Regularity and sparse approximation of the recursive first moment equations for the lognormal Darcy problem

Francesca Bonizzoni, Fabio Nobile

Angaben zur Veröffentlichung / Publication details:

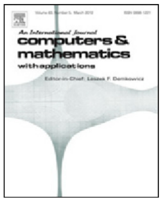
Bonizzoni, Francesca, and Fabio Nobile. 2020. "Regularity and sparse approximation of the recursive first moment equations for the lognormal Darcy problem." *Computers and Mathematics with Applications* 80 (12): 2925–47.
https://doi.org/10.1016/j.camwa.2020.10.014.

Computers and Mathematics with Applications 80 (2020) 2925–2947

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa



Regularity and sparse approximation of the recursive first moment equations for the lognormal Darcy problem

Francesca Bonizzoni [a,*], Fabio Nobile [b]

[a] Faculty of Mathematics, University of Vienna, Austria
[b] CSQI – MATH, Ecole Polytechnique Fédérale de Lausanne, Switzerland

ARTICLE INFO

Article history:
Received 15 May 2020
Received in revised form 16 October 2020
Accepted 18 October 2020
Available online 13 November 2020

Keywords:
Uncertainty quantification
Lognormal distribution
Perturbation technique
Moment equations
Hölder spaces with mixed regularity
Sparse approximation

ABSTRACT

We study the Darcy boundary value problem with lognormal permeability field. We adopt a perturbation approach, expanding the solution in Taylor series around the nominal value of the coefficient, and approximating the expected value of the stochastic solution of the PDE by the expected value of its Taylor polynomial. The recursive deterministic equation satisfied by the expected value of the Taylor polynomial (first moment equation) is formally derived. Well-posedness and regularity results for the recursion are proved to hold in Sobolev space-valued Hölder spaces with mixed regularity. The recursive first moment equation is then discretized by means of a sparse approximation technique, and the convergence rates are derived.

1. Introduction

In many applications, the input parameters of the mathematical model describing the system behavior are unavoidably affected by uncertainty, as a consequence of the incomplete knowledge or the intrinsic variability of certain phenomena. *Uncertainty Quantification* (UQ) conveniently incorporates the input variability or lack of knowledge inside the model, often by describing the uncertain parameters as random variables or random fields, and aims to infer the uncertainty in the solution of the model, or the specific output quantities of interest, by computing their statistical moments.

The physical phenomenon we are interested in this work is the single-phase flow of a fluid in a bounded heterogeneous saturated porous medium. In particular, we consider the following stochastic partial differential equation (PDE), named the Darcy problem, posed in the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and in the bounded physical domain $D \subset \mathbb{R}^d$ $(d = 2, 3)$:

$$-\operatorname{div}\left(e^{Y(\omega,x)}\nabla u(\omega, x)\right) = f(x) \quad \text{for } x \in D \text{ and a.e. } \omega \in \Omega \tag{1}$$

endowed with suitable boundary conditions on $\partial\Omega$, where $u(\omega, x)$ represents the hydraulic head, the forcing term $f(x) \in L^2(D)$ is deterministic, and the permeability coefficient $\mathrm{e}^{Y(\omega,x)}$ is modeled as a *lognormal random field*, $Y(\omega, x)$ being a centered Gaussian random field with small standard deviation. The lognormal diffusion problem (1) is widely used in geophysical applications (see, e.g., [1–4] and the references there), and has been studied mathematically, e.g., in [5–8].

Under suitable assumptions on the covariance of the random field $Y(\omega, x)$, it is possible to show that the Darcy problem is well-posed (see [7]).

Given complete statistical information on the Gaussian random field $Y(\omega, x)$, and assuming that each realization $Y(\omega, \cdot)$ is almost surely Hölder continuous with parameter γ, the aim of the present work is to construct an approximation for

* Corresponding author.
E-mail addresses: francesca.bonizzoni@univie.ac.at (F. Bonizzoni), fabio.nobile@epfl.ch (F. Nobile).

https://doi.org/10.1016/j.camwa.2020.10.014

the expected value of the stochastic solution $\mathbb{E}[u]$. To this end, we adopt a *perturbation approach*, in which the stochastic solution u is viewed as the map $u : \mathcal{C}^{0,\gamma}\left(\bar{D}\right) \to H^1(D)$ which associates to each realization $Y(\omega,\cdot) \in \mathcal{C}^{0,\gamma}\left(\bar{D}\right)$, the unique solution $u(\omega,\cdot)$ of (2), and is expanded in Taylor series w.r.t. Y, i.e., $\sum_{k=0}^{+\infty} \frac{D^k u(0)[Y]^k}{k!}$, $D^k u(0)[Y]^k$ being the kth Gateaux derivative of u in $Y=0$ evaluated along the vector $\underbrace{(Y,\ldots,Y)}_{k \text{ times}}$. The expected value of u is then approximated as

$$\mathbb{E}[u](x) \simeq \mathbb{E}\left[T^K u\right](x) = \sum_{k=0}^{K} \frac{\mathbb{E}\left[D^k u(0)[Y]^k\right](x)}{k!},$$

where $T^K u(Y,x)$ denotes the Kth degree Taylor polynomial. We refer to $\mathbb{E}\left[D^k u(0)[Y]^k\right]$ as the kth order correction to the expected value of u, and to $\mathbb{E}\left[T^K u\right]$ as the Kth degree approximation of the expected value of u.

In [6,9,10] the authors show that, as K goes to infinity, the Kth order approximation of the expected value of u may actually diverge, for any positive value of the standard deviation $\sigma := \sqrt{\frac{1}{|D|}\int_D \mathbb{E}\left[Y^2\right](x)dx}$ of the random field $Y(\omega,x)$. Nevertheless, for σ and K small enough, $\mathbb{E}\left[T^K u\right]$ provides a good approximation of $\mathbb{E}[u]$. The work [6] also provides an estimate of the optimal degree of the Taylor polynomial achieving minimal error, for any given $\sigma > 0$.

If a finite-dimensional approximation of the random field $Y(\omega,x)$ via N random variables is available (e.g., by using the Karhunen–Loève (KL) expansion), then the (multi-variate) Taylor polynomial can be explicitly computed (see, e.g., the geophysical literature [11–14]). However, this approach entails the computation of $\begin{pmatrix} N+K \\ K \end{pmatrix}$ derivatives. To alleviate the curse of dimensionality, adaptive algorithms have been proposed in [15,16] for the case of uniform random variables.

In the present paper we consider the entire field $Y(\omega,x)$, and not a finite dimensional approximation of it, hence the Taylor polynomial cannot be directly computed. Following [17–19], we adopt the *moment equations* approach, that is, we solve the deterministic equations satisfied by $\mathbb{E}\left[D^k u(0)[Y]^k\right]$, for $k \geq 0$.

In [20] the authors derive analytically the recursive problem solved by $\mathbb{E}\left[D^k u(0)[Y]^k\right]$, which requires the recursive computation of the $(i+1)$-points correlations $\mathbb{E}\left[D^{k-i}u(0)[Y]^{k-i} \otimes Y^{\otimes i}\right]$, with $i = k, k-1, \ldots, 1$. These functions being high dimensional, a full tensor product finite element discretization is impractical and suffer the curse of dimensionality. To overcome this issue, in [20] the authors have proposed a low rank approximation of the fully (tensor product) discrete problem, using the Tensor Train format. The effectiveness of the method is shown with both one and two-dimensional numerical examples.

The present paper complements the above-mentioned results. The main achievement consists in the well-posedness and regularity results for the recursive first moment equation. These results are developed in the framework of p-integrable Lebesgue spaces. In particular, the key tool consists in showing that the diagonal trace of functions in the $L^p(D)$ space-valued mixed γ-Hölder space, belongs to $L^p(D)$, whenever $p > \frac{2d}{\gamma}$. We also address the discretization of the moment equations. Differently from [20], to alleviate the curse of dimensionality we propose here a sparse approximation method based on the Smolyak construction, which is more amenable to error analysis. We present then a complete convergence analysis of the proposed discretization method.

The paper is organized as follows: in Section 2 we recall the recursion solved by the kth order correction $\mathbb{E}\left[D^k u(0)[Y]^k\right]$ under the assumption that every quantity is well-defined, and every problem is well-posed. In Section 3, we first introduce the Banach space-valued maps with mixed Hölder regularity, and then study the Hölder regularity of the diagonal trace of Sobolev space-valued mixed Hölder maps. These technical results will be needed in Section 4 to study the well-posedness and regularity of the recursion for $\mathbb{E}\left[D^k u(0)[Y]^k\right]$. Section 5 is dedicated to the sparse discretization of the recursion and its error analysis. Finally, we draw some conclusions in Section 6.

2. Analytical derivation of the first moment equation

The weak formulation of the Darcy PDE (1) endowed with homogeneous Dirichlet boundary conditions reads:

$$\int_D e^{Y(\omega,x)} \nabla u(\omega,x) \cdot \nabla v(x)\, dx = \int_D f(x)v(x)\, dx, \quad \forall v \in H_0^1(D), \text{ a.s. in } \Omega. \tag{2}$$

We assume here that the random field $Y \in L^\theta\left(\Omega; \mathcal{C}^{0,\gamma}\left(\bar{D}\right)\right)$ $(0 < \gamma < 1/2)$ for all $1 \leq \theta < +\infty$. Then, for any $f \in L^p(D)$, $1 < p < +\infty$, the boundary value problem (2) admits a unique solution $u \in L^p\left(\Omega; H^1(D)\right)$, which depends continuously on the data (see [7]). In particular, using the Kolmogorov–Chentsov continuity theorem (see, e.g., [21]), it has been proven that the Hölder regularity assumption $Y \in L^\theta\left(\Omega; \mathcal{C}^{0,\gamma}\left(\bar{D}\right)\right)$ $(0 < \gamma < 1/2)$ for all $1 \leq \theta < +\infty$, is fulfilled if the covariance function $Cov_Y \in \mathcal{C}^{0,t}\left(\overline{D \times D}\right)$ for some $2\gamma < t \leq 1$ (see [6,10]). The mentioned well-posedness result extends to the case of uniform/non-uniform Neumann as well as mixed Dirichlet–Neumann boundary conditions. In particular, the limit situation of Neumann boundary conditions on ∂D leads to the uniqueness of the solution $u(\omega,x)$ up to a constant. For clarity of presentation, in this work we restrict to the case of homogeneous Dirichlet boundary conditions in the rest of the paper.

In this section we recall (see [10,20]) the structure of the problem solved by $\mathbb{E}\left[D^k u(0)[Y]^k\right]$ – the kth order correction of the expected value of u – assuming that every quantity is well-defined and every problem is well-posed. We will detail these theoretical aspects in the next sections.

Let $D \subset \mathbb{R}^d$, be such that $\partial D \in C^1$. Let p, q be real numbers such that $1 < p, q < \infty$, with $\frac{1}{p} + \frac{1}{q} = 1$ and $p > \frac{2d}{\gamma}$, where γ is the Hölder regularity of the random field Y. The requirement $p > \frac{2d}{\gamma}$ will be clarified later (see Proposition 11). Given $f \in L^p(D)$, $1 < p < +\infty$, we define the linear form $\mathcal{F} \in (W_0^{1,q}(D))^\star$ as

$$\mathcal{F}(v) := \int_D f v \; dx \quad \forall v \in W_0^{1,q}(D),$$

where $(W_0^{1,q}(D))^\star$ denotes the dual space of $W_0^{1,q}$. The correction of order 0, $u^0 := u|_{Y=0}$, is deterministic and is the *unique weak solution* of the following problem: find $u^0 \in W_0^{1,p}(D)$ such that

$$\int_D \nabla u^0 \cdot \nabla v \; dx = \mathcal{F}(v) \quad \forall v \in W_0^{1,q}(D), \tag{3}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Moreover, it exists $C = C(D) > 0$ such that

$$\left\| u^0 \right\|_{W^{1,p}(D)} \leq C \left\| f \right\|_{L^p(D)}, \tag{4}$$

We refer to [22, Chapter 7] for the proof of existence and uniqueness of weak solutions for the Laplace–Dirichlet problem in $W^{1,p}$ spaces.

To lighten the notations, let $u^k := D^k u(0)[Y]^k$. For $k \geq 1$, the kth order correction $\mathbb{E}\left[u^k\right]$ satisfies the following problem,

k-th order correction BVP

$$\int_D \nabla \mathbb{E}\left[u^k\right] \cdot \nabla v \; dx = -\sum_{j=1}^{k} \binom{k}{j} \int_D \mathbb{E}\left[\nabla u^{k-j} Y^j\right] \cdot \nabla v \; dx \quad \forall v \in W_0^{1,q}(D). \tag{5}$$

Eq. (5) is obtained in two steps: (i) derive the problem satisfied by u^k, by taking derivatives with respect to Y of the stochastic equation (2) (see [6] and the references therein); (ii) apply the expected value to both sides of the obtained equation.

The function $\mathbb{E}\left[\nabla u^{k-i} Y^i\right]$ appearing in the r.h.s. of (5) is the diagonal of the $(i+1)$-points correlation function $\mathbb{E}\left[\nabla u^{k-i} \otimes Y^{\otimes i}\right]$, where $\otimes$ denotes the tensor product. In particular, it holds

$$\mathbb{E}\left[u^{k-i} Y^i\right](x) := \left(\mathrm{Tr}_{|1:i+1} \mathbb{E}\left[u^{k-i} \otimes Y^{\otimes i}\right]\right)(x) = \mathbb{E}\left[u^{k-i} \otimes Y^{\otimes i}\right](\underbrace{x, \ldots, x}_{(i+1)-\text{times}}),$$

where

- Tr is the diagonal trace operator (it will be formally defined in Definition 10);
- $\mathbb{E}\left[u^{k-i} \otimes Y^{\otimes i}\right](x, y_1, \ldots, y_i)$ is the $(i+1)$-point correlation function defined as

$$\mathbb{E}\left[u^{k-i} \otimes Y^{\otimes i}\right](x, y_1, \ldots, y_i) := \int_\Omega u^{k-i}(\omega, x) \otimes Y(\omega, y_1) \otimes \cdots \otimes Y(\omega, y_i) \mathrm{d}\mathbb{P}(\omega).$$

In the same way,

$$\mathbb{E}\left[\nabla u^{k-i} Y^i\right](x) := \left(\mathrm{Tr}_{|1:i+1} \mathbb{E}\left[\nabla u^{k-i} \otimes Y^{\otimes i}\right]\right)(x) = \mathbb{E}\left[\nabla u^{k-i} \otimes Y^{\otimes i}\right](\underbrace{x, \ldots, x}_{(i+1)-\text{times}}),$$

where $\mathbb{E}\left[\nabla u^{k-i} \otimes Y^{\otimes i}\right] = \nabla \otimes \mathrm{Id}^{\otimes i} \mathbb{E}\left[u^{k-i} \otimes Y^{\otimes i}\right]$, that is, the linear operator $\nabla \otimes \mathrm{Id}^{\otimes i}$ applies the gradient operator to the first variable x and the identity operator to all other variables y_j for $j = 1, \ldots, i$.

The correlation functions themselves satisfy the following recursion:

Recursion on the correlations

Given all lower order terms $\mathbb{E}\left[u^{k-i-j} \otimes Y^{\otimes(i+j)}\right]$ for $j = 1, \ldots, k-i$, find $\mathbb{E}\left[u^{k-i} \otimes Y^{\otimes i}\right]$ s.t.

$$\begin{aligned} &\int_D \left(\nabla \otimes \mathrm{Id}^{\otimes i}\right) \mathbb{E}\left[u^{k-i} \otimes Y^{\otimes i}\right](x, y_1, \ldots, y_i) \cdot \nabla v(x) \, \mathrm{d}x \\ &= -\sum_{j=1}^{k-i} \binom{k-i}{j} \int_D \mathrm{Tr}_{|1:j+1} \mathbb{E}\left[\nabla u^{k-i-j} \otimes Y^{\otimes(i+j)}\right](x, y_1, \ldots, y_i) \cdot \nabla v(x) \, \mathrm{d}x \end{aligned} \tag{6}$$

$\forall v \in W_0^{1,q}(D)$, for all $y_1, \ldots, y_i \in D$.

Table 1
K-th order approximation of the mean. The first column contains the input terms $\mathbb{E}\left[u^0 \otimes Y^{\otimes k}\right]$ and the first row contains the kth order corrections $\mathbb{E}\left[u^k\right]$, for $k = 0, \ldots, K$. To compute $\mathbb{E}\left[T^K u(Y,x)\right]$, we need all the elements in the top left triangular part, that is, all elements in the kth diagonal, for $k = 0, \ldots, K$.

u^0	0	$\mathbb{E}\left[u^2\right]$	0	$\mathbb{E}\left[u^4\right]$	0
0	$\mathbb{E}\left[u^1 \otimes Y\right]$	0	$\mathbb{E}\left[u^3 \otimes Y\right]$	0	$\iddots$
$u^0 \otimes \mathbb{E}\left[Y^{\otimes 2}\right]$	0	$\mathbb{E}\left[u^2 \otimes Y^{\otimes 2}\right]$	0	$\iddots$	0
0	$\mathbb{E}\left[u^1 \otimes Y^{\otimes 3}\right]$	0	$\iddots$	0	$\iddots$
$u^0 \otimes \mathbb{E}\left[Y^{\otimes 4}\right]$	0	$\iddots$	0	$\iddots$	0

Note that problem (5) is a particular case of (6) for $i = 0$, since $\mathbb{E}\left[u^{k-0} \otimes Y^{\otimes 0}\right] = \mathbb{E}\left[u^k\right]$. Moreover, observe that $\mathbb{E}\left[u^0 \otimes Y^{\otimes k}\right] = u^0 \otimes \mathbb{E}\left[Y^{\otimes k}\right]$, since u^0 is deterministic, and it is fully characterized by the mean solution u^0 and the covariance structure of Y, which is an input of the problem.

The computation of the kth order correction of the expected value of u relies on the *recursive* solution of BVPs of the type (6), as summarized in Algorithm 1.

Algorithm 1 Computation of the k-th order correction $\mathbb{E}\left[u^k\right]$

1: **for** $k = 0, \ldots, K$ **do**
2: Compute $u^0 \otimes \mathbb{E}\left[Y^{\otimes k}\right]$.
3: **for** $i = k-1, k-2, \ldots, 0$ **do**
4: Compute the $(i+1)$-point correlation function $\mathbb{E}\left[u^{k-i} \otimes Y^{\otimes i}\right]$ (Eq. (6)).
5: **end for**
6: The result for $i = 0$ is the k-th order correction $\mathbb{E}\left[u^k\right]$ to the mean $\mathbb{E}\left[u\right]$.
7: **end for**

Table 1 illustrates the computational flow of the presented algorithm. Each non-zero correlation $\mathbb{E}\left[u^{k-i} \otimes Y^{\otimes i}\right]$, with $i < k$, can be obtained only when all the preceding terms in the kth diagonal have been already computed. As a consequence, to derive the Kth order approximation $\mathbb{E}\left[T^K u\right]$, it is necessary to compute all the elements in the top left triangular part of the table. Notice that, since we assumed $\mathbb{E}\left[Y\right](x) = 0$, all the $(2k+1)$-point correlations of Y vanish, and all odd diagonals are zero.

