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Note on axiomatic properties of apportionment methods for proportional representation systems

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Abstract
Apportionment methods are used in proportional representation systems for the apportionment of parliamentary seats among political parties proportionately to their vote counts, or for the allocation of parliamentary seats between geographical districts proportionately to their population figures. From an axiomatic viewpoint apportionment methods ought to satisfy six basic principles: anonymity, balancedness, concordance, decency, exactness, and fairness. It is well-known that the first two principles are implied by the last four. In this note it is shown that the last four principles are logically independent of each other.

Keywords Apportionment rules · Divisor rules of apportionment · Jump point sequences · Seat allocation procedures

Mathematics Subject Classification 65K05 · 62P25

1 Introduction

Proportional representation systems apply apportionment methods to assign the seats of a parliament to those entitled to fill the seats. In North America the problem occurs when, after a census, the seats of the United States' House of Representatives are allocated between the states of the Union proportionately to the states' population

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figures. In Europe many countries have adopted a proportional electoral system by which, after a general election, the seats of the country's parliament are apportioned among the political parties in proportion to the parties' vote counts.

Over time many apportionment methods have been proposed in theory and many have been implemented in practice. It is one of the lasting achievements of the seminal monograph of Balinski and Young [3] to propose a systematic view of the multitude of methods available. What makes their approach so unique is their insistence on axiomatic foundations, first to single out persuasive principles and then to study their logical consequences. As Balinski and Young announce in their preface:

> "The aim of our book is to establish a solid logical foundation for choosing among the available methods of apportioning power in representative systems.. . .
>
> The choice of principles to follow and compromises to accept is, of course, ultimately a question of political legitimacy and should be made by a nation's legislators. Our intent is to clarify the consequences of these choices."

One of their major results is the Coherence Theorem. It states that the class of apportionment rules narrows down to a smaller subclass, divisor methods, when insisting that six basic principles be satisfied. These are anonymity, balancedness, concordance, decency, exactness, and fairness. The last property, fairness, is also known under a synonymous label, coherence; hence the name Coherence Theorem. The history of the Coherence Theorem, its forerunners, and the pertinent literature are reviewed by Palomares, Pukelsheim and Ramírez [6].

The current note grew out of the authors' curiosity whether the six basic principles qualify as logically independent axioms. The answer is in the negative. In fact it is well-known that the first two, anonymity and balancedness, are implied by the last four, concordance, decency, exactness and fairness (Sect. 4). On the positive side, this note shows that the last four principles are independent of each other (Sect. 5).

Usually the terms apportionment method and divisor method are used only for procedures satisfying one or more of the six basic principles. In order to handle procedures that do not necessarily enjoy any of the properties, Sect. 2 introduces the notions of an apportionment rule and a divisor rule. The precise definitions of the basic principles are given in Sect. 3. The paper concludes with an empirical example illustrating that different methods lead to rather different solutions (Sect. 6).

2 Apportionment rules and divisor rules

Typically the setting is the following. At an election of a parliament of house size h, $\ell \geq 2$ political parties finish with vote counts $v_1, \ldots, v_\ell$. Practically, the vote counts will be whole numbers; generally, the mathematical setting allows them to be arbitrary positive quantities (in which case it is preferable to speak of vote weights rather than vote counts). The electoral law stipulates an apportionment rule that assigns x_j seats to party $j = 1, \ldots, \ell$. The rule should be such that the seat contingents are close to what ideal proportionality would promise,

$$x_1 \approx w_1 h, \ldots, x_\ell \approx w_\ell h,$$

where w_j designates the vote share of party j, $w_j = v_j/(v_1 + \cdots + v_\ell) > 0$. Since seats are assigned to human beings who are indivisible, the seat contingents x_j must be natural numbers. The integer requirement prevents the seat contingents from generally being exactly equal to the ideal shares $w_j h$.

The number of parties contesting an election usually varies from one election to the other. Hence for an apportionment rule to be suitable for an electoral law, its formulation must not involve the size of the party system, ℓ. To this end we define a *vote vector* $(v_1, v_2, \ldots)$ to be an infinite sequence with finitely many terms positive and all other terms zero. Similarly, a *seat vector* $(x_1, x_2, \ldots)$ is taken to be a sequence of nonnegative integers that terminates with a tail of zeros. This convention allows a specification of apportionment rules without reference to the system size ℓ, see Hylland [4, page 5].

