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The Regularized Free Fall I. Index Computations

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Abstract. The main results are, firstly, a generalization of the Conley–Zehnder index from ODEs to the delay equation at hand and, secondly, the equality of the Morse index and the clockwise normalized Conley–Zehnder index μ^{CZ}. We consider the nonlocal Lagrangian action functional $\mathcal{B}$ discovered by Barutello, Ortega, and Verzini [7] with which they obtained a new regularization of the Kepler problem. Critical points of this functional are regularized periodic solutions x of the Kepler problem. In this article, we look at *period 1 only* and at dimension one (gravitational free fall). Via a nonlocal Legendre transform regularized periodic Kepler orbits x can be interpreted as periodic solutions (x, y) of a Hamiltonian delay equation. In particular, regularized 1-periodic solutions of the free fall are represented variationally in two ways: as critical points x of a nonlocal Lagrangian action functional and as critical points (x, y) of a nonlocal Hamiltonian action functional. As critical points of the Lagrangian action, the 1-periodic solutions have a finite Morse index which we compute first. As critical points of the Hamiltonian action $\mathcal{A}_{\mathcal{H}}$, one encounters the obstacle, due to nonlocality, that the 1-periodic solutions are not generated any more by a flow on the phase space manifold. Hence, the usual definition of the Conley–Zehnder index as the intersection number with a Maslov cycle is not available. In the local case, Hofer, Wysocki, and Zehnder [10] gave an alternative definition of the Conley–Zehnder index by assigning a winding number to each eigenvalue of the Hessian of $\mathcal{A}_{\mathcal{H}}$ at critical points. In this article, we show how to generalize the Conley–Zehnder index to the nonlocal case at hand. On one side, we discover how properties from the local case generalize to this delay equation, and on the other side, we see a new phenomenon arising. In contrast to the local case, the winding number is no longer monotone as a function of the eigenvalues.

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1. INTRODUCTION

In celestial mechanics, as well as in the theory of atoms, collisions play an intriguing role. There are many geometric ways of regularizing collisions, see, for instance, [11, 14]. In both regularizations, Levi-Civita and Moser, one has to change time. A quite new approach to the regularization of collisions was discovered in the recent article [7] by Barutello, Ortega, and Verzini, where the change of time leads to a *delayed* functional (meaning that the critical point equation is a delay equation). In Section 2, we explain, starting from the physics 1-dimensional Kepler problem, how one arrives at this functional

$$\mathcal{B}:\ W^{1,2}_{\times} := W^{1,2}(\mathbb{S}^1,\mathbb{R})\setminus\{0\}\to(0,\infty),\quad x\mapsto 4\|x\|^2\tfrac{1}{2}\|x'\|^2+\frac{1}{\|x\|^2},$$

where $\|\cdot\|$ is the L^2 norm associated to the L^2 inner product $\langle\cdot\,,\,\cdot\rangle$. One might interpret the functional $\mathcal{B}$ as a **nonlocal Lagrangian action functional**, a nonlocal mechanical system, consisting of kinetic minus potential energy

$$\mathcal{B}(x)=\tfrac{1}{2}\langle x',\,x'\rangle_x+\frac{1}{\|x\|^2}.$$

Here we use the following metric on the tangent bundle of the loop space

$$\langle\cdot\,,\,\cdot\rangle_x := 4\|x\|^2\langle\cdot\,,\,\cdot\rangle,$$

which from the perspective of the manifold is nonlocal, since it depends on the whole loop. Also the potential $x\mapsto -\frac{1}{\|x\|^2}$ is only defined on the loop space.

The set $\operatorname{Crit}\mathcal{B}$ of critical points of the nonlocal action consists of the smooth solutions $x\in C^\infty(\mathbb{S}^1,\mathbb{R})\setminus\{0\}$ of the second order delay, or nonlocal, equation

$$x''=\alpha x,\qquad \alpha=\alpha_x:=\left(\frac{\|x'\|^2}{\|x\|^2}-\frac{1}{2\|x\|^6}\right)<0. \tag{1}$$

Nevertheless, this nonlocal Lagrangian action functional admits a Legendre transform which leads to a nonlocal Hamiltonian that lives on (the L^2 extension of) the cotangent bundle of the loop space, namely,

$$\mathcal{H}: W^{1,2}_\times \times L^2 \to \mathbb{R}, \quad (x,y) \mapsto \tfrac{1}{2}\langle y,\, y\rangle^x - \frac{1}{\|x\|^2},$$

where we use the dual metric on the cotangent bundle of the loop space

$$\langle \cdot\,, \cdot\rangle^x := \tfrac{1}{4\|x\|^2}\langle \cdot\,, \cdot\rangle.$$

Associated to the nonlocal Hamiltonian $\mathcal{H}$ there is the **nonlocal Hamiltonian action functional**, namely,

$$\mathcal{A}_\mathcal{H} := \mathcal{A}_0 - \mathcal{H}\colon W^{1,2}_\times \times L^2 \to \mathbb{R} \quad (x,y) \mapsto \int_0^1 y(\tau)x'(\tau)\,d\tau - \mathcal{H}(x,y).$$

The natural base point projection and the following injection

$$\begin{aligned} \pi: W^{1,2}_\times \times L^2 &\to W^{1,2}_\times, & \iota: W^{1,2}_\times &\to W^{1,2}_\times \times L^2 \\ (x,y) &\mapsto x, & x &\mapsto \big(x, 4\|x\|^2 x'\big) \end{aligned} \tag{2}$$

are *along the sets of critical points* inverses of one another – which of course explains the choice of the factor $4\|x\|^2$. Therefore, the sets

$$\operatorname{Crit}\mathcal{A}_\mathcal{H} \quad \underset{\iota}{\overset{\pi}{\underset{\longleftarrow}{\overset{\longrightarrow}{\scriptstyle 1:1}}}} \quad \operatorname{Crit}\mathcal{B}$$

are in one-to-one correspondence. Whereas $\mathcal{B}$ at any critical point has a finite dimensional Morse index, in sharp contrast, both the Morse index and coindex are infinite at the critical points of $\mathcal{A}_\mathcal{H}$.

Since the action functional $\mathcal{A}_\mathcal{H}$ is nonlocal, its critical points cannot be interpreted as fixed points of a flow. Therefore, the usual definition of the Conley–Zehnder index as a Maslov index does not make sense. However, in [10], Hofer, Wysocki, and Zehnder gave a different characterization of the Conley–Zehnder index in terms of winding numbers of the eigenvalues of the Hessian.

We show in this article that this definition of the Conley–Zehnder index makes sense, too, for the critical points of the *nonlocal* functional $\mathcal{A}_\mathcal{H}$. Our main result is equality of Morse and Conley–Zehnder index of corresponding critical points.

Theorem 1. *For each critical point (x,y) of $\mathcal{A}_\mathcal{H}$, the canonical (clockwise normalized) Conley–Zehnder index*

$$\mu^{\mathrm{CZ}}(x,y) = \operatorname{Ind}(x)$$

is equal to the Morse index of $\mathcal{B}$ at x.

Proof. Proposition 2 and Proposition 4.

The theorem generalizes the local result, see [17], to this *nonlocal* case. Note that [17, Th. 1.2] uses the counter-clockwise normalized Conley–Zehnder index $\mu_{\mathrm{CZ}} = -\mu^{\mathrm{CZ}}$. Sign conventions are discussed at large in the introduction to [18]. The local result was a crucial ingredient in the proof that Floer homology of the cotangent bundle is the homology of the loop space [5, 15, 16].

Notation 1. We define $\mathbb{S}^1 := \mathbb{R}/\mathbb{Z}$ and consider functions $x\colon \mathbb{S}^1 \to \mathbb{R}$ as functions defined on $\mathbb{R}$ that satisfy $x(\tau+1) = x(\tau)$ for every $\tau \in \mathbb{R}$. Throughout $\langle \cdot\,, \cdot\rangle$ is the standard L^2 inner product on $L^2(\mathbb{S}^1, \mathbb{R})$ and $\|\cdot\|$ is the induced L^2 norm.

Outlook. Since 2018, the first steps have been taken to study Floer homology for delay equations, see [1–3]. The present article is the first in a series of four dealing with the free fall – the simplest instance which might already exhibit all novelties that occur in comparison to ODE Floer homology of the cotangent bundle (which still represents the homology of a space, loop space). The other parts will deal with II "Homology computation via heat flow," III "Floer homology," and IV "Floer homology and heat flow homology."

2. FREE GRAVITATIONAL FALL

Section 2 is to motivate and explain regularization. Readers familiar with regularization can directly go to subsequent sections.

2.1. Classical

For $r > 0$ and $\mathfrak{v} \in \mathbb{R}$, let $L(r, \mathfrak{v}) := \frac{1}{2}|\mathfrak{v}|^2 - V(r)$, where $V(r) := -1/r$. Then

$$d_{\mathfrak{v}} L(r, \mathfrak{v}) = \mathfrak{v}, \qquad d_r L(r, \mathfrak{v}) = -1/r^2.$$

Due to the potential, one cannot allow $r = 0$. Thus the classical action functional

$$\mathcal{S}_L(q) := \int_0^1 L(q(t), \dot{q}(t))\, dt = \int_0^1 \left(\tfrac{1}{2}|\dot{q}(t)|^2 + \frac{1}{q(t)} \right) dt \tag{3}$$

is defined on the space

$$W^{1,2}_+ := W^{1,2}(\mathbb{S}^1, (0, \infty)), \tag{4}$$

that consists of absolutely continuous positive functions $q\colon [0,1] \to (0, \infty)$ which are periodic, that is $q(1) = q(0)$, and whose derivative is L^2 integrable, that is $\|\dot{q}^2\| := \int_0^1 \dot{q}(t)^2\, dt < \infty$.
The Euler–Lagrange (or critical point) equation is given by the second order ODE

$$\frac{d}{dt} d_2 L(q, \dot{q}) = d_1 L(q, \dot{q}) \qquad \Leftrightarrow \qquad \ddot{q} = -\frac{1}{q^2} \tag{5}$$

pointwise at t. Unfortunately, there are no periodic solutions of this equation. In other words, there are no critical points of $\mathcal{S}_L$ on the space $W^{1,2}_+$, in symbols

$$\operatorname{Crit} \mathcal{S}_L = \emptyset. \tag{6}$$

All solutions q with zero initial velocity v_0 end up in collision with the origin. Set $q_0 := q(0)$ and $v_0 := \dot{q}(0)$. Multiplying (5) by $2\dot{q}$ and integrating, we obtain

$$\dot{q}(t)^2 - v_0^2 = \int_0^t \underbrace{\ddot{q} 2\dot{q}}_{\frac{d}{ds}\dot{q}^2}\, ds \overset{(5)}{=} \int_0^t \underbrace{-\frac{2\dot{q}}{q^2}}_{2\frac{d}{ds}\dot{q}^{-1}}\, ds = \frac{2}{q(t)} - \frac{2}{q_0},$$

hence,

$$\dot{q}(t) = \pm\sqrt{\frac{2}{q(t)} + 2E}, \qquad E := \frac{v_0^2}{2} - \frac{1}{q_0} \equiv \frac{\dot{q}(t)^2}{2} - \frac{1}{q(t)}.$$

This shows that collision and bouncing off happen at infinite velocity. If the particle starts with zero initial velocity, collision happens in finite time since the acceleration $\ddot{q}$ is bounded away from zero.

2.2. Nonlocal Regularization

In this section, we explain why Barutello, Ortega, and Verzini [7] discovered their functional $\mathcal{B}$ defined on the larger (than $W^{1,2}_+$) space

$$W^{1,2}_\times := W^{1,2}_\times(\mathbb{S}^1, \mathbb{R}) \setminus \{0\} \quad \supset \quad W^{1,2}(\mathbb{S}^1, (0, \infty)) =: W^{1,2}_+.$$

It consists of absolutely continuous maps $x\colon [0,1] \to \mathbb{R}$ which are periodic $x(1) = x(0)$, not identically zero $x \not\equiv 0$, and whose weak derivative x' is L^2 integrable.

The key observation is that if on the subset $W^{1,2}_+$ – the domain (4) of the classical action $\mathcal{S}_L$ – one defines an operation $\mathcal{R}$ that takes the square and appropriately rescales time, then the composition $(\mathcal{S}_L \circ \mathcal{R})(x)$ is given by a formula, let's call it $\mathcal{B}(x)$, see Fig. 1, that makes sense perfectly on the ambient space $W^{1,2}_\times$ whose elements are real-valued and so allow for origin traversing.

Whereas the classical functional $\mathcal{R}^*\mathcal{S}_L$ is defined on loops that take values in $(0, \infty)$, thereby not allowing for collisions at 0, has no critical points, the rescaled functional has critical points when considered on the extended domain $W^{1,2}_\times(\mathbb{S}^1, \mathbb{R}) \setminus \{0\}$. These critical points correspond to classical solutions having collisions, that is running into the origin 0.

The key fact is that $\mathcal{B}$ admits critical points x on the ambient space $W^{1,2}_\times$, see (26), in which case $q_x := \mathcal{R}(x)$ solves by Proposition 1 the classical free fall equation (5) away from a finite set $T_x \subset \mathbb{S}^1$ of collision times.

