-, Z(c[\bullet])\big)$ denote the functor which takes an ordered simplicial complex K to the set of semi-simplicial maps $K \to Z(c[\bullet])$. We must show that, for any inclusion $i : L \hookrightarrow K$ of simplicial complexes of rank at most 1, the diagram of pointed sets

$$\begin{array}{ccc} \mathrm{Map}\big(K, Z(c[\bullet])\big) & \longrightarrow & Z(c[K]) \\ \big\downarrow{\scriptstyle Map(i,Z(c[\bullet])} & & \big\downarrow{\scriptstyle Z(i^*)} \\ \mathrm{Map}\big(L, Z(c[\bullet])\big) & \longrightarrow & Z(c[L]) \end{array}$$

commutes, where the horizontal functions denote the correspondence just described. The cases where $K = \emptyset$ or $L = \emptyset$ are trivial and the case where both K and L have rank 1 is immediate from the definition of the face maps of the semi-simplicial set $Z(c[\bullet])$.

Next, let K be a given ordered simplicial complex with $rk(K) = n \geq 2$, and suppose that the statement of the proposition holds for all simplicial complexes with rank less than n. Denote by $K' \subset K$ the simplicial subcomplex of K given by the union $\cup_{i=1,\dots,n-1}\sigma_i$. Furthermore, set $\sigma := \sigma_n$ and $K'' := \sigma \cap K'$.

Then, by the universal property of the pushout a map $K \to Z(c[\bullet])$ is equivalent to a pair of maps $\big(K' \to Z(c[\bullet]), \sigma \to Z(c[\bullet])\big)$ which agree over K''. Since K'' is equal to the union of simplices $\cup_{i=1,\dots,n-1}\sigma \cap \sigma_i$, it clearly follows that $\mathrm{rk}(K'') \leq n-1$. By the induction hypothesis we deduce that every map $K \to Z(c[\bullet])$ corresponds to a pair $(x, y) \in Z(c[K']) \times Z(c[\sigma])$ such that the elements x and y restrict to the same element in $Z(c[K''])$. Note that such a pair (x, y) corresponds uniquely to an element in $Z(c[K' \cup \sigma]) = Z(c[K])$, by the assumption that Z is discrete and local. Analogous reasoning, using locality of Z to reduce to the inductive case, shows that the construction is independent of the choice of maximal simplex σ_n and natural with respect to simplicial inclusions. $\square$

Remark 4.5.2. Proposition 4.5.1 is based on similar observations [LM14, Remark 15.4], [Ran92, p. 142] in the case of the closed-objects functor of ad theories and algebraic L-theory, respectively.

Definition 4.5.3. Let $(\mathcal{C}, p, \mu, \alpha)$ be a category with extended parametrisation structure and $Z : \mathcal{C} \to \mathbf{Set}_*$ be a functor.

1. We say Z is *surjective on expansions* if, for all objects c in $\mathcal{C}$ and elementary expansions $B \to E$ of ball complexes, the induced map of sets
$$Z(c[E]) \to Z(c[B])$$
is surjective.

2. We say Z is a *Kan* functor if, for every object $c \in \mathcal{C}$, the semi-simplicial set $Z(c[\bullet])$ is Kan (see [RS71, p. 329]).

We will now apply Proposition 4.5.1 to formulate conditions on a category with extended parametrisation structure $\mathcal{C}$ and discrete functor Z from $\mathcal{C}$ which imply that Z satisfies the Kan condition and, moreover, that its parametric realisation is absolute.

Lemma 4.5.4. *Let $(\mathcal{C}, p, \mu, \alpha)$ be a category with extended parametrisation structure and $Z : \mathcal{C} \to \mathbf{Set}_*$ be a functor. Suppose that Z is local and surjective on expansions. Then, Z is a Kan functor and its parametric realisation PZ is absolute.*

Proof. We first prove that Z is a Kan functor. Let c be an object in $\mathcal{C}$. By the previous Proposition 4.5.1, the Kan condition for the semi-simplicial set $Z(c[\bullet])$ is equivalent to the statement that, for all n, k, where $n > 0$ and $0 \leq k \leq n$, the restriction map

$$Z(c[\Delta^n]) \to Z(c[\Lambda^n_k])$$

is surjective, where Λ^n_k denotes the k^{th} horn of Δ^n (see [RS71, p. 323]). The latter statement follows directly from the fact that the inclusions $\Lambda^n_k \hookrightarrow \Delta^n$ are all elementary expansions, and the assumption that Z is surjective on expansions.

We next claim that the functor PZ is reduced. Let c be an object in $\mathcal{C}$ and K be a ball complex. By associativity of the parametrisation, there are homeomorphisms

$$Z((c[K, K])[\Delta^n]) \xrightarrow[\alpha_c^{(K,K),\Delta^n}]{\cong} Z(c[K \times \Delta^n, K \times \Delta^n]),$$

for every $n \geq 0$. Since Z is discrete and reduced, the latter spaces must be singeltons. Hence, the semi-simplicial set $Z(c[K, K][\bullet])$ is isomorphic to the terminal semi-simplicial set $*$, and thus, upon applying geometric realisation, we obtain a homeomorphism $PZ(c[K, K]) \cong ||*||$. The claim is now immediate since the geometric realisation of $*$ is contractible (cf. Example 1.3.2).

We now turn to proving that the functor PZ is absolute. Let $i : A \hookrightarrow X$ be an inclusion of ball complexes and c be a given object of $\mathcal{C}$. We must show that the square

$$\begin{tikzcd} PZ(c[X, A]) \arrow[r] \arrow[d] & PZ(c[X]) \arrow[d, "PZ(i^*)"] \\ PZ(c[A, A]) \arrow[r] & PZ(c[A]) \end{tikzcd} \tag{4.2}$$

is a homotopy pullback. First note that, by the assumption that Z is absolute and discrete, the underlying square of pointed semi-simplicial sets

$$\begin{tikzcd} Z(c[X, A][\bullet]) \arrow[r, "\subseteq"] \arrow[d] & Z(c[X][\bullet]) \arrow[d, "PZ(i^*)"] \\ Z(c[A, A][\bullet]) \cong * \arrow[r, "\subseteq"] & Z(c[A][\bullet]) \end{tikzcd}$$

is a pullback square. Moreover, the horizontal arrows are inclusions, up to isomorphism. Since the geometric realisation preserves inclusions, it follows that Square (4.2) is a strict pullback square of pointed spaces. It therefore suffices to show (cf. [MV15, Proposition 3.2.13]) that the map $PZ(i^*)$ is a Serre-fibration. We will prove that the underlying map of semi-simplicial sets

$$Z(i^*[\bullet]) : Z(c[X][\bullet]) \to Z(c[A][\bullet])$$

of $PZ(i^*)$ is a Kan fibration, whence the result follows since Kan fibrations geometrically realise to Serre-fibrations by a well known result of Quillen (see [Qui67]).

For fixed $n \geq 0$ and $1 \leq k \leq n$, consider the following commutative diagram:

$$\begin{tikzcd}
Z((c[X])[\Delta^n]) \arrow[r, "r_{n,k}"] \arrow[d, "\cong"', "\alpha"] & Z((c[X])[\Lambda^n_k] \times_{Z((c[A])[\Lambda^n_k]} Z((c[A])[\Delta^n]) \arrow[d, "\cong"', "\alpha"] \\
Z(c[X \times \Delta^n]) \arrow[r] \arrow[dr, "r'_{n,k}"'] & Z(c[X \times \Lambda^n_k]) \times_{Z(c[A\times\Lambda^n_k])} Z(c[A \times \Delta^n]) \\
& Z(c[X \times \Lambda^n_k \cup_{A\times\Lambda^n_k} A \times \Delta^n]). \arrow[u, "\cong", "\pi"']
\end{tikzcd} \tag{4.3}$$

The horizontal arrows and the arrows labelled by α in Diagram 4.3 are induced by functoriality of Z and the associativity morphisms of the parametrisation, respectively. Furthermore, the map π is induced by functoriality of Z. Note that π is a homeomorphism by locality of the discrete functor Z. From Lemma 4.5.1 we see that the Kan condition for the map $Z(i^*[\bullet])$ is equivalent to the statement that the maps $r_{k,n}$ are surjective. By commutativity of Diagram 4.3, it thus suffices to show that the maps $r'_{n,k}$ are surjective for all ball complex pairs (X, A). We may reduce the proof via cellular induction to checking just two cases:

Case I: $X = A \sqcup \sigma$, where $\sigma \cong \Delta^0$ is a 0-dimensional ball. In this case, $X \times \Delta^n = A \times \Delta^n \sqcup \sigma \times \Delta^n$ and $X \times \Lambda^n_k \cup_{A\times\Lambda^n_k} A \times \Delta^n = A \times \Delta^n \sqcup \sigma \times \Lambda^n_k$. Then, by locality of Z, the maps $r_{k,n}$ are surjective if and only if the maps

$$Z(c[\sigma \times \Delta^n]) \to Z(c[\sigma \times \Lambda^n_k])$$

are surjective, for all n and k. However, the latter lifting problem is equivalent to the statement that $Z(c[\sigma][\bullet])$ is a Kan semi-simplicial set, by associativity of the parametrisation operator on $\mathcal{C}$, and thus the claim follows.

Case II: $X = A \cup \sigma$ and $A \cap \sigma = \partial\sigma$ for a ball σ of dimension greater than 0. Observe that in this case the inclusion map

$$(X \times \Lambda^n_k) \cup_{(A\times\Lambda^n_k)} (A \times \Delta^n) \hookrightarrow X \times \Delta^n$$

is an elementary expansion of ball complexes. The result is then immediate from the assumption that Z is surjective on expansions. $\square$

Remark 4.5.5. The proof of the Kan property for the functor Z given in the previous proposition is based on a similar argument the case of ad theories described in the proof of [LM14, Lemma 15.12].

In our next lemma, we give conditions on a category with parametrisation structure and discrete functor such that parametric realisation yields a stable functor.

Lemma 4.5.6. *Let $(\mathcal{C}, p, \mu, \alpha)$ be a category with extended parametrisation structure whose parametrisation operator p is combinatorial and deloopable. Let $Z : \mathcal{C} \to \mathbf{Set}_*$ be an absolute and Kan functor. Then the parametric realisation PZ of Z is stable. In particular PZ vanishes on null bordisms.*

Proof. We have already shown that PZ is reduced in the previous proposition. We must then show that, for every object c in $\mathcal{C}$, the following square is homotopy cartesian:

$$\begin{tikzcd} PZ(c[\Omega]) \arrow[r] \arrow[d] & PZ(c[\partial]) \arrow[d] \\ PZ(c[\overline{\{0\}}]) \arrow[r] & PZ(c), \end{tikzcd} \tag{4.4}$$

and moreover that the space $PZ(c[\partial])$ is contractible.

The proof strategy is to construct a commutative cube of semi-simplicial sets:

$$\begin{tikzcd} & Z(c[\Omega][\bullet]) \arrow[rr] \arrow[dl, "\phi"', "\cong"] \arrow[dd] & & Z(c[\partial][\bullet]) \arrow[dl, "\psi"', "\cong"] \arrow[dd] \\ \Omega Z(c[\bullet]) \arrow[rr] \arrow[dd] & & \Lambda Z(c[\bullet]) \arrow[dd] & \\ & Z(c[\overline{\{0\}}][\bullet]) \arrow[rr] \arrow[dl, "\cong"] & & Z(c[\bullet]) \arrow[dl, "="] \\ * \arrow[rr] & & Z(c[\bullet]) & \end{tikzcd}$$

where $\Omega Z(c[\bullet])$ and $\Lambda Z(c[\bullet])$ denote the semi-simplicial loop and path space of the semi-simplicial set $Z(c[\bullet])$ (see [Nic82, p. 13] for definitions). The maps $\Omega Z(c[\bullet]) \hookrightarrow \Lambda Z(c[\bullet])$ and $\Lambda Z(c[\bullet]) \to Z(c[\bullet])$ in the cube are the inclusion and path evaluation map, respectively.

The existence of such a cube suffices since the realisation of the back face yields Square 4.4, and the realisation of the front face is homotopy cartesian by the assumption that $Z(c[\bullet])$ is a Kan semi-simplicial set (cf. [Nic82, Proposition 1.3.5]).

We turn to constructing the isomorphisms of semi-simplicial sets ϕ and ψ: For any $n \in \mathbb{N}$, let S_n^Ω and T_n^Ω denote the ball complex pairs $(\Delta^{n+1}, d_{n+1}\Delta^{n+1} \cup \{n+1\})$ and $(\Delta^n \times \Delta^1, \Delta^n \times \partial\Delta^1)$, respectively; moreover, let S_n^∂ and T_n^∂ denote the ball complex pairs $(\Delta^{n+1}, \{n+1\})$ and $(\Delta^n \times \Delta^1, \Delta^n \times \{1\})$, respectively. Note that we have inclusions $S_n^\partial \hookrightarrow S_n^\Omega$ and $T_n^\partial \hookrightarrow T_n^\Omega$ for all $n \in \mathbb{N}$. The n^{th} components of the semi-simplicial maps ϕ and ψ are then defined as the compositions,

$$\phi_n : Z(c[\Omega][\bullet])_n = Z((c[\Omega])[\Delta^n]) \overset{\alpha}{\cong} Z(c[T_n^\Omega]) \overset{Z(\tilde{\phi}_n^*)}{\cong} Z(c[S_n^\Omega]) \overset{\pi_n}{\cong} \Omega Z(c[\bullet])_n,$$

$$\psi_n : Z(c[\partial][\bullet])_n = Z((c[\partial])[\Delta^n]) \overset{\alpha'}{\cong} Z(c[T_n^\partial]) \overset{Z(\tilde{\psi}_n^*)}{\cong} Z(c[S_n^\partial]) \overset{\pi_n'}{\cong} \Lambda Z(c[\bullet])_n,$$

for arbitrary $n \in \mathbb{N}$, where:

- α and α' denote the corresponding associativity morphisms.
- The isomorphisms π_n and π_n' are determined by the assumption that Z is absolute and discrete. To be more precise, note that for all ball complex

pairs (K, L), the diagram of pointed sets

$$\begin{tikzcd} Z(c[K,L]) \arrow[r] \arrow[d] & Z(c[K]) \arrow[d] \\ pt = Z(c[L,L]) \arrow[r] & Z(c[L]) \end{tikzcd}$$

is a pullback diagram. Considering the cases $(K, L) = S^{\Omega}_n$ and $(K, L) = S^{\partial}_n$ yield the isomorphisms π_n and π'_n, respectively.

