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# An equilibrated a posteriori error estimator for an Interior Penalty Discontinuous Galerkin approximation of the $p$-Laplace problem

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**Abstract:** We are concerned with an Interior Penalty Discontinuous Galerkin (IPDG) approximation of the $p$-Laplace equation and an equilibrated a posteriori error estimator. The IPDG method can be derived from a discretization of the associated minimization problem involving appropriately defined reconstruction operators. The equilibrated a posteriori error estimator provides an upper bound for the discretization error in the broken $W^{1,p}$ norm and relies on the construction of an equilibrated flux in terms of a numerical flux function associated with the mixed formulation of the IPDG approximation. The relationship with a residual-type a posteriori error estimator is established as well. Numerical results illustrate the performance of both estimators.

**Keywords:** Interior Penalty Discontinuous Galerkin method, $p$-Laplace problem, a posteriori error estimation, equilibration.

**MSC 2010:** 65N30, 65N15, 65N50

Adaptive finite element methods for the $p$-Laplace problem and generalizations thereof such as problems with power growth functionals based on either residual-type a posteriori error estimators or on error estimators derived by duality theory have been developed, analyzed, and implemented in [5, 7, 15, 18, 24, 34, 45, 50] (see also the monograph [47]), whereas Discontinuous Galerkin (DG) methods for such problems have been considered in [6, 13, 14, 21, 23, 33, 37, 40, 43, 44, 48] (see also [22] for the $p(x)$-Laplacian).

On the other hand, equilibrated a posteriori error estimators for adaptive finite element approximations of linear and nonlinear second and fourth order elliptic boundary value problems have been suggested in [8–11, 16, 17, 20, 31, 32]; see also Chapter 12 in [52].

In the present paper, we consider an Interior Penalty Discontinuous Galerkin (IPDG) method for the $p$-Laplace problem similar to the ones in [5, 13, 14]. The method is the optimality condition for the minimization of an IPDG approximation of the primal energy functional associated with the $p$-Laplacian. We also consider a two-field formulation of the IPDG approximation which allows to specify a numerical flux function such that the IPDG method falls within the approach taken in [3]. The equilibrated a posteriori error estimator is based on a general approach from [46] which enables to estimate the global discretization error in terms of primal and dual energy functionals. It requires the construction of an equilibrated flux in Brezzi–Douglas–Marini finite element spaces involving the two-field formulation and the numerical flux function.

The paper is organized as follows: In Section 1, we provide some basic notations and auxiliary results. Then, in Section 2 we consider the $p$-Laplace problem and its associated primal and dual energy functionals. Section 3 is devoted to the IPDG approximation and its related two-field formulation. The equilibrated a posteriori error estimator based on the result from [46] is dealt with in Section 4, whereas Section 5 addresses the construction of an equilibrated flux using Brezzi–Douglas–Marini finite elements. Section 6 provides a comparison with a residual-based a posteriori error estimator for the IPDG approximation. Finally, in Section 7

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we present a documentation of numerical results for two examples that illustrates the performance of the suggested approach.

# 1 Basic notations and auxiliary results

We use standard notation from Lebesgue and Sobolev space theory (see, e.g., [49]). In particular, for a bounded domain $\Omega \subset \mathbb{R}^d, d \in \mathbb{N}$, we refer to $L^p(\Omega;\mathbb{R}^d), 1 < p < \infty$, as the Banach space of $p$th power Lebesgue integrable functions on $\Omega$ with norm $\|\cdot\|_{L^p(\Omega;\mathbb{R}^d)}$. In case $d = 1$ we will write $L^p(\Omega)$ instead of $L^p(\Omega;\mathbb{R})$. We denote by $W^{s,p}(\Omega), s \in \mathbb{R}_+, 1 < p < \infty$, the Sobolev spaces with norms $\|\cdot\|_{W^{s,p}(\Omega)}$ and by $W_0^{s,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{s,p}(\Omega)}$. Functions $u \in W^{1,p}(\Omega)$ have a trace $u|_\Gamma$ on the boundary $\Gamma = \partial\Omega$ with $u|_\Gamma \in W^{1-1/p,p}(\Gamma)$. If $\Gamma = \overline{\Gamma}_D \cup \overline{\Gamma}_N, \Gamma_D \cap \Gamma_N = \emptyset$, the space $W_{00}^{1-1/p,p}(\Gamma_D)$ denotes the space of functions on $\Gamma_D$ whose extension by zero to $\Gamma$ belongs to $W^{1-1/p,p}(\Gamma)$ (for examples of functions that do belong to $W_{00}^{1-1/p,p}(\Gamma_D)$ and those that do not as well as for further discussion we refer to [35]). For $u_D \in W_{00}^{1-1/p}(\Gamma_D)$ we set

$$W^{1,p}_{u_D,\Gamma_D}(\Omega) := \{v \in W^{1,p}(\Omega) \mid v|_{\Gamma_D} = u_D\}.$$

Further, we define $\underline{\mathbf{H}}^{(p)}(\mathrm{div},\Omega), 1 < p < \infty$, as the Banach space

$$\underline{\mathbf{H}}^{(p)}(\mathrm{div},\Omega) = \{\underline{\tau} \in L^p(\Omega)^2 \mid \nabla\cdot\underline{\tau} \in L^p(\Omega)\}$$

with the graph norm

$$\|\underline{\tau}\|_{\underline{\mathbf{H}}^{(p)}(\mathrm{div},\Omega)} := \left(\|\underline{\tau}\|^p_{L^p(\Omega;\mathbb{R}^d)} + \|\nabla\cdot\underline{\tau}\|^p_{L^p(\Omega)}\right)^{1/p}.$$

We refer to $\underline{\mathbf{H}}^{(p)}_{0,\Gamma_D}(\mathrm{div},\Omega)$ as the subspace

$$\underline{\mathbf{H}}^{(p)}_{0,\Gamma_D}(\mathrm{div},\Omega) = \{\underline{\tau} \in \underline{\mathbf{H}}^{(p)}(\mathrm{div},\Omega) \mid \mathbf{n}_{\Gamma_D}\cdot\underline{\tau} = 0 \text{ on } \Gamma_D\}.$$

For further properties of $\underline{\mathbf{H}}^{(p)}(\mathrm{div},\Omega)$ we refer to [1].

For later use we recall Young's inequality

$$\prod_{i=1}^{2} a_i \leq \frac{\varepsilon}{p}a_1^p + \frac{\varepsilon^{-q/p}}{q}a_2^q \tag{1.1}$$

for $a_i > 0, 1 \leq i \leq 2$, and $1 < p, q < \infty, 1/p + 1/q = 1$, and any $\varepsilon > 0$, as well as the following inequality: Let $w_i \in \mathbb{R}, 1 \leq i \leq 2$, and $0 \leq r < \infty$. Then there exists a constant $C_r > 0$ such that it holds

$$(|w_1| + |w_2|)^r \leq C_r\left(|w_1|^r + |w_2|^r\right), \quad C_r^{1/r}C_{1/r} = 1. \tag{1.2}$$

# 2 The $p$-Laplace problem and the associated primal and dual energy functionals

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain with boundary $\Gamma = \partial\Omega, \Gamma = \overline{\Gamma}_D \cup \overline{\Gamma}_N, \Gamma_D \cap \Gamma_N = \emptyset, \Gamma_D \neq \emptyset$, and exterior unit normal vectors $\mathbf{n}_\Gamma, \mathbf{n}_{\Gamma_D}, \mathbf{n}_{\Gamma_N}$. Further, let $1 < p, q < \infty, 1/p + 1/q = 1$, and $f \in L^q(\Omega), u_D \in W_{00}^{1-1/p,p}(\Gamma_D), u_N \in L^q(\Gamma_N)$. The $p$-Laplace problem with inhomogeneous Dirichlet and Neumann boundary conditions reads as follows:

$$-\nabla\cdot\left(|\nabla u|^{p-2}\nabla u\right) = f \quad \text{in } \Omega \tag{2.1a}$$

$$u = u_D \quad \text{on } \Gamma_D \tag{2.1b}$$

$$\mathbf{n}_{\Gamma_N}\cdot\left(|\nabla u|^{p-2}\nabla u\right) = u_N \quad \text{on } \Gamma_N. \tag{2.1c}$$

The variational formulation of (2.1) requires the computation of $u \in W^{1,p}_{u_D,\Gamma_D}(\Omega)$ such that for all $v \in W^{1,p}_{0,\Gamma_D}(\Omega)$ it holds

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx = \ell(v) \tag{2.2a}$$

where the functional $\ell : V \to \mathbb{R}$ is given by

$$\ell(v) := \int_\Omega f v \, dx + \int_{\Gamma_N} u_N v \, ds. \tag{2.2b}$$

It is well known that (2.2) admits a unique solution (see, e.g., [19]). Moreover, (2.2) represents the necessary and sufficient optimality condition for the minimization problem

$$J_P(u) = \inf_{v \in W^{1,p}_{u_D,\Gamma_D}(\Omega)} J_P(v) \tag{2.3a}$$

where the objective functional $J_P$ is given by

$$J_P(v) := \frac{1}{p}\int_\Omega |\nabla v|^p \, dx - \int_\Omega f v \, dx - \int_{\Gamma_N} u_N v \, ds. \tag{2.3b}$$

The dual problem of (2.3) is given by (see Chapter 4, Section 2.2 in [29]):

$$J_D(\underline{\mathbf{p}}) = \inf_{\underline{\mathbf{q}} \in \underline{\mathbf{H}}^{(q)}(\mathrm{div};\Omega)} J_D(\underline{\mathbf{q}}) \tag{2.4a}$$

subject to the equilibrium conditions

$$-\nabla \cdot \underline{\mathbf{p}} = f \quad \text{in } L^q(\Omega), \qquad \mathbf{n}_{\Gamma_N} \cdot \underline{\mathbf{p}} = u_N \quad \text{in } L^q(\Gamma_N) \tag{2.4b}$$

$$\tag{2.4c}$$

where the objective functional $J_D$ is given by

$$J_D(\underline{\mathbf{q}}) := \frac{1}{q}\int_\Omega |\underline{\mathbf{q}}|^q \, dx - \int_{\Gamma_D} u_D \, \mathbf{n}_{\Gamma_D} \cdot \underline{\mathbf{q}} \, ds. \tag{2.4d}$$

# 3 IPDG approximation of the $p$-Laplace problem

Let $\mathcal{T}_h$ be a geometrically conforming, locally quasi-uniform, simplicial triangulation of the computational domain $\Omega$ which on $\Gamma$ aligns with $\Gamma_D$ and $\Gamma_N$. Given $D \subset \overline{\Omega}$, we denote by $\mathcal{N}_h(D)$ and $\mathcal{E}_h(D)$ the set of vertices and edges of $\mathcal{T}_h$ in $D$, and we refer to $P_k(D)$, $k \in \mathbb{N}$, as the set of polynomials of degree $\leqslant k$ on $D$. Moreover, $h_K, K \in \mathcal{T}_h$, and $h_E$, $E \in \mathcal{E}_h$, stand for the diameter of $K$ and the length of $E$, respectively. We define $h := \min\{h_K \mid K \in \mathcal{T}_h\}$. Due to the local quasi-uniformity of the triangulation there exist constants $0 < c_R \leqslant C_R$ such that for all $K \in \mathcal{T}_h$ it holds

$$c_R h_K \leqslant h_E \leqslant C_R h_K, \quad E \in \mathcal{E}_h(\partial K). \tag{3.1}$$

For two quantities $a, b \in \mathbb{R}$ we will write $a \lesssim b$, if there exists a constant $C > 0$, independent of $h$, such that $a \leqslant Cb$.

We will further use the following trace inequality (see, e.g., [25]): For $1 \leqslant p \leqslant \infty$ there exists a constant $C_T > 0$, only depending on $p$, the polynomial degree $k$, and the local geometry of the triangulation, such that for $v_h \in P_k(K)$ and $K \in \mathcal{T}_h$ it holds

$$\|v_h\|_{L^p(\partial K)} \leqslant C_T h_K^{-1/p} \|v_h\|_{L^p(K)}. \tag{3.2}$$

For $E \in \mathcal{E}_h(\Omega)$, $E = K_+ \cap K_-$, $K_\pm \in \mathcal{T}_h(\Omega)$, and $v_h \in V_h$, we denote the average and jump of $v_h$ across $E$ by $\{v_h\}_E$ and $[v_h]_E$, i.e.,

$$\{v_h\}_E := \frac{1}{2}\Big(v_h|_{E\cap K_+} + v_h|_{E\cap K_-}\Big), \quad [v_h]_E := v_h|_{E\cap K_+} - v_h|_{E\cap K_-}$$

whereas for $E \in \mathcal{E}_h(\Gamma)$ we set

$$\{v_h\}_E := v_h|_E, \quad [v_h]_E := v_h|_E.$$

The averages $\{\nabla v_h\}_E$, $\{\underline{\tau}_h\}_E$ and jumps $[\nabla v_h]_E$, $[\underline{\tau}_h]_E$ of vector-valued functions $\nabla v_h$ and $\underline{\tau}_h$ are defined analogously. For $E \in \mathcal{E}_h(\Omega)$ it holds

$$\int_E u_h\, v_h \, ds = \int_E (\{u_h\}_E\, [v_h]_E + [u_h]_E\, \{v_h\}_E)\, ds. \tag{3.3}$$

We further denote by $\mathbf{n}_E$, $E \in \mathcal{E}_h(\Omega)$, with $E = K_+ \cap K_-$ the unit normal on $E$ pointing from $K_+$ to $K_-$ and by $\mathbf{n}_E$, $E \in \mathcal{E}_h(\Gamma)$, the exterior unit normal on $E$.

We define the broken $W^{1,p}$-space $W^{1,p}(\Omega; \mathcal{T}_h)$, $1 < p < \infty$, by

$$W^{1,p}(\Omega; \mathcal{T}_h) := \{v_h \in L^p(\Omega) \mid v_h|_K \in W^{1,p}(K), K \in \mathcal{T}_h\} \tag{3.4}$$

equipped with the norm

$$\|v_h\|_{W^{1,p}(\Omega;\mathcal{T}_h)} := \Big(\sum_{K\in\mathcal{T}_h} \|v_h\|^p_{W^{1,p}(K)}\Big)^{1/p} \tag{3.5}$$

and the broken $\underline{\mathbf{H}}$-space $\underline{\mathbf{H}}^{(p)}(\mathrm{div}, \Omega; \mathcal{T}_h)$ by

$$\underline{\mathbf{H}}^{(p)}(\mathrm{div}, \Omega; \mathcal{T}_h) := \big\{\underline{\mathbf{q}}_h \in L^p(\Omega; \mathbb{R}^2) \mid \underline{\mathbf{q}}_h|_K \in \underline{\mathbf{H}}^{(p)}(\mathrm{div}, K), K \in \mathcal{T}_h\big\} \tag{3.6}$$

equipped with the norm

$$\|\underline{\mathbf{q}}_h\|_{\underline{\mathbf{H}}^{(p)}(\mathrm{div},\Omega;\mathcal{T}_h)} := \Big(\sum_{K\in\mathcal{T}_h} \|\underline{\mathbf{q}}_h\|^p_{\underline{\mathbf{H}}^{(p)}(\mathrm{div},K)}\Big)^{1/p}. \tag{3.7}$$

We redefine the primal energy functional (2.3b) according to

$$J_P(v) := \frac{1}{p} \sum_{K\in\mathcal{T}_h} \int_K |\nabla v|^p \, dx - \int_\Omega f v \, dx - \int_{\Gamma_N} u_N v \, ds, \quad v \in W^{1,p}(\Omega; \mathcal{T}_h) \tag{3.8}$$

and note that it reduces to (2.3b) for $v \in W^{1,p}(\Omega)$.

We consider the finite element approximation with the DG spaces

$$V_h := \{v_h : \overline{\Omega} \to \mathbb{R} \mid v_h|_K \in P_k(K),\ K \in \mathcal{T}_h\} \tag{3.9a}$$

$$\underline{\mathbf{V}}_h := \{\underline{\mathbf{q}}_h : \overline{\Omega} \to \mathbb{R}^2 \mid \underline{\mathbf{q}}_h|_K \in P_k(K)^2,\ K \in \mathcal{T}_h\}. \tag{3.9b}$$

We note that $V_h \subset W^{1,p}(\Omega; \mathcal{T}_h)$. Moreover, for $\underline{\mathbf{q}}_h \in \underline{\mathbf{V}}_h$, we have $(\nabla \cdot \underline{\mathbf{q}}_h)|_K \in P_{k-1}(K)$, $K \in \mathcal{T}_h$, and $\mathbf{n}_E \cdot \underline{\mathbf{q}}_h|_E \in P_k(E)$, $E \in \mathcal{E}_h(\Gamma)$.