3. Banach space-valued mixed Hölder maps, and trace results

Within this section, we introduce the notion of V-valued Hölder maps with mixed regularity, V being a Banach space, and we prove some useful properties. In particular, we study the regularity of the diagonal trace of Hölder mixed regular maps when V is a Sobolev space. These properties will be needed later in Section 4 to analyze the well-posedness of the equations in the recursion (6). Since the proofs of the Propositions in this Section are tedious and not essential for the later developments, they have been postponed to Appendix.

3.1. Banach space-valued mixed Hölder spaces

Definition 1 (*Banach Space-Valued Hölder Space*). Let V be a Banach space, $0 < \gamma \leq 1$ be real, and $k \geq 1$ integer. The V-valued Hölder space with exponent γ, $\mathcal{C}^{0,\gamma}_{\mathbf{y}}\left(\bar{D}^{\times k}; V\right)$, consists of all continuous maps $\varphi : \bar{D}^{\times k} \to V$ with Hölder γ-regularity. It is a Banach space with the norm

$$\|\varphi\|_{\mathcal{C}^{0,\gamma}_{\mathbf{y}}(\bar{D}^{\times k};V)} := \max\left\{\|\varphi\|_{\mathcal{C}^{0}_{\mathbf{y}}(\bar{D}^{\times k};V)}, |\varphi|_{\mathcal{C}^{0,\gamma}_{\mathbf{y}}(\bar{D}^{\times k};V)}\right\}$$

with

$$\|\varphi\|_{\mathcal{C}^{0}_{\mathbf{y}}(\bar{D}^{\times k};V)} := \sup_{\mathbf{y}\in\bar{D}^{\times k}} \|\varphi(\mathbf{y})\|_V$$

and

$$|\varphi|_{\mathcal{C}^{0,\gamma}_{\mathbf{y}}(\bar{D}^{\times k};V)} := \sup_{\substack{\mathbf{y}\in\bar{D}^{\times k},\mathbf{h}\neq\mathbf{0} \\ \text{s.t. } \mathbf{y}+\mathbf{h}\in\bar{D}^{\times k}}} \frac{\|\varphi(\mathbf{y}+\mathbf{h})-\varphi(\mathbf{y})\|_V}{\|\mathbf{h}\|^{\gamma}},$$

where

$$\mathbf{h} := (h_1, \ldots, h_k) = (h_{1,1}, \ldots, h_{1,d}; h_{2,1}, \ldots, h_{2,d}; \ldots; h_{k,1}, \ldots, h_{k,d}) \in \mathbb{R}^{kd},$$

that is h_j is a vector of d components for each $j = 1, \ldots, k$, and $\|\cdot\|$ denotes the Euclidean norm.

Note that, even if it is not standard, we prefer to specify the subscript $\mathbf{y}$ in the notation of the Banach space-valued Hölder space $\mathcal{C}^{0,\gamma}_{\mathbf{y}}\left(\bar{D}^{\times k};V\right)$, in view of the rest of the paper, where the Banach space V will be a Sobolev space of functions of the spatial variable $x\in D$.

Definition 2. Let $h_j\neq 0$. The one-dimensional difference quotient D^{γ}_{j,h_j} along the direction j and with exponent $0<\gamma\leq 1$ of the function $v:\bar{D}^{\times k}\to\mathbb{R}$ is defined as

$$D^{\gamma}_{j,h_j}v(y_1,\dots,y_k):=\frac{v(y_1,\dots,y_j+h_j,\dots,y_k)-v(y_1,\dots,y_k)}{\left\|h_j\right\|^{\gamma}}. \tag{7}$$

Definition 3. Given $\mathbf{h}=(h_1,\dots,h_k)\in\mathbb{R}^{kd}$, we introduce $\mathbf{i}=\mathbf{i}(\mathbf{h})$ as the vector containing the (non repeated) indices corresponding to the non-zero entries h_j of $\mathbf{h}$, and $\mathbf{i}(\mathbf{h})^c=\{1,\dots,k\}\setminus\mathbf{i}(\mathbf{h})$ (i.e., $h_j\neq(0,\dots,0)$ for all $j\in\mathbf{i}(\mathbf{h})$, and $h_j=(0,\dots,0)$ for all $j\in\mathbf{i}(\mathbf{h})^c$). The mixed difference quotient $D^{\gamma,mix}_{\mathbf{i},\mathbf{h}}$ is defined as

$$D^{\gamma,mix}_{\mathbf{i},\mathbf{h}}:=\prod_{j=1}^{\|\mathbf{h}\|_0}D^{\gamma}_{i_j,h_{i_j}} \tag{8}$$

where $\|\mathbf{h}\|_0:=\#\mathbf{i}(\mathbf{h})$.

In the following, when no confusion arises, we will denote the one-dimensional difference quotient also as D^{γ}_{j}, and the mixed different quotient as $D^{\gamma,mix}_{\mathbf{i}}$, omitting to specify the increment $\mathbf{h}$.

Definition 4 (*Banach Space-Valued Mixed Hölder Space*)**.** Let V be a Banach space, $0<\gamma\leq 1$ be real, and $k\geq 1$ integer. The V-valued mixed Hölder space with exponent γ, $\mathcal{C}^{0,\gamma,mix}_{\mathbf{y}}\left(\bar{D}^{\times k};V\right)$, consists of all continuous maps $\varphi=\varphi(y_1,\dots,y_k):\bar{D}^{\times k}\to V$ with Hölder γ-regularity in each variable y_j, $j=1,\dots,k$, separately. It is a Banach space with the norm

$$\|\varphi\|_{\mathcal{C}^{0,\gamma,mix}_{\mathbf{y}}(\bar{D}^{\times k};V)}:=\max\left\{\|\varphi\|_{\mathcal{C}^{0}_{\mathbf{y}}(\bar{D}^{\times k};V)},|\varphi|_{\mathcal{C}^{0,\gamma,mix}_{\mathbf{y}}(\bar{D}^{\times k};V)}\right\} \tag{9}$$

where $\|\cdot\|_{\mathcal{C}^{0}_{\mathbf{y}}(\bar{D}^{\times k};V)}$ is as in Definition 1, and

$$|\varphi|_{\mathcal{C}^{0,\gamma,mix}_{\mathbf{y}}(\bar{D}^{\times k};V)}:=\max_{j=1,\dots,k}\sup_{\substack{\mathbf{y}\in\bar{D}^{\times k},\ \mathbf{h}\neq\mathbf{0},\\ \text{s.t. } \|\mathbf{h}\|_0=j\\ \text{and } \mathbf{y}+\mathbf{h}\in\bar{D}^{\times k}}}\left\|D^{\gamma,mix}_{\mathbf{i}}\varphi(\mathbf{y})\right\|_V, \tag{10}$$

$D^{\gamma,mix}_{\mathbf{i}}$ being introduced in Definition 3.

Note that, for $k=1$, it holds

$$\mathcal{C}^{0,\gamma,mix}_{y}\left(\bar{D};V\right)=\mathcal{C}^{0,\gamma}_{y}\left(\bar{D};V\right). \tag{11}$$

3.1.1. Banach space-valued Hölder spaces with higher regularity

Let V, k and γ as in Definition 1, and let $n\geq 1$ integer. Moreover, given a vector, denote as $|\cdot|$ its ℓ_1-norm. We introduce

$$\mathcal{C}^{n}_{\mathbf{y}}\left(\bar{D}^{\times k};V\right)=\left\{\begin{array}{c}\varphi:D^{\times k}\to V\text{ s.t. }\forall\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_k)\in\mathbb{N}^{kd}\text{ with}\\ \alpha_\ell=(\alpha_{\ell,1},\dots,\alpha_{\ell,d})\in\mathbb{N}^d\text{ and }\sum_{\ell=1}^{k}|\alpha_\ell|\leq n,\\ \partial^{\boldsymbol{\alpha}}\varphi=\partial^{\alpha_1}_{y_1}\cdots\partial^{\alpha_k}_{y_k}\varphi\in\mathcal{C}^{0}_{\mathbf{y}}\left(\bar{D}^{\times k};V\right)\end{array}\right\}, \tag{12}$$

which is a Banach space with the norm

$$\|\varphi\|_{\mathcal{C}^{n}_{\mathbf{y}}(\bar{D}^{\times k};V)}:=\max_{|\boldsymbol{\alpha}|=0,\dots,n}\left\|\partial^{\boldsymbol{\alpha}}\varphi\right\|_{\mathcal{C}^{0}_{\mathbf{y}}(\bar{D}^{\times k};V)}. \tag{13}$$

We define the Banach space-valued Hölder space with regularity n and exponent γ as

$$\mathcal{C}^{n,\gamma}_{\mathbf{y}}\left(\bar{D}^{\times k};V\right)=\left\{\begin{array}{c}\varphi\in\mathcal{C}^{n}_{\mathbf{y}}\left(\bar{D}^{\times k};V\right)\text{ s.t. }\forall\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_k)\in\mathbb{N}^{kd}\\ \text{with }|\boldsymbol{\alpha}|=|\alpha_1|+\cdots+|\alpha_k|=n\\ \partial^{\boldsymbol{\alpha}}\varphi=\partial^{\alpha_1}_{y_1}\cdots\partial^{\alpha_k}_{y_k}\varphi\in\mathcal{C}^{0,\gamma}_{\mathbf{y}}\left(\bar{D}^{\times k};V\right)\end{array}\right\}. \tag{14}$$

The space $\mathcal{C}^{n,\gamma}_{\mathbf{y}}\left(\bar{D}^{\times k};V\right)$ is a Banach space with seminorm

$$|\varphi|_{\mathcal{C}^{n,\gamma}_{\mathbf{y}}(\bar{D}^{\times k};V)}:=\max_{|\boldsymbol{\alpha}|=n}\left\|\partial^{\boldsymbol{\alpha}}\varphi(\mathbf{y})\right\|_{\mathcal{C}^{0,\gamma}_{\mathbf{y}}(\bar{D}^{\times k};V)} \tag{15}$$

and norm

$$\|\varphi\|_{\mathcal{C}^{n,\gamma}_{\mathbf{y}}(\bar{D}^{\times k};V)} := \max\left\{\|\varphi\|_{\mathcal{C}^{n}_{\mathbf{y}}(\bar{D}^{\times k};V)}, |\varphi|_{\mathcal{C}^{n,\gamma}_{\mathbf{y}}(\bar{D}^{\times k};V)}\right\}. \tag{16}$$

Moreover, we introduce the space

$$\mathcal{C}^{n,mix}_{\mathbf{y}}\left(\bar{D}^{\times k};V\right) = \left\{\begin{array}{c} \varphi : D^{\times k} \to V \text{ s.t. } \forall\, \boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_k) \in \mathbb{N}^{kd} \text{ with} \\ \alpha_\ell = (\alpha_{\ell,1},\dots,\alpha_{\ell,d}) \in \mathbb{N}^d \text{ and } 0 \le |\alpha_\ell| \le n\ \forall\, \ell, \\ \partial^{\boldsymbol{\alpha}}\varphi = \partial^{\alpha_1}_{y_1}\cdots\partial^{\alpha_k}_{y_k}\varphi \in \mathcal{C}^0_{\mathbf{y}}\left(\bar{D}^{\times k};V\right) \end{array}\right\}, \tag{17}$$

which is a Banach space with the norm

$$\|\varphi\|_{\mathcal{C}^{n,mix}_{\mathbf{y}}(\bar{D}^{\times k};V)} := \max_{\substack{(\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_k)\in\mathbb{N}^{kd}\\ 0\le|\boldsymbol{\alpha}_\ell|\le n}} \left\|\partial^{\boldsymbol{\alpha}}\varphi\right\|_{\mathcal{C}^0_{\mathbf{y}}(\bar{D}^{\times k};V)}. \tag{18}$$

Finally, generalizing Definition 4, we introduce the space $\mathcal{C}^{n,\gamma,mix}_{\mathbf{y}}\left(\bar{D}^{\times k};V\right)$ as follows:

$$\mathcal{C}^{n,\gamma,mix}_{\mathbf{y}}\left(\bar{D}^{\times k};V\right) = \left\{\varphi \in \mathcal{C}^{n,mix}_{\mathbf{y}}\left(\bar{D}^{\times k};V\right) \text{ s.t. } |\varphi|_{\mathcal{C}^{n,\gamma,mix}_{\mathbf{y}}(\bar{D}^{\times k};V)} < +\infty\right\}. \tag{19}$$

It is a Banach space with the norm

$$\|\varphi\|_{\mathcal{C}^{n,\gamma,mix}_{\mathbf{y}}(\bar{D}^{\times k};V)} := \max\left\{\|\varphi\|_{\mathcal{C}^{n,mix}_{\mathbf{y}}(\bar{D}^{\times k};V)}, |\varphi|_{\mathcal{C}^{n,\gamma,mix}_{\mathbf{y}}(\bar{D}^{\times k};V)}\right\}, \tag{20}$$

where the seminorm is defined as

$$|\varphi|_{\mathcal{C}^{n,\gamma,mix}_{\mathbf{y}}(\bar{D}^{\times k};V)} := \max_{j=1,\dots,k} \sup_{\substack{\mathbf{y}\in\bar{D}^{\times k},\ \mathbf{h}\neq 0,\\ \text{s.t. } \|\mathbf{h}\|_0=j\\ \text{and } \mathbf{y}+\mathbf{h}\in\bar{D}^{\times k}}} \max_{\substack{\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_k)\in\mathbb{N}^{kd}\\ 0\le|\alpha_\ell|\le n,\ \forall\,\ell=1,\dots,k\\ |\alpha_\ell|=n,\ \ell\in\mathbf{i}(\mathbf{h})}} \left\|D^{\gamma,mix}_{\mathbf{i},\mathbf{h}}\partial^{\boldsymbol{\alpha}}\varphi(\mathbf{y})\right\|_V. \tag{21}$$

Note that, for $k = 1$, it holds $\mathcal{C}^{n,\gamma,mix}_{y}\left(\bar{D};V\right) = \mathcal{C}^{n,\gamma}_{y}\left(\bar{D};V\right)$.

3.1.2. Properties of Banach space-valued mixed Hölder spaces

In this section we prove some properties of Banach space-valued mixed Hölder spaces. We refer to Appendix for the proofs.

Proposition 5. *Let V be a Banach space, and $0 < \gamma \le 1$. Then,*

$$\mathcal{C}^{0,\gamma}_{\mathbf{y}}\left(\bar{D}^{\times k};V\right) \subset \mathcal{C}^{0,\gamma/k,mix}_{\mathbf{y}}\left(\bar{D}^{\times k};V\right) \tag{22}$$

for all $k \ge 2$.

Proposition 6. *The spaces $\mathcal{C}^{0,\gamma}_{y_2}\left(\bar{D};\mathcal{C}^{0,\gamma}_{y_1}\left(\bar{D};V\right)\right)$ and $\mathcal{C}^{0,\gamma}_{y_1}\left(\bar{D};\mathcal{C}^{0,\gamma}_{y_2}\left(\bar{D};V\right)\right)$ are isomorphic to the space $\mathcal{C}^{0,\gamma,mix}_{y_1,y_2}\left(\overline{D\times D};V\right)$ for all $n \ge 0$ integer.*

Remark 7. With small modifications to the proof, it is possible to prove that Proposition 5 holds for Hölder spaces with higher regularity, yielding

$$\mathcal{C}^{n,\gamma}_{\mathbf{y}}\left(\bar{D}^{\times k};V\right) \subset \mathcal{C}^{n,\gamma/k,mix}_{\mathbf{y}}\left(\bar{D}^{\times k};V\right) \tag{23}$$

for all $k \ge 2$. Moreover, Proposition 6 generalizes to higher regularity and higher dimension, yielding

$$\mathcal{C}^{n,\gamma,mix}_{\mathbf{y}}\left(\bar{D}^{\times k};V\right) \sim \mathcal{C}^{n,\gamma,mix}_{\mathbf{y}_{\sim i}}\left(\bar{D}^{\times(k-1)};\mathcal{C}^{n,\gamma}_{y_i}\left(\bar{D};V\right)\right) \quad \forall i = 1,\dots,k+1, \tag{24}$$

where $\mathbf{y}_{\sim i} = (y_1,\dots,y_{i-1},y_{i+1},\dots,y_k)$.

Proposition 8. *Denote with $\mathcal{C}^{0,\gamma,mix}\left(\bar{D}^{\times k}\right)$ the space $\mathcal{C}^{0,\gamma,mix}_{\mathbf{y}}\left(\bar{D}^{\times k};\mathbb{R}\right)$. Then, it holds*

$$\|u\|_{\mathcal{C}^{0,\gamma,mix}(\bar{D}^{\times k})} = \prod_{\ell=1}^{k}\|u_\ell\|_{\mathcal{C}^{0,\gamma}(\bar{D})}, \tag{25}$$

for all $u(y_1,\dots,y_k) := u_1(y_1)\otimes\cdots\otimes u_k(y_k) \in \mathcal{C}^{0,\gamma,mix}\left(\bar{D}^{\times k}\right)$.

Remark 9. With small modifications to the proof, it is possible to prove that Proposition 8 holds for Hölder spaces with higher regularity, yielding to:

$$\|u\|_{\mathcal{C}^{n,\gamma,mix}(\bar{D}^{\times k})} = \prod_{\ell=1}^{k} \|u_\ell\|_{\mathcal{C}^{n,\gamma}(\bar{D})}, \tag{26}$$

for all $u(y_1, \dots, y_k) := u_1(y_1) \otimes \cdots \otimes u_k(y_k) \in \mathcal{C}^{n,\gamma,mix}\left(\bar{D}^{\times k}\right)$.

3.2. Diagonal trace of Sobolev space-valued mixed Hölder maps

In this section we focus on Sobolev space-valued maps with mixed Hölder regularity, namely maps in $(k+1)$ variables $\varphi = \varphi(x, y_1, \dots, y_k) : \bar{D}^{\times(k+1)} \to \mathbb{R}$, which we interpret as $W^{m,p}(D)$-valued maps in k variables $\bar{D}^{\times k} \ni (y_1, \dots, y_k) \mapsto \varphi(\cdot, y_1, \dots, y_k) \in W^{m,p}(D)$. From clarity, we will use the subscript x in the notation $\mathcal{C}^{n,\gamma,mix}_{\mathbf{y}}\left(\bar{D}^{\times k}; W^{m,p}_x(D)\right)$.

Definition 10 (*Diagonal Trace*)**.** Let p, q, N be positive integers satisfying $1 \le p \le q \le N$, and let v be a function of N variables. Then the diagonal trace function $\mathrm{Tr}_{|p:q} v$ is a function of $N - (q-p)$ variables, defined as

$$\left(\mathrm{Tr}_{|p:q}\right) v(x_1, \dots, x_p, x_{q+1}, \dots, x_N) := v(x_1, \dots, x_{p-1}, \underbrace{x_p, \dots, x_p}_{(q-p+1)-\text{times}}, x_{q+1}, \dots, x_N).$$

In the following proposition we state the regularity of the diagonal trace of Sobolev space-valued mixed Hölder maps. We refer to Appendix for the proof.