- The isomorphisms $\tilde{\phi}^*_n$ and $\tilde{\psi}^*_n$ are induced by the incidence compatible 0-isomorphisms of cell posets

$$\tilde{\phi}_n : \mathrm{cell}(S^{\Omega}_n) \cong \mathrm{cell}(T^{\Omega}_n)$$
$$\tilde{\psi}_n : \mathrm{cell}(S^{\partial}_n) \cong \mathrm{cell}(T^{\partial}_n),$$

suggested by the homeomorphisms of topological spaces:

$$\Delta^n \times \Delta^1/\Delta^n \times \{1\} \cong \Delta^{n+1}$$
$$(u, s) \to ((1-s)u, s).$$

We present an explicit formula for precision. Let o denote the orientation of Δ^n defined via the standard incidence numbers

$$[\langle v_0, \dots v_k\rangle, \langle v_0, \dots, \widehat{v_i}, \dots, v_k\rangle] = (-1)^i,$$

for all pairs $(\langle v_0, \dots v_k\rangle, \langle v_0, \dots, \widehat{v_i}, \dots, v_k\rangle)$ consisting of a k-dimensional face $\langle v_0, \dots v_k\rangle$ and its i^{th} subface $\langle v_0, \dots, \hat{v}_i, \dots, v_k\rangle$.

Then the isomorphism of cell posets $\tilde{\psi}_n : \mathrm{cell}(S^{\partial}_n) \cong \mathrm{cell}(T^{\partial}_n)$ is given by

$$(\langle v_0, \dots, v_k\rangle, o) \mapsto \begin{cases} (\langle v_0, \dots, v_{k-1}\rangle \times \langle 0, 1\rangle, o \times o), & \text{if } v_k = n+1 \\ (\langle v_0, \dots, v_k\rangle \times \langle 0\rangle, o \times o), & \text{if } v_k \neq n+1, \end{cases}$$

where $\langle v_0, \dots v_k\rangle$ is a k-dimensional face of Δ^{n+1}. Furthermore, the restriction of $\tilde{\psi}_n$ along the evident inclusion of posets $\mathrm{cell}(S^{\Omega}_n) \hookrightarrow \mathrm{cell}(S^{\partial}_n)$ determines the isomorphism $\tilde{\phi}_n$.

The fact that the maps ϕ_n and ψ_n indeed assemble to maps of semi-simplicial sets follows from the naturality of the associativity morphisms α together with Condition (3) of combinatoriality (see Definition 4.4.1) of the parametrisation operator p. $\square$

Combining the results of Lemmas 4.5.4, 4.5.6 and 4.4.9 yields our main result regarding properties of parametric realisation of a discrete functor:

Theorem 4.5.7. *Let $(\mathcal{C}, p, \mu, \alpha)$ be a category with extended parametrisation structure and $Z : \mathcal{C} \to \mathbf{Set}_*$ be a functor. Suppose that the parametrisation operator p is combinatorial and deloopable, and that the functor Z is local and surjective on expansions. Then the parametric realisation PZ of Z is linear, i.e., stable, local and homotopy invariant.*

Proof. By assumption, the category with extended parametrisation structure $(\mathcal{C}, p, \mu, \alpha)$ and functor Z satisfy the conditions of both Lemmas 4.5.4 and 4.5.6. Hence, the parametric realisation of Z must be absolute and stable. Since stable functors also vanish on null bordisms by definition, the result follows immediately from the criterium for linearity described in Lemma 4.4.9. $\square$

Remark 4.5.8. Theorem 4.5.7 is inspired by work of S. Buoncristiano, C. P. Rourke and B. J. Sanderson on the construction of homotopy invariant functors $\mathbf{Ball}_2 \to \mathbf{Set}$ (see [BRS76, §§6-7]). In particular, the conditions of locality and surjectivity on expansions are based on the glueing and extension axioms (Axioms E and G) defined there.

As a corollary of Theorem 4.5.7, we obtain a specialised version of the Abstract Universality Theorem in the case of discrete functors defined on a category with extended parametrisation structure:

Theorem 4.5.9. *Let $(\mathcal{C}, p, \mu, \alpha)$ be a category with extended parametrisation structure and $Z : \mathcal{C} \to \mathbf{Set}_*$ a functor. Suppose that the parametrisation operator p is combinatorial and deloopable and that the functor Z is local and surjective on expansions. Then the pair (PZ, ι_Z) is a universal bordism characteristic of Z.*

Moreover, every natural transformation $\eta : PZ \Rightarrow F$ to a stable functor $F : \mathcal{C} \to \mathbf{Top}_*$ extends to a natural transformation $\underline{\eta} : \underline{PZ} \Rightarrow \underline{F}$ of Ω-spectrum-valued functors.

Proof. By Theorem 4.5.7 we deduce that the parametric realisation of Z is linear. In particular, Proposition 4.3.13 implies that PZ must be bordism invariant. Universality of the pair (PZ, ι_Z) now follows directly from Theorem 4.1.7.

The second statement of the theorem is an immediate consequence of Proposition 4.3.7 and the fact that linear functors are stable, by definition. $\square$

Theorem 4.5.9 should be regarded as an exemplar of a result describing conditions directly on a given functor Z and the parametrisation structure such that parametric realisation of Z yields a universal bordism characteristic whose target naturally takes values in infinite loop spaces, and, moreover, such that maps from the target to stable functors respect the delooping.

Chapter 5

Example I: Properties of Quinn's Bordism Machine

In this chapter we review the example of ad theories and Quinn's bordism machine from Chapter 3 in terms of the theory of extended parametrisation structures.

Our main goal is to show that the conditions of the specialised universality Theorem 4.5.9 are satisfied in the case of the closed-objects functor $Z = \mathrm{cl}^0$, thereby highlighting properties of cl^0 that lead to universality of Quinn's bordism machine $\mathcal{Q}$. In parallel, we additionally deduce that Quinn's bordism machine $\mathcal{Q}$ is a linear functor in the sense of Definition 4.3.11. Our results turn out to provide an alternative interpretation of Quinn's bordism machine and properties of bordism groups of ad theories established in [LM14]. The comparison is given at the end of the chapter.

We have divided our analysis into two sections: In Section 5.1, we establish the special properties of the parametrisation structure on the category of ad theories **Ad**. More precisely, we verify that the parametrisation structure is in fact extended, and then prove that the parametrisation operator is combinatorial and deloopable in the sense of Definitions 4.3.1 and 4.4.1, respectively. In Section 5.2, we prove that the closed-objects functor cl^0 is both local and surjective on expansions, and then verify Theorem 4.5.9.

5.1 Further Properties of Parametrisation

Recall from Section 3.2.2, Section 3.2.3, and Proposition 3.2.1, the definitions of the parametrisation operator $\hat{p} : \mathbf{Ball}_2^{op} \times \mathbf{Ad} \to \mathbf{Ad}$, unit $\mu : \mathcal{A}[\Delta^0] \cong \mathcal{A}$, and associativity morphisms $\alpha : (\mathcal{A}[K, L])[M, N] \cong \mathcal{A}[(M, N) \times (K, L)]$, respectively.

Proposition 5.1.1. *The quadruple $(\mathbf{Ad}, \hat{p}, \mu^{-1}, \alpha)$ defines a category with extended parametrisation structure.*

Proof. The proof is a straightforward inspection. In particular, the argument for compatibility of α with the unit μ is analogous to the one given at the end of the proof of Proposition 3.2.8. $\square$

Our next goal is to show that the parametrisation operator $\hat{p}$ on **Ad** is deloopable and combinatorial. The proofs depend on the observation, described below in Lemma 5.1.3, that the reindexing axiom for ad theories admits a stronger reformulation. We introduce the notion of a *k-morphism* in the category of ad theories to formulate the result. Recall the definition of the shift functors Σ from Definition 3.3.3.

Definition 5.1.2. Let $k \in \mathbb{Z}$. A morphism of ad theories $\mathcal{A} \to \Sigma^k \mathcal{B}$, equivalently, $\Sigma^{-k}\mathcal{A} \to \mathcal{B}$, is called a *k-morphism* of ad theories $\mathcal{A} \to \mathcal{B}$.

Lemma 5.1.3. *Let $\mathcal{A}$ be an ad theory. Then, for every incidence compatible k-isomorphism $\theta : (K, L) \to (K', L')$ in $\mathbf{CBall}_2$, the induced k-isomorphism*

$$\theta^* : \mathcal{A}[K', L'] \to \mathcal{A}[K, L]$$

maps ads isomorphically to ads, and thus defines a morphism of ad theories. Moreover, the isomorphisms θ^* are natural in the variable $\mathcal{A}$.

Proof. Let (M, N) be a ball complex and $\theta : (K, L) \to (K', L')$ be an incidence compatible k-isomorphism. Then θ induces an incidence compatible k-isomorphism

$$\begin{aligned} \mathrm{id} \times \theta :& (M, N) \times (K, L) \to (M, N) \times (K', L') \\ & (\sigma, o) \times (\sigma', o') \mapsto i^{k|\sigma|}((\sigma, o) \times \theta(\sigma, o')). \end{aligned}$$

Moreover, the following diagram can be checked to commute by inspecting definitions (the signs of Definition 3.1.16 are needed here):

$$\begin{tikzcd} \mathrm{pre}_{\mathcal{A}[K',L']}(M, N) \arrow[r, "\mathrm{pre}_{\theta^*}(M,N)"] \arrow[d, "\alpha"] & \mathrm{pre}_{\mathcal{A}[K,L]}(M, N) \arrow[d, "\alpha"] \\ \mathrm{pre}_{\mathcal{A}}((M, N) \times (K', L')) \arrow[r, "(\mathrm{id} \times \theta)^*"] & \mathrm{pre}_{\mathcal{A}}((M, N) \times (K, L)). \end{tikzcd}$$

The statement of the lemma would follow if the map $\mathrm{pre}_{\theta^*}(M, N)$ preserves ads, but this is immediate from the reindexing axiom for the ad theory $\mathcal{A}$. $\square$

Corollary 5.1.4. *The parametrisation operator $\hat{p}$ on* **Ad** *is combinatorial.*

Proof. By the previous lemma, each 0-morphism $\theta : (K, L) \cong (K', L')$ of ball complexes induces a natural isomorphism $\theta^* : \mathcal{A}[K, L] \cong \mathcal{A}[K', L']$ of ad theories. It is straightforward to check that the conditions of Definition 4.4.1 are satisfied. $\square$

Lemma 5.1.5. *The parametrisation operator $\hat{p}$ on* **Ad** *is deloopable, i.e., the functor*

$$\begin{aligned} \mathbf{Ad} &\to \mathbf{Ad}, \\ \mathcal{A} &\mapsto \mathcal{A}[\Omega], \end{aligned}$$

is an equivalence of categories. An inverse is given by the shift functor $\Sigma : \mathbf{Ad} \to \mathbf{Ad}$.

Proof. We define an incidence compatible (-1)-isomorphism of cell posets:

$$\begin{aligned} s : \mathrm{cell}(\Delta^0) &\to \mathrm{cell}(\Delta^1, \partial\Delta^1) \\ (\langle 0 \rangle, o) &\mapsto (\langle 0, 1 \rangle, o). \end{aligned}$$

$$\begin{tikzcd} & & s(a) & \\ a & \emptyset_0 \arrow[ur] & & \emptyset_0 \arrow[ul] \end{tikzcd}$$

Figure 5.1: A partial diagram of $\mathrm{cell}(\Delta^0)$ and $\mathrm{cell}(\Delta^1, \partial\Delta^1)$, showing the cell $a := (\langle 0 \rangle, o) \in \mathrm{cell}(\Delta^0)$ and its image $s(a)$ in $\mathrm{cell}(\Delta^1, \partial\Delta^1)$ under s. The basepoint $\emptyset_0$ and arrow $\emptyset_0 \to s(a)$ in $\mathrm{cell}(\Delta^1, \partial\Delta^1)$ are drawn doubled and underneath $s(a)$ to indicate the dimension shift and to suggest the interpration of $s(a)$ as a 'bordism with empty boundary'.

See Figure 5.1 for a geometric illustration. By Lemma 5.1.3, we obtain a natural isomorphism

$$\Sigma(\mathcal{A}[\Omega]) \overset{s^*}{\cong} \mathcal{A}[\Delta^0] \overset{\mu}{\cong} \mathcal{A}.$$

The result now follows from the observation that the shift functors commute with the parametrisation operator $\hat{p}$ in the sense that

$$(\Sigma^k \mathcal{A})[K, L] = \Sigma^k(\mathcal{A}[K, L])$$

as ad theories, for every ad theory $\mathcal{A}$, ball complex pair (K, L) and integer k.

