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Omega-CVaR portfolio optimization and its worst case analysis

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Abstract This paper presents a novel framework for optimizing portfolios using distribution dependent thresholds in Omega ratio to control the downside risk. Portfolios resulting from the maximization of the classical Omega ratio simultaneously maximize the probability of superior performance compared to a threshold point set by an investor and minimize the probability of a worse performance compared to the same threshold. However, there is no mandatory rule or mechanism to choose this threshold point in the Omega ratio optimization model yet. In this paper, we redefine the Omega ratio for a loss averse investor by taking the distribution dependent threshold point as the conditional value-at-risk at an α confidence level (CVaR$_\alpha$) of the benchmark market. The α-value reflects the attitude of an investor towards losses. We then embed this new Omega-CVaR$_\alpha$ model in a robust portfolio optimization framework and present its worst case analysis under three uncertainty sets. The robustness is introduced both in the Omega measure and the CVaR$_\alpha$ measure. We show that the worst case Omega-CVaR$_\alpha$ robust optimization models are linear programs for mixed and box uncertainty sets and a second order cone program under ellipsoidal sets, and hence tractable in all three cases. We conduct a comprehensive empirical investigation of the classical CVaR$_\alpha$ model, the STARR$_\alpha$ model, the Omega-CVaR$_\alpha$ model, and

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robust Omega-CVaR$_\alpha$ model under a mixed uncertainty set for listed stocks of the S&P 500. The optimal portfolios resulting from the Omega-CVaR$_\alpha$ model exhibit a superior performance compared to the classical CVaR$_\alpha$ model in the sense of higher expected returns, Sharpe ratios, modified Sharpe ratios, and lesser losses in terms of VaR$_\alpha$ and CVaR$_\alpha$ values. The robust Omega-CVaR$_\alpha$ model under mixed uncertainty set is shown to dominate the Omega-CVaR$_\alpha$ model in terms of all performance measures. Furthermore, both the Omega-CVaR$_\alpha$ and robust Omega-CVaR$_\alpha$ model under a mixed uncertainty set yield significantly lower risk compared to STARR$_\alpha$ model in terms of CVaR$_\alpha$ and variance values.

Keywords Omega ratio optimization · Value-at-risk · Conditional value-at-risk · Robust portfolio optimization · Asset allocation

1 Introduction

The modern portfolio theory (MPT) has now become a giant tree whose seed was farmed by Markowitz (1952) and later nourished and disseminated by a number of academics and practitioners. Markowitz's MPT foundation is laid on the fact that an investor always aims for the best trade-off between portfolio return and its risk. The portfolio return is assumed to follow a symmetric distribution and is thus characterized by its first two moments, mean and variance.[1] This theory gives birth to the popular mean-variance model. However, the model fails to capture heavy tails in return distributions.

Over the decades, the assumption of symmetric return distribution has been relaxed and other statistical measures have been proposed to represent risk (Roman and Mitra 2009 and references therein). Nevertheless, standard deviation is still a popular measure to quantify risk for a large number of risk seeking investors in practice. The standard deviation and mean absolute deviation (Konno and Yamazaki 1991) are examples of symmetric risk measures that penalize the deviations above as well as below the threshold point (which is the mean of the return distribution). Yet, many investors wish to amplify the threshold point and thus choose an asymmetric risk measure (Markowitz 1959; Fishburn 1977) that minimizes only the points below the threshold point. Other risk measures called "downside risk measures" focus on the right tail of the loss distribution (or the left tail of the return distribution) to minimize large losses. The Value-at-Risk VaR$_\alpha$ (Linsmeier and Pearson 1996) and the Conditional Value-at-Risk CVaR$_\alpha$ (Rockafellar and Uryasev 2002), at an α confidence level, are two popular downside risk measures designed for loss averse investors. The VaR$_\alpha$ is the realization of the loss distribution from which all realizations over the time horizon are smaller with an α probability and thus a higher value of α is desirable. CVaR$_\alpha$ is the average of those realizations that are larger than the VaR$_\alpha$. Thus, CVaR$_\alpha$ contains additional information of the loss distribution compared to the VaR$_\alpha$.

[1] To be precise, variance is the second central moment of the distribution. However, it can be computed by the first two moments.

Beside an accurate determination of a risk measure of a return distribution, investors also wish to evaluate the performance of their portfolios with respect to a benchmark index or have the desire to rank different portfolio strategies. Performance indices are one of the tools available in finance to assess this task (Treynor 1965 ratio; Sharpe 1966, 1994 ratio; Jensen 1967 alpha). Apart from ranking the portfolios, they are also used to construct optimal portfolios (Mansini et al. 2003).

The development of numerous risk measures succeeding the Sharpe ratio leads to the emergence of new performance indices such as the modified Sharpe ratio (Gregoriou and Gueyie 2003), Sortino ratio (Sortino and Price 1994), STARR (Martin et al. 2003), and Rachev ratio and generalized Rachev ratio (Biglova et al. 2004). Keating and Shadwick (2002) introduce the Omega ratio as a performance index to measure the growth of the upside deviation relative to the downside deviation of portfolio return from a constant threshold point. Stoyanov et al. (2007) consider the maximization of the STARR and Rachev ratio.

The notable advantage of using the Omega ratio in a portfolio selection model is that this ratio neither requires a specific type of utility function nor assumes any specific distribution of portfolio return. Also, contrary to the Sharpe and Treynor ratios wherein an investor needs to estimate the expected return and risk in stocks returns, the Omega ratio does not suffer from such estimation errors. Moreover, while the Rachev ratio (Biglova et al. 2004) focuses on extreme gains and extreme losses by considering only the ratio of the upper and the lower tails of the return distribution, the Omega ratio is a performance measure to build portfolios based on the entire return distribution, thereby innate more information.

Similar to other performance indices, the Omega ratio can be optimized over a set of admissible portfolios to construct an optimal portfolio. The theoretical beauty of maximizing the Omega ratio allows us to convert a non-convex non differentiable problem into a linear programming problem (LPP) under specific conditions on the upper bound of the threshold point. Nevertheless, the Omega ratio, along with many desirable features, has some weak points such as that it is sensitive to changes in the threshold point. Moreover, there is no formal rule to guide an investor regarding an appropriate choice of a threshold point. Consequently, the Omega ratio optimization model is not conducive to capturing heavy tails in portfolio return distribution (Mausser et al. 2006; Sharma and Mehra 2015). Thus, the Omega ratio optimization in its present shape with an arbitrary but constant threshold point is not suitable for loss averse investors.

In this paper, we attempt to improve the Omega ratio optimization model in the sense of these weaknesses. We present a systematic approach to select distribution dependent threshold points to reflect loss aversion of an investor. We incorporate the loss aversion behavior into our model by employing the CVaR_α of a benchmark market portfolio (BM) as a distribution dependent threshold point in the Omega ratio. We name the new Omega ratio optimization model by the Omega-CVaR_α model. By maximizing the Omega-CVaR_α model, an investor maximizes expected losses below the threshold CVaR_α (i.e. gains) and simultaneously minimizes expected losses in excess of the same CVaR_α.[2] To account for this, we redefine the original definition of

[2] Note that the underlying distribution is the loss distribution in which expected losses below the threshold indicate smaller losses than the CVaR_α.

Omega ratio and denote it as $\Omega_\alpha(\ell)$ by replacing the portfolio return r by the portfolio loss ℓ to make the study compatible with the concept of CVaR_α.

We address the situation of an investor who wishes to avoid portfolio losses higher than the CVaR_α of the BM z.[3] Our model has its practical application in mutual funds which try to replicate an index (such as the S&P 500, MSCI World), and enhanced index tracking (Guastaroba et al. 2016), for instance. Therefore, we construct a portfolio by maximizing the Omega ratio $\Omega_\alpha(\ell)$ from a subset of stocks x of the BM z. In our empirical study, we take the BM z as being the entire set of those stocks listed on the S&P 500 on June 2015 and whose data has been available for at least 10 years. Our investment universe x contains the top 50 stocks from z with respect to the market capitalization. The proposed methodology aims to maximize the Omega-CVaR_α model using the subset of stocks x wherein the threshold point is obtained by minimizing the CVaR_α model for z. In other words, the proposed Omega-CVaR_α model requires no extra effort to find the benchmark threshold as it gets computed within the model itself from the BM z-data, and subsequently the model construct an optimal portfolio by optimizing the Omega ratio $\Omega_\alpha(\ell)$ from x-data.

To solve the Omega-CVaR_α model, an investor should have a true knowledge of the loss distributions of z and x which is generally not available. Therefore, the main challenge is to find a portfolio which always remains close to optimality or feasibility, independent of whatever future scenario persists in the market. Several studies (Ben-Tal and Nemirovski 2000; Beyer and Sendhoff 2007) develop robust optimization as a possible attempt to this uncertainty. To ascertain the robustness in the outcomes from the proposed Omega-CVaR_α model, we employ the worst case approach in the Omega ratio as well as its threshold CVaR_α when only partial information on the underlying distributions is available.[4] We present the worst case analysis of the Omega-CVaR_α model in the robust portfolio optimization framework under mixed, box, and ellipsoidal uncertainty sets for discrete distribution for x and z. We prove that Omega-CVaR_α robust portfolio optimization models can be converted into linear programs under mixed and box uncertainty sets and into a second order cone problem (SOCP) under an ellipsoidal uncertainty set which makes them tractable in all three cases.[5]

To test the performance of the proposed Omega-CVaR_α model and its worst case Omega-CVaR_α robust optimization version under the mixed uncertainty set, we perform a one month rolling window strategy for four different values of $\alpha = 0.97, 0.95, 0.93, 0.90$. The observation period of the study covers 25.5 years from Jan 1990 to June 2015 with 306 closing monthly returns of all stocks listed on the S&P 500 as of June 2015. Our in-sample estimates are based on 20 years (240 months) and

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[3] With respect to the definition of the Omega ratio, the value-at-risk is not the natural choice for the threshold. In fact, the Omega ratio for the value of risk at a certain confidence level is constant (when distribution to compute Omega ratio is similar to the distribution to compute value-at-risk) due to the definition of the value-at-risk.

[4] Thus, in this framework, we are able to reduce the sensitivity of the Omega ratio to its threshold point, which now is also chosen by the robust optimization approach.

[5] For computational purposes, we analyze the worst case of Omega-CVaR_α under the mixed uncertainty set.

the corresponding out-of-sample period covers the subsequent month. For a comparative analysis, we optimize the classical CVaR_α model using the sample of stocks x. Moreover, we relate the performance of our proposed model to three further benchmark portfolios. First, we include the STARR optimization model in our empirical analysis. The STARR_α, at an α level, is the ratio of the excess mean return from the benchmark index to the CVaR_α value of the return, and thus conducive for our comparative analysis. Second, DeMiguel et al. (2009) find that no portfolio strategy was able to statistically outperform the naïve diversification $1/m$ portfolio, so we decide to entail the $1/m$ portfolio strategy as an additional benchmark strategy. The $1/m$ portfolio strategy stems from the allocation of a fraction $1/m$ of a budget to each of the available assets from S&P 500 index in each in-sample period. Third, we also take a market capitalization value weighted portfolio (MCWP) from S&P 500 index in our comparison analysis; it is formulated by allocating the normalized market capitalization weights instead of equal weights as in the $1/m$ portfolio strategy.

Our empirical results show that the portfolios of Omega-CVaR_α and robust Omega-CVaR_α (under the mixed uncertainty set) exhibit higher expected returns, Sharpe ratios, modified Sharpe ratios, lower values of VaR_α, and CVaR_α for all four values of α compared to the classical CVaR_α model, and outcomes are significant for the first two performance measures for most values of α. Both the models outperform the $1/m$ portfolio strategy statistically in terms of CVaR_α values for all values of α. Moreover, the portfolios of robust Omega-CVaR_α under the mixed uncertainty set outperform those of the Omega-CVaR_α model in terms of expected returns, Sharpe ratio, modified Sharpe ratio, VaR_α, and CVaR_α values for all four values of α and thus signifies the utility of robust optimization. Finally, portfolios of the STARR_α model exhibit significantly higher variance values and CVaR_α values compared to portfolios of the Omega-CVaR_α and robust Omega-CVaR_α model under a mixed uncertainty set for all four values of α. Nevertheless, optimal STARR_α model portfolios earn higher average returns than the Omega-CVaR_α and robust Omega-CVaR_α under the mixed uncertainty set only for $\alpha = 0.97$.

The remainder of the paper is organized as follows. Section 2 presents a brief overview on the Omega ratio, the CVaR_α concept, and their optimization. Section 3 explains the proposed Omega-CVaR_α model. Section 4 extends the proposed model to its robust Omega-CVaR_α model under three different uncertainty sets. Section 5 presents the empirical analysis. Section 6 concludes the paper.

2 Optimization models

In this section, we introduce and discuss the Omega optimization model as well as the general shape of the CVaR_α portfolio model. In the remainder of the paper we use the following notations:

Γ	time horizon of investment
$z = (z_1, \ldots, z_m) \in \mathbb{R}^m$	vector of benchmark market portfolio of m stocks; z_j is the weight of the jth stock
$x = (x_1, \ldots, x_n) \in \mathbb{R}^n$	vector of a subset portfolio of $n \leq m$ stocks of the benchmark market; x_j is the weight of the jth stock

$\hat{T}$	discrete case: investment time Γ is divided into $\hat{T}$ scenarios for z
T	discrete case: investment time Γ is divided into T scenarios for x
ℓ_{ij}	loss realization of jth asset at ith time point, $i = 1, \ldots, T$
$\hat{\ell}_z^i = \sum_{j=1}^m \hat{\ell}_{ij} z_j$	random loss in portfolio z at ith time point, $i = 1, \ldots, \hat{T}$
$\ell_x^i = \sum_{j=1}^n \ell_{ij} x_j$	random loss in portfolio x at ith time point, $i = 1, \ldots, T$

ℓ_x and $\hat{\ell}_z$ denote the loss for portfolios x and z, respectively, and are distributed over the respective finite scenarios $(\ell_x^1, \ldots, \ell_x^T)$ and $(\hat{\ell}_z^1, \ldots, \hat{\ell}_z^{\hat{T}})$. Also, note that all necessary mathematical symbols associated with the benchmark market z are denoted using the *hat* sign on top such as $\hat{\ell}_z$ and $\hat{T}$, while all those associated with a subset of stocks $x \subset z$ are denoted by simple characters such as ℓ and T.