Proposition 11. *Let $\varphi = \varphi(x, y_1, \dots, y_k) \in \mathcal{C}^{n,\gamma,mix}_{y_1,\dots,y_k}\left(\bar{D}^{\times k}; W^{m,p}_x(D)\right)$, with $D \subset \mathbb{R}^d$, $k \ge 1$ integer, $n \ge m \ge 0$ integers, $\gamma \in (0,1]$ and $p > \frac{2d}{\gamma}$. Then, for all $j = 2, \dots, k+1$, and for all $(y_j, \dots, y_k) \in D^{\times(k-j+1)}$, $(\mathrm{Tr}_{|1:j}\varphi)(x; y_j, \dots, y_k) \in W^{m,p}_x(D)$. In particular, there exists $C_{tr} > 0$ such that*

$$\begin{aligned} &\left\|(\mathrm{Tr}_{|1:j}\varphi)(x; y_j, \dots, y_k)\right\|_{W^{m,p}_x(D)} \\ &\quad \le C_{tr}^{j-1} \left\|\varphi(x, y_1, \dots, y_{j-1}; y_j, \dots, y_k)\right\|_{\mathcal{C}^{n,\gamma,mix}_{y_1,\dots,y_{j-1}}\left(\bar{D}^{\times(j-1)}; W^{m,p}_x(D)\right)}, \end{aligned} \tag{27}$$

for all $(y_j, \dots, y_k) \in D^{\times(k-j+1)}$.

Moreover, $\mathrm{Tr}_{|1:j}\varphi \in \mathcal{C}^{n,\gamma,mix}_{y_j,\dots,y_k}\left(\bar{D}^{\times(k-j+1)}; W^{m,p}_x(D)\right)$, and

$$\left\|\mathrm{Tr}_{|1:j}\varphi\right\|_{\mathcal{C}^{n,\gamma,mix}_{y_j,\dots,y_k}\left(\bar{D}^{\times(k-j+1)}; W^{m,p}_x(D)\right)} \le C_{tr}^{j-1} \|\varphi\|_{\mathcal{C}^{n,\gamma,mix}_{y_1,\dots,y_k}\left(\bar{D}^{\times k}; W^{m,p}_x(D)\right)} \tag{28}$$

for all $j = 2, \dots, k+1$.

4. Recursion on the correlations — analytical results

This section is organized as follows. We first study the mixed Hölder regularity of the input of the recursion (6) , i.e., the $(k+1)$-points correlation function $\mathbb{E}\left[u^0 \otimes Y^{\otimes k}\right]$ (see Corollary 13). Then, in Section 4.2, we prove the well-posedness and regularity of the recursion itself.

4.1. Mixed Hölder regularity of the input of the recursion

The following proposition states the mixed Hölder regularity of the $(k+1)$-points correlation function $\mathbb{E}\left[v \otimes Y^{\otimes k}\right]$, where v belongs to a Banach space V.

Proposition 12. *Let V be a Banach space of functions on D, and Y be a centered Gaussian random field such that $Y \in L^\theta\left(\Omega; \mathcal{C}^{n,\gamma}\left(\bar{D}\right)\right)$, $n \ge 0$, for all $1 \le \theta < +\infty$. Then, for every $v \in V$ and every positive integer k, the $(k+1)$-points correlation $\mathbb{E}\left[v \otimes Y^{\otimes k}\right]$ belongs to the Hölder space with mixed regularity $\mathcal{C}^{n,\gamma,mix}_{\mathbf{y}}\left(\bar{D}^{\times k}; V_x\right)$. Moreover, it holds:*

$$\left\|\mathbb{E}\left[v \otimes Y^{\otimes k}\right]\right\|_{\mathcal{C}^{n,\gamma,mix}_{\mathbf{y}}(\bar{D}^{\times k}; V_x)} = \|v\|_V \left\|\mathbb{E}\left[Y^{\otimes k}\right]\right\|_{\mathcal{C}^{n,\gamma,mix}(\bar{D}^{\times k})}. \tag{29}$$

Proof. We prove that $\mathbb{E}\left[v \otimes Y^{\otimes k}\right] \in \mathcal{C}^{n,\gamma,mix}_{\mathbf{y}}\left(\bar{D}^{\times k}; V_x\right)$ in two steps.

Step 1: $\mathbb{E}\left[Y^{\otimes k}\right] \in \mathcal{C}^{n,\gamma,mix}\left(\bar{D}^{\times k}\right)$

We have to show that

(i) $\mathbb{E}\left[Y^{\otimes k}\right] \in \mathcal{C}^{n,mix}\left(\bar{D}^{\times k}\right)$, i.e., $\partial^{\boldsymbol{\alpha}}\mathbb{E}\left[Y^{\otimes k}\right] = \partial^{\alpha_1}_{x_1} \cdots \partial^{\alpha_k}_{x_k}\mathbb{E}\left[Y^{\otimes k}\right] \in \mathcal{C}^0\left(\bar{D}^{\times k}\right)$ for all $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_k) \in \mathbb{N}^{kd}$ with $0 \le \left|\alpha_j\right| \le n$, for $j = 1, \dots, k$.

(ii) $\partial^{\boldsymbol{\alpha}}\mathbb{E}\left[Y^{\otimes k}\right] \in \mathcal{C}^{0,\gamma,mix}\left(\bar{D}^{\times k}\right)$, for all $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_k) \in \mathbb{N}^{kd}$ with $\left|\alpha_j\right| = n$, for some $j = 1, \dots, k$.

Let us start with (i). Fix $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_k) \in \mathbb{N}^{kd}$ with $0 \le \left|\alpha_j\right| \le n$, for $j = 1, \dots, k$. Then,

$$
\begin{aligned}
\left\|\partial^{\boldsymbol{\alpha}}\mathbb{E}\left[Y^{\otimes k}\right]\right\|_{\mathcal{C}^0(\bar{D}^{\times k})} &= \max_{\mathbf{y}\in\bar{D}^{\times k}} \left|\partial^{\boldsymbol{\alpha}}\mathbb{E}\left[Y^{\otimes k}\right](\mathbf{y})\right| \\
&= \max_{\mathbf{y}\in\bar{D}^{\times k}} \left|\partial^{\alpha_1}_{y_1}\cdots\partial^{\alpha_k}_{y_k}\mathbb{E}\left[Y^{\otimes k}\right](y_1,\dots,y_k)\right| \\
&= \max_{\mathbf{y}\in\bar{D}^{\times k}} \left|\mathbb{E}\left[\partial^{\alpha_1}_{y_1}Y(y_1)\otimes\cdots\otimes\partial^{\alpha_k}_{y_k}Y(y_k)\right]\right| \\
&\le \max_{\mathbf{y}\in\bar{D}^{\times k}} \mathbb{E}\left[\left|\partial^{\alpha_1}_{y_1}Y(y_1)\otimes\cdots\otimes\partial^{\alpha_k}_{y_k}Y(y_k)\right|\right].
\end{aligned}
\tag{30}
$$

Using the Hölder inequality, we get

$$
(30) \le \max_{\mathbf{y}\in\bar{D}^{\times k}} \prod_{i=1}^{k}\left(\mathbb{E}\left[\left|\partial^{\alpha_i}_{y_i}Y(y_i)\right|^k\right]\right)^{1/k} \le \prod_{i=1}^{k}\max_{y_i\in\bar{D}}\left(\mathbb{E}\left[\left|\partial^{\alpha_i}_{y_i}Y(y_i)\right|^k\right]\right)^{1/k}.
$$

Observe that

$$
\begin{aligned}
&\max_{y_i\in\bar{D}}\left(\mathbb{E}\left[\left|\partial^{\alpha_i}_{y_i}Y(y_i)\right|^k\right]\right)^{1/k} = \left(\max_{y_i\in\bar{D}}\mathbb{E}\left[\left|\partial^{\alpha_i}_{y_i}Y(y_i)\right|^k\right]\right)^{1/k} \\
&\quad\le \left(\mathbb{E}\left[\max_{y_i\in\bar{D}}\left|\partial^{\alpha_i}_{y_i}Y(y_i)\right|^k\right]\right)^{1/k} = \left(\mathbb{E}\left[\left(\max_{y_i\in\bar{D}}\left|\partial^{\alpha_i}_{y_i}Y(y_i)\right|\right)^k\right]\right)^{1/k} \\
&\quad\le \left(\mathbb{E}\left[\|Y\|^k_{\mathcal{C}^n(\bar{D})}\right]\right)^{1/k} = \|Y\|_{L^k(\Omega;\mathcal{C}^n(\bar{D}))}.
\end{aligned}
$$

We conclude that

$$
\prod_{i=1}^{k}\max_{y_i\in\bar{D}}\left(\mathbb{E}\left[\left|\partial^{\alpha_i}_{y_i}Y(y_i)\right|^k\right]\right)^{1/k} \le \|Y\|^k_{L^k(\Omega;\mathcal{C}^n(\bar{D}))} < +\infty.
$$

We prove now (ii). Let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_k)$ with $\left|\alpha_j\right| = n$ for some $j = 1, \dots, k$. Using Definitions 2 and 4, we have

$$
\begin{aligned}
&\left|\partial^{\boldsymbol{\alpha}}\mathbb{E}\left[Y^{\otimes k}\right]\right|_{\mathcal{C}^{0,\gamma,mix}(\bar{D}^{\times k})} = \max_{j=1,\dots,k}\sup_{\substack{\mathbf{y},\mathbf{h}\\ \|\mathbf{h}\|_0=j}}\left|D^{\gamma,mix}_{\mathbf{i}}\partial^{\boldsymbol{\alpha}}\mathbb{E}\left[Y^{\otimes k}\right]\right| \\
&\quad= \max_{j=1,\dots,k}\sup_{\substack{\mathbf{y},\mathbf{h}\\ \|\mathbf{h}\|_0=j}}\left|D^{\gamma}_{i_j}\cdots D^{\gamma}_{i_1}\partial^{\boldsymbol{\alpha}}\mathbb{E}\left[Y^{\otimes k}\right]\right| \\
&\quad= \max_{j=1,\dots,k}\sup_{\substack{\mathbf{y},\mathbf{h}\\ \|\mathbf{h}\|_0=j}}\left|D^{\gamma}_{i_j}\cdots D^{\gamma}_{i_1}\mathbb{E}\left[\partial^{\alpha_1}_{y_1}Y(y_1)\otimes\cdots\otimes\partial^{\alpha_k}_{y_k}Y(y_k)\right]\right| \\
&\quad= \max_{j=1,\dots,k}\sup_{\substack{\mathbf{y},\mathbf{h}\\ \|\mathbf{h}\|_0=j}}\left|\mathbb{E}\left[\bigotimes_{\ell\in\mathbf{i}(\mathbf{h})}\frac{\partial^{\alpha_\ell}_{y_\ell}Y(y_\ell+h_\ell)-\partial^{\alpha_\ell}_{y_\ell}Y(y_\ell)}{\|h_\ell\|^\gamma}\cdot\bigotimes_{\ell'\in\mathbf{i}(\mathbf{h})^c}\partial^{\alpha_{\ell'}}_{y_{\ell'}}Y(y_{\ell'})\right]\right|.
\end{aligned}
\tag{31}
$$

Proceeding as in the proof of (i), we conclude

$$
\begin{aligned}
(31) &\le \max_{j=1,\dots,k}\prod_{\ell\in\mathbf{i}(\mathbf{h})}\left(\mathbb{E}\left[\sup_{\substack{\mathbf{y},\mathbf{h}\\ \|\mathbf{h}\|_0=j}}\left|\frac{\partial^{\alpha_\ell}_{y_\ell}Y(y_\ell+h_\ell)-\partial^{\alpha_\ell}_{y_\ell}Y(y_\ell)}{\|h_\ell\|^\gamma}\right|^k\right]\right)^{1/k} \\
&\qquad\prod_{\ell'\in\mathbf{i}(\mathbf{h})^c}\left(\mathbb{E}\left[\left|\partial^{\alpha_{\ell'}}_{y_{\ell'}}Y(y_{\ell'})\right|^k\right]\right)^{1/k} \\
&\le \|Y\|^k_{L^k(\Omega;\mathcal{C}^{n,\gamma}(\bar{D}))} < +\infty.
\end{aligned}
$$

Step 2: $\mathbb{E}\left[v\otimes Y^{\otimes k}\right] \in \mathcal{C}^{n,\gamma,mix}_{\mathbf{y}}\left(\bar{D}^{\times k}; V_x\right)$
It is enough to observe that

$$
\begin{aligned}
\left|\mathbb{E}\left[v\otimes Y^{\otimes k}\right]\right|_{\mathcal{C}^{n,\gamma,mix}_{\mathbf{y}}(\bar{D}^{\times k};V_x)} &= \left|v\otimes\mathbb{E}\left[Y^{\otimes k}\right]\right|_{\mathcal{C}^{n,\gamma,mix}_{\mathbf{y}}(\bar{D}^{\times k};V_x)} \\
&= \|v\|_V\left|\mathbb{E}\left[Y^{\otimes k}\right]\right|_{\mathcal{C}^{n,\gamma,mix}(\bar{D}^{\times k})} < +\infty.
\end{aligned}
$$

It remains us to show equality (29). By definition, it holds:

$$\left\|\mathbb{E}\left[v\otimes Y^{\otimes k}\right]\right\|_{\mathcal{C}_{\mathbf{y}}^{n,mix}(\bar{D}^{\times k};V_x)} = \max_{\boldsymbol{\alpha}}\max_{\mathbf{y}}\left\|\partial^{\boldsymbol{\alpha}}\mathbb{E}\left[v\otimes Y^{\otimes k}\right](\cdot,\mathbf{y})\right\|_{V_x}$$
$$= \max_{\boldsymbol{\alpha}}\max_{\mathbf{y}}\left\|v(\cdot)\otimes\partial^{\boldsymbol{\alpha}}\mathbb{E}\left[Y^{\otimes k}\right](\mathbf{y})\right\|_{V_x} = \max_{\boldsymbol{\alpha}}\max_{\mathbf{y}}\|v\|_{V_x}\left|\partial^{\boldsymbol{\alpha}}\mathbb{E}\left[Y^{\otimes k}\right]\right|$$
$$= \|v\|_{V_x}\left\|\partial^{\boldsymbol{\alpha}}\mathbb{E}\left[Y^{\otimes k}\right]\right\|_{\mathcal{C}_{\mathbf{y}}^{n,mix}(\bar{D}^{\times k})}.$$

In the same way, it is possible to show that

$$\left|\mathbb{E}\left[v\otimes Y^{\otimes k}\right]\right|_{\mathcal{C}_{\mathbf{y}}^{n,\gamma,mix}(\bar{D}^{\times k};V_x)} = \|v\|_{V_x}\left|\partial^{\boldsymbol{\alpha}}\mathbb{E}\left[Y^{\otimes k}\right]\right|_{\mathcal{C}_{\mathbf{y}}^{n,\gamma,mix}(\bar{D}^{\times k})},$$

and equality (29) follows. □

Corollary 13. *Applying Proposition 12 with $v = u^0 \in W^{1,p}(D)$, we have*

$$\mathbb{E}\left[u^0\otimes Y^{\otimes k}\right] \in \mathcal{C}_{\mathbf{y}}^{n,\gamma,mix}\left(\bar{D}^{\times k}; W_x^{1,p}(D)\right).$$

4.2. Well-posedness and regularity of the recursion

To lighten the notation, from now on we denote the kth order correction $\mathbb{E}\left[u^k\right]$ with E^k, and the $(i+1)$-points correlation $\mathbb{E}\left[u^{k-i}\otimes Y^{\otimes i}\right](x, y_1, \ldots, y_i)$ with $E^{k-i,i}$.

Theorem 14 (*Well-Posedness of the Recursion*)**.** *Let $D \subset \mathbb{R}^d$, such that $\partial D \in C^1$, and $Y \in L^\theta\left(\Omega; \mathcal{C}^{0,\gamma}\left(\bar{D}\right)\right)$ for all $1 \le \theta < +\infty$. Let $f \in L^p(D)$ for $p > \frac{2d}{\gamma}$, and $1 < q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, for any $k \ge 0$ and for any $i = 0, \ldots, k-1$, the Laplace–Dirichlet problem: Given $E^{k-i-j,i+j}$ for $j = 1, \ldots, k-i$, find $w(\cdot, \mathbf{y}) \in W_{0,x}^{1,p}(D)$ such that, for all $\mathbf{y} := (y_1, \ldots, y_i) \in D^{\times i}$,*

$$\int_D \left(\nabla\otimes \mathrm{Id}^{\otimes i}\right) w(x,\mathbf{y})\cdot\nabla v(x)\,dx = \mathcal{L}_{\mathbf{y}}(v) \quad \forall v \in W_0^{1,q}(D) \tag{32}$$

has a unique solution, with

$$\|w(\cdot,\mathbf{y})\|_{W_{0,x}^{1,p}(D)} \le C\left\|\mathcal{L}_{\mathbf{y}}\right\|_{(W_0^{1,q})^\star}, \tag{33}$$

where $C > 0$ is independent of $\mathbf{y}$, and the linear form $\mathcal{L}_{\mathbf{y}} : W_0^{1,q}(D) \to \mathbb{R}$ is defined as

$$\mathcal{L}_{\mathbf{y}}(v) := -\sum_{j=1}^{k-i}\binom{k-i}{j}\int_D \mathrm{Tr}_{|1:j+1}\nabla_x E^{k-i-j,i+j}(x,\mathbf{y})\cdot\nabla v(x)\,dx. \tag{34}$$

Moreover, the unique solution belongs to the space $\mathcal{C}_{y_1,\ldots,y_i}^{0,\gamma,mix}\left(\bar{D}^{\times i}; W_{0,x}^{1,p}(D)\right)$ and coincides with $E^{k-i,i}$.

Proof. We prove the theorem by induction. Let $k = 2$ and $i = 1$. The problem we handle with is: given $E^{0,2}$, find $w(\cdot, y) \in W_{0,x}^{1,p}(D)$ s.t., for all $y \in D$,

$$\int_D \left(\nabla\otimes \mathrm{Id}^{\otimes i}\right) w(x,y)\cdot\nabla v(x)\,\mathrm{d}x = \mathcal{L}_y(v) \quad \forall v \in W_0^{1,q}(D), \tag{35}$$

where $\mathcal{L}_y(v) := -\int_D \mathrm{Tr}_{|1:2}\nabla_x E^{0,2}(x,y)\cdot\nabla v(x)\,\mathrm{d}x$.