We will make this observation explicit for completeness. Let $\mathcal{A}$ be an ad theory, (K, L) a ball complex pair and $k \in \mathbb{Z}$. Notice that the underlying $\mathbb{Z}$-graded categories of $(\Sigma^k \mathcal{A})[K, L]$ and $\Sigma^k(\mathcal{A}[K, L])$ are equal. Indeed, for each $n \in \mathbb{Z}$, we have equalites of sets:

$$\begin{aligned} \mathrm{ob}^n\big((\Sigma^k \mathcal{A})[K, L]\big) &= \{(K, L)\text{-preads in } \Sigma^k \mathcal{A} \text{ of dimension } n\} \\ &= \{(K, L)\text{-preads in } \mathcal{A} \text{ of dimension } n - k\} \\ &= \mathrm{ob}^{n-k}\big(\mathcal{A}[K, L]\big) \\ &= \mathrm{ob}^n\big(\Sigma^k(\mathcal{A}[K, L])\big). \end{aligned}$$

Moreover, the morphisms sets in both cases are just the natural transformations. Finally, for any ball complex (M, N), observe that:

$$\begin{aligned} \mathrm{ad}^n_{(\Sigma^k \mathcal{A})[K,L]}(M, N) &\overset{\alpha}{\cong} \mathrm{ad}^n_{\Sigma^k \mathcal{A}}((M, N) \times (K, L)) \\ &\overset{\mathrm{def}}{=} \mathrm{ad}^{n+k}_{\mathcal{A}}((M, N) \times (K, L)) \\ &\overset{\alpha}{\cong} \mathrm{ad}^{n+k}_{\mathcal{A}[K,L]}(M, N) \\ &\overset{\mathrm{def}}{=} \mathrm{ad}^n_{\Sigma^k(\mathcal{A}[K,L])}(M, N). \end{aligned}$$

Hence, the ad structures, $\mathrm{ad}^n_{(\Sigma^k \mathcal{A})[K,L]}$ and $\mathrm{ad}^n_{\Sigma^k(\mathcal{A}[K,L])}$, agree. $\square$

5.2 Properties of Closed Objects and Quinn's Machine

We next study properties of the closed-objects functor cl^0.

Lemma 5.2.1. *The functor* cl^0 *is local.*

Proof. Let $\mathcal{A}$ be an ad theory and consider a ball complex K. The inclusion $\emptyset \hookrightarrow (K, K)$ evidently induces an isomorphism of cell posets $\mathrm{cell}(K, K) \cong \mathrm{cell}(\emptyset)$. Hence, the ad structure of the categories $\mathcal{A}[K, K]$ is trivial. It follows that the set $\mathrm{cl}^0(\mathcal{A}[K, K])$ consists of a point, and thus the functor cl^0 is reduced.

Next, let $j : L \to K$ be an inclusion of ball complexes and consider the following square:

$$\begin{array}{ccc}
\mathrm{cl}^0(\mathcal{A}[K, L]) & \xrightarrow{\mathrm{cl}^0(\varepsilon^*_{(K,L)})} & \mathrm{cl}^0(\mathcal{A}[K]) \\
\downarrow & & \downarrow{\scriptstyle \mathrm{cl}^0(j^*)} \\
\mathrm{cl}^0(\mathcal{A}[L, L]) & \longrightarrow & \mathrm{cl}^0(\mathcal{A}[L]).
\end{array}$$

It follows from the absolute axiom for the ad theory $\mathcal{A}$ and the assumption that basepoints in $\mathcal{A}$ are initial that the square is a pullback square. In particular, the square is homotopy cartesian since all spaces involved are discrete.

Finally, let X and Y be ball complexes. We claim that the restriction map

$$r : \mathrm{cl}^0(\mathcal{A}[X \cup Y]) \to \mathrm{cl}^0(\mathcal{A}[X]) \times_{\mathrm{cl}^0(\mathcal{A}[X \cap Y])} \mathrm{cl}^0(\mathcal{A}[Y])$$

is an isomorphism. First, note that the map r is injective since any functor

$$F : \mathrm{cell}(X \cup Y) \to \mathcal{A}$$

is evidently determined by its restriction to the subposets $\mathrm{cell}(X)$ and $\mathrm{cell}(Y)$. For surjectivity, let (x, y) be a given element of $\mathrm{cl}^0(\mathcal{A}[X]) \times_{\mathrm{cl}^0(\mathcal{A}[X \cap Y])} \mathrm{cl}^0(\mathcal{A}[Y])$, and let

$$z : \mathrm{cell}(X \cup Y) \to \mathcal{A}$$

be the $(X \cup Y)$-pread defined by:

$$\begin{aligned}
z(\sigma, o) &= x(\sigma, o), \text{ if } (\sigma, o) \in \mathrm{cell}(X), \\
z(\sigma, o) &= y(\sigma, o), \text{ if } (\sigma, o) \in \mathrm{cell}(Y), \\
z(f) &= x(f), \text{ if } f \in \mathrm{Mor}(\mathrm{cell}(X)), \\
z(f) &= y(f), \text{ if } f \in \mathrm{Mor}(\mathrm{cell}(Y)).
\end{aligned}$$

Observe that the pread z is an ad in $\mathcal{A}$, by the locality axiom for the ad theory $\mathcal{A}$. A preimage of (x, y) under r may now be obtained from z via the identification $\mathrm{ad}^0_{\mathcal{A}}(X \cup Y) \cong \mathrm{cl}^0(\mathcal{A}[X \cup Y])$ given by the following composition:

$$\mathrm{ad}^0_{\mathcal{A}}(X \cup Y) \cong \mathrm{ad}^0_{\mathcal{A}}(\Delta^0 \times (X \cup Y)) \overset{\alpha}{\cong} \mathrm{ad}^0_{\mathcal{A}[X \cup Y]}(\Delta^0) = \mathrm{cl}^0(\mathcal{A}[X \cup Y]),$$

where the first isomorphism is induced by the projection $\Delta^0 \times (X \cup Y)) \cong X \cup Y$. $\square$

It was proven in [LM14, Lemma 14.7] that, for every ad theory $\mathcal{A}$ and every elementary expansion $(K_1, L_1) \to (K, L)$, the restriction of sets of ads

$$\mathrm{ad}^k(K, L) \to \mathrm{ad}^k(K_1, L_1)$$

is surjective, for every integer $k \in \mathbb{Z}$. As a direct corollary, we obtain the following lemma:

Lemma 5.2.2. *The functor* cl^0 *is surjective on expansions.*

Corollary 5.1.4, Lemmas 5.2.1 and 5.2.2, and Proposition 5.1.5 imply that the conditions of Theorem 4.5.7 and Theorem 4.5.9 are satisfied for the category with extended parametrisation $(\mathbf{Ad}, \hat{p}, \mu^{-1}, \alpha)$ and the closed-objects functor cl^0. The following results are thus immediate consequences:

Theorem 5.2.3. *The pair* $(\mathcal{Q}, \iota_{\mathrm{cl}^0})$ *is a universal bordism characteristic of* cl^0. *Moreover, every natural transformation* $\eta : \mathcal{Q} \Rightarrow F$ *to a stable functor* $F : \mathbf{Ad} \to \mathbf{Top}_*$ *extends to a natural transformation* $\underline{\eta} : \underline{\mathcal{Q}} \Rightarrow \underline{F}$ *of spectrum-valued functors.*

Theorem 5.2.4. *Quinn's bordism machine* $\mathcal{Q}$ *is linear, i.e., local, stable and homotopy invariant.*

Theorem 5.2.3 should be regarded as an extension of Theorem 3.3.9 with the extra property that maps from Quinn's bordism machine to a stable functor extend to maps of their associated spectra. In the following remarks, we explain how Theorem 5.2.4 compares to results already established in [LM14]:

Remark 5.2.5. It is proven in [LM14, Proposition 15.9] that the functor $\mathcal{Q}$ extends to an Ω-spectrum-valued functor via a direct construction. Theorem 5.2.4 reinforces this result and, in addition, shows that the stability of the functor $\mathcal{Q}$ can be deduced as a consequence of formal properties of the functor cl^0 and the parametrisation structure on **Ad**.

Remark 5.2.6. Associated to any ad theory are abelian groups, called bordism groups and denoted by $T^k(K, L)$, for any ball complex pairs (K, L) and integer k (see [LM14, §14]). These were shown to form a cohomology theory on the category of finite CW pairs (see [LM14, Theorem 14.11]).

It is not hard to see that the homotopy groups of the spaces $\mathcal{Q}(\mathcal{A}[K, L])$ are naturally isomorphic to the bordism groups of an ad theory $\mathcal{A}$; an argument in the case $(K, L) = (\Delta^0, \emptyset)$ is illustrated in [BLM19, Remark 3.1], and the general case is similar. The idea is to exploit the Kan description of homotopy groups (e.g., see [Nic82, p. 11]) and the combinatorial axiom for ad theories. We may therefore translate the condition of homotopy invariance of the functor $\mathcal{Q}$ as the homotopy invariance of the bordism groups $T^*(K, L)$. The latter statement had been proven in [LM14, Proposition 14.2] and Theorem 5.2.4 recovers this fact.

Remark 5.2.7. The property that $\mathcal{Q}$ is local, in particular absolute, implies the existence of a long exact sequence of bordism groups for an arbitrary ball complex pair (K, L). A direct construction of such a sequence was given in [LM14, Definition 14.10]. The additional insight made here is that the existence of the sequence comes from a fibre square and that the maps

$\mathcal{Q}(\mathcal{A}[K]) \to \mathcal{Q}(\mathcal{A}[L])$ are fibrations with fibre $\mathcal{Q}(\mathcal{A}[K, L])$ (compare the proof of Lemma 4.5.4).

Chapter 6

Example II: Symmetric L-Theory of Additive Categories with Chain Duality

This final chapter is devoted to a second illustration of the theory of extended parametrisation structures. Specifically, we return to consider symmetric L-theory, but now our analysis takes place in the algebraic setting of *additive categories with chain duality* (ACCDs) introduced by A. A. Ranicki. For background material, we refer the reader to [Ran92] and [AFM18]. We note that our exposition has also benefited from [Wei09], [RW12], [KMM13] and [CLM21].

The setting of ACCDs has two advantages over the more general setting of Waldhausen categories with duality considered in Chapter 2: Firstly, it is more concrete in the sense that every ACCD is an example of a WCD (see [WW98, Example 1.A.1]). Secondly, a description of an extended parametrisation structure on the category of ACCDs has already been partially developed in [AFM18]. Our analysis is intended to be complementary to [AFM18], and consists of a summary of partial results aimed toward showing that symmetric L-theory of additive categories with chain duality is the universal bordism characteristic of symmetric Poincaré complexes.

We have organised the chapter as follows: Section 6.1 describes the setting of additive categories with chain dualities and the notion of symmetric Poincaré complexes based on [Ran92].

Section 6.2 is dedicated to the development of an extended parametrisation structure on the category of ACCDs and its properties. We build on results of [AFM18]. Therein, a construction of ACCDs parametrised over any oriented ball complex X is given. However, functoriality of the construction is not discussed. We show the construction is functorial and record the result in Theorem 6.2.17. The proof requires careful attention to orientations and is the main technical part of this chapter. We have devoted Subsections 6.2.1 and 6.2.2 to the setup and proof. In addition to establishing the existence of a parametrisation operator on the category of ACCDs, we will also show that it yields an extended parametrisation structure such that the parametrisation operator is deloopable (Theorem 6.2.28). This latter result is the subject of Section 6.2.3 and should be regarded as the main result of this chapter.

In the final Section 6.3, we prove that the functor taking symmetric Poincaré complexes of an additive category with chain duality is absolute in the sense of Definition 4.3.10. In addition, we provide an outlook on the universality of symmetric L-theory in the context of ACCDs.

6.1 Chain Dualitites and Symmetric Poincaré Complexes

We define a category of additive categories with duality based on [Ran92, Definitions 3.2 and 3.7].

Notation 6.1.1. For an additive category $\mathbb{A}$, we denote by $\mathbb{B}(\mathbb{A})$ the additive category of bounded chain complexes in $\mathbb{A}$ together with chain maps. In addition, we let $\zeta_{\mathbb{A}} : \mathbb{A} \to \mathbb{B}(\mathbb{A})$ denote inclusion into degree 0.

Definition 6.1.2.

1. An *additive category with chain duality* (ACCD) is a triple $(\mathbb{A}, T, e)$ where:
   - $\mathbb{A}$ is an additive category,
   - $T : \mathbb{A} \to \mathbb{B}(\mathbb{A})$ is a contravariant additive functor, and
   - $e : T^2 \Rightarrow \zeta_{\mathbb{A}} : \mathbb{A} \to \mathbb{B}(\mathbb{A})$ is a natural transformation

   such that the following conditions hold for all objects M of $\mathbb{A}$:

   (a) $e(T(M)) \circ T(e(M)) = \mathrm{id}_{T(M)} : T(M) \to T^2(T(M)) \to T(M)$.
   (b) $e : T^2(M) \to M$ is a chain equivalence.

   The pair (T, e) is called a *chain duality* on $\mathbb{A}$. Furthermore, the chain complex $T(C)$ is called the *dual* of the chain complex C for any chain complex $C \in \mathbb{B}(\mathbb{A})$.