Definition 1 An *apportionment rule* A maps a house size h and a vote vector $v = (v_1, v_2, \ldots)$ into a nonempty *solution set* $A(h; v)$ that consists of seat vectors $x = (x_1, x_2, \ldots)$ having component sum h and inheriting all zeros of v: $v_j = 0 \Rightarrow x_j = 0$.

An apportionment rule is defined to be a set-valued mapping so that it may accommodate tied situations. A prototype tie shows up when just one seat is available for two parties that are equally strong, $v_1 = v_2 = v_0$ say. The seat may be apportioned either to the first party or else to the second party. As both options are equally justified, the solution set comprises both: $A\big(1; (v_0, v_0)\big) = \big\{(1, 0), (0, 1)\big\}$. Note that vote vectors and seat vectors, which a minute ago were agreed to be infinite sequences, are jotted down as vectors of finite length simply by omitting their vanishing tails.

The abstract notion of apportionment rules embraces procedures clearly unfit for concrete use. For instance all seats could be allocated to the party listed first, $A(h; v) = \{(h, 0, 0, \ldots)\}$, irrespective of the vote counts v. Or vote counts are neglected by stolidly proposing an abundance of ties, $A\big(h; (v_1, \ldots, v_\ell)\big) = \mathbb{N}^\ell(h)$, where $\mathbb{N}^\ell(h)$ is the grid of vectors $(x_1, \ldots, x_\ell)$ with nonnegative integer components summing to h. The set $\mathbb{N}^\ell(h)$ abounds with the same $\binom{h+\ell-1}{\ell-1}$ seat vectors whatever the vote vector $(v_1, \ldots, v_\ell)$.

Divisor rules are a subfamily of apportionment rules that turns out to be sensible and of practical relevance. The idea is to scale the vote counts v_j by a divisor $d > 0$ in the hope that the interim quotients v_j/d are of a magnitude which facilitates their rounding to whole numbers x_j.

To this end a *jump point sequence* is defined to be a nonconstant sequence that starts at zero and is non-decreasing, $t(0) = 0 \leq t(1) \leq t(2) \leq \ldots$.

Given a jump point sequence, the induced rounding rule R is determined by means of

$$R\left(\frac{v_j}{d}\right) = \begin{cases} \{n\} & \text{in case } \frac{v_j}{d} \in \big(t(n), t(n+1)\big), \\ \{m-1, \ldots, n\} & \text{in case } t(m-1) < \frac{v_j}{d} = t(m) = \cdots = t(n) < t(n+1), \\ \{0\} & \text{in case } \frac{v_j}{d} = 0. \end{cases}$$

That is, a quotient that is zero (whence v_j must be zero, as the divisor d is positive) is always rounded to zero. If a positive quotient hits successive jump points $t(m) = \cdots = t(n)$ then it may be rounded to any integer between $m-1$ and n. If a quotient comes to lie above the jump point $t(n)$ and below its successor, it is unambiguously rounded to n.

Definition 2 The *divisor rule D with rounding rule R* maps a house size h and a vote vector $v = (v_1, \ldots, v_\ell)$ into the set of seat vectors $x = (x_1, \ldots, x_\ell)$ given by

$$D(h; v) = \left\{x \in \mathbb{N}^\ell(h) \,\middle|\, x_1 \in R\left(\frac{v_1}{d}\right), \ldots, x_\ell \in R\left(\frac{v_\ell}{d}\right) \text{ for some } d > 0\right\}.$$

That is, the seat contingent x_j of party j is obtained by scaling its vote count v_j by some divisor $d > 0$ and rounding the interim quotient v_j/d to the whole number x_j. The divisor serves as a sliding controller to ensure that all seats are meted out, $x_1 + \cdots + x_\ell = h$.