Step 1: well-posedness of problem (35)

We have to show that $\mathcal{L}_y \in (W_0^{1,q})^\star$. Since $\partial D \in C^1$ and $f \in L^p(D)$, then $u^0 \in W^{1,p}(D)$, as stated in Section 2. Applying Proposition 12 with $n = 0$, we have $\nabla_x E^{0,2} \in \mathcal{C}_{y_1,y_2}^{0,\gamma,mix}\left(\overline{D\times D}; L_x^p(D)\right)$. Applying Proposition 11 with $n = 0$, we get $\mathrm{Tr}_{|1:2}\nabla_x E^{0,2} \in \mathcal{C}_{y_2}^{0,\gamma}\left(D; L_x^p(D)\right)$, and, in particular,

$$C_{\mathcal{L}} := \sup_{y_2\in D}\left\|\mathrm{Tr}_{|1:2}\nabla_x E^{0,2}\right\|_{L_x^p(D)} < \infty.$$

Hence, by the Hölder inequality, we have

$$\left|\mathcal{L}_y(v)\right| \le \left\|\mathrm{Tr}_{|1:2}\nabla_x E^{0,2}\right\|_{L_x^p(D)}\|\nabla v\|_{L_x^q(D)} \le C_{\mathcal{L}}\|v\|_{W_x^{1,q}(D)},$$

so that $\mathcal{L}_y \in (W_0^{1,q})^\star$ for all $y \in D$. Thanks to [22, Chapter 7], we conclude that problem (35) has a unique solution $w(\cdot, y) \in W_{0,x}^{1,p}(D)$ for every $y \in D$. Moreover, there exists a positive constant $C = C(p, d, D)$ such that

$$\|w(\cdot,y)\|_{W_{0,x}^{1,p}(D)} \le C\left\|\mathcal{L}_y\right\|_{(W_0^{1,q}(D))^\star} \le C\,C_{\mathcal{L}}.$$

Step 2: Hölder regularity of $w(x,\cdot)$

Let us consider the difference between problem (35) in $y+h$ and in y:

$$\begin{aligned}&\int_D (\nabla\otimes \mathrm{Id})(w(x,y+h)-w(x,y))\cdot\nabla v(x)\,dx\\&\quad=\int_D(\mathrm{Tr}_{|1:2}\nabla_x E^{0,2}(x,y+h)-\mathrm{Tr}_{|1:2}\nabla_x E^{0,2}(x,y))\cdot\nabla v(x)\,dx,\end{aligned}\tag{36}$$

for all $v\in W_0^{1,q}(D)$. Following the same procedure as in Step 1, we conclude that problem (36) is well-posed, and

$$\|w(\cdot,y+h)-w(\cdot,y)\|_{W^{1,p}_{0,x}(D)}\le C\left\|\mathcal{L}_{y+h}-\mathcal{L}_y\right\|_{(W_0^{1,q}(D))^\star}.\tag{37}$$

Hence, we have

$$\begin{aligned}|w|_{\mathcal{C}_y^{0,\gamma}\left(\bar D;W^{1,p}_{0,x}(D)\right)}&=\sup_{y,h}\frac{1}{\|h\|^\gamma}\|w(\cdot,y+h)-w(\cdot,y)\|_{W^{1,p}_{0,x}(D)}\\&\overset{(37)}{\le} C\sup_{y,h}\frac{1}{\|h\|^\gamma}\left\|\mathcal{L}_{y+h}-\mathcal{L}_y\right\|_{(W_0^{1,q}(D))^\star}\\&=C\sup_{y,h}\frac{1}{\|h\|^\gamma}\sup_{\substack{v\in W_0^{1,q}(D)\\ \|v\|_{W_0^{1,q}(D)}=1}}\left|\int_D(\mathrm{Tr}_{|1:2}\nabla_xE^{0,2}(x,y+h)-\mathrm{Tr}_{|1:2}\nabla_xE^{0,2})\cdot\nabla v(x)\,dx\right|\\&\le C\sup_{y,h}\frac{1}{\|h\|^\gamma}\left\|\mathrm{Tr}_{|1:2}\nabla_xE^{0,2}(\cdot,y+h)-\mathrm{Tr}_{|1:2}\nabla_xE^{0,2}(\cdot,y)\right\|_{L^p_x(D)}\\&=C\sup_{y,h}\left\|D^\gamma_{y,h}\mathrm{Tr}_{|1:2}\nabla_xE^{0,2}(\cdot,y)\right\|_{L^p_x(D)}\le C\left|\mathrm{Tr}_{|1:2}E^{0,2}\right|_{\mathcal{C}_y^{0,\gamma}\left(\bar D;W^{1,p}_{0,x}(D)\right)}\\&\overset{(A.7)}{\le} C\,C_{tr}\left\|E^{0,2}\right\|_{\mathcal{C}^{0,\gamma,mix}_{y_1,y_2}\left(\overline{D\times D};W^{1,p}_{0,x}(D)\right)}<+\infty,\end{aligned}$$

so that $w\in\mathcal{C}_y^{0,\gamma}\left(\bar D;W^{1,p}_{0,x}(D)\right)$. Moreover, since $E^{1,1}$ solves problem (35) for every $y\in D$, then $E^{1,1}\in\mathcal{C}_y^{0,\gamma}\left(\bar D;W^{1,p}_{0,x}(D)\right)$ is the unique solution of (35).

We perform now the induction step. Let $k\ge 2$ and $0\le i\le k-1$ be fixed, and assume that $E^{k-i-j,i+j}\in\mathcal{C}^{0,\gamma,mix}_{y_1,\dots,y_{i+j}}\left(\bar D^{\times(i+j)};W^{1,p}_{0,x}(D)\right)$, for $j=1,\dots,k-i$.

Step 1: well-posedness of problem (32)

We have to show that $\mathcal{L}_{\mathbf{y}}$ as in (34) is in $(W_0^{1,q})^\star$. Since $E^{k-i-j,i+j}\in\mathcal{C}^{0,\gamma,mix}_{y_1,\dots,y_{i+j}}\left(\bar D^{\times(i+j)};W^{1,p}_{0,x}(D)\right)$, then $\mathrm{Tr}_{|1:j+1}\nabla_xE^{k-i-j,i+j}\in\mathcal{C}^{0,\gamma,mix}_{y_1,\dots,y_i}\left(\bar D^{\times i};L^p_x(D)\right)$, and, in particular,

$$C_{\mathcal{L},j}:=\sup_{y_1,\dots,y_i\in D^{\times i}}\left\|\mathrm{Tr}_{|1:j+1}\nabla_xE^{k-i-j,i+j}\right\|_{L^p_x(D)}<\infty.$$

Hence, by the Hölder inequality, we have $\left|\mathcal{L}_{\mathbf{y}}(v)\right|\le C_{\mathcal{L}}\|v\|_{W^{1,q}(D)}$, with $C_{\mathcal{L}}:=\sum_{j=1}^{k-i}\binom{k-i}{j}C_{\mathcal{L},j}$, so that $\mathcal{L}_{\mathbf{y}}\in(W_0^{1,q})^\star$. Thanks to [22, Chapter 7], we conclude that problem (35) has a unique solution $w(\cdot,\mathbf{y})\in W^{1,p}_{0,x}(D)$ for a.e. $\mathbf{y}\in D^{\times i}$. Moreover, it holds

$$\|w(\cdot,\mathbf{y})\|_{W^{1,p}_{0,x}(D)}\le C\left\|\mathcal{L}_{\mathbf{y}}\right\|_{(W_0^{1,q}(D))^\star}\le C\,C_{\mathcal{L}}.$$

Step 2: Hölder regularity of $w(x,\cdot)$

By considering the problem solved by $D^{\gamma,mix}_{\mathbf{i}}w(x,\mathbf{y})$, we have

$$\begin{aligned}&\left\|D^{\gamma,mix}_{\mathbf{i}}w(\cdot,\mathbf{y})\right\|_{W^{1,p}_{0,x}(D)}\\&\quad\le C\sup_{\substack{v\in W_0^{1,q}(D)\\ \|v\|_{W_0^{1,q}(D)}=1}}\left|\sum_{j=1}^{k-i}\binom{k-i}{j}\int_D D^{\gamma,mix}_{\mathbf{i}}\mathrm{Tr}_{|1:j+1}\nabla E^{k-i-j,i+j}\cdot\nabla v\,dx\right|\\&\quad\le C\sum_{j=1}^{k-i}\binom{k-i}{j}\left\|D^{\gamma,mix}_{\mathbf{i}}\mathrm{Tr}_{|1:j+1}\nabla E^{k-i-j,i+j}(\cdot,\mathbf{y})\right\|_{L^p_x}.\end{aligned}\tag{38}$$

Hence, we have

$$\begin{aligned}|w|_{\mathcal{C}^{0,\gamma,mix}_{\mathbf{y}}\left(\bar{D}^{\times i};W^{1,p}_{0,x}(D)\right)} &= \max_{\ell=1,\dots,i}\ \sup_{\mathbf{y},\mathbf{h},\|\mathbf{h}\|_0=\ell} \left\| D^{\gamma,mix}_{\mathbf{i}} w(\cdot,\mathbf{y})\right\|_{W^{1,p}_{0,x}(D)} \\ &\overset{(38)}{\leq} C \max_{\ell=1,\dots,i}\ \sup_{\mathbf{y},\mathbf{h},\|\mathbf{h}\|_0=\ell} \sum_{j=1}^{k-i} \binom{k-i}{j} \left\| D^{\gamma,mix}_{\mathbf{i}} \mathrm{Tr}_{|_{1:j+1}} \nabla E^{k-i-j,i+j}(\cdot,\mathbf{y})\right\|_{L^p_x} \\ &\leq C \sum_{j=1}^{k-i} \binom{k-i}{j} \left|\mathrm{Tr}_{|_{1:j+1}} \nabla E^{k-i-j,i+j}\right|_{\mathcal{C}^{0,\gamma,mix}_{\mathbf{y}}\left(\bar{D}^{\times i};L^p_x(D)\right)} \\ &\leq C \sum_{j=1}^{k-i} \binom{k-i}{j} C^j_{tr} \left\| E^{k-i-j,i+j}\right\|_{\mathcal{C}^{0,\gamma,mix}_{y_1,\dots,y_{i+j}}\left(\bar{D}^{\times(i+j)};W^{1,p}_x(D)\right)} < +\infty.\end{aligned}$$

In particular, since $E^{k-i,i}$ solves problem (32) for a.e. $\mathbf{y}\in D^{\times i}$, then $E^{k-i,i}\in \mathcal{C}^{0,\gamma,mix}_{\mathbf{y}}\left(\bar{D}^{\times i}; W^{1,p}_{0,x}(D)\right)$ is the unique solution of (32). □

Theorem 15 (*Regularity of the Recursion*)**.** *Let $D\subset\mathbb{R}^d$ such that $\partial D\in C^{2+r}$, $r\geq 0$. Let $f\in W^{r,p}(D)$, and $Y\in L^\theta\left(\Omega;\mathcal{C}^{n,\gamma}\left(\bar{D}\right)\right)$, for all $1\leq\theta<\infty$ and $n\geq r+1$. Then the correlation $E^{k-i,i}\in\mathcal{C}^{n,\gamma,mix}_{y_1,\dots,y_i}\left(\bar{D}^{\times i}; W^{2+r,p}_x(D)\cap W^{1,p}_{0,x}(D)\right)$ for all $i=k,k-1,\dots,0$. Moreover, there exists a positive constant C_{reg} independent of $\mathbf{y}=(y_1,\dots,y_i)$, such that*

$$\left\|E^{k-i,i}(\cdot,\mathbf{y})\right\|_{W^{2+r,p}(D)}\leq C_{reg}\left\|\mathcal{L}_{\mathbf{y}}\right\|_{(W^{r,q})^*}, \tag{39}$$

where $\mathcal{L}_{\mathbf{y}}$ has been introduced in (34).

Proof. We prove the theorem by induction. Let $k=2$ and $i=1$. Since $f\in W^{r,p}(D)$, we have $u^0\in W^{1,p}_0(D)\cap W^{2+r,p}(D)$ (see [22, Chapter 9]). Using the assumption $Y\in L^\theta\left(\Omega;\mathcal{C}^{n,\gamma}\left(\bar{D}\right)\right)$ and Proposition 12, we have

$$E^{0,2}\in\mathcal{C}^{n,\gamma,mix}_{y_1,y_2}\left(\overline{D\times D}; W^{1,p}_{0,x}(D)\cap W^{2+r,p}_x(D)\right),$$

so that $\nabla_x E^{0,2}\in\mathcal{C}^{n,\gamma,mix}_{y_1,y_2}\left(\overline{D\times D}; W^{1+r,p}_x(D)\right)$. Applying Proposition 11, we have $\mathrm{Tr}_{|_{1:2}}\nabla E^{0,2}\in\mathcal{C}^{n,\gamma}_y\left(\bar{D}; W^{1+r,p}_x(D)\right)$. Following the same reasoning as in the proof of Theorem 14, we have that $E^{1,1}\in\mathcal{C}^{n,\gamma}_y\left(\bar{D}; W^{2+r,p}_x(D)\right)$. Finally, since we have already shown that $E^{1,1}\in\mathcal{C}^{n,\gamma}_y\left(\bar{D}; W^{1,p}_{0,x}(D)\right)$, we conclude the thesis.

We perform now the induction step. Assume that the correlation

$$E^{k-i-j,i+j}\in\mathcal{C}^{n,\gamma,mix}_{y_1,\dots,y_{i+j}}\left(\bar{D}^{\times(i+j)}; W^{1,p}_{0,x}(D)\cap W^{r+2,p}_x(D)\right),$$

for $j=1,\dots,k-i$. Applying Proposition 11, we have

$$\mathrm{Tr}_{|_{1:j+1}}\nabla E^{k-i-j,i+j}\in\mathcal{C}^{n,\gamma,mix}_{y_1,\dots,y_i}\left(\bar{D}^{\times i}; W^{r+1,p}_x(D)\right).$$

Following the same reasoning as in the proof of Theorem 14, we conclude that

$$E^{k-i,i}\in\mathcal{C}^{n,\gamma,mix}_{y_1\dots,y_i}\left(\bar{D}^{\times i}; W^{2+r,p}_x(D)\right).$$

Finally, since we have already shown that $E^{k-i,i}\in\mathcal{C}^{n,\gamma,mix}_{y_1\dots,y_i}\left(\bar{D}^{\times i}; W^{1,p}_{0,x}(D)\right)$, we conclude the thesis.

Finally, the upper bound (39) follows from [22, Chapter 9], observing that $\mathcal{L}_{\mathbf{y}}\in(W^{1+r,p})^*$. □

Proposition 16. *Under the assumptions of Theorem 15, it holds*

$$\left\|E^{k-i,i}\right\|_{\mathcal{C}^{n,\gamma,mix}_{y_1,\dots,y_i}\left(\bar{D}^{\times i};W^{2+r,p}_x(D)\right)}\leq\lambda_{k-i}\left\|E^{0,k}\right\|_{\mathcal{C}^{n,\gamma,mix}_{y_1,\dots,y_k}\left(\bar{D}^{\times k};W^{2+r,p}_x(D)\right)} \tag{40}$$

for all $i\leq k$, where the coefficients $\{\lambda_{k-i}\}_{i=1}^k$ are defined by recursion as $\lambda_0:=1$ and $\lambda_{k-i}:=C_{reg}\sum_{j=1}^{k-i}\binom{k-i}{j}C^j_{tr}\lambda_{k-i-j}$ for $i<k$, the constants C_{reg}, C_{tr} being introduced in Theorem 15 and Proposition 11, respectively.

Proof. Let k be fixed. We prove the Theorem by induction on i. If $i=k$, bound (40) holds as an equality. Let now $i<k$ fixed. By induction, we assume

$$\left\|E^{k-\ell,\ell}\right\|_{\mathcal{C}^{n,\gamma,mix}_{y_1,\dots,y_\ell}\left(\bar{D}^{\times \ell};W^{2+r,p}_{0,x}(D)\right)}\leq\lambda_{k-\ell}\left\|E^{0,k}\right\|_{\mathcal{C}^{n,\gamma,mix}_{y_1,\dots,y_k}\left(\bar{D}^{\times k};W^{2+r,p}_{0,x}(D)\right)}, \tag{41}$$

for all $i+1 \leq \ell \leq k-1$. Thanks to (39) and Proposition 11, it holds:

$$\begin{aligned}&\left\|E^{k-i,i}\right\|_{\mathcal{C}^{n,\gamma,mix}_{y_1,\ldots,y_i}\left(\bar{D}^{\times i};W^{2+r,p}_{0,x}(D)\right)}\\&\leq C_{reg}\sum_{j=1}^{k-i}\binom{k-i}{j}C_{tr}^{j}\left\|E^{k-i-j,i+j}\right\|_{\mathcal{C}^{n,\gamma,mix}_{y_1,\ldots,y_{i+j}}\left(\bar{D}^{\times(i+j)};W^{2+r,p}_{0,x}(D)\right)}.\end{aligned}$$

Using the assumption (41), we have

$$\begin{aligned}&\left\|E^{k-i,i}\right\|_{\mathcal{C}^{n,\gamma,mix}_{y_1,\ldots,y_i}\left(\bar{D}^{\times i};W^{2+r,p}_{0,x}(D)\right)}\\&\leq C_{reg}\sum_{j=1}^{k-i}\binom{k-i}{j}C_{tr}^{j}\,\lambda_{k-i-j}\left\|E^{0,k}\right\|_{\mathcal{C}^{n,\gamma,mix}_{y_1,\ldots,y_k}\left(\bar{D}^{\times k};W^{2+r,p}_{0,x}(D)\right)}\\&=\lambda_{k-i}\left\|E^{0,k}\right\|_{\mathcal{C}^{n,\gamma,mix}_{y_1,\ldots,y_k}\left(\bar{D}^{\times k};W^{2+r,p}_{0,x}(D)\right)}. \quad \square\end{aligned}$$

5. Recursion on the correlations — sparse discretization

Within this section we aim at deriving a discretization for the recursive problem (6). In particular, the differential operator in the spatial variable x will be discretized by a Galerkin method, whereas the parametric dependence on the variable $\mathbf{y}$ will be approximated by a sparse interpolation technique. To this end, we first introduce some assumptions on the considered Galerkin projector π_h, and the sparse interpolant operator $\widehat{P}_L$.

5.1. Galerkin projector

Let $\{W_h\}_{h\geq 0}$ be a sequence of nested finite dimensional subspaces of $W^{1,\infty}_0(D)$, h being the discretization parameter. Assume that

$$\min_{w_h\in W_h}\|u-w_h\|_{W^{1,p}(D)} \leq C_{fem}h^{\beta}\,|u|_{W^{2+r,p}(D)} \quad \forall\, u\in W^{1,p}_0(D)\cap W^{2+r,p}(D), \tag{42}$$

for some $\beta\in(0,r+1)$. Moreover, we assume the following discrete inf–sup condition: there exists $\nu>0$ independent of h such that

$$\inf_{u_h\in W_h}\sup_{w_h\in W_h}\frac{\int_D \nabla u_h\cdot\nabla w_h\,dx}{|u_h|_{W^{1,q}(D)}\,|w_h|_{W^{1,q}(D)}}\geq \nu. \tag{43}$$

Then, we can infer

$$\|u-\pi_h u\|_{W^{1,p}(D)}\leq C_{\pi_h}h^{\beta}\,|u|_{W^{2+r,p}(D)} \quad \forall u\in W^{1,p}_0(D)\cap W^{2+r,p}(D), \tag{44}$$

where $\pi_h : W^{1,p}_0(D)\cap W^{2+r,p}(D) \to W_h$ is the Galerkin projector, i.e., the operator which associates u to its finite dimensional approximation via the Galerkin method, and $C_{\pi_h}>0$ is independent of h.

Remark 17. The discrete inf–sup condition (43), as well as (42), hold, for instance, for D convex $\mathcal{C}^1$, and continuous $\mathbb{P}^1$ finite elements (with $\beta=1$ and $r=0$). We refer to [23] for further cases.