For $u_h \in V_h$ we define the broken gradient $\nabla_h u_h$ by means of

$$\nabla_h u_h|_K := \nabla u_h|_K, \quad K \in \mathcal{T}_h. \tag{3.10}$$

Further, let $u_D^*$ be chosen according to

$$u_D^* \in W^{1,p}(\Omega) \text{ such that } u_D^*|_{\Gamma_D} = u_D \text{ and } u_D^*|_{\Gamma_N} = 0. \tag{3.11}$$

Following [13, 23], we define recovery operators $\underline{\mathbf{R}}_{h,i} : V_h \oplus W^{1,p}(\Omega) \to \underline{\mathbf{V}}_h$, $1 \leqslant i \leqslant 2$, according to

$$\int_\Omega \underline{\mathbf{R}}_{h,1}(u)\cdot\underline{\mathbf{q}}_h\,dx = \sum_{E\in\mathcal{E}_h(\Omega\cup\Gamma_D)}\int_E [u]_E\,\mathbf{n}_E\cdot\{\underline{\mathbf{q}}_h\}_E\,ds,\quad \underline{\mathbf{q}}_h\in\underline{\mathbf{V}}_h \tag{3.12a}$$

$$\int_\Omega \underline{\mathbf{R}}_{h,2}(u)\cdot\underline{\mathbf{q}}_h\,dx = \sum_{E\in\mathcal{E}_h(\Gamma_D)}\int_E u\,\mathbf{n}_E\cdot\underline{\mathbf{q}}_h\,ds,\quad \underline{\mathbf{q}}_h\in\underline{\mathbf{V}}_h. \tag{3.12b}$$

We define the broken DG gradients $\nabla_{DG,i}u_h$, $1\leqslant i\leqslant 2$, as follows:

$$\nabla_{DG,1}u_h := \nabla_h u_h - \underline{\mathbf{R}}_{h,1}(u_h) \tag{3.13a}$$

$$\nabla_{DG,2}u_h := \nabla_{DG,1}u_h + \underline{\mathbf{R}}_{h,2}(u_D^*). \tag{3.13b}$$

The following auxiliary result from [13] will enable us to estimate the $L^p$ norm of $\underline{\mathbf{R}}_{h,1}(u_h)-\underline{\mathbf{R}}_{h,2}(u_D^*)$ for $u_h\in V_h$ (see Lemma A2 in [13]).

**Lemma 3.1.** *For each $p,q\in(1,\infty)$ such that $1/p+1/q=1$ there exists a constant $C_{IS}>0$, independent of $h$, such that it holds*

$$\inf_{u_h\in V_h}\sup_{\underline{\mathbf{q}}_h\in\underline{\mathbf{V}}_h}\frac{\int_\Omega(\underline{\mathbf{R}}_{h,1}(u_h)-\underline{\mathbf{R}}_{h,2}(u_D^*))\cdot\underline{\mathbf{q}}_h\,dx}{\|u_h\|_{L^p(\Omega)}\|\underline{\mathbf{q}}_h\|_{L^q(\Omega;\mathbb{R}^2)}}\geqslant C_{IS}. \tag{3.14}$$

**Theorem 3.1.** *Under the assumptions of Lemma 3.1 there exists a constant $C_{\rm rec}>0$, independent of $h$, such that for $u_h\in V_h$ it holds*

$$\|\underline{\mathbf{R}}_{h,1}(u_h)-\underline{\mathbf{R}}_{h,2}(u_D^*)\|_{L^p(\Omega;\mathbb{R}^2)}\leqslant C_{\rm rec}\bigg(\Big(\sum_{E\in\mathcal{E}_h(\Omega)}h_E^{-p/q}\int_E|[u_h]_E|^p\,ds\Big)^{1/p}+\Big(\sum_{E\in\mathcal{E}_h(\Gamma_D)}h_E^{-p/q}\int_E|u_h-u_D|^p\,ds\Big)^{1/p}\bigg) \tag{3.15a}$$

$$\|\underline{\mathbf{R}}_{h,2}(u_D^*)\|_{L^p(\Omega;\mathbb{R}^2)}\leqslant C_{\rm rec}\Big(\sum_{E\in\mathcal{E}_h(\Gamma_D)}h_E^{-p/q}\int_E|u_D|^p\,ds\Big)^{1/p}. \tag{3.15b}$$

*Proof.* We have

$$\|\underline{\mathbf{R}}_{h,1}(u_h)\|_{L^p(\Omega;\mathbb{R}^2)} = \sup_{\underline{\mathbf{q}}\in L^q(\Omega;\mathbb{R}^2)}\frac{\int_\Omega\underline{\mathbf{R}}_{h,1}(u_h)\cdot\underline{\mathbf{q}}\,dx}{\|\underline{\mathbf{q}}\|_{L^q(\Omega;\mathbb{R}^2)}}\geqslant\sup_{\underline{\mathbf{q}}_h\in\underline{\mathbf{V}}_h}\frac{\int_\Omega\underline{\mathbf{R}}_{h,1}(u_h)\cdot\underline{\mathbf{q}}_h\,dx}{\|\underline{\mathbf{q}}_h\|_{L^q(\Omega;\mathbb{R}^2)}}. \tag{3.16}$$

The inf-sup property (3.14) implies

$$\|\underline{\mathbf{R}}_{h,1}(u_h)\|_{L^p(\Omega;\mathbb{R}^2)}\leqslant C_{IS}^{-1}\sup_{\underline{\mathbf{q}}_h\in\underline{\mathbf{V}}_h}\frac{\int_\Omega\underline{\mathbf{R}}_{h,1}(u_h)\cdot\underline{\mathbf{q}}_h\,dx}{\|\underline{\mathbf{q}}_h\|_{L^q(\Omega;\mathbb{R}^2)}}. \tag{3.17}$$

Now, observing (3.12), setting $E_1:=E_+$, $E_2:=E_-$ for $E\in\mathcal{E}_h(\Omega)$ and using (1.2), (3.1), the trace inequality (3.2) as well as Hölder's inequality and the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}\int_\Omega(\underline{\mathbf{R}}_{h,1}(u_h)-\underline{\mathbf{R}}_{h,2}(u_D^*))\cdot\underline{\mathbf{q}}_h\,dx &\leqslant \sum_{E\in\mathcal{E}_h(\Omega)}\int_E h_E^{-1/q}|[u_h]_E|\,h_E^{1/q}|\{\underline{\mathbf{q}}_h\}_E|\,ds\\ &\quad+\sum_{E\in\mathcal{E}_h(\Gamma_D)}\int_E h_E^{-1/q}|u_h-u_D|\,h_E^{1/q}|\{\underline{\mathbf{q}}_h\}_E|\,ds\\ &\leqslant\frac{1}{2}\sum_{E\in\mathcal{E}_h(\Omega)}\Big(\int_E h_E^{-p/q}|[u_h]_E|^p\,ds\Big)^{1/p}\Big(\int_E h_E|\underline{\mathbf{q}}_h|_{E_+}+\underline{\mathbf{q}}_h|_{E_-}|^q\,ds\Big)^{1/q}\\ &\quad+\sum_{E\in\mathcal{E}_h(\Gamma_D)}\Big(\int_E h_E^{-p/q}|u_h-u_D|^p\,ds\Big)^{1/p}\Big(\int_E h_E|\underline{\mathbf{q}}_h|^q\,ds\Big)^{1/q}\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{C_q^{1/q}}{2} \sum_{E\in\mathcal{E}_h(\Omega\cup\Gamma_D)} \Big(\int_E h_E^{-p/q}|[u_h]_E|^p\,ds\Big)^{1/p} \Big(\int_E h_E(|\underline{\mathbf{q}}_h|_{E_+}|^q + |\underline{\mathbf{q}}_h|_{E_-}|^q)\,ds\Big)^{1/q} \\
&\quad + \sum_{E\in\mathcal{E}_h(\Gamma_D)} \Big(\int_E h_E^{-p/q}|u_h - u_D|^p\,ds\Big)^{1/p} \Big(\int_E h_E|\underline{\mathbf{q}}_h|^q\,ds\Big)^{1/q} \\
&\leqslant \frac{C_q^{1/q}}{2} \Big(\sum_{E\in\mathcal{E}_h(\Omega)} \int_E h_E^{-p/q}|[u_h]_E|^p\,ds\Big)^{1/p} \Big(\sum_{E\in\mathcal{E}_h(\Omega)} \int_E h_E \sum_{i=1}^2 |\underline{\mathbf{q}}_h|_{E_i}|^q\,ds\Big)^{1/q} \\
&\quad + \Big(\sum_{E\in\mathcal{E}_h(\Gamma_D)} \int_E h_E^{-p/q}|u_h - u_D|^p\,ds\Big)^{1/p} \Big(\sum_{E\in\mathcal{E}_h(\Gamma_D)} \int_E h_E|\underline{\mathbf{q}}_h|^q\,ds\Big)^{1/q} \\
&\leqslant C_R^{1/q} C_q^{1/q} \Big(\sum_{E\in\mathcal{E}_h(\Omega)} \int_E h_E^{-p/q}|[u_h]_E|^p\,ds\Big)^{1/p} \Big(\sum_{K\in\mathcal{T}_h} \int_{\partial K} h_K\, |\underline{\mathbf{q}}_h|_{\partial K}|^q\,ds\Big)^{1/q} \\
&\quad + C_R^{1/q} \Big(\sum_{E\in\mathcal{E}_h(\Gamma_D)} \int_E h_E^{-p/q}|u_h - u_D|^p\,ds\Big)^{1/p} \Big(\sum_{K\in\mathcal{T}_h} \int_{\partial K} h_K\, |\underline{\mathbf{q}}_h|_{\partial K}|^q\,ds\Big)^{1/q} \\
&\leqslant C_R^{1/q} C_q^{1/q} C_T \Big(\sum_{E\in\mathcal{E}_h(\Omega)} \int_E h_E^{-p/q}|[u_h]_E|^p\,ds\Big)^{1/p} \|\underline{\mathbf{q}}_h\|_{L^q(\Omega;\mathbb{R}^2)} \\
&\quad + C_R^{1/q} C_T \Big(\sum_{E\in\mathcal{E}_h(\Gamma_D)} \int_E h_E^{-p/q}|u_h - u_D|^p\,ds\Big)^{1/p} \|\underline{\mathbf{q}}_h\|_{L^q(\Omega;\mathbb{R}^2)}.
\end{aligned}
\tag{3.18}
$$

Using (3.18) in (3.17) gives (3.15a). The proof of (3.15b) follows along the same lines. □

We refer to $\Pi_k^{(p)}$ as a contractive $L^p$ projection of $L^p(\Omega)$ onto $\{v_h \in L^p(\Omega) \mid v_h|_K \in P_k(K), K \in \mathcal{T}_h\}$, which can be defined elementwise by

$$
\begin{aligned}
\int_\Omega \Pi_k^{(p)}(v)\, v_h\,dx &= \sum_{K\in\mathcal{T}_h} \int_K \Pi_k^{(p)}(v)\, v_h\,dx, \quad v \in L^p(\Omega) \\
\int_K \Pi_k^{(p)}(v)\, p_k\,dx &= \int_K v\, p_k\,dx, \quad p_k \in P_k(K),\ K \in \mathcal{T}_h.
\end{aligned}
\tag{3.19}
$$

We note that $\Pi_k^{(q)} := (\Pi_k^{(p)})^*$ is a contractive $L^q$ projection of $L^q(\Omega)$ onto $\{v_h \in L^q(\Omega) \mid v_h|_K \in P_k(K), K \in \mathcal{T}_h\}$ (see, e.g., [2]). We further refer to $\underline{\Pi}_k^{(p)}$ as a contractive $L^p$ projection of $L^p(\Omega;\mathbb{R}^2)$ onto $\underline{\mathbf{V}}_h$. We also denote by $\Pi_k^{(p)}$ a contractive $L^p$ projection of $L^p(\Gamma')$, $\Gamma' = \Gamma_D$ or $\Gamma' = \Gamma_N$, onto $\{v_h \in L^p(\Gamma') \mid v_h|_E \in P_k(E), E \in \mathcal{E}_h(\Gamma')\}$.

We define approximations $f_h$, $u_{h,D}$, and $u_{h,N}$ of $f$, $u_D$, and $u_N$ such that

$$
\begin{aligned}
f_h|_K \in P_{k-1}(K), &\quad \int_K (f - f_h)p_0\,dx = 0, \quad p_0 \in P_0(K),\ K \in \mathcal{T}_h \\
u_{h,D}|_E \in P_k(E), &\quad \int_E (u_D - u_{h,D})p_0\,ds = 0, \quad p_0 \in P_0(E),\ E \in \mathcal{E}_h(\Gamma_D) \\
u_{h,N}|E \in P_k(E), &\quad \int_E (u_N - u_{h,N})p_0\,ds = 0, \quad p_0 \in P_0(E),\ E \in \mathcal{E}_h(\Gamma_N).
\end{aligned}
\tag{3.20}
$$

**Remark 3.1.** We may choose $f_h|_K$ and $u_{h,N}|_E$ as contractive $L^q$ projections onto $P_{k-1}(K)$ and $P_k(E)$ and $u_{h,D}|E$ as a contractive $L^p$ projection onto $P_k(E)$.

We consider the discrete minimization problem

$$
J_{h,P}(u_h) = \inf_{v_h\in V_h} J_{h,P}(v_h) \tag{3.21a}
$$

where the objective functional $J_{h,P}$ is given by

$$J_{h,P}(v_h) := \frac{1}{p} \sum_{K\in\mathcal{T}_h} \int_K |\nabla_{DG,2} v_h|^p \,\mathrm{d}x - \int_\Omega f_h v_h \,\mathrm{d}x - \int_{\Gamma_N} u_{h,N}\, v_h \,\mathrm{d}s$$
$$+ \frac{\alpha}{p}\Big( \sum_{E\in\mathcal{E}_h(\Omega)} h_E^{-p/q} \int_E |[v_h]_E|^p \,\mathrm{d}s + \sum_{E\in\mathcal{E}_h(\Gamma_D)} h_E^{-p/q} \int_E |v_h - u_D|^p \,\mathrm{d}s\Big) \tag{3.21b}$$

and $\alpha > 0$ is a penalization parameter. The existence and uniqueness of a solution of (3.21) follows by standard arguments from the calculus of variations. The necessary and sufficient optimality condition gives rise to a discrete variational equation which represents the IPDG approximation of the $p$-Laplace problem (2.1a)–(2.1c): Find $u_h \in V_h$ such that for all $v_h \in V_h$ it holds

$$a_h^{DG}(u_h, v_h) = \ell_h(v_h) \tag{3.22}$$

where, observing $\underline{\Pi}_K^{(p)}(\nabla_{DG,1} v_h) = \nabla_{DG,1} v_h$, the semilinear IPDG form $a_h^{DG}(\cdot,\cdot) : V_h \times V_h \to \mathbb{R}$ is given by

$$a_h^{DG}(u_h, v_h) := \sum_{K\in\mathcal{T}_h} \int_K |\nabla_{DG,2} u_h|^{p-2} \nabla_{DG,2} u_h \cdot \underline{\Pi}_K^{(p)}(\nabla_{DG,1} v_h) \,\mathrm{d}x$$
$$+ \alpha \sum_{E\in\mathcal{E}_h(\Omega\cup\Gamma_D)} h_E^{-p/q} \int_E |[u_h - u_D^*]_E|^{p-2} [u_h - u_D^*]_E \,[v_h]_E \,\mathrm{d}s$$
$$= \sum_{K\in\mathcal{T}_h} \int_K \underline{\Pi}_K^{(q)}(|\nabla_{DG,2} u_h|^{p-2} \nabla_{DG,2} u_h) \cdot \nabla_{DG,1} v_h \,\mathrm{d}x$$
$$+ \alpha \sum_{E\in\mathcal{E}_h(\Omega\cup\Gamma_D)} h_E^{-p/q} \int_E |[u_h - u_D^*]_E|^{p-2} [u_h - u_D^*]_E \,[v_h]_E \,\mathrm{d}s \tag{3.23}$$

and $\ell_h(\cdot) : V_h \to \mathbb{R}$ stands for the linear functional

$$\ell_h(v_h) := \sum_{K\in\mathcal{T}_h} \int_K f_h v_h \,\mathrm{d}x + \sum_{E\in\mathcal{E}_h(\Gamma_N)} \int_E u_{h,N} v_h \,\mathrm{d}s. \tag{3.24}$$