We need to verify that a divisor rule is an apportionment rule. Indeed, all solution vectors lie in the set $\mathbb{N}^\ell(h)$ and hence are required to have component sum h. Moreover, by virtue of the rounding rule, they inherit all zeros of v. The crucial issue is whether the solution sets $D(h; v)$ are nonempty. Nonconstancy of the jump point sequence secures a last jump point $t(k_t)$ that is zero, $t(0) = \cdots = t(k_t) = 0 < t(k_t + 1)$. Now a large divisor d forces all quotients v_j/d into the interval $\big(0, t(k_t+1)\big)$, whence $x = (k_t, \ldots, k_t)$ is a solution for house size $h = k_t\ell$. As the divisor decreases, the quotients increase beyond $t(k_t + 1)$ and subsequent jump points, thereby yielding solutions for $h > k_t\ell$. If $k_t = 0$, all solution sets $D(h; v)$ are nonempty. Therefore a divisor rule qualifies for an apportionment rule provided $t(1) > 0$. Hylland [4, page 32] restricts the notion of divisor rules to the cases when the jump point $t(1)$ is positive.

Yet divisor rules with $t(1) = 0 < t(2)$ (that is, $k_t = 1$) exist and may be practically relevant, such as the divisor method with upward rounding (Adams method) which is mentioned at the end of Sect. 3. For instance, when allocating seats to geographical districts, a rule with $t(1) = 0 < t(2)$ awards every district at least one seat. No district finishes without representation. However, a jump point sequence with $k_t \geq 1$ is troublesome for our terminology since the solution sets for $h = 1, \ldots, k_t\ell - 1$ are empty, $D\big(h; (v_1, \ldots, v_\ell)\big) = \emptyset$. In theory, the case $k_t \geq 1$ and $h < k_t\ell$ defies our objective of specifying apportionment rules independent of the system size ℓ. In practice, we overlook such troubles and continue to include divisor rules with $k_t \geq 1$ into our discussion of apportionment rules.

3 The six basic principles

In the universe of all apportionment rules some rules appear more attractive than others. Practical needs suggest six basic principles to assess the performance of apportionment rules: anonymity, balancedness, concordance, decency, exactness, and fairness. The naming of the properties has changed over time, see the literature review in Palomares, Pukelsheim and Ramírez [6, Sect. 2]. We discuss the six principles one after the other, and check their consequences for divisor rules.

Definition 3 An apportionment rule is called *anonymous* when every rearrangement of the vote vector induces the same rearrangement of the resulting seat vectors.

With an anonymous apportionment rule it does not matter whether a party is listed first or last, its seat contingent stays the same.

Anonymity allows a vote vector to be (re-)arranged so that all positive components are assembled in the initial section $(v_1, \ldots, v_\ell)$, while all vanishing components are moved to positions beyond the last positive component v_ℓ. Sometimes a vote vector is arranged by decreasing vote counts, $v_1 \geq \cdots \geq v_\ell > 0$, which also is in accord with anonymity.

Evidently all divisor rules are anonymous.

Definition 4 An apportionment rule is called *balanced* when any two equally strong parties differ by at most one seat: $v_i = v_j \Rightarrow |x_i - x_j| \leq 1$.

In view of a possible occurrence of ties it is unrealistic to require equally strong parties to have equal seat contingents, but a difference of two or more seats will not be tolerated.

A divisor rule is balanced if and only if the underlying jump point sequence is strictly increasing, $t(1) < t(2) <$ etc., except for the initialization $t(0) = 0 \leq t(1)$.

Indeed, two equally strong parties can cause trouble only when the (common) interim quotient hits a positive jump point, $v_i/d = v_j/d = t(n) > 0$ say. If the jump points are strictly increasing, neighboring jump points do not interfere and $x_i, x_j \in \{n-1, n\}$. The difference is at most one seat, whence follows balancedness. Conversely, if the jump points are not strictly increasing and two jump points (or more) are equal, $t(n) = t(n+1)$, then the interim quotients may be rounded to $n-1$ or to n or to $n+1$, whence $x_i = n - 1$ and $x_j = n + 1$ are two seats apart rather than being balanced.

For a balanced divisor rule it is instructive to include the jump points $t(n)$ visibly in the definition and to circumvent thereby explicit reference to the rounding rule R:

$$D\big(h; (v_1, \ldots, v_\ell)\big) = \Big\{(x_1, \ldots, x_\ell) \in \mathbb{N}^\ell(h) \;\Big|\; t(x_1) \leq \frac{v_1}{d} \leq t(x_1 + 1), \ldots, t(x_\ell) \leq \frac{v_\ell}{d} \leq t(x_\ell + 1) \text{ for some } d > 0\Big\}.$$

Definition 5 An apportionment rule A is called *concordant* when of any two parties the stronger party is allotted at least as many seats as the weaker party: $v_i > v_j \Rightarrow x_i \geq x_j$.