5.2. Sparse interpolant operator

Let $\{V_\ell\}_{l\geq 0}$ be a dense sequence of nested finite dimensional subspaces of $\mathcal{C}^{0,\gamma}\left(\bar{D}\right)$, and let the discretization parameter h_ℓ of V_ℓ be $h_\ell := \frac{h_{\ell-1}}{2}$, so that $h_\ell = h_0\,2^{-\ell}$. Denote with $\{a^\ell_j\}_{j=1}^{N_\ell}\subset D$ a set of interpolation points unisolvent in V_ℓ, and with $\{\xi^\ell_j\}_{j=1}^{N_\ell}$ the Lagrangian basis of V_ℓ such that $\xi^\ell_j(a^\ell_i)=\delta_{i,j}$ for all $i,j=1,\ldots,N_\ell$. Moreover, let $P_\ell : \mathcal{C}^{0,\gamma}\left(\bar{D}\right)\to V_\ell$ be the Lagrangian interpolation operator, that is $P_\ell(v)=\sum_{j=1}^{N_\ell}v(a^\ell_j)\xi^\ell_j$, and note that P_ℓ is a projector, too. Assume that P_ℓ satisfies the following property:

$$\|u-P_\ell u\|_{\mathcal{C}^{0,\gamma}(\bar{D})}\leq C\,h_\ell^{s}\,\|u\|_{\mathcal{C}^{n,\gamma}(\bar{D})} \quad \forall u\in\mathcal{C}^{n,\gamma}\left(\bar{D}\right), \tag{45}$$

where $C>0$ is independent of h_ℓ, and $s>0$.

Following [24], we define the sparse interpolation operator as follows. Let $\Delta_\ell := P_\ell - P_{\ell-1}$ be the difference operator. Given k, L positive integers, the *sparse interpolation operator of level L* is defined as:

$$\widehat{P}_{L,k} := \sum_{\substack{\boldsymbol{\ell}=(\ell_1,\ldots,\ell_k)\in\mathbb{N}^k\\|\boldsymbol{\ell}|\leq L}}\bigotimes_{j=1}^{k}\Delta_{\ell_j}. \tag{46}$$

The sparse interpolation operator $\widehat{P}_{L,k}$ maps the Hölder space with mixed regularity $\mathcal{C}^{0,\gamma,mix}\left(\bar{D}^{\times k}\right)$ onto the sparse tensor product space $\widehat{V}_{L,k}$, defined as

$$\widehat{V}_{L,k} := \bigcup_{\substack{\boldsymbol{\ell}=(\ell_1,\ldots,\ell_k)\in\mathbb{N}^k \\ |\boldsymbol{\ell}|\leq L}} \bigotimes_{j=1}^{k} V_{\ell_j}. \tag{47}$$

The application of the sparse interpolation operator $\widehat{P}_{L,k}$ to a function implies the evaluation of the function itself in a finite set of points – the *sparse grid* – denoted as $\mathcal{H}_{L,k}$. To lighten the notation, the sparse interpolation operator $\widehat{P}_{L,k}$ will be simply denoted as $\widehat{P}_L$, when no confusion occurs.

Proposition 18. *Let k be a positive integer and W a Banach space of functions on D. Then it holds:*

$$\left\|\widehat{P}_{L,k}u - u\right\|_{\mathcal{C}^{0,\gamma,mix}_{\mathbf{y}}(\bar{D}^{\times k};W_x)} \leq C_{\widehat{P}_{L,k}}\, h_L^{s(1-\tau)} \left\|u\right\|_{\mathcal{C}^{n,\gamma,mix}_{\mathbf{y}}(\bar{D}^{\times k};W_x)} \tag{48}$$

for all $u = u(x,\mathbf{y}) \in \mathcal{C}^{n,\gamma,mix}_{\mathbf{y}}\left(\bar{D}^{\times k};W_x\right)$*, with* $\mathbf{y} = (y_1,\ldots,y_k) \in \bar{D}^{\times k}$*, where* $0 < \tau < 1$*, and* $C_{\widehat{P}_{L,k}}$ *is a positive constant independent of* h_L *(but blowing up when* $\tau \to 0$*).*

Proof. The bound (48) is derived by standard computations (see, e.g., [25]). For completeness, we report here all the steps.

Denote with $\hat{\mathbf{y}}_i \in \bar{D}^{\times(k-1)}$ the vector $(y_1,\ldots,y_{i-1},y_{i+1},\ldots,y_k)$. We start giving an upper bound for the norm $\left\|\Delta_\ell \otimes \mathrm{Id}^{\otimes(k-1)} u(\cdot,\hat{\mathbf{y}}_1)\right\|_{\mathcal{C}^{0,\gamma,mix}_{y_1}(\bar{D};W_x)}$. Using the triangular inequality and (45), we have

$$\begin{aligned}
&\left\|\Delta_\ell \otimes \mathrm{Id}^{\otimes(k-1)} u(\cdot,\hat{\mathbf{y}}_1)\right\|_{\mathcal{C}^{0,\gamma,mix}_{y_1}(\bar{D};W_x)} \\
&\quad\leq \left\|(P_\ell - \mathrm{Id}) \otimes \mathrm{Id}^{\otimes(k-1)} u(\cdot,\hat{\mathbf{y}}_1)\right\|_{\mathcal{C}^{0,\gamma,mix}_{y_1}(\bar{D};W_x)} \\
&\qquad + \left\|(P_{\ell-1} - \mathrm{Id}) \otimes \mathrm{Id}^{\otimes(k-1)} u(\cdot,\hat{\mathbf{y}}_1)\right\|_{\mathcal{C}^{0,\gamma,mix}_{y_1}(\bar{D};W_x)} \\
&\quad\leq \left(Ch_\ell^s + Ch_{\ell-1}^s\right) \left\|u(\cdot,\hat{\mathbf{y}}_1)\right\|_{\mathcal{C}^{n,\gamma,mix}_{y_1}(\bar{D};W_x)} \\
&\quad\leq 2Ch_{\ell-1}^s \left\|u(\cdot,\hat{\mathbf{y}}_1)\right\|_{\mathcal{C}^{n,\gamma,mix}_{y_1}(\bar{D};W_x)} \\
&\quad\leq 2Ch_0^s\, 2^{-s(\ell-1)} \left\|u(\cdot,\hat{\mathbf{y}}_1)\right\|_{\mathcal{C}^{n,\gamma,mix}_{y_1}(\bar{D};W_x)},
\end{aligned} \tag{49}$$

where we have used that $h_\ell \leq h_{\ell-1}$, and $h_{\ell-1} = h_0\, 2^{-(\ell-1)}$.

Using (49), it follows:

$$\begin{aligned}
&\left\|\Delta_{\ell_1} \otimes \Delta_{\ell_2} \otimes \mathrm{Id}^{\otimes(k-2)} u\right\|_{\mathcal{C}^{0,\gamma,mix}_{y_1,y_2}(\overline{D\times D};W_x)} \\
&\quad= \left\|(\Delta_{\ell_1} \otimes \mathrm{Id}^{\otimes(k-1)})(\mathrm{Id} \otimes \Delta_{\ell_2} \otimes \mathrm{Id}^{\otimes(k-2)}) u\right\|_{\mathcal{C}^{0,\gamma,mix}_{y_1,y_2}(\overline{D\times D};W_x)} \\
&\quad\leq 4C^2 h_0^{2s}\, 2^{-s(\ell_1-1)} 2^{-s(\ell_2-1)} \left\|u\right\|_{\mathcal{C}^{n,\gamma,mix}_{y_1,y_2}(\overline{D\times D};W_x)}.
\end{aligned}$$

By recursion, we have

$$\begin{aligned}
&\left\|\Delta_{\ell_1} \otimes \cdots \otimes \Delta_{\ell_k} u\right\|_{\mathcal{C}^{0,\gamma,mix}_{\mathbf{y}}(\bar{D};W_x)} \\
&\quad\leq 2^k C^k \left(\frac{h_0}{2}\right)^{sk} 2^{-s|\boldsymbol{\ell}|} \left\|u\right\|_{\mathcal{C}^{n,\gamma,mix}_{\mathbf{y}}(\bar{D};W_x)}.
\end{aligned} \tag{50}$$

By (50), it follows that the series $\sum_{\boldsymbol{\ell}\in\mathbb{N}^k} \otimes_{n=1}^k \Delta_{\ell_n} u$ is absolutely convergent, and that $\sum_{|\boldsymbol{\ell}|\leq L} \otimes_{n=1}^k \Delta_{\ell_n} u$ converges to u as $L \to \infty$ in $\mathcal{C}^{0,\gamma,mix}_{\mathbf{y}}\left(\bar{D}^{\times k};W_x\right)$.

Finally, we have

$$\begin{aligned}
\left\|\widehat{P}_L u - u\right\|_{\mathcal{C}^{0,\gamma,mix}_{\mathbf{y}}(\bar{D}^{\times k};W_x)} &= \left\|\sum_{|\boldsymbol{\ell}|\leq L} \bigotimes_{n=1}^{k} \Delta_{\ell_n} u - u\right\|_{\mathcal{C}^{0,\gamma,mix}_{\mathbf{y}}(\bar{D}^{\times k};W_x)} \\
&= \left\|\sum_{|\boldsymbol{\ell}|> L} \bigotimes_{n=1}^{k} \Delta_{\ell_n} u\right\|_{\mathcal{C}^{0,\gamma,mix}_{\mathbf{y}}(\bar{D}^{\times k};W_x)} \leq \sum_{|\boldsymbol{\ell}|> L} \left\|\bigotimes_{n=1}^{k} \Delta_{\ell_n} u\right\|_{\mathcal{C}^{0,\gamma,mix}_{\mathbf{y}}(\bar{D}^{\times k};W_x)} \\
&\leq 2^k C^k \left(\frac{h_0}{2}\right)^{sk} \left(\sum_{|\boldsymbol{\ell}|> L} 2^{-s|\boldsymbol{\ell}|}\right) \left\|u\right\|_{\mathcal{C}^{n,\gamma,mix}_{\mathbf{y}}(\bar{D};W_x)}.
\end{aligned}$$

In [26, Lemma 6.10] the authors prove that

$$\sum_{|\boldsymbol{\ell}|>L} 2^{-s|\boldsymbol{\ell}|} \leq \left(\frac{1}{1-2^{-s\tau}}\right)^k 2^{-Ls(1-\tau)} = \left(\frac{1}{1-2^{-s\tau}}\right)^k h_0^{s(\tau-1)} h_L^{s(1-\tau)},$$

with $0<\tau<1$. Hence, we conclude (48) with

$$C_{\widehat{P}_{L,k}} = 2^k \left(\frac{h_0}{2}\right)^{sk} h_0^{s(\tau-1)} C^k \left(\frac{1}{1-2^{-s\tau}}\right)^k. \quad \square$$

Remark 19. The result proved in Proposition 18 holds whenever P_ℓ is any operator fulfilling (45).

5.3. Sparse discretization of the recursion

As highlighted in Section 2, the input of the recursion – at the continuous level – is the $(k+1)$-points correlation $E^{0,k}$. In the same way – at the discrete level – we start giving a (sparse) discretization of $E^{0,k}$. It is obtained in two consecutive steps. First, we define the finite dimensional approximation of u^0 by applying the Galerkin projector π_h to u^0, i.e., $u_h^0 := \pi_h u^0$. The fully-discrete sparse approximation of $E^{0,k}$, denoted as $E_{L,h}^{0,k}$, is then obtained by applying the sparse interpolant operator $\widehat{P}_{L,k}$ to the semi-discrete correlation $E_h^{0,k} := u_h^0 \otimes E^k$, i.e.,

$$E_{L,h}^{0,k} := \widehat{P}_{L,k} E_h^{0,k} = \pi_h u^0 \otimes \widehat{P}_{k,L} E^k.$$

Note that the semi-discrete correlation $E_h^{0,k}$ is an element of $\mathcal{C}_{\mathbf{y}}^{n,\gamma,mix}\left(\bar{D}^{\times k}; W_h\right)$, whereas the fully-discrete correlation $E_{L,h}^{0,k}$ is an element of the tensor product space $W_h \otimes \widehat{V}_{L,k}$.

Let $i = k-1, \ldots, 0$ fixed. The fully-discrete sparse approximation of the correlation $E^{k-i,i}$ is obtained as

$$E_{L,h}^{k-i,i} := \widehat{P}_{L,i} E_h^{k-i,i},$$

where the semi-discrete correlation $E_h^{k-i,i}$ is defined as the unique solution of the following recursive problem: given all lower order terms $E_{L,h}^{k-i-j,i+j} \in W_h \otimes \widehat{V}_{L,i+j}$ for $j=1,\ldots,k-i$, find $E_h^{k-i,i} \in \mathcal{C}_{\mathbf{y}}^{n,\gamma,mix}\left(\bar{D}^{\times i}; W_h\right)$ such that

$$\int_D \nabla E_h^{k-i,i}(x,\mathbf{y}) \cdot \nabla \varphi_h(x)\, dx$$
$$= -\sum_{j=1}^{k-1} \begin{pmatrix} k-i \\ j \end{pmatrix} \int_D (\mathrm{Tr}_{|_{1:j+1}} \nabla E_{L,h}^{k-i-j,i+j})(x,\mathbf{y}) \cdot \nabla \varphi_h(x)\, dx \tag{51}$$

for all $\varphi \in W_h$.

In the next theorem we analyze the discretization error.

Theorem 20. *Let* (42), (44) and (45) *hold. Moreover, let the assumptions of* Theorem 15 *be satisfied. Then, it holds*

$$\left\|(E^{k-i,i} - E_{L,h}^{k-i,i})(x,\mathbf{y})\right\|_{W_x^{1,p}(D)} = O(\min\{h^\beta, h_L^{s(1-\tau)}\}), \tag{52}$$

where $0<\tau<1$ *has been introduced in* Proposition 18.

To prove Theorem 20, we need to show some preliminary results.

Lemma 21. *Let the assumptions of* Theorem 20 *hold, and define*

$$\theta_{n,m} = \begin{cases} 1, & \text{if } n = m \\ 0, & \text{if } n < m \\ C_S \sum_{j=1}^{n-m} \begin{pmatrix} n \\ j \end{pmatrix} C_{tr}^j \theta_{n-j,m}, & \text{if } n > m, \end{cases} \tag{53}$$

C_{tr} *being as in* Proposition 11. *Then, it holds:*

$$\left\|(E^{k-i,i} - E_{L,h}^{k-i,i})(x,\mathbf{y})\right\|_{W_x^{1,p}(D)}$$
$$\leq C_S\, C_{fem}\, h^\beta \sum_{m=i}^{k-1} \theta_{k-i,k-m} \left\|E^{k-m,m}(x,\mathbf{y}^{(m)};\mathbf{y})\right\|_{\mathcal{C}_{\mathbf{y}^{(m)}}^{0,\gamma,mix}\left(\bar{D}^{\times(m-i)}; W_x^{2+r,p}(D)\right)}$$

$$
\begin{aligned}
&+\sum_{m=i}^{k-1}\theta_{k-i,k-m}\left\|(E_h^{k-m,m}-E_{L,h}^{k-m,m})(x,\mathbf{y}^{(m)};\mathbf{y})\right\|_{\mathcal{C}^{0,\gamma,mix}_{\mathbf{y}^{(m)}}\left(\bar{D}^{\times(m-i)};W_x^{1,p}(D)\right)}\\
&+\theta_{k-i,0}\left\|(E^{0,k}-E_{L,h}^{0,k})(x,\mathbf{y}^{(k)};\mathbf{y})\right\|_{\mathcal{C}^{0,\gamma,mix}_{\mathbf{y}^{(k)}}\left(\bar{D}^{\times(k-i)};W_x^{1,p}(D)\right)} \qquad (54)
\end{aligned}
$$

for all $\mathbf{y}:=(y_{k-i+1},\dots,y_k)\in\bar{D}^{\times i}$, *where* $\mathbf{y}^{(m)}:=(y_{k-m+1},\dots,y_{k-i})\in\bar{D}^{\times(m-i)}$, *for* $m=i,\dots,k$.

Proof. Let k be fixed. We prove the result by induction on i. If $i=k$, then (54) holds as an equality.

Let $i<k$, and assume, by induction, that (54) holds for all $i+1\le j\le k$. Denote $e_{k-i,i}:=E^{k-i,i}-E_{L,h}^{k-i,i}$, and $f_{k-i,i}:=E_h^{k-i,i}-E_{L,h}^{k-i,i}$. By triangular inequality we have:

$$
\left\|e_{k-i,i}(x,\mathbf{y})\right\|_{W_x^{1,p}}\le\left\|E^{k-i,i}-E_h^{k-i,i}(x,\mathbf{y})\right\|_{W_x^{1,p}}+\left\|f_{k-i,i}(\cdot,\mathbf{y})\right\|_{W_x^{1,p}}. \qquad (55)
$$

The discrete inf–sup condition (43) implies the Strang's Lemma in the spaces $W^{1,p}(D)$-$W^{1,q}(D)$. Then, there holds:

$$
\begin{aligned}
&\left\|(E^{k-i,i}-E_h^{k-i,i})(x,\mathbf{y})\right\|_{W_x^{1,p}}\\
&\le C_S\left(\inf_{\varphi_h\in W_h}\left\|E^{k-i,i}(x,\mathbf{y})-\varphi_h(x)\right\|_{W_x^{1,p}}+\sup_{\varphi_h\in W_h}\frac{\left|\mathcal{L}_{\mathbf{y}}(\varphi_h)-\mathcal{L}_h(\varphi_h)\right|}{\|\varphi_h\|_{W_x^{1,q}}}\right), \qquad (56)
\end{aligned}
$$

where $\mathcal{L}_{\mathbf{y}},\mathcal{L}_h:W_h\to\mathbb{R}$ are the functionals defining the right-hand side of problems (6) and (51), respectively. The bound on the first term on the right-hand side of (56) follows from the approximation property (42):

$$
\inf_{\varphi_h\in W_h}\left\|E^{k-i,i}(x,\mathbf{y})-\varphi_h(x)\right\|_{W_x^{1,p}}\le C_{fem}\,h^\beta\left|E^{k-i,i}(x,\mathbf{y})\right|_{W_x^{2+r,p}}, \qquad (57)
$$

with $\beta\in(0,r+1)$. We bound now the second term on the right-hand side of (56). Using the Hölder inequality and Proposition 11, we have:

$$
\begin{aligned}
&\left|\mathcal{L}_{\mathbf{y}}(\varphi_h)-\mathcal{L}_h(\varphi_h)\right|\\
&\le\sum_{l=1}^{k-i}\binom{k-i}{l}\left|\int_D\left(\mathrm{Tr}_{|_{1:l+1}}\nabla(E^{k-i-l,i+l}-E_{L,h}^{k-i-l,i+l})\right)(x,\mathbf{y})\cdot\nabla\varphi_h(x)\,dx\right|\\
&\le\sum_{l=1}^{k-i}\binom{k-i}{l}\left\|\left(\mathrm{Tr}_{|_{1:l+1}}\nabla(E^{k-i-l,i+l}-E_{L,h}^{k-i-l,i+l})\right)(x,\mathbf{y})\right\|_{L_x^p}\|\nabla\varphi_h\|_{L_x^q}\\
&\le\sum_{l=1}^{k-i}\binom{k-i}{l}C_{tr}^l\left\|e_{k-i-l,i+l}(x,\mathbf{y}^{(i+l)};\mathbf{y})\right\|_{\mathcal{C}^{0,\gamma,mix}_{\mathbf{y}^{(i+l)}}\left(\bar{D}^{\times l};W_x^{1,p}\right)}\|\varphi_h\|_{W_x^{1,q}}. \qquad (58)
\end{aligned}
$$