2.1 The Omega optimization model

The Omega ratio is a young performance index. It ranks different strategies and, therefore, is used to determine an optimal portfolio (Mausser et al. 2006; Gilli et al. 2011; Kirilyuk 2013; Kapos et al. 2014; Sharma and Mehra 2015; Guastaroba et al. 2016). Mausser et al. (2006) apply the Charnes and Cooper (1962) transformation to convert the Omega ratio optimization model into a linear program in the case of the optimal Omega ratio being greater than 1. For lower values of optimal Omega ratios it is advisable to use global optimization (Glover and Laguna 1997), heuristics (Reeves 1993), and integer programming approaches. Dembo and Mausser (2000) and Kapos et al. (2014) optimize the Omega ratio in the risk-reward framework, in which risk is the downside deviation and reward is taken as the upside deviation (Dembo and Mausser 2000) or the mean return (Kapos et al. 2014). Sharma and Mehra (2015) optimize the Omega ratio model with an additional constraint involving the traditional risk measure to reduce some component of risk which otherwise is not captured solely by the Omega ratio.[6] Moreover, Avouyi-Dovi et al. (2004) and Kane et al. (2009) optimize the Omega ratio as a non-convex program using the threshold acceptance technique (Dueck and Tobias 1990) and NAG library implementation MCS method (Huyer and Neumaier 1999), respectively. Guastaroba et al. (2016) study an application of the Omega ratio in enhanced index tracking problem in two frameworks, one when the threshold point is a constant, and the second when it is random. The authors show that the optimal portfolio obtained from the Omega ratio optimization with random threshold generally outperforms the one obtained with the constant threshold. This also supports the idea of considering the distribution dependent threshold point in Omega ratio in our study.

In the remainder of this subsection, we update the primary definition of the Omega ratio for the case of a loss distribution and reshape the corresponding optimization

[6] Heavy tails and dispersion around the mean return (as, at mean return, the Omega ratio is a constant) are controlled by setting upper bounds on risk measures CVaR_α, minimax (Young 1998), and semi-mean absolute deviation SemiMAD (Ogryczak and Ruszczynski 1999) in constraints in the Omega ratio model while maintaining linearity in the resulting three hybrid models.

model. For a loss ℓ_x in a portfolio $x \in \mathbb{R}^n$ and a fixed threshold point, L, the Omega ratio is defined as

$$\Omega_L(\ell_x) = \frac{\displaystyle\int_{-\infty}^{L} \Pr(\ell_x < \ell)\mathrm{d}\ell}{\displaystyle\int_{L}^{\infty} \Pr(\ell_x > \ell)\mathrm{d}\ell} = \frac{\displaystyle\int_{-\infty}^{L} F_{\ell_x}(\ell)\mathrm{d}\ell}{\displaystyle\int_{L}^{\infty} \big(1 - F_{\ell_x}(\ell)\big)\mathrm{d}\ell} = \frac{E_p(L-\ell_x)^+}{E_p(\ell_x - L)^+},$$

where $F_{\ell_x}(\ell)$ is the probability distribution and p the probability density functions of the random variable ℓ_x, $E_p(\cdot)$ is the expected function under p, and y^+ denotes the maximum of zero and y. We assume that $E_p(\ell_x - L)^+ > 0$. This assumption holds in a situation in which an investor can generate portfolios with higher losses ℓ_x than the threshold L. Indeed, this obviously holds for a reasonable selection of L.

For a given threshold point L, ranking under the Omega ratio optimization is taken in the spirit that a portfolio x^1 outperforms a portfolio x^2 if and only if $\Omega_L(\ell_{x^1}) \geq \Omega_L(\ell_{x^2})$. Note that $\Omega_L(\ell_x)$ as a function of L is strictly increasing and so is the function $\Omega_L^* = \max_x \Omega_L(\ell_x)$. The Omega ratio equals one for $L = E(\ell_x)$.

Rachev et al. (2008) introduce an aggressive-coherent ratio as a function $G(X)$ defined on real numbers X which satisfy the following three properties:

(I) It admits the form $G(X) = \nu(X)/\eta(X)$, where $\nu(X)$ and $\eta(X)$ are, respectively, the reward and the risk measures, $\eta(X)$ posses same sign as of $\nu(X)$, for all X.
(II) In the ratio representation, both the reward and the risk measures are either concave or convex functions.
(III) If $X \geq Y$, then $G(X) \geq G(Y)$ provided that the reward and risk measures are both strictly positive, and $G(X) \leq G(Y)$ when both are strictly negative.

In context of the above properties, it can easily be verified that the classical Omega ratio is an aggressive-coherent ratio by taking $\nu(X)$ and $\eta(X)$ as $E_p(L-\ell_x)^+$ and $E_p(\ell_x - L)^+$, respectively. Then, both functions are convex and positive. Also, for a fixed L, the higher value of Omega ratio is preferable for portfolio X preference over portfolio Y, resulting in property (III). Thus, the $\Omega_L(\ell_x)$ is an aggressive-coherent ratio, $\forall\, L$, and hence for $L = \text{CVaR}_\alpha$.

Next, we determine the Omega ratio optimization model using the loss distribution as

$$P_1 \quad \max \quad \Omega_L(\ell_x) = \frac{E_p(L-\ell_x)^+}{E_p(\ell_x - L)^+} \quad \text{subject to :} \quad \sum_{j=1}^{n} x_j = 1, \;\; x_j \geq 0, \;\; j = 1, \ldots, n,$$

where $\sum_{j=1}^{n} x_j = 1$ is the normalized budget constraint and $x_j \geq 0$ prohibits short selling. A computationally convenient and natural method to represent uncertainty is through its finite scenarios.[7] With this motivation, we approximate the $\Omega_L(\ell_x)$ function by taking T finite number of scenarios of ℓ_x (using sampling techniques) with the probability vector $p = \big((p_1, \ldots, p_T)^t;\;\; p^t e = 1,\;\; p_i \geq 0,\; \forall\, i = 1, \ldots, T\big)$. P_1 can then be rewritten as the following fractional program:

[7] A scenario is a particular realization of the uncertain data.

$$P_2 \qquad \max \quad \Omega_L(\ell_x) = \frac{p^t u}{p^t d}$$

$$\text{subject to: } Bx + u - d = Le \tag{1}$$

$$u.d = 0, \quad u,\, d \in \mathbb{R}^T_+ \tag{2}$$

$$x^t e = 1, \quad x \in \mathbb{R}^n_+, \tag{3}$$

where $B = [\ell_{ij}]_{T\times n}$ is the loss matrix of portfolio x. Throughout the paper, $\mathbb{R}^n_+$ is the non-negative orthant of $\mathbb{R}^n$ and e is the vector of 1's in an appropriate dimensional space in the context, $u = ((u_1, \ldots, u_T)^t;\ u_i = (L - \ell^i_x)^+,\ \forall i = 1, \ldots, T)$ and $d = ((d_1, \ldots, d_T)^t;\ d_i = (\ell^i_x - L)^+,\ \forall i = 1, \ldots, T)$ are the respective upside and downside deviations vectors reflecting overachievement and underachievement of the ith realization $\ell^i_x = \sum_{j=1}^{n} \ell_{ij} x_j$ from the threshold point L. In (2), $u.d$ is the point-wise product, i. e. $u_i d_i = 0,\ \forall i = 1, \ldots, T$.

The complementarity constraints (2) depict that, in every scenario, the portfolio loss is either less than or greater than the threshold point L. Because of these constraints, P_2 is a nonconvex nonlinear problem and thus requires an efficient nonlinear solver (Huyer and Neumaier 1999; Dueck and Tobias 1990; Glover and Laguna 1997) to find a solution close to its global optimum. Following Mausser et al. (2006), we apply the Charnes and Cooper (1962) transformation technique to convert P_2 into the following program, which we refer to as Omega.

$$\text{Omega} \qquad \max \ \widetilde{\Omega_L(\ell_x)} = p^t \widetilde{u}$$

$$\begin{aligned} \text{subject to: } B\widetilde{x} + \widetilde{u} - \widetilde{d} &= \widetilde{L}e \\ \widetilde{x}^t e &= \gamma \\ p^t \widetilde{d} &= 1 \\ \widetilde{u},\ \widetilde{d} \in \mathbb{R}^T_+, \quad \widetilde{x} &\in \mathbb{R}^n_+, \end{aligned}$$

where $\gamma > 0$ is a homogenization variable, $\widetilde{x} = x\gamma,\ \widetilde{u} = u\gamma,\ \widetilde{d} = d\gamma,\ \widetilde{L} = L\gamma$. The resulting model Omega is a LPP in $\widetilde{u},\ \widetilde{x},\ \widetilde{d},\ \gamma$. An optimal solution for P_2 can be obtained from an optimal solution of Omega only if $\max\ \widetilde{\Omega_L(\ell_x)} > 1$ or equivalently $L > \min_{x^t e = 1,\ x \in \mathbb{R}^n_+} E_p(\ell_x)$. In this case the complementarity constraints (2) hold naturally in Omega (Mausser et al. 2006). Most of the optimal portfolio strategies in real practice do not let the optimal Omega ratio value become less than or equal to 1. As a consequence optimizing the Omega ratio is equivalent to solving the linear program Omega ensuring a global optimal solution for P_2. Earlier studies (Avouyi-Dovi et al. 2004; Kane et al. 2009; Mausser et al. 2006) quote some efficient nonlinear solvers to solve P_2 in case it remains a nonconvex nonlinear model.

2.2 The CVaR$_\alpha$ model

Worse scenarios on financial market associated with high losses are reasons for the right skewness of loss distributions of investments. Risk managers try to gain control of these losses by quantifying downside risk measures to capture these unfavorable occurrences. The Value-at-Risk VaR_α is the first downside risk measure indicating the amount of a possible maximum loss at a given confidence level α. The foundation of VaR_α provides a new perspective to risk managers when analyzing losses in many fields such as investment banking, insurance, and gold mining companies. But it has been noticed that the VaR_α measure fails to have desirable properties of being a coherent risk measure (Artzner et al. 1999). Moreover, VaR_α optimization is computationally cumbersome and does not reach the exact value (Ghaoui et al. 2003).

The Conditional Value-at-Risk CVaR_α, an alternative downside risk measure, which accounts for all of those losses that are greater than VaR_α, surpasses the Value-at-Risk measure in crucial aspects. For example, it accommodates more information in the interest of risk managers, it is a coherent risk measure, and its optimization can be approximated by a LPP for continuous scenarios and is exactly the same for a discrete scenario. Therefore, we propose using the CVaR_α measure to determine the threshold in the Omega ratio optimization model. Following this approach, we take the performance of a corresponding benchmark market into account and present a framework to compute a portfolio which exhibits the highest Omega ratio with respect to the CVaR_α measure of the BM, i.e. which prevents an investor to fail the expected shortfall of a BM at a given confidence level α. For a random loss $\hat{\ell}_z$ of the BM $z \in \mathbb{R}^m$, the $\mathrm{VaR}_\alpha(\hat{\ell}_z)$ is the loss point from which all BM losses over the time horizon are smaller with α probability. The $\mathrm{CVaR}_\alpha(\hat{\ell}_z)$ is the average of those BM losses that are larger than $\mathrm{VaR}_\alpha(\hat{\ell}_z)$ (Rockafellar and Uryasev 2000).

Rockafellar and Uryasev (2000) introduce the primary $\mathrm{CVaR}_\alpha(\hat{\ell}_z)$ portfolio optimization model. They prove that the $\mathrm{CVaR}_\alpha(\hat{\ell}_z)$ minimization with a continuous distribution can be approximated by a linear program using sampling techniques. Let $\hat{T}$ be the finite number of scenarios of $\hat{\ell}_z$ (using sampling techniques) with probability vector $q = ((q_1, \ldots, q_{\hat{T}})^t;\ q^t e = 1,\ q_i \geq 0,\ \forall\, i = 1, \ldots, \hat{T})$, the $\mathrm{CVaR}_\alpha(\hat{\ell}_z)$ minimization is equivalent to solving the following LPP:

$$P_{\mathrm{CVaR}_\alpha(\hat{\ell}_z)} \qquad \min \quad \eta + \frac{1}{1-\alpha} q^t \hat{u}$$

$$\text{subject to: } \hat{u} + \eta e - \hat{B} z \geq 0, \quad \eta \in \mathbb{R}, \quad \hat{u} \in \mathbb{R}_+^{\hat{T}} \tag{4}$$

$$e^t z = 1, \qquad z \in \mathbb{R}_+^m, \tag{5}$$

where $\hat{B} = [\hat{\ell}_{ij}]_{\hat{T} \times m}$ is the loss matrix of portfolio z and $\hat{u} = ((\hat{u}_1, \ldots, \hat{u}_{\hat{T}})^t;\ \hat{u}_i = (\hat{B}_i z - \eta)^+,\ \hat{B}_i$ is the ith row of matrix $\hat{B},\ i = 1, \ldots, \hat{T})$.

3 The Omega-CVaR_α model

3.1 Definition of the Omega-CVaR_α model

In this section, we propose the Omega ratio optimization by replacing L by $L(\alpha) := \mathrm{CVaR}^*_\alpha(\hat{\ell}_z)$ as the optimal objective value from $P_{\mathrm{CVaR}_\alpha(\hat{\ell}_z)}$ model. We establish the new Omega ratio for a fixed confidence level $\alpha,\ 0 < \alpha < 1$, as

$$\Omega_{L(\alpha)}(\ell_x) = \frac{E_p(L(\alpha) - \ell_x)^+}{E_p(\ell_x - L(\alpha))^+}.$$

With $L(\alpha) = \mathrm{CVaR}^*_\alpha(\hat{\ell}_z)$, we focus on designing the portfolios for investors who do not wish to fall beyond this critical value at a fixed confidence level α. The motive is to control heavy losses in the distribution of portfolio returns by minimizing lower deviation from $L(\alpha) = \mathrm{CVaR}^*_\alpha(\hat{\ell}_z)$ and still advancing towards the positive rewards by maximizing the upper deviation from the same $L(\alpha)$.

