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An enhanced algorithmic scheme for relaxed incremental variational damage formulations at finite strains

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Relaxed damage formulations enable to overcome the mesh dependency induced by a loss of ellipticity. However, such approaches require the construction of a convex hull of the associated energy potential. Within this contribution, an improved one-dimensional, relaxed damage formulation of [1] with a successive, computationally optimal convexification scheme, see [2], is presented. Instead of solving a non-convex optimization problem at the material point level, we iterate over a discrete grid and build consecutively linear Taylor expansions to construct the convex hull. Due to the reduced computational effort, integration of the one-dimensional mechanical model over orientation distributions to obtain the three-dimensional response as proposed in [3] becomes more feasible.

1 Introduction

Deterioration of materials plays a crucial role in engineering science since it is a vital part of the description of mechanical behavior. Phenomenological descriptions of this phenomenon often work in the framework of continuum damage mechanics. However, such formulations may lead to a loss of ellipticity of the field equations yielding mesh-dependent solutions. Different methods exist to overcome the underlying issues. Within this contribution, we focus on the relaxation approach, where instead of the nonconvex incremental stress potential the convex hull is considered in the nonconvex regime. The nonconvex incremental stress potential is given by

$$W(\boldsymbol{F}_{k+1}) = \psi(\boldsymbol{F}_{k+1}, D_{k+1}) - \psi(\boldsymbol{F}_k, D_k) + \beta D_{k+1} - \beta_k D_k - \tilde{D}_{k+1} + \tilde{D}_k \tag{1}$$

where ψ, $\boldsymbol{F}$, $D(\beta)$, $\tilde{D}$ correspond to the effective energy, deformation gradient, damage function and antiderivative of the damage function, respectively. The variable β is the thermodynamic force. A detailed derivation of the potential and its derivatives can be found in [1]. There, also a relaxed model including the rank-one convex hull as well as its one-dimensional reduction was presented. For the physically one-dimensional model, an at least two-dimensional non-convex optimization problem needs to be solved at the material point level. The solution of this optimization problem yields the rank-one convex hull with two supporting points, F^+ and F^-. These points mark the beginning and the end of the nonconvex regime. Attached to these two points two different phases, namely a weakly and a strongly damaged phase, were associated and in between a linear interpolation was assumed. The interpolation between the two supporting points of the convex hull can be mathematically expressed as:

$$F := \xi F^+ + (1-\xi)F^- \quad \text{with} \quad \begin{cases} F^+ & := & F(1+(1-\xi)d), \\ F^- & := & F(1-\xi d). \end{cases} \tag{2}$$

Hence, the convex hull of the initially non-convex energy is obtained by

$$W \Leftarrow W_{\mathrm{C}}(F) = \inf_{\xi,d}[\bar{W}(F,d,\xi)] \quad \text{with} \quad \bar{W} = \xi W(F^+) + (1-\xi)W(F^-). \tag{3}$$

2 Convexification

Instead of solving the nonconvex problem of equation (3), an algorithm for the consecutive construction of the convex hull is presented. The convexification algorithm is a slight variation of the convexification algorithm of [2] (Section 9.3.4, Algorithm 9.3) and is given in Algorithm 1. To the authors' best knowledge the algorithm is applied to continuum damage mechanics for the first time.

The algorithm works in the following way: a discrete one-dimensional grid of deformation gradients is spanned over the area of interest, named x in Algorithm 1. All points of the grid are inserted in the original nonconvex incremental stress potential eq. (1), which forms an array of incremental stress potential values, f in Algorithm 1. A copy of this list is created, labeled as g, and an iteration over every point of the list, except the first two, is started. If the tangent of grid point j w.r.t. $j-1$ is smaller than the tangent of $j-1$ w.r.t. $j-2$, the point, where this condition held true for the last time, is sought. As

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soon as this point, with index $j-k$, is found, all points from j to $j-k$ are overwritten by a linear Taylor expansion. The algorithm is further optimized by deleting all points from g and x which lie over the convex hull instead of overwriting them. Due to this modification, the complexity is reduced to $\mathcal{O}(n)$.