Inserting (56), (57) and (58) into (55), we have:

$$
\begin{aligned}
&\left\|e_{k-i,i}(x,\mathbf{y})\right\|_{W_x^{1,p}}\le C_S\,C_{fem}\,h^\beta\left|E^{k-i,i}(x,\mathbf{y})\right|_{W_x^{2+r,p}}\\
&+C_S\sum_{l=1}^{k-i}\binom{k-i}{l}C_{tr}^l\left\|e_{k-i-l,i+l}(\cdot,\mathbf{y}^{(i+l)};\mathbf{y})\right\|_{\mathcal{C}^{0,\gamma,mix}_{\mathbf{y}^{(i+l)}}\left(\bar{D}^{\times l};W_x^{1,p}\right)}\\
&+\left\|f_{k-i,i}(x,\mathbf{y})\right\|_{W_x^{1,p}}. \qquad (59)
\end{aligned}
$$

Using the inductive assumption on $e_{k-i-l,i+l}$, we get:

$$
\begin{aligned}
&\left\|e_{k-i,i}(x,\mathbf{y})\right\|_{W_x^{1,p}}\le C_S\,C_{fem}\,h^\beta\left|E^{k-i,i}(x,\mathbf{y})\right|_{W_x^{2+r,p}}\\
&+C_S\sum_{l=1}^{k-i}\binom{k-i}{l}C_{tr}^l\\
&\Bigg(C_S\,C_{fem}\,h^\beta\sum_{m=i+l}^{k-1}\theta_{k-i-l,k-m}\left\|E^{k-m,m}(x,\mathbf{y}^{(m)};\mathbf{y})\right\|_{\mathcal{C}^{0,\gamma,mix}_{\mathbf{y}^{(m)}}\left(\bar{D}^{\times(m-i)};W_x^{2+r,p}\right)}\\
&+\sum_{m=i+l}^{k-1}\theta_{k-i-l,k-m}\left\|f_{k-m,m}(x,\mathbf{y}^{(m)};\mathbf{y})\right\|_{\mathcal{C}^{0,\gamma,mix}_{\mathbf{y}^{(m)}}\left(\bar{D}^{\times(m-i)};W_x^{1,p}\right)}
\end{aligned}
$$

$$
\left. + \theta_{k-i-l,0} \left\| e_0(x, \mathbf{y}^{(k)}; \mathbf{y}) \right\|_{\mathcal{C}^{0,\gamma,mix}_{\mathbf{y}^{(k)}}\left(\bar{D}^{\times(k-i)}; W^{1,p}_x\right)} \right)
+ \left\| f_{k-i,i}(x, \mathbf{y}) \right\|_{W^{1,p}_x} . \tag{60}
$$

Observe that, by definition of $\theta_{k-i,0}$, we have

$$
C_S \sum_{l=1}^{k-i} \binom{k-i}{l} C^l_{tr} \theta_{k-i-l,0} = \theta_{k-i,0}. \tag{61}
$$

Moreover, by switching the sum in l and m, and using that $\theta_{k-i,k-i} = 1$, we have

$$
\begin{aligned}
C_S \sum_{l=1}^{k-i} \sum_{m=i+l}^{k-1} \binom{k-i}{l} C^l_{tr}\, \theta_{k-i-l,k-m} &= C_S \sum_{m=i+1}^{k-1} \sum_{l=1}^{m-i} \binom{k-i}{l} C^l_{tr} \theta_{k-i-l,k-m} \\
&= \sum_{m=i+1}^{k-1} \theta_{k-i,k-m},
\end{aligned}
$$

so that

$$
\begin{aligned}
& C_S \sum_{l=1}^{k-i} \sum_{m=i+l}^{k-1} \binom{k-i}{l} C^l_{tr}\, \theta_{k-i-l,k-m} \left\| f_{k-m,m}(x, \mathbf{y}^{(m)}; \mathbf{y}) \right\|_{\mathcal{C}^{0,\gamma,mix}_{\mathbf{y}^{(m)}}\left(\bar{D}^{\times(m-i)}; W^{1,p}_x\right)} \\
& \quad + \left\| f_{k-i,i}(x, \mathbf{y}) \right\|_{W^{1,p}_x} \\
& = \sum_{m=i+1}^{k-1} \theta_{k-i,k-m} \left\| f_{k-m,m}(x, \mathbf{y}^{(m)}; \mathbf{y}) \right\|_{\mathcal{C}^{0,\gamma,mix}_{\mathbf{y}^{(m)}}\left(\bar{D}^{\times(m-i)}; W^{1,p}_x\right)} + \left\| f_{k-i,i}(x, \mathbf{y}) \right\|_{W^{1,p}_x} \\
& = \sum_{m=i}^{k-1} \theta_{k-i,k-m} \left\| f_{k-m,m}(x, \mathbf{y}^{(m)}; \mathbf{y}) \right\|_{\mathcal{C}^{0,\gamma,mix}_{\mathbf{y}^{(m)}}\left(\bar{D}^{\times(m-i)}; W^{1,p}_x\right)} ,
\end{aligned} \tag{62}
$$

and

$$
\begin{aligned}
& C_S^2\, C_{fem}\, h^\beta \sum_{l=1}^{k-i} \sum_{m=i+l}^{k-1} \binom{k-i}{l} C^l_{tr}\, \theta_{k-i-l,k-m} \left\| E^{k-m,m}(x, \mathbf{y}^{(m)}; \mathbf{y}) \right\|_{\mathcal{C}^{0,\gamma,mix}_{\mathbf{y}^{(m)}}\left(\bar{D}^{\times(m-i)}; W^{2+r,p}_x\right)} \\
& \quad + C_S\, C_{fem}\, h^\beta \left\| E^{k-i,i}(x, \mathbf{y}) \right\|_{W^{2+r,p}_x} \\
& = C_S\, C_{fem}\, h^\beta \sum_{m=i+1}^{k-1} \theta_{k-i,k-m} \left\| E^{k-m,m}(x, \mathbf{y}^{(m)}; \mathbf{y}) \right\|_{\mathcal{C}^{0,\gamma,mix}_{\mathbf{y}^{(m)}}\left(\bar{D}^{\times(m-i)}; W^{2+r,p}_x\right)} \\
& \quad + C_S\, C_{fem}\, h^\beta \left\| E^{k-i,i}(x, \mathbf{y}) \right\|_{W^{2+r,p}_x} \\
& = C_S\, C_{fem}\, h^\beta \sum_{m=i}^{k-1} \theta_{k-i,k-m} \left\| E^{k-m,m}(x, \mathbf{y}^{(m)}; \mathbf{y}) \right\|_{\mathcal{C}^{0,\gamma,mix}_{\mathbf{y}^{(m)}}\left(\bar{D}^{\times(m-i)}; W^{2+r,p}_x\right)} .
\end{aligned} \tag{63}
$$

Inserting (61), (62) and (63) into (60), we conclude the bound (54). □

Lemma 22. *Under the assumptions of Theorem 20 it holds:*

$$
\begin{aligned}
& \left\| E^{0,k} - E^{0,k}_{L,h} \right\|_{\mathcal{C}^{0,\gamma,mix}_{y_1,\ldots,y_k}\left(\bar{D}^{\times k}; W^{1,p}_x(D)\right)} \\
& \quad \le \left(C_{\pi_h} h^\beta + C_{\widehat{P}_{L,k}}\, h_L^{s(1-\tau)} (C_\pi\, h^\beta + 1) \right) \left\| E^{0,k} \right\|_{\mathcal{C}^{0,\gamma,mix}_{y_1,\ldots,y_k}\left(\bar{D}^{\times k}; W^{2+r,p}_x(D)\right)} .
\end{aligned} \tag{64}
$$

Proof. Using the triangular inequality, we have

$$
\begin{aligned}
& \left\| E^{0,k} - E^{0,k}_{L,h} \right\|_{\mathcal{C}^{0,\gamma,mix}_{y_1,\ldots,y_k}\left(\bar{D}^{\times k}; W^{1,p}_x(D)\right)} \\
& \quad \le \left\| E^{0,k} - E^{0,k}_h \right\|_{\mathcal{C}^{0,\gamma,mix}_{y_1,\ldots,y_k}\left(\bar{D}^{\times k}; W^{1,p}_x(D)\right)} + \left\| E^{0,k}_h - E^{0,k}_{L,h} \right\|_{\mathcal{C}^{0,\gamma,mix}_{y_1,\ldots,y_k}\left(\bar{D}^{\times k}; W^{1,p}_x(D)\right)} .
\end{aligned} \tag{65}
$$

We bound the two terms at the right hand side of (65) separately. Using (44), we have

$$\begin{aligned}
&\left\| E^{0,k} - E_h^{0,k} \right\|_{\mathcal{C}^{0,\gamma,mix}_{y_1,\dots,y_k}\left(\bar{D}^{\times k}; W_x^{1,p}(D)\right)} \\
&= \left\| u^0 - \pi_h u^0 \right\|_{W^{1,p}(D)} \left\| E^k \right\|_{\mathcal{C}^{0,\gamma,mix}(\bar{D}^{\times k})} \\
&\le C_{\pi_h} h^\beta \left| u^0 \right|_{W^{2+r,p}(D)} \left\| E^k \right\|_{\mathcal{C}^{0,\gamma,mix}(\bar{D}^{\times k})} \\
&= C_{\pi_h} h^\beta \left\| E^{0,k} \right\|_{\mathcal{C}^{0,\gamma,mix}(\bar{D}^{\times k}; W^{2+r,p}(D))} .
\end{aligned} \tag{66}$$

Moreover, applying Proposition 18, the triangular inequality, and (44), we have

$$\begin{aligned}
&\left\| E_h^{0,k} - E_{L,h}^{0,k} \right\|_{\mathcal{C}^{0,\gamma,mix}_{y_1,\dots,y_k}\left(\bar{D}^{\times k}; W_x^{1,p}(D)\right)} \le C_{\widehat{P}_{L,k}} h_L^{s(1-\tau)} \left\| E_h^{0,k} \right\|_{\mathcal{C}^{n,\gamma,mix}_{y_1,\dots,y_k}\left(\bar{D}^{\times k}; W_x^{1,p}(D)\right)} \\
&\le C_{\widehat{P}_{L,k}} h_L^{s(1-\tau)} \left(\left\| E^{0,k} - E_h^{0,k} \right\|_{\mathcal{C}^{n,\gamma,mix}_{y_1,\dots,y_k}\left(\bar{D}^{\times k}; W_x^{1,p}(D)\right)} + \left\| E^{0,k} \right\|_{\mathcal{C}^{n,\gamma,mix}_{y_1,\dots,y_k}\left(\bar{D}^{\times k}; W_x^{1,p}(D)\right)} \right) \\
&\le C_{\widehat{P}_{L,k}} h_L^{s(1-\tau)} (C_\pi h^\beta + 1) \left\| E^{0,k} \right\|_{\mathcal{C}^{n,\gamma,mix}_{y_1,\dots,y_k}\left(\bar{D}^{\times k}; W_x^{1,p}(D)\right)} .
\end{aligned} \tag{67}$$

The result is then proved inserting (66) and (67) into (65). □

Theorem 20. To prove (52) we bound each term at the right-hand side of (54), separately. Applying Proposition 16, we have:

$$\begin{aligned}
&C_S\, C_{fem}\, h^\beta \sum_{m=i}^{k-1} \theta_{k-i,k-m} \left\| E^{k-m,m}(x, \mathbf{y}^{(m)}; \mathbf{y}) \right\|_{\mathcal{C}^{0,\gamma,mix}_{\mathbf{y}^{(m)}}\left(\bar{D}^{\times(m-i)}; W_x^{2+r,p}(D)\right)} \\
&\le C_S\, C_{fem}\, h^\beta \left(\sum_{m=i}^{k-1} \theta_{k-i,k-m}\, \lambda_{k-m} \right) \left\| E^{0,k}(x, \mathbf{y}^{(k)}; \mathbf{y}) \right\|_{\mathcal{C}^{0,\gamma,mix}_{\mathbf{y}^{(k)}}\left(\bar{D}^{\times(k-i)}; W_x^{2+r,p}(D)\right)} .
\end{aligned} \tag{68}$$

Applying Proposition 18, the triangular inequality, and Proposition 16, we have:

$$\begin{aligned}
&\sum_{m=i}^{k-1} \theta_{k-i,k-m} \left\| (E_h^{k-m,m} - E_{L,h}^{k-m,m})(x, \mathbf{y}^{(m)}; \mathbf{y}) \right\|_{\mathcal{C}^{0,\gamma,mix}_{\mathbf{y}^{(m)}}\left(\bar{D}^{\times(m-i)}; W_x^{1,p}(D)\right)} \\
&\le \sum_{m=i}^{k-1} \theta_{k-i,k-m} C_{\widehat{P}_{L,m}} h_L^{s(1-\tau)} \left\| E_h^{k-m,m}(x, \mathbf{y}^{(m)}; \mathbf{y}) \right\|_{\mathcal{C}^{n,\gamma,mix}_{\mathbf{y}^{(m)}}\left(\bar{D}^{\times(m-i)}; W_x^{1,p}(D)\right)} \\
&\le h_L^{s(1-\tau)} \sum_{m=i}^{k-1} \theta_{k-i,k-m} C_{\widehat{P}_{L,m}} \Bigg(\left\| E^{k-m,m}(x, \mathbf{y}^{(m)}; \mathbf{y}) \right\|_{\mathcal{C}^{n,\gamma,mix}_{\mathbf{y}^{(m)}}\left(\bar{D}^{\times(m-i)}; W_x^{1,p}(D)\right)} \\
&\quad + \left\| (E^{k-m,m} - E_h^{k-m,m})(x, \mathbf{y}^{(m)}; \mathbf{y}) \right\|_{\mathcal{C}^{n,\gamma,mix}_{\mathbf{y}^{(m)}}\left(\bar{D}^{\times(m-i)}; W_x^{1,p}(D)\right)} \Bigg) \\
&\le h_L^{s(1-\tau)} \sum_{m=i}^{k-1} \theta_{k-i,k-m} C_{\widehat{P}_{L,m}} \Bigg(\lambda_{k-m} \left\| E^{0,k}(x, \mathbf{y}^{(k)}; \mathbf{y}) \right\|_{\mathcal{C}^{n,\gamma,mix}_{\mathbf{y}^{(k)}}\left(\bar{D}^{\times(k-i)}; W_x^{2+r,p}(D)\right)} \\
&\quad + C_{\pi_h} h^\beta \left\| E^{k-m,m}(x, \mathbf{y}^{(m)}; \mathbf{y}) \right\|_{\mathcal{C}^{n,\gamma,mix}_{\mathbf{y}^{(m)}}\left(\bar{D}^{\times(m-i)}; W_x^{2+r,p}(D)\right)} \Bigg) \\
&\le h_L^{s(1-\tau)} (C_{\pi_h} h^\beta + 1) \sum_{m=i}^{k-1} \theta_{k-i,k-m} C_{\widehat{P}_{L,m}} \lambda_{k-m} \left\| E^{0,k}(x, \mathbf{y}^{(k)}; \mathbf{y}) \right\|_{\mathcal{C}^{n,\gamma,mix}_{\mathbf{y}^{(k)}}\left(\bar{D}^{\times(k-i)}; W_x^{2+r,p}(D)\right)} .
\end{aligned} \tag{69}$$

The result follows by applying Lemma 22, and inserting (68), and (69) into (54). □

Remark 23. The finite dimensional spaces W_h and V_ℓ are defined on the same physical domain D. It is then natural to take $V_\ell = W_h$ – $\mathcal{T}_\ell$ having discretization parameter h_ℓ – and $h = h_L$. Then, (52) becomes:

$$\left\| (E^{k-i,i} - E_{L,h}^{k-i,i})(x, \mathbf{y}) \right\|_{W_x^{1,p}(D)} = O(h^{\min\{\beta, s(1-\tau)\}}).$$

6. Conclusions

This paper addresses the computation of an approximation for the expected value of the unique stochastic solution u to the Darcy problem with lognormal permeability coefficient. In particular, we adopt the perturbation method –

approximating the solution by its Taylor polynomial $T^K u$ – in combination with the moment equation technique – approximating $\mathbb{E}[u]$ by $\mathbb{E}\left[T^K u\right]$. The first moment equation is recalled, and its recursive structure is explained. In particular, for each $k = 0, \dots, K$, a recursion on the $(i+1)$-points correlation $E^{k-i,i}$, $i = 1, \dots, k$, is needed. Well-posedness and regularity results for the recursion satisfied by $E^{k-i,i}$ are proved. In particular we show that $E^{k-i,i} \in \mathcal{C}^{n,\gamma,mix}\left(\bar{D}^{\times i}; W^{2+r,p}(D)\right)$, under the assumptions $Y \in \mathcal{C}^{n,\gamma}\left(\bar{D}\right)$ a.s. and $u^0 \in W^{2+r,p}(D) \cap W_0^{1,p}(D)$. Finally, a sparse discretization for the recursion is analyzed, and the convergence of the sparse discretization error is proved.

The procedure proposed in this paper can be used also to approximate higher moments of u. In particular, we refer to [20] for the recursion on the two-points correlation of u, $\mathbb{E}[u \otimes u]$. Moreover, the bounds on sparse grid approximations derived in this work could also be useful to establish convergence estimates for low rank approximations as the Tensor Train considered in [20] (see, e.g., [27]).

Acknowledgments

The author F. Bonizzoni acknowledges partial support from the Austrian Science Fund (FWF) through the project F 65, and has been supported by the FWF Firnberg-Program Austria, grant T998.

Appendix

Proposition 5. Let V be a Banach space, and $0 < \gamma \leq 1$. Then,

$$\mathcal{C}_{\mathbf{y}}^{0,\gamma}\left(\bar{D}^{\times k}; V\right) \subset \mathcal{C}_{\mathbf{y}}^{0,\gamma/k,mix}\left(\bar{D}^{\times k}; V\right) \tag{A.1}$$

for all $k \geq 2$.