In contrast to the classical Omega ratio model, the optimization of the Omega ratio with $L(\alpha) = \mathrm{CVaR}^*_\alpha(\hat{\ell}_z)$ involves an additional inner minimization problem $P_{\mathrm{CVaR}_\alpha(\hat{\ell}_z)}$ in the constraints. We can solve this inner minimization problem by considering behavior of $\Omega_{L(\alpha)}(\ell_x)$ with respect to $L(\alpha)$ and the existence of the zero duality gap in $P_{\mathrm{CVaR}_\alpha(\hat{\ell}_z)}$. For a fix value of α, $\Omega_{L(\alpha)}(\ell_x)$ is an increasing function of $L(\alpha)$, i.e. the maximum value of $\Omega_{L(\alpha)}(\ell_x)$ is attained at the upper bound of $L(\alpha)$. Using this fact along with the zero duality gap in $P_{\mathrm{CVaR}_\alpha(\hat{\ell}_z)}$ model, we solve the inner minimization problem by taking its corresponding dual maximization problem in optimizing $\Omega_{L(\alpha)}(\ell_x)$.

Combining P_2 and $P_{\mathrm{CVaR}_\alpha(\hat{\ell}_z)}$, the Omega ratio optimization with $L(\alpha) = \mathrm{CVaR}^*_\alpha(\hat{\ell}_z)$ is given as follows:

$$
\begin{aligned}
P_3 \qquad & \max \quad \Omega_{L(\alpha)}(\ell_x) = \frac{p^t u}{p^t d} \\
& \text{subject to: constraints} \quad (1) - (3) \\
& L(\alpha) \le \min_{\{\text{constraints (4) and (5)}\}} \eta + \frac{1}{1-\alpha} q^t \hat{u}. \qquad (6)
\end{aligned}
$$

We solve the inner minimization problem in P_3 (constraint 6) by taking its dual problem which is stated as follows:

$$
\begin{aligned}
P_{\mathrm{DCVaR}_\alpha(\hat{\ell}_z)} \qquad & \max \quad \vartheta \\
& \text{subject to: } v - \frac{q}{1-\alpha} \le 0 \qquad (7) \\
& -\hat{B}^t v + \vartheta e \le 0 \qquad (8) \\
& v^t e = 1, \quad v \in \mathbb{R}^{\hat{T}}_+, \quad \vartheta \in \mathbb{R}, \qquad (9)
\end{aligned}
$$

where $\vartheta \in \mathbb{R}$ is the dual variable corresponding to constraint (5) in its primal problem $P_{\mathrm{CVaR}_\alpha(\hat{\ell}_z)}$. Using $P_{\mathrm{DCVaR}_\alpha(\hat{\ell}_z)}$ in P_3, we obtain the following reduced problem P_4:

$$
\begin{aligned}
P_4 \qquad & \max \quad \Omega_{L(\alpha)}(\ell_x) = \frac{p^t u}{p^t d} \\
& \text{subject to: } \quad L(\alpha) \le \vartheta, \quad \text{constraints} \quad (1) - (3) \text{ and } (7) - (9).
\end{aligned}
$$

Applying the Charnes and Cooper (1962) transformation in P_4, we receive the following linear program named Omega-CVaR_α

$$\text{Omega} - \text{CVaR}_\alpha \quad \max \quad \widetilde{\Omega_{L(\alpha)}(\ell_x)} = p^t \widetilde{u}$$

$$\text{subject to: } B\widetilde{x} + \widetilde{u} - \widetilde{d} = \widetilde{L(\alpha)}e$$
$$p^t \widetilde{d} = 1, \quad \widetilde{L(\alpha)} \le \widetilde{\vartheta}$$
$$\widetilde{v} - \frac{\gamma}{1-\alpha} q \le 0, \quad -\hat{B}^t \widetilde{v} + \widetilde{\vartheta} e \le 0$$
$$\widetilde{x}^t e = \gamma, \quad \widetilde{v}^t e = \gamma, \quad \widetilde{v} \in \mathbb{R}_+^{\hat{T}}, \quad \widetilde{\vartheta} \in \mathbb{R}, \quad \widetilde{x} \in \mathbb{R}_+^n,$$

where $\gamma > 0$ is a homogenization variable, and for every other variable ϕ, $\widetilde{\phi} = \gamma\phi$ and Omega-CVaR$_\alpha$ is linear in $\widetilde{x}$, $\widetilde{u}$, $\widetilde{d}$, γ, $\widetilde{v}$, $\widetilde{\vartheta}$. Analogously to the Omega model, the optimal solution for P_4 can be calculated from Omega-CVaR$_\alpha$ only if $\max \widetilde{\Omega_{L(\alpha)}(\ell_x)} > 1$ or equivalently if $L(\alpha) > \min_{x^t e=1,\ x \in \mathbb{R}_+^n} E_p(\ell_x)$ (see, Mausser et al. 2006; Sharma and Mehra 2015). For sufficiently large values of α, the average of all losses beyond VaR$_\alpha$ in BM z is generally greater than the minimum expected loss of market x (i.e. $\text{CVaR}_\alpha^*(\hat{\ell}_z) > \min_{x^t e=1,\ x \in \mathbb{R}_+^n} E_p(\ell_x)$).[8] Hence, the Omega-CVaR$_\alpha$ model remains a LPP and can be solved using any standard LPP solver.

3.2 Omega-CVaR$_\alpha$ model and related ratios

Besides $\Omega_{L(\alpha)}(\ell_x)$, the other two prominent ratios involving CVaR-measure to control the extreme losses in the return distribution with respect to the benchmark index (or benchmark market) are STARR (Martin et al. 2003) and Rachev ratio along with the generalized Rachev ratio (Rachev et al. 2008; Biglova et al. 2004). Rachev ratio is the ratio of the average of $(1-\alpha)\%$ of most extreme gains from the benchmark index over the average of $(1-\beta)\%$ of the most extreme losses from the same benchmark index. In short, it is the ratio of CVaR$_\alpha$ value (to maximize) of the excess return of benchmark index from the portfolio return to the CVaR$_\beta$ value (to minimize) of its negative series. It is defined as follows:

$$\text{Rachev}_{\alpha,\beta}(r_X) = \frac{\text{CVaR}_\alpha(r_M - r_X)}{\text{CVaR}_\beta(r_X - r_M)},$$

where $0 < \alpha,\ \beta < 1$, and r_M and r_x denote the benchmark index returns and portfolio X returns, respectively. The following characteristics of Rachev ratio in relation with the Omega ratio are noteworthy.

(I) In the case of $\alpha + \beta \neq 1$: Rachev ratio does not extract complete information of the distribution of stock returns and only emphasize on the tail parts of the distribution (according to the values of α and β) of the excess return from the benchmark index. However, the Omega ratio bifurcates the entire return

[8] In other words, it conveys that value of CVaR$_\alpha$ (which accounts for the right tail of loss distribution of z) is larger than the minimum expected value of the loss function (which accounts for the left tail of loss distribution of x)

distribution into two parts according to the threshold point L. By replacing L by $\text{CVaR}^*_\alpha(\hat{\ell}_z)$, we account to minimize all losses exceeding to the CVaR_α value of the benchmark market z and maximizing all losses lesser than the same benchmark threshold, and hence do not miss any information in the distribution.

(II) In the case of $\alpha + \beta = 1$: Rachev ratio acts similar (although not exactly equal) to the Omega ratio for $L = \text{VaR}_\alpha$ and $\alpha = 1 - \beta$ in the Omega ratio, and $r_M = 0$ in Rachev ratio.

(III) Rachev ratio as well as the Omega ratio are aggressive-coherent ratios irrespective of the values of the input parameters (α, β, and L).

(IV) The resultant optimization model for Rachev ratio is a mixed integer linear program (see, Stoyanov et al. 2007). It is computationally challenging to solve especially for 50 or more scenarios. However, the optimization model for the Omega ratio is an LPP for a suitably chosen threshold point L, and hence easily tractable.

Another similar ratio is STARR_α (Martin et al. 2003) which, for a fixed confidence level α, $0 < \alpha < 1$, is the ratio of the excess mean return to a benchmark index over its CVaR_α value. It is given as follows:

$$\text{STARR}_\alpha(r_X) = \frac{E_p(r_X) - E(r_M)}{\text{CVaR}_\alpha(r_X - r_M)}.$$

Note that STARR_α is the special case of the $\text{Rachev}_{\alpha,\beta}$ ratio in which $\text{Rachev}_{1,\,0.97}(r_X) = \text{STARR}_{0.97}(r_X)$. Also, STARR_α is a coherent ratio (Rachev et al. 2008).

The optimization model for STARR_α is stated as follows:

$$\begin{aligned}
\text{STARR}_\alpha(\ell_x) \qquad \max \quad & -E_p(\ell)\tilde{x} + \gamma E(\ell_M) \\
\text{subject to:}\ \ & \tilde{\tau}_1 + \frac{1}{1-\alpha} p^t \tilde{o} \le 1 \\
& \tilde{o} + \tilde{\tau}_1 e - \hat{B}\tilde{x} + \gamma \ell_M \ge 0 \\
& e^t \tilde{x} = \gamma \\
& \tilde{o} \in \mathbb{R}^T_+, \quad \tilde{\tau}_1 \in \mathbb{R}, \quad \tilde{x} \in \mathbb{R}^n_+,
\end{aligned}$$

where $\gamma > 0$ is a homogenization variable, and $E_p(\ell) = (\sum_{j=1}^T \ell_{ij} p_j;\ i = 1, \ldots, n)^t$ is the vector of expected losses. In our empirical analysis, $\ell_M = \hat{\ell}_z$, where z is the BM of the entire S&P 500 stocks, $\hat{B}[\hat{\ell}_{ij}]_{\hat{T}\times m}$ is the loss matrix of z, and $E(\hat{\ell}_z) = \frac{1}{T}(\frac{1}{m}\sum_{t=1}^T \sum_{i=1}^m \ell_{ti} z_i)$. Due to the computational benefits of the $\text{STARR}_\alpha(\ell_x)$ model, we propose to include it in our empirical analysis.

4 Robust optimization

In the presence of two unknown distributions in the Omega-CVaR_α model, namely p and q, an optimal solution from it can suffer severely from errors in the distribution estimation which misleads an investor about the portfolio outcomes. To account for the risk due to uncertainty in the underlying distributions, we present a worst case analysis of the Omega-CVaR$_\alpha$ model.

The randomness in modeling can be handled in two ways: either by robust optimization (RO) such as in Ben-Tal and Nemirovski (2000); Beyer and Sendhoff (2007) or by distribution dependent optimization (Ruszczynski and Shapiro 2003; Birge and Louveaux 2011). The latter framework requires true knowledge of the underlying probability distribution which is not usually available for future stock prices. RO overcomes this shortcoming by relaxing the assumption on the underlying distribution and generates a solution even if only partial or minuscule information is available on the probability distribution. RO is a young research field which deals with the problem of data uncertainty by guaranteeing feasibility and optimality of a solution for the worst case of the parameters in an optimization problem. RO is useful in many situations, for instance, when the model involves uncertainty in probability distribution or it requires parameters estimation and hence carries estimation errors. RO is a set-based approach wherein randomness in the parameters is assumed to be in the form of an uncertainty set; different uncertainty sets lead to different RO models. An investor constructs a solution that is feasible for any realization of the uncertainty in a given set. A significant advantage of RO is that the resulting optimization problems continue to remain computationally tractable for many popular classes of the uncertainty sets.

RO technique has been widely applied to problems in engineering, supply chain, hub location problems, and portfolio optimization (see, Bertsimas et al. 2011). Ghaoui et al. (2003) optimize the worst case of VaR$_\alpha$ when only partial information of the distribution are known, i.e. the bounds on the mean and covariance matrix are available. They show that the resulting problem is a semi-definite program. Zhu and Fukushima (2009) propose the worst case analysis of CVaR$_\alpha$ under mixed, box, and ellipsoidal uncertainty sets and show that the models remain computationally tractable. In the spirit of Zhu and Fukushima (2009), Kapos et al. (2014) optimize the worst case of Omega ratio under same uncertainty sets and perform empirical analysis for artificial as well as real market data for three asset classes, S&P 500 index, US government bonds, and gold asset. Moon and Yao (2011) study the robust optimization of mean absolute deviation model by considering a type of uncertainty following Bertsimas and Sim (2004) in estimating the expected returns based on real market data of 100 stocks randomly selected from NYSE, NASDAQ, and AMEX.

For the first time, Fliege and Werner (2014) consider the robust counterpart for a multi-objective programming problem and examine the relationship between the robust efficient frontier with the original nominal efficient frontier. They then investigate the approach to derive the robust mean variance optimization (Markowitz 1952) model. Chen and Kwon (2012) obtain a robust portfolio in application of index tracking using the approach of Bertsimas and Sim (2004). Experimental analysis is performed for tracking the S&P 500 index and claimed that the tracking error is lesser in the robust model than to its nominal model. Ben-Tal et al. (2009) provide more details on the specific formulations and tractability issues in RO. The survey paper of Bertsimas et al. (2011) highlights the prominent theoretical results and applications of robust theory in various areas including portfolio optimization.

We ascertain the robustness in portfolio outcomes from the proposed model. The worst case optimization is considered both in maximizing the Omega ratio as well as in minimizing the CVaR$_\alpha$ measure (for a fixed α) in Omega-CVaR$_\alpha$ model. Hence, the resulting portfolio becomes robust with respect to the threshold point as well as the distribution considered in the maximization of the Omega ratio. We formulate three robust variants for Omega-CVaR$_\alpha$ model corresponding to the mixed, box, and ellipsoidal uncertainty sets, respectively, for discrete distribution of ℓ_x and $\hat{\ell}_z$. An uncertainty set is considered to be a selection of a range of distributions for which the underlying distribution possibly belongs to. We follow the worst case descriptions of Kapos et al. (2014) for the Omega ratio and Zhu and Fukushima (2009) for the CVaR$_\alpha$ measure for the three mentioned uncertainty sets.