2. A *morphism* $(\mathbb{A}, T, e) \to (\mathbb{A}', T', e')$ *of additive categories with chain duality* consists of a pair (F, G), where
   - $F : \mathbb{A} \to \mathbb{A}'$ is an additive functor, and
   - $G : T'F \Rightarrow FT : \mathbb{A} \to \mathbb{B}(\mathbb{A}')$ is a natural transformation,

   such that for all objects M in $\mathbb{A}$,

   (a) the chain map
   $$G(M) : T'F(M) \xrightarrow{\simeq} FT(M)$$
   is a chain equivalence, and
   (b) the following diagram commutes in $\mathbb{B}(\mathbb{A}')$:
   $$\begin{tikzcd} T'FT(M) \arrow[r, "GT(M)"] \arrow[d, "T'G(M)"'] & FT^2(M) \arrow[d, "Fe(M)"] \\ T'^2F(M) \arrow[r, "e'F(M)"'] & F(M). \end{tikzcd}$$

The composition of two morphisms $(F_1, G_1) : (\mathbb{A}, T, e) \to (\mathbb{A}', T', e')$ and $(F_2, G_2) : (\mathbb{A}', T', e') \to (\mathbb{A}'', T'', e'')$ is defined as the pair (F_3, G_3) where

- $F_3 = F_2 \circ F_1 : \mathbb{A} \to \mathbb{A}''$ and
- $G_3(M) = F_2G_1(M) \circ G_2F_1(M) : T''F_3(M) \Rightarrow F_3T(M)$, for all $M \in \mathrm{ob}(\mathbb{A})$.

We denote the category of additive categories with chain duality by **ACCD**.

Remark 6.1.3. To keep notation simple, we use the same notation for a functor $T : \mathbb{A} \to \mathbb{B}(\mathbb{A})$ and its extension $T : \mathbb{B}(\mathbb{A}) \to \mathbb{B}(\mathbb{A})$ (as defined on [Ran92, p. 25]); likewise for natural transformations thereof (cf. [AFM18, Proposition 3.1]).

Remark 6.1.4. Morphisms between ACCDs are usually considered in the more general context of algebraic bordism categories (see [Ran92, Ch. 3]). We only consider ACCDs here for simplicity.

Note also that, in contrast to [Ran92], we include the natural transformation G as *data* in a morphism of ACCDs. This is necessary in order to define a *functor* of symmetric Poincaré complexes (see Proposition 6.1.10 below).

Proposition 6.1.5. *The composition of morphisms in* **ACCD** *is well defined.*

Proof. Consider two arbitrary morphisms $(F_1, G_1) : (\mathbb{A}, T, e) \to (\mathbb{A}', T', e')$ and $(F_2, G_2) : (\mathbb{A}', T', e') \to (\mathbb{A}'', T'', e'')$. Property $2(a)$ for the composite (F_3, G_3) is clearly satisfied. To establish Property $2(b)$ we must show that, for any fixed $M \in \mathrm{ob}(\mathbb{A})$, the outer square of the following diagram commutes:

$$\begin{tikzcd}
T''F_2F_1T(M) \arrow[r, "G_2F_1T(M)"] \arrow[d, "T''F_2G_1(M)"'] & F_2T'F_1T(M) \arrow[r, "F_2G_1T(M)"] \arrow[d, "F_2T'G(M)"] & F_2F_1T^2(M) \arrow[dd, "F_2F_1e(M)"] \\
T''F_2T'F_1(M) \arrow[r, "G_2T'F_1(M)"] \arrow[d, "T''G_2F_1(M)"'] & F_2(T')^2F_1(M) \arrow[dr, "F_2e'F_1(M)"] & \\
(T'')^2F_2F_1(M) \arrow[rr, "e''F_2F_1(M)"] & & F_2F_1(M).
\end{tikzcd}$$

Indeed, the top-left square commutes by naturality of G_2, and the other inner squares commute as applications of Property $2(b)$ to the morphisms (F_1, G_2) and (F_2, G_2). □

The concept of chain duality in an additive category gives rise to the notion of symmetric Poincaré complexes in an additive category $\mathbb{A}$. We briefly recall the construction here and refer to [AFM18, pp. 6-8] and [Ran92, pp. 27-30] for more details.

Let $(\mathbb{A}, T, e)$ be an ACCD and let **Ab** denote the additive category of abelian groups. The functor T induces a pairing

$$- \otimes_{\mathbb{A}} - : \mathbb{B}(\mathbb{A}) \times \mathbb{B}(\mathbb{A}) \to \mathbb{B}(\mathbf{Ab})$$

given by

$$C \otimes_{\mathbb{A}} D := \mathrm{Hom}_{\mathbb{A}}(T(C), D),$$

where the chain complex of abelian groups $\mathrm{Hom}_{\mathbb{A}}(T(C), D)$ is defined as in [Ran92, p. 26]. Furthermore, the pair (T, e) yields natural switch isomorphisms

$$\tau_{C,D} : C \otimes_{\mathbb{A}} D \to D \otimes_{\mathbb{A}} C$$

given by the composition of natural isomorphisms

$$C \otimes D \xrightarrow{T} D \otimes T^2(C) \xrightarrow{id_D \otimes e(C)} D \otimes C.$$

The switch isomorphisms satisfy $\tau_{C,D}^{-1} = \tau_{D,C}$, for every chain complexes C, D in $\mathbb{A}$. In particular, there are canonical involutions $\tau_{C,C}$ on the chain complexes of abelian groups $C \otimes C$, for every bounded chain complex C in $\mathbb{A}$.

Notation 6.1.6.

1. Let $\mathbb{Z}_2 = \{1, t\}$ denote the cyclic group of order 2 and $\mathbb{Z}[\mathbb{Z}_2]$ denote the group ring of $\mathbb{Z}_2$ over the ring of integers $\mathbb{Z}$. Furthermore, let W denote the standard free $\mathbb{Z}[\mathbb{Z}_2]$-module resolution of $\mathbb{Z}$, i.e.,

$$W : \cdots \to \mathbb{Z}[\mathbb{Z}_2] \overset{1-t}{\to} \mathbb{Z}[\mathbb{Z}_2] \overset{1+t}{\to} \mathbb{Z}[\mathbb{Z}_2] \overset{1-t}{\to} \mathbb{Z}[\mathbb{Z}_2].$$

2. Let $\phi_0 : T(C) \to C$ denote the underlying chain map of a 0-cycle $\phi \in \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes_{\mathbb{A}} C)$, i.e,

$$\phi_0 := \phi(1) \in \mathrm{Hom}(T(C), C)_0,$$

   where $1 \in W_0 = \mathbb{Z}[\mathbb{Z}_2]$ is the unit.

Definition 6.1.7. A *symmetric complex* in an additive category with chain duality $(\mathbb{A}, T, e)$ is a pair (C, ϕ) consisting of:

- a finite chain complex C in $\mathbb{A}$, and
- a 0-cycle $\phi \in \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes_{\mathbb{A}} C)_0$, called the *symmetric structure* on C.

If the underlying chain map of ϕ is a chain equivalence $\phi_0 : T(C) \overset{\simeq}{\to} C$, we call the pair (C, ϕ) a *symmetric Poincaré complex*.

Remark 6.1.8. A symmetric Poincaré complex in the sense of the definition above is usually called a 0-dimensional symmetric Poincaré complex in the literature (e.g., see [Ran92, Definition 1.6]). We have dropped the dimension prefix for simplicity as it does not play role in our work.

Definition 6.1.9. Let $(\mathbb{A}, T, e)$ be an additive category with chain duality. We denote the pointed set of symmetric Poincaré complexes in $\mathbb{A}$ by $\mathrm{sp}(\mathbb{A})$, where the basepoint is given by the zero object in $\mathbb{B}(\mathbb{A})$ together with its unique symmetric structure.

In the next proposition, we show that the construction of the sets $\mathrm{sp}(\mathbb{A})$ are functorial with respect to morphisms of ACCDs. This fact was proven as part of the proof of [Ran92, Proposition 3.8]. More precisely, it was shown there that a morphism of algebraic bordism categories induces a map on L-groups. We recall the proof in order to demonstrate the need of the natural transformation G as data in a morphism (F, G) of ACCDs.

Proposition 6.1.10. *The assignment*

$$(\mathbb{A}, T, e) \mapsto \mathrm{sp}(\mathbb{A})$$

induces a functor

$$\mathbf{ACCD} \to \mathbf{Set}_* \, .$$

Proof. Let $(F, G) : (\mathbb{A}, T, e) \to (\mathbb{A}', T', e')$ be a morphism of ACCDs. The map

$$\mathrm{sp}(F, G) : \mathrm{sp}(\mathbb{A}) \to \mathrm{sp}(\mathbb{A}')$$

of pointed sets is induced by a natural transformation

$$(F_* G^*)_{C,D} : C \otimes_{\mathbb{A}} D \to F(C) \otimes_{\mathbb{A}'} F(D) : \mathbb{B}(\mathbb{A}) \times \mathbb{B}(\mathbb{A}) \to \mathbb{B}(\mathbf{Ab})$$

defined as follows: For every objects M, N in $\mathbb{A}$, let

$$(F_* G^*)_{M,N} : M \otimes_{\mathbb{A}} N \to F(M) \otimes_{\mathbb{A}'} F(N)$$

be the map of abelian group chain complexes given by

$$\begin{aligned} &(F_* G^*)_{M,N} : M \otimes_{\mathbb{A}} N \to F(M) \otimes_{\mathbb{A}'} F(N) \\ &\qquad (\phi : T(M) \to N) \mapsto (F(\phi) \circ G(M) : T' F(M) \to FT(M) \to F(N)). \end{aligned}$$

It can be seen that the natural transformation $F_* G^*$ commutes with the switch isomorphisms so that the morphism $(F_* G^*)_{C,C}$ induces a well-defined morphism of chain complexes

$$(F_* G^*)_{C,C} : \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes_{\mathbb{A}} C) \to \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, F(C) \otimes_{\mathbb{A}'} F(C))$$

for any chain complex C in $\mathbb{B}(\mathbb{A})$. The map $\mathrm{sp}(F, G)$ is now given by sending a symmetric Poincaré complex (C, ϕ) in $\mathbb{A}$ to the symmetric Poincaré complex $(F(C), (F_* G^*)_{C,C}(\phi))$. ☐

6.2 The Extended Parametrisation Structure

The main goal of this section is to prove that there exists an extended parametrisaion operator on the category **ACCD**. Our results expand upon the construction of ACCDs parametrised over *oriented* ball complexes given in [AFM18]. The role of orientations is suppressed in [AFM18], however it is clear that different choices of orientation lead to isomorphic definitions. On the contrary, we will take orientation into account in order to give a careful proof of functoriality of the parametrisation construction.

We have divided this section into three parts: In Section 6.2.1, we recall the description of ACCDs parametrised over a ball complex, highlighting the dependancy on orientations. Then, we extend the construction to ball complex pairs. In Section 6.2.2, we prove that the construction is functorial. Lastly in Section 6.2.3, we show that the operation of parametrisation equips the category **ACCD** with an extended parametrisation structure, whose parametrisation operator is deloopable.

6.2.1 ACCDs Parametrised over a Ball Complex

Recall (see Definition 3.1.5) the notation $\mathcal{F}(X)$ for the face poset of a ball complex X. We define additive categories parametrised over a ball complex following [AFM18, Definitions 4.5 and 4.2]:

Definition 6.2.1. Let X be a ball complex and $\mathbb{A}$ an additive category.

1. An object M in $\mathbb{A}$ is X*-based* if it is expressed as a direct sum
$$M = \bigoplus_{\sigma \in \mathcal{F}(X)} M(\sigma)$$
of objects $M(\sigma)$ in $\mathbb{A}$.

2. We define $\mathbb{A}^*(X)$ (resp. $\mathbb{A}_*(X)$) to be the additive categories of X-based objects M in $\mathbb{A}$, where a morphism $f : M \to N$ is a collection of morphisms
$$f = \{f_{\tau,\sigma} : M(\sigma) \to N(\tau) : \tau, \sigma \in \mathcal{F}(X)\}$$
in $\mathbb{A}$, such that $f_{\tau,\sigma} : M(\sigma) \to N(\tau)$ is zero unless $\tau \leq \sigma$ (resp. $\tau \geq \sigma$). The composition of two morphisms $f : L \to M, g : M \to N$ is the morphism $g \circ f : L \to N$ defined by
$$(g \circ f)_{\rho,\sigma} := \bigoplus_{\tau \in \mathcal{F}(X)} g_{\rho,\tau} f_{\tau,\sigma} : L(\sigma) \to N(\rho).$$

Remark 6.2.2. The morphisms in $\mathbb{A}^*(X)$ (resp. $\mathbb{A}_*(X)$) can be regarded as upper (resp. lower) triangular matrices with composition given by matrix multiplication.

Definition 6.2.3. Let $\mathbb{A}$ be an additive category and X be a ball complex pair. We let $\mathbb{A}^*[X]$ (resp. $\mathbb{A}_*[X]$) denote the additive category whose objects are covariant (resp. contravariant) functors
$$M : \mathcal{F}(X) \to \mathbb{A},$$
and morphisms are natural transformations of such functors.

Fix an ACCD $(\mathbb{A}, T, e)$ and oriented ball complex (X, o). The duality functor $T_{(X,o)} : \mathbb{A}^*(X) \to \mathbb{B}(\mathbb{A}^*(X))$ on $\mathbb{A}^*(X)$ is defined in [AFM18, Definition 5.1] as a composition of functors:
$$T_{(X,o)} : \mathbb{A}^*(X) \xrightarrow{[-]} \mathbb{A}^*[X] \xrightarrow{T_*} \mathbb{B}(\mathbb{A})_*[X] \xrightarrow{sh_{(X,o)}} \mathbb{B}(\mathbb{A}^*(X)).$$

We will recall the definition of each of these functors following [AFM18, Definitions 4.3, 4.14 and 4.25].

Definition 6.2.4. Let $\mathbb{A}$ be an additive category and X a ball complex. The covariant functor
$$[-] : \mathbb{A}^*(X) \to \mathbb{A}^*[X]$$

assigns to an object M of $\mathbb{A}^*(X)$ the functor $[M] : \mathcal{F}(X) \to \mathbb{A}$ given by

$$[M](\sigma) := \bigoplus_{\tau \leq \sigma} M(\tau),$$
$$[M](\sigma \to \sigma') := [M](\sigma) \hookrightarrow [M](\sigma'),$$

for all $\sigma \in \mathcal{F}(X)$ and $\sigma \leq \sigma' \in \mathcal{F}(X)$.