Proof. We first prove (A.1) for $k = 2$. Let $\varphi \in \mathcal{C}_{y_1,y_2}^{0,\gamma}\left(\overline{D \times D}; V\right)$. Then,

$$|\varphi|_{\mathcal{C}_{y_1,y_2}^{0,\gamma/2,mix}(\overline{D\times D};V)} = \max\left\{\sup_{y_1,y_2,h_1}\left\|D_1^{\gamma/2}\varphi(y_1,y_2)\right\|_V, \sup_{y_1,y_2,h_2}\left\|D_2^{\gamma/2}\varphi(y_1,y_2)\right\|_V, \right.$$
$$\left.\sup_{y_1,y_2,h_1,h_2}\left\|D_1^{\gamma/2}D_2^{\gamma/2}\varphi(y_1,y_2)\right\|_V\right\}. \tag{A.2}$$

We bound the three terms in (A.2) separately. Observe that

$$\begin{aligned}\sup_{y_1,y_2,h_1}\left\|D_1^{\gamma/2}\varphi(y_1,y_2)\right\|_V &= \sup_{y_1,y_2,h_1}\frac{\|\varphi(y_1+h_1,y_2)-\varphi(y_1,y_2)\|_V}{\|h_1\|^{\gamma/2}}\\ &= \sup_{y_1,y_2,h_1}\|h_1\|^{\gamma/2}\frac{\|\varphi(y_1+h_1,y_2)-\varphi(y_1,y_2)\|_V}{\|h_1\|^{\gamma}}\\ &\leq \max\{1, diam(D)^{\gamma/2}\}\,|\varphi|_{\mathcal{C}_{y_1,y_2}^{0,\gamma}(\overline{D\times D};V)},\end{aligned} \tag{A.3}$$

which is bounded by assumption, and the same holds for $\sup_{y_1,y_2,h_2}\left\|D_2^{\gamma/2}\varphi(y_1,y_2)\right\|_V$. We focus now on the third term in (A.2). Define

$$\begin{aligned}w(y_1,y_2;h_1,h_2) &:= D_1^{\gamma/2}D_2^{\gamma/2}\varphi(y_1,y_2)\,\|h_1\|^{\gamma/2}\,\|h_2\|^{\gamma/2}\\ &= \varphi(y_1+h_1,y_2+h_2)-\varphi(y_1+h_1,y_2)-\varphi(y_1,y_2+h_2)+\varphi(y_1,y_2).\end{aligned}$$

Hence, we have

$$\begin{aligned}\sup_{y_1,y_2,h_1,h_2}\left\|D_1^{\gamma/2}D_2^{\gamma/2}\varphi(y_1,y_2)\right\|_V &= \sup_{y_1,y_2,h_1,h_2}\frac{\|w(y_1,y_2;h_1,h_2)\|_V}{\|h_1\|^{\gamma/2}\,\|h_2\|^{\gamma/2}}\\ &\leq \max\left\{\sup_{\mathbf{y},\|h_1\|<\|h_2\|}\frac{\|w(y_1,y_2;h_1,h_2)\|_V}{\|h_1\|^{\gamma/2}\,\|h_2\|^{\gamma/2}}, \sup_{\mathbf{y},\|h_1\|\geq\|h_2\|}\frac{\|w(y_1,y_2;h_1,h_2)\|_V}{\|h_1\|^{\gamma/2}\,\|h_2\|^{\gamma/2}}\right\}.\end{aligned}$$

We start considering

$$\begin{aligned}
&\sup_{\mathbf{y},\|h_1\|<\|h_2\|} \frac{\|w(y_1,y_2;h_1,h_2)\|_V}{\|h_1\|^{\gamma/2}\,\|h_2\|^{\gamma/2}} \\
&\leq \sup_{\mathbf{y},\|h_1\|<\|h_2\|} \frac{1}{\|h_1\|^{\gamma/2}\,\|h_2\|^{\gamma/2}} \left(\|h_1\|^{\gamma} \frac{\|\varphi(y_1+h_1,y_2+h_2)-\varphi(y_1,y_2+h_2)\|_V}{\|h_1\|^{\gamma}} \right. \\
&\quad \left. + \|h_1\|^{\gamma} \frac{\|\varphi(y_1+h_1,y_2)-\varphi(y_1,y_2)\|_V}{\|h_1\|^{\gamma}} \right) \\
&\leq \sup_{\mathbf{y},\|h_1\|<\|h_2\|} \frac{\|h_1\|^{\gamma/2}}{\|h_2\|^{\gamma/2}} \left(\left\|D_1^{\gamma}\varphi(y_1,y_2+h_2)\right\|_V + \left\|D_1^{\gamma}\varphi(y_1,y_2)\right\|_V \right) \\
&\leq 2\,\|\varphi\|_{C^{0,\gamma}_{y_1,y_2}(\overline{D\times D};V)}\,.
\end{aligned}$$

The case $\|h_1\| \geq \|h_2\|$ is analogous. Hence, we conclude (A.1) for $k=2$.

In the general case, given $\mathbf{h}=(h_1,\ldots,h_k)$, let $\mathbf{i}(\mathbf{h})$ as in Definition 3, and $i^* \in \{\mathbf{i}(\mathbf{h})\}$ such that $\|h_{i^*}\| \leq \left\|h_{i_j}\right\|$ for all j such that $i^* \neq i_j$. Moreover, define $w(\cdot,\mathbf{y};\mathbf{h}) := D_{\mathbf{i}}^{\gamma,mix}\varphi(\cdot,\mathbf{y})\prod_{\ell=1}^{j}\left\|h_{i_\ell}\right\|^{\gamma/j}$. We bound each term of the seminorm (10) as follows:

$$\begin{aligned}
&\sup_{\mathbf{y},\|h_{i^*}\|\leq\left\|h_{i_j}\right\|} \frac{\|w(\mathbf{y};\mathbf{h})\|_V}{\prod_{\ell=1}^{j}\left\|h_{i_\ell}\right\|^{\gamma/j}} \leq \sup_{\mathbf{y},\|h_{i^*}\|\leq\left\|h_{i_j}\right\|} \frac{\|h_{i^*}\|^{\gamma}}{\prod_{\ell=1}^{j}\left\|h_{i_\ell}\right\|^{\gamma/j}} \frac{\|w(\mathbf{y};\mathbf{h})\|_V}{\|h_{i^*}\|^{\gamma}} \\
&\leq 2^{j-1}\,|\varphi|_{C^{0,\gamma}_{\mathbf{y}}(\bar{D}^{\times k};V_x)}\,,
\end{aligned}$$

and the inclusion (A.1) is then proved. □

Proposition 6. The spaces $C^{0,\gamma}_{y_2}\left(\bar{D};C^{0,\gamma}_{y_1}\left(\bar{D};V\right)\right)$ and $C^{0,\gamma}_{y_1}\left(\bar{D};C^{0,\gamma}_{y_2}\left(\bar{D};V\right)\right)$ *are* isomorphic to the space $C^{0,\gamma,mix}_{y_1,y_2}\left(\overline{D\times D};V\right)$ for all $n\geq 0$ integer.

Proof. According to definition (20), we have

$$\begin{aligned}
&\|\varphi\|_{C^{0,\gamma,mix}_{y_1,y_2}(\overline{D\times D};V)} = \max\left\{ \|\varphi\|_{C^{0,mix}_{y_1,y_2}(\overline{D\times D};V)}\,,\, |\varphi|_{C^{0,\gamma,mix}_{y_1,y_2}(\overline{D\times D};V)} \right\} \\
&= \max\Big\{ \max_{y_1,y_2}\|\varphi(y_1,y_2)\|_V\,,\, \sup_{(y_1,y_2),h_1}\left\|D_1^{\gamma}\varphi(y_1,y_2)\right\|_V\,, \\
&\qquad \sup_{(y_1,y_2),h_2}\left\|D_2^{\gamma}\varphi(y_1,y_2)\right\|_V\,,\, \sup_{(y_1,y_2),(h_1,h_2)}\left\|D_2^{\gamma}D_1^{\gamma}\varphi(y_1,y_2)\right\|_V \Big\}.
\end{aligned}$$

On the other hand, we have

$$\|\varphi\|_{C^{0,\gamma}_{y_2}\left(\bar{D};C^{0,\gamma}_{y_1}(\bar{D};V)\right)} = \max\left\{ \|\varphi\|_{C^{0}_{y_2}\left(\bar{D};C^{0,\gamma}_{y_1}(\bar{D};V)\right)}\,,\, |\varphi|_{C^{0,\gamma}_{y}\left(\bar{D};C^{0,\gamma}_{y_1}(\bar{D};V)\right)} \right\},$$

where

$$\begin{aligned}
&\|\varphi\|_{C^{0}_{y_2}\left(\bar{D};C^{0,\gamma}_{y_1}(\bar{D};V)\right)} = \max_{y_2}\|\varphi(\cdot,y_2)\|_{C^{0,\gamma}_{y_1}(\bar{D};V)} \\
&= \max_{y_2}\max\left\{ \|\varphi(\cdot,y_2)\|_{C^{0}_{y_1}(\bar{D};V)}\,,\, |\varphi(\cdot,y_2)|_{C^{0,\gamma}_{y_1}(\bar{D};V)} \right\} \\
&= \max\left\{ \max_{y_1,y_2}\|\varphi(y_1,y_2)\|_V\,,\, \max_{y_2}\sup_{y_1,h_1}\left\|D_1^{\gamma}\varphi(y_1,y_2)\right\|_V \right\},
\end{aligned}$$

and

$$\begin{aligned}
&|\varphi|_{C^{0}_{y_2}\left(\bar{D};C^{0,\gamma}_{y_1}(\bar{D};V)\right)} = \sup_{y_2,h_2}\left\|D_2^{\gamma}\varphi(\cdot,y_2)\right\|_{C^{0,\gamma}_{y_1}(\bar{D};V)} \\
&= \sup_{y_2,h_2}\max\left\{ \left\|D_2^{\gamma}\varphi(\cdot,y_2)\right\|_{C^{0}_{y_1}(\bar{D};V)}\,,\, \left|D_2^{\gamma}\varphi(\cdot,y_2)\right|_{C^{0,\gamma}_{y_1}(\bar{D};V)} \right\} \\
&= \max\left\{ \max_{y_1}\sup_{y_2,h_2}\left\|D_2^{\gamma}\varphi(y_1,y_2)\right\|_V\,,\, \sup_{y_2,h_2}\sup_{y_1,h_1}\left\|D_1^{\gamma}D_2^{\gamma}\varphi(y_1,y_2)\right\|_V \right\}.
\end{aligned}$$

Hence, we conclude that $\|\varphi\|_{C^{0,\gamma,mix}_{y_1,y_2}(\overline{D\times D};V)} = \|\varphi\|_{C^{0,\gamma}_{y_2}\left(\bar{D};C^{0,\gamma}_{y_1}(\bar{D};V)\right)}$. In the same way, it is possible to show that $\|\varphi\|_{C^{0,\gamma,mix}_{y_1,y_2}(\overline{D\times D};V)} = \|\varphi\|_{C^{0,\gamma}_{y_1}\left(\bar{D};C^{0,\gamma}_{y_2}(\bar{D};V)\right)}$. □

Proposition 8. *Denote with* $\mathcal{C}^{0,\gamma,mix}\left(\bar{D}^{\times k}\right)$ *the space* $\mathcal{C}^{0,\gamma,mix}_{\mathbf{y}}\left(\bar{D}^{\times k};\mathbb{R}\right)$*. Then, it holds*

$$\|u\|_{\mathcal{C}^{0,\gamma,mix}(\bar{D}^{\times k})} = \prod_{\ell=1}^{k} \|u_\ell\|_{\mathcal{C}^{0,\gamma}(\bar{D})}, \tag{A.4}$$

for all $u(y_1,\ldots,y_k) := u_1(y_1)\otimes\cdots\otimes u_k(y_k) \in \mathcal{C}^{0,\gamma,mix}\left(\bar{D}^{\times k}\right)$.

Proof. Using (9), we have:

$$\|u\|_{\mathcal{C}^{0,\gamma,mix}(\bar{D}^{\times k})} = \max\left\{\|u_1\otimes\cdots\otimes u_k\|_{\mathcal{C}^{0}(\bar{D}^{\times k})}, |u_1\otimes\cdots\otimes u_k|_{\mathcal{C}^{0,\gamma,mix}(\bar{D}^{\times k})}\right\}.$$

Observe that:

$$\|u_1\otimes\cdots\otimes u_k\|_{\mathcal{C}^{0}(\bar{D}^{\times k})} = \max_{(y_1,\ldots,y_k)\in\bar{D}^{\times k}} |u_1(y_1)\otimes\cdots\otimes u_k(y_k)|$$

$$= \max_{(y_1,\ldots,y_k)\in\bar{D}^{\times k}} \prod_{\ell=1}^{k} |u_\ell(y_\ell)| = \prod_{\ell=1}^{k} \max_{y_\ell\in\bar{D}} |u_\ell(y_\ell)| = \prod_{\ell=1}^{k} \|u_\ell\|_{\mathcal{C}^{0}(\bar{D})}.$$

We focus now on the seminorm of u:

$$|u_1\otimes\cdots\otimes u_k|_{\mathcal{C}^{0,\gamma,mix}(\bar{D}^{\times k})} = \max_{j=1,\ldots,k} \sup_{\substack{\mathbf{y}\in\bar{D}^{\times k},\ \mathbf{h}\neq 0,\\ \|\mathbf{h}\|_0=j,\ \mathbf{y}+\mathbf{h}\in\bar{D}^{\times k}}} \left|D^{\gamma,mix}_{\mathbf{i}} u(y_1,\ldots,y_k)\right|$$

$$= \max_{j=1,\ldots,k} \sup_{\substack{\mathbf{y}\in\bar{D}^{\times k},\ \mathbf{h}\neq 0,\\ \|\mathbf{h}\|_0=j,\ \mathbf{y}+\mathbf{h}\in\bar{D}^{\times k}}} \prod_{\ell\in\{1,\ldots,k\}\setminus\{\mathbf{i}\}} |u_\ell(y_\ell)| \prod_{\ell\in\{\mathbf{i}\}} \left|D^{\gamma}_{\ell} u_\ell(y_\ell)\right|$$

$$= \max_{j=1,\ldots,k} \prod_{\substack{\ell\in\{1,\ldots,k\}\setminus\{\mathbf{i}\}\\ \|\mathbf{i}\|_0=j}} \sup_{y_\ell\in\bar{D}} |u_\ell(y_\ell)| \prod_{\substack{\ell\in\{\mathbf{i}\}\\ \|\mathbf{i}\|_0=j}} \sup_{\substack{y_\ell\in\bar{D},h_\ell\neq 0\\ y_\ell+h_\ell\in\bar{D}}} \left|D^{\gamma}_{\ell} u_\ell(y_\ell)\right|. \tag{A.5}$$

Choosing $j=k$, we have

$$\text{(A.5)} \geq \prod_{\ell=1}^{k} \sup_{\substack{y_\ell\in\bar{D},h_\ell\neq 0\\ y_\ell+h_\ell\in\bar{D}}} \left|D^{\gamma}_{\ell} u_\ell(y_\ell)\right| = \prod_{\ell=1}^{k} |u_\ell|_{\mathcal{C}^{0,\gamma}(\bar{D})}.$$

On the other hand, given $j^\star$ the index which realizes the maximum, we have

$$\text{(A.5)} = \prod_{\substack{\ell\in\{1,\ldots,k\}\setminus\{\mathbf{i}\}\\ \|\mathbf{i}\|_0=j^\star}} \sup_{x_\ell\in\bar{D}} |u_\ell(y_\ell)| \prod_{\substack{\ell\in\{\mathbf{i}\}\\ \|\mathbf{i}\|_0=j^\star}} \sup_{\substack{y_\ell\in\bar{D},h_\ell\neq 0\\ y_\ell+h_\ell\in\bar{D}}} \left|D^{\gamma}_{\ell} u_\ell(y_\ell)\right|$$

$$= \prod_{\substack{\ell\in\{1,\ldots,k\}\setminus\{\mathbf{i}\}\\ \|\mathbf{i}\|_0=j^\star}} \|u_\ell\|_{\mathcal{C}^{0}(\bar{D})} \prod_{\substack{\ell\in\{\mathbf{i}\}\\ \|\mathbf{i}\|_0=j^\star}} |u_\ell|_{\mathcal{C}^{0,\gamma}(\bar{D})} \leq \prod_{\ell=1}^{k} \|u_\ell\|_{\mathcal{C}^{0,\gamma}(\bar{D})}.$$

Hence, we have proved:

$$\|u\|_{\mathcal{C}^{0,\gamma,mix}(\bar{D}^{\times k})} \geq \max\left\{\prod_{\ell=1}^{k} \|u_\ell\|_{\mathcal{C}^{0}(\bar{D})}, \prod_{\ell=1}^{k} |u_\ell|_{\mathcal{C}^{0,\gamma}(\bar{D})}\right\} = \prod_{\ell=1}^{k} \|u_\ell\|_{\mathcal{C}^{0,\gamma}(\bar{D})},$$

and

$$\|u\|_{\mathcal{C}^{0,\gamma,mix}(\bar{D}^{\times k})} \leq \max\left\{\prod_{\ell=1}^{k} \|u_\ell\|_{\mathcal{C}^{0}(\bar{D})}, \prod_{\ell=1}^{k} \|u_\ell\|_{\mathcal{C}^{0,\gamma}(\bar{D})}\right\} = \prod_{\ell=1}^{k} \|u_\ell\|_{\mathcal{C}^{0,\gamma}(\bar{D})},$$

and (A.4) follows. □

Proposition 11. *Let* $\varphi = \varphi(x,y_1,\ldots,y_k) \in \mathcal{C}^{n,\gamma,mix}_{y_1,\ldots,y_k}\left(\bar{D}^{\times k}; W^{m,p}_x(D)\right)$*, with* $D\subset\mathbb{R}^d$*,* $k\geq 1$ *integer,* $n\geq m\geq 0$ *integers,* $\gamma\in(0,1]$ *and* $p>\frac{2d}{\gamma}$*. Then, for all* $j=2,\ldots,k+1$*, and for all* $(y_j,\ldots,y_k)\in D^{\times(k-j+1)}$*,* $(\mathrm{Tr}_{|1:j}\varphi)(x;y_j,\ldots,y_k)\in W^{m,p}_x(D)$*. In particular, there exists* $C_{tr}>0$ *such that*

$$\left\|(\mathrm{Tr}_{|1:j}\varphi)(x;y_j,\ldots,y_k)\right\|_{W^{m,p}_x(D)}$$
$$\leq C_{tr}^{j-1}\left\|\varphi(x,y_1,\ldots,y_{j-1};y_j,\ldots,y_k)\right\|_{\mathcal{C}^{n,\gamma,mix}_{y_1,\ldots,y_{j-1}}\left(\bar{D}^{\times(j-1)};W^{m,p}_x(D)\right)}, \tag{A.6}$$

for all $(y_j,\ldots,y_k)\in D^{\times(k-j+1)}$.

Moreover, $\mathrm{Tr}_{|1:j}\varphi \in \mathcal{C}^{n,\gamma,mix}_{y_j,\ldots,y_k}\left(\bar{D}^{\times(k-j+1)}; W^{m,p}_x(D)\right)$*, and*

$$\left\|\mathrm{Tr}_{|1:j}\varphi\right\|_{\mathcal{C}^{n,\gamma,mix}_{y_j,\ldots,y_k}\left(\bar{D}^{\times(k-j+1)};W^{m,p}_x(D)\right)} \le C^{j-1}_{tr}\left\|\varphi\right\|_{\mathcal{C}^{n,\gamma,mix}_{y_1,\ldots,y_k}\left(\bar{D}^{\times k};W^{m,p}_x(D)\right)} \tag{A.7}$$

for all $j = 2,\ldots,k+1$.

Proof. We prove the results in three steps.