4.1 Worst case Omega-CVaR$_\alpha$ model

For general uncertainty sets $\mathcal{P}$ and $\mathcal{Q}$ of the unknown distributions of ℓ_x and $\hat{\ell}_z$, respectively, the worst case values of $\Omega_{L(\alpha)}(\ell_x)$ and CVaR$_\alpha(\hat{\ell}_z)$ are defined as follows:

$$\min_{p\in\mathcal{P}} \frac{E_p(L-\ell_x)^+}{E_p(\ell_x-L)^+} \quad \text{and} \quad \max_{q\in\mathcal{Q}} \min_{\tau} \; \tau + \frac{1}{1-\alpha} E_q(\hat{\ell}_z-\tau)^+,$$

and their worst case portfolio optimization problems are given, respectively, as follows:

$$\text{WOmega} \qquad \max_{x} \min_{p\in\mathcal{P}} \frac{E_p(L-\ell_x)^+}{E_p(\ell_x-L)^+}$$

$$\text{WCVaR}_\alpha \qquad \min_{z,\tau} \max_{q\in\mathcal{Q}} \; \tau + \frac{1}{1-\alpha} E_q(\hat{\ell}_z-\tau)^+.$$

We present the worst case of the Omega-CVaR$_\alpha$ model under mixed, box, and ellipsoidal uncertainty sets for discrete case of loss distributions resulting in robust Omega-CVaR$_\alpha$ models that are tractable. Indeed, the assumption of discrete distribution is reasonable and also applies in practice to the prices of the financial instruments are observed at discrete time points (daily, weakly, monthly, etc).

The worst case portfolio optimization problem for Omega-CVaR$_\alpha$ is described as follows:

$$\text{WOmega-CVaR}_\alpha \qquad \max_{x} \min_{p\in\mathcal{P}} \frac{E_p(L(\alpha)-\ell_x)^+}{E_p(\ell_x-L(\alpha))^+}, \text{ where}$$

$$L(\alpha) \le \min_{z,\tau} \max_{q\in\mathcal{Q}} \; \tau + \frac{1}{1-\alpha} E_q(\hat{\ell}_z-\tau)^+. \qquad (10)$$

For fix uncertainty sets $\mathcal{P}$ and $\mathcal{Q}$, we solve the inner minimization programming problem (10) associated with $L(\alpha)$ in WOmega-CVaR$_\alpha$ model by taking its dual under $\mathcal{Q}$.

4.2 Mixed uncertainty set

In the scope of financial markets, the three possible likelihood scenarios of stock returns are bearish (when price continuously fall), bullish (when price continuously rise), and nominal (when the changes in price is not deep). The resulting worst case optimal solution does not hurt investors if any of the scenario (specifically the bearish one) or a combination of the these scenarios occurs in the future.

The mixed uncertainty set is the set of convex combination (or mixed) of a finite number of likelihood functions to estimate the underlying distribution. It is defined as follows:

$$\mathcal{P}_M = \left\{ p = \sum_{k=1}^{s} w_k p^k : w \in \Lambda \right\},$$
$$\Lambda = \left\{ w = (w_1, \ldots, w_s) : \sum_{k=1}^{s} w_k = 1,\ w_k \geq 0,\ k = 1, \ldots, s \right\}, \qquad (11)$$

where p^k is the kth likelihood density function of portfolio loss ℓ_x. The mixed uncertainty set $\mathcal{Q}$ for q is similarly defined to derive the worst case of $P_{\text{CVaR}_\alpha(\hat{\ell}_z)}$ model (see, Appendix B).

For a continuous density function p with the uncertainty set described in (11), we have the following equivalence condition:

$$\max_{\tilde{x}} \min_{w \in \Lambda} \sum_{k=1}^{s} w_k G_1^k \Longleftrightarrow \max_{\tilde{x}} \min_{k=1,\ldots,s} G_1^k, \quad \text{and}$$
$$\sum_{k=1}^{s} w_k G_2^k = 1 \ \forall\, w \in \Lambda \Longleftrightarrow G_2^k = 1,\ k = 1, \ldots, s,$$

where $G_1^k = \int_\ell (L\gamma - \ell_{\tilde{x}})^+ p^k(\ell)\, \mathrm{d}\ell,\ G_2^k = \int_\ell (\ell_{\tilde{x}} - L\gamma)^+ p^k(\ell)\, \mathrm{d}\ell, \quad \gamma > 0$ (see, Appendix A).

Thus, optimizing WOmega becomes $\max_{\tilde{x}} \{\theta : \theta \leq G_1^k,\ G_2^k = 1, \quad k = 1, \ldots, s\}$ for the continuous likelihood density function p. For the derivation of WCVaR_α for the continuous case under the mixed uncertainty set, one can refer to Zhu and Fukushima (2009).

Just like P_2 is approximated from P_1 using a sampling technique, analogously we approximate G_1^k and G_2^k with T^k number of scenarios of ℓ_x with the kth likelihood probability vector $p^k = ((p_1^k, \ldots, p_{T^k}^k)^t;\ (p^k)^t e = 1,\ p_i^k \geq 0, \forall\, i = 1, \ldots, T^k), k = 1, \ldots, s$, when $L(\alpha) > \min_{x^t e = 1,\, x \in \mathbb{R}_+^n} E_{p^k}(\ell_x^k),\ \forall\, k = 1, \ldots, s$ (Mausser et al. 2006). Then the robust optimization model for the worst case of Omega-CVaR_α model is an LPP under $\mathcal{P}_M$ given by

$$P_5 \qquad \max \quad \theta$$

$$\text{subject to:} \quad (p^k)^t \widetilde{u}^k - \theta \geq 0 \qquad k = 1, \ldots, s \tag{12}$$

$$(p^k)^t \widetilde{d}^k = 1, \; \widetilde{u}^k, \widetilde{d}^k \in \mathbb{R}_+^{T^k} \qquad k = 1, \ldots, s \tag{13}$$

$$\widetilde{x}^t e = \gamma, \quad \widetilde{x} \in \mathbb{R}_+^n \tag{14}$$

$$B^k \widetilde{x} + \widetilde{u}^k - \widetilde{d}^k = L(\alpha)\gamma e, \qquad k = 1, \ldots, s$$

$$L(\alpha) \leq \vartheta, \tag{15}$$

where $\gamma > 0$ is a homogenization variable and for every variable ϕ, $\widetilde{\phi} = \gamma\phi$, $B^k = [\ell_{ij}^k]_{T^k \times n}$ is loss matrix of portfolio x corresponding to the kth likelihood probability density p^k. And ϑ is the optimal value of the following problem:

$$\max \quad \vartheta$$

$$\text{subject to:} \; \vartheta e - \sum_{k=1}^{\hat{s}} (\hat{B}^k)^t h^k \leq 0$$

$$v^t e = \gamma, \quad -v^t e + \sum_{k=1}^{\hat{s}} (h^k)^t e = 0$$

$$-\frac{v_k}{1-\alpha} q^k + h^k \leq 0, \quad h^k \in \mathbb{R}_+^{\hat{T}^k}, \; v \in \mathbb{R}_+^{\hat{s}}, \qquad k = 1, \ldots, \hat{s}.$$

The above problem is dual $P_{\text{DMCVaR}_\alpha(\hat{\ell}_z)}$ of $P_{\text{MCVaR}_\alpha(\hat{\ell}_z)}$ model, a worst case model of $P_{\text{CVaR}_\alpha}(\hat{\ell}_z)$ under mixed uncertainty set (Zhu and Fukushima 2009), see Appendix B. Since $P_{\text{MCVaR}_\alpha(\hat{\ell}_z)}$ is an LPP, due to the strong duality, the duality gap is zero in $P_{\text{DMCVaR}_\alpha(\hat{\ell}_z)}$.

Proposition 1 *For the discrete mixed uncertainty sets $\mathcal{P}_M$ and $\mathcal{Q}_M$, such a choice of $L(\alpha)$ in constraint (15) ensures that P_5 model is an LPP.*

Proof Here, it is sufficient to show that $L(\alpha) > \min_{x^t e=1,\, x\in\mathbb{R}_+^n} E_{p^k}(\ell_x^k)$, $\forall k = 1, \ldots, s$, in P_5. Since $L(\alpha)$ is the worst case of $P_{\text{CVaR}_\alpha(\hat{\ell}_z)}$ model under uncertainty set $\mathcal{Q}_M$ it implies $L(\alpha) \geq \text{CVaR}_\alpha^{k*}(\hat{\ell}_z)$, $\forall k = 1, \ldots, \hat{s}$, where $\text{CVaR}_\alpha^{k*}(\hat{\ell}_z)$ is the optimal value of $P_{\text{MCVaR}_\alpha(\hat{\ell}_z)}$ (and hence of $P_{\text{DMCVaR}_\alpha(\hat{\ell}_z)}$) at the kth likelihood density function q^k, $k=1, \ldots, \hat{s}$. Consequently, $L(\alpha) > \min_{x^t e=1,\, x\in\mathbb{R}_+^n} E_{p^k}(\ell_x^k)$, $\forall k=1, \ldots, s$. □

The worst case of Omega-CVaR_α model under the mixed uncertainty set is thus proposed as the following LPP:

$$\text{MOmega}-\text{CVaR}_\alpha \quad \max \quad \theta$$

$$\text{subject to: constraints} \quad (12)-(14)$$

$$B^k \widetilde{x} + \widetilde{u}^k - \widetilde{d}^k = \widetilde{L(\alpha)} e, \qquad k = 1, \ldots, s$$

$$\widetilde{L(\alpha)} \leq \widetilde{\vartheta}$$

$$\widetilde{\vartheta} e - \sum_{k=1}^{\hat{s}} (\hat{B}^k)^t \widetilde{h}^k \leq 0$$
$$\widetilde{v}^t e = \gamma$$
$$-\widetilde{v}^t e + \sum_{k=1}^{\hat{s}} (\widetilde{h}^k)^t e = 0$$
$$-\frac{\widetilde{v}_k}{1-\alpha} q^k + \widetilde{h}^k \leq 0 \qquad k = 1, \ldots, \hat{s}$$
$$\widetilde{v} \in \mathbb{R}_+^{\hat{s}}, \ \widetilde{h}^k \in \mathbb{R}_+^{\hat{T^k}} \qquad k = 1, \ldots, \hat{s},$$

where $\widetilde{v}_k, \ k = 1, \ldots, \hat{s}$, is the kth component of vector $\widetilde{v}$.

The MOmega-CVaR$_\alpha$ model optimizes the worst case of $P_{\text{CVaR}_\alpha(\hat{\ell}_z)}$ model over the mixed uncertainty set and subsequently optimizes the worst case of the Omega ratio again over mixed uncertainty set for such a robust outcome of threshold value. In this manner, we are able to impact robustness to the Omega ratio as well as its threshold value.

4.3 Box uncertainty set

In robust optimization, one is required to make a trade-off between 'full' robustness and the length of the underline uncertainty set. A box uncertainty set that contains a vast range of unknown parameters is the most robust choice and thus most suited to pessimistic investors. The simplest case of box uncertainty is when the only information about an unknown parameter Υ is that $|| \Upsilon ||_\infty \leq 1$. If more information becomes readily available, e.g. bounds on the moments, or if the probability distribution is symmetric or unimodal, smaller box uncertainty sets become available. Other than simplicity in defining the box uncertainty sets, a large number of robust optimization problems result in computationally tractable problems under box uncertainty sets. The box uncertainty set for the distribution of portfolio loss ℓ_x is

$$\mathcal{P}_B = \{p = p^0 + \pi; \ \pi^t e = 0, \ \underline{\pi} \leq \pi \leq \bar{\pi}\},$$

where p^0 is the most likely distribution of ℓ_x and $\underline{\pi}, \bar{\pi} \in \mathbb{R}^T$ are constant parameter vectors to decide size/width of the box uncertainty set; larger width indicates higher robustness. In other words, the bounds of the box uncertainty indicate attitude of an investor towards risk.

The worst case of Omega-CVaR$_\alpha$ model under the box uncertainty $\mathcal{P}_B$ is described as follows:

$$P_6 \qquad \max_{\widetilde{x}, \widetilde{u}, \widetilde{d}, L} \quad \theta$$
$$\text{subject to: } \theta \leq \min_{p = p^0 + \pi; \ \pi^t e = 0, \ \underline{\pi} \leq \pi \leq \bar{\pi}} p^t \widetilde{u} \tag{16}$$

$$\widetilde{x}^t e = \gamma,\ \widetilde{x} \in \mathbb{R}^n_+,\ \ \widetilde{u},\ \widetilde{d} \in \mathbb{R}^T_+ \tag{17}$$

$$p^t \widetilde{d} = 1 \tag{18}$$

$$B\widetilde{x} + \widetilde{u} - \widetilde{d} = L(\alpha)\gamma e \tag{19}$$

$$L(\alpha) \leq \vartheta, \tag{20}$$

where $B = [\ell_{ij}]_{T\times n}$ is the loss matrix of portfolio x. And ϑ is the optimal value of the following problem:

$$\begin{aligned} &\max \quad \vartheta \\ &\text{subject to: } \hat{B}^t v - \vartheta e - \lambda_5 = 0 \\ &\qquad e^t v = \gamma, \quad -e^t \lambda_6 = 0 \\ &\qquad \frac{q^0}{1-\alpha} + \lambda_6 - v - \lambda_7 = 0 \\ &\qquad \frac{\bar{\hat{\pi}}}{1-\alpha} - \lambda_6 - \lambda_8 = 0 \\ &\qquad \frac{\underline{\hat{\pi}}}{1-\alpha} - \lambda_6 - \lambda_9 = 0 \\ &\vartheta \in \mathbb{R}, \quad v, \lambda_7, \lambda_8, \lambda_9 \in \mathbb{R}^{\hat{T}}_+, \quad \lambda_5 \in \mathbb{R}^m_+, \lambda_6 \in \mathbb{R}^T, \end{aligned}$$

which is the dual $P_{\text{DBCVaR}_\alpha(\hat{\ell}_z)}$ of $P_{\text{BCVaR}_\alpha(\hat{\ell}_z)}$ model, the worst case model of $P_{\text{BCVaR}_\alpha(\hat{\ell}_z)}$ under the box uncertainty set (Zhu and Fukushima 2009), see Appendix C. Again $P_{\text{BCVaR}_\alpha(\hat{\ell}_z)}$ being an LPP, due to the strong duality, the duality gap is zero in $P_{\text{DBCVaR}_\alpha(\hat{\ell}_z)}$.