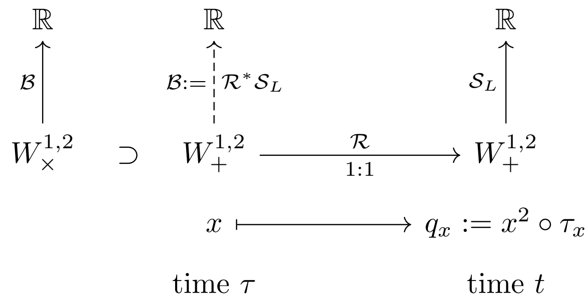

Fig. 1. Pull-back $\mathcal{R}^*\mathcal{S}_L$ gives a formula $\mathcal{B}$ that makes sense on a larger space on which critical points exist and represent time rescaled physical trajectories.

Time rescaling on the small space $W^{1,2}_+$. Pick a loop $x \in W^{1,2}_+$, see (4). We call the variable of the map $x : [0,1] \to (0,\infty)$ the **regularized time**, usually denoted by τ. We call **classical time** we call the values of the map $t_x : [0,1] \to [0,1]$ defined by

$$t_x(\tau) := \frac{\int_0^\tau x(s)^2\, ds}{\|x\|^2}. \tag{7}$$

Classical time has the following properties

$$t_x'(\tau) = \frac{x(\tau)^2}{\|x\|^2} > 0, \quad t_x \in C^1, \quad t_{rx} = t_x, \quad t_x(0) = 0, \quad t_x(1) = 1 \tag{8}$$

for every real $r > 0$. Since, moreover, the map $t_x : [0,1] \to [0,1]$ is strictly increasing, it is a bijection and we denote its inverse by

$$\tau_x : [0,1] \to [0,1].$$

The inverse $\tau_x := {t_x}^{-1}$ of classical time inherits the property $\tau_{rx} = \tau_x\ \forall r > 0$ and

$$\dot\tau_x(t) = \frac{1}{t_x'(\tau_x(t))} = \frac{\|x\|^2}{x(\tau_x(t))^2}, \quad \tau_x \in C^1, \quad \tau_x(0) = 0, \quad \tau_x(1) = 1. \tag{9}$$

Definition 1. The **rescale-square operation** is defined by

$$\mathcal{R} : W^{1,2}_+ \to W^{1,2}_+, \quad x \mapsto x^2 \circ \tau_x.$$

We abbreviate $q_x := \mathcal{R}(x)$. Note that $\mathcal{R}(rx) = r^2\mathcal{R}(x)$ for $r > 0$.

Lemma 1 (Well defined bijection)**.** *For* $x \in W^{1,2}_+$*, the image* $\mathcal{R}(x)$ *lies in* $W^{1,2}_+$*, too, and the map* $\mathcal{R} : W^{1,2}_+ \to W^{1,2}_+$ *is bijective with inverse* (11).

Proof. We must show that both are in L^2, namely, (a) $q_x(t) = x^2(\tau_x(t))$ and (b)

$$\dot q_x(t) = 2x(\tau_x(t))\, x'(\tau_x(t))\, \dot\tau_x(t) \overset{(9)}{=} 2\|x\|^2 \frac{x'(\tau_x(t))}{x(\tau_x(t))}.$$

Use the fact that x is continuous and hence $\|x\|_{L^\infty} < \infty$, in order to obtain a)

$$\|q_x\|^2 = \int_0^1 x^4(\tau_x(t))\, dt = \|x\|^4_{L^\infty} < \infty.$$

In what follows, we change the variable to $\sigma := \tau_x(t)$ and use (9) to get (b)

$$\|\dot q_x\|^2 = \int_0^1 \left(2\|x\|^2 \frac{x'(\tau_x(t))}{x(\tau_x(t))}\right)^2 \frac{x^2(\sigma)\, d\sigma}{\|x\|^2} = 4\|x\|^2\langle x', x'\rangle < \infty. \tag{10}$$

Here the value is finite since x is of class $W^{1,2}$. This proves that $\mathcal{R}(x) \in W^{1,2}_+$.

Surjective. Given $q \in W^{1,2}_+$, set

$$\mathcal{Q}(q) := x_q := q^{\frac{1}{2}} \circ \tau_{1/\sqrt{q}}. \tag{11}$$

Then for $\tau \in [0,1]$, we obtain

$$t_{x_q}(\tau) \stackrel{\text{def.}}{=} \frac{\int_0^\tau {x_q}^2(\sigma)\, d\sigma}{\int_0^1 {x_q}^2(\sigma)\, d\sigma} = \frac{\int_0^\tau q \circ \overbrace{\tau_{1/\sqrt{q}}(\sigma)}^{=:s}\, d\sigma}{\int_0^1 q \circ \tau_{1/\sqrt{q}}(\sigma)\, d\sigma} = \frac{\int_0^{\tau_{1/\sqrt{q}}(\tau)} \frac{ds}{\|1/\sqrt{q}\|^2}}{\int_0^1 \frac{ds}{\|1/\sqrt{q}\|^2}} = \tau_{1/\sqrt{q}}(\tau)$$

by change of variables. In view of this result, we see that

$$\big(\mathcal{R} \circ \underbrace{\mathcal{Q}(q)}_{x_q}\big)(t) \stackrel{\text{def. }\mathcal{R}}{=} x_q^2 \circ \tau_{x_q}(t) \stackrel{\text{def. } x_q^2}{=} q \circ \underbrace{t_{1/\sqrt{q}}}_{t_{x_q}} \circ \underbrace{\tau_{x_q}}_{{t_{x_q}}^{-1}}(t) = q(t).$$

Injective. For $x \in W^{1,2}_+$, set $q_x := \mathcal{R}(x) := x^2 \circ \tau_x$. Then for $t \in [0,1]$, we can write

$$\int_0^t \frac{1}{q_x(s)}\, ds = \int_0^t \frac{1}{x^2 \circ \tau_x(s)}\, ds = \int_0^{\tau_x(t)} \frac{1}{x^2(\sigma)} \frac{x^2(\sigma)\, d\sigma}{\|x\|^2} = \frac{\tau_x(t)}{\|x\|^2}$$

by change of variables $\sigma = \tau_x(s)$ using (9). Pick $t = 1$ to obtain

$$\int_0^1 \frac{1}{q_x(t)}\, dt = \frac{1}{\|x\|^2}. \tag{12}$$

Therefore, for $t \in [0,1]$ we have the formula

$$\tau_x(t) = \|x\|^2 \int_0^t \frac{1}{q_x(s)}\, ds \stackrel{(12)}{=} \frac{\int_0^t \frac{1}{q_x(s)}\, ds}{\int_0^1 \frac{1}{q_x(s)}\, ds} \stackrel{(7)}{=:} t_{1/\sqrt{q_x}}(t).$$

In view of this result, we obtain

$$\big(\mathcal{Q} \circ \underbrace{\mathcal{R}(x)}_{q_x}\big)(\tau) \stackrel{\text{def. }\mathcal{Q}}{=} \sqrt{q_x} \circ \tau_{1/\sqrt{q_x}}(\tau) \stackrel{\text{def. } q_x}{=} x \circ \underbrace{\tau_x}_{t_{1/\sqrt{q_x}}} \circ \underbrace{t_{1/\sqrt{q_x}}}_{{\tau_{1/\sqrt{q_x}}}^{-1}}(\tau) = x(\tau).$$

Lemma 2 (Pull-back yields an extendable formula)**.** *Given $x \in W^{1,2}_+$, then*

$$\mathcal{S}_L(\mathcal{R}(x)) = \tfrac{1}{2}\langle x',\, x'\rangle_x + \frac{1}{\|x\|^2}$$

as illustrated in Fig. 1.

Note that in the previous formula, $x(\tau)$ may take on the value zero at will, even along intervals, as long as $\|x\| \neq 0$, i.e., as long as x is not constantly zero.

Proof. Set $q_x := \mathcal{R}(x)$. Definition (3) of $\mathcal{S}_L$ tells us that

$$\mathcal{S}_L(q_x) \stackrel{(3)}{=} \tfrac{1}{2}\|\dot{q}_x\|^2 + \int_0^1 \frac{1}{q_x(t)}\, dt = \tfrac{1}{2}\, 4\|x\|^2\rangle x',\, x'\rangle + \frac{1}{\|x\|^2}, \tag{13}$$

where in the second equality, we used (10) and (12).

2.3. Correspondence of Solutions

Whereas the classical action $\mathcal{S}_L$ does not admit, see (6), any critical points on the small space $W^{1,2}_+$, i.e. it admits no 1-periodic free fall solutions, the formula (13) for $\mathcal{S}_L$ evaluated on the regularization $q_x := \mathcal{R}(x)$ of a loop $x \in W^{1,2}_+$ makes perfectly sense on the larger Sobolev Hilbert space $W^{1,2}_\times := W^{1,2}(\mathbb{S}^1, \mathbb{R}) \setminus \{0\}$ with the origin removed. In this way one arrives, in case of the free fall, at the Barutello–Ortega–Verzini [7], nonlocal functional

$$\mathcal{B} \colon W^{1,2}_\times := W^{1,2}(\mathbb{S}^1, \mathbb{R}) \setminus \{0\} \to (0,\infty), \quad x \mapsto \tfrac{1}{2}\langle x',\, x'\rangle_x + \frac{1}{\|x\|^2}. \tag{14}$$

For this functional $\mathcal{B}$, eventual zeros of x cause no problem at all. It has even lots of critical points – one circle worth of critical points for each natural number $k \in \mathbb{N}$, see Lemma 4.

But are the critical points $x \colon \mathbb{S}^1 \to \mathbb{R}$ of $\mathcal{B}$ related to free fall solutions $q \colon \operatorname{dom} q \to (0,\infty)$? If so, what is the domain of q? Next we prepare to give the answers in Proposition 1 below. From now on,

- we only consider critical points $x \in W^{1,2}_\times$ of $\mathcal{B}$ and
- we identify $\mathbb{S}^1$ with $[0,1]/\{0,1\}$.

A priori such $x : [0,1] \to \mathbb{R}$ might have zeros and these might obstruct bijectivity of the classical time t_x defined as before in (7). Indeed the derivative $t'_x(\tau) = x(\tau)^2/\|x\|^2 \geqslant 0$, more properties in (8), might now be zero at certain times – precisely the times of collision of the solution x with the origin. By continuity of the map $x : [0,1] \to \mathbb{R}$, the zero set $T_x := x^{-1}(0)$ is closed, thus compact. On the other hand, the set is discrete because being a critical point $x \not\equiv 0$ solves a second order delay equation, see Corollary 1. We denote the finite set of **regularized collision times** by

$$\mathcal{T}_x := \{\tau \in [0,1] \mid x(\tau) = 0\} = \{\tau_1, \dots, \tau_N\}.$$

Thus, $t_x : [0,1] \to [0,1]$ is still strictly increasing, hence, a bijection. We denote the inverse again by $\tau_x := {t_x}^{-1}$ and the set of **classical collision times**, this terminology will become clear in a moment, by $T_{q_x} := t_x(\mathcal{T}_x)$, i.e.,

$$T_{q_x} := \{t_1, \dots, t_N\}, \quad t_i := t_x(\tau_i).$$

The derivative of $\tau_x : [0,1] \to [0,1]$ is still given by

$$\dot\tau_x(t) = \frac{\|x\|^2}{x(\tau_x(t))^2}, \tag{15}$$

but now only at *noncollision times* t, i.e., $t \in \mathbb{S}^1 \setminus T_{q_x}$.

The following proposition is a special case of a theorem due to Barutello, Ortega, and Verzini [7].

Proposition 1. *Given a critical point $x \in W^{1,2}_\times$ of $\mathcal{B}$, namely a solution of* (1), *then the rescale-squared map $q = q_x := \mathcal{R}(x)$ is a **physical solution** in the sense that q solves the free fall equation at all times*

$$\ddot q(t) = -\frac{1}{q(t)^2}, \qquad \forall t \in \mathbb{S}^1 \setminus T_{q_x}$$

except at the finitely many collision times *which form the set $T_{q_x} = \{t_1, \dots, t_N\}$.*

Think of the critical points x of $\mathcal{B}$ as the regularized versions, "the regularizations" of the physical solutions q_x. Note that when the regularization x runs through the big mass sitting at the origin the physical solution q bounces back.

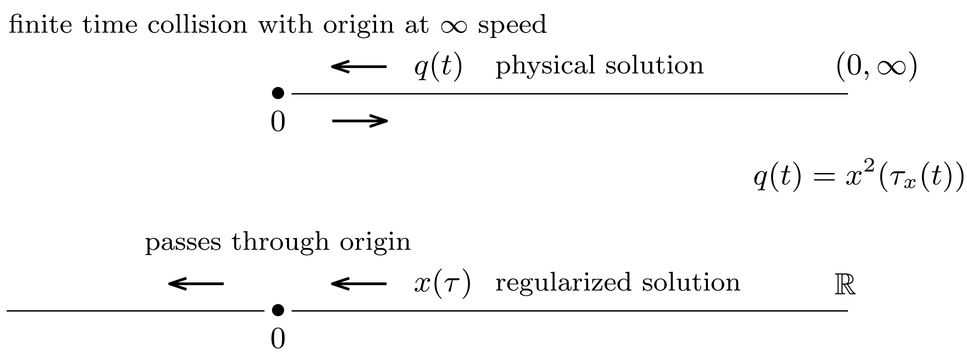

Fig. 2. Physical solution q and regularized solution x.