**Lemma 3.2.** *The IPDG approximation* (3.22) *is consistent with the p-Laplace problem* (2.1a)–(2.1c) *in the sense that if $f = f_h$ and $u_N = u_{h,N}$ in* (2.2b) *and u satisfies* (2.1a)–(2.1c) *pointwise almost everywhere, then for all $v_h \in V_h$ it holds*

$$a_h^{DG}(u, v_h) = \ell_h(v_h). \tag{3.25}$$

*Proof.* Since $[u]_E = [u_D^*]_E = 0, E \in \mathcal{E}_h(\Omega)$, and $(u - u_D^*)|_E = (u - u_D)|_E = 0, E \in \mathcal{E}_h(\Gamma_D)$, we have $\underline{\mathbf{R}}_{h,1}(u) - \underline{\mathbf{R}}_{h,2}(u_D^*) = 0$. It follows that

$$a_h^{DG}(u, v_h) = \sum_{K\in\mathcal{T}_h} \int_K \underline{\Pi}_K^{(q)}(|\nabla u|^{p-2}\nabla u) \cdot (\nabla v_h - \underline{\mathbf{R}}_{h,1}(v_h)) \,\mathrm{d}x$$
$$= \sum_{K\in\mathcal{T}_h} \int_K \underline{\Pi}_K^{(q)}(|\nabla u|^{p-2}\nabla u) \cdot \nabla v_h \,\mathrm{d}x - \sum_{E\in\mathcal{E}_h(\Omega\cup\Gamma_D)} \int_E \mathbf{n}_E \cdot \{\underline{\Pi}_K^{(q)}(|\nabla u|^{p-2}\nabla u)\}_E \,[v_h]_E \,\mathrm{d}s$$
$$= \sum_{K\in\mathcal{T}_h} \int_K |\nabla u|^{p-2}\nabla u \cdot \underline{\Pi}_K^{(p)}(\nabla v_h) \,\mathrm{d}x - \sum_{E\in\mathcal{E}_h(\Omega\cup\Gamma_D)} \int_E \Pi_K^{(q)}(\mathbf{n}_E \cdot \{|\nabla u|^{p-2}\nabla u\}_E) \,[v_h]_E \,\mathrm{d}s$$
$$= \sum_{K\in\mathcal{T}_h} \int_K |\nabla u|^{p-2}\nabla u \cdot \underline{\Pi}_K^{(p)}(\nabla v_h) \,\mathrm{d}x - \sum_{E\in\mathcal{E}_h(\Omega\cup\Gamma_D)} \int_E \mathbf{n}_E \cdot \{|\nabla u|^{p-2}\nabla u\}_E \,\Pi_K^{(p)}([v_h]_E) \,\mathrm{d}s$$
$$= \sum_{K\in\mathcal{T}_h} \int_K |\nabla u|^{p-2}\nabla u \cdot \nabla v_h \,\mathrm{d}x - \sum_{E\in\mathcal{E}_h(\Omega\cup\Gamma_D)} \int_E \mathbf{n}_E \cdot \{|\nabla u|^{p-2}\nabla u\}_E \,[v_h]_E \,\mathrm{d}s. \tag{3.26}$$

An application of Green's formula gives

$$\int_K |\nabla u|^{p-2}\nabla u \cdot \nabla v_h \,\mathrm{d}x = -\int_K \nabla \cdot (|\nabla u|^{p-2}\nabla u) v_h \,\mathrm{d}x + \int_{\partial K} \mathbf{n}_{\partial K} \cdot (|\nabla u|^{p-2}\nabla u) v_h \,\mathrm{d}s.$$

Summing over all $K \in \mathcal{T}_h$ and observing (3.3) yields

$$\sum_{K\in\mathcal{T}_h}\int_K |\nabla u|^{p-2}\nabla u\cdot\nabla v_h\,dx = \sum_{K\in\mathcal{T}_h}\int_K (-\nabla\cdot(|\nabla u|^{p-2}\nabla u)\,v_h\,dx + \sum_{E\in\mathcal{E}_h(\Omega\cup\Gamma_D)}\int_E \mathbf{n}_E\cdot\{|\nabla u|^{p-2}\nabla u\}_E\,[v_h]_E\,ds$$
$$+ \sum_{E\in\mathcal{E}_h(\Gamma_N)}\int_E \mathbf{n}_E\cdot(|\nabla u|^{p-2}\nabla u)\,v_h\,ds. \tag{3.27}$$

Using (3.27) in (3.26) and observing (2.1a)–(2.1c) results in

$$a_h^{DG}(u,v_h) = \sum_{K\in\mathcal{T}_h}\int_K(-\nabla\cdot(|\nabla u|^{p-2}\nabla u)\,v_h\,dx + \sum_{E\in\mathcal{E}_h(\Gamma_N)}\int_E \mathbf{n}_E\cdot|\nabla u|^{p-2}\nabla u\,v_h\,ds = \int_\Omega f_h v_h\,dx + \int_{\Gamma_N} u_{h,N}v_h\,ds$$

which is the assertion. □

Observing (3.13a) and (3.12a), for the first term on the right-hand side of (3.23) we find

$$\begin{aligned}
\sum_{K\in\mathcal{T}_h}\int_K \underline{\Pi}_k^{(q)}(|\nabla_{DG,2}u_h|^{p-2}\nabla_{DG,2}u_h)\cdot\nabla_{DG,1}v_h\,dx &= \sum_{K\in\mathcal{T}_h}\int_K \underline{\Pi}_k^{(q)}(|\nabla_{DG,2}u_h|^{p-2}\nabla_{DG,2}u_h)\cdot\nabla v_h\,dx\\
&\quad - \sum_{K\in\mathcal{T}_h}\int_K \underline{\Pi}_k^{(q)}(|\nabla_{DG,2}u_h|^{p-2}\nabla_{DG,2}u_h)\cdot\underline{\mathbf{R}}_{h,1}(v_h)\,dx\\
&= \sum_{K\in\mathcal{T}_h}\int_K |\nabla_{DG,2}u_h|^{p-2}\nabla_{DG,2}u_h\cdot\underline{\Pi}_k^{(p)}(\nabla v_h)\,dx\\
&\quad - \sum_{K\in\mathcal{T}_h}\int_K \underline{\Pi}_k^{(q)}(|\nabla_{DG,2}u_h|^{p-2}\nabla_{DG,2}u_h)\cdot\underline{\mathbf{R}}_{h,1}(v_h)\,dx\\
&= \sum_{K\in\mathcal{T}_h}\int_K |\nabla_{DG,2}u_h|^{p-2}\nabla_{DG,2}u_h\cdot\nabla v_h\,dx\\
&\quad - \sum_{E\in\mathcal{E}_h(\Omega\cup\Gamma_D)}\int_E [v_h]_E\,\mathbf{n}_E\cdot\{\underline{\Pi}_k^{(q)}(|\nabla_{DG,2}u_h|^{p-2}\nabla_{DG,2}u_h)\}_E\,ds\\
&= \sum_{K\in\mathcal{T}_h}\int_K |\nabla_{DG,2}u_h|^{p-2}\nabla_{DG,2}u_h\cdot\nabla v_h\,dx\\
&\quad - \sum_{E\in\mathcal{E}_h(\Omega\cup\Gamma_D)}\int_E [v_h]_E\,\Pi_k(\mathbf{n}_E\cdot\{|\nabla_{DG,2}u_h|^{p-2}\nabla_{DG,2}u_h\}_E)\,ds\\
&= \sum_{K\in\mathcal{T}_h}\int_K |\nabla_{DG,2}u_h|^{p-2}\nabla_{DG,2}u_h\cdot\nabla v_h\,dx - \sum_{E\in\mathcal{E}_h(\Omega\cup\Gamma_D)}\int_E [v_h]_E\,\mathbf{n}_E\cdot\{|\nabla_{DG,2}u_h|^{p-2}\nabla_{DG,2}u_h\}_E\,ds
\end{aligned}$$

and hence, we obtain

$$a_h^{DG}(u_h,v_h) = \sum_{K\in\mathcal{T}_h}\int_K |\nabla_{DG,2}u_h|^{p-2}\nabla_{DG,2}u_h\cdot\nabla v_h\,dx - \sum_{E\in\mathcal{E}_h(\Omega\cup\Gamma_D)}\int_E [v_h]_E\,\mathbf{n}_E\cdot\{|\nabla_{DG,2}u_h|^{p-2}\nabla_{DG,2}u_h\}_E\,ds$$
$$+ \alpha\sum_{E\in\mathcal{E}_h(\Omega\cup\Gamma_D)} h_E^{-p/q}\int_E |[u_h-u_D^*]_E|^{p-2}[u_h-u_D^*]_E\,[v_h]_E\,ds. \tag{3.28}$$

**Remark 3.2.** In case $p = 2$, i.e., for the Poisson problem, we have

$$a_h^{DG}(u_h,v_h) = \sum_{K\in\mathcal{T}_h}\int_K \nabla u_h\cdot\nabla v_h\,dx - \sum_{E\in\mathcal{E}_h(\Omega\cup\Gamma_D)}\int_E [v_h]_E\,\mathbf{n}_E\cdot\{\nabla u_h\}_E\,ds - \sum_{E\in\mathcal{E}_h(\Omega\cup\Gamma_D)}\int_E [u_h-u_D^*]_E\,\mathbf{n}_E\cdot\{\nabla v_h\}_E\,ds$$
$$+ \alpha\sum_{E\in\mathcal{E}_h(\Omega\cup\Gamma_D)} h_E^{-p/q}\int_E |[u_h-u_D^*]_E|^{p-2}[u_h-u_D^*]_E\,[v_h]_E\,ds$$
$$+ \sum_{K\in\mathcal{T}_h}\int_K(\underline{\mathbf{R}}_{h,1}(u_h)-\underline{\mathbf{R}}_{h,2}(u_D^*))\cdot\underline{\mathbf{R}}_{h,1}(v_h)\,dx$$

which, except for the last term, coincides with the IPDG approximation for the Poisson problem from [3].

**Remark 3.3.** We note that $u_h \notin W^{1,p}(\Omega)$, but a conforming finite element function $u_h^c \in V_h^c := V_h \cap W^{1,p}(\Omega)$ can be obtained from $u_h \in V_h$ by postprocessing in the following way (see [38]): Let $\mathcal{N}^L$ be the set of Lagrangian nodal points for the elements in $V_h^c$ and let Let $\varkappa_i$ be the number of triangles that share the nodal point $x_i \in \mathcal{N}^L$. We have $\varkappa_i = 1$, if $x_i$ is contained in the interior of an element, while $\varkappa_i > 1$, if $x_i \in \mathcal{N}^L \cap \mathcal{E}_h(\Omega)$. The multiplicity $\varkappa_i$ is bounded, since the triangulation is locally quasi-uniform. Denoting by $\mathcal{N}(x_i)$ the set $\mathcal{N}(x_i) := \bigcup\{K \in \mathcal{T}_h(\Omega) \mid x_i \in K\}$, the associated conforming element is defined by its nodal values

$$u_h^c(x_i) := \begin{cases} \dfrac{1}{\varkappa_i} \sum\limits_{K \in \mathcal{N}(x_i)} u_h|_K(x_i), & x_i \in \Omega \cup \Gamma_N \\ u_{h,D}, & x_i \in \Gamma. \end{cases} \tag{3.29}$$

By a generalization of Theorem 2.2 in [38] to the case $p \neq 2$ there exists a constant $C_c > 0$, only depending on the local geometry of the triangulation, such that

$$\|u_h - u_h^c\|^p_{W^{1,p}(\Omega;\mathcal{T}_h)} \leq C_c\Big( \sum_{E \in \mathcal{E}_h(\Omega)} h_E^{-p/q} \int_E |[u_h]_E|^p \, ds + \sum_{E \in \mathcal{E}_h(\Gamma_D)} h_E^{-p/q} \int_E |u_h - u_{h,D}|^p \, ds \Big). \tag{3.30}$$

Next, we consider a two-field formulation of the IPDG approximation (3.22). We set

$$\underline{\mathbf{p}}_h := \underline{\Pi}_k^{(q)}(|\nabla_{DG,2} u_h|^{p-2} \nabla_{DG,2} u_h) \tag{3.31a}$$

$$-\nabla_h \cdot \underline{\mathbf{p}}_h = f_h. \tag{3.31b}$$

We consider (3.31a) elementwise for each $K \in \mathcal{T}_h$, multiply by $\underline{\mathbf{q}}_h|_K$, $\underline{\mathbf{q}}_h \in \underline{\mathbf{V}}_h$, integrate over $K$, and finally sum over all $K \in \mathcal{T}_h$. Observing $\underline{\Pi}_k^{(q)}(\underline{\mathbf{q}}_h) = \underline{\mathbf{q}}_h$, we thus obtain

$$\sum_{K \in \mathcal{T}_h} \int_K \underline{\mathbf{p}}_h \cdot \underline{\mathbf{q}}_h \, dx = \sum_{K \in \mathcal{T}_h} \int_K |\nabla_{DG,2} u_h|^{p-2} \nabla_{DG,2} u_h \cdot \underline{\mathbf{q}}_h \, dx. \tag{3.32a}$$

Likewise, we consider (3.31b) elementwise for each $K \in \mathcal{T}_h$, multiply by $v_h|_K$, $v_h \in V_h$, integrate over $K$, and finally sum over all $K \in \mathcal{T}_h$. An elementwise application of Green's formula gives

$$-\sum_{K \in \mathcal{T}_h} \int_K \nabla \cdot \underline{\mathbf{p}}_h \, v_h \, dx = \sum_{K \in \mathcal{T}_h} \int_K \underline{\mathbf{p}}_h \cdot \nabla v_h \, dx - \sum_{K \in \mathcal{T}_h} \int_{\partial K} \mathbf{n}_{\partial K} \cdot \underline{\mathbf{p}}_h|_{\partial K} \, v_h \, ds = \sum_{K \in \mathcal{T}_h} \int_K f_h v_h \, dx. \tag{3.32b}$$

We replace $\underline{\mathbf{p}}_h|_{\partial K}$ in (3.32b) by a numerical flux function $\hat{\underline{\mathbf{p}}}_{\partial K}$. We thus obtain the following system of discrete variational equations: Find $(u_h, \underline{\mathbf{p}}_h) \in V_h \times \underline{\mathbf{V}}_h$ such that for all $(v_h, \underline{\mathbf{q}}_h) \in V_h \times \underline{\mathbf{V}}_h$ it holds

$$\sum_{K \in \mathcal{T}_h} \int_K \underline{\mathbf{p}}_h \cdot \underline{\mathbf{q}}_h \, dx = \sum_{K \in \mathcal{T}_h} \int_K |\nabla_{DG,2} u_h|^{p-2} \nabla_{DG,2} u_h \cdot \underline{\mathbf{q}}_h \, dx \tag{3.33a}$$

$$\sum_{K \in \mathcal{T}_h} \int_K \underline{\mathbf{p}}_h \cdot \nabla v_h \, dx - \sum_{K \in \mathcal{T}_h} \int_{\partial K} \mathbf{n}_{\partial K} \cdot \hat{\underline{\mathbf{p}}}_{\partial K} \, v_h \, ds = \sum_{K \in \mathcal{T}_h} \int_K f_h v_h \, dx. \tag{3.33b}$$

In particular, for the two-field formulation of the IPDG approximation (3.22) the numerical flux function $\hat{\underline{\mathbf{p}}}_{\partial K}$ is chosen as follows:

$$\hat{\underline{\mathbf{p}}}_{\partial K}|_E := \begin{cases} \{\underline{\mathbf{z}}_h\}_E - \alpha h_E^{-p/q} |[u_h]_E|^{p-2} [u_h]_E \, \mathbf{n}_E, & E \in \mathcal{E}_h(\Omega) \\ \underline{\mathbf{z}}_h - \alpha h_E^{-p/q} |u_h - u_D|^{p-2} (u_h - u_D) \, \mathbf{n}_E, & E \in \mathcal{E}_h(\Gamma_D) \\ u_N \mathbf{n}_E, & E \in \mathcal{E}_h(\Gamma_N) \end{cases} \tag{3.34}$$

where $\underline{\mathbf{z}}_h := |\nabla_{DG,2} u_h|^{p-2} \nabla_{DG,2} u_h$.

**Theorem 3.2.** *The two-field formulation* (3.33) *is equivalent with* (3.22). *In particular, if $u_h \in V_h$ is the solution of* (3.22), *there exists $\underline{\mathbf{p}}_h \in \underline{\mathbf{V}}_h$ such that the pair $(\underline{\mathbf{p}}_h, u_h) \in \underline{\mathbf{V}}_h \times V_h$ satisfies* (3.33). *Conversely, if the pair $(\underline{\mathbf{p}}_h, u_h) \in \underline{\mathbf{V}}_h \times V_h$ satisfies* (3.33), *then $u_h \in V_h$ solves* (3.22).