Proposition 2 *For the discrete box uncertainty sets $\mathcal{P}_B$ and $\mathcal{Q}_B$, such a choice of $L(\alpha)$ in constraint (20) ensures that P_6 is an LPP.*

Proof Problem P_6 involves two inner optimization problem (constraint (16) and (20)). Since the inner maximization problem in constraint (20) is an LPP, it remains to show that dual of the inner minimization problem in constraint (16) is also an LPP with $L(\alpha) > \min_{x^t e=1,\ x\in\mathbb{R}^n_+} E_{\mathcal{P}_B}(\ell_x)$.

The inner minimization problem in constraint (16) is an LPP and thus it can be solved by its corresponding dual maximization problem. Moreover, for $L(\alpha) = \text{CVaR}^{B*}_\alpha(\hat{\ell}_z)$, where $\text{CVaR}^{B*}_\alpha(\hat{\ell}_z)$ is the optimal value of $P_{\text{BCVaR}_\alpha(\hat{\ell}_z)}$ (and hence for $P_{\text{DBCVaR}_\alpha(\hat{\ell}_z)}$), we have, $L(\alpha) > \min\limits_{x^t e=1,\ x\in\mathbb{R}^n_+} E_{\mathcal{P}_B}(\ell_x)$. □

The dual of the inner linear minimization problem in constraint (16), for fixed values of $\widetilde{u}$, $\widetilde{d}$, and γ, or in other words, the dual of the following problem P_7

$$P_7 \qquad \begin{aligned} &\min\ \widetilde{u}^t(p^0 + \pi) \\ &\text{subject to: } \widetilde{d}^t(p^0 + \pi) = 1, \quad e^t\pi = 0, \quad \underline{\pi} \leq \pi \leq \bar{\pi} \end{aligned}$$

is an LPP given by

$$P_8 \qquad \max\ \widetilde{u}^t p^0 + \lambda_2^t \underline{\pi} - \lambda_3^t \bar{\pi} - \lambda_4 \widetilde{d}^t p^0 + \lambda_4$$

$$\text{subject to:}\quad \widetilde{u} - \lambda_1 e - \lambda_2 + \lambda_3 - \lambda_4 \widetilde{d} = 0, \quad \lambda_1, \lambda_4 \in \mathbb{R}, \quad \lambda_2, \lambda_3 \in \mathbb{R}_+^T. \tag{21}$$

Using P_8 in constraint (16), we have the following simpler LPP to solve

$$\text{BOmega}-\text{CVaR}_\alpha \qquad \max_{\widetilde{x},\widetilde{u},\widetilde{d},\gamma}\ \theta$$

$$\begin{aligned}
\text{subject to:}\ & \widetilde{u}^t p^0 + \lambda_2^t \underline{\pi} - \lambda_3^t \bar{\pi} - \lambda_4 \widetilde{d}^t p^0 + \lambda_4 \geq \theta \\
& \text{Constraints } (17) - (18) \text{ and } (21) \\
& B\widetilde{x} + \widetilde{u} - \widetilde{d} = \widetilde{L(\alpha)} e \\
& \widetilde{L(\alpha)} \leq \widetilde{\vartheta} \\
& \hat{B}^t \widetilde{v} - \widetilde{\vartheta} e - \widetilde{\lambda}_5 = 0 \\
& e^t \widetilde{v} = \gamma, \quad e^t \widetilde{\lambda}_6 = 0 \\
& \frac{\gamma}{1-\alpha} q^0 + \widetilde{\lambda}_6 - \widetilde{v} - \widetilde{\lambda}_7 = 0 \\
& \frac{\gamma}{1-\alpha} \hat{\bar{\pi}} - \widetilde{\lambda}_6 - \widetilde{\lambda}_8 = 0 \\
& \frac{\gamma}{1-\alpha} \underline{\hat{\pi}} - \widetilde{\lambda}_6 - \widetilde{\lambda}_9 = 0 \\
& \widetilde{\vartheta} \in \mathbb{R}, \quad \widetilde{v}, \widetilde{\lambda}_7, \widetilde{\lambda}_8, \widetilde{\lambda}_9 \in \mathbb{R}_+^{\hat{T}}, \quad \widetilde{\lambda}_5 \in \mathbb{R}_+^m, \quad \widetilde{\lambda}_6 \in \mathbb{R}^{\hat{T}},
\end{aligned}$$

where $\hat{B} = [\hat{\ell}_{ij}]_{\hat{T}\times m}$ is the loss matrix of portfolio z.

4.4 Ellipsoidal uncertainty set

When the box uncertainty is found to be too pessimistic for a given problem, a wise choice of the uncertainty set afterwards is an ellipsoidal uncertainty set. The ellipsoidal uncertainty set of an unknown parameter Υ is given as $||\Upsilon||_2 \leq a_0,\ a_0 \geq 0$ with $||\cdot||_2$ being the L^2-norm. The ellipsoidal uncertainty set is the intersection of a finite number of ellipsoid sets resulting in a finite number of convex quadratic inequalities and thus robust optimization model under which it becomes a second order cone program (SOCP). In many practical instances, the applicability of the ellipsoidal uncertainty set has statistical reasons (see, Ben-Tal and Nemirovski 1999).

The ellipsoidal uncertainty set is described as follows:

$$\mathcal{P}_E = \{p = p^0 + A\pi;\ e^t A\pi = 0,\ p^0 + A\pi \geq 0,\ ||\pi||_2 \leq \omega\},$$

where $A \in \mathbb{R}^{T\times T}$ is the scaling matrix of the ellipsoidal uncertainty set, $\pi \in \mathbb{R}^T$, and $p^0 \in \mathbb{R}^T$ is a nominal distribution and the center of the ellipsoid, $\omega \in \mathbb{R}$ decides the size of the ellipsoidal uncertainty set and its choice depends upon the behavior of an investor towards the robustness. Specifically, we take $\omega = 1$ in our present study.

The Omega-CVaR$_\alpha$ under the ellipsoidal uncertainty set is as follows:

$$
\begin{aligned}
P_9 \qquad & \max_{\widetilde{x},\widetilde{u},\widetilde{d},\gamma} \quad \theta \\
& \text{subject to:} \quad \text{constraints (17)} - (19) \\
& \theta \le \min_{p=p^0+A\pi;\ e^tA\pi=0,\ p^0+A\pi\ge 0,\ ||\pi||_2\le 1} p^t\widetilde{u} \qquad (22) \\
& L(\alpha) \le \vartheta, \qquad (23)
\end{aligned}
$$

and ϑ is the optimal value of the following problem:

$$
\begin{aligned}
P_{10} \qquad & \max \quad \vartheta \\
& \text{subject to:} \quad \hat{B}^t v - \vartheta\, e - \lambda_6 = 0 \\
& v^t e = 1, \qquad e^t \hat{A}\lambda_7 = 0 \\
& \frac{q^0}{1-\alpha} + \hat{A}\lambda_7 - v - \lambda_8 = 0 \\
& \frac{q^0}{1-\alpha} + \hat{A}\lambda_7 - \lambda_9 = 0 \\
& ||\lambda_7||_2 \le \frac{1}{1-\alpha} \\
& \vartheta \in \mathbb{R}, \quad \lambda_7, \lambda_{10} \in \mathbb{R}^{\hat{T}}, \quad v, \lambda_8, \lambda_9 \in \mathbb{R}_+^{\hat{T}}, \quad \lambda_6 \in \mathbb{R}_+^m.
\end{aligned}
$$

P_{10} is the dual $P_{\text{DECVaR}_\alpha}(\hat{\ell}_z)$ of $P_{\text{ECVaR}_\alpha}(\hat{\ell}_z)$ model, the worst case model of $P_{\text{CVaR}_\alpha}(\hat{\ell}_z)$ under the ellipsoidal uncertainty set $\mathcal{Q}_E$ described by the parameters $\hat{A} \in \mathbb{R}^{\hat{T}\times\hat{T}}$ and $q^0 \in \mathbb{R}^{\hat{T}}$, as explained in Appendix D.

Remark 1 The worst case of $P_{\text{CVaR}_\alpha}(\hat{\ell}_z)$ model under ellipsoidal uncertainty set is an SCOP, namely $P_{\text{ECVaR}_\alpha}(\hat{\ell}_z)$ (see, Appendix D). Under some mild condition (Lobo et al. 1998; Calafiore and Ghaoui 2014), the duality gap is zero in SOCP. Consequently, we can find ϑ in constraint (23) by solving problem P_{10}. A similar argument can be given for the constraint (22) in problem P_9.

Proposition 3 *For the discrete ellipsoidal uncertainty sets $\mathcal{P}_E$ and $\mathcal{Q}_E$, such a choice of $L(\alpha)$ in constraint* (23) *ensures that P_9 is an SOCP.*

Proof It follows on similar lines as proofs of Propositions 1 and 2, and in light of Remark 1. □

Now, the inner minimization problem in constraint (22) is resolved by taking dual of the following problem P_{11} (Calafiore and Ghaoui 2014) (see, Appendix D, problem P_{12}).

$$
\begin{aligned}
P_{11} \quad & \min \quad \widetilde{u}^t p^0 + \widetilde{u}^t A\pi \\
& \text{subject to}: \quad \widetilde{d}^t p^0 + \widetilde{d}^t A\pi = 1, \quad e^t A\pi = 0, \quad p^0 + A\pi \ge 0, \quad ||\pi||_2 \le 1.
\end{aligned}
$$

Therefore, the worst case of the Omega-CVaR$_\alpha$ model under the ellipsoidal uncertainty set is the following SOCP:

$$
\begin{aligned}
\text{EOmega}-\text{CVaR}_\alpha \qquad & \max \quad \theta \\
& \text{subject to: } \widetilde{u}^t p^0 - \lambda_1 \widetilde{d}^t p^0 + \lambda_1 - \lambda_2^t p^0 - \lambda_3 \geq \theta \\
& \quad \text{constraints (17), (18), and (26)} - \text{(28)} \\
& B\widetilde{x} + \widetilde{u} - \widetilde{d} = \widetilde{L(\alpha)}e, \quad \widetilde{u},\ \widetilde{d} \in \mathbb{R}_+^T \\
& \widetilde{L(\alpha)} \leq \widetilde{\vartheta} \\
& \hat{B}^t \widetilde{v} - \widetilde{\vartheta}\, e - \widetilde{\lambda}_6 = 0 \\
& \widetilde{v}^t e = \gamma \\
& e^t \hat{A} \widetilde{\lambda}_7 = 0 \\
& \frac{\gamma}{1-\alpha} q^0 + \hat{A}\widetilde{\lambda}_7 - \widetilde{v} - \widetilde{\lambda}_8 = 0 \\
& \frac{\gamma}{1-\alpha} q^0 + \hat{A}\widetilde{\lambda}_7 - \widetilde{\lambda}_9 = 0 \\
& ||\widetilde{\lambda}_7||_2 \leq \frac{\gamma}{1-\alpha} \\
& \widetilde{\vartheta} \in \mathbb{R}, \quad \widetilde{\lambda}_6 \in \mathbb{R}_+^m, \quad \widetilde{\lambda}_7 \in \mathbb{R}^{\hat{T}}, \ \widetilde{v}, \widetilde{\lambda}_8, \widetilde{\lambda}_9 \in \mathbb{R}_+^{\hat{T}},
\end{aligned}
$$

where constraints (26)–(28) are described in Appendix D.

Zhu and Fukushima (2009) address the issues in specifying the input parameters (p^k, $k = 1, \ldots, s$) in mixed, (p^0, $\underline{\pi}$ and $\overline{\pi}$) in box, and (p^0 and matrix A) in the ellipsoidal uncertainty sets. We apply the same type of uncertainty sets to bring in robustness in both the Omega ratio and the CVaR$^*_\alpha(\hat{\ell}_z)$ measures. However, a further development is to consider different possible permutations between the three sets in two measures to derive different worst cases of the Omega-CVaR$_\alpha$ model in the robust portfolio optimization framework. This is reasonable, especially when the benchmark market z is different to the market of the actual portfolio x construction in maximizing the Omega ratio. However, if both z and x are drawn from the same market then our approach of considering the same uncertainty set sounds appropriate.

5 Empirical applications and models comparison

In this section, we show the financial benefits of the Omega-CVaR$_\alpha$ model and its worst case robust optimization model using mixed uncertainty sets in a comprehensive empirical study based on a real-world data set.

5.1 Sample data and methodology

We choose the US based index S&P 500 as the benchmark in our analysis and consider a risk-averse investor who decides to invest in a portfolio of 50 constituents of the S&P 500 with the highest market capitalization to mimic the developments of the

S&P 500. An investor uses the CVaR_α of the S&P 500 at a certain confidence level α as the threshold. We take the whole S&P 500 to calculate the threshold value L in the Omega ratio by solving $P_{\mathrm{CVaR}_\alpha(\hat{\ell}_z)}$ model, and on the other hand take a subset of 50 stocks from S&P 500 as the possible investment universe for which we maximize the Omega ratio. The choice of the 50 stocks follows as a result of several studies (Newbould and Poon 1996; Shawky and Smith 2005) showing that this quantity is a reasonable number of stocks for an investor to achieve a well-diversified portfolio and to keep transaction and management costs low. Following the notation in the body of our paper, the threshold value $L(\alpha) = \mathrm{CVaR}^*_\alpha(\hat{\ell}_z)$ is estimated from the portfolio z comprising a weight vector of at most $m = 500$ stocks[9] to display the expected shortfall of the S&P 500 portfolio. The portfolio x is the vector of the portfolio weights of $n = 50$ stocks.

We collect time series of monthly adjusted stock prices and market capitalization data from Thomson Reuters Datastream for all constituents of the S&P 500 as on June 2015. We use the 3 months US Treasury Bill for the risk free rate, which we also obtain from Thomson Reuters Datastream on a monthly basis. The period of the historical data ranges from January 1990 to June 2015. For each stock $j = 1, \ldots, m$, we calculate monthly, discrete returns during the entire period if stock prices are available. We follow the rolling window approach with an estimation window of 240 months starting in Jan 1990 and one month out-of-sample period. Therefore, we obtain 66 one month out-of-sample periods during the period between Jan 2010 and June 2015. We delete all stocks whose stock price data is missing for at least 120 months within the estimation period, and we thereby are left with S&P 500 portfolios including at least 412 stocks and at most 450 stocks (an average of 435 stock in each in-sample period) to estimate the threshold value $L(\alpha) = \mathrm{CVaR}^*_\alpha(\hat{\ell}_z)$. We compare the performance of 412 to 450 stocks S&P 500 portfolios with the real S&P 500 (market capitalization value weighted) index performance during the entire period and find a correlation of 98.9 %, which is highly significant. Therefore, dropping at most 88 stocks, because of a lack of sufficient data availability does not change the performance of the slightly reduced portfolios compared to the real S&P 500 portfolio and supports the fact that those portfolios are an appropriate proxy for the S&P 500 index.