Algorithm 1 Pseudocode for convexifying f

Input: $f_{j=0,\dots,L}$, $g_{j=0,\dots,L}$, $x_{j=0,\dots,L}$
for $j \geq 2$ to L **do**
  if $\frac{g_j - g_{j-1}}{x_j - x_{j-1}} < \frac{g_{j-1} - g_{j-2}}{x_{j-1} - x_{j-2}}$ **then**
    find k, s.t. $\frac{g_j - g_{j-k}}{x_j - x_{j-k}} \geq \frac{g_{j-k} - g_{j-k-1}}{x_{j-k} - x_{j-k-1}}$
    for $m = 1$ to k **do**
      successive linear Taylor expansion
      $g_{j-m} = g_{j-k} + (x_{j-m} - x_{j-k}) \frac{g_j - g_{j-k}}{x_j - x_{j-k}}$
    end for
  end if
end for

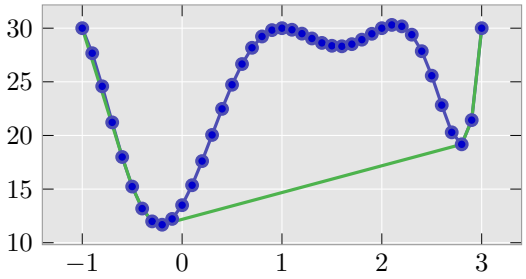

Fig. 1: Test problem for convexification algorithm, non-convex benchmark function $W(F) = (F^3 - 4)((F-2)^2 - 1)(F^2 - 1)((F-0.5)^2 - 3)0.5 + 30$

3 Benchmark

A simple boundary value problem is used as a benchmark problem. Consider a unit cube discretized with fourty linear displacement elements, ten in x_1 direction and two in x_2 and x_3, respectively. The face at $x_1 = 0$ is fixed in all degress of freedom and at the opposite face, i.e. at $x_1 = 1$, a displacement in x_1 direction is prescribed. 1000 incremental steps are used. The material model is generalized to three dimensions by including it in a microsphere approach. This concept applied to the relaxed model of [1] is described in detail in [3]. Within the benchmark, we measure the memory allocations and time used by one fiber of the microsphere of one Gauss point. The results show a significant speed up compared to the non-convex optimization scheme. A single fiber has an approximate speed-up factor of 270 within the assembly routine. These results are very promising and make more complex modeling by the microsphere approach feasible.

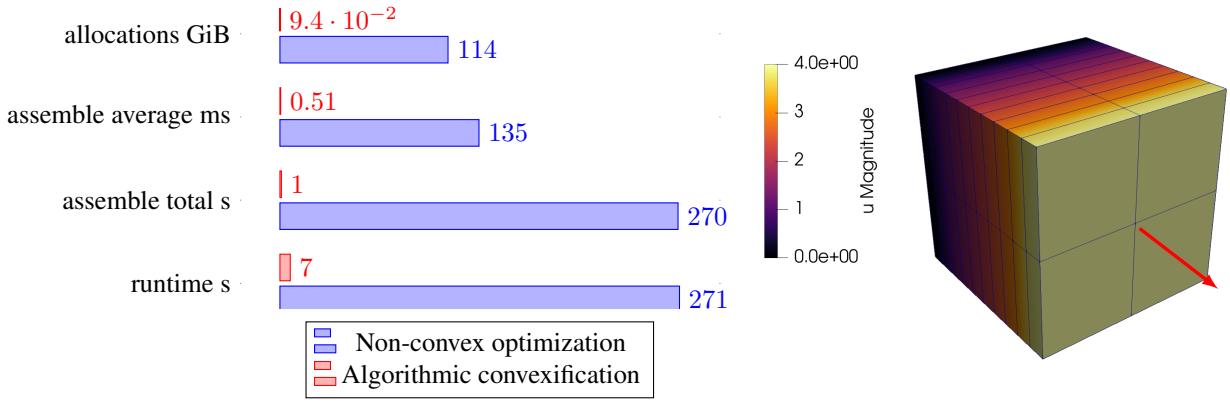

Fig. 2: Benchmark measurements for one fiber (a) and boundary value problem of the benchmark (b)

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