Moreover, $[-]$ assigns to a morphism $f : M \to N$ in $\mathbb{A}^*(X)$ the natural transformation

$$[f] : [M] \Rightarrow [N]$$

given by

$$[f](\sigma) = \bigoplus_{\rho \leq \tau \leq \sigma} f_{\tau,\rho} : \bigoplus_{\rho \leq \sigma} M(\rho) \to \bigoplus_{\tau \leq \sigma} N(\tau)$$

for all $\sigma \in \mathcal{F}(X)$.

Remark 6.2.5. The functor $[-]$ is denoted $\mathcal{I}_{X,\mathbb{A}}$ in [AFM18]. We prefer to use the square bracket notation from [Ran92].

Definition 6.2.6. Let $(\mathbb{A}, T, e)$ be an ACCD and X be a ball complex. Define $T_* : \mathbb{A}^*[X] \to \mathbb{B}(\mathbb{A})_*[X]$ to be the contravariant functor given by postcomposition with T.

The definition of the functor $sh_{(X,o)}$ involves the cellular chain complex with $\mathbb{Z}$-coefficients $C(X;\mathbb{Z})$ of an oriented ball complex (X, o). We define the latter first in terms of the suspension functor on chain complexes.

Notation 6.2.7. Let S^k denote the signed suspension functor of Ranicki (cf. [Ran92, p. 25]) defined by assigning to a chain complex (C, d) in an arbitrary additive category $\mathbb{A}$ the chain complex given by $(S^k C)_n := C_{n-k}$ and $d^n_{S^k C} = (-1)^k d^{n-k}$, for all $n \in \mathbb{Z}$.

We regard $C(X;\mathbb{Z})$ as an object in $\mathbb{B}(\mathbf{Ab}^*(X))$ with $C(X;\mathbb{Z})(\sigma) = S^{|\sigma|}\mathbb{Z}$, for all $\sigma \in \mathcal{F}(X)$, and differentials given by incidence numbers with respect to the given orientation on X.

Definition 6.2.8. (cf. [AFM18, Example 4.26]) Let (X, o) be an oriented ball complex. The corresponding *shift functor*

$$sh_{(X,o)} : \mathbb{B}(\mathbb{A})_*[X] \to \mathbb{B}(\mathbb{A}^*(X))$$

assigns to a given contravariant functor $C : \mathcal{F}(X) \to \mathbb{B}(\mathbb{A})$ the chain complex $sh_{(X,o)}(C)$ in $\mathbb{B}(\mathbb{A}^*(X))$ with components given by

$$sh_{(X,o)}(C)_n(\sigma) := C(\sigma)_{n-|\sigma|},$$

for all $n \in \mathbb{Z}$ and $\sigma \in \mathcal{F}(X)$, and differentials $(d_{sh_{(X,o)}(C)})^n_{\tau,\sigma}$ equal to

$$(d_{C(X;\mathbb{Z}))})^{|\sigma|}_{\tau,\sigma} \otimes_{\mathbb{Z}} C(\tau \to \sigma)_{n-|\sigma|} + (-1)^{|\sigma|} (\mathrm{id}_{C(X,\mathbb{Z})})^{|\sigma|}_{\tau,\sigma} \otimes_{\mathbb{Z}} d^{n-|\sigma|}_{C(\sigma)},$$

for all $n \in \mathbb{Z}$ and $\tau \leq \sigma \in \mathcal{F}(X)$.

Furthermore, the functor $sh_{(X,o)}$ assigns to every morphism $f : C \to D$ in $\mathbb{B}(\mathbb{A})_*[X]$ the morphism

$$sh_{(X,o)}(f) : sh_{(X,o)}(C) \to sh_{(X,o)}(D)$$

with components

$$(sh_{(X,o)}(f))^n_{\tau,\sigma} := (\mathrm{id}_{C(X,\mathbb{Z})})^{|\sigma|}_{\tau,\sigma} \otimes_{\mathbb{Z}} f(\sigma)_{n-|\sigma|},$$

for all $n \in \mathbb{Z}$ and $\tau \leq \sigma \in \mathcal{F}(X)$.

Remark 6.2.9. The tensor product $\otimes_{\mathbb{Z}}$ is defined in [AFM18, Definition 4.21] and depends on a choice of basis for the $\mathbb{Z}$-modules $C(X;\mathbb{Z})_n(\sigma)$. This is the reason we work with explicit orientations of X. It is well known (e.g., see [CF67, p. 54, Theorem 5.4]) that other choices of orientation of X lead to isomorphic cellular chain complexes. From this fact, it can be easily seen that the corresponding shift functors are isomorphic (cf. [AFM18, Proposition 4.22]).

The duality functors $T_{(X,o)}$ yield pairings

$$- \otimes_{\mathbb{A}^*(X)} - : \mathbb{B}(\mathbb{A}^*(X)) \times \mathbb{B}(\mathbb{A}^*(X)) \to \mathbb{B}(\mathbf{Ab}_*(X))$$

and corresponding switch isomorphisms

$$\tau_{M,T_{(X,o)}(M)} : M \otimes_{\mathbb{A}^*(X)} T_{(X,o)}(M) \to T_{(X,o)}(M) \otimes_{\mathbb{A}^*(X)} M,$$

where $M \in \mathbb{A}^*(X)$ and (X,o) is an arbitrary oriented ball complex. We refer to Definition 4.5 and Proposition 5.7 of [AFM18], respectively, for the explicit constructions.

Definition 6.2.10. Let $(\mathbb{A}, T, e)$ be an ACCD and let (X,o) be an oriented ball complex. The ACCD of X-based objects in $\mathbb{A}$ with respect to the orientation o is defined as the triple $(\mathbb{A}^*(X), T_{(X,o)}, e_{(X,o)})$, where the natural transformation $e_{(X,o)} : T^2_{(X,o)} \Rightarrow \iota_{\mathbb{A}^*(X)}$ is given by

$$e_{(X,o)}(M) = \tau_{M,T_{(X,o)}(M)}(\mathrm{id}_{T_{(X,o)}(M)}), \text{ for all } M \in \mathrm{ob}(\mathbb{A}^*(X)).$$

Remark 6.2.11. The definition of the natural transformation $e_{(X,o)}$ in the previous definition is based on [AFM18, Propostion 5.8]

In [AFM18, §5], it is shown that the triples $(\mathbb{A}^*(X), T_{(X,o)}, e_{(X,o)})$ do indeed define additive categories with chain duality, for every ACCD $\mathbb{A}$ and oriented ball complex (X,o).

We will now extend the previous definition to pairs, in analogy to the case of ordered simplicial complex pairs described in [Ran92, Definition 13.3]. Note, by an orientation of a ball complex pair (X,A), we will simply mean an orientation on the ball complex X.

Definition 6.2.12. Let $(\mathbb{A}, T, e)$ be an additive category with chain duality and let (X,A,o) be an oriented ball complex pair. The ACCD of (X,A)-based objects in $\mathbb{A}$ is defined to be the triple $(\mathbb{A}^*(X,A), T_{(X,A,o)}, e_{(X,A,o)})$ where:

- $\mathbb{A}^*(X,A) \subset \mathbb{A}^*(X)$ is the full subcategory of $\mathbb{A}^*(X)$ generated on objects M such that
$$M(\sigma) = 0, \text{ if } \sigma \in \mathcal{F}(A),$$
and
- $T_{(X,A,o)}$ and $e_{(X,A,o)}$ are the restrictions of $T_{(X,o)}$ and $e_{(X,o)}$, respectively, to $\mathbb{A}^*(X,A)$.

We will employ the more compact notation $\mathbb{A}^*(X,A,o)$ to denote the triples $(\mathbb{A}^*(X), T_{(X,A,o)}, e_{(X,A,o)})$.

We will next present explicit formulas for the chain duality $(T_{(X,o)}, e_{(X,o)})$ of $\mathbb{A}^*(X,o)$, for any fixed ACCD $(\mathbb{A},T,e)$ and oriented ball complex (X,o), in preparation for computations in the proceeding section.

The following formula for $T_{(X,o)}$ can be immediately obtained from its definition: For an object $M = \oplus_\sigma M(\sigma)$ in $\mathbb{A}^*(X)$, the r-chains of the chain complex $T_{(X,o)}(M)$ in $\mathbb{B}(\mathbb{A}^*(X))$ are given by

$$T_{(X,o)}(M)_n := \bigoplus_{\sigma \in \mathcal{F}(X)} T_{(X,o)}(M)_n(\sigma),$$

where

$$T_{(X,o)}(M)_n(\sigma) := T\Big(\bigoplus_{\sigma' \leq \sigma} M(\sigma')\Big)_{n-|\sigma|}.$$

Moreover, the differentials

$$(d_{T_{(X,o)}(M)})^n_{\tau,\sigma} : T_X(M)_n(\sigma) \to T_{(X,o)}(M)_{n-1}(\tau),$$

with respect to the cells τ, σ in $\mathcal{F}(X)$, are given by

$$\begin{aligned}
&(d_{T_{(X,o)}(M)})^n_{\tau,\sigma} = (-1)^{|\sigma|} \bigoplus_{\sigma' \leq \sigma} d^{n-|\sigma|}_{TM(\sigma')}, && \text{if } \tau = \sigma,\\
&(d_{T_{(X,o)}(M)})^n_{\tau,\sigma} = [\sigma,\tau] T\big([M](\tau) \hookrightarrow [M](\sigma)\big)_{n-|\sigma|}, && \text{if } \tau < \sigma, |\tau| = |\sigma| - 1,\\
&(d_{T_{(X,o)}(M)})^n_{\tau,\sigma} = 0, && \text{else,}
\end{aligned}$$

where $[\sigma,\tau]$ denotes the incidence number of the oriented cells σ and τ of X.

The components of the map of chain complexes in $\mathbb{B}(\mathbb{A}^*(X))$,

$$e_{(X,o)}(M) : T^2_X(M) \to M,$$

for a given object M in $\mathbb{A}^*(X)$, may be obtained by examing expression (5.1) in the proof of [AFM18, Proposition 5.9] for the case $C = M$, and by using additivity of e. To state the result, denote by

$$\pi_{(X,o)}(M)_{\tau,\sigma} : T^2_{(X,o)}(M)_0(\sigma) \twoheadrightarrow T^2(M(\tau))_0 \tag{6.1}$$

for arbitrary $\tau \leq \sigma \in \mathcal{F}(X)$, the signed projection map in $\mathbb{A}$

$$\bigoplus_{k,l \in \mathbb{Z}} \pi(M)^{kl}_{\tau,\sigma} : \bigoplus_{k \in \mathbb{Z}} T_{(X,o)}(T_{(X,o)}(M)_k(\sigma))_k \twoheadrightarrow \bigoplus_{l \in \mathbb{Z}} T(T(M(\tau))_l)_l$$

whose components are given by

$$\pi(M)^{kl}_{\tau,\sigma} := \begin{cases} \epsilon(\sigma,k)T(j(M)^k_{\sigma,\tau})_{k-|\sigma|}, & \text{if } l = k-|\sigma|, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\epsilon(\sigma,k) = (-1)^{|\sigma|(k-|\sigma|)+\frac{1}{2}|\sigma|(|\sigma|-1)} \tag{6.2}$$

and $j(M)^k_{\sigma,\tau}$ denotes the composite inclusion

$$j(M)^k_{\sigma,\tau} : T(M(\tau))_{k-|\sigma|} \hookrightarrow T_{(X,o)}(M)_k(\sigma) \hookrightarrow [T_{(X,o)}(M)](\sigma)_k. \tag{6.3}$$

Then, the chain map $e_{(X,o)}(M) : T^2_X(M) \to M$ has components

$$e_{(X,o)}(M)^n_{\tau,\sigma} = e(M(\tau))_0 \circ \pi_{(X,o)}(M)_{\tau,\sigma} : T^2_{(X,o)}(M)_0(\sigma) \to M(\tau), \tag{6.4}$$

if $n = 0$ and $\tau \leq \sigma \in \mathcal{F}(X)$, and $e_{(X,o)}(M)^n_{\tau,\sigma} = 0$, otherwise.

Remark 6.2.13. Note that the formulas for $e_{(X,o)}(M)$ do not depend on the differentials of $T_{(X,o)}(M)$, and thus are independent of the orientation o.

6.2.2 Functoriality of Parametrisation

In this section we will show that the assignment

$$(X, A, o) \mapsto \mathbb{A}^*(X, o)$$

is functorial with respect to morphisms of ball complex pairs. In order to take into account the orientations we introduce the following category equivalent to $\mathbf{Ball}_2$ whose objects are *oriented* ball complex pairs.

Definition 6.2.14.

1. Let $\mathbf{oBall}_2$ be the category whose objects are oriented ball complex pairs (X, A, o) and morphism sets are given by

   $$\mathrm{Mor}_{\mathbf{oBall}_2}\big((X, A, o_X), (Y, B, o_Y)\big) = \mathrm{Mor}_{\mathbf{Ball}_2}\big((X, A), (Y, B)\big).$$

   Also, let $\mathbf{oBall} \subset \mathbf{oBall}_2$ denote the full subcategory on absolute objects i.e., on objects of the form $(X, \emptyset, o)$.

2. A morphism $(X, A, o_X) \to (Y, B, o_Y)$ is called an *inclusion* in $\mathbf{oBall}_2$, if the underlying map of ball complexes is an inclusion $(X, A) \hookrightarrow (Y, B)$ of ball complex pairs, and the orientation o_Y on Y agrees with o_X over the subcomplex X.