We compare the performance of portfolios resulting from four different portfolio optimization models and two index based portfolios in our analysis. The classical $P_{\mathrm{CVaR}_\alpha(\ell_x)}$ model (when $\hat{\ell}_z$ is same as ℓ_x), the Omega-CVaR_α model, the MOmega-CVaR_α, the $\mathrm{STARR}_\alpha(\ell_x)$ model, the naïve $1/m$ portfolio strategy, and the market capitalization value weighted portfolio (MCWP), using all available m stocks of the S&P 500,[10] the latter three act as benchmark strategies. The weight of the ith stock in the $1/m$ portfolio and MCWP portfolio is calculated as $w_i = \frac{1}{m},\ i = 1, \ldots, m,$ and $w_i = \frac{\mathrm{MC}_i}{\sum_{i=1}^m \mathrm{MC}_i},\ i = 1, \ldots, m,$ respectively, where MC_i denotes the market capitalization of the ith stock. To illustrate the performance of portfolios of the models stated in

[9] We take z as all those stocks listed on the S&P 500 whose monthly return data is available for more than 10 years during each in-sample period.

[10] The naïve $1/m$ and the MCWP portfolios are continuously updated as soon as any stock on the S&P 500 index listed as of June 2015 declares its initial public offering (IPO).

this paper, we present the results of the Omega-CVaR$_\alpha$ model and the MOmega-CVaR$_\alpha$ model. We control for the sensitivity of the results according to different levels of loss aversion by applying optimization for four values of $\alpha = 0.97, 0.95, 0.93,$ and 0.90. All models except the naïve $1/m$ portfolio strategy and the MCWP are based on the set of 50 stocks and the threshold point $L(\alpha)$ used in proposed models (nominal as well as robust) is calculated from the S&P 500 stocks on the basis of availability of the data for at least 10 years.

Whereas we compute the optimal solutions of the classical $P_{\text{CVaR}_\alpha(\hat{\ell}_x)}$ model, the STARR$_\alpha(\ell_x)$ model, and the Omega-CVaR$_\alpha$ model by taking $\hat{T} = T = 240$ simulations for random variables in ℓ_x and $\hat{\ell}_z$ with equally likely probabilities $q_j = p_j = \frac{1}{240}, \quad j = 1, \ldots, 240$, the procedure to solve the MOmega-CVaR$_\alpha$ model requires additional explanation. We consider the mixture of three likelihood functions in line with Zhu and Fukushima (2009) to optimize robust counterpart of P_{CVaR_α} model for any distribution. We divide each 240 months estimation period into three equal sizes, each of 80 months, as three phases of the market for both stock samples (the investment universe and the filtered S&P 500 sample). That is, for both the loss portfolios ℓ_x and $\hat{\ell}_z$ in MOmega-CVaR$_\alpha$, the length of the estimation windows $T^k = \hat{T}^k = 80$ months, $k = 1, 2, 3$, with likelihood mass function $p_i^k = q_i^k = \frac{1}{80}$, $i = 1, \ldots, 80$, $k = 1, 2, 3$. Due to possible differences between the returns series in three phases it is not reasonable to assume that the portfolio loss follows a uniform distribution in the entire period. Therefore, we take the mixed distribution (convex combination) of these three phases whereby we are able to find such a solution which remains feasible and optimal in the three phases each of 80 months. Hence, we consider the mixed set as the convex combination of three uniform distributions to optimize MOmega-CVaR$_\alpha$. Indeed, we assume the portfolio loss follows a uniform distribution in each of the three phases. We apply this methodology for each of the 66 in-sample periods.

5.2 Results

Table 1 reports upon descriptive statistics along with the Sharpe ratio, the modified Sharpe ratio, the VaR, and the CVaR of the out-of-sample return from the rolling window approach for $P_{\text{CVaR}_\alpha(\ell_x)}$, STARR$_\alpha(\ell_x)$, Omega-CVaR$_\alpha$, MOmega-CVaR$_\alpha$ models, the MCWP, and the naïve $1/m$ portfolio strategy for four values of $\alpha = 0.97, 0.95, 0.93, 0.90$ (over 66 out-of-sample monthly returns). The modified Sharpe ratio (MSR) is defined in Gregoriou and Gueyie (2003) as follows

$$\text{MSR}_\alpha := \frac{\mu - r_f}{\text{MVaR}_\alpha} \text{ with } \text{MVaR}_\alpha = \mu - Z\sigma, \text{ and}$$
$$Z = \left(z_\alpha + \frac{1}{6}(z_\alpha^2 - 1)S + \frac{1}{24}(z_\alpha^3 - 3z_\alpha)K - \frac{1}{36}(2z_\alpha^3 - 5z_\alpha)S^2 \right),$$

where μ, σ, and r_f are mean, standard deviation, and risk free returns, S and K are the skewness and kurtosis of the portfolio, and z_α is the αth quantile of the portfolio assuming the latter follows normal distribution. The Sharpe (1994) ratio SR is the ratio

Table 1 Descriptive statistics of the one month out-of-sample returns for four values of $\alpha = 0.97, 0.95, 0.93, 0.90$

	$P_{\mathrm{CVaR}_\alpha(\ell_x)}$	$\mathrm{STARR}_\alpha(\ell_x)$	Omega-CVaR$_\alpha$	MOmega-CVaR$_\alpha$	MCWP	$1/m$ strategy
Panel $\alpha = 0.97$						
Mean	0.0072	0.0144	0.0096	0.0101	0.0133	0.0130
St. Dev.	0.0331	0.0435	0.0331	0.0312	0.0366	0.0392
Median	0.0052	0.0175	0.0115	0.0102	0.0178	0.0184
Kurtosis	2.3586	2.6105	2.5520	2.6536	3.1650	3.4125
Skewness	−0.1114	−0.2499	−0.1741	−0.3273	−0.0897	−0.0403
Minimum	−0.0715	−0.0945	−0.0802	−0.0807	−0.0793	−0.0848
Maximum	0.0836	0.1009	0.0756	0.0658	0.1143	0.1289
SR	0.1882	0.3085	0.2586	0.2912	0.3353	0.3063
$\mathrm{MSR}_{0.97}$	0.1191	0.1872	0.1614	0.1782	0.2131	0.1925
$\mathrm{VaR}_{0.97}$	0.0592	0.0804	0.0552	0.0551	0.0645	0.0694
$\mathrm{CVaR}_{0.97}$	0.0654	0.0875	0.0678	0.0681	0.0720	0.0772
Panel $\alpha = 0.95$						
Mean	0.0081	0.0139	0.0097	0.0102	0.0133	0.0130
St. Dev.	0.0324	0.0433	0.0330	0.0311	0.0369	0.0395
Median	0.0085	0.0436	0.0118	0.0106	0.0178	0.0184
Kurtosis	2.5877	2.4717	2.635	2.6702	3.1650	3.4125
Skewness	−0.2653	−0.1795	−0.2418	−0.3317	−0.0897	−0.0403
Minimum	−0.0823	−0.0915	−0.0839	−0.0806	−0.0793	−0.0848
Maximum	0.0710	0.0974	0.0726	0.0661	0.1143	0.1289
SR	0.2198	0.2979	0.2639	0.2972	0.3353	0.3062
$\mathrm{MSR}_{0.95}$	0.1747	0.2406	0.2133	0.2409	0.2893	0.2621
$\mathrm{VaR}_{0.95}$	0.0472	0.0783	0.0433	0.0408	0.0517	0.0601
$\mathrm{CVaR}_{0.95}$	0.0631	0.0849	0.0621	0.0584	0.0654	0.0724
Panel $\alpha = 0.93$						
Mean	0.0089	0.0126	0.0098	0.0104	0.0133	0.0130
St. Dev.	0.0325	0.0445	0.0328	0.0310	0.0369	0.0395
Median	0.0128	0.0183	0.0124	0.0106	0.0178	0.0184
Kurtosis	2.6492	2.377	2.6459	2.7417	3.1650	3.4125
Skewness	−0.3135	−0.2152	−0.2702	−0.3509	−0.0897	−0.0403
Minimum	−0.0853	−0.0945	−0.0845	−0.0826	−0.0793	−0.0848
Maximum	0.0671	0.1026	0.0709	0.0679	0.1143	0.1289
SR	0.2446	0.2625	0.2695	0.3034	0.3353	0.3063
$\mathrm{MSR}_{0.93}$	0.2407	0.2537	0.2707	0.3095	0.3706	0.3361
$\mathrm{VaR}_{0.93}$	0.0339	0.0729	0.0340	0.0339	0.0513	0.0565
$\mathrm{CVaR}_{0.93}$	0.0563	0.0839	0.0552	0.0511	0.0614	0.0684

Table 1 continued

	$P_{\text{CVaR}_\alpha(\ell_x)}$	$\text{STARR}_\alpha(\ell_x)$	Omega-$CVaR_\alpha$	MOmega-$CVaR_\alpha$	MCWP	$1/m$ strategy
Panel $\alpha = 0.90$						
Mean	0.0097	0.0127	0.0099	0.0107	0.0133	0.0130
St. Dev.	0.0335	0.0453	0.0321	0.0310	0.0369	0.0395
Median	0.0118	0.0178	0.0128	0.0085	0.0178	0.0184
Kurtosis	2.6077	2.4696	2.4914	2.8992	3.1650	3.4125
Skewness	−0.2563	−0.2580	−0.2203	−0.3509	−0.0897	−0.0403
Minimum	−0.0837	−0.0972	−0.0770	−0.0842	−0.0793	−0.0848
Maximum	0.0736	0.1027	0.0690	0.0762	0.1142	0.1289
SR	0.2584	0.2588	0.2770	0.3134	0.3353	0.3063
$\text{MSR}_{0.90}$	0.3414	0.3266	0.3722	0.4442	0.5144	0.4658
$\text{VaR}_{0.90}$	0.0315	0.0785	0.0295	0.0205	0.0320	0.0320
$\text{CVaR}_{0.90}$	0.0503	0.088	0.0461	0.0454	0.0554	0.0604

of expected excess mean return from the risk free return to the standard deviation of the excess returns, σ:

$$SR := \frac{\mu - r_f}{\sigma}.$$

Note that a reasonable comparative analysis based on the modified Sharpe and the Sharpe ratios is limited to the case of $r_f > \mu$.

We find that the proposed Omega-CVaR_α model earns, on average (over $\alpha = 0.97, 0.95, 0.93, 0.90$), a monthly return of 0.0097, a monthly SR of 0.2652, and a monthly MSR_α of 0.2514 which are, respectively, higher than the average monthly return of 0.0085, monthly SR of 0.226, and a monthly MSR_α of 0.2164 for the existing $P_{\text{CVaR}_\alpha(\ell_x)}$ model. We also depict that the proposed model generates an average VaR_α and CVaR_α as 0.0405 and 0.0578, respectively, whereas $P_{\text{CVaR}_\alpha(\ell_x)}$ produces 0.0430 and 0.0588 and thus the proposed scheme Omega-CVaR_α improves $P_{\text{CVaR}_\alpha(\ell_x)}$ in all performance measures considered in this study.

Furthermore, the MOmega-CVaR_α model has an average monthly return of 0.0103, a monthly SR of 0.299, a monthly MSR_α of 0.2665, a monthly VaR_α of 0.0387, and a monthly CVaR_α of 0.0558. STARR_α appears to have higher average return of 0.0134 (whereas a monthly SR of 0.2819, a monthly MSR_α of 0.2520) which is associated with higher extreme events monthly VaR_α of 0.0775, and a monthly CVaR_α of 0.0860. Clearly MOmega-CVaR_α model shows an improvement over Omega-CVaR_α model and $\text{STARR}_\alpha(\ell_x)$ model in the sense of all performance measures (except in average monthly return). Due to the robust target value $\text{MCVaR}^*_\alpha(\hat{\ell}_z)$ and robust return distribution in the MOmega-CVaR_α model, the portfolio from it has the least values of losses in terms of VaR_α and CVaR_α while it also produces comparatively large returns.

The MCWP and the naïve $1/m$ portfolio strategy earn higher (average monthly returns, monthly SR, monthly MSR_α of (0.0133, 0.3328, 0.3468) and (0.013, 0.3039, 0.3142), respectively, but generate higher losses (VaR_α, CVaR_α), (0.0499, 0.0636) at

MCWP and (0.0545, 0.0696) at the naïve $1/m$ portfolio strategy. In other words, all models except $\text{STARR}_\alpha(\ell_x)$ improve over the MCWP and the naïve $1/m$ portfolio strategy in terms of having less losses. Therefore, for a loss averse investor we consider in our study, we are able to successfully develop portfolio strategies which produce low losses through the Omega-CVaR_α and MOmega-CVaR_α models. The proposed model enhances the $P_{\text{CVaR}_\alpha(\ell_x)}$ model, MCWP, and the naïve $1/m$ portfolio strategy in terms of losses and thus most suitable for loss averse investors. Also, robust optimization under the mixed uncertainty set produces promising results compared to its nominal counterpart and thus encourages investors to apply it in finance modeling.

The descriptive statistics in Table 1 display the behavior of the return series for each of the five portfolios. Due to presence of a higher magnitude of negative returns than positive returns, all models possess negative skewness. The kurtosis which explains steepness in distribution relative to the normal distribution is higher for the MCWP and the naïve $1/m$ portfolio strategy than for all other models. Among all models, portfolios from MOmega-CVaR_α produce the highest kurtosis indicating a comparatively large mass of returns around the mean value than the other two robust models. The range (difference of maximum and minimum values) shows highest the amount of uncertainty in the naïve $1/m$ portfolio strategy (an average of 0.2138) and minimum in the MOmega-CVaR_α (an average of 0.151).