*Proof.* Let $u_h \in V_h$ be the solution of (3.22). We define $\underline{\mathbf{p}}_h \in \underline{\mathbf{V}}_h$ by means of (3.33a). Then, choosing $\underline{\mathbf{q}}_h = \nabla v_h$ in (3.33a) and observing (3.28) and (3.34) yields (3.33b). Conversely, if the pair $(\underline{\mathbf{p}}_h, u_h) \in \underline{\mathbf{V}}_h \times V_h$ satisfies (3.33), we choose $\underline{\mathbf{q}}_h = \nabla v_h$ in (3.33a) and insert (3.33a) into (3.33b). Taking (3.34) into account this shows that $u_h \in V_h$ satisfies (3.22). □

## 4 An a posteriori error estimator for the global discretization error

Given reflexive Banach spaces $V$, $Q$ with norms $\|\cdot\|_V$, $\|\cdot\|_Q$, convex and coercive objective functionals $C : V \to \mathbb{R}$, $D : Q \to \mathbb{R}$, and a bounded linear operator $\Lambda : V \to Q$, we consider the minimization problem

$$\inf_{u \in V} J(u) \tag{4.1}$$

for the objective functional

$$J(u) := C(u) + D(\Lambda u). \tag{4.2}$$

An abstract approach to the a posteriori error control for (4.1) has been provided in [46]. The a posteriori error control relies on the dual formulation of (4.1):

$$\sup_{q \in Q} J^*(q) \quad \text{or} \quad \inf_{q \in Q} (-J^*(q)) \tag{4.3}$$

in terms of the Fenchel conjugate $J^*$ of $J$ as given by

$$J^*(q) = -C^*(-\Lambda^* q) - D^*(q) \tag{4.4}$$

where $C^*$ and $D^*$ are the Fenchel conjugates of $C$ and $D$ and $\Lambda^*$ stands for the adjoint of $\Lambda$.

Given some approximation $u_h \in V$ of the minimizer $u$ of (4.1), the a posteriori error estimate from [46] states that for any admissible function $q \in Q$ it holds

$$\|u - u_h\|_V^2 \lesssim C(u_h) + C^*(-\Lambda^* q) + D(\Lambda u_h) + D^*(q). \tag{4.5}$$

Now, let $u_h^c \in V_h^c \subset W^{1,p}(\Omega)$ be the conforming finite element function obtained from the solution $u_h \in V_h$ of (3.22) by postprocessing according to Remark 3.3. Then it holds

$$\|u - u_h\|^2_{W^{1,p}(\Omega;\mathcal{T}_h)} \leqslant 2\Big(\|u - u_h^c\|^2_{W^{1,p}(\Omega)} + \|u_h - u_h^c\|^2_{W^{1,p}(\Omega;\mathcal{T}_h)}\Big). \tag{4.6}$$

In order to estimate the first term on the right-hand side of (4.6) we apply (4.5) with $V = W^{1,p}(\Omega)$, $Q := L^p(\Omega;\mathbb{R}^2)$, $\Lambda = \nabla$, and

$$C(u_h^c) := -\int_\Omega f u_h^c \, dx - \int_{\Gamma_N} u_N u_h^c \, ds \tag{4.7a}$$

$$D(\nabla u_h^c) := \frac{1}{p} \sum_{K \in \mathcal{T}_h} \int_K |\nabla u_h^c|^p \, dx + I_{K_1}(u_h^c) \tag{4.7b}$$

where $I_{K_1}$ is the indicator function of the closed convex set

$$K_1 := \left\{ v \in W^{1,p}(\Omega) \mid v = u_D \text{ on } \Gamma_D \right\}. \tag{4.7c}$$

We obtain

$$C^*(-\Lambda^* \underline{\mathbf{q}}) := I_{K_2}(\underline{\mathbf{q}}), \quad \underline{\mathbf{q}} \in \underline{\mathbf{H}}^{(q)}(\mathrm{div}, \Omega) \tag{4.8a}$$

$$D^*(\underline{\mathbf{q}}) := \frac{1}{q} \int_\Omega |\underline{\mathbf{q}}|^q \, dx - \sum_{E \in \mathcal{E}_h(\Gamma_D)} \int_E u_D \, \mathbf{n}_E \cdot \underline{\mathbf{q}} \, ds, \quad \underline{\mathbf{q}} \in \underline{\mathbf{H}}^{(q)}(\mathrm{div}, \Omega) \tag{4.8b}$$

where $I_{K_2}$ is the indicator function of the closed convex set

$$K_2 := \{ \underline{\mathbf{q}} \in \underline{\mathbf{H}}^{(q)}(\mathrm{div}; \Omega) \mid -\nabla \cdot \underline{\mathbf{q}} = f \text{ in } \Omega, \ \mathbf{n}_{\Gamma_N} \cdot \underline{\mathbf{q}} = u_N \text{ on } \Gamma_N \}. \tag{4.8c}$$

We call $\underline{\mathbf{p}}_h^{\mathrm{eq}} \in \underline{\mathbf{V}}_h$ an equilibrated flux, if

$$\underline{\mathbf{p}}_h^{\mathrm{eq}} \in \underline{\mathbf{H}}^{(q)}(\mathrm{div}; \Omega) \tag{4.9a}$$

and $\underline{\mathbf{p}}_h^{\mathrm{eq}}$ satisfies the equilibrium conditions

$$-\nabla\cdot\underline{\mathbf{p}}_h^{\mathrm{eq}} = f_h \qquad \text{in } \Omega \tag{4.9b}$$

$$\mathbf{n}_{\Gamma_N}\cdot\underline{\mathbf{p}}_h^{\mathrm{eq}} = u_{h,N} \qquad \text{on } \Gamma_N. \tag{4.9c}$$

Moreover, we choose $\underline{\mathbf{p}}_c \in \{\underline{\mathbf{q}} \in \underline{\mathbf{H}}_{0,\Gamma_D}^{(q)}(\mathrm{div},\Omega) \mid \mathbf{n}_{\Gamma_N}\cdot\underline{\mathbf{q}} \in L^q(\Gamma_N)\}$ such that

$$-\nabla\cdot\underline{\mathbf{p}}_c = f - f_h \text{ in } \Omega, \qquad \mathbf{n}_{\Gamma_N}\cdot\underline{\mathbf{p}}_c = u_N - u_{h,N} \text{ on } \Gamma_N. \tag{4.10}$$

It follows that $\underline{\mathbf{p}}_h^{\mathrm{eq}} + \underline{\mathbf{p}}_c \in K_2$, i.e., $I_{K_2}(\underline{\mathbf{p}}_h^{\mathrm{eq}} + \underline{\mathbf{p}}_c) = 0$, and hence, (4.5) reads as follows

$$\|u - u_h^c\|^2_{W^{1,p}(\Omega)} \lesssim J_P(u_h^c) + I_{K_1}(u_h^c) + J_D(\underline{\mathbf{p}}_h^{\mathrm{eq}} + \underline{\mathbf{p}}_c). \tag{4.11}$$

In view of (2.4d) and $\mathbf{n}_{\Gamma_D}\cdot\underline{\mathbf{p}}_c = 0$ on $\Gamma_D$ we have

$$J_D(\underline{\mathbf{p}}_h^{\mathrm{eq}} + \underline{\mathbf{p}}_c) = \frac{1}{q}\sum_{K\in\mathcal{T}_h}\int_K |\underline{\mathbf{p}}_h^{\mathrm{eq}} + \underline{\mathbf{p}}_c|^q\,dx - \sum_{E\in\mathcal{E}_h(\Gamma_D)}\int_E u_D\,\mathbf{n}_{\Gamma_D}\cdot\underline{\mathbf{p}}_h^{\mathrm{eq}}\,ds. \tag{4.12}$$

Using (1.2), we find

$$\frac{1}{q}\sum_{K\in\mathcal{T}_h}\int_K |\underline{\mathbf{p}}_h^{\mathrm{eq}} + \underline{\mathbf{p}}_c|^q\,dx \leq \frac{1}{q}C_q\Big(\sum_{K\in\mathcal{T}_h}\int_K |\underline{\mathbf{p}}_h^{\mathrm{eq}}|^q\,dx + \sum_{K\in\mathcal{T}_h}\int_K |\underline{\mathbf{p}}_c|^q\,dx\Big). \tag{4.13}$$

In order to estimate the second term on the right-hand side of (4.13) we use the Poincaré–Friedrichs inequalities

$$\|v - |K|^{-1}\int_K v\,dx\|_{L^p(K)} \leq C_{PF}^{(1)}(p)\,h_K\|\nabla v\|_{L^p(K)}, \quad v \in W^{1,p}(K),\ K\in\mathcal{T}_h \tag{4.14a}$$

$$\|v - |E|^{-1}\int_E v\,ds\|_{L^p(E)} \leq C_{PF}^{(2)}(p)\,h_E\|\nabla v\|_{L^p(E)}, \quad v \in W^{1,p}(E),\ E\in\mathcal{E}_h(\Gamma_N) \tag{4.14b}$$

where $C_{PF}^{(i)}(p)$, $1 \leq i \leq 2$, are positive constants depending only on $p$ (see, e.g., [28]).

**Lemma 4.1.** *Suppose that the following regularity assumption is satisfied: For $\{\underline{\boldsymbol{\tau}} \in \underline{\mathbf{H}}_{0,\Gamma_D}^{(p)}(\mathrm{div},\Omega) \mid \mathbf{n}_{\Gamma_N}\cdot\underline{\boldsymbol{\tau}} \in L^p(\Gamma_N)\}$ and the weak solution $z$ of the elliptic boundary value problem*

$$-\Delta z = -\nabla\cdot\underline{\boldsymbol{\tau}} \qquad \textit{in } \Omega \tag{4.15a}$$

$$\mathbf{n}_{\Gamma_N}\cdot\nabla z = 0 \qquad \textit{on } \Gamma_D \tag{4.15b}$$

$$\mathbf{n}_{\Gamma_N}\cdot\nabla z = \mathbf{n}_{\Gamma_N}\cdot\underline{\boldsymbol{\tau}} \qquad \textit{on } \Gamma_N \tag{4.15c}$$

*there exists a constant $C_z > 0$ such that*

$$\nabla z|_{\Gamma_N} \in L^p(\Gamma_N), \quad \|\nabla z\|_{L^p(\Gamma_N;\mathbb{R}^2)} \leq C_z. \tag{4.16}$$

*Then for $\underline{\mathbf{p}}_c \in \{\underline{\mathbf{H}}_{0,\Gamma_D}^{(q)}(\mathrm{div},\Omega) \mid \mathbf{n}_{\Gamma_N}\cdot\underline{\boldsymbol{\tau}} \in L^q(\Gamma_N)\}$ it holds*

$$\|\underline{\mathbf{p}}_c\|^q_{L^q(\Omega;\mathbb{R}^2)} \leq C_q\Big(C_{PF}^{(1)}(p)^q\,osc_{h,1} + C_z^q C_{PF}^{(2)}(p)^q\,osc_{h,2}\Big) \tag{4.17}$$

*where $osc_{h,1}$ and $osc_{h,2}$ refer to the data oscillations*

$$osc_{h,1} := \sum_{K\in\mathcal{T}_h} osc_{K,1}, \qquad osc_{K,1} := h_K^q\int_K |f - f_h|^q\,dx \tag{4.18a}$$

$$osc_{h,2} := \sum_{K\in\mathcal{T}_h} osc_{K,2}, \qquad osc_{k;2} := \sum_{E\in\mathcal{E}_h(\partial K\cap\Gamma_N)} h_E^q\int_E |u_N - u_{h,N}|^q\,ds. \tag{4.18b}$$

*Proof.* We have

$$\|\underline{\mathbf{p}}_c\|_{L^q(\Omega;\mathbb{R}^2)} = \sup\Big\{ \int_\Omega \underline{\mathbf{p}}_c \cdot \underline{\boldsymbol{\tau}}\,\mathrm{d}x \mid \underline{\boldsymbol{\tau}} \in \underline{\mathbf{H}}^{(p)}_{0,\Gamma_D}(\mathrm{div},\Omega),\ \|\underline{\boldsymbol{\tau}}\|_{L^p(\Omega;\mathbb{R}^2)} \le 1\Big\}.$$

For $\underline{\boldsymbol{\tau}} \in \{\underline{\mathbf{H}}^{(p)}_{0,\Gamma_D} \mid \mathbf{n}_{\Gamma_N}\cdot\underline{\boldsymbol{\tau}} \in L^p(\Gamma_N)\}$ there exists $z \in W^{1,p}(\Omega)$ such that $\underline{\boldsymbol{\tau}} = \nabla z$. In fact, $z$ can be chosen as the weak solution of the boundary value problem (4.15). Hence, we have

$$\|\underline{\mathbf{p}}_c\|_{L^q(\Omega,\mathbb{R}^2)} \le \sup_{\|\nabla z\|_{L^p(\Omega;\mathbb{R}^2)}\le 1} \int_\Omega \underline{\mathbf{p}}_c \cdot \nabla z\,\mathrm{d}x. \tag{4.19}$$

Applying Green's formula locally on each $K \in \mathcal{T}_h$ and taking (3.20) into account, we get

$$\begin{aligned}\int_\Omega \underline{\mathbf{p}}_c\cdot\nabla z\,\mathrm{d}x &= \sum_{K\in\mathcal{T}_h}\int_K \underline{\mathbf{p}}_c\cdot\nabla z\,\mathrm{d}x = -\sum_{K\in\mathcal{T}_h}\int_K \nabla\cdot\underline{\mathbf{p}}_c\, z\,\mathrm{d}x + \sum_{E\in\mathcal{E}_h(\Gamma_N)}\int_E \mathbf{n}_E\cdot\underline{\mathbf{p}}_c\, z\,\mathrm{d}s\\ &= \sum_{K\in\mathcal{T}_h}\int_K (f-f_h)(z-p_{0,1})\,\mathrm{d}x + \sum_{E\in\mathcal{E}_h(\Gamma_N)}\int_E (u_N-u_{h,N})(z-p_{0,2})\,\mathrm{d}s\end{aligned} \tag{4.20}$$

where $p_{0,1} := |K|^{-1}\int_K z\,\mathrm{d}x$ and $p_{0,2} := |E|^{-1}\int_E z\,\mathrm{d}s$. Using Hölder's inequality, the Cauchy–Schwarz inequality, and the Poincaré–Friedrichs inequalities (4.14), we obtain

$$\begin{aligned}&\Big|\sum_{K\in\mathcal{T}_h}\int_K (f-f_h)(z-p_{0,1})\,\mathrm{d}x + \sum_{E\in\mathcal{E}_h(\Gamma_N)}\int_E (u_N-u_{h,N})(z-p_{0,2})\,\mathrm{d}s\Big|\\ &\le \sum_{K\in\mathcal{T}_h}\Big(\int_K |f-f_h|^q\,\mathrm{d}x\Big)^{1/q}\Big(\int_K |z-p_{01}|^p\,\mathrm{d}x\Big)^{1/p}\\ &\quad + \Big(\sum_{E\in\mathcal{E}_h(\Gamma_N)}\int_E |u_N-u_{h,N}|^q\,\mathrm{d}s\Big)^{1/q}\Big(\sum_{E\in\mathcal{E}_h(\Gamma_N)}\int_E |z-p_{0,2}|^p\,\mathrm{d}s\Big)^{1/p}\\ &\le C^{(1)}_{PF}(p)\Big(\sum_{K\in\mathcal{T}_h} h_K^q\int_K |f-f_h|^q\,\mathrm{d}x\Big)^{1/q}\Big(\sum_{K\in\mathcal{T}_h}\int_K |\nabla z|^p\,\mathrm{d}x\Big)^{1/p}\\ &\quad + C^{(2)}_{PF}(p)\Big(\sum_{E\in\mathcal{E}_h(\Gamma_N)} h_E^q\int_E |u_N-u_{h,N}|^q\,\mathrm{d}s\Big)^{1/q}\Big(\sum_{E\in\mathcal{E}_h(\Gamma_N)}\int_E |\nabla z|^p\,\mathrm{d}s\Big)^{1/p}.\end{aligned} \tag{4.21}$$

Using (4.20), (4.21), and (4.16) in (4.19), it follows that

$$\|\underline{\mathbf{p}}_c\|_{L^q(\Omega,\mathbb{R}^2)} \le C^{(1)}_{PF}(p)\, osc_{h,1}^{1/q} + C_z C^{(2)}_{PF}(p)\, osc_{h,2}^{1/q}. \tag{4.22}$$