A discordant result, which would allot the stronger party fewer seats than the weaker party, is rejected.

All divisor rules are concordant. This follows from the fact that every rounding rule R is set-monotonic in the sense that $q < Q$ implies $n \leq N$ for all $n \in R(q)$ and $N \in R(Q)$.

Definition 6 An apportionment rule A is called *decent* when scalings of the vote vector do not affect the solutions: $A(h; \frac{1}{c}v) = A(h; v)$ for all $c > 0$.

Decency entails that absolute vote counts v_j and relative vote shares $w_j = v_j/(v_1 + v_2 + \cdots)$ yield identical solutions.

All divisor rules are decent. Clearly, a scaling of the vote vector is instantly absorbed into the divisor, $v_j/d = (v_j/c)/(d/c)$.

Decency has repercussions on the underlying jump point sequence. Since the inequalities $c\,t(n) \leq v_j/d$ and $t(n) \leq (v_j/c)/d$ are equivalent, multiplication of the jump points by a constant $c > 0$ has the same effect as scaling the vote counts by c. Therefore decency indicates that the underlying jump point sequence is determined only up to a constant. In fact this is the only degree of freedom available.

Proposition 1 *Two jump point sequences that define the same divisor rule cannot differ other than by a multiplicative constant.*

Proof Hylland [4, page 34] presents an indirect proof. Here we argue directly. Let $s(0) = 0 \leq s(1) \leq s(2) \leq \cdots$ be a second jump point sequence for the same divisor rule D.

The initial sections where jump points are zero must be identical. To see this, let $t(k_t) = 0$ and $s(k_s) = 0$ be the last zeros of the respective sequences. Then the smallest house size with a nonempty solution set is $h = k_t\ell = k_s\ell$, whence follows $k_t = k_s$.

For indices n and m with positive jump points $t(n)$ and $t(m)$ let $V(n, m)$ be the set of vote ratios of two parties of which the first has n seats and the second $m - 1$ seats:

$$V(n, m) = \left\{ \frac{v_1}{v_2} \,\middle|\, (n, m-1) \in D\big(n + m - 1, (v_1, v_2)\big) \right\}.$$

The set $V(n, m)$ contains the ratio $t(n)/t(m)$. Indeed, if $v_1 = t(n) > 0$ and $v_2 = t(m) > 0$ then the divisor $d = 1$ yields $x_1 \in \{n - 1, n\}$ and $x_2 \in \{m - 1, m\}$, whence $(n, m - 1)$ is a solution in the set $D\big(n + m - 1; \big(t(n), t(m)\big)\big)$. Moreover the ratio $t(n)/t(m)$ is a lower bound for $V(n, m)$. For if v_1 and v_2 are such that $(n, m - 1) \in D\big(n + m - 1; (v_1, v_2)\big)$ then there is a divisor $d > 0$ satisfying $t(n) \leq v_1/d \leq t(n + 1)$ and $t(m - 1) \leq v_2/d \leq t(m)$. The last and first inequalities imply $v_2/t(m) \leq d \leq v_1/t(n)$. Hence $t(n)/t(m) \leq v_1/v_2$ actually is the minimum of $V(n, m)$ and so is, with the same reasoning, $s(n)/s(m)$:

$$\frac{t(n)}{t(m)} = \min V(n, m) = \frac{s(n)}{s(m)}.$$

With constant $c = s(m)/t(m)$ we realize $s(n) = c\, t(n)$.

□

The fifth principle, exactness, is of a more technical nature. It links the continuum nature of the input domain, the quadrant of vote vectors $[0, \infty)^\ell$, to the discrete character of the output range, the grid of seat vectors $\mathbb{N}^\ell(h)$.