Recall that every morphism in $\mathbf{Ball}_2$ factors as the composition of an isomorphism followed by an inclusion. Similarly, note that we may factor every morphism $f : (X, A, o_X) \to (Y, B, o_Y)$ in $\mathbf{oBall}_2$ uniquely as a composition

$$(X, A, o) \underset{\bar{f}}{\overset{\cong}{\to}} (f(X), f(A), o_{f(X)}) \underset{j}{\hookrightarrow} (Y, B, o_Y),$$

where $\bar{f}$ is an isomorphism and j is an inclusion in $\mathbf{oBall}_2$.

The proof of functoriality of the construction $(\mathbb{A}, (X, A, o)) \mapsto \mathbb{A}^*(X, A, o)$ with respect to morphisms of ball complexes relies crucially on the fact that the assignment of cellular chain complexes

$$(X, o) \mapsto C(X; \mathbb{Z})$$

is functorial with respect to morphisms of ball complexes. The analogous statement for arbitrary CW complexes is well known (e.g., see [Geo08, Proposition 2.3.4]); the point being that cellular chain complexes admit a description independent of orientations in terms of singular homology (e.g., see [Geo08, §2.3]).

We will denote the chain map induced from a morphism of oriented ball complex pairs $f : (X, o_X) \to (Y, o_Y)$ by

$$f_\# : C(X; \mathbb{Z}) \to C(Y; \mathbb{Z}).$$

Moreover, we denote by $[f : \sigma, \tau]$ the integer components of the linear maps

$$(f_\#)_n : C(X; \mathbb{Z})_n \to C(Y; \mathbb{Z})_n \ (n \in \mathbb{Z})$$

with respect to any oriented n-cells σ and τ of X and Y, respectively, and call them *mapping degrees* in accordance with the literature (e.g., see [LW69, p. 165, Definition 3.7])

Remark 6.2.15. Note that mapping degrees are fully determined in the case of a map of ball complexes (e.g., see [LW69, p. 172, Theorem 4.7]).

We require the following proposition about mapping degrees of morphisms in $\mathbf{oBall}_2$, as preparation for the proof of functoriality:

Proposition 6.2.16. *Let $f : (X, o_X) \to (Y, o_Y)$ be a map of oriented ball complexes and denote by $f = j \circ \bar{f}$ its canonical factorisation into an isomorphism followed by an inclusion of ball complexes. Then*

$$[f : \sigma, \tau] = \delta_{\tau f(\sigma)}[\bar{f} : \sigma, \bar{f}(\sigma)],$$

for all oriented n-cells σ and τ of X and Y, respectively, where $\delta_{\tau f(\sigma)}$ denotes the Kronecker delta function.

Proof. Since f maps every n-cell σ isomorphically onto the n-cell $f(\sigma)$, the mapping degrees $[f : \sigma, \tau]$ vanish, for all $\tau \neq f(\sigma)$ (cf. [LW69, p. 166, Corollary 3.9]). Furthermore, the equality $f_\# = j_\# \circ \bar{f}_\#$ implies that

$$[f : \sigma, \tau] = \sum_{\tau'} [j : \tau', \tau][\bar{f} : \sigma, \tau'],$$

for all oriented n-cells σ and τ of X and Y, respectively, where the sum runs over all n-cells τ' of $f(X)$. It is clear the mapping degrees of an inclusion j satisfy

$$[j : \tau', \tau] = \delta_{\tau'\tau},$$

for all oriented n-cells τ of $f(X)$ and τ' of Y. Hence, it follows that

$$[f : \sigma, f(\sigma)] = [\bar{f} : \sigma, \bar{f}(\sigma)]$$

for all oriented n-cells σ of X, completing the proof. □

Lemma 6.2.17. *The assignment*

$$((\mathbb{A},T,e),(X,A,o)) \to \mathbb{A}^*(X,A,o)$$

induces a functor

$$\hat{p} : \mathbf{ACCD} \times \mathbf{oBall}_2^{op} \to \mathbf{ACCD}.$$

Proof. We will demonstrate functoriality of $\hat{p}$ over the full subcategory $\mathbf{ACCD} \times \mathbf{oBall} \subset \mathbf{ACCD} \times \mathbf{oBall}_2$. The extension to ball complex pairs follows trivially by restriction.

First consider an ACCD $(\mathbb{A},T,e)$ and a morphism of oriented ball complexes $f : (X,o_X) \to (Y,o_Y)$. We define an additive functor

$$f^* : \mathbb{A}^*(Y) \to \mathbb{A}^*(X)$$

to be given by

$$f^*\left(\bigoplus_{\sigma \in \mathcal{F}(Y)} M(\sigma')\right) := \bigoplus_{\sigma\in\mathcal{F}(X)} M(f(\sigma))$$
$$f^*(\phi)_{\tau,\sigma} := \phi_{f(\tau),f(\sigma)},$$

for all $M \in \mathrm{ob}(\mathbb{A}^*(Y))$, $\phi \in \mathrm{Mor}(\mathbb{A}^*(Y))$ and $\tau,\sigma \in \mathcal{F}(X)$.

We also let $f^\square_{\mathbb{A}} : \mathbb{A}^*[Y] \to \mathbb{A}^*[X]$ and $f^\square_{\mathbb{B}(\mathbb{A})} : \mathbb{B}(\mathbb{A})_*[Y] \to \mathbb{B}(\mathbb{A})_*[X]$ denote the additive functors induced by precomposition with the map of posets $\mathcal{F}(f) : \mathcal{F}(X) \to \mathcal{F}(Y)$ determined by f. Now, consider the following diagram of additive categories:

$$\begin{tikzcd}
\mathbb{A}^*(Y) \arrow[r, "{[-]}"] \arrow[d, "f^*"] & \mathbb{A}^*[Y] \arrow[r, "T_*"] \arrow[d, "f^\square_{\mathbb{A}}"] & \mathbb{B}(\mathbb{A})_*[Y] \arrow[r, "sh_{(Y,o_Y)}"] \arrow[d, "f^\square_{\mathbb{B}(\mathbb{A})}"] & \mathbb{B}(\mathbb{A}^*(Y)) \arrow[d, "f^*"] \\
\mathbb{A}^*(X) \arrow[r, "{[-]}"] & \mathbb{A}^*[X] \arrow[r, "T_*"] & \mathbb{B}(\mathbb{A})_*[X] \arrow[r, "sh_{(X,o_X)}"] & \mathbb{B}(\mathbb{A}^*(X)).
\end{tikzcd} \tag{6.5}$$

It is easy to see that the left and middle square in Diagram 6.5 commute. We claim that the right square commutes up to natural isomorphism.

First, consider the case that f is an inclusion. Then the right square in Diagram 6.5 commutes on the nose, and, hence, we have an equality of functors:

$$G(f) : T_X f^* = f^* T_Y : \mathbb{A}^*(Y) \to \mathbb{B}(\mathbb{A}^*(X)).$$

More generally, the map f may be factored uniquely as a composition of an isomorphism $\bar{f} : (X,o_X) \to (Z,o_Z)$ and an inclusion $j : (Z,o_Z) \hookrightarrow (Y,o_Y)$. The chain isomorphism $\bar{f}_\# : C(X;\mathbb{Z}) \cong C(Z;\mathbb{Z})$ of $\mathbb{Z}$-modules is used to define a chain isomorphism $H(\bar{f})(C) : sh_{(X,o_X)}\bar{f}^*(C) \cong \bar{f}^* sh_{(Z,o_Z)}(C)$ in $\mathbb{B}(\mathbb{A}^*(X))$, for all C in $\mathbb{B}(\mathbb{A})_*[Z]$, as follows: The components of $H(\bar{f})(C)$ are given by

$$H(\bar{f})(C)^n_{\tau,\sigma} = \delta_{\tau\sigma}[\bar{f} : \sigma, f(\sigma)]\, \mathrm{id}^{n-|\sigma|}_{C(f(\sigma))},$$

for all $n \in \mathbb{Z}$ and $\sigma \in \mathcal{F}(X)$. Let $j^\square_{\mathbb{B}(\mathbb{A})} : \mathbb{B}(\mathbb{A})_*[Y] \to \mathbb{B}(\mathbb{A})_*[Z]$ be defined in analogy to $f^\square_{\mathbb{B}(\mathbb{A})}$. We then define $G(f) : T_X f^* \cong f^* T_Y : \mathbb{A}^*(Y) \to \mathbb{B}(\mathbb{A}^*(X))$ to be the natural transformation with components for all $M \in \mathrm{ob}(\mathbb{A}^*(Y))$ given by

$$G(f)(M) := H(\bar{f})(j^\square_{\mathbb{B}(\mathbb{A})} T_*[M]) : T_X f^*(M) \cong f^* T_Y(M).$$

We claim that the pair $(f^*, G(f))$ is a well-defined morphism of ACCDs. To ease readability, we will suppress the orientation of ball complexes from notation in the following calculations. Using Formula 6.4 for the components of e_X and e_Y, we compute that

$$\begin{aligned} f^*e_Y(M)^0_{\tau,\sigma} &= e_Y(M)^0_{f(\tau),f(\sigma)} \\ &= e(M(f(\tau))^0 \circ \pi_Y(M)_{f(\tau),f(\sigma)} \\ &= e(f^*M(\tau))^0 \circ \pi_X(f^*M)_{\tau,\sigma} \\ &= e_X(f^*(M))^0_{f(\tau),f(\sigma)} \end{aligned}$$

for all $M \in \mathrm{ob}(\mathbb{A}^*(Y))$ and $\tau \le \sigma \in \mathcal{F}(X)$. Moreover, a straightforward computation shows that

$$\pi_Y(M)_{f(\tau),f(\sigma)} \circ G(f)(T_Y(M))^0_{\sigma,\sigma} = \pi_X(f^*M)_{\tau,\sigma} \circ T_X(G(f)(M))^0_{\sigma,\sigma},$$

as morphisms $T_X f^* T_Y(M)_0(\sigma) \to T^2(M(f(\tau))_0$ in $\mathbb{A}$ for all $M \in \mathrm{ob}(\mathbb{A}^*(Y))$ and $\tau \le \sigma \in \mathcal{F}(X)$. It follows that the following diagram commutes in $\mathbb{B}(\mathbb{A}^*(X))$ for all $M \in \mathrm{ob}(\mathbb{A}^*(Y))$:

$$\begin{array}{ccc} T_X f^* T_Y(M) & \xrightarrow{G(T_Y(M))} & f^*T_Y^2(M) \\ {\scriptstyle T_X(G(M))}\big\downarrow & & \big\downarrow{\scriptstyle f^*e_Y(M)} \\ T_X^2 f^*(M) & \xrightarrow[e_X(f^*(M))]{} & f^*(M). \end{array}$$

Finally, we claim that composition of the induced morphisms is well-defined. Let $g : (Y, o_Y) \to (Z, o_Z)$ be a morphism of oriented ball complexes and let $h : (X, o_X) \to (Z, o_Z)$ denote its composition with f. In order to see that the composite of the maps $(f^*, G(f))$ and $(g^*, G(g))$ of ACCDs is equal to the map $(h^*, G(h))$, observe that, for all $M \in \mathrm{ob}(\mathbb{A}^*(Z))$, $n \in \mathbb{Z}$, and $\tau, \sigma \in \mathcal{F}(X)$, the components of the chain map $G(h) : T_{(Z,o_Z)}h^* \to h^*T_{(X,o_X)}$ are given by

$$G(h)(M)^n_{\tau,\sigma} = [\bar{h} : \sigma, \bar{h}(\sigma)]\, \mathrm{id}^{n-|\sigma|}_{T([M](h(\sigma))} .$$

Similarly, one may compute that

$$(f^*G(g)(M) \circ G(f)(g^*M))^n_{\tau,\sigma} = [\bar{g} : f(\sigma), \bar{g}(f(\sigma))][\bar{f} : \sigma, \bar{f}(\sigma)]\, \mathrm{id}^{n-|\sigma|}_{T([M](g(f(\sigma)))} .$$

The claim then follows from the relation of mapping degrees

$$[\bar{g} : f(\sigma), \bar{g}(f(\sigma))][\bar{f} : \sigma, \bar{f}(\sigma)] = [\bar{h} : \sigma, \bar{h}(\sigma)], \text{ for all } \sigma \in \mathcal{F}(X),$$

which can be seen to hold by Proposition 6.2.16 and the equality of chain maps $h_\# = g_\# \circ f_\#$.

Next, we consider functoriality of $\hat{p}$ with respect to morphisms of ACCDs. We will again suppress orientation from notation; a fixed orientation on any given ball complex being implicit. Let $(F, G) : (\mathbb{A}, T, e) \to (\mathbb{A}', T', e')$ be a morphism of ACCDs and X be an oriented ball complex. Since F is additive, it restricts to a well-defined additive functor

$$F : \mathbb{A}^*(X) \to (\mathbb{A}')^*(X).$$

Consider now the following diagram of additive categories:

$$
\begin{tikzcd}
\mathbb{A}^*(X) \arrow[r, "{[-]}"] \arrow[d, "F"] & \mathbb{A}^*[X] \arrow[r, "T_*"] \arrow[d, "F_*"] & \mathbb{B}(\mathbb{A})_*[X] \arrow[r, "sh_X"] \arrow[d, "F_*"] & \mathbb{B}(\mathbb{A}^*(X)) \arrow[d, "F"] \\
(\mathbb{A}')^*(X) \arrow[r, "{[-]}"] & (\mathbb{A}')^*[X] \arrow[r, "T'_*"] & \mathbb{B}(\mathbb{A}')_*[X] \arrow[r, "sh_X"] & \mathbb{B}((\mathbb{A}')^*(X)).
\end{tikzcd}
\tag{6.6}
$$

The left and right hand squares in Diagram 6.6 can be seen to commute using additivity of F, and the middle square commutes up to the natural transformation

$$G_\diamond : T'_*F_* \Rightarrow F_*T_* : \mathbb{A}^*[X] \to \mathbb{B}(\mathbb{A}')_*[X],$$

obtained by applying G pointwise. Hence, there is a natural transformation

$$G_X : T'_X F \Rightarrow FT_X : \mathbb{A}^*(X) \to \mathbb{B}((\mathbb{A}')^*(X))$$

given by

$$G_X(M) := sh_X(G_\diamond([M])) : T'_X F(M) \to FT_X(M),$$

for all $M \in \mathrm{ob}(\mathbb{A}^*(X))$.