Hence, using (1.2), we find

$$\|\underline{\mathbf{p}}_c\|^q_{L^q(\Omega,\mathbb{R}^2)} \le C_q\Big(C^{(1)}_{PF}(p)^q\, osc_{h,1} + C_z^q C^{(2)}_{PF}(p)^q\, osc_{h,2}\Big) \tag{4.23}$$

which is the assertion. □

Moreover, as far as $J_P(u_h^c)$ is concerned, we have

$$J_P(u_h^c) = J_P(u_h) + \frac{1}{p}\sum_{K\in\mathcal{T}_h}\int_K \Big(|\nabla u_h^c|^p - |\nabla u_h|^p\Big)\mathrm{d}x + \sum_{K\in\mathcal{T}_h}\int_K f(u_h-u_h^c)\,\mathrm{d}x + \sum_{E\in\mathcal{E}_h(\Gamma_N)}\int_E u_N(u_h-u_h^c)\,\mathrm{d}s. \tag{4.24}$$

**Lemma 4.2.** *Let $u_h \in V_h$ be the solution of (3.22) and let $u_h^c \in V_h^c$ be its postprocessed finite element function. Then it holds*

$$\Big|J_P(u_h^c) - J_P(u_h)\Big| \le \sum_{K\in\mathcal{T}_h} \varkappa_K^{\mathrm{eq}} \tag{4.25}$$

*where*

$$\varkappa_K^{\mathrm{eq}} := \|u_h-u_h^c\|^p_{W^{1,p}(K)} + \Big(\sum_{K\in\mathcal{T}_h}|\nabla u_h|^{p/q}_{DG,K} + \|f\|_{L^q(K)} + \sum_{E\in\mathcal{E}_h(\partial K\cap\Gamma_N)}\|u_N\|_{L^q(E)}\Big)\|u_h-u_h^c\|_{W^{1,p}(K)} \tag{4.26}$$

*and $|\nabla u_h|_{DG,K}$ is given by*

$$|\nabla u_h|_{DG,K} := \Big(\int_K |\nabla u_h|^p\,\mathrm{d}x\Big)^{1/p}. \tag{4.27}$$

*Proof.* By Taylor expansion and using (1.2) as well as Hölder's inequality we find

$$
\begin{aligned}
&\Big|\frac{1}{p}\sum_{K\in\mathcal{T}_h}\int_K\Big(|\nabla u_h^c|^p-|\nabla u_h|^p\Big)\Big|\\
&\quad=\Big|\sum_{K\in\mathcal{T}_h}\int_K\int_0^1|\nabla u_h+\lambda\nabla(u_h^c-u_h)|^{p-2}(\nabla u_h+\lambda\nabla(u_h^c-u_h)\,d\lambda\cdot\nabla(u_h^c-u_h)\,dx\Big|\\
&\quad\leqslant\sum_{K\in\mathcal{T}_h}\int_K\int_0^1|\nabla u_h+\lambda\nabla(u_h^c-u_h)|^{p-1}|\nabla(u_h^c-u_h)|\,d\lambda\,dx\\
&\quad\leqslant C_{p-1}\sum_{K\in\mathcal{T}_h}\int_K\int_0^1\Big(|\nabla u_h|^{p-1}+\lambda^{p-1}|\nabla(u_h-u_h^c)|^{p-1}|\nabla(u_h-u_h^c)|\,d\lambda\,dx\\
&\quad\leqslant C_{p-1}\sum_{K\in\mathcal{T}_h}\Big(\int_K|\nabla u_h|^p\,dx\Big)^{1/q}\Big(\int_K|\nabla(u_h-u_h^c)|^p\,dx\Big)^{1/p}+\frac{1}{p}C_{p-1}\sum_{K\in\mathcal{T}_h}\int_K|\nabla(u_h-u_h^c)|^p\,dx.
\end{aligned}
\tag{4.28}
$$

Moreover, we have

$$
\begin{aligned}
&\Big|\sum_{K\in\mathcal{T}_h}\int_K f(u_h-u_h^c)\,dx+\sum_{E\in\mathcal{E}_h(\Gamma_N)}\int_E u_N(u_h-u_h^c)\,ds\Big|\\
&\quad\leqslant\sum_{K\in\mathcal{T}_h}\Big(\int_K|f|^q\,dx\Big)^{1/q}\Big(\int_K|u_h-u_h^c|^p\,dx\Big)^{1/p}+\sum_{E\in\mathcal{E}_h(\Gamma_N)}\Big(\int_E|u_N|^q\,ds\Big)^{1/q}\Big(\int_E|u_h-u_h^c|^p\,ds\Big)^{1/p}.
\end{aligned}
\tag{4.29}
$$

The assertion now follows from (4.24) and (4.28), (4.29). □

For practical purposes, we further replace $I_{K_1}(u_h^c)$ by the penalty term

$$\alpha\sum_{E\in\mathcal{E}_h(\Gamma_D)}h_E^{-p/q}\int_E|u_h^c-u_D|^p\,ds. \tag{4.30}$$

In view of (3.29) we have $u_h^c|_E=u_{h,D}$ on $E\in\mathcal{E}_h(\Gamma_D)$ and hence, (4.30) gives rise to the data oscillation

$$\alpha\,osc_{h,3}:=\alpha\sum_{K\in\mathcal{T}_h}osc_{K,3} \tag{4.31a}$$

$$osc_{K,3}:=\sum_{E\in\mathcal{E}_h(\partial K\cap\Gamma_D)}h_E^{-p/q}\int_E|u_D-u_{h,D}|^p\,ds. \tag{4.31b}$$

Using Lemma 4.1 and Lemma 4.2 in (4.11) yields

$$\|u-u_h\|^2_{W^{1,p}(\Omega;\mathcal{T}_h)}\lesssim\eta_{h,1}^{eq}+\eta_{h,2}^{eq}. \tag{4.32a}$$

Here, $\eta_{h,1}^{eq}$ and $\eta_{h,2}^{eq}$ are given by

$$\eta_{h,1}^{eq}:=\sum_{K\in\mathcal{T}_h}\eta_{K,1}^{res},\qquad\eta_{h,2}^{eq}:=\sum_{K\in\mathcal{T}_h}\eta_{K,2}^{res} \tag{4.32b}$$

where $\eta_{K,i}^{res}$, $1\leqslant i\leqslant 2$, read as follows:

$$\eta_{K,1}^{eq}:=\frac{1}{p}\int_K|\nabla u_h|^p\,dx-\int_K fu_h\,dx-\sum_{E\in\mathcal{E}_h(\partial K\cap\Gamma_N)}\int_E u_N\,u_h\,ds+\frac{1}{q}\int_K|\underline{\mathbf{p}}_h^{eq}|^q\,dx-\sum_{E\in\mathcal{E}_h(\partial K\cap\Gamma_D)}\int_E u_D\,\mathbf{n}_E\cdot\underline{\mathbf{p}}_h^{eq}\,ds \tag{4.32c}$$

$$\eta_{K,2}^{eq}:=\|u_h-u_h^c\|^2_{W^{1,p}(K)}+\varkappa_K^{eq}+\sum_{i=1}^{3}osc_{K,i}. \tag{4.32d}$$

The right-hand side in (4.32) is then a computable and localizable quantity for the a posteriori estimation of the global discretization error. It gives rise to the following equilibrated a posteriori error estimator

$$\eta_h^{\mathrm{eq}} := \sum_{K \in \mathcal{T}_h(\Omega)} \eta_K^{\mathrm{eq}} \tag{4.33}$$

where the local contributions $\eta_K^{\mathrm{eq}}$, $K \in \mathcal{T}_h(\Omega)$, read as follows

$$\eta_K^{\mathrm{eq}} := \eta_{K,1}^{\mathrm{eq}} + \eta_{K,2}^{\mathrm{eq}}.$$

The construction of an equilibrated flux will be dealt with in the subsequent section.

# 5 Construction of an equilibrated flux

We construct an equilibrated flux $\underline{\mathbf{p}}_h^{\mathrm{eq}} \in \underline{\mathbf{H}}^{(q)}(\mathrm{div}, \Omega)$ which allows us to apply the equilibrated a posteriori error estimator (4.33). The construction will be done locally by an interpolation on each element. In particular, we denote by $\mathrm{BDM}_k(K)$, $k \in \mathbb{N}$, the Brezzi–Douglas–Marini element $\mathrm{BDM}_k(K) := P_k(K)^2$ (see, e.g., [12]), and recall the following result.

**Lemma 5.1.** *Let $k \geqslant 1$. Any vector field $\underline{\mathbf{q}} \in \mathrm{BDM}_k(K)$ is uniquely defined by the following degrees of freedom*

$$\int_E \mathbf{n}_E \cdot \underline{\mathbf{q}}\, p_k \,\mathrm{d}s, \quad p_k \in P_k(E),\ E \in \mathcal{E}_h(\partial K) \tag{5.1a}$$

$$\int_K \underline{\mathbf{q}} \cdot \nabla p_{k-1} \,\mathrm{d}x, \quad p_{k-1} \in P_{k-1}(K) \tag{5.1b}$$

$$\int_K \underline{\mathbf{q}} \cdot \mathbf{curl}(b_K p_{k-2}) \,\mathrm{d}x, \quad p_{k-2} \in P_{k-2}(K) \tag{5.1c}$$

*where $b_K$ in (5.1c) is the element bubble function on $K$ given by $b_K = \prod_{i=1}^3 \lambda_i^K$ and $\lambda_i^K$, $1 \leqslant i \leqslant 3$, are the barycentric coordinates of $K$.*

*Proof.* We refer to [12]. □

**Lemma 5.2.** *Let $K \in \mathcal{T}_h$. There exists a constant $C_E > 0$, depending only on $k$ and the local geometry of the triangulation, such that for any $\underline{\mathbf{q}} \in P_k(K)^2$ it holds*

$$\begin{aligned}\int_K |\underline{\mathbf{q}}|^q \,\mathrm{d}x \leqslant C_E\Bigg( &\sum_{E \in \mathcal{E}_h(\partial K)} h_E \int_E |\mathbf{n}_E \cdot \underline{\mathbf{q}}|^q \,\mathrm{d}s + h_K^q \int_K |\nabla \cdot \underline{\mathbf{q}}|^q \,\mathrm{d}x \\ &+ h_K^q \max\Big\{ \int_K |\underline{\mathbf{q}} \cdot \mathbf{curl}(b_K p_{k-2})|^q \,\mathrm{d}x \mid p_{k-2} \in P_{k-2}(K),\ \max_{x \in K} |p_{k-2}(x)| \leqslant 1 \Big\} \Bigg).\end{aligned} \tag{5.2}$$

*Proof.* The assertion can be proved by standard scaling arguments. □

We construct an equilibrated flux $\underline{\mathbf{p}}_h^{\mathrm{eq}}|_K \in \mathrm{BDM}_k(K)$ by the specifications

$$\int_E \mathbf{n}_E \cdot \underline{\mathbf{p}}_h^{\mathrm{eq}}\, p_k \,\mathrm{d}s = \int_E \mathbf{n}_E \cdot \underline{\hat{\mathbf{p}}}_{\partial K}|_E\, p_k \,\mathrm{d}s, \quad p_k \in P_k(E),\ E \in \mathcal{E}_h(\partial K) \tag{5.3a}$$

$$\int_K \underline{\mathbf{p}}_h^{\mathrm{eq}} \cdot \nabla p_{k-1} \,\mathrm{d}x = \int_K \underline{\mathbf{p}}_h \cdot \nabla p_{k-1} \,\mathrm{d}x, \quad p_{k-1} \in P_{k-1}(K) \tag{5.3b}$$

$$\int_K \underline{\mathbf{p}}_h^{\mathrm{eq}} \cdot \mathbf{curl}(b_K p_{k-2}) \,\mathrm{d}x = \int_K (\underline{\Pi}_k^{(q)}(|\nabla_{DG,2} u_h|^{p-2} \nabla_{DG,2} u_h) \cdot \mathbf{curl}(b_K p_{k-2}) \,\mathrm{d}x, \quad p_{k-2} \in P_{k-2}(K) \tag{5.3c}$$

where $\underline{\mathbf{p}}_h \in \underline{\mathbf{V}}_h$ satisfies the two-field formulation (3.33).

**Theorem 5.1.** *The flux $\underline{\mathbf{p}}_h^{\mathrm{eq}}$ as given by (5.3) is an equilibrated flux, i.e., $\underline{\mathbf{p}}_h^{\mathrm{eq}} \in \underline{\mathbf{H}}^{(q)}(\mathrm{div}, \Omega)$ and it satisfies (4.9b), (4.9c).*

*Proof.* Due to (5.3a) the normal components of $\underline{\mathbf{p}}_h^{\mathrm{eq}}$ on $E \in \mathcal{E}_h(\Omega)$ are continuous across $E$, and hence, $\underline{\mathbf{p}}_h^{\mathrm{eq}} \in \underline{\mathbf{H}}^{(q)}(\mathrm{div};\Omega)$, i.e., (4.9a) is satisfied.

Further, it follows from Gauss's theorem and (5.3a) that for $p_{k1} \in P_{k-1}(K)$, $K \in \mathcal{T}_h$, it holds

$$\int_K \nabla\cdot(\underline{\mathbf{p}}_h^{\mathrm{eq}}\, p_{k-1})\,\mathrm{d}x = \int_{\partial K} \mathbf{n}_{\partial K}\cdot\underline{\mathbf{p}}_h^{\mathrm{eq}} p_{k-1}\,\mathrm{d}s = \int_{\partial K} \mathbf{n}_{\partial K}\cdot\hat{\underline{\mathbf{p}}}_{\partial K} p_{k-1}\,\mathrm{d}s. \tag{5.4}$$

On the other hand, from (5.3b) we deduce

$$\int_K \nabla\cdot(\underline{\mathbf{p}}_h^{\mathrm{eq}}\, p_{k-1})\,\mathrm{d}x = \int_K \nabla\cdot\underline{\mathbf{p}}_h^{\mathrm{eq}}\, p_{k-1}\,\mathrm{d}x + \int_K \underline{\mathbf{p}}_h^{\mathrm{eq}}\cdot\nabla p_{k-1}\,\mathrm{d}x = \int_K \nabla\cdot\underline{\mathbf{p}}_h^{\mathrm{eq}}\, p_{k-1}\,\mathrm{d}x + \int_K \underline{\mathbf{p}}_h\cdot\nabla p_{k-1}\,\mathrm{d}x$$

Hence, observing (5.4) and using (3.33b) with $v_h|_K = p_{k-1}$ and $v_h|_{K'} = 0$, $K' \in \mathcal{T}_h$, $K' \neq K$, we obtain

$$0 = \int_K \nabla\cdot\underline{\mathbf{p}}_h^{\mathrm{eq}}\, p_{k-1}\,\mathrm{d}x + \int_K \underline{\mathbf{p}}_h\cdot\nabla p_{k-1}\,\mathrm{d}x - \int_{\partial K} \mathbf{n}_{\partial K}\cdot\hat{\underline{\mathbf{p}}}_{\partial K} p_{k-1}\,\mathrm{d}s = \int_K \nabla\cdot\underline{\mathbf{p}}_h^{\mathrm{eq}}\, p_{k-1}\,\mathrm{d}x + \int_K f_h p_{k-1}\,\mathrm{d}x.$$

Since $\nabla\cdot\underline{\mathbf{p}}_h^{\mathrm{eq}}$ and $f_h|_K$ are contained in $P_{k-1}(K)$, we readily deduce that (4.9b) holds true. Moreover, for $E \in \mathcal{E}_h(\Gamma_N)$ it follows from (5.3a) and (3.34) that

$$\int_E \mathbf{n}_E\cdot\underline{\mathbf{p}}_h^{\mathrm{eq}}\, p_k\,\mathrm{d}s = \int_E \mathbf{n}_E\cdot\hat{\underline{\mathbf{p}}}_{\partial K}|_E\, p_k\,\mathrm{d}s = \int_E u_{h,N}\, p_k\,\mathrm{d}s. \tag{5.5}$$

Since both $\mathbf{n}_E\cdot\underline{\mathbf{p}}_h^{\mathrm{eq}}$ and $u_{h,N}|_E$ are polynomials of degree $k$ on $E$, it follows from (5.5) that (4.9c) is satisfied. □