Next, we perform some statistical tests over winsorized data[11] of 66 out-of-sample monthly returns to provide evidence regarding the performance differences between the benchmark portfolio and the proposed model Omega-CVaR_α as well as its robust counterpart MOmega-CVaR_α. To test whether the out-of-sample average returns of two strategies s_1 and s_2 are statistically different, we apply a one-sided t_μ test with hypothesis $H_0 : \mu_{s_1} - \mu_{s_2} = 0$ and $H_a : \mu_{s_1} - \mu_{s_2} > 0$.[12]

Furthermore, we test whether the out-of-sample Sharpe ratio of two strategies s_1 and s_2 are statistically different. We apply a one-sided z_{SR} test with the hypothesis $H_0 : \text{SR}_{s_1} - \text{SR}_{s_2} = 0$ and $H_a : \text{SR}_{s_1} - \text{SR}_{s_2} > 0$ (Jobson and Korkie 1981; DeMiguel et al. 2009).[13]

---

[11] Winsorizing data at β percentile means replacing the extreme values of a data set with their corresponding β percentile values to limit the effect of the extreme values on the test. We take $\beta = 0.90$ in our study.

[12] Given two strategies s_1 and s_2, with μ_{s_1}, μ_{s_2} as their sample means and $\sigma(s_1 - s_2)$ as the standard deviation of the difference of two strategies over a sample period of size n ($n = 66$ in our case). We evaluate the p values of the difference using the t test statistic:

$$t_\mu := \frac{\mu_{s_1} - \mu_{s_2}}{\sigma(s_1 - s_2)/\sqrt{n}}.$$

[13] Given two strategies s_1 and s_2, with $\mu_{s_1}, \mu_{s_2}, \sigma_{s_1}, \sigma_{s_2}, \sigma_{s_1,s_2}$ as their sample means, standard deviations, and the covariance of two strategies over a sample period n. We evaluate the p values by calculating the z test statistic:

$$z_{\text{SR}} := \frac{\sigma_{s_2}\mu_{s_1} - \sigma_{s_1}\mu_{s_2}}{\sqrt{\Upsilon}}$$

with

$$\Upsilon = \frac{1}{n}\left(2\sigma_{s_1}^2\sigma_{s_2}^2 - 2\sigma_{s_1}\sigma_{s_2}\sigma_{s_1,s_2} + 0.5\mu_{s_1}^2\sigma_{s_2}^2 + 0.5\mu_{s_2}^2\sigma_{s_1}^2 - \frac{\mu_{s_1}\mu_{s_2}}{\sigma_{s_1}\sigma_{s_2}}\sigma_{s_1,s_2}^2\right).$$

Table 2 This table reports upon the statistical inference for the differences between the portfolio of Omega-CVaR$_\alpha$ and MOmega-CVaR$_\alpha$ from the target portfolio of $P_{\mathrm{CVaR}_\alpha(\ell_x)}$ one month out-of-sample returns (t-statistics) and in the out-of-sample Sharpe ratios SR (z-statistics) for four values of $\alpha = 0.97, 0.95, 0.93, 0.90$

	Omega-CVaR$_\alpha$	MOmega-CVaR$_\alpha$
Panel $\alpha = 0.97$		
t_μ	2.0004* (0.0237)	1.9109* (0.0290)
z_{SR}	1.8215* (0.0344)	2.1129** (0.0174)
Panel $\alpha = 0.95$		
t_μ	1.8569** (0.0327)	1.6099* (0.0548)
z_{SR}	1.5000* (0.0668)	1.8206** (0.0344)
Panel $\alpha = 0.93$		
t_μ	1.4460* (0.0751)	1.1004 (0.1365)
z_{SR}	1.0771 (0.142)	1.4416* (0.0749)
Panel $\alpha = 0.90$		
t_μ	0.21481 (0.4151)	0.6358 (0.2629)
z_{SR}	0.7135 (0.2389)	1.3366* (0.0918)

The values in parentheses are p value and the significance levels are 0.01, 0.05 and 0.1 which are displayed by ***, **, and *, respectively

The third measure is to test whether the out-of-sample CVaR$_\alpha$ of one strategy s_1 is different to the CVaR$_\alpha$ (c) of the target portfolio using the one-sided z_{CVaR_α} test with hypothesis $H_0 : c - \mathrm{CVaR}_{\alpha_{s_1}} = 0$ and $H_a : c - \mathrm{CVaR}_{\alpha_{s_1}} > 0$ following Vekas (2015).[14]

In Table 2, we report values of t_μ and z_{SR} (and their respective p values in parentheses) for the differences amongst expected returns and Sharpe ratios of Omega-CVaR$_\alpha$, and MOmega-CVaR$_\alpha$ models from the $P_{\mathrm{CVaR}_\alpha(\ell_x)}$ model, for four values of $\alpha = 0.97, 0.95, 0.93, 0.90$.

For a value of α, this is accomplished through calculating μ_{s_2} and σ_{s_2} of the 66 out-of-sample monthly return series of portfolios from $P_{\mathrm{CVaR}_\alpha(\ell_x)}$ model and using them to derive t_μ and z_{SR} test values for portfolios from Omega-CVaR$_\alpha$ and MOmega-CVaR$_\alpha$ models.

We find that Omega-CVaR$_\alpha$ exhibits statistically significant improvement over $P_{\mathrm{CVaR}_\alpha(\ell_x)}$ in expected return for $\alpha = 0.97, 0.95, 0.93$ and Sharpe ratio for $\alpha = 0.97, 0.95$, all within 90 % level of confidence. MOmega-$CVaR_\alpha$ also dominates $P_{CVaR_\alpha(\ell_x)}$ statistically, on expected returns for $\alpha = 0.97, 0.95$ and in Sharpe ratio

[14] Given a strategy s_1 and a target portfolio s^*, with $y_1, \ldots, y_n$ as the return series of s_1 sorted from lowest to highest, $\widehat{\mathrm{CVaR}}_\alpha$, $\widehat{\mathrm{VaR}}_\alpha$ as their sample CVaR$_\alpha$ and CVaR$_\alpha$ values over sample period n, we evaluate the p values by calculating the z test statistic:

$$z_{\mathrm{CVaR}_\alpha} := \frac{\sqrt{n(1-\alpha)}(c - \widehat{\mathrm{CVaR}}_\alpha)}{\sqrt{\sum_{i=n\alpha+1}^{n} (y_i - \widehat{\mathrm{CVaR}}_\alpha)^2 / \big(n(1-\alpha)\big) + \alpha\big(\widehat{\mathrm{CVaR}}_\alpha - \widehat{\mathrm{VaR}}_\alpha\big)^2}}, \quad \text{where} \quad \widehat{\mathrm{VaR}}_\alpha := y_{n\alpha},$$

and

$$\widehat{\mathrm{CVaR}}_\alpha := \frac{1}{n(1-\alpha)} \sum_{i=n\alpha+1}^{n} y_i .$$

Table 3 Out-of-sample statistical analysis for z_{CVaR_α} when the target is the naïve $1/m$ portfolio

	$P_{CVaR_\alpha(\ell_x)}$	Omega-$CVaR_\alpha$	MOmega-$CVaR_\alpha$	MCWP
Panel $\alpha = 0.97$				
$z_{\mathrm{CVaR}_{0.97}}$	11.7148*** (0.000)	7.2624*** (0.0000)	2.1151** (0.0191)	0.856 (0.1975)
Panel $\alpha = 0.95$				
$z_{\mathrm{CVaR}_{0.95}}$	2.2285** (0.0146)	2.1563** (0.0173)	2.5064*** (0.0073)	1.0897 (0.1399)
Panel $\alpha = 0.93$				
$z_{\mathrm{CVaR}_{0.93}}$	2.13** (0.0184)	1.9415** (0.0282)	3.0414*** (0.0016)	0.9675 (0.1684)
Panel $\alpha = 0.90$				
$z_{\mathrm{CVaR}_{0.90}}$	1.643* (0.0427)	2.1466** (0.0162)	2.4148*** (0.008)	0.5468 (0.2981)

The values in parentheses are p values and the significance levels are 0.01, 0.05 and 0.1 which are displayed by ***, **, and *, respectively

for $\alpha = 0.97, 0.95, 0.093$, all within 92 % level of confidence. These results statistically support the beneficial properties of the proposed models over the classical $P_{CVaR_\alpha(\ell_x)}$ model.

In Table 3, we report upon values of z_{CVaR_α} for portfolios from $P_{\mathrm{CVaR}_\alpha(\ell_x)}$, Omega-CVaR$_\alpha$, MOmega-CVaR$_\alpha$ models, for four values of $\alpha = 0.97, 0.95, 0.93, 0.90$, and the MCWP, to test the significant improvement over the naïve $1/m$ portfolio in terms of CVaR$_\alpha$. For this, we take $c = \mathrm{CVaR}_\alpha$ of the 66 out-of-sample monthly return series of the naïve $1/m$ portfolio in calculating the z_{CVaR_α} test values for each portfolio.

From Table 3, we observe that portfolios from $P_{\mathrm{CVaR}_\alpha(\ell_x)}$, Omega-CVaR$_\alpha$ and MOmega-CVaR$_\alpha$ models generate statistically significant lower values of CVaR$_\alpha$ than the naïve $1/m$ portfolio.

Lastly, in Table 4, we report the statistical analysis of Omega-CVaR$_\alpha$, and MOmega-CVaR$_\alpha$ models vis a vis STARR$_\alpha$ model on the basis of three out-of-sample statistics namely, t_μ, z_{CVaR_α}, and the F test for variance (F_{σ^2}). The F test is used to find if the out-of-sample variance of the two strategies s_1 and s_2 is statistically different, we apply a one-tailed F_{σ^2} test with null hypothesis $H_0 : \sigma^2_{s_1} \geq \sigma^2_{s_2}$ against an alternative hypothesis $H_a : \sigma^2_{s_1} < \sigma^2_{s_2}$.[15]

Since the average returns of the portfolios of STARR$_\alpha$ model are found to be greater than that of portfolios of Omega-CVaR$_\alpha$ and MOmega-CVaR$_\alpha$ models (see, Table 1), so strategy s_1 is taken as the portfolio from STARR$_\alpha$ model in computing t_μ statistics. On the other hand, variance and CVaR$_\alpha$ of portfolios of STARR$_\alpha$ are lower than that of portfolios of Omega-CVaR$_\alpha$ and MOmega-CVaR$_\alpha$ (again see, Table 1), hence strategy s_2 is taken as the portfolio from STARR$_\alpha$ model in computing the z_{CVaR_α} and F_{σ^2} statistics. We observe that both Omega-CVaR$_\alpha$ and MOmega-CVaR$_\alpha$ models surpass the STARR$_\alpha$ model significantly in terms of risks (CVaR$_\alpha$ as well as variance) while the STARR$_\alpha$ model earn higher average return than Omega-CVaR$_\alpha$ model only for α = 0.97, 0.95, and MOmega-CVaR$_\alpha$ only for α = 0.97.

[15] Given two strategies s_1 and s_2, with $\bar{\sigma}^2_{s_1}, \bar{\sigma}^2_{s_2}$ as their sample variances over a sample period of size n ($n = 66$ in our case), we evaluate the p values using F test statistic: $F_{\sigma^2} := \frac{\bar{\sigma}^2_{s_2}}{\bar{\sigma}^2_{s_1}}$ (Table 4).

Table 4 Out-of-sample statistical analysis for t_μ, z_{CVaR_α}, and F_{σ^2} when the target are the portfolios from STARR_α model

	Omega-$CVaR_\alpha$	MOmega-$CVaR_\alpha$
Panel $\alpha = 0.97$		
t_μ	1.6401* (0.0529)	1.3163* (0.0963)
F_{σ^2}	1.7791** (0.0107)	2.0316*** (0.0024)
$z_{\mathrm{CVaR}_{0.97}}$	12.8629*** (0.0000)	3.7353*** (0.0001)
Panel $\alpha = 0.95$		
t_μ	1.3421* (0.0921)	1.1019 (0.1372)
F_{σ^2}	1.8043*** (0.0093)	2.0477*** (0.0021)
$z_{\mathrm{CVaR}_{0.97}}$	2.925*** (0.0023)	3.2403*** (0.0009)
Panel $\alpha = 0.93$		
t_μ	0.8195 (0.2077)	0.6470 (0.2599)
F_{σ^2}	1.8785*** (0.0060)	2.1386*** (0.0012)
$z_{\mathrm{CVaR}_{0.97}}$	2.4896*** (0.0076)	3.7534*** (0.0001)
Panel $\alpha = 0.90$		
t_μ	0.7974 (0.2140)	0.5523 (0.2912)
F_{σ^2}	2.0087*** (0.0027)	2.2783*** (0.0005)
$z_{\mathrm{CVaR}_{0.97}}$	3.7831*** (0.0001)	4.0107*** (0.0000)

The values in parentheses are p values and the significance levels are 0.01, 0.05 and 0.1 which are displayed by ***, **, and *, respectively

6 Conclusions

In this paper, we propose to take the threshold point L in Omega ratio as $L(\alpha) = \mathrm{CVaR}_\alpha$ of the loss distribution of the benchmark market z at α confidence level where α is a measure of loss averse attitude of an investor. To accomplish this, we first re-defined the Omega ratio for loss distribution to make this study compatible with the concepts of CVaR_α. The Omega-CVaR_α model involves maximization of $L(\alpha)$ while $L(\alpha)$ is described in terms of CVaR_α which requires minimization. To bridge this gap in our study, we use the zero duality gap in the scenario based CVaR_α minimization problem (an inner problem in the proposed Omega-CVaR_α model) and convert it into a maximization problem. The optimal maximum value of the dual problem act as an upper bound for $L(\alpha)$ in the proposed Omega ratio maximization model.