Proof. At $t \in \mathbb{S}^1 \setminus T_{q_x}$, set $\tau(t) := \tau_x(t)$, the derivative of $q = q_x$ is given by

$$\dot q(t) = 2\, x(\tau(t))\, x'(\tau(t)) \underbrace{\dot\tau(t)}_{(15)} = 2\|x\|^2 \frac{x'(\tau(t))}{x(\tau(t))} \tag{16}$$

and via a change of variables from t to $s := \tau(t)$ we obtain, using (15), that

$$\|\dot q\|^2 = 4\|x\|^2\|x'\|^2 \qquad \text{equivalently} \qquad \langle \dot q_x, \dot q_x\rangle = \langle x', x'\rangle_x. \tag{17}$$

To calculate the second derivative $\ddot{q}$, we use the critical point equation (1)

$$\begin{aligned}\ddot{q}(t) &= \frac{2\|x\|^4}{x(\tau(t))^3}\left(x''(\tau(t)) - \frac{x'(\tau(t))^2}{x(\tau(t))}\right) = \frac{1}{q(t)}\left(2\|x\|^2\|x'\|^2 - \frac{1}{\|x\|^2}\right) - \frac{\dot{q}(t)^2}{2q(t)} \\ &= \frac{1}{q(t)}\left(\frac{1}{2}\|\dot{q}\|^2 - \int_0^1 \frac{ds}{q(s)} - \frac{1}{2}\dot{q}(t)^2\right).\end{aligned} \tag{18}$$

To obtain equation (3), we used the identities (17) and (12). Let $t_-, t_+ \in T_{q_x}$ be neighboring collision times in the sense that the interval (t_-, t_+) lies in the complement of the collision time set T_{q_x}.
Similarly to Barutello, Ortega, and Verzini [7, Eq. (3.12)], we consider

$$\beta = \beta_q := \frac{\ddot{q}}{q}$$

as a function on the open noncollision interval (t_-, t_+). Differentiate the identity $q^2\beta = \left(\frac{1}{2}\|\dot{q}\|^2 - \int_0^1 \frac{dt}{q(t)} - \frac{1}{2}\dot{q}^2\right)$ to obtain that $2q\dot{q}\beta + q^2\dot{\beta} = -\dot{q}\ddot{q} = -q\dot{q}\beta$ or equivalently $q^2\dot{\beta} = -3\beta q\dot{q}$. Hence, the logarithmic derivatives satisfy

$$\frac{\dot{\beta}}{\beta} = -\frac{3\dot{q}}{q}$$

and, therefore,

$$\beta = \frac{\mu}{q^3}$$

for some constant $\mu \in \mathbb{R}$ that a priori might depend on the interval (t_-, t_+). By definition of β, we conclude that

$$\ddot{q} = \frac{\mu}{q^2}$$

on the interval (t_-, t_+). By (18), for the constant μ we find the expression

$$\mu = q(t)\left(\frac{1}{2}\|\dot{q}\|^2 - \int_0^1 \frac{ds}{q(s)}\right) - \frac{1}{2}q(t)\dot{q}(t)^2 = q(t)\left(\frac{1}{2}\|\dot{q}\|^2 - \int_0^1 \frac{ds}{q(s)}\right) - 2\|x\|^4 x'(\tau(t))^2,$$

where, in the second equation, we used (16). Taking the limits $t \to t_\pm$, we obtain

$$\mu = -2\|x\|^4 x'(\tau(t_\pm))^2 \leqslant 0.$$

Thus the constant μ takes the same value on the boundary of adjacent intervals. Hence, the constant μ is independent of the interval.

To see that $\mu = -1$, we multiply (18) by q and use $q'' = \mu/q^2$ to see that

$$\frac{\mu}{q} = \frac{1}{2}\|\dot{q}\|^2 - \int_0^1 \frac{ds}{q(s)} - \frac{1}{2}\dot{q}(t)^2.$$

We take the mean value of this equation, obtaining

$$\mu\int_0^1 \frac{dt}{q(t)} = \frac{1}{2}\|\dot{q}\|^2 - \int_0^1 \frac{dt}{q(t)} - \frac{1}{2}\|\dot{q}\|^2 = -\int_0^1 \frac{dt}{q(t)}.$$

This shows that $\mu = -1$.

3. NONLOCAL LAGRANGIAN MECHANICS

3.1. Nonlocal Lagrangian Action $\mathcal{B}$ and L^2_x Inner Product

In the novel approach [7] to the regularization of collisions discovered recently by Barutello, Ortega, and Verzini, the change of time leads to a *delayed* action functional $\mathcal{B}$. Section 2 above explains this for the free fall. The delay, that is the nonlocal term, is best incorporated into the L^2 inner product on the loop space. More precisely, we introduce a metric on the following Hilbert space without the origin, namely,

$$W^{1,2}_\times := W^{1,2} \setminus \{0\}, \qquad W^{1,2} := W^{1,2}(\mathbb{S}^1, \mathbb{R}).$$

Given a point $x \in W^{1,2}_\times$ and two tangent vectors

$$\xi_1, \xi_2 \in T_x W^{1,2}_\times = W^{1,2},$$

we define the $\boldsymbol{L^2_x}$ **inner product** by

$$\langle \xi_1, \xi_2 \rangle_x := 4\|x\|^2 \langle \xi_1, \xi_2 \rangle, \qquad \text{where } \langle \xi_1, \xi_2 \rangle := \int_0^1 \xi_1(\tau)\xi_2(\tau)\, d\tau, \tag{19}$$

where $\|x\| := \sqrt{\langle x, x \rangle}$ is the L^2 norm associated to the L^2 inner product. In our case of the 1-dimensional Kepler problem, the functional then acquires the form

$$\mathcal{B} : W^{1,2}_\times := W^{1,2}(\mathbb{S}^1, \mathbb{R}) \setminus \{0\} \to (0, \infty), \quad x \mapsto \tfrac{1}{2}\langle x', x' \rangle_x + \frac{1}{\|x\|^2}.$$

One might interpret this functional as a nonlocal mechanical system consisting of kinetic minus potential energy.

3.2. Critical Points and Hessian Operator

Straightforward calculation provides

Lemma 3 (Differential of $\mathcal{B}$)**.** *The differential $d\mathcal{B} : W^{1,2}_\times \times W^{1,2} \to \mathbb{R}$ is*

$$d\mathcal{B}(x, \xi) = 4\langle x, \xi \rangle \|x'\|^2 + 4\|x\|^2 \langle x', \xi' \rangle - 2\frac{\langle x, \xi \rangle}{\|x\|^4} = -\langle x'', \xi \rangle_x + \underbrace{\left(\frac{\|x'\|^2}{\|x\|^2} - \frac{1}{2\|x\|^6} \right)}_{=:\alpha<0} \langle x, \xi \rangle_x,$$

where identity (2) is valid whenever x is of better regularity than $W^{2,2}(\mathbb{S}^1, \mathbb{R}) \setminus \{0\}$.

Note that $\alpha = \alpha_x < 0$. Indeed, $\alpha > 0$ is impossible by the periodicity requirement for solutions and $\alpha = 0$ is impossible, because otherwise, x would vanish identically which is excluded by assumption.

Corollary 1 ($\mathrm{Crit}\,\mathcal{B}$)**.** *The set of critical points consists of the smooth solutions $x \in C^\infty(\mathbb{S}^1, \mathbb{R}) \setminus \{0\}$ of the second order delay equation $x'' = \alpha x$, see* (1).

By Lemma 3 the $\boldsymbol{L^2_x}$ **gradient of** $\boldsymbol{\mathcal{B}}$ at a loop $x \in W^{2,2}_\times$ is given by

$$\mathrm{grad}^x \mathcal{B}(x) = -x'' + \alpha x, \qquad x \in W^{2,2}_\times := W^{2,2}(\mathbb{S}^1, \mathbb{R}) \setminus \{0\}. \tag{20}$$

Solutions of the critical point equation. To find the solutions $x : \mathbb{S}^1 \to \mathbb{R}$ of equation (1), we set $\beta := -\alpha > 0$ to get the second order ODE and its solution

$$x'' = -\beta x, \qquad x(\tau) = C \cos \sqrt{\beta}\tau + D \sin \sqrt{\beta}\tau,$$

where C and D are constants. Thus

$$x' = -C\sqrt{\beta} \sin \sqrt{\beta}\tau + D\sqrt{\beta} \cos \sqrt{\beta}\tau.$$

Since our solutions x is periodic with period 1, we must have

$$\sqrt{\beta} = 2\pi k, \quad k \in \mathbb{N}. \tag{21}$$

Thus $x(\tau) = C \cos 2\pi k\tau + D \sin 2\pi k\tau$ and

$$x'(\tau) = -2\pi k C \sin 2\pi k\tau + 2\pi k D \cos 2\pi k\tau.$$

Note that integrating the identity $1 = \cos^2 + \sin^2$, one obtains

$$1 = \int_0^1 1\, d\tau = \int_0^1 \cos^2\, d\tau + \int_0^1 \sin^2\, d\tau = 2\int_0^1 \cos^2(2\pi k\tau)\, d\tau \tag{22}$$

as is well known from the theory of Fourier series. Moreover, from the theory of Fourier series, it is known that cosine is orthogonal to sine, i.e.,

$$0 = \int_0^1 \cos(2\pi k\tau)\sin(2\pi k\tau)\,d\tau.$$

Therefore, $\|x\|^2 = \frac{C^2+D^2}{2}$ and $\|x'\|^2 = \frac{C^2+D^2}{2}(2\pi k)^2$ and so

$$(2\pi k)^2 \overset{(21)}{=} \beta^2 \overset{(1)}{=} \frac{1}{2\|x\|^6} - \frac{\|x'\|^2}{\|x\|^2} = \frac{4}{(C^2+D^2)^3} - (2\pi k)^2, \quad k \in \mathbb{N}. \tag{23}$$

We fix the parametrization of our solution by requiring that, at time zero, the solution be maximal. Therefore, $D = 0$ and we abbreviate $c_k := C(k) > 0$. Then c_k is uniquely determined by k via the above equation which becomes

$$\frac{4}{c_k^6} = 8\pi^2 k^2, \qquad \frac{1}{c_k^6} = 2(\pi k)^2. \tag{24}$$

Hence,

$$c_k = \frac{1}{2^{\frac{1}{6}}(\pi k)^{\frac{1}{3}}} \in (0,1), \qquad c_k^2 = \frac{1}{2^{\frac{1}{3}}(\pi k)^{\frac{2}{3}}}. \tag{25}$$

Note that $c_k < 1$. This proves the following lemma.

Lemma 4 (Critical points of $\mathcal{B}$)**.** *The maps*

$$x_k(\tau) = c_k \cos 2\pi k\tau, \qquad k \in \mathbb{N}, \tag{26}$$

are solutions of (1)*. Each map x_k is worth a circle of solutions via time shift $\sigma_* x_k := x_k(\cdot + \sigma)$, where $\sigma \in \mathbb{S}^1$. All solutions of* (1) *are given by*

$$\operatorname{Crit}\mathcal{B} = \bigcup_{k\in\mathbb{N}} \{\sigma_* x_k \mid \sigma \in \mathbb{S}^1\}.$$

The Hessian operator with respect to the L^2_x inner product. The **Hessian operator** A_x of the Lagrange functional $\mathcal{B}$ is the derivative of the L^2_x gradient equation $0 = -x'' + \alpha x$ at a critical point x. Varying this equation with respect to x in the direction ξ, we obtain

$$\begin{aligned} A_x\xi &= -\ddot{\xi} + \alpha\xi + \left(-\frac{2\langle x'', \xi\rangle}{\|x\|^2} - \frac{2\|x'\|^2\langle x, \xi\rangle}{\|x\|^4} + \frac{3\langle x, \xi\rangle}{\|x\|^8}\right) x \\ &= -\ddot{\xi} + \alpha\xi - \left(\frac{2\alpha}{\|x\|^2} + \frac{2\|x'\|^2}{\|x\|^4} - \frac{3}{\|x\|^8}\right)\langle x, \xi\rangle x \\ &= -\ddot{\xi} + \alpha\xi - \frac{2}{\|x\|^2}\left(2\alpha - \frac{1}{\|x\|^6}\right)\langle x, \xi\rangle x, \end{aligned}$$

where, in the second step, we used the critical point equation $x'' = \alpha x$ and, in the third step, we replaced $2\|x'\|^2$ by $2\alpha\|x\|^2 + \frac{1}{\|x\|^4}$ according to the definition of α. This proves