# 6 Relationship with a residual type a posteriori error estimator

A residual-type a posteriori error estimator for the IPDG approximation of the $p$-Laplace equation with homogeneous Dirichlet boundary conditions in case $p = 2$ has been derived and analyzed in [36, 38, 39]. Its generalization to arbitrary $1 < p < \infty$ reads as follows:

$$\eta_h^{\mathrm{res}} = \sum_{i=1}^{5}(\eta_{h,i}^{\mathrm{res}} + \tilde{\eta}_{h,i}^{\mathrm{res}}) + \sum_{i=3}^{4}\hat{\eta}_{h,i}^{\mathrm{res}}. \tag{6.1a}$$

Here, the element residuals $\eta_{h,1}^{\mathrm{res}}$ and the edge residuals $\eta_{h,i}^{\mathrm{res}}$, $2 \leqslant i \leqslant 5$, are given by

$$\eta_{h,1}^{\mathrm{res}} := \sum_{K\in\mathcal{T}_h} h_K^q \int_K |f + \nabla\cdot\underline{\Pi}_k^{(q)}\big(|\nabla_{DG,2}u_h|^{p-2}(\nabla_{DG,2}u_h)\big)|^q\,\mathrm{d}x \tag{6.1b}$$

$$\eta_{h,2}^{\mathrm{res}} := \sum_{E\in\mathcal{E}_h(\Omega)} h_E \int_E |\mathbf{n}_E\cdot[|\nabla_{DG,2}u_h|^{p-2}\nabla_{DG,2}u_h]_E|^q\,\mathrm{d}s \tag{6.1c}$$

$$\eta_{h,3}^{\mathrm{res}} := \sum_{E\in\mathcal{E}_h(\Omega)} h_E^{-p/q} \int_E |[u_h]_E|^p\,\mathrm{d}s \tag{6.1d}$$

$$\eta_{h,4}^{\mathrm{res}} := \sum_{E\in\mathcal{E}_h(\Gamma_D)} h_E^{-p/q} \int_E |u_h - u_D|^p\,\mathrm{d}s \tag{6.1e}$$

$$\eta_{h,5}^{\mathrm{res}} := \sum_{E\in\mathcal{E}_h(\Gamma_N)} h_E \int_E |u_N - \mathbf{n}_E\cdot|\nabla_{DG,2}u_h|^{p-2}\nabla_{DG,2}u_h|^q\,\mathrm{d}s. \tag{6.1f}$$

The residuals $\tilde{\eta}_{h,i}^{\mathrm{res}}$, $1 \leqslant i \leqslant 5$, and $\hat{\eta}_{h,i}^{\mathrm{res}}$, $3 \leqslant i \leqslant 4$, read as follows:

$$\tilde{\eta}_{h,i}^{\mathrm{res}} := (\eta_{h,i}^{\mathrm{res}})^{1/q}\, |\nabla_{DG,2}u_h|_{DG,\Omega}, \quad 1 \leqslant i \leqslant 5 \tag{6.1g}$$

$$\hat{\eta}_{h,i}^{\mathrm{res}} := (\eta_{h,i}^{\mathrm{res}})^{1/p}\, |\nabla u_h|_{DG,\Omega}^{1/q}, \quad 3 \leqslant i \leqslant 4 \tag{6.1h}$$

where

$$|\nabla_{DG,2}u_h|_{DG,\Omega} := \Big(\sum_{K\in\mathcal{T}_h}\int_K |\nabla_{DG,2}u_h|^p\,dx\Big)^{1/p} \tag{6.2}$$

$$|\nabla u_h|_{DG,\Omega} := \sum_{K\in\mathcal{T}_h}\int_K |\nabla u_h|^p\,dx. \tag{6.3}$$

In addition to (4.18) and (4.31) we define data oscillations $\widetilde{osc}_{h,i}$, $1 \leqslant i \leqslant 2$, according to

$$\widetilde{osc}_{h,1} := (osc_{h,1})^{1/q}\,|\nabla u_h|_{DG,\Omega}^{1/p} \tag{6.4a}$$

$$\widetilde{osc}_{h,2} := (osc_{h,2})^{1/q}\,\|\nabla u_h\|_{DG,\Gamma_D}^{1/p} \tag{6.4b}$$

where

$$|\nabla v_h|_{DG,\Gamma'} := \sum_{E\in\mathcal{E}_h(\Gamma')}\int_E |\nabla v_h|^p\,ds, \quad \Gamma' \in \{\Gamma_D, \Gamma_N\}. \tag{6.5}$$

**Remark 6.1.** Residual-type a posteriori error estimates for P1 conforming finite element approximations of the $p$-Laplace problem with homogeneous Dirichlet boundary conditions have been considered in [5] (see also [41, 42]). We note that in this case the residuals $\tilde{\eta}_{h,i}^{\mathrm{res}}$, $3 \leqslant i \leqslant 5$, and $\eta_{h,i}^{\mathrm{res}}$, $3 \leqslant i \leqslant 4$, as well as the data oscillations $osc_{h,i}$, $2 \leqslant i \leqslant 3$, and $\widetilde{osc}_{h,2}$ vanish. The residuals $\eta_{h,i}^{\mathrm{res}}$, $1 \leqslant i \leqslant 2$, reduce to

$$\begin{aligned}\eta_{h,1}^{\mathrm{res}} &= \sum_{K\in\mathcal{T}_h} h_K^q \int_K |f|^q\,dx\\ \eta_{h,2}^{\mathrm{res}} &= \sum_{E\in\mathcal{E}_h(\Omega)} h_E \int_E |\mathbf{n}_E\cdot[|\nabla u_h|^{p-2}\nabla u_h]_E|^q\,ds.\end{aligned} \tag{6.6}$$

Using the relationships between the $\|\cdot\|_{W^{1,p}(\Omega)}$ norm and quasi-norms provided in [4], it can be shown that (6.6) and the data oscillation $\widetilde{osc}_{h,1}$ are closely related to those in [5].

The goal of this section is to establish the relationship between the equilibrated a posteriori error estimator $\eta_h^{\mathrm{eq}}$ and the residual-based a posteriori error estimator $\eta_h^{\mathrm{res}}$. For notational convenience, throughout this section we set $\underline{\mathbf{z}}_h := |\nabla_{DG,2}u_h|^{p-2}\nabla_{DG,2}u_h$. In view of the definitions (2.3a) and (2.4a) of the primal and dual energies we have

$$\eta_{h,1}^{\mathrm{eq}} = \frac{1}{p}\sum_{K\in\mathcal{T}_h}\int_K |\nabla u_h|^p\,dx + \frac{1}{q}\sum_{K\in\mathcal{T}_h}\int_K |\underline{\mathbf{p}}_h^{\mathrm{eq}}|^q\,dx - \sum_{K\in\mathcal{T}_h}\int_K f u_h\,dx - \sum_{E\in\mathcal{E}_h(\Gamma_N)}\int_E u_N u_h\,ds - \sum_{E\in\mathcal{E}_h(\Gamma_D)}\int_E u_D\,\mathbf{n}_E\cdot\underline{\mathbf{p}}_h^{\mathrm{eq}}\,ds. \tag{6.7}$$

For the first two terms on the right-hand side of (6.7) we obtain

$$\begin{aligned}\frac{1}{p}\sum_{K\in\mathcal{T}_h}\int_K |\nabla u_h|^p\,dx + \frac{1}{q}\sum_{K\in\mathcal{T}_h}\int_K |\underline{\mathbf{p}}_h^{\mathrm{eq}}|^q\,dx &= \frac{1}{p}\sum_{K\in\mathcal{T}_h}\int_K |\nabla u_h|^p\,dx - \frac{1}{p}\sum_{K\in\mathcal{T}_h}\int_K |\nabla_{DG,2}u_h|^p\,dx\\ &\quad + \Big(1-\frac{1}{q}\Big)\sum_{K\in\mathcal{T}_h}\int_K |\nabla_{DG,2}u_h|^p\,dx + \frac{1}{q}\sum_{K\in\mathcal{T}_h}\int_K |\underline{\mathbf{p}}_h^{\mathrm{eq}}|^q\,dx.\end{aligned} \tag{6.8}$$

Using (6.8) in (6.7) it follows that

$$\begin{aligned}\eta_{h,1}^{\mathrm{eq}} &= \sum_{K\in\mathcal{T}_h}\int_K \underline{\mathbf{z}}_h\cdot\nabla_{DG,2}u_h\,dx - \sum_{K\in\mathcal{T}_h}\int_K f u_h\,dx - \sum_{E\in\mathcal{E}_h(\Gamma_D)}\int_E u_D\,\mathbf{n}_E\cdot\underline{\mathbf{p}}_h^{\mathrm{eq}}\,ds - \sum_{E\in\mathcal{E}_h(\Gamma_N)}\int_E u_N u_h\,ds\\ &\quad + \frac{1}{p}\sum_{K\in\mathcal{T}_h}\int_K |\nabla u_h|^p\,dx - \frac{1}{p}\sum_{K\in\mathcal{T}_h}\int_K |\nabla_{DG,2}u_h|^p\,dx + \frac{1}{q}\sum_{K\in\mathcal{T}_h}\int_K |\underline{\mathbf{p}}_h^{\mathrm{eq}}|^q\,dx - \frac{1}{q}\sum_{K\in\mathcal{T}_h}\int_K |\nabla_{DG,2}u_h|^p\,dx.\end{aligned} \tag{6.9}$$

We will estimate the terms on the right-hand side in (6.9) by a series of Lemmas.

**Lemma 6.1.** *There exists a constant $\hat{C}_{\mathrm{rel}}^{(1)} > 0$, depending only on $\alpha$, $C_p$, $C_{PF}^{(i)}(p)$, $1 \leqslant i \leqslant 2$, and on $p, q$, such that it holds*

$$\Big|\sum_{K\in\mathcal{T}_h}\int_K |\nabla_{DG,2}u_h|^{p-2}\nabla_{DG,2}u_h\cdot\nabla_{DG,2}u_h\,\mathrm{d}x - \sum_{K\in\mathcal{T}_h}\int_K fu_h\,\mathrm{d}x - \sum_{E\in\mathcal{E}_h(\Gamma_D)}\int_E u_D\,\mathbf{n}_E\cdot\underline{\mathbf{p}}_h^{\mathrm{eq}}\,\mathrm{d}s - \sum_{E\in\mathcal{E}_h(\Gamma_N)}\int_E u_N u_h\,\mathrm{d}s\Big|$$
$$\leqslant \hat{C}_{\mathrm{rel}}^{(1)}\Big(\eta_{h,4}^{\mathrm{res}} + osc_{h,3} + \widetilde{osc}_{h,1} + \widetilde{osc}_{h,2}\Big). \tag{6.10}$$

*Proof.* In view of (3.12b), (3.13b), and (3.22) we get

$$\begin{aligned}\sum_{K\in\mathcal{T}_h}\int_K \underline{\mathbf{z}}_h\cdot\nabla_{DG,2}u_h\,\mathrm{d}x &= \sum_{K\in\mathcal{T}_h}\int_K \underline{\mathbf{z}}_h\cdot\underline{\Pi}_k^{(p)}(\nabla_{DG,2}u_h)\,\mathrm{d}x = \sum_{K\in\mathcal{T}_h}\int_K \underline{\Pi}_k^{(q)}(\underline{\mathbf{z}}_h)\cdot\nabla_{DG,2}u_h\,\mathrm{d}x\\
&= \sum_{K\in\mathcal{T}_h}\int_K \underline{\Pi}_k^{(q)}(\underline{\mathbf{z}}_h)\cdot\nabla_{DG,1}u_h\,\mathrm{d}x + \sum_{K\in\mathcal{T}_h}\int_K \underline{\Pi}_k^{(q)}(\underline{\mathbf{z}}_h)\cdot\underline{\mathbf{R}}_{h,2}(u_D^*)\,\mathrm{d}x\\
&= \sum_{K\in\mathcal{T}_h}\int_K f_h u_h\,\mathrm{d}x + \sum_{E\in\mathcal{E}_h(\Gamma_N)}\int_E u_{h,N}u_h\,\mathrm{d}s - \alpha\sum_{E\in\mathcal{E}_h(\Omega)} h_E^{-p/q}\int_E |[u_h]_E|^{p-2}\,[u_h]_E\,[u_h]_E\,\mathrm{d}s\\
&\quad - \alpha\sum_{E\in\mathcal{E}_h(\Gamma_D)} h_E^{-p/q}\int_E |u_h-u_D|^{p-2}(u_h-u_D)\,u_h\,\mathrm{d}s + \sum_{E\in\mathcal{E}_h(\Gamma_D)}\mathbf{n}_E\cdot\underline{\mathbf{z}}_h\,u_{h,D}\,\mathrm{d}s\end{aligned} \tag{6.11}$$

where we have used that

$$\sum_{E\in\mathcal{E}_h(\Gamma_D)}\mathbf{n}_E\cdot\underline{\Pi}_k^{(q)}(\underline{\mathbf{z}}_h)\,u_D\,\mathrm{d}s = \sum_{E\in\mathcal{E}_h(\Gamma_D)}\Pi_k^{(q)}(\mathbf{n}_E\cdot\underline{\mathbf{z}}_h)\,u_D\,\mathrm{d}s = \sum_{E\in\mathcal{E}_h(\Gamma_D)}\mathbf{n}_E\cdot\underline{\mathbf{z}}_h\,\Pi_k^{(p)}(u_D)\,\mathrm{d}s = \sum_{E\in\mathcal{E}_h(\Gamma_D)}\mathbf{n}_E\cdot\underline{\mathbf{z}}_h\,u_{h,D}\,\mathrm{d}s.$$

On the other hand, observing (5.3a), (3.34), and $\mathbf{n}_E\cdot\underline{\mathbf{p}}_h^{\mathrm{eq}} = \Pi_k(\mathbf{n}_E\cdot\hat{\underline{\mathbf{p}}}_{\partial K}|_E)$, it follows that

$$\begin{aligned}\sum_{E\in\mathcal{E}_h(\Gamma_D)}\int_E u_D\,\mathbf{n}_E\cdot\underline{\mathbf{p}}_h^{\mathrm{eq}}\,\mathrm{d}s &= \sum_{E\in\mathcal{E}_h(\Gamma_D)}\int_E u_D\,\Pi_k(\mathbf{n}_E\cdot\hat{\underline{\mathbf{p}}}_{\partial K}|_E)\,\mathrm{d}s\\
&= \sum_{E\in\mathcal{E}_h(\Gamma_D)}\int_E \Pi_k(u_D)\,\mathbf{n}_E\cdot\hat{\underline{\mathbf{p}}}_{\partial K}|_E\,\mathrm{d}s = \sum_{E\in\mathcal{E}_h(\Gamma_D)}\int_E u_{h,D}\,\mathbf{n}_E\cdot\hat{\underline{\mathbf{p}}}_{\partial K}|_E\,\mathrm{d}s\\
&= \sum_{E\in\mathcal{E}_h(\Gamma_D)}\int_E u_{h,D}\,\mathbf{n}_E\cdot\underline{\mathbf{z}}_h\,\mathrm{d}s - \alpha\sum_{E\in\mathcal{E}_h(\Gamma_D)} h_E^{-p/q}\int_E |u_h-u_D|^{p-2}(u_h-u_D)\,u_{h,D}\,\mathrm{d}s.\end{aligned} \tag{6.12}$$

From (6.11) and (6.12) we deduce

$$\begin{aligned}&\Big|\sum_{K\in\mathcal{T}_h}\int_K \underline{\mathbf{z}}_h\cdot\nabla_{DG,2}u_h\,\mathrm{d}x - \sum_{K\in\mathcal{T}_h}\int_K fu_h\,\mathrm{d}x - \sum_{E\in\mathcal{E}_h(\Gamma_D)}\int_E u_D\,\mathbf{n}_E\cdot\underline{\mathbf{p}}_h^{\mathrm{eq}}\,\mathrm{d}s - \sum_{E\in\mathcal{E}_h(\Gamma_N)}\int_E u_N u_h\,\mathrm{d}s\Big|\\
&\leqslant \Big|\sum_{K\in\mathcal{T}_h}\int_K (f_h-f)u_h\,\mathrm{d}x\Big| + \Big|\sum_{E\in\mathcal{E}_h(\Gamma_N)}\int_E (u_{h,N}-u_N)u_h\,\mathrm{d}s\Big|\\
&\quad + \alpha\Big|\sum_{E\in\mathcal{E}_h(\Gamma_D)} h_E^{-p/q}\int_E |u_h-u_D|^{p-2}(u_h-u_D)\,(u_{h,D}-u_h)\,\mathrm{d}s\Big|.\end{aligned} \tag{6.13}$$