We claim that the pair (F, G_X) is a morphism of ACCDs. First, notice that the maps $G_X(M) : T'_X F(M) \to FT_X(M)$ are chain equivalences for every $M \in \mathrm{ob}(\mathbb{A}^*(X))$ since all the diagonal components $G_X(M)_{\sigma,\sigma}$ are chain equivalences (cf. [AFM18, Proposition 4.13]). It remains to check Property (b) of Definition 6.1.2.

Let $M \in \mathrm{ob}(\mathbb{A}^*(X))$, $\tau \leq \sigma \in \mathcal{F}(X)$ be given, and consider the following diagram in $\mathbb{A}$:

$$
\begin{tikzcd}
T'_X FT_X(M)_0(\sigma) \arrow[rr, "G_X(T_X(M))^0_{\sigma,\sigma}"] \arrow[dr, two heads, "q(M)_{\tau,\sigma}"] \arrow[dd, "T'_X(G_X(M))^0_{\sigma,\sigma}"'] & & FT^2_X(M)_0(\sigma) \arrow[d, two heads, "F(\pi(M)_{\tau,\sigma})"] \\
& T'FT(M(\tau))_0 \arrow[r, "G(T(M(\tau)))_0"] \arrow[d, "T'G(M(\tau))_0"] & FT^2(M(\tau))_0 \arrow[d, "Fe(M(\tau))_0"] \\
(T'_X)^2F(M)_0(\sigma) \arrow[r, two heads, "\pi(F(M))_{\tau,\sigma}"'] & (T')^2F(M(\tau))_0 \arrow[r, "e'F(M(\tau))_0"'] & F(M(\tau)).
\end{tikzcd}
\tag{6.7}
$$

The projection map

$$q(M)_{\tau,\sigma} = \bigoplus_{k,l\in\mathbb{Z}} q(M)^{kl} : \bigoplus_{k\in\mathbb{Z}} T'_X(FT_X(M)_k(\sigma))_k \to \bigoplus_{l\in\mathbb{Z}} T'(FT(M(\tau))_l)_l$$

in Diagram 6.7 is given by

$$q(M)^{kl} := \begin{cases} \epsilon(\sigma,k)T'F(j(M)^k_{\tau,\sigma})_{k-|\sigma|}, & \text{if } l = k - |\sigma| \\ 0, & \text{otherwise,} \end{cases}$$

where the sign $a(\sigma, k)$ and morphisms $j(M)^k_{\tau,\sigma}$ are defined as in Equations 6.2 and 6.3, respectively. The lower right inner square of Diagram 6.7 then commutes by the assumption that (F, G) is a morphism of ACCDs. Moreover, the other inner squares can be seen to commute using the additivity of G and T'. Thus, the outer square of Diagram 6.7 commutes implying that the pair (F, G_X) is well-defined morphism of ACCDs.

The final assertions that the assignment $(F,G) \mapsto (F,G_X)$ is compatible with composition, and that it commutes with the assignment $f \mapsto (f^*, G(f))$ are straightforward.

$\square$

Remark 6.2.18. Functoriality of the construction of the ACCDs $\mathbb{A}^*(X)$ with respect to inclusions of simplicial complexes was proven in [Ran92, Proposition 5.6]. The argument given there inspired our construction of the induced maps f^* for an inclusion of ball complexes. Note that functoriality with respect to morphisms of ACCDs has not, to the author's knowledge, been thoroughly examined in the literature.

A direct consequence of Lemma 6.2.17 is that there exists a functor

$$\hat{p}(\omega) : \mathbf{ACCD} \times \mathbf{Ball}_2^{op} \to \mathbf{ACCD}$$

for every choice of section $\omega : \mathbf{Ball}_2 \to \mathbf{oBall}_2$. For the purposes of demonstration, we will work with a *fixed* section ω in the rest of this chapter. We denote the corresponding parametrisation operator simply by p. Moreover, we introduce the convention that the additive categories $\mathbb{A}^*(X,A)$ are assumed to be equipped with the chain duality prescribed by the fixed section ω unless otherwise specified. Lastly, for convenience, we will also make the following assumptions on the section ω:

1. ω is *absolute*: For every ball complex pair (X,A), the map $\omega(X) \to \omega(X,A)$ is an inclusion of oriented ball complexes, i.e., the orientation on $\omega(X,A)$ agrees with the orientation on $\omega(X)$.
2. ω is *simplicial*: If X is a simplicial complex, then the orientation on $\omega(X)$ is given by the standard simplicial incidence numbers:

   $$[\langle v_0, \dots, v_k\rangle, \langle v_0, \dots, \hat{v_i}, \dots, v_k\rangle] = (-1)^i,$$

   for all k-cells $\langle v_0, \dots, v_k\rangle$ of X and their i^{th} subfaces $\langle v_0, \dots, \hat{v_i}, \dots, v_k\rangle$.

Remark 6.2.19. The first assumption ensures that the chain duality of the ACCD $\mathbb{A}^*(\omega(X,A))$ is precisely the restriction of that of $\mathbb{A}^*(\omega(X))$; not just up to isomorphism. The second convention ensures compatibility with [Ran92].

6.2.3 Properties of Parametrisation

In this subsection we will show that the parametrisation operator p is part of an extended parametrisation structure (p, μ, α) on the category **ACCD**, and, moreover, that it is deloopable.

The construction of the unit μ is straightforward. Denote by

$$\mu_{\mathbb{A}} : \mathbb{A} \cong \mathbb{A}^*(\Delta^0)$$

the natural isomorphism of additive categories given by considering objects as sums over the singleton set $\mathcal{F}(\Delta^0)$. It is then routine to see that we have an equality of functors

$$T_{\Delta^0}\mu_{\mathbb{A}} = \mu_{\mathbb{A}} T : \mathbb{A} \to \mathbb{B}(\mathbb{A}^*(\Delta^0)).$$

and moreover, that the following proposition holds:

Proposition 6.2.20. *The collection of morphisms*

$$\mu := \{(\mu_{\mathbb{A}}, \mathrm{id}) : (\mathbb{A}, T, e) \cong (\mathbb{A}^*(\Delta^0), T_{\Delta^0}, e_{\Delta^0})\},$$

where the index runs over all ACCD $(\mathbb{A}, T, e)$*, assemble to a natural isomorphism of functors.*

The more difficult task is the construction of associativity morphisms. Our strategy makes use of an auxiliary construction that is independent of any choice of preferred section $\omega : \mathbf{Ball}_2 \to \mathbf{oBall}_2$ and described in Lemma 6.2.22. The construction relies crucially on the existence of a canonical choice of orientation on products of oriented ball complexes (recall Formula 3.1) and the subsequent facts about mapping degrees of morphisms of ball complexes, whose proof can be easily checked using the characterising properties of mapping degrees (e.g., see [LW69, p. 172, Theorem 4.7]).

Proposition 6.2.21. *In the following statements, we assume that products of ball complexes are equipped with the canonical product orientation.*

1. *Let* $f : X \to X'$ *be a map of oriented ball complexes and* Y *be an oriented ball complex. The mapping degrees of the induced maps* $f \times \mathrm{id} : X \times Y \to X' \times Y$ *and* $\mathrm{id} \times f : Y \times X \to Y \times X'$ *satisfy*

$$[f \times \mathrm{id} : \sigma \times \tau, \sigma' \times \tau'] = \delta_{\sigma' f(\sigma)} \delta_{\tau\tau'} [f : \sigma, \sigma'] = [\mathrm{id} \times f : \tau \times \sigma, \tau' \times \sigma']$$

for all oriented n*-cells* σ *of* X*,* σ' *of* X' *and* τ *and* τ' *of* Y*.*

2. *Let* X *be an oriented ball complex. The mapping degrees of the projections* $\pi_1 : X \times \Delta^0 \to X$ *and* $\pi_2 : \Delta^0 \times X \to X$ *satisfy*

$$[\pi_1 : \rho, \rho'] = \delta_{\rho' \pi_1(\rho)}$$
$$[\pi_2 : \nu, \nu'] = \delta_{\nu' \pi_2(\nu)}$$

for all oriented n*-cells* ρ *of* $(X \times \Delta^0)$*,* ν *of* $(\Delta^0 \times X)$ *and* ρ' *and* ν' *of* X*.*

Lemma 6.2.22. *There are natural isomorphisms*

$$\tilde{\alpha}_{\mathbb{A}}^{(X,A,o_X),(Y,B,o_Y)} : (\mathbb{A}^*(Y, B, o_Y))^*(X, A, o_X) \to \mathbb{A}^*((X, A) \times (Y, B), o_X \times o_Y)$$

of functors $\mathbf{ACCD} \times \mathbf{oBall}_2^{op} \times \mathbf{oBall}_2^{op} \to \mathbf{ACCD}$ *that are compatible with* μ*.*

Proof. Let (X, o_X) and (Y, o_Y) be oriented ball complexes and denote the product $X \times Y$ equipped with the product orientation by (W, o_W). The isomorphism of face posets

$$\mathcal{F}(W) \to \mathcal{F}(X) \times \mathcal{F}(Y)$$
$$(\sigma \times \tau) \mapsto (\sigma, \tau)$$

induces a natural isomorphism of additive categories

$$\tilde{\alpha}_{\mathbb{A}}^{X,Y} : (\mathbb{A}^*(Y))^*(X) \to \mathbb{A}^*(W)$$

given by

$$\tilde{\alpha}_{\mathbb{A}}^{X,Y}\left(\bigoplus_{\sigma\in\mathcal{F}(X)}\bigoplus_{\tau\in\mathcal{F}(Y)} M(\sigma)(\tau)\right) := \left(\bigoplus_{\sigma\times\tau\in\mathcal{F}(W)} M(\sigma)(\tau)\right),$$
$$\tilde{\alpha}_{\mathbb{A}}^{X,Y}(\phi)_{\sigma\times\tau,\sigma'\times\tau'} := ((\phi)_{\sigma,\sigma'})_{\tau,\tau'}$$

for all $M \in \mathrm{ob}((\mathbb{A}^*(Y))^*(X))$, $\phi \in \mathrm{Mor}((\mathbb{A}^*(Y))^*(X))$, $\tau, \tau' \in \mathcal{F}(Y)$, and $\sigma, \sigma' \in \mathcal{F}(X)$.

A straightforward calculation using the incidence relations 3.1 shows that, for all $M \in \mathrm{ob}((\mathbb{A}^*(Y))^*(X))$, the following equality of chain complexes in $\mathbb{B}(\mathbb{A}^*(W))$ holds:

$$T_{(W,o_W)}\tilde{\alpha}_{\mathbb{A}}^{X,Y}(M) = \tilde{\alpha}_{\mathbb{A}}^{X,Y}(T_{(Y,o_Y)})_{(X,o_X)}(M).$$

Furthermore, it can be seen using Formula 6.4 that the triangle

$$\begin{tikzcd}
T^2_{(W,o_W)}(\tilde{\alpha}_{\mathbb{A}}^{X,Y}(M)) \arrow[rr, "="] \arrow[dr, "e_{(W,o_W)}(\tilde{\alpha}_{\mathbb{A}}^{X,Y}(M))"'] & & \tilde{\alpha}_{\mathbb{A}}^{X,Y}(T_{(Y,o_Y)})^2_{(X,o_X)}(M) \arrow[dl, "\tilde{\alpha}_{\mathbb{A}}^{X,Y}(e_{(Y,o_Y)})_{(X,o_X)}(M)"] \\
& \tilde{\alpha}_{\mathbb{A}}^{X,Y}(M) &
\end{tikzcd}$$

commutes in $\mathbb{B}(\mathbb{A}^*(X \times Y))$, for every $M \in \mathrm{ob}((\mathbb{A}^*(Y))^*(X))$. Hence, the pair $(\tilde{\alpha}_{\mathbb{A}}^{X,Y}, \mathrm{id})$ defines a morphism of ACCDs. We claim that the collection $\alpha = \{(\tilde{\alpha}_{\mathbb{A}}^{X,Y}, \mathrm{id})\}$ defines a natural transformation of functors $\mathbf{ACCD} \times \mathbf{oBall} \times \mathbf{oBall} \to \mathbf{ACCD}$.

Naturality in the variable $\mathbb{A}$ is immediate from definitions and additivity of F. Moreover, naturality $\tilde{\alpha}_{\mathbb{A}}^{X,Y}$ in the variables X and Y, and its compatibility with μ can be checked with the aid of Propositions 6.2.16 and 6.2.21.