Lemma 5. *The Hessian operator of $\mathcal{B}$ at a critical point x is given by*

$$A_x : W^{2,2}(\mathbb{S}^1, \mathbb{R}) \to L^2(\mathbb{S}^1, \mathbb{R}), \quad \xi \mapsto -\ddot{\xi} + \alpha\xi - \frac{2}{\|x\|^2}\left(2\alpha - \frac{1}{\|x\|^6}\right)\langle x, \xi\rangle x.$$

Recall that, by (26), the critical points of $\mathcal{B}$ are of the form

$$x_k(\tau) = c_k \cos 2\pi k\tau, \qquad k \in \mathbb{N}, \tag{27}$$

with c_k given by (25). Taking two τ derivatives, we conclude that

$$x_k'' = -(2\pi k)^2 x_k.$$

Since $x_k'' = \alpha x_k$, we obtain

$$\alpha = \alpha(x_k) = -(2\pi k)^2 = -\frac{2}{c_k^6},$$

where the last equality is (24). The formula of the Hessian operator A_{x_k} involves the L^2 norm of x_k and, in addition, the formula of the nonlocal Lagrange functional $\mathcal{B}$ involves $\|x_k'\|^2$. By (26) and (22), we obtain that

$$\|x_k\|^2 = \frac{c_k^2}{2} = \frac{1}{2^{\frac{4}{3}}(\pi k)^{\frac{2}{3}}}, \qquad \|x_k'\|^2 = (2\pi k)^2 \frac{c_k^2}{2} = 2^{\frac{2}{3}}(\pi k)^{\frac{4}{3}}.$$

Thus

$$\mathcal{B}(x_k) = 2\|x_k\|^2\|x_k'\|^2 + \frac{1}{\|x_k\|^2} = 2^{\frac{1}{3}}3(\pi k)^{\frac{2}{3}}.$$

To calculate the formula of A_{x_k}, we write ξ as a Fourier series

$$\xi = \xi_0 + \sum_{n=1}^{\infty} (\xi_n \cos 2\pi n\tau + \xi^n \sin 2\pi n\tau)$$

and we use the orthogonality relation

$$\langle \cos 2\pi n\cdot, \, \xi\rangle = \frac{1}{2}\xi_n$$

to calculate the product

$$\langle x_k, \, \xi\rangle = \tfrac{1}{2}c_k\xi_k.$$

Putting everything together, we obtain the following lemma.

Lemma 6 (Critical values and Hessian)**.** *The critical points of* $\mathcal{B}$ *are of the form* $x_k(\tau) = c_k \cos 2\pi k\tau$ *for* $k \in \mathbb{N}$, *see* (27). *At any such* x_k, *the value of* $\mathcal{B}$ *is*

$$\mathcal{B}(x_k) = 2^{\frac{1}{3}}3(\pi k)^{\frac{2}{3}}$$

and the Hessian operator of $\mathcal{B}$ *is*

$$A_{x_k}\xi = -\ddot{\xi} - (2\pi k)^2\xi + 12(2\pi k)^2\xi_k \cos 2\pi k\tau$$

for every $\xi \in W^{2,2}(\mathbb{S}^1, \mathbb{R})$.

3.3. Eigenvalue Problem and Morse Index

Recall that $k \in \mathbb{N}$ is fixed, since we consider the critical point x_k. We are looking for solutions of the eigenvalue problem

$$A_{x_k}\xi = \mu\xi$$

for $\mu = \mu(\xi; k) \in \mathbb{R}$ and $\xi \in W^{2,2}(\mathbb{S}^1, \mathbb{R}) \setminus \{0\}$. Observe that

$$-\ddot{\xi} = \sum_{n=1}^{\infty} (2\pi n)^2 \left(\xi_n \cos 2\pi n\tau + \xi^n \sin 2\pi n\tau\right),$$

$$-(2\pi k)^2\xi = -(2\pi k)^2\xi_0 - (2\pi k)^2 \sum_{n=1}^{\infty} \left(\xi_n \cos 2\pi n\tau + \xi^n \sin 2\pi n\tau\right).$$

Comparing coefficients in the eigenvalue equation $A_{x_k}\xi = \mu\xi$, we obtain

$$\begin{aligned} \cos 2\pi n\tau \quad & \begin{cases} \mu\xi_n = 4\pi^2\left(n^2 - k^2\right)\xi_n, & \forall n \in \mathbb{N}_0 \setminus \{k\}, \\ \mu\xi_k = 12(2\pi k)^2\xi_k, & n = k, \end{cases} \\ \sin 2\pi n\tau \quad & \begin{cases} \mu\xi^n = 4\pi^2\left(n^2 - k^2\right)\xi^n, & \forall n \in \mathbb{N}. \end{cases} \end{aligned}$$

This proves the following lemma.

Lemma 7 (Eigenvalues)**.** *The eigenvalues of the Hessian* A_{x_k} *are given by*

$$\mu_n := 4\pi^2(n^2 - k^2), \qquad n \in \mathbb{N} \setminus \{k\},$$

and by

$$\mu_0 := -4\pi^2k^2, \qquad \mu_k := 0, \qquad \widehat{\mu}_k := 12(2\pi k)^2.$$

Moreover, their multiplicity (the dimension of the eigenspace) is given by

$$m(\mu_n) = 2, \quad n \in \mathbb{N} \setminus \{k\}, \qquad m(\mu_0) = m(\mu_k) = m(\mu_k) = 1.$$

Observe that the eigenvalue $\widehat{\mu}_k \neq \mu_n$ is different from μ_n for every $n \in \mathbb{N}_0$. Indeed, suppose by contradiction that $\widehat{\mu}_k = \mu_n$ for some $n \in \mathbb{N}_0$, i.e.,

$$48\pi^2 k^2 = 4\pi^2(n^2 - k^2) \quad \Leftrightarrow \quad 13k^2 = n^2,$$

which contradicts the fact that n is an integer.

Proposition 2 (Morse index of x_k)**.** *The number of negative eigenvalues of the Hessian A_{x_k}, called the* ***Morse index*** *of $x_k \in \operatorname{Crit} \mathcal{B}$, is odd and is given by*

$$\operatorname{Ind}(x_k) = 2k - 1, \qquad k \in \mathbb{N}.$$

Note that, by the lemma, the bounded below functional $\mathcal{B} > 0$ has no critical point of index zero, in other words the functional $\mathcal{B} > 0$ has no minimum.

4. CONLEY ZEHNDER INDEX VIA SPECTRAL FLOW

In the local case, the Conley–Zehnder index can be defined as a version of the celebrated Maslov index [4, 13]. In the nonlocal case, we do not have a flow and, therefore, the interpretation as an intersection number of the linearized flow trajectory with the Maslov cycle is not available.

In this section, we recall the approach in the local case of Hofer, Wysocki, and Zehnder [10] to the Conley–Zehnder index via winding numbers of eigenvalues of the Hessian. In Section 5, we will see that this theory can be generalized to the nonlocal case, namely, for the Hessian of the nonlocal functional $\mathcal{A}_{\mathcal{H}}$.

In what follows we identify the unit circle $\mathbb{S}^1 \subset \mathbb{R}^2$ with $\mathbb{R}/\mathbb{Z}$ and $\mathbb{R}^2 \simeq \mathbb{C}$ via $(x, y) \mapsto x+iy$. Multiplication by i on $\mathbb{C}$ is expressed by J_0 on $\mathbb{R}^2$, i.e.,

$$J_0 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Moreover, we think of maps f with domain $\mathbb{S}^1$ as 1-periodic maps defined on $\mathbb{R}$, that is $f(t+1) = f(t)$, for all $t \in \mathbb{R}$.

To a continuous path[1] of symmetric 2×2 matrices $S : [0, 1] \to \mathbb{R}^{2\times 2}$, $t \mapsto S(t) = S(t)^T$, we associate the operator

$$L_S : L^2(\mathbb{S}^1, \mathbb{R}^2) \supset W^{1,2} \to L^2(\mathbb{S}^1, \mathbb{R}^2), \quad \zeta \mapsto -J_0\dot{\zeta} - S\zeta, \tag{28}$$

where $\dot{\zeta} := \frac{d}{dt}\zeta$. This operator is an unbounded self-adjoint operator whose resolvent is compact (due to the compactness of the embedding $W^{1,2} \hookrightarrow L^2$). Therefore, the spectrum is discrete and consists of real eigenvalues of finite multiplicity. Given an eigenvalue $\lambda \in \operatorname{spec} L_S$, then an eigenvector ζ corresponding to λ is a nonconstantly vanishing solution $\zeta \not\equiv 0$ of the first order ODE

$$\dot{\zeta} = J_0(S + \lambda \mathbb{I})\zeta$$

for absolutely continuous 1-periodic maps $\zeta : \mathbb{R} \to \mathbb{R}^2$. In particular, since by definition an eigenvector does not vanish identically, it does not vanish anywhere, in symbols $\zeta(t) \neq 0$ for every t. Therefore, we can associate to eigenvectors ζ a **winding number** $w(\zeta) \in \mathbb{Z}$ by looking at the degree of the map

$$\mathbb{S}^1 \to \mathbb{S}^1, \quad t \mapsto \frac{\zeta(t)}{|\zeta(t)|}.$$

This winding number only depends on the eigenvalue λ and not on the particular eigenvector for λ. Indeed, if the geometric multiplicity of λ is 1, then a different eigenvector for λ is of the form $r\zeta$ for some nonzero real $r \neq 0$. If the geometric multiplicity of λ is grater than 1, then the space E_λ of eigenvectors to λ is a vector space of dimension at least 2 minus the origin, in particular, it is path connected. The winding number has to be constant on E_λ, because it is discrete and depends continuously on the eigenvector. In view of these findings, we write

$$w(\lambda) = w(\lambda; S)$$

for the **winding number associated to an eigenvalue $\boldsymbol{\lambda}$** of the operator L_S associated to the family S of symmetric matrices.

[1]Because the elements of the target L^2 of L_S are not necessarily continuous, a requirement to close up after time 1 would be meaningless, hence, paths $S : [0, 1] \to \mathbb{R}^{2\times 2}$ are fine.

Remark 1 (Case $S=0$)**.** Each integer $\ell \in \mathbb{Z}$ is realized as the winding number of the eigenvalue $\lambda = 2\pi\ell$ of L_0 of geometric multiplicity 2. To see this, note that the spectrum of $L_0 = -J_0\frac{d}{dt}$ is $2\pi\mathbb{Z}$. Indeed, pick an integer $\ell \in \mathbb{Z}$. Then the eigenvectors of L_0 corresponding to $2\pi\ell$ are of the form $t \mapsto ze^{2\pi i\ell t}$, where $z \in \mathbb{C} \setminus \{0\}$. Therefore, the winding number associated to ℓ is

$$w(\ell; 0) = \ell.$$

The geometric multiplicity of each eigenvalue ℓ is 2 (pick $z=1$ or $z=i$).

Remark 2 (General S)**.** Each integer $\ell \in \mathbb{Z}$ is realized *either* as the winding number of two different eigenvalues of L_S of geometric multiplicity 1 each

$$\ell = w(\lambda_1; S) = w(\lambda_2; S)$$

or as the winding number of a single eigenvalue of geometric multiplicity 2. To see this, pick $\ell \in \mathbb{Z}$. Consider the family $\{rS\}_{r\in[0,1]}$. By Kato's perturbation theory, we can choose continuous functions $\lambda_1(r)$ and $\lambda_2(r)$, $r \in [0,1]$, such that

(i) $\lambda_1(0) = 2\pi\ell$ and $\lambda_2(0) = 2\pi\ell$,

(ii) $\lambda_1(r)$, $\lambda_2(r) \in \operatorname{spec} L_{rS}$ and the total geometric multiplicity remains 2, i.e.,

$$\dim\left(E_{\lambda_1(r)} \oplus E_{\lambda_2(r)}\right) = 2,$$

where $E_{\lambda_i(r)}$ is the eigenspace of the eigenvalue $\lambda_i(r)$,

(iii) $w(\lambda_1(r); rS) = \ell$ and $w(\lambda_2(r); rS) = \ell$.

The third assertion follows from the fact that the function $r \mapsto w(\lambda_1(r); rS)$ is continuous and takes value in the discrete set $\mathbb{Z}$ and, because we have $w(\ell; 0) = \ell$ by Remark 1.

Moreover, the winding number continues to be monotone in the eigenvalue as the next lemma shows.

Lemma 8 (Monotonicity of w)**.** $\lambda_1 < \lambda_2 \in \operatorname{spec} L_S \Rightarrow w(\lambda_1) \leqslant w(\lambda_2)$.

Proof. By contradiction, suppose that there are $\lambda_1 < \lambda_2 \in \operatorname{spec} L_S$ such that their winding numbers satisfy

$$\ell_1 := w(\lambda_1; S) > w(\lambda_2; S) =: \ell_2.$$

Because eigenvalues depend continuously on the operator L_S by Kato's perturbation theory, looking at the family $\{rS\}_{r\in[0,1]}$, we can choose continuous functions $\lambda_1(r)$ and $\lambda_2(r)$ with $r \in [0,1]$ having the following properties for every $r \in [0,1]$

(i) $\lambda_1(1) = \lambda_1$ and $\lambda_2(1) = \lambda_2$,

(ii) $\lambda_1(r)$, $\lambda_2(r) \in \operatorname{spec} L_{rS}$,

(iii) $w(\lambda_1(r); rS) = \ell_1$ and $w(\lambda_2(r); rS) = \ell_2$.