Applying Hölder's inequality, the Cauchy–Schwarz inequality, observing (3.20) with $p_0 = \int_K u_h\,\mathrm{d}x$, and the Poincaré–Friedrichs inequality (4.14), the first term on the right-hand side of (6.13) can be estimated from above as follows:

$$\begin{aligned}\Big|\sum_{K\in\mathcal{T}_h}\int_K (f_h-f)u_h\,\mathrm{d}x\Big| &= \Big|\sum_{K\in\mathcal{T}_h}\int_K (f_h-f)(u_h-p_0)\,\mathrm{d}x\Big| \leqslant \sum_{K\in\mathcal{T}_h}\Big(\int_K |f-f_h|^q\,\mathrm{d}x\Big)^{1/q}\Big(\int_K |u_h-p_0|^p\,\mathrm{d}x\Big)^{1/p}\\
&\leqslant C_{PF}^{(1)}(q)\sum_{K\in\mathcal{T}_h}\Big(\int_K |f-f_h|^q\,\mathrm{d}x\Big)^{1/q} h_K\Big(\int_K |\nabla u_h|^p\,\mathrm{d}x\Big)^{1/p}\\
&\leqslant C_{PF}^{(1)}(p)\Big(\sum_{K\in\mathcal{T}_h} h_K^q\int_K |f-f_h|^q\,\mathrm{d}x\Big)^{1/q}|\nabla u_h|_{DG,\Omega} = C_{PF}^{(1)}(p)\,\widetilde{osc}_{h,1}.\end{aligned} \tag{6.14}$$

Likewise, with $p_0 = \int_E u_h ds$, $E \in \mathcal{E}_h(\Gamma_N)$, we obtain

$$\Big|\sum_{E\in\mathcal{E}_h(\Gamma_N)}\int_E (u_{h,N}-u_N)u_h\,\mathrm{d}s\Big| = \Big|\sum_{E\in\mathcal{E}_h(\Gamma_N)}\int_E (u_{h,N}-u_N)(u_h-p_0)\,\mathrm{d}s\Big|$$
$$\leqslant C_{PF}^{(2)}(p)\Big(\sum_{E\in\mathcal{E}_h(\Gamma_N)} h_E^q\int_E |u_N-u_{h,N}|^q\,\mathrm{d}x\Big)^{1/q}|\nabla u_h|_{DG,\Gamma_N} = C_{PF}^{(2)}(p)\,\widetilde{osc}_{h,2}. \tag{6.15}$$

Likewise, using the same arguments and (1.2), for the last term on the right-hand side of (6.13) we obtain

$$\alpha\Big|\sum_{E\in\mathcal{E}_h(\Gamma_D)} h_E^{-p/q}\int_E |u_h-u_D|^{p-2}(u_h-u_D)\,(u_{h,D}-u_h)\,\mathrm{d}s\Big|$$
$$\leqslant \alpha\sum_{E\in\mathcal{E}_h(\Gamma_D)} h_E^{-p/q}\Big(\int_E \big||u_h-u_D|^{p-2}(u_h-u_D)\big|^q\,\mathrm{d}s\Big)^{1/q}\Big(\int_E |u_h-u_{h,D}|^p\,\mathrm{d}s\Big)^{1/p}$$
$$\leqslant \alpha\Big(\sum_{E\in\mathcal{E}_h(\Gamma_D)} h_E^{-p/q}\int_E |u_h-u_D|^p\,\mathrm{d}s\Big)^{1/q}\Big(\sum_{E\in\mathcal{E}_h(\Gamma_D)} h_E^{-p/q}\int_E |u_h-u_{h,D}|^p\,\mathrm{d}s\Big)^{1/p}$$
$$\leqslant \alpha\Big(\frac{1}{q}\Big(\frac{1}{2}\Big)^{-q/p}\eta_{h,4}^{\mathrm{res}} + \frac{1}{2p}C_p(\eta_{h,4}^{\mathrm{res}} + osc_{h,3})\Big). \tag{6.16}$$

The assertion now follows from (6.11)–(6.16). □

**Lemma 6.2.** *There exists a constant $\hat{C}_{\mathrm{rel}}^{(2)} > 0$, depending only on $C_{p-1}$, $C_p$, $C_{\mathrm{rec}}$, and on $p, q$, such that it holds*

$$\Big|\frac{1}{p}\sum_{K\in\mathcal{T}_h}\int_K |\nabla u_h|^p\,\mathrm{d}x - \frac{1}{p}\sum_{K\in\mathcal{T}_h}\int_K |\nabla_{DG,2}u_h|^p\,\mathrm{d}x\Big| \leqslant \hat{C}_{\mathrm{rel}}^{(2)}\big(\eta_{h,3}^{\mathrm{res}} + \hat{\eta}_{h,3}^{\mathrm{res}} + \eta_{h,4}^{\mathrm{res}} + \hat{\eta}_{h,4}^{\mathrm{res}}\big). \tag{6.17}$$

*Proof.* By Taylor expansion, observing (3.13), applying Hölder's inequality, the Cauchy–Schwarz inequality, and using (1.2) as well as (3.15), we find

$$\Big|\frac{1}{p}\sum_{K\in\mathcal{T}_h}\int_K |\nabla_{DG,2}u_h|^p\,\mathrm{d}x - \frac{1}{p}\sum_{K\in\mathcal{T}_h}\int_K |\nabla u_h|^p\,\mathrm{d}x\Big|$$
$$= \Big|\sum_{K\in\mathcal{T}_h}\int_K\int_0^1 \big(|\nabla u_h + \lambda(\nabla_{DG,2}u_h - \nabla u_h)|^{p-2}(\nabla u_h + \lambda(\nabla_{DG,2}u_h - \nabla u_h)\,\mathrm{d}\lambda\cdot(\nabla_{DG,2}u_h - \nabla u_h)\,\mathrm{d}x\Big|$$
$$\leqslant \sum_{K\in\mathcal{T}_h}\int_K\int_0^1 |\nabla u_h + \lambda(\nabla_{DG,2}u_h - \nabla u_h)|^{p-1}|\nabla_{DG,2}u_h - \nabla u_h|\,\mathrm{d}\lambda\,\mathrm{d}x$$
$$\leqslant C_{p-1}\Big(\sum_{K\in\mathcal{T}_h}\int_K |\nabla u_h|^{p-1}||\nabla_{DG,2}u_h - \nabla u_h|\,\mathrm{d}x + \frac{1}{p}\sum_{K\in\mathcal{T}_h}\int_K |\nabla_{DG,2}u_h - \nabla u_h|^p\,\mathrm{d}x\Big)$$
$$\leqslant C_{p-1}\Big(\sum_{K\in\mathcal{T}_h}\int_K |\nabla u_h|^p\,\mathrm{d}x\Big)^{1/q}\Big(\sum_{K\in\mathcal{T}_h}\int_K |\underline{\mathbf{R}}_{h,1}(u_h) - \underline{\mathbf{R}}_{h,2}(u_D^*)|^p\,\mathrm{d}x\Big)^{1/p}$$
$$+ \frac{1}{p}C_{p-1}\sum_{K\in\mathcal{T}_h}\int_K |\underline{\mathbf{R}}_{h,1}(u_h) - \underline{\mathbf{R}}_{h,2}(u_D^*)|^p\,\mathrm{d}x$$
$$\leqslant C_{p-1}C_{\mathrm{rec}}(\hat{\eta}_{h,3}^{\mathrm{res}} + \hat{\eta}_{h,4}^{\mathrm{res}}) + \frac{1}{p}C_{p-1}C_pC_{\mathrm{rec}}(\eta_{h,3}^{\mathrm{res}} + \eta_{h,4}^{\mathrm{res}}). \tag{6.18}$$

This completes the proof. □

**Lemma 6.3.** *There exists a constant $\hat{C}_{\mathrm{rel}}^{(3)} > 0$, depending only on $\alpha$, $c_R$, $C_q$, $C_E$, and on $p, q$, such that it holds*

$$\frac{1}{q}\sum_{K\in\mathcal{T}_h}\int_K |\underline{\mathbf{p}}_h^{\mathrm{eq}}|^q\,\mathrm{d}x - \frac{1}{q}\sum_{K\in\mathcal{T}_h}\int_K |\nabla_{DG,2}u_h|^p\,\mathrm{d}x \leqslant \hat{C}_{\mathrm{rel}}^{(3)}\Big(\sum_{i=1}^5(\eta_{h,i}^{\mathrm{res}} + \tilde{\eta}_{h,i}^{\mathrm{res}}) + osc_{h,1} + \widetilde{osc}_{h,1}\Big). \tag{6.19}$$

*Proof.* We have

$$\frac{1}{q}\sum_{K\in\mathcal{T}_h}\int_K |\underline{\mathbf{p}}_h^{\text{eq}}|^q\,dx - \frac{1}{q}\sum_{K\in\mathcal{T}_h}\int_K |\nabla_{DG,2}u_h|^p\,dx = \frac{1}{q}\sum_{K\in\mathcal{T}_h}\int_K |\underline{\mathbf{p}}_h^{\text{eq}}|^q\,dx - \frac{1}{q}\sum_{K\in\mathcal{T}_h}\int_K |\underline{\Pi}_k^{(q)}(\underline{\mathbf{z}}_h)|^q\,dx$$
$$+\frac{1}{q}\sum_{K\in\mathcal{T}_h}\int_K |\underline{\Pi}_k^{(q)}(\underline{\mathbf{z}}_h)|^q\,dx - \frac{1}{q}\sum_{K\in\mathcal{T}_h}\int_K |\nabla_{DG,2}u_h|^p\,dx. \tag{6.20}$$

For the first two terms on the right-hand side of (6.20), applying Taylor expansion and using (1.2) we obtain

$$\Big|\frac{1}{q}\sum_{K\in\mathcal{T}_h}\int_K |\underline{\mathbf{p}}_h^{\text{eq}}|^q\,dx - \frac{1}{q}\sum_{K\in\mathcal{T}_h}\int_K |\underline{\Pi}_k^{(q)}(\underline{\mathbf{z}}_h)|^q\,dx\Big|$$
$$= \Big|\sum_{K\in\mathcal{T}_h}\int_K\int_0^1 |\underline{\Pi}_k^{(q)}(\underline{\mathbf{z}}_h)+\lambda(\underline{\mathbf{p}}_h^{\text{eq}}-\underline{\Pi}_k^{(q)}(\underline{\mathbf{z}}_h)|^{q-2}(\underline{\Pi}_k^{(q)}(\underline{\mathbf{z}}_h)+\lambda(\underline{\mathbf{p}}_h^{\text{eq}}-\underline{\Pi}_k^{(q)}(\underline{\mathbf{z}}_h))\,d\lambda\cdot(\underline{\mathbf{p}}_h^{\text{eq}}-\underline{\Pi}_k^{(q)}(\underline{\mathbf{z}}_h))\,dx\Big|$$
$$\leq \sum_{K\in\mathcal{T}_h}\int_K\int_0^1 |\underline{\Pi}_k^{(q)}(\underline{\mathbf{z}}_h)+\lambda(\underline{\mathbf{p}}_h^{\text{eq}}-\underline{\Pi}_k^{(q)}(\underline{\mathbf{z}}_h)|^{q-1}|\underline{\mathbf{p}}_h^{\text{eq}}-\underline{\Pi}_k^{(q)}(\underline{\mathbf{z}}_h)|\,d\lambda\,dx$$
$$\leq C_{q-1}\Big(\sum_{K\in\mathcal{T}_h}\int_K |\underline{\Pi}_k^{(q)}(\underline{\mathbf{z}}_h)|^{q-1}|\underline{\mathbf{p}}_h^{\text{eq}}-\underline{\Pi}_k^{(q)}(\underline{\mathbf{z}}_h)|\,dx + \frac{1}{q}\sum_{K\in\mathcal{T}_h}\int_K |\underline{\mathbf{p}}_h^{\text{eq}}-\underline{\Pi}_k^{(q)}(\underline{\mathbf{z}}_h)|^q\,dx\Big)$$
$$\leq C_{q-1}\bigg(\Big(\Big(\sum_{K\in\mathcal{T}_h}\int_K |\underline{\Pi}_k^{(q)}(\underline{\mathbf{z}}_h)|^q|\,dx\Big)^{1/p}\sum_{K\in\mathcal{T}_h}\int_K |\underline{\mathbf{p}}_h^{\text{eq}}-\underline{\Pi}_k^{(q)}(\underline{\mathbf{z}}_h)|^q\,dx\Big)^{1/q}$$
$$+\frac{1}{q}\sum_{K\in\mathcal{T}_h}\int_K |\underline{\mathbf{p}}_h^{\text{eq}}-\underline{\Pi}_k^{(q)}(\underline{\mathbf{z}}_h)|^q\,dx\bigg). \tag{6.21}$$

In view of (5.2) and (5.3c) we deduce

$$\sum_{K\in\mathcal{T}_h}\int_K |\underline{\mathbf{p}}_h^{\text{eq}}-\underline{\Pi}_k^{(q)}(\underline{\mathbf{z}}_h)|^q\,dx \leq C_E\Big(\sum_{K\in\mathcal{T}_h} h_K\int_{\partial K} |\mathbf{n}_{\partial K}\cdot(\underline{\mathbf{p}}_h^{\text{eq}}-\underline{\Pi}_k(\underline{\mathbf{z}}_h)|^q\,ds + \sum_{K\in\mathcal{T}_h} h_K^q\int_K |\nabla\cdot(\underline{\mathbf{p}}_h^{\text{eq}}-\underline{\Pi}_k(\underline{\mathbf{z}}_h))|^q\,dx\Big). \tag{6.22}$$

Since (5.3a) implies $\mathbf{n}_{\partial K}\cdot\underline{\mathbf{p}}_h^{\text{eq}}|_E = \Pi_k^{(q)}(\mathbf{n}_{\partial K}\cdot\hat{\underline{\mathbf{p}}}_{\partial K})$, we get

$$\sum_{K\in\mathcal{T}_h} h_K\int_{\partial K} |\mathbf{n}_{\partial K}\cdot(\underline{\mathbf{p}}_h^{\text{eq}}-\underline{\Pi}_k(|\nabla_{DG,2}u_h|^{p-2}\nabla_{DG,2}u_h))|^q\,ds$$
$$= \sum_{K\in\mathcal{T}_h} h_K\int_{\partial K\cap\Omega} |\Pi_k^{(q)}(\mathbf{n}_{\partial K}\cdot(\hat{\underline{\mathbf{p}}}_{\partial K}-|\nabla_{DG,2}u_h|^{p-2}\nabla_{DG,2}u_h))|^q\,ds$$
$$+\sum_{K\in\mathcal{T}_h} h_K\int_{\partial K\cap\Gamma} |\Pi_k^{(q)}(\mathbf{n}_{\partial K}\cdot(\hat{\underline{\mathbf{p}}}_{\partial K}-|\nabla_{DG,2}u_h|^{p-2}\nabla_{DG,2}u_h))|^q\,ds. \tag{6.23}$$

Now, using (3.1), (3.34) and observing $1-p=-p/q$ it follows that

$$\Big|\sum_{K\in\mathcal{T}_h} h_K\int_{\partial K\cap\Omega} |\Pi_k^{(q)}(\mathbf{n}_{\partial K}\cdot(\hat{\underline{\mathbf{p}}}_{\partial K}-\underline{\mathbf{z}}_h|^q\,ds\Big|$$
$$\leq c_R^{-1}\sum_{E\in\mathcal{E}_h(\Omega)} h_E\int_E\Big(|\{\underline{\mathbf{z}}_h\}_E-\underline{\mathbf{z}}_h|_{E_+}|^q+|\{\underline{\mathbf{z}}_h\}_E-(\underline{\mathbf{z}}_h|_{E_-}|^q\Big)\,ds$$
$$+\,c_R^{-1}\alpha^q\sum_{E\in\mathcal{E}_h(\Omega)} h_E\int_E |h_E^{-p/q}|[u_h]_E|^{p-2}[u_h]_E|^q\,ds$$
$$\leq c_R^{-1}\Big(\sum_{E\in\mathcal{E}_h(\Omega)} h_E\int_E |[\underline{\mathbf{z}}_h]_E|^q\,ds + \alpha^q\sum_{E\in\mathcal{E}_h(\Omega)} h_E^{-p/q}\int_E |[u_h]_E|^p\,ds\Big) = c_R^{-1}\Big(\eta_{h,2}^{\text{res}}+\alpha^q\eta_{h,3}^{\text{res}}\Big). \tag{6.24}$$