We next formulate the robust variants of the Omega-CVaR_α model in which robustness is introduced in maximizing the Omega ratio as well as in minimizing CVaR_α (with the aim to derive the robust threshold point) under the mixed, box, and ellipsoidal uncertainty sets. We observe that the robust Omega-CVaR_α remains a linear program under mixed and box uncertainty sets while it becomes a second order conic program in case of ellipsoidal uncertainty set. We then perform a comparative analysis between the $P_{\mathrm{CVaR}_\alpha(\ell_x)}$ model, the STARR_α model, the proposed Omega-CVaR_α, and a robust variant of Omega-CVaR_α under the mixed uncertainty set over the sample period from Jan 1990 to June 2015. We use the S&P 500 index as the benchmark market to derive the threshold CVaR_α in Omega-CVaR_α and select the top 50 stocks from the S&P 500 on the basis of high market capitalization to optimize the standard nominal CVaR_α model and to maximize the Omega ratio in the proposed nominal Omega-CVaR_α model. We find that the Omega-CVaR_α model improves $P_{\mathrm{CVaR}_\alpha(\ell_x)}$ model in expected returns, Sharpe ratio, and modified Sharpe ratio while controlling the losses in terms of having smaller values of VaR_α and CVaR_α, where results are significant for the first two measures. We also perceive that the robust Omega-CVaR_α upgrades the Omega-CVaR_α model by having a comparatively large expected return, Sharpe ratio and modified Sharpe ratio and lesser value of losses measured by VaR_α and CVaR_α.

The Omega-CVaR_α as well as the robust counterpart under mixed uncertainty set generate statistically significant lower values of CVaR_α than the naïve $1/m$ portfolio strategy. Moreover, the two proposed models Omega-CVaR_α and its robust model under mixed uncertainty set outperform the STARR_α model in terms of losses by producing statistically significant lower CVaR_α and variance. These results show the financial benefits of our proposed model and its robust counterpart over the classical CVaR_α model in terms of both return and risk of loss.

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Appendix A: Worst Omega ratio in continuous case under mixed uncertainty set

Define

$$H = \frac{E_p(L-\ell_x)^+}{E_p(\ell_x-L)^+} = \frac{\int_\ell (L-\ell_x)^+ p(\ell)\,\mathrm{d}\ell}{\int_\ell (\ell_x-L)^+ p(\ell)\,\mathrm{d}\ell} = \frac{\int_\ell \sum_{k=1}^{s} w_k (L-\ell_x)^+ p^k(\ell)\,\mathrm{d}\ell}{\int_\ell \sum_{k=1}^{s} w_k (\ell_x-L)^+ p^k(\ell)\,\mathrm{d}\ell}.$$

Applying the Charnes and Cooper (1962) transformation, with $\gamma > 0$ as a homogenization variable, we have,

$$\begin{aligned} H &= \int_\ell \sum_{k=1}^{s} w_k (L\gamma - \ell_{\tilde{x}})^+ p^k(\ell)\,\mathrm{d}\ell \text{ with } \int_\ell \sum_{k=1}^{s} w_k (\ell_{\tilde{x}} - L\gamma)^+ p^k(\ell)\,d\ell = 1,\quad \tilde{x}=x\gamma, \\ &= \sum_{k=1}^{s} w_k G_1^k \quad \text{with} \quad \sum_{k=1}^{s} w_k G_2^k = 1, \end{aligned}$$

where $G_1^k = \int_\ell (L\gamma - \ell_{\tilde{x}})^+ p^k(\ell)\,\mathrm{d}\ell,\; G_2^k = \int_\ell (\ell_{\tilde{x}} - L\gamma)^+ p^k(\ell)\,\mathrm{d}\ell.$

Appendix B: Mixed uncertainty set

The mixed uncertainty set for the distribution of benchmark market loss ℓ_z is as follows:

$$\mathcal{Q}_M = \left\{ q = \sum_{k=1}^{\hat{s}} \hat{w}_k q^k;\; \sum_{k=1}^{\hat{s}} \hat{w}_k = 1,\; \hat{w}_k \geq 0,\; k = 1, \ldots, \hat{s} \right\},$$

where q^k is the kth likelihood density function of portfolio loss $\hat{\ell}_z$. The worst case analysis of $P_{\text{CVaR}_\alpha(\hat{\ell}_z)}$ under the mixed uncertainty set for continuous case is already discussed in Zhu and Fukushima (2009). Here we re-state its discrete version for reader comprehension. Let $\hat{T}^k$ be the finite number of scenarios of $\hat{\ell}_z$ (using sampling techniques) with the kth, $k = 1, \ldots, \hat{s}$ likelihood probability vector $q^k = ((q_1^k, \ldots, q_{\hat{T}^k}^k)^t;\ (q^k)^t e = 1,\ q_i^k \geq 0,\ \forall\, i = 1, \ldots, \hat{T}^k)$. Then following Zhu and Fukushima (2009), the $P_{\text{CVaR}_\alpha(\hat{\ell}_z)}$ model under $\mathcal{Q}_M$ as follows:

$$
\begin{aligned}
P_{\text{MCVaR}_\alpha(\hat{\ell}_z)} \qquad & \min \quad \hat{\theta} \\
& \text{subject to: } \tau + \frac{1}{1-\alpha}(q^k)^t \hat{u}^k \leq \hat{\theta} \qquad k = 1, \ldots, \hat{s} \\
& \hat{u}^k + \tau e - \hat{B}^k z \geq 0, \quad k = 1, \ldots, \hat{s} \\
& z^t e = 1, \quad \tau \in \mathbb{R}, \quad z \in \mathbb{R}_+^m, \ \hat{u}^k \in \mathbb{R}_+^{\hat{T}^k},
\end{aligned}
$$

where $\hat{B}^k = [\hat{\ell}_{ij}^k]_{\hat{T}^k \times m}$ is the loss matrix of portfolio z corresponding to the likelihood probability function q^k.

Therefore, the dual of $P_{\text{MCVaR}_\alpha(\hat{\ell}_z)}$ is derived as follows:

$$
\begin{aligned}
P_{\text{DMCVaR}_\alpha(\hat{\ell}_z)} \qquad & \max \quad \vartheta \\
& \text{subject to: } \vartheta e - \sum_{k=1}^{\hat{s}} (\hat{B}^k)^t h^k \leq 0 \\
& v^t e = \gamma \\
& -v^t e + \sum_{k=1}^{\hat{s}} (h^k)^t e = 0 \\
& -\frac{v_k}{1-\alpha} q^k + h^k \leq 0 \qquad k = 1, \ldots, \hat{s} \\
& h^k \in \mathbb{R}_+^{\hat{T}^k}, \ v \in \mathbb{R}_+^{\hat{s}}, \qquad k = 1, \ldots, \hat{s},
\end{aligned}
$$

where v_k, $k = 1, \ldots, \hat{s}$, is the kth component of vector v.

Appendix C: Box uncertainty set

Let $\mathcal{Q}_B = \{q = q^0 + \hat{\pi};\ \hat{\pi}^t e = 0,\ \underline{\hat{\pi}} \leq \hat{\pi} \leq \bar{\hat{\pi}}\}$ be a box uncertainty set for the distribution of benchmark market loss $\hat{\ell}_z$, then $P_{\text{CVaR}_\alpha(\hat{\ell}_z)}$ under $\mathcal{Q}_B$ is given according to Zhu and Fukushima (2009):

$$
\begin{aligned}
P_{\text{BCVaR}_\alpha(\hat{\ell}_z)} \qquad & \min \quad \hat{\theta} \\
& \text{subject to: } \tau + \frac{1}{1-\alpha}(q^0)^t \hat{u} + \frac{1}{1-\alpha}(\bar{\hat{\pi}}^t \xi + \underline{\hat{\pi}}^t \varrho) \leq \hat{\theta}
\end{aligned}
$$

$$\epsilon e + \xi - \varrho = \hat{u}$$
$$\hat{u} + \tau e - \hat{B} z \geq 0$$
$$e^t z = 1, \quad \tau \in \mathbb{R}, \quad z \in \mathbb{R}_+^{\hat{m}}$$
$$\epsilon \in \mathbb{R}, \ \hat{u}, \ \xi, \ \varrho \in \mathbb{R}_+^{\hat{T}},$$

where $\hat{B} = [\hat{\ell}_{ij}]_{\hat{T}\times m}$ is the loss matrix of portfolio z. The dual of $P_{\mathrm{BCVaR}_\alpha(\hat{\ell}_z)}$ is as follows:

$$\begin{aligned}
P_{\mathrm{DBCVaR}_\alpha(\hat{\ell}_z)} \qquad & \max \quad \vartheta \\
\text{subject to: } & \hat{B}^t v - \vartheta e - \lambda_5 = 0 \\
& e^t v = \gamma \\
& -e^t \lambda_6 = 0 \\
& \frac{q^0}{1-\alpha} + \lambda_6 - v - \lambda_7 = 0 \\
& \frac{\bar{\hat{\pi}}}{1-\alpha} - \lambda_6 - \lambda_8 = 0 \\
& \frac{\hat{\pi}}{1-\alpha} - \lambda_6 - \lambda_9 = 0 \\
& \vartheta \in \mathbb{R}, \quad v, \lambda_7, \lambda_8, \lambda_9 \in \mathbb{R}_+^{\hat{T}}, \quad \lambda_5 \in \mathbb{R}_+^m, \quad \lambda_6 \in \mathbb{R}^T .
\end{aligned}$$

Appendix D: Ellipsoidal uncertainty set

For fixed values of $\theta, \widetilde{u}, \widetilde{d}$, here, we first derive the dual of P_{10} in the following steps: The Lagrange of P_{10} with Lagrange multipliers $\lambda_1, \ \lambda_2, \ \lambda_3, \ \lambda_4, \ \lambda_5$, is given as follows:

$$\begin{aligned}
L(.) &= \widetilde{u}^t p^0 + u^t B\pi - \lambda_1(\widetilde{d}^t p^0 + \widetilde{d}^t A\pi - 1) - \lambda_2^t(p^0 + A\pi) - \lambda_4(e^t A\pi) + \lambda_5 \pi - \lambda_3. \\
&= (A^t\widetilde{u} - \lambda_1 A^t \widetilde{d} - \lambda_4 A^t e - A^t \lambda_2 + \lambda_5)\pi + \widetilde{u}^t p^0 - \lambda_1 \widetilde{d}^t p^0 + \lambda_1 - \lambda_2^t p^0 - \lambda_3.
\end{aligned} \tag{24}$$

Using the minimax representation of the primal problem P_{11} as

$$\min_{\pi} \max_{\{||\lambda_5||_2 \leq \lambda_3, \lambda_1, \lambda_2, \lambda_4\}} L(.) = \min_{\{||\lambda_5||_2 \leq \lambda_3, \lambda_1, \lambda_2, \lambda_4\}} \max_{\pi} L(.), \tag{25}$$

the inner problem of the latter one is affine in π and can be solved by taking its derivative with respect to π which leads to the following dual constraint derived from Eq. (24):

$$A^t \widetilde{u} - \lambda_1 A^t \widetilde{d} - \lambda_4 A^t e - A^t \lambda_2 + \lambda_5 = 0$$

Therefore, the dual of P_{11} is given as follows:

$$
\begin{aligned}
P_{12} \qquad & \max \quad \widetilde{u}^t p^0 - \lambda_1 \widetilde{d}^t p^0 + \lambda_1 - \lambda_2^t p^0 - \lambda_3 \\
& \text{subject to: } A^t \widetilde{u} - \lambda_1 A^t \widetilde{d} - \lambda_4 A^t e - A^t \lambda_2 + \lambda_5 = 0 && (26)\\
& ||\lambda_5||_2 \le \lambda_3 && (27)\\
& \lambda_1, \lambda_3, \lambda_4 \in \mathbb{R}; \quad \lambda_2 \in \mathbb{R}_+^T, \quad \lambda_5 \in \mathbb{R}^T. && (28)
\end{aligned}
$$

P_{12} is an SOCP problem.

Dual of worst case of CVaR$_\alpha$ *model under ellipsoidal case*

For the ellipsoidal set $\mathcal{Q}_E = \{q = q^0 + \hat{A}\hat{\pi};\ e^t \hat{A}\hat{\pi} = 0,\ q^0 + \hat{A}\hat{\pi} \ge 0,\ ||\hat{\pi}||_2 \le 1\}$ of portfolio z, the worst case of $P_{\text{CVaR}_\alpha(\ell_z)}$ is as follows (Zhu and Fukushima 2009):

$$
\begin{aligned}
P_{\text{ECVaR}_{\hat{\alpha}}(\ell_z)} \qquad & \min \quad \hat{\theta} \\
& \text{subject to: } \tau + \frac{1}{1-\alpha}(q^0)^t \hat{u} + \frac{1}{1-\alpha}(\zeta + (\hat{P}^0)^t \Upsilon_1) \le \hat{\theta} \\
& -\xi - \hat{A}^t \Omega + \hat{A}^t e \epsilon = \hat{A}^t \hat{u} \\
& ||\xi||_2 \le \zeta \\
& \hat{u} + \tau - \hat{B} z \ge 0 \\
& e^t z = 1, \quad z \in \mathbb{R}_+^m \\
& \epsilon,\ \tau \in \mathbb{R}, \quad \hat{u} \in \mathbb{R}_+^{\hat{T}}\ \xi \in \mathbb{R}^{\hat{T}}, \quad \Upsilon_1 \in \mathbb{R}_+^{\hat{T}}.
\end{aligned}
$$

$P_{\text{ECVaR}_\alpha(\hat{\ell}_z)}$ is an SOCP problem. Analogously to the dual derivation P_{12} from P_{11}, we can also obtain the dual of $P_{\text{ECVaR}_\alpha(\hat{\ell}_z)}$ as follows:

$$
\begin{aligned}
P_{\text{DECVaR}_\alpha(\hat{\ell}_z)} \qquad & \max \quad \vartheta \\
& \text{subject to: } \hat{B}^t v - \vartheta e - \lambda_6 = 0 \\
& v^t e = 1 \\
& e^t \hat{A} \lambda_7 = 0 \\
& \frac{q^0}{1-\alpha} + \hat{A}\lambda_7 - v - \lambda_8 = 0 \\
& \frac{q^0}{1-\alpha} + \hat{A}\lambda_7 - \lambda_9 = 0 \\
& ||\lambda_7||_2 \le \frac{1}{1-\alpha} \\
& \vartheta \in \mathbb{R}, \quad \lambda_7,\ \lambda_{10} \in \mathbb{R}^{\hat{T}}, \quad v, \lambda_8,\ \lambda_9 \in \mathbb{R}_+^{\hat{T}}, \quad \lambda_6 \in \mathbb{R}_+^m.
\end{aligned}
$$

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