From the case $S=0$, it follows that $\lambda_1(0) = w(\lambda_1(0); 0)$, which is ℓ_1 by (iii), and that $\lambda_2(0) = w(\lambda_2(0); 0)$, which is ℓ_2. Thus $\lambda_1(0) = \ell_1 > \ell_2 = \lambda_2(0)$. Because ℓ_1 is different from ℓ_2, it follows from (iii) as well that $\lambda_1(r)$ is different from $\lambda_2(r)$ for every $r \in [0,1]$. Therefore, by continuity, we must have $\lambda_1(r) > \lambda_2(r)$ for every $r \in [0,1]$. Hence, by (i), we have that $\lambda_1 > \lambda_2$. Contradiction.

Definition 2. Denote the **largest winding number among *negative* eigenvalues** of L_S by

$$\alpha(S) := \max\{w(\lambda) \mid \lambda \in (-\infty, 0) \cap \operatorname{spec} L_S\} \in \mathbb{Z}.$$

If both eigenvalues of L_S (counted with multiplicity) whose winding number is $\alpha(S)$ are negative, then S has **parity** $p(S) := 1$. Otherwise, define $p(S) := 0$.

Definition 3 (Conley–Zehnder index of path S)**.** According to Hofer, Wysocki, and Zehnder [10], the counter-clockwise normalized Conley–Zehnder index μ_{CZ} of a continuous family of symmetric 2×2 matrices $S : [0,1] \to \mathbb{R}^{2\times 2}$ is

$$\mu_{\mathrm{CZ}}(S) := 2\alpha(S) + p(S).$$

5. NONLOCAL HAMILTONIAN MECHANICS

5.1. Nonlocal Hamiltonian Equations

Recall from (14) that the functional $\mathcal{B} : W^{1,2}_\times \to \mathbb{R}$, $x \mapsto \frac{1}{2}\langle x', x'\rangle_x + \frac{1}{\|x\|^2}$, extends the classical action $\mathcal{S}_L : W^{1,2}_+ \to \mathbb{R}$. Then the functional defined by

$$\mathcal{L} : W^{1,2}_\times \times L^2 \to \mathbb{R}, \quad (x, \xi) \mapsto \tfrac{1}{2}\langle \xi, \xi\rangle_x + \frac{1}{\|x\|^2}, \tag{29}$$

naturally extends $\mathcal{B}(x) = \mathcal{L}(x, x')$ in the same way as in the classical case $\mathcal{S}_L(q) = \mathcal{L}_L(q, \dot{q})$ is extended by a corresponding functional $\mathcal{L}_L(q, v)$.

In the classical case, a fiberwise strictly convex Lagrange function L on the tangent bundle determines a function H on the cotangent bundle: let us solve $\mathfrak{p} := d_\mathfrak{v} L(r, \mathfrak{v})$ for $\mathfrak{v}$ and substitute the obtained $\mathfrak{v} = \mathfrak{v}(\mathfrak{p})$ in the Legendre identity

$$\langle \mathfrak{p}, \mathfrak{v}\rangle = L(r, \mathfrak{v}) + H(r, \mathfrak{p}).$$

Returning to the nonlocal situation where the manifold is loop space and (x, ξ) and (x, y) are pairs of loops, an analogous approach yields

$$y := d_\xi \mathcal{L}(x, \xi) = 4\|x\|^2 \xi, \qquad \xi = \tfrac{1}{4\|x\|^2} y, \qquad d_{\xi\xi}\mathcal{L}(x, \xi) = 4\|x\|^2 > 0.$$

The **nonlocal Hamiltonian function** is then defined by

$$\mathcal{H}(x, y) := \langle y, \xi\rangle - \mathcal{L}(x, \xi)$$

and given by the formula

$$\mathcal{H} : W^{1,2}_\times \times L^2 \to \mathbb{R}, \quad (x, y) \mapsto \frac{1}{2}\langle y, y\rangle^x - \frac{1}{\|x\|^2} = \tfrac{1}{4\|x\|^2}\left(\tfrac{1}{2}\|y\|^2 - 4\right).$$

The Hamiltonian equations of the nonlocal Hamiltonian $\mathcal{H}$ are the following

$$\begin{cases} x' = \partial_y \mathcal{H} = \dfrac{1}{4\|x\|^2}\, y, \\ y' = -\partial_x \mathcal{H} = \dfrac{x}{\|x\|^4}\left(\dfrac{\|y\|^2}{4} - 2\right) \end{cases} \tag{30}$$

for smooth solutions $x, y : \mathbb{S}^1 \to \mathbb{R}$ with $x \not\equiv 0$.

5.2. Hamiltonian and Euler–Lagrange Solutions Correspond

Lemma 9 (Correspondence of Hamiltonian and Lagrangian solutions)**.**

(a) *If* (x, y) *solves the Hamiltonian equations* (30)*, then* x *solves the Lagrangian equations* (1).

(b) *Vice versa, if* x *solves the Lagrangian equations* (1)*, then* (x, y_x)*, where*

$$y_x := 4\|x\|^2\, x'$$

solves the Hamiltonian equations (30).

Proof. (a) The first equation in (30) leads to $\|x'\|^2 = \|y\|^2/16\|x\|^4$, which we solve for $\|y\|^2$ and then plug it into the second equation to obtain

$$y' = \frac{x}{\|x\|^4}\left(4\|x\|^4\|x'\|^2 - 2\right).$$

Now take the time τ derivative of the first equation to obtain indeed

$$x'' = \frac{1}{4\|x\|^2}\, y' = \frac{1}{4\|x\|^2}\,\frac{x}{\|x\|^4}\left(4\|x\|^4\|x'\|^2 - 2\right) = \frac{\|\dot{x}\|^2}{\|x\|^2}\, x - \frac{1}{2\|x\|^6}\, x = \alpha x,$$

(b) Suppose x solves (1) and define $y = y_x := 4\|x\|^2 x'$, hence, $\|y\|^2 = 16\|x\|^2\|x'\|^2$. Solve for x', we obtain the first of the Hamilton equations (30). To obtain the second equation, take the time τ derivative of y and use the Lagrange equation for x'' and conclude that

$$y' = 4\|x\|^2 x'' = 4\|x\|^2\left(\frac{\|x'\|^2}{\|x\|^2} - \frac{1}{2\|x^6}\right) x = \left(\frac{\|y\|^2}{4\|x\|^4} - \frac{2}{\|x\|^4}\right) x = \frac{x}{\|x\|^4}\left(\frac{\|y\|^2}{4} - 2\right),$$

where, in the third step, we replaced $\|x'\|^2$ according to $\|y\|^2 = 16\|x\|^2\|x'\|^2$.

5.3. Nonlocal Hamiltonian Action $\mathcal{A}_{\mathcal{H}}$

The **symplectic area functional** is defined by

$$\mathcal{A}_0 : W^{1,2} \times L^2 \to \mathbb{R}, \quad (x, y) \mapsto \int_0^1 y(\tau) x'(\tau)\, d\tau$$

and the **nonlocal Hamiltonian action functional** by

$$\mathcal{A}_{\mathcal{H}} := \mathcal{A}_0 - \mathcal{H} : W^{1,2}_{\times} \times L^2 \to \mathbb{R}, \quad (x, y) \mapsto \int_0^1 y(\tau) x'(\tau)\, d\tau - \mathcal{H}(x, y).$$

The derivative $d\mathcal{A}_{\mathcal{H}}(x, y) : W^{1,2} \times L^2 \to \mathbb{R}$ is given by

$$\begin{aligned} d\mathcal{A}_{\mathcal{H}}(x, y)(\xi, \eta) &= \int_0^1 (y\xi' + \eta x')\, d\tau - \frac{\langle y, \eta\rangle}{4\|x\|^2} + \left(\tfrac{1}{4}\|y\|^2 - 2\right) \frac{\langle x, \xi\rangle}{\|x\|^4} \\ &= -\langle y', \xi\rangle + \langle x', \eta\rangle - \frac{\langle y, \eta\rangle}{4\|x\|^2} + \left(\tfrac{1}{4}\|y\|^2 - 2\right) \frac{\langle x, \xi\rangle}{\|x\|^4} \\ &= \left\langle x' - \frac{y}{4\|x\|^2}, \eta \right\rangle + \left\langle -y' + \frac{\frac{1}{4}\|y\|^2 - 2}{\|x\|^4} x, \xi \right\rangle \end{aligned} \tag{31}$$

and this proves the following lemma.

Lemma 10. *The critical points of $\mathcal{A}_{\mathcal{H}}$ are precisely the 1-periodic solutions of the Hamiltonian equations (30) of $\mathcal{H}$.*

In view of Lemmas 9 and 10, we have a one-to-one correspondence between critical points of $\mathcal{B}$ and $\mathcal{A}_{\mathcal{H}}$, namely, $x \mapsto (x, y_x)$. Under this correspondence, the values of the two functionals coincide along critical points, see Lemma 11.

5.4. Lagrangian Action $\mathcal{B}$ Dominates Hamiltonian One $\mathcal{A}_{\mathcal{H}}$

Lemma 11 (Lagrangian domination)**.** *There are the identities*

$$\mathcal{B}(x) = \mathcal{A}_{\mathcal{H}}(x, y) + \frac{1}{2} \left\| 2\|x\|\, x' - \frac{y}{2\|x\|} \right\|^2 = \mathcal{A}_{\mathcal{H}}(x, y) + \frac{1}{2} \left\| 4\|x\|^2\, x' - y \right\|^2_{\frac{1}{4\|x\|^2}}$$

for every pair of loops $(x, y) \in W^{1,2}_{\times} \times W^{1,2}$.

Proof. Using the definition of $\mathcal{A}_{\mathcal{H}}$ and multiplying out the inner product, we obtain

$$\begin{aligned} &\mathcal{A}_{\mathcal{H}}(x, y) + \frac{1}{2} \left\| 2\|x\| x' - \frac{y}{2\|x\|} \right\|^2 \\ &\quad = \langle y, x'\rangle - \frac{\|y\|^2}{8\|x\|^2} + \frac{1}{\|x\|^2} + \frac{1}{2} \left\langle 2\|x\| x' - \frac{y}{2\|x\|} \right\rangle 2\|x\| x' - \frac{y}{2\|x\|} \\ &\quad = 2\|x\|\|x'\|^2 + \frac{1}{\|x\|^2} = \mathcal{B}(x). \end{aligned}$$

Corollary 2 (Equal values on critical points)**.** *The two functionals coincide*

$$\mathcal{B}(x) = \mathcal{A}_{\mathcal{H}}(x, y_x), \qquad y_x := 4\|x\|^2\, x'$$

on critical points, i.e., the solutions x of (1)*, equivalently (x, y_x) of* (30)*.*

In terms of the projection π and injection ι in (2), the corollary tells that

$$\mathcal{B} = \mathcal{A}_{\mathcal{H}} \circ \iota, \qquad \mathcal{B} \circ \pi = \mathcal{A}_{\mathcal{H}}$$

along critical points of $\mathcal{B}$, respectively of $\mathcal{A}_{\mathcal{H}}$.

The diffeomorphism $\mathbb{L}$. Given the functional $\mathcal{L}$ in (29), define the nonlocal analogue of the diffeomorphism introduced in [6, p. 1891], in the local context, by the formula

$$\mathbb{L}:\,(TL^2_\times)_0 = W^{1,2}_\times \times L^2 \to W^{1,2}_\times \times L^2 = (T^*L^2_\times)_0, \quad (x,\xi)\mapsto (x, d_\xi\mathcal{L}(x,x'+\xi)) =: (x,y),$$

where $(TL^2_\times)_0 = (L^2_\times)_1 \times (L^2)_0$ is scale calculus notation, cf. [8, Sec. 4], and

$$y := d_\xi\mathcal{L}(x,x'+\xi) = 4\|x\|^2\,(x'+\xi).$$

Note that since the inverse is given by

$$\mathbb{L}^{-1}(x,y) = \left(x, \frac{y}{4\|x\|^2} - x'\right) =: (x,\xi),$$

the solutions (x,y) of the Hamiltonian equations (30) are zeros of $\mathbb{L}^{-1}$.

As in the ODE case [6] also in the present delay equation situation, both functionals are related through the maps ι and π (Lemma 11) in the form

$$\mathcal{B}\circ\pi(x,y) = \mathcal{A}_{\mathcal{H}}(x,y) + \mathcal{U}^*(x,y), \qquad \mathcal{U}^*(x,y) := \tfrac{1}{2}\left\|\iota(x)-y\right\|^2 \tfrac{1}{4\|x\|^2}$$

for every $(x,y)\in W^{2,2}_\times \times W^{1,2}$. Observe that the map $\mathcal{U}^* \geqslant 0$ vanishes precisely along the critical points.