Using (3.1) and (3.34) again, we obtain

$$\begin{aligned}
&\Big|\sum_{K\in\mathcal{T}_h} h_K \int\limits_{\partial K\cap\Gamma} |\Pi_k^{(q)}(\mathbf{n}_{\partial K}\cdot(\underline{\hat{\mathbf{p}}}_{\partial K}-\underline{\mathbf{z}}_h)|^q\,ds\Big| \\
&\quad\leqslant c_R^{-1}\sum_{E\in\mathcal{E}_h(\Gamma_D)} h_E\int\limits_E |\mathbf{n}_E\cdot(\underline{\hat{\mathbf{p}}}_{\partial K}|_E-\underline{\mathbf{z}}_h)|^q\,ds + c_R^{-1}\sum_{E\in\mathcal{E}_h(\Gamma_N)} h_E\int\limits_E |\mathbf{n}_E\cdot(\underline{\hat{\mathbf{p}}}_{\partial K}|_E-\underline{\mathbf{z}}_h)|^q\,ds \\
&\quad\leqslant \alpha^q c_R^{-1}\sum_{E\in\mathcal{E}_h(\Gamma_D)} h_E\int\limits_E |h_E^{-p/q}|u_h-u_D|^{p-2}(u_h-u_D)|^q\,ds + c_R^{-1}\sum_{E\in\mathcal{E}_h(\Gamma_N)} h_E\int\limits_E |(u_N-\mathbf{n}_E\cdot\underline{\mathbf{z}}_h)|^q\,ds \\
&\quad\leqslant \alpha^q c_R^{-1}\sum_{E\in\mathcal{E}_h(\Gamma_D)} h_E^{-p/q}\int\limits_E |u_h-u_D|^p|\,ds + c_R^{-1}\sum_{E\in\mathcal{E}_h(\Gamma_N)} h_E\int\limits_E |(u_N-\mathbf{n}_E\cdot\underline{\mathbf{z}}_h)|^q\,ds = c_R^{-1}(\alpha^q\eta_{h,4}^{\mathrm{res}}+\eta_{h,5}^{\mathrm{res}}).
\end{aligned} \tag{6.25}$$

Moreover, due to (1.2) and (4.9a) we have

$$\begin{aligned}
&\sum_{K\in\mathcal{T}_h} h_K^q\int\limits_K |\nabla\cdot(\underline{\mathbf{p}}_h^{\mathrm{eq}}-\underline{\Pi}_k^{(q)}(\underline{\mathbf{z}}_h))|^q\,dx = \sum_{K\in\mathcal{T}_h} h_K^q\int\limits_K |f_h+\nabla\cdot\underline{\Pi}_k(\underline{\mathbf{z}}_h)|^q\,dx \\
&\quad\leqslant C_q\sum_{K\in\mathcal{T}_h} h_K^q\int\limits_k |f-f_h|^q\,dx + C_q\sum_{K\in\mathcal{T}_h} h_K^q\int\limits_k |f+\nabla\cdot\underline{\Pi}_k(\underline{\mathbf{z}}_h)|^q\,dx = C_q(\eta_{h,1}^{\mathrm{res}}+osc_{h,1}).
\end{aligned} \tag{6.26}$$

Finally, we have

$$\sum_{K\in\mathcal{T}_h}\int\limits_K |\underline{\Pi}_k^{(q)}(\underline{\mathbf{z}}_h)|^q\,dx \leqslant \sum_{K\in\mathcal{T}_h}\int\limits_K |\nabla_{DG,2}u_h|^p\,dx$$

and hence,

$$\frac{1}{q}\sum_{K\in\mathcal{T}_h}\int\limits_K |\underline{\Pi}_k^{(q)}(\underline{\mathbf{z}}_h)|^q\,dx - \frac{1}{q}\sum_{K\in\mathcal{T}_h}\int\limits_K |\nabla_{DG,2}u_h|^p\,dx \leqslant 0. \tag{6.27}$$

The assertion now follows from (6.20)–(6.27). □

The following result establishes the relationship between the equilibrated and the residual a posteriori error estimator.

**Theorem 6.1.** *Let $u_h \in V_h$ be the IPDG approximation as given by (3.22) and let $\eta_{h,1}^{\mathrm{eq}}, \eta_{h,i}^{\mathrm{res}}, \bar{\eta}_{h,i}^{\mathrm{res}}, 1 \leqslant i \leqslant 5, \hat{\eta}_{h,i}^{\mathrm{res}}, 3 \leqslant i \leqslant 4$, and $osc_{h,i}, 1 \leqslant i \leqslant 3, \overline{osc}_{h,i}, 1 \leqslant i \leqslant 2$, be the equilibrated and the residual a posteriori error estimators as well as the data oscillations as given by (4.32b), (6.1), and (4.18), (4.31), (6.4). Then there exists a constant $C_{res} > 0$, depending on $\alpha$, $c_R$, $C_{p-1}$, $C_p$, $C_q$, $C_{\mathrm{rec}}$, $C_E$, $C_{PF}^{(i)}(p)$, $1 \leqslant i \leqslant 2$, and on $p$, $q$, such that*

$$\eta_{h,1}^{\mathrm{eq}} \leqslant C_{res}\Big(\eta_h^{\mathrm{res}} + \sum_{i=1}^{3} osc_{h,i} + \sum_{i=1}^{2}\overline{osc}_{h,i}\Big). \tag{6.28}$$

*Moreover, if we use (3.30) in (4.32b), then $\eta_{h,2}^{\mathrm{eq}}$ can be estimated from above in terms of the residuals $\eta_{h,3}^{\mathrm{res}}$, $\eta_{h,4}^{\mathrm{res}}$, and the data oscillations $osc_{h,i}$, $1 \leqslant i \leqslant 3$.*

*Proof.* The estimate (6.28) follows from (6.9) and Lemmas 6.1, 6.2, and 6.3, whereas the second assertion can be established by means of the definition of $\eta_{h,2}^{\mathrm{eq}}$ in (4.32b) and (3.30). □

**Remark 6.2.** Theorem 6.1 implies the reliability of the residual-type a posteriori error estimator via the reliability of the equilibrated error estimator. The constants in the estimate are specified in the theorem and are computable. In general, the reliability estimate for residual-type error estimators involves interpolation constants which can be computed as well. The number of constants increases with the polynomial degree $k$. The computations involve eigenvalue-type problems whose solutions may require a substantial amount of computational time (see, for example, [47] for details).

The efficiency of the residual-type error estimator can be established using techniques as in [51] involving suitably chosen bubble functions. The constants in the efficiency estimate depend on the local geometry of the triangulation and can be computed as well.

# 7 Numerical results

We have implemented the IPDG approximation (3.22) with the penalty parameter $\alpha$ chosen as $\alpha = 12\ k^2$. Further, we have implemented the adaptive algorithm based on the equilibrated error estimator $\eta_h^{\mathrm{eq}}$ by Dörfler marking [27], i.e., given a bulk parameter $\Theta \in (0,1)$, we have selected a set $\mathcal{M}_h \subset \mathcal{T}_h$ according to

$$\Theta \sum_{K\in\mathcal{T}_h} \eta_K^{\mathrm{eq}} \leqslant \sum_{K\in\mathcal{M}_h} \eta_K^{\mathrm{eq}}$$

and we have refined elements $K \in \mathcal{M}_h$ by newest vertex bisection. We note that for the evaluation of $\eta_{h,2}^{\mathrm{eq}}$ we have not computed $u_h^c$ by postprocessing according to (3.29) but used the estimate (3.30) instead. In case of the residual-based error estimator $\eta_h^{\mathrm{res}}$ we have implemented the adaptive refinement likewise.

As numerical examples, we have chosen $\Omega$ as the L-shaped domain $\Omega := (-1,+1)^2 \setminus ([0,1)\times(-1,0])$ with Dirichlet boundary $\Gamma_D = (\{0\}\times(-1,0))\cup((0,1)\times\{0\})$ and Neumann boundary $\Gamma_N = \partial\Omega\setminus\bar{\Gamma}_D$ with interior angle $\varphi = 3\pi/2$ at the origin. In [26] (see also [5]) it has been shown that in polar coordinates $(r,\varphi)$ the solution $u$ of the $p$-Laplace problem (2.1) behaves as $u \sim r^\gamma$ with

$$\gamma = \sigma(p) - \sqrt{\sigma^2(p) - 4/3}, \qquad \sigma(p) = \frac{7p-6}{6(p-1)}.$$

We have considered the cases $p = 1.5$ and $p = 3.0$ with the solution given by

$$u(r,\varphi) = r^\gamma \sin\Big(\frac{2}{3}\varphi\Big)$$

and the right-hand side $f$ in (2.1a), the Dirichlet data $u_D$ in (2.1b), and the Neumann data $u_N$ in (2.1c) given accordingly. This results in homogeneous Dirichlet data $u_D = 0$. We note that $u \in W^{1+\gamma-\varepsilon,p}(\Omega)$ for any $\varepsilon > 0$ and $u$ has a singularity at the origin. The same applies to the solution $z$ of the boundary value problem (4.15). However, since $\Gamma_N$ is located off the singularity at the origin, the trace of $\nabla z$ on $\Gamma_N$ is more regular and the regularity assumption (4.16) is satisfied.

We have performed computations for $p = 1.5$ and $p = 3.0$ and the polynomial degrees $k = 1$ and $k = 3$. The discrete data $f_h|K$, $K \in \mathcal{T}_h$, and $u_{h,N}|_E$, $E \in \mathcal{E}_h(\Gamma_N)$, have been obtained according to Remark 3.1. Moreover, the numerical solution of the nonlinear IPDG approximation (3.22) has been done by Newton's method with a relative tolerance of tol $= 10^{-3}$ as termination criterion for the Newton iterates. As outlined in [5], the expected convergence rate for the discretization error in the broken $W^{1,p}$ norm is 0.5.

Figure 1 shows the adaptively generated mesh in case $p = 1.5$ and bulk parameter $\Theta = 0.5$ for polynomial degree $k = 1$ (left) and polynomial degree $k = 3$ (right) where the adaptive mesh refinements were based on the equilibrated error estimator. As expected, we observe a pronounced refinement around the reentrant corner and substantially less refinement off the singularity for the higher polynomial degree $k = 3$. The meshes in case $p = 3.0$ and bulk parameter $\Theta = 0.5$ for $k = 1$ and $k = 3$ as well as the meshes obtained by the residual-based error estimator look similarly and are therefore omitted.

Figure 2 displays the discretization error in the broken $W^{1,p}$ norm, the equilibrated error estimator $\eta_h^{\mathrm{eq}}$, and the residual-based error estimator $\eta_h^{\mathrm{res}}$ as a function of the total number of degrees of freedom (DOFs) on a logarithmic scale. As expected by the theory, the convergence rate appears to be 0.5. The equilibrated error estimator is smaller than the residual-based error estimator by approximately 1/2 of an order of magnitude. Figure 3 shows the corresponding results for the polynomial degree $k = 3$. Here, we observe that in a pre-asymptotic phase the error decays faster, but asymptotically approaches the predicted convergence rate of 0.5.

Figures 4 and 5 contain the corresponding results for $p = 3.0$, $\Theta = 0.5$, and the polynomial degrees $k = 1$ and $k = 3$. We observe a similar behavior as for $p = 1.5$.

**Remark 7.1.** As far as the robustness of the a priori error estimators is concerned, in our numerical examples the equilibrated error estimator turned out to be robust with respect to the termination criterion for the approximate solution by Newton's method and the choice of the penalty parameter $\alpha$. In contrast, the

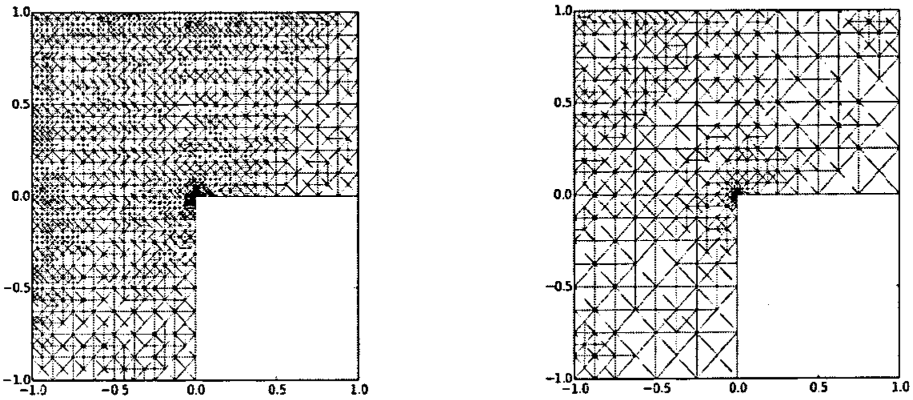


**Fig. 1:** $p = 1.5$, $\theta = 0.5$: Adaptively generated mesh (equilibrated error estimator) for polynomial degree $k = 1$ (left) and $k = 3$ (right).

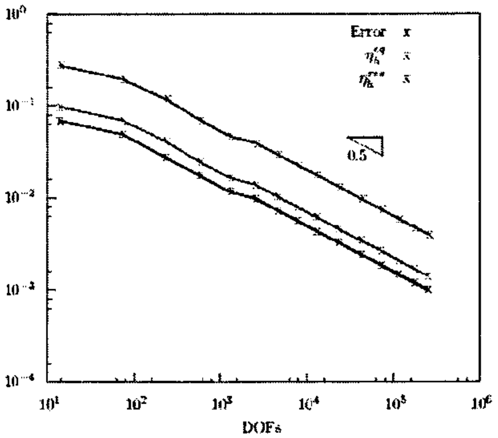


**Fig. 2:** $p = 1.5$, $\theta = 0.5$, polynomial degree $k = 1$: The error in the broken $W^{1,p}$ norm (black), the equilibrated error estimator $\eta_h^{eq}$ (red), and the residual-based error estimator $\eta_h^{res}$ (blue).

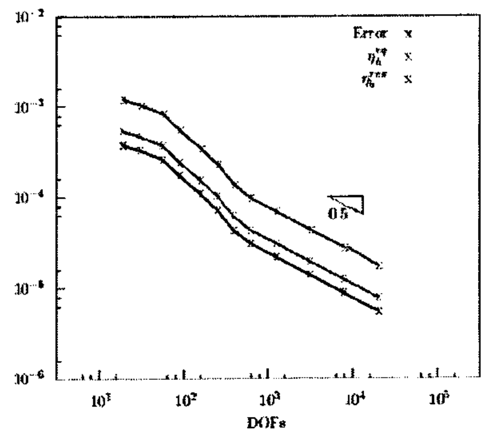


**Fig. 3:** $p = 1.5$, $\theta = 0.5$, polynomial degree $k = 3$: The error in the broken $W^{1,p}$ norm (black), the equilibrated error estimator $\eta_h^{eq}$ (red), and the residual-based error estimator $\eta_h^{res}$ (blue).

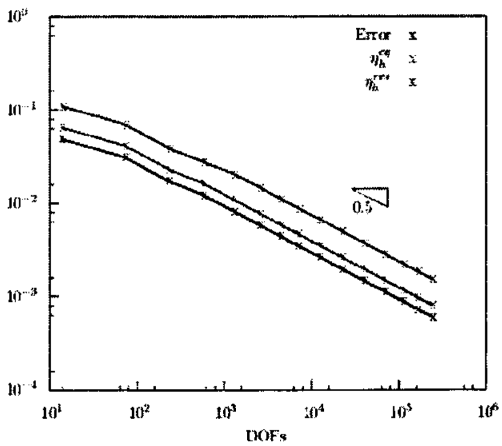


**Fig. 4:** $p = 3.0$, $\theta = 0.5$, polynomial degree $k = 1$: The error in the broken $W^{1,p}$ norm (black), the equilibrated error estimator $\eta_h^{eq}$ (red), and the residual-based error estimator $\eta_h^{res}$ (blue).

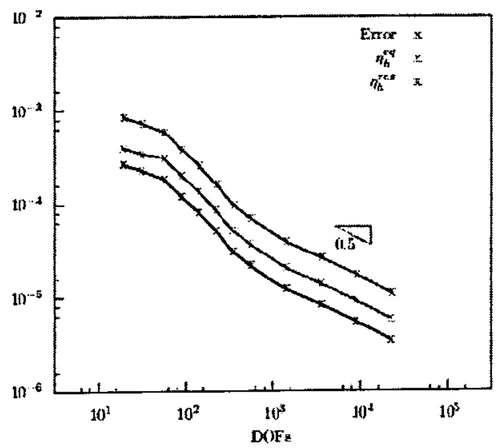


**Fig. 5:** $p = 3.0$, $\theta = 0.5$, polynomial degree $k = 3$: The error in the broken $W^{1,p}$ norm (black), the equilibrated error estimator $\eta_h^{eq}$ (red), and the residual-based error estimator $\eta_h^{res}$ (blue).

residual-type estimator has shown less robustness with the efficiency index (estimated error versus true error) increasing slightly with a reduction of the tolerance in the termination criterion and for choosing larger penalty parameters.

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