Lastly, we define the morphisms $\tilde{\alpha}_{\mathbb{A}}^{(X,A,o_X),(Y,B,o_Y)}$, for arbitrary ball complex pairs $((X, A, o_X)$ and $(Y, B, o_Y))$, by restriction of the morphisms $\tilde{\alpha}_{\mathbb{A}}^{X,Y}$ along the inclusions $(\mathbb{A}^*(Y, B))^*(X, A) \subset (\mathbb{A}^*(Y))^*(X)$. It may be seen that these are well-defined and extend the naturality of α, by using the additivity of $\tilde{\alpha}_{\mathbb{A}}^{X,Y}$ and the functoriality of the extended functor $\hat{p}$ of Proposition 6.2.17, respectively. □

Corollary 6.2.23. *There are natural isomorphisms*

$$\alpha_{\mathbb{A}}^{(X,A),(Y,B)} : (\mathbb{A}^*(Y,B))^*(X,A)) \to \mathbb{A}^*((X,A) \times (Y,B))$$

of functors $\mathbf{ACCD} \times \mathbf{Ball}_2^{op} \times \mathbf{Ball}_2^{op} \to \mathbf{ACCD}$*, compatible with the unit* μ.

Proof. Recall we have fixed a section $\omega : \mathbf{Ball}_2 \to \mathbf{oBall}_2$. For a given pair of ball complex pairs (X, A) and (Y, B), let o_X, o_Y and $o_{X\times Y}$ denote the orientations of $\omega(X, A)$, $\omega(Y, B)$ and $\omega((X, A) \times (Y, B))$, respectively. The isomorphisms $\alpha_{\mathbb{A}}^{(X,A),(Y,B)}$ are then defined as the composition of the isomorphisms $\tilde{\alpha}_{\mathbb{A}}^{(X,A,o_X),(Y,B,o_Y)}$ described in Lemma 6.2.22 with the isomorphism of ACCDs

$$(\mathrm{id}^*, G(\mathrm{id})) : \mathbb{A}^*((X,A) \times (Y,B), o_X \times o_Y) \cong \mathbb{A}^*((X,A) \times (Y,B)),$$

induced from the morphism of oriented ball complex pairs

$$\mathrm{id} : ((X,A) \times (Y,B), o_X \times o_Y) \cong ((X,A) \times (Y,B), o_{X\times Y}).$$

The conditions of naturality of α and compatibility with μ hold by the corresponding properties of $\tilde{\alpha}$ and the functoriality of the parametrisation operator $\hat{p}$. $\square$

Lastly, we will prove that the parametrisation operator p is deloopable. In analogy with the example of ad theories, it will be helpful to introduce automorphisms $\Sigma^k : \mathbf{ACCD} \to \mathbf{ACCD}$ on the category of additive categories with chain duality that shift the grading in chain duality by an integer k.

Definition 6.2.24. Let $(\mathbb{A}, T, e)$ be an additive category with chain duality. For $k \in \mathbb{Z}$, we let $\Sigma^k\mathbb{A}$ be the additive category with chain duality $(\mathbb{A}, T_k, e)$, where

$$T_k := S^{-k} \circ T : \mathbb{A} \to \mathbb{B}(\mathbb{A}).$$

Moreover, Σ^k assigns to a morphism of ACCDs $(F, G) : (\mathbb{A}, T, e) \to (\mathbb{A}', T', e')$ the morphism $(F, S^{-k}G) : \Sigma^k\mathbb{A} \to \Sigma^k\mathbb{A}'$.

Remark 6.2.25. The minus sign in the definition $T_k = S^{-k}T$ is introduced in order that the notation Σ agrees with that of Chapter 4; cf. Remark 4.3.9.

Proposition 6.2.26. *The functors*

$$\Sigma^k : \mathbf{ACCD} \to \mathbf{ACCD}$$

are well defined.

Proof. We will begin by showing that the triple $(\mathbb{A}, T_k, e)$ defined above is indeed an additive category with duality, for any given additive category with duality $(\mathbb{A}, T, e)$ and $k \in \mathbb{Z}$. Only property (a) of Definition 6.1.2 is not immediately clear. Let M be an object of $\mathbb{A}$. Then, by definition of the chain duality (T, e),

$$e(T(M)) \circ T(e(M)) = \mathrm{id}_{T(M)}$$

is an equality in of morphisms in $\mathbb{B}(\mathbb{A})$. Applying the chain suspension functor S^{-k} to the previous equation yields

$$S^{-k}\big(e(T(M)) \circ T(e(M))\big) = S^{-k}\big(\,\mathrm{id}_{T(M)}\,\big).$$

Moreover, functoriality of S^{-k} implies that

$$S^{-k}e(T(M)) \circ S^{-k}T(e(M)) = \mathrm{id}_{S^{-k}T(M)}\,.$$

It is easy to see that S^{-k} commutes with e. Hence, by associativity of composition, we deduce that

$$e(T_k(M)) \circ T_k(e(M)) = \mathrm{id}_{T_k(M)},$$

as required.

Next, we consider functoriality of Σ^k. Let $(F, G) : (\mathbb{A}, T, e) \to (\mathbb{A}', T', e')$ be a morphism of ACCDs. For any object M in $\mathbb{A}$, the chain map

$$S^{-k}G(M) : S^{-k}TF(M) \Rightarrow S^{-k}FT(M) = FS^{-k}T(M)$$

is a weak equivalence since S^{-k} preserves weak equivalences.

Moreover, by the assumption that (T, e) is a chain duality, and the observation that the functor S^{-k} anticommutes with contravariant functors and natural transformations thereof (cf. [AFM18, p. 22]), it follows that the following square commutes in $\mathbb{B}(\mathbb{A})$, for every $M \in \mathrm{ob}(\mathbb{A})$:

$$\begin{tikzcd}
S^{-k}T'S^{-k}FT(A) \arrow[r, "S^{-k}GS^{-k}T(M)"] \arrow[d, "S^{-k}T'G(M)"'] & FT^2(M) \arrow[d, "Fe(M)"] \\
T'^2F(M) \arrow[r, "e'F(M)"'] & F(M).
\end{tikzcd}$$

□

We come to the proof of deloopability of the parametrisation operator p on **ACCD**.

Proposition 6.2.27. *There is a natural isomorphism*

$$(K, \kappa) : \Sigma^{-1}\mathbb{A} \xrightarrow{\cong} \big(\mathbb{A}^*(\Delta^1, \partial\Delta^1), T_{(\Delta^1,\partial\Delta^1)}, e_{(\Delta^1,\partial\Delta^1)}\big)$$

of functors **ACCD** $\to$ **ACCD**. *In particular, the parametrisation operator p is deloopable.*

Proof. Fix an ACCD $(\mathbb{A}, T, e)$. Let E denote the unique 1-dimensional cell of Δ^1. We define an isomorphism of additive categories

$$K : \mathbb{A} \xrightarrow{\cong} \mathbb{A}^*(\Delta^1, \partial\Delta^1)$$

by formal "suspension", i.e., K is given by

- $K(a) = \oplus_{\sigma\in\mathcal{F}(\Delta^1)}F(a)(\sigma)$ where

$$\begin{aligned} K(a)(\sigma) &:= a, && \text{if, } \sigma = E, \\ K(a)(\sigma) &:= 0, && \text{else} \end{aligned}$$

  for all $a \in \mathrm{ob}(\mathbb{A})$.

- $K(f) : K(a) \to K(a')$, where

$$\begin{aligned} K(f)(\tau, \tau') &:= f && \text{if } \tau = \tau' = E, \\ K(f)(\tau, \tau') &:= 0, && \text{else} \end{aligned}$$

  for all morphisms $f : a \to a'$ in $\mathbb{A}$.

Now, for all $M \in \mathrm{ob}(\mathbb{A})$, let

$$\kappa(M) : T_{(\Delta^1,\partial\Delta^1)}K(M) \cong KT_{-1}(M)$$

be the isomorphism of chain complexes in $\mathbb{A}^*(\Delta^1, \partial\Delta^1)$, whose components are given by

$$\kappa(M)^n_{E,E} = (-1)^{n(n-1)/2}\,\mathrm{id}_{ST(M)_{n-1}},$$

for all $n \in \mathbb{Z}$. A straightforward inspection then shows that the following diagram:

$$\begin{tikzcd}
T_{\Delta^1}KT_{-1}(M) \arrow[r, "\kappa T_{-1}(M)"] \arrow[d, "T_{\Delta^1}\kappa(M)"'] & KT^2_{-1}(M) \arrow[d, "Ke(M)"] \\
T^2_{\Delta^1}K(M) \arrow[r, "e_{\Delta^1}K(M)"'] & K(M)
\end{tikzcd}$$

commutes in $\mathbb{B}(\mathbb{A}^*(\Delta^1, \partial\Delta^1))$, for all $M \in \mathrm{ob}(\mathbb{A})$. $\square$

In summary, Lemmas 6.2.17, 6.2.27, Proposition 6.2.20 and Corollary 6.2.23 together imply that the category **ACCD** admits the structure of a category with extended parametrisation structure. We record the result in the following theorem:

Theorem 6.2.28. *The quadruple* $(\mathbf{ACCD}, p, \mu, \alpha)$ *defines a category with extended parametrisation structure with deloopable parametrisation operator.*

6.3 Outlook on Symmetric L-Theory of ACCDs

We define symmetric L-theory of additive categories with chain duality in analogy to Definition 2.3.4 for the case of Waldhausen categories with duality. Note that this definition agrees with the geometric realisation of Ranicki's symmetric L-theory functor L^0 defined in [Ran92, Definition 13.2] over the category **ACCD**.

Definition 6.3.1. *Symmetric L-theory* of additive categories with chain duality $L : \mathbf{ACCD} \to \mathbf{Top}_*$ is defined as the parametric realisation of the symmetric-Poincaré-complexes functor sp, i.e.,

$$L := P\,\mathrm{sp}\,.$$

Remark 6.3.2. We expect that the pair (L, ι_{sp}) is a universal bordism characteristic, and indeed this would follow from [Ran92, Proposition 13.7] which states that the assignment $K \mapsto L(\mathbb{A}^*(K))$, for any fixed ACCD $\mathbb{A}$, is a cohomology theory over the category of ordered simplicial complexes; bordism invariance of L is immediate from the homotopy invariance of L.

However, the proof of [Ran92, Proposition 13.7] appears to have a gap: In [Ran92, Errata p. 140] Ranicki cites [LM14, Remark 16.2] as an ingredient in the proof. The latter depends on the application of [LM14, Theorem 16.1] to suitable ad theories of symmetric Poincaré complexes, yet only the case of additive categories of R-modules for a ring R is considered in [LM14] leaving the statement of [Ran92, Proposition 13.7] for the general case, i.e., arbitrary $\mathbb{A}$, open.

In closing, we summarise a number of formal properties of the symmetric-Poincaré-complexes functor sp towards a proof of universality of the pair (L, ι_{sp}) following the strategy developed in Chapter 4.

Proposition 6.3.3. *The functor* $\mathrm{sp} : \mathbf{ACCD} \to \mathbf{Set}_*$ *is absolute.*

Proof. Let K be a ball complex and $(\mathbb{A}, T, e)$ be an ACCD. The additive category $\mathbb{A}^*(K, K)$ is by definition trivial, implying that $\mathrm{sp}(\mathbb{A}^*(K, K))$ is a point. Hence, the functor sp is reduced.

Next, consider the following commutative square of pointed sets for a given inclusion $j : A \hookrightarrow X$ of ball complexes:

$$\begin{tikzcd} \mathrm{sp}(\mathbb{A}^*(X,A)) \arrow[r] \arrow[d] & \mathrm{sp}(\mathbb{A}^*(X)) \arrow[d, "\mathrm{sp}(j^*)"] \\ pt = \mathrm{sp}(\mathbb{A}^*(A,A)) \arrow[r] & \mathrm{sp}(\mathbb{A}^*(A)). \end{tikzcd}$$

We claim that the square is a homotopy pullback square. Since the spaces in the square are discrete and $\mathrm{sp}(\mathbb{A}^*(A, A)) = pt$, all we need to show is that $\mathrm{sp}(\mathbb{A}^*(X, A))$ is equal to the fibre of the restriction map

$$\mathrm{sp}(j^*) : \mathrm{sp}(\mathbb{A}^*(X)) \to \mathrm{sp}(\mathbb{A}^*(A)).$$

Firstly, notice that $\mathrm{sp}(\mathbb{A}^*(X, A)) \subseteq \mathrm{sp}(\mathbb{A}^*(X))$ because the chain duality on $\mathbb{A}^*(X, A)$ is defined to be the restriction of the chain duality on $\mathbb{A}^*(X)$. Secondly, a chain complex $C \in \mathbb{B}(\mathbb{A}^*(X))$ satisfies $j^*C = 0$ as chain complexes in $\mathbb{A}^*(A)$, precisely when $C \in \mathbb{B}(\mathbb{A}^*(X, A))$. It follows that (C, ϕ) maps to the basepoint in $\mathrm{sp}(\mathbb{A}^*(A))$ if and only if $(C, \phi) \in \mathrm{sp}(\mathbb{A}^*(X, A))$. $\square$

It is well known that the functor sp is Kan (e.g., see [Ran92, Prop. 13.4]). The proof of the Kan condition of the semi-simplicial sets $[n] \mapsto \mathrm{sp}(\mathbb{A}^*(\Delta^n))$ for a given ACCD $\mathbb{A}$ is verified there by observing that its set of n-simplices, $\mathrm{sp}(\mathbb{A}^*(\Delta^n))$, is isomorphic to the set of semi-simplicial maps $\Delta^n \to \mathrm{sp}(\mathbb{A}^*(\Delta^\bullet))$, for all $n \in \mathbb{N}$ (cf. Proposition 4.5.1), and by showing that the restriction maps

$$\mathrm{sp}(\mathbb{A}^*(\Delta^n)) \to \mathrm{sp}(\mathbb{A}^*(\Lambda^n_k))$$

induced by the elementary expansions $\Lambda^n_k \to \Delta^n$ are surjective, for all $n \geq 0$ and $0 \leq k \leq n$.

Based on this observation and the previous proposition, we expect that the symmetric-Poincaré-complexes functor sp is both local and surjective on expansions.

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