With the nonnegative functional $\mathcal{U}$ defined and given by

$$\mathcal{U}(x,\xi) := \mathcal{U}^*\circ\mathbb{L}(x,\xi) = \tfrac{1}{2}\left\langle \xi,\,\xi\right\rangle_x \geqslant 0,$$

the functionals $\mathcal{A}_{\mathcal{H}}$ and $\mathcal{B}$ are related by the formula

$$\mathcal{A}_{\mathcal{H}}\circ\mathbb{L}(x,\xi) = \mathcal{B}(x) - \mathcal{U}(x,\xi) = \tfrac{1}{2}\left\langle x',\,x'\right\rangle_x + \frac{1}{\|x\|^2} - \tfrac{1}{2}\left\langle \xi,\,\xi\right\rangle_x.$$

5.5. Critical Points and Hessian

Defining

$$a := \frac{1}{4\|x\|^2} > 0, \qquad b := \left(2 - \frac{\|y\|^2}{4}\right)\frac{1}{\|x\|^4} > 0,$$

we obtain

$$\begin{cases} x' = ay, \\ y' = -bx. \end{cases} \tag{32}$$

We shall prove that $b>0$. Since our solution has to be periodic, we conclude that b has to be nonnegative. We claim that b has to be actually positive. Otherwise, we have the ODE $x'=ay$ and $y'=0$. Thus $x''=0$, so x is linear. But as the solution must be periodic x has to be constant. This implies that x' and y are zero, in particular, $\|y\|^2=0$. Thus $b\neq 0$. Contradiction. This shows that $b>0$.

We get the second order ODE and its solution

$$x'' = -abx, \qquad q(\tau) = c\cos\sqrt{ab}\tau + d\sin\sqrt{ab}\tau.$$

Thus

$$x' = -c_1\sqrt{ab}\sin\sqrt{ab}\tau + c_2\sqrt{ab}\cos\sqrt{ab}\tau$$

and

$$y(\tau) = \frac{x'(\tau)}{a} = -c_1\sqrt{b/a}\sin\sqrt{ab}\tau + c_2\sqrt{b/a}\cos\sqrt{ab}\tau.$$

Since our solutions x,y are periodic with period 1, we must have

$$\sqrt{ab} = 2\pi k, \quad k\in\mathbb{N}. \tag{33}$$

Thus

$$x(\tau) = C\cos 2\pi k\tau + D\sin 2\pi k\tau, \tag{34}$$

where C and D are constants and

$$y(\tau) = -C\frac{2\pi k}{a}\sin 2\pi k\tau + D\frac{2\pi k}{a}\cos 2\pi k. \tag{35}$$

Note that integrating the identity $1 = \cos^2 + \sin^2$ we can write

$$1 = \int_0^1 1\, d\tau = \int_0^1 \cos^2\, d\tau + \int_0^1 \sin^2\, d\tau = 2\int_0^1 \cos^2(2\pi k\tau)\, d\tau, \tag{36}$$

which is well known from the theory of Fourier series. Moreover, from the theory of Fourier series, it is known that cosine is orthogonal to sine, that is

$$0 = \int_0^1 \cos(2\pi k\tau)\sin(2\pi k\tau)\, d\tau.$$

Therefore, $\|x\|^2 = (C^2 + D^2)/2$ and so for a we obtain the value

$$a = \frac{1}{2\,(C^2 + D^2)}.$$

Similarly, $\|y\|^2 = \frac{C^2+D^2}{2}\left(\frac{2\pi k}{a}\right)^2 = 2(C^2 + D^2)^3(2\pi k)^2$ and so for b we have

$$b = \frac{8}{(C^2 + D^2)^2} - 2(C^2 + D^2)(2\pi k)^2.$$

Using (33), we see that

$$(2\pi k)^2 = ab = \frac{4}{(C^2 + D^2)^3} - (2\pi k)^2, \quad k \in \mathbb{N}. \tag{37}$$

We fix the parametrization of our solution by requiring that, at time zero, the solution be maximal. Therefore, $D = 0$ and we abbreviate $c_k := C(k) > 0$. Then c_k is uniquely determined by k via the above equation, which becomes

$$\frac{4}{c_k^6} = 8\pi^2 k^2, \qquad \frac{1}{c_k^6} = 2(\pi k)^2. \tag{38}$$

Hence,

$$c_k = \frac{1}{2^{\frac{1}{6}}\,(\pi k)^{\frac{1}{3}}} \in (0,1), \qquad c_k^2 = \frac{1}{2^{\frac{1}{3}}\,(\pi k)^{\frac{2}{3}}}. \tag{39}$$

Note that $c_k < 1$. Thus, for each $k \in \mathbb{N}$, there is a solution of (32), namely

$$\begin{cases} x_k(\tau) = c_k \cos 2\pi k\tau, \\ y_k(\tau) = -2c_k^3(2\pi k)\sin 2\pi k\tau, \end{cases} \qquad k \in \mathbb{N}. \tag{40}$$

Linearization. Linearizing the Hamilton equations (30) at a solution (x, y), we obtain

$$\begin{cases} \xi' = -\dfrac{y}{2\|x\|^4}\langle x,\, \xi\rangle + \dfrac{\eta}{4\|x\|^2}, \\ \eta' = \left(\dfrac{\xi}{\|x\|^4} - \dfrac{4x}{\|x\|^6}\,\langle x,\, \xi\rangle\right)\left(\dfrac{\|y\|^2}{4} - 2\right) + \dfrac{1}{2}\dfrac{x}{\|x\|^4}\,\langle y,\, \eta\rangle \end{cases} \tag{41}$$

for function pairs

$$\zeta = (\xi, \eta) \in W^{1,2}(\mathbb{S}^1, \mathbb{R}^2).$$

Consider the linear operator defined by

$$\mathcal{S} = \mathcal{S}_{(x,y)} : W^{1,2}(\mathbb{S}^1, \mathbb{R}^2) \to W^{1,2}(\mathbb{S}^1, \mathbb{R}^2), \quad \begin{pmatrix} \xi \\ \eta \end{pmatrix} \mapsto \begin{pmatrix} \left(\frac{\xi}{\|x\|^4} - \frac{4x}{\|x\|^6}\,\langle x,\, \xi\rangle\right)\left(\frac{\|y\|^2}{4} - 2\right) + \frac{1}{2}\frac{x}{\|x\|^4}\,\langle y,\, \eta\rangle \\ \frac{y}{2\|x\|^4}\langle x,\, \xi\rangle - \frac{\eta}{4\|x\|^2} \end{pmatrix}$$

and the linear operator defined by

$$A_{(x,y)} = L_{\mathcal{S}} : L^2(\mathbb{S}^1, \mathbb{R}^2) \supset W^{1,2} \to L^2(\mathbb{S}^1, \mathbb{R}^2), \quad \zeta := \begin{pmatrix} \xi \\ \eta \end{pmatrix} \mapsto -J_0\zeta' - I\mathcal{S}_{(x,y)}\zeta,$$

where $I : W^{1,2}(\mathbb{S}^1, \mathbb{R}^2) \hookrightarrow L^2(\mathbb{S}^1, \mathbb{R}^2)$ is the compact operator given by inclusion. The kernel of the operator $L_{\mathcal{S}}$ is composed of the solutions to the linearized equations (41). If $(x, y) \in \operatorname{Crit} \mathcal{A}_{\mathcal{H}}$ is a critical point, then $L_{\mathcal{S}}$ is equal to the Hessian operator $A_{(x,y)}$ of $\mathcal{A}_{\mathcal{H}}$ at (x, y).

5.6. Eigenvalue Problem and Conley–Zehnder Index

Fix $k \in \mathbb{N}$ and let (x_k, y_k) be the solution (40) of the Hamiltonian equation (30). The square of the L^2 norm of the solution is given by

$$\|x_k\|^2 = \frac{c_k^2}{2} = \frac{1}{(4\pi k)^{\frac{2}{3}}}, \qquad \|y_k\|^2 = 2(2\pi k)^2 c_k^6 = 4. \tag{42}$$

Abbreviating $(x, y) := (x_k, y_k)$, we look for reals λ and functions $\zeta = (\xi, \eta)$ satisfying

$$L_{\mathcal{S}}\zeta := -J_0\zeta' - I\mathcal{S}\zeta = \lambda\zeta, \qquad \mathcal{S} = \mathcal{S}_{(x_k, y_k)}.$$

Apply J_0 to both sides of the eigenvalue problem to obtain equivalently

$$\begin{aligned}\begin{pmatrix} -\lambda\eta \\ \lambda\xi \end{pmatrix} &= J_0\lambda\zeta = J_0 L_{\mathcal{S}}\zeta = \left(\partial_\tau - J_0 I\mathcal{S}_{(x,y)}\right)\zeta \\ &= \begin{pmatrix} \xi' + \frac{y}{2\|x\|^4}\langle x, \xi\rangle - \frac{\eta}{4\|x\|^2} \\ \eta' - \left(\frac{\xi}{\|x\|^4} - \frac{4x}{\|x\|^6}\langle x, \xi\rangle\right)\underbrace{\left(\frac{\|y\|^2}{4} - 2\right)}_{=-1} - \frac{1}{2}\frac{x}{\|x\|^4}\langle y, \eta\rangle \end{pmatrix}.\end{aligned}$$

Resolving for the first order terms and substituting $\|y\|^2 = 4$ the ODE becomes

$$\zeta' = \begin{pmatrix} \xi' \\ \eta' \end{pmatrix} = \begin{pmatrix} -\lambda\eta - \frac{y}{2\|x\|^4}\langle x, \xi\rangle + \frac{\eta}{4\|x\|^2} \\ \lambda\xi - \left(\frac{\xi}{\|x\|^4} - \frac{4x}{\|x\|^6}\langle x, \xi\rangle\right) + \frac{1}{2}\frac{x}{\|x\|^4}\langle y, \eta\rangle \end{pmatrix} = J_0\lambda\zeta + J_0 I\mathcal{S}\zeta.$$

Substitute first $\|x\|^2 = \frac{c_k^2}{2}$ and then $(x, y) := (x_k, y_k)$ we obtain in view of (40).

$$\begin{aligned}\begin{pmatrix} \xi' \\ \eta' \end{pmatrix} &= \begin{pmatrix} -\lambda\eta - \frac{2y}{c_k^4}\langle x, \xi\rangle + \frac{\eta}{2c_k^2} \\ \lambda\xi - \left(\frac{4\xi}{c_k^4} - \frac{2^5 x}{c_k^6}\langle x, \xi\rangle\right) + \frac{2x}{c_k^4}\langle y, \eta\rangle \end{pmatrix} \\ &= \begin{pmatrix} -\left(\lambda - \frac{1}{2c_k^2}\right)\eta + 8\pi k \sin 2\pi k\tau \langle \cos 2\pi k\,\cdot, \xi\rangle \\ \left(\lambda - \frac{4}{c_k^4}\right)\xi + \frac{2^5 \cos 2\pi k\tau}{c_k^4}\langle \cos 2\pi k\,\cdot, \xi\rangle - 8\pi k \cos 2\pi k\tau \langle \sin 2\pi k\cdot, \eta\rangle \end{pmatrix}.\end{aligned}$$

We write the periodic absolutely continuous maps $\xi, \eta : \mathbb{S}^1 \to \mathbb{R}$ as Fourier series

$$\begin{cases} \xi = \xi_0 + \sum_{n=1}^\infty \left(\xi_n \cos 2\pi n\tau + \xi^n \sin 2\pi n\tau\right), \\ \eta = \eta_0 + \sum_{n=1}^\infty \left(\eta_n \cos 2\pi n\tau + \eta^n \sin 2\pi n\tau\right). \end{cases}$$

We set $\xi^0 = \eta^0 = 0$. Taking the derivative, we can write

$$\begin{cases} \xi' = \sum_{n=1}^\infty \left(-2\pi n \cdot \xi_n \sin 2\pi n\tau + 2\pi n \cdot \xi^n \cos 2\pi n\tau\right), \\ \eta' = \sum_{n=1}^\infty \left(-2\pi n \cdot \eta_n \sin 2\pi n\tau + 2\pi n \cdot \eta^n \cos 2\pi n\tau\right). \end{cases}$$

By the orthogonality relation and (36), we have

$$\langle \cos 2\pi n\,\cdot, \xi\rangle = \frac{1}{2}\xi_n, \qquad \langle \sin 2\pi n\,\cdot, \eta\rangle = \frac{1}{2}\eta^n, \qquad n \in \mathbb{N}.$$

Let $n \in \mathbb{N}_0$. Comparing coefficients, from the **first equations** above, we obtain

$$\begin{aligned} &\sin 2\pi n\tau \quad \begin{cases} -2\pi n \cdot \xi_n = -\left(\lambda - \frac{1}{2c_k^2}\right)\eta^n, & n \neq k, \\ -2\pi k \cdot \xi_k = -\left(\lambda - \frac{1}{2c_k^2}\right)\eta^k + 4\pi k \cdot \xi_k, & n = k, \end{cases} \\ &\cos 2\pi n\tau \quad \left\{ \; 2\pi n \cdot \xi^n = -\left(\lambda - \frac{1}{2c_k^2}\right)\eta_n, \qquad \forall n, \right. \end{aligned}$$

and from the **second equations**,

$$\begin{aligned} &\cos 2\pi n\tau \quad \begin{cases} 2\pi n \cdot \eta^n = \left(\lambda - \frac{4}{c_k^4}\right)\xi_n, & n \neq k, \\ 2\pi k \cdot \eta^k = \left(\lambda - \frac{4}{c_k^4}\right)\xi_k + \frac{2^4}{c_k^4}\xi_k - 4\pi k\eta^k, & n = k,. \end{cases} \\ &\sin 2\pi n\tau \quad \left\{ -2\pi n \cdot \eta_n = \left(\lambda - \frac{4}{c_k^4}\right)\xi^n, \qquad \forall n. \right. \end{aligned}$$

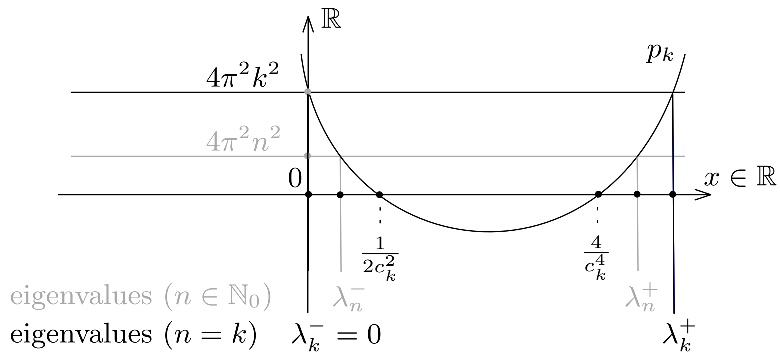

Fig. 3. The parabola p_k and the eigenvalues $\lambda_n^{\mp}$ for each $n \in \mathbb{N}_0$.