Step 1: inequality (A.6) for $k = 1$ and $j = 2$

Let $\xi = \xi(x,y) \in \mathcal{C}^{n,\gamma}_y\left(\bar{D}; W^{m,p}_x(D)\right)$, i.e., $x \mapsto \xi(x,\cdot) \in \mathcal{C}^{n,\gamma}\left(\bar{D}\right)$ a.e., and $y \mapsto \xi(\cdot,y) \in W^{m,p}(D)$. Denote with $g(x) := \left(\mathrm{Tr}_{|1:2}\xi\right)(x)$ for all $x = (x_1,\ldots,x_d) \in D$. We want to show that $g \in W^{m,p}(D)$, i.e., $\partial^\alpha_x g = \frac{\partial^{|\alpha|}g}{\partial^{\alpha_1}x_1\cdots\partial^{\alpha_d}x_d} \in L^p(D)$ for all $\alpha = (\alpha_1,\ldots,\alpha_d) \ge 0$ with $|\alpha| = \alpha_1+\cdots+\alpha_d \le m$.

Let α be such that $|\alpha| \le m$, and let $x^{(i)} = (x^{(i)}_1,\ldots,x^{(i)}_d) \in D$ for $i = 1,2$. Then, it holds

$$\begin{aligned}\left\|\partial^\alpha_x g(x)\right\|_{L^p_x(D)} &= \left\|\frac{\partial^{|\alpha|}g}{\partial^{\alpha_1}x_1\cdots\partial^{\alpha_d}x_d}\right\|_{L^p_x(D)} \\ &= \left\|\sum_{t_1=0}^{\alpha_1}\cdots\sum_{t_d=0}^{\alpha_d}\binom{\alpha_1}{t_1}\cdots\binom{\alpha_d}{t_d}\frac{\partial^{|\alpha|}\xi(x^{(1)},x^{(2)})}{\partial^{t_1}x^{(1)}_1\partial^{\alpha_1-t_1}x^{(2)}_1\cdots\partial^{t_d}x^{(1)}_d\partial^{\alpha_d-t_d}x^{(2)}_d}\Big|_{(x,x)}\right\|_{L^p_x(D)} \\ &\le \sum_{t_1=0}^{\alpha_1}\cdots\sum_{t_d=0}^{\alpha_d}\binom{\alpha_1}{t_1}\cdots\binom{\alpha_d}{t_d}\left\|\frac{\partial^{|\alpha|}\xi(x^{(1)},x^{(2)})}{\partial^{t_1}x^{(1)}_1\partial^{\alpha_1-t_1}x^{(2)}_1\cdots\partial^{t_d}x^{(1)}_d\partial^{\alpha_d-t_d}x^{(2)}_d}\Big|_{(x,x)}\right\|_{L^p_x(D)}.\end{aligned}$$

Denote $\partial^\alpha_t\xi(x^{(1)},x^{(2)}) := \frac{\partial^{|\alpha|}\xi(x^{(1)},x^{(2)})}{\partial^{t_1}x^{(1)}_1\partial^{\alpha_1-t_1}x^{(2)}_1\cdot\partial^{t_d}x^{(1)}_d\partial^{\alpha_d-t_d}x^{(2)}_d}$, where $t = (t_1,\ldots,t_d)$. Using the triangular inequality, we have

$$\left\|\partial^\alpha_t\xi(x^{(1)},x^{(2)})\big|_{(x,x)}\right\|_{L^p_x(D)} = \left\|\partial^\alpha_t\xi(x,x)\right\|_{L^p_x(D)}$$

$$\le \sup_{y\in D}\left\|\partial^\alpha_t\xi(x,x) - \partial^\alpha_t\xi(x,y)\right\|_{L^p_x(D)} \tag{A.8}$$

$$+ \sup_{y\in D}\left\|\partial^\alpha_t\xi(x,y)\right\|_{L^p_x(D)}. \tag{A.9}$$

We bound first the term (A.8). According to the Sobolev embedding theorem, if $sp > d$, then $W^{s,p}(D) \hookrightarrow \mathcal{C}^{0,\beta}(D)$ for all $0 < \beta < s - \frac{d}{p}$. Hence, there exists a positive constant C_s such that

$$\left|\partial^\alpha_t\xi(x,y_1) - \partial^\alpha_t\xi(x,y_2)\right| \le C_s\left|y_1-y_2\right|^\beta\left(\int_D\int_D\frac{\left|\partial^\alpha_t\xi(x,z_1)-\partial^\alpha_t\xi(x,z_2)\right|^p}{|z_1-z_2|^{d+sp}}dz_1dz_2\right)^{1/p}. \tag{A.10}$$

Using (A.10), we have

$$\begin{aligned}&\sup_{y\in D}\left\|\partial^\alpha_t\xi(x,x)-\partial^\alpha_t\xi(x,y)\right\|^p_{L^p_x(D)} \\ &= \sup_{y\in D}\int_D\left|\partial^\alpha_t\xi(x,x)-\partial^\alpha_t\xi(x,y)\right|^p dx \\ &\le C^p_s\int_D\sup_{y\in D}|x-y|^{\beta p}\left(\int_D\int_D\frac{\left|\partial^\alpha_t\xi(x,z_1)-\partial^\alpha_t\xi(x,z_2)\right|^p}{|z_1-z_2|^{d+sp}}dz_1dz_2\right)dx \\ &\le C^p_s|D|^{\beta p}\int_D\int_D\int_D\left(\frac{\left|\partial^\alpha_t\xi(x,z_1)-\partial^\alpha_t\xi(x,z_2)\right|}{|z_1-z_2|^{d/p+s}}\right)^p dz_1dz_2dx \\ &= C^p_s|D|^{\beta p}\int_D\int_D\frac{1}{|z_1-z_2|^{d-\varepsilon}}\int_D\left(\frac{\left|\partial^\alpha_t\xi(x,z_1)-\partial^\alpha_t\xi(x,z_2)\right|}{|z_1-z_2|^{\varepsilon/p+s}}\right)^p dxdz_1dz_2 \\ &\le C^p_s|D|^{\beta p}\left|\partial^\alpha_t\xi\right|^p_{\mathcal{C}^{0,\varepsilon/p+s}(\bar{D};L^p(D))}\int_D\int_D\frac{1}{|z_1-z_2|^{d-\varepsilon}}dz_1dz_2 \\ &\le C_1(\varepsilon)C^p_s|D|^{\beta p}\left|\partial^\alpha_t\xi\right|^p_{\mathcal{C}^{0,\varepsilon/p+s}_y\left(\bar{D};L^p_x(D)\right)}\end{aligned}$$

for all $0 < \varepsilon < d$, with $C_1(\varepsilon) := \int_D\int_D\frac{1}{|z_1-z_2|^{d-\varepsilon}}dz_1dz_2 < +\infty$. Hence, we have shown that

$$\sup_{y\in D}\left\|\partial^\alpha_t\xi(x,x)-\partial^\alpha_t\xi(x,y)\right\|_{L^p_x(D)} \le (C_1(\varepsilon))^{1/p}C_s|D|^{s-d/p}\left|\partial^\alpha_t\xi\right|_{\mathcal{C}^{0,\tilde{\gamma}}_y\left(\bar{D};L^p_x(D)\right)}, \tag{A.11}$$

for any $s > \frac{d}{p}$, with $\tilde{\gamma} = \varepsilon/p + s$. Since $p > \frac{2d}{\gamma}$ and $\varepsilon < d$, by taking $s = \frac{\gamma}{2} > \frac{d}{p}$, we have $\tilde{\gamma} < \gamma$.

We bound now the second term (A.9). Since $|\alpha| = \alpha_1 + \cdots + \alpha_d \le m \le n$, then

$$\text{(A.9)} \le \|\xi\|_{\mathcal{C}^n_y\left(\bar{D};W^{m,p}_x(D)\right)} \le \|\xi\|_{\mathcal{C}^{n,\gamma}_y\left(\bar{D};W^{m,p}_x(D)\right)}. \tag{A.12}$$

Putting together (A.11) and (A.12), we conclude (A.6) (for $k = 1$ and $j = 2$) with constant $C_{tr} = 2^m(C_1(\varepsilon)^{1/p}C_s\,|D|^{s-d/p} + 1)$.

Step 2: inequality (A.6) for $k > 1$ and $j = 2, \ldots, k+1$

Let $\varphi \in \mathcal{C}^{n,\gamma,mix}_{y_1,\ldots,y_k}\left(\bar{D}^{\times k}; W^{m,p}_x(D)\right)$, with $k > 1$ and $n \ge m$. We prove the proposition by induction on j. In Step 1, we have shown that the result holds for $j = 2$, namely, for all $(y_2, \ldots, y_k) \in D^{\times(k-1)}$, $\mathrm{Tr}_{|1:2}\varphi(x; y_2, \ldots, y_k) \in W^{m,p}_x(D)$. In particular,

$$\left\|\mathrm{Tr}_{|1:2}\varphi(x; y_2, \ldots, y_k)\right\|_{W^{m,p}_x(D)} \le C_{tr}\,\|\varphi(x, y_1; y_2, \ldots, y_k)\|_{\mathcal{C}^{n,\gamma}_{y_1}\left(\bar{D};W^{m,p}_x(D)\right)},$$

for all $(y_2, \ldots, y_k) \in D^{\times(k-1)}$.

By induction, we assume that

$$\mathrm{Tr}_{|1:\ell}\varphi(x; y_\ell, \ldots, y_k) \in W^{m,p}_x(D)$$
$$\left\|\mathrm{Tr}_{|1:\ell}\varphi(x; y_\ell, \ldots, y_k)\right\|_{W^{m,p}_x(D)} \le C_{tr}\left\|\mathrm{Tr}_{|1:\ell-1}\varphi(x; y_{\ell-1}, \ldots, y_k)\right\|_{\mathcal{C}^{n,\gamma}_{y_{\ell-1}}\left(\bar{D};W^{m,p}_x(D)\right)}$$

for all $\ell = 3, \ldots, j$, and for all $(y_\ell, \ldots, y_k) \in D^{\times(k-\ell+1)}$. Then, it holds

$$\left\|\mathrm{Tr}_{|1:j}\varphi(x; y_j, \ldots, y_k)\right\|_{W^{m,p}_x(D)} \le C^{j-1}_{tr}\left\|\varphi(x, y_1, \ldots, y_{j-1}; y_j, \ldots, y_k)\right\|_{\mathcal{C}^{n,\gamma,mix}_{y_1,\ldots,y_{j-1}}\left(\bar{D}^{\times j};W^{m,p}_x(D)\right)}, \tag{A.13}$$

where we have used the isomorphism (24).

Denote with $\mathbf{y} = (y_{j+1}, \ldots, y_k)$. We bound $\left\|\mathrm{Tr}_{|1:j+1}\varphi(x; \mathbf{y})\right\|_{W^{m,p}_x(D)}$ as follows:

$$\begin{aligned}\left\|\mathrm{Tr}_{|1:j+1}\varphi(x; \mathbf{y})\right\|_{W^{m,p}_x(D)} &= \left\|\mathrm{Tr}_{|1:j}\varphi(x, x; \mathbf{y})\right\|_{W^{m,p}_x(D)}\\ &\le \sup_{y_j\in D}\left\|\mathrm{Tr}_{|1:j}\varphi(x, x; \mathbf{y}) - \mathrm{Tr}_{|1:j}\varphi(x, y_j; \mathbf{y})\right\|_{W^{m,p}_x(D)} + \sup_{y_j\in D}\left\|\mathrm{Tr}_{|1:j}\varphi(x, y_j; \mathbf{y})\right\|_{W^{m,p}_x(D)}.\end{aligned} \tag{A.14}$$

Using inequality (A.13) we bound the second term on the right hand side of (A.14) as:

$$\begin{aligned}\sup_{y_j\in D}\left\|\mathrm{Tr}_{|1:j}\varphi(x, y_j; \mathbf{y})\right\|_{W^{m,p}_x(D)} &\le C^{j-1}_{tr}\sup_{y_j\in D}\left\|\varphi(x, y_1, \ldots, y_j; \mathbf{y})\right\|_{\mathcal{C}^{n,\gamma,mix}_{y_1,\ldots,y_{j-1}}\left(\bar{D}^{\times(j-1)};W^{m,p}_x(D)\right)}\\ &\le C^{j-1}_{tr}\left\|\varphi(x, y_1, \ldots, y_j; \mathbf{y})\right\|_{\mathcal{C}^{n,\gamma,mix}_{y_1,\ldots,y_j}\left(\bar{D}^{\times j};W^{m,p}_x(D)\right)}.\end{aligned}$$

We bound the first term on the right hand side of (A.14) by proceeding as in the case $k = 1$ and $j = 2$:

$$\begin{aligned}&\sup_{y_j\in D}\left\|\mathrm{Tr}_{|1:j}\varphi(x, x; \mathbf{y}) - \mathrm{Tr}_{|1:j}\varphi(x, y_j; \mathbf{y})\right\|_{W^{m,p}_x(D)}\\ &\quad\le C_{tr}\left\|\mathrm{Tr}_{|1:j}\varphi(x, y_j; \mathbf{y})\right\|_{\mathcal{C}^{n,\gamma}_{y_j}\left(D;W^{m,p}_x(D)\right)}\\ &\quad\overset{(A.13)}{\le} C^{j}_{tr}\left\|\varphi(x, y_1, \ldots, y_j; \mathbf{y})\right\|_{\mathcal{C}^{n,\gamma,mix}_{y_1,\ldots,y_j}\left(\bar{D}^{\times j};W^{m,p}_x(D)\right)},\end{aligned}$$

and the conclusion holds.

Step 3: mixed Hölder regularity of $\mathrm{Tr}_{|1:j}\varphi$

Let $\xi(x, y_1, y_2) \in \mathcal{C}^{n,\gamma,mix}_{y_1,y_2}\left(\bar{D}^{\times 2}; W^{m,p}_x(D)\right)$. By applying the same steps as in Step 2 to the increment in the variable y_2 of the trace of ξ, $D^\gamma_2\left(\mathrm{Tr}_{|1:2}\xi\right)(x; y_2)$, we conclude (A.7) in the case $k = 2$ and $j = 2$. Then, by induction, we conclude (A.7) for any k and j. □

References

[1] J. Bear, A.H.-D. Cheng, Modeling Groundwater Flow and Contaminant Transport, Vol. 23, Springer Science & Business Media, 2010.
[2] G. Dagan, Flow and Transport in Porous Formations, Springer Science & Business Media, 2012.
[3] C.-A. Xia, A. Guadagnini, B.X. Hu, M. Riva, P. Ackerer, Grid convergence for numerical solutions of stochastic moment equations of groundwater flow, Stoch. Environ. Res. Risk Assess. 33 (8) (2019) 1565–1579.
[4] L. Li, H.A. Tchelepi, D. Zhang, Perturbation-based moment equation approach for flow in heterogeneous porous media: applicabiliy range and analysis of high-order terms, J. Comput. Phys. 188 (1) (2003) 296–317.
[5] J. Charrier, Numerical analysis of the advection-diffusion of a solute in porous media with uncertainty, SIAM/ASA J. Uncertain. Quantif. 3 (1) (2015) 650–685.
[6] F. Bonizzoni, F. Nobile, Perturbation analysis for the Darcy problem with log-normal permeability, SIAM/ASA J. Uncertain. Quantif. 2 (1) (2014) 223–244.
[7] J. Charrier, A. Debussche, Weak truncation error estimates for elliptic PDEs with lognormal coefficients, Stoch. Partial Differ. Equ. Anal. Comput. 1 (1) (2013) 63–93.
[8] C.J. Gittelson, Stochastic Galerkin discretization of the lognormal isotropic diffusion problem, Math. Models Methods Appl. Sci. 20 (02) (2010) 237–263.

[9] F. Bonizzoni, F. Nobile, Perturbation analysis for the stochastic Darcy problem, in: Proceeding in ECCOMAS 2012, ISBN: 9783950353709, 2012.
[10] F. Bonizzoni, Analysis and Approximation of Moment Equations for PDEs with Stochastic Data (Ph.D. thesis), Department of Mathematics, Politecnico di Milano, Italy, 2013.
[11] D. Zhang, Z. Lu, An efficient, high-order perturbation approach for flow in random porous media via Karhunen-Loève and polynomial expansions, J. Comput. Phys. 194 (2) (2004) 773–794.
[12] Z. Lu, D. Zhang, Conditional simulations of flow in randomly heterogeneous porous media using a KL-based moment-equation approach, Adv. Water Resour. 27 (9) (2004) 859–874.
[13] Z. Lu, D. Zhang, Stochastic simulations for flow in nonstationary randomly heterogeneous porous media using a KL-based moment-equation approach, Multiscale Model. Simul. 6 (1) (2007) 228–245.
[14] G. Liu, Z. Lu, D. Zhang, Stochastic uncertainty analysis for solute transport in randomly heterogeneous media using a Karhunen-Loève-based moment equation approach, Water Resour. Res. 43 (7) (2007).
[15] A. Chkifa, A. Cohen, R. DeVore, C. Schwab, Sparse adaptive Taylor approximation algorithms for parametric and stochastic elliptic PDEs, ESAIM Math. Model. Numer. Anal. 47 (1) (2013) 253–280.
[16] A. Cohen, R. DeVore, Approximation of high-dimensional parametric PDEs, Acta Numer. 24 (2015) 1–159.
[17] A. Guadagnini, S.P. Neuman, Nonlocal and localized analyses of conditional mean steady state flow in bounded, randomly nonuniform domains: 1. Theory and computational approach, Water Resour. Res. 35 (10) (1999) 2999–3018.
[18] A. Guadagnini, S.P. Neuman, Nonlocal and localized analyses of conditional mean steady state flow in bounded, randomly nonuniform domains: 2. Computational examples, Water Resour. Res. 35 (10) (1999) 3019–3039.
[19] R.A. Todor, Sparse Perturbation Algorithms for Elliptic PDE'S with Stochastic Data (Ph.D. thesis), ETH Zurich, Switzerland, 2005.
[20] F. Bonizzoni, F. Nobile, D. Kressner, Tensor train approximation of moment equations for elliptic equations with lognormal coefficient, Comput. Methods Appl. Mech. Engrg. 308 (2016) 349–376.
[21] N.N. Chentsov, Weak convergence of stochastic processes whose trajectories have no discontinuities of the second kind and the heuristic approach to the Kolmogorov-Smirnov tests, Theory Probab. Appl. 1 (1) (1956) 140–144.
[22] C.G. Simader, On Dirichlet'S Boundary Value Problem, in: Lecture Notes in Mathematics, vol. 268, Springer-Verlag, Berlin-New York, 1972, An L^p-theory based on a generalization of Gårding's inequality.
[23] P. Houston, I. Muga, S. Roggendorf, K.G. van der Zee, The convection-diffusion-reaction equation in non-Hilbert Sobolev spaces: A direct proof of the inf-sup condition and stability of Galerkin's method, Comput. Methods Appl. Math. 19 (3) (2019) 503–522.
[24] F. Nobile, R. Tempone, C.G. Webster, A sparse grid stochastic collocation method for partial differential equations with random input data, SIAM J. Numer. Anal. 46 (5) (2008) 2309–2345.
[25] H.J. Bungartz, M. Griebel, Sparse grids, Acta Numer. 13 (2004) 147–269.
[26] F. Bonizzoni, A. Buffa, F. Nobile, Moment equations for the mixed formulation of the Hodge Laplacian with stochastic loading term, IMA J. Numer. Anal. 34 (4) (2014) 1328–1360.
[27] R. Schneider, A. Uschmajew, Approximation rates for the hierarchical tensor format in periodic Sobolev spaces, J. Complexity 30 (2) (2014) 56–71.