Simplifying, the **first equations**, we can write

$$\sin 2\pi n\tau \qquad \begin{cases} \text{(a) } 2\pi n \cdot \xi_n = \left(\lambda - \frac{1}{2c_k^2}\right)\eta^n, & n \neq k, \\ \text{(b) } 6\pi k \cdot \xi_k = \left(\lambda - \frac{1}{2c_k^2}\right)\eta^k, & n = k, \end{cases}$$

$$\cos 2\pi n\tau \qquad \Big\{ \text{ (c) } 2\pi n \cdot \xi^n = -\left(\lambda - \tfrac{1}{2c_k^2}\right)\eta_n, \quad \forall n,$$

and simplifying the **second equations**,

$$\sin 2\pi n\tau \qquad \Big\{ \text{ (d) } -2\pi n \cdot \eta_n = \left(\lambda - \tfrac{4}{c_k^4}\right)\xi^n, \quad \forall n,$$

$$\cos 2\pi n\tau \qquad \begin{cases} \text{(e) } 2\pi n \cdot \eta^n = \left(\lambda - \frac{4}{c_k^4}\right)\xi_n, & n \neq k, \\ \text{(f) } 6\pi k \cdot \eta^k = \left(\lambda + \frac{12}{c_k^4}\right)\xi_k, & n = k. \end{cases}$$

Eigenvalues. Equations (c) and (d) imply

$$\underbrace{\left(\lambda_n - \frac{1}{2c_k^2}\right)\left(\lambda_n - \frac{4}{c_k^4}\right)}_{\text{polynomial } p_k(x) \text{ in variable } x = \lambda_n} = 4\pi^2 n^2, \qquad n \in \mathbb{N}_0. \tag{43}$$

The polynomial $p_k(x)$ is illustrated by Fig. 3. As in (43), we obtain the quadratic equation for λ_n given by

$$\lambda_n^2 - \beta_k \lambda_n + \gamma_{k,n} = 0,$$

where

$$\beta_k := \frac{4}{c_k^4} + \frac{1}{2c_k^2} = \frac{8 + c_k^2}{2c_k^4}, \qquad \gamma_{k,n} := \frac{2}{c_k^6} - 4\pi^2 n^2 = 4\pi^2\left(k^2 - n^2\right).$$

The solutions are

$$\lambda_n^- = \frac{\beta_k}{2} - \frac{1}{2}\sqrt{\beta_k^2 - 4\gamma_{k,n}}, \qquad \lambda_n^+ = \frac{\beta_k}{2} + \frac{1}{2}\sqrt{\beta_k^2 - 4\gamma_{k,n}}.$$

Note that $\gamma_{k,k} = 0$ and, therefore,

$$\lambda_k^- = 0 \tag{44}$$

is zero as well and $\lambda_k^+ = \beta_k$. In the case $n = 0$, the quadratic equation (43) is already factorized, so we read off

$$\lambda_0^- = \frac{1}{2c_k^2}, \qquad \lambda_0^+ = \frac{4}{c_k^4}, \qquad \lambda_0^- < \lambda_0^+,$$

both of which are real numbers. So the argument of the square root is positive for $n = 0$ and, therefore, for all n (since $-\gamma_{k,n}$ is monotone increasing in n). Thus

Lemma 12 (Monotonicity)**.** *The sequence $(\lambda_n^-)_{n \in \mathbb{N}_0}$ is strictly monotone decreasing and $(\lambda_n^+)_{n \in \mathbb{N}_0}$ is strictly monotone increasing.*

Eigenvectors. Recall that we had fixed $k \in \mathbb{N}$, in other words, the solution (q_k, p_k) given by (40) of the Hamiltonian equation (30). We assume in addition[2] $n \neq 0$, that is, $n \in \mathbb{N}$. Eigenvectors to the eigenvalues $\lambda_n^{\pm}$, notation $u_n^{\pm}$, can be found by setting $\eta_n := 1$, then according to equation (c) we define $\xi_{\pm}^n := -\frac{1}{2\pi n}\left(\lambda_n^{\pm} - 1/2c_k^2\right)$. The other Fourier coefficients we define to be equal to 0. With these choices, an eigenvector for $\lambda_n^{\pm}$ is given by the function

$$u_n^{\pm} : \mathbb{S}^1 \to \mathbb{R}^2, \quad \tau \mapsto \Bigg(\underbrace{-\frac{1}{2\pi n}\left(\lambda_n^{\pm} - \frac{1}{2c_k^2}\right)}_{\xi_{\pm}^n} \sin 2\pi n\tau, \cos 2\pi n\tau\Bigg).$$

Note that the coefficient ξ_+^n is strictly negative and ξ_-^n is strictly positive since

$$\lambda_n^+ - \frac{1}{2c_k^2} > \lambda_0^+ - \frac{1}{2c_k^2} > \lambda_0^- - \frac{1}{2c_k^2} = 0$$

and similarly

$$\lambda_n^- - \frac{1}{2c_k^2} < \lambda_0^- - \frac{1}{2c_k^2} = 0.$$

Since $\xi_+^n > 0$, we see that the eigenvector u_n^+ winds n times counter-clockwise around the origin, while u_n^- winds n times clockwise around the origin since $\xi_-^n < 0$. Therefore, the winding numbers equal $\pm n$, in symbols

$$w(u_n^{\pm}) = \pm n.$$

Note that, in the ODE (local) case, Lemma 12 would tell us that $w(\lambda_n^+) = n$, but in the nonlocal case, we cannot yet conclude independence of the choice of an eigenvector.

Remark 3 (Case $n = 0$)**.**
Eigenvalue $\lambda_0^- = 1/2c_k^2$. In (c), we choose $\eta_0 := 1$ and set all other Fourier coefficients zero. Then $u_0^- = (0, 1)$ is an eigenvector to the eigenvalue λ_0^-. Since the function u_0^- is constant, its winding number vanishes, in symbols $w(u_0^-) = 0$.

Eigenvalue $\lambda_0^+ = 4/c_k^4$. By (e), we can choose $\xi_0 := 1$ and all other Fourier coefficients equal zero. For these choices $u_0^+ = (1, 0)$ is an eigenvector to the eigenvalue λ_0^+. By constancy, the winding number is 0, in symbols $w(u_0^+) = 0$.

Remark 4 (Geometric multiplicity of eigenvalues $\lambda_n^{\pm}$ is $\geqslant 2$ for $n \neq 0, k$)**.** Instead of using (c) and (d), one can use (a) and (e). Setting $\eta^n := 1$ equation (a) motivates to define $\xi_n^{\pm} := -\frac{1}{2\pi n}\left(\lambda_n^+ - 1/2c_k^2\right)$. With these choices, a further eigenvector for $\lambda_n^{\pm}$ is given by the function

$$v_n^{\pm} : \mathbb{S}^1 \to \mathbb{R}^2, \quad \tau \mapsto \Bigg(\underbrace{\frac{1}{2\pi n}\left(\lambda_n^{\pm} - \frac{1}{2c_k^2}\right)}_{\xi_n^{\pm}} \cos 2\pi n\tau, \sin 2\pi n\tau\Bigg).$$

We observe that, just as above, the winding number of the eigenvector v_n^+ is n, and of v_n^- it is $-n$, in symbols

$$w(v_n^{\pm}) = \pm n.$$

Case $n = k$ and the eigenvalues $\widehat{\lambda}_k^{\pm}$. In the case $n = k$ we obtain from equations (b) and (f) the quadratic equation

$$\underbrace{\left(\widehat{\lambda}_k - \frac{1}{2c_k^2}\right)\left(\widehat{\lambda}_k + \frac{12}{c_k^4}\right)}_{\text{polynomial } \widehat{p}_k(x) \text{ in variable } x = \widehat{\lambda}_k} = 36\pi^2 k^2. \tag{45}$$

The polynomial $\widehat{p}_k(x)$ is illustrated by Figure 4. Eigenvalues. Equivalently we obtain the quadratic equation for $\widehat{\lambda}_k$ given by

$$\widehat{\lambda}_k^2 - B_k \widehat{\lambda}_k + C_k = 0,$$

[2]New phenomena appear in the case $n = 0$. For instance, the geometric multiplicities of $\lambda_0^{\mp}$ are 1, as opposed to 2 in the case $n > 0$.

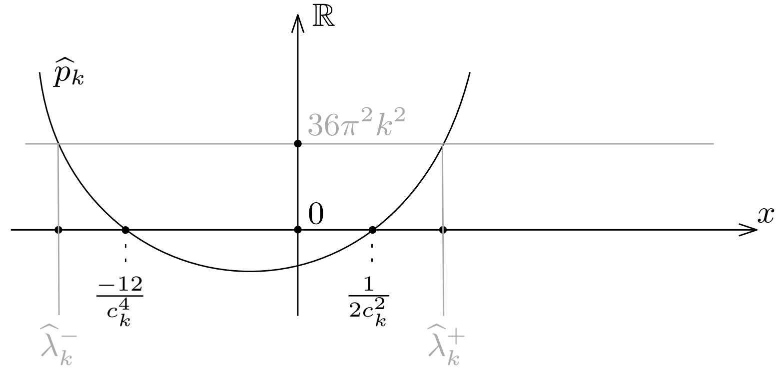

Fig. 4. Parabola $\widehat{p}_k$ in the variable $x = \widehat{\lambda}_k$ given by (45).

where

$$B_k := \frac{1}{2c_k^2} - \frac{12}{c_k^4} = \beta_k - \frac{16}{c_k^4}, \qquad \beta_k := \frac{4}{c_k^4} + \frac{1}{2c_k^2} = \frac{8 + c_k^2}{2c_k^4},$$

$$C_k := -\frac{6}{c_k^6} - 36\pi^2k^2 = -48\pi^2k^2,$$

with $c_k < 1$ given by (39). The solutions are

$$\widehat{\lambda}_k^- = \frac{B_k}{2} - \frac{1}{2}\sqrt{B_k^2 - 4C_k}, \qquad \widehat{\lambda}_k^+ = \frac{B_k}{2} + \frac{1}{2}\sqrt{B_k^2 - 4C_k}.$$

Eigenvectors. Eigenvectors to the eigenvalues $\widehat{\lambda}_k^\pm$, notation $u_k^\pm$, can be found by setting $\eta^k := 1$, then equation (b) motivates to define

$$\xi_k := \frac{1}{6\pi k}\left(\widehat{\lambda}_k^\pm - 1/2c_k^2\right).$$

The other Fourier coefficients we define to be equal 0. With these choices, an eigenvector for $\widehat{\lambda}_k^\pm$ is given by the function

$$v_k^\pm : \mathbb{S}^1 \to \mathbb{R}^2, \quad \tau \mapsto \left(\frac{1}{6\pi k}\left(\widehat{\lambda}_k^\pm - \frac{1}{2c_k^2}\right)\cos 2\pi k\tau, \sin 2\pi k\tau\right).$$

As one sees from Figure 4 the following inequalities hold

$$\widehat{\lambda}_k^- < -\frac{12}{c_k^4} < 0 < \frac{1}{2c_k^2} < \widehat{\lambda}_k^+.$$

Therefore, $\widehat{\lambda}_k^+ - \frac{1}{2c_k^2} > 0$ and $\widehat{\lambda}_k^- - \frac{1}{2c_k^2} < 0$ and hence the winding number are

$$w(v_k^\pm) = \pm k.$$

5.7. Disjoint Families and Winding Numbers

Consider the two quadratic polynomials p_k and $\widehat{p}_k$ in the variable $x = \lambda_n$ given by the left-hand sides of (43) and (45), namely,

$$p_k(x) := \left(x - \frac{1}{2c_k^2}\right)\left(x - \frac{4}{c_k^4}\right) \quad \text{and} \quad \widehat{p}_k(x) := \left(x - \frac{1}{2c_k^2}\right)\left(x + \frac{12}{c_k^4}\right).$$

These two polynomials have a common zero at $x = 1/2c_k^2$, they are sketched in Fig. 5. For $n = 3k$, we have equality $4\pi^2n^2 = 36\pi^2k^2$ and the intersection of $\widehat{p}_k$ and p_k with the horizontal line $\{36\pi^2k^2\}$ consists of 4 points whose x-coordinates are the following eigenvalues in the following order

$$\widehat{\lambda}_k^- < \lambda_{3k}^- < \widehat{\lambda}_k^+ < \lambda_{3k}^+.$$

Proposition 3. *For any $n \in \mathbb{N}_0$, the $\lambda_n^\mp$ are different from $\widehat{\lambda}_k^-$ and from $\widehat{\lambda}_k^+$; in symbols:*

$$\lambda_n^\mp \neq \widehat{\lambda}_k^-, \qquad \lambda_n^\mp \neq \widehat{\lambda}_k^+, \qquad n \in \mathbb{N}_0.$$

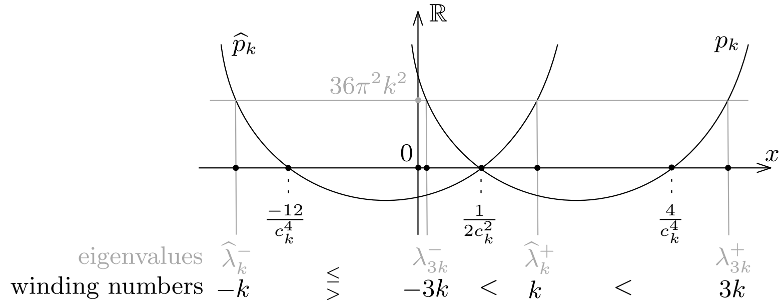

Fig. 5. The parabolas $\widehat{p}_k$ and p_k, eigenvalues and winding numbers.

Proof. Note that $\widehat{\lambda}_k^-$ is negative. On the other hand $\lambda_n^+ > 0$, for $n \in \mathbb{N}_0$, as well as $\lambda_0^- > 0$ are all positive. Therefore, it suffices to show that $\widehat{\lambda}_k^- \neq \lambda_n^-$ for every $n \in \mathbb{N}$.

Suppose by contradiction that there are $i, j \in \{+, -\}$ such that $\widehat{\lambda}_k^i = \lambda_n^j =: \lambda$ for some $n \in \mathbb{N}_0$. The idea is to construct two polynomials $P(z)$ and $Q(z)$ which have a common zero at $z = c_k^2$ and then use algebra to show that there cannot be two such polynomials.

Step 0. It is useful to consider the field extension $\mathbb{Q}(\pi)$ of $\mathbb{Q}$, which is a subfield of $\mathbb{R}$, i.e., $\mathbb{Q} \subset \mathbb{Q}(\pi) \subset \mathbb{R}$. Elements of $\mathbb{Q}(\pi)$ have the following form. Given rational polynomials $p, q \in \mathbb{Q}[x]$ with $q \not\equiv 0$ not the zero polynomial, the numbers in $\mathbb{Q}(\pi)$ are given by $p(\pi)/q(\pi)$. Note that by a theorem of Lindemann [12], see [9] for an elegant proof by Hilbert, the number π is transcendental[3] and, therefore, $q(\pi) \neq 0$ is nonzero.

Step 1. The definition of the polynomial

$$Q(z) := z^3 + a_0, \qquad a_0 := -(c_k^2)^3 \overset{(39)}{=} -\frac{1}{2\pi^2 k^2} \in \mathbb{Q}(\pi), \tag{46}$$

is motivated by the goal that it has a zero at the point $z = c_k^2$ given by (39). Here $\mathbb{Q}(\pi)$ is the field extension of $\mathbb{Q}$ by adjoining π from Step 0. The field extension $\mathbb{Q}(\pi)$ is isomorphic to the field $\mathbb{Q}(x)$ of rational functions.

Step 2. Divide the polynomial identity $\widehat{p}_k(\lambda) = 36\pi^2 k^2$ by $p_k(\lambda) = 4\pi^2 n^2$ to get that

$$\frac{\lambda + \frac{12}{c_k^4}}{\lambda - \frac{4}{c_k^4}} = \frac{36\pi^2 k^2}{4\pi^2 n^2} = 9\frac{k^2}{n^2}.$$

Now multiplication by the denominator yields

$$\lambda + \frac{12}{c_k^4} = 9\frac{k^2}{n^2}\left(\lambda - \frac{4}{c_k^4}\right).$$

Resolving for λ gives

$$\lambda = \frac{4}{c_k^4} \cdot \frac{9\frac{k^2}{n^2} + 3}{9\frac{k^2}{n^2} - 1} = \frac{4}{c_k^4} \cdot \frac{9k^2 + 3n^2}{9k^2 - n^2}.$$

Consequently,

$$\lambda \in \frac{1}{c_k^4}\mathbb{Q}. \tag{47}$$

Now evaluate $\widehat{p}_k$ at λ, obtaining (since $\lambda := \widehat{\lambda}_k^i$)

$$0 = \widehat{p}_k(\lambda) - 36\pi^2 k^2 = \left(\lambda - \frac{1}{2c_k^2}\right)\left(\lambda + \frac{12}{c_k^4}\right) - 36\pi^2 k^2. \tag{48}$$

[3] A real number is called **transcendental** if it is not a zero of a polynomial with rational coefficients. Transcendental implies irrational. Note that $\sqrt{2}$ is irrational, but not transcendental (zero of $x^2 - 2$).

Multiplication by ${c_k}^8$ and division by $-36\pi^2 k^2$ leads to

$$
\begin{aligned}
0 &= \frac{\lambda^2 c_k^8 + \lambda c_k^4 \left(12 - c_k^2/2\right) - 6c_k^2}{-36\pi^2 k^2} + c_k^8 \\
&= -\frac{4}{9\pi^2 k^2}\left(\frac{9k^2+3n^2}{9k^2-n^2}\right)^2 + \frac{1}{18\pi^2 k^2}\left(\frac{9k^2+3n^2}{9k^2-n^2}\right) c_k^2 \\
&\quad - \frac{4}{3\pi^2 k^2}\left(\frac{9k^2+3n^2}{9k^2-n^2}\right) + \frac{1}{18\pi^2 k^2}\left(\frac{9k^2+3n^2}{9k^2-n^2}\right) c_k^2 + c_k^8 \\
&= P(c_k^2),
\end{aligned} \tag{49}
$$

where the polynomial P is given by

$$P(z) = z^4 + b_1 z + b_0, \qquad b_0, b_1 \in \mathbb{Q}(\pi),$$

and the coefficients b_0 and b_1 – according to (49) – by

$$
\begin{aligned}
b_0 &= -\frac{4}{3\pi^2 k^2}\frac{9k^2+3n^2}{9k^2-n^2}\left(\frac{3k^2+n^2}{9k^2-n^2}+1\right) = -\frac{48}{\pi^2}\frac{3k^2+n^2}{(9k^2-n^2)^2} < 0 \\
b_1 &= \frac{1}{9\pi^2 k^2}\left(\frac{9k^2+3n^2}{9k^2-n^2}\right).
\end{aligned}
$$

We define a linear polynomial in z with coefficients in the field $\mathbb{Q}(\pi)$ by the formula

$$R(z) := P(z) - zQ(z) = (b_1 - a_0)z + b_0.$$

Since $z = c_k^2$ is a zero of P by (49) and of Q by (46), it is a zero of the linear polynomial R. Since $b_0 \neq 0$, it follows that $b_1 - a_0 \neq 0$: otherwise, $R \equiv b_0 \neq 0$ would not have a zero at all. Since $0 = R(c_k^2) = (b_1 - a_0)c_k^2 + b_0$, we see that $c_k^2 = -b_0/(b_1 - a_0) \in \mathbb{Q}(\pi)$, i.e., c_k^2 is of the form $p(\pi)/q(\pi)$ where $p, q \in \mathbb{Q}[z]$.

This leads to a contradiction: evaluate (46) at $z = \frac{p(\pi)}{q(\pi)} = c_k^2 \in (0,1)$, obtaining

$$0 = Q(c_k^2) = \frac{p(\pi)^3}{q(\pi)^3} - \frac{1}{2\pi^2 k^2},$$

multiply the identity by $q(\pi)^3\left(2\pi^2 k^2\right)$, observe that

$$0 = \underbrace{p(\pi)^3\left(2\pi^2 k^2\right)}_{\deg = 2 \mod 3} - \underbrace{q(\pi)^3}_{\deg = 0 \mod 3}.$$

Consider the polynomial $s := p^3 r - q^3 \in \mathbb{Q}[z]$, where $r(z) := 2k^2 z^2$.

Claim. $s \not\equiv 0$

Proof of the claim. It follows from considering the degrees. Note that

$$\deg r = 2 \mod 3, \qquad \deg p^3 = 0 \mod 3, \qquad \deg q^3 = 0 \mod 3.$$

Therefore,

$$\deg p^3 r = \deg p^3 + \deg r = 2 \mod 3, \qquad \deg q^3 = 0 \mod 3.$$

Hence, $p^3 r \neq q^3$ and consequently $s \not\equiv 0$. This proves the claim.

According to the claim, we have found a nonzero polynomial s with rational coefficients possessing the property that $s(\pi) = 0$. But this contradicts the theorem of Lindemann as explained earlier.[4]

Corollary 3 (Well-defined winding number)**.** *For every $n \in \mathbb{N}_0$, we define the winding numbers*

$$w(\lambda_n^-) := -n, \quad w(\lambda_n^+) := n, \quad w(\widehat{\lambda}_k^-) := -k, \quad w(\widehat{\lambda}_k^+) := k.$$

In view of Proposition 3, these winding numbers are well defined and, in view of the discussion before, correspond to the winding number of an arbitrary eigenvector of the eigenvalue.

spec $L_{\mathcal{S}}$	λ_n^-	$\lambda_k^- = 0$ $\widehat{\lambda}_k^-$	λ_0^-	λ_0^+	λ_k^+ $\widehat{\lambda}_k^+$	λ_n^+
m	2	1	1	1	1	2
w	$-n$	$-k$	0	0	k	n

Fig. 6. Multiplicities m and winding numbers w of eigenvalues, $n \in \mathbb{N} \setminus \{k\}$.

Proposition 4. *At a critical point* (x_k, y_k) *of* $\mathcal{A}_{\mathcal{H}}$*, see* (40)*, the following relations hold*

$$\alpha(\mathcal{S}_{(x_k,y_k)}) = w(\widehat{\lambda}_k^-) = -k, \qquad p(\mathcal{S}_{(x_k,y_k)}) = 1;$$

using the definitions $\mu_{\mathrm{CZ}} := 2\alpha + p$ *and* $\mu^{\mathrm{CZ}} := -\mu_{\mathrm{CZ}}$ *imply*

$$\mu_{\mathrm{CZ}}(x_k, y_k) = -2k + 1, \qquad \mu^{\mathrm{CZ}}(x_k, y_k) = 2k - 1.$$

Proof. Setting $\mathcal{S} := \mathcal{S}_{(x_k,y_k)}$, we recall the definition of α, namely,

$$\alpha(\mathcal{S}) := \max\{w(\lambda) \mid \lambda \in (-\infty, 0) \cap \operatorname{spec} L_{\mathcal{S}}\} \in \mathbb{Z}.$$

Observe that $\widehat{\lambda}_k^- < 0$, see Fig. 4, and $w(\widehat{\lambda}_k^-) = -k$, see Fig. 6. Therefore, $\alpha(\mathcal{S}) \geqslant -k$. To show the reverse inequality $\alpha(\mathcal{S}) \leqslant -k$, we need to check the nonnegativity of all eigenvalues with winding number $> -k$. By Fig. 6, the eigenvalues of winding number $> -k$ are of three types:

(i) λ_n^- for $n < k$: In this case $\lambda_n^- > \lambda_k^- = 0$ for every $n < k$ by monotonicity, see Lemma 12, and (44),

(ii) λ_n^+ for all $n \in \mathbb{N}_0$: In this case $\lambda_n^+ > 0$ for every $n \in \mathbb{N}_0$ by Figure 3,

(iii) $\widehat{\lambda}_k^+$: In this case $\widehat{\lambda}_k^+ > 0$ by Fig. 4.

This proves that $\alpha(\mathcal{S}) = -k$.

Because there exists a nonnegative eigenvalue, namely $\lambda_k^- = 0$, with the same winding number $-k$ as the negative eigenvalue $\widehat{\lambda}_k^- < 0$ that realizes the maximal winding number $\alpha(\mathcal{S}_{(x_k,y_k)})$ among negative eigenvalues, we conclude that

$$p(\mathcal{S}_{(x_k,y_k)}) = 1.$$

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