Definition 7 An apportionment rule A is called *exact* when for every sequence of vote vectors $v(n)$, $n \geq 1$, that converges to a seat vector x and that satisfies $v_j(n) = 0$ whenever $x_j = 0$ there exists some n_0 such that $A\big(h; v(n)\big) = \{x\}$ for all $n \geq n_0$.

In cases when all seat contingents x_j are nonzero, as practically happens more often than not, exactness guarantees that *every* sequence of vote vectors which tends to x eventually reproduces x as the unique apportionment solution.

Moreover, when the sequence of vote vectors is constant and coincides with the seat vector x, $v(n) = x$, an exact apportionment rule reproduces x, $A(h; x) = \{x\}$. This gives rise to the notion of weak-exactness which Balinski and Ramírez [2, page 111] paraphrase through the pun:

"If there is no 'problem' then there is no problem!"

Definition 8 An apportionment rule A is called *weakly-exact* when every vote vector that is a seat vector reproduces itself: $A(h; x) = \{x\}$ whenever $x_1 + x_2 + \cdots = h$.

Weak-exactness does not imply exactness. An example is the quota method with residual fit by strongest parties, see Palomares, Pukelsheim and Ramírez [6, page 13].

For divisor rules, exactness gives rise to a proper subclass of jump point sequences.

By definition, a *signpost sequence* $s(0) \leq s(1) \leq s(2) \leq \cdots$ is required to fulfill three properties. Firstly it is initialized by $s(0) = 0$. Secondly it is *localized*, in that the n-th signpost lies in the n-th interval with integer endpoints, $s(n) \in [n-1, n]$ for $n \geq 1$. Thirdly the sequence obeys a *left-right disjunction*: If there is a signpost hitting the left endpoint of its localization interval then all signposts stay away from their right endpoints, and if there is a signpost hitting the right endpoint then all signposts stay away from their left endpoints,

$$\begin{aligned} s(m) = m-1 \text{ for some } m \geq 2 \quad &\Longrightarrow \quad s(n) < n \text{ for all } n \geq 1, \\ s(n) = n \text{ for some } n \geq 1 \quad &\Longrightarrow \quad s(m) > m-1 \text{ for all } m \geq 2. \end{aligned}$$

It follows from the definition that signposts are strictly increasing, $s(1) < s(2) < s(3) < \cdots$, with the exception of $0 = s(0) \leq s(1)$. Indeed, assume otherwise. Nonstrictness $s(n) = s(n+1)$ for some $n \geq 1$ would imply $s(n) = n = s(n+1)$, because of the localization property. Thus $s(n)$ would hit its right limit, and $s(n+1)$ its left limit. This constellation is ruled out by the left-right disjunction.

Proposition 2 *A divisor rule is weakly-exact if and only if its underlying jump point sequence is equal to a signpost sequence or can be scaled into a signpost sequence.*

Proof For the direct part of the proof assume the divisor rule D to be weakly-exact. Let $t(0), t(1), t(2), \ldots$ be the jump point sequence underlying D. Since the seat vector $x = (1, \ldots, 1)$ is a solution in the set $D(\ell; x)$ the second jump point cannot be zero, $t(2) > 0$. Hence the jump points $t(n)$ and $t(m)$ are positive for $n \geq 2$ and $m \geq 2$, in case of $t(1) > 0$ actually for $n \geq 1$. By exactness we have $(n, m-1) \in D\big(n+m-1; (n, m-1)\big)$. With set $V(n, m)$ as in the proof of Proposition 1 we get $n/(m-1) \in V(n, m)$ and

$$\frac{t(n)}{t(m)} \leq \frac{n}{m-1} \qquad \text{for all } n \geq 1,\ m \geq 2.$$

In case of $t(1) = 0$ the inclusion of the missed index $n = 1$ is trivially permitted.

This leads to a string of inequalities:

$$\limsup_{n\to\infty} \frac{t(n)}{n} \leq \sup_{n\geq 1} \frac{t(n)}{n} \leq \inf_{m\geq 2} \frac{t(m)}{m-1} \leq \liminf_{m\to\infty} \frac{t(m)}{m-1} = \liminf_{n\to\infty} \frac{t(n)}{n}.$$

Therefore the quotients $t(n)/n$ are convergent, $\lim_{n\to\infty} t(n)/n = L$ say. The limit is equal to the supremum of $t(n)/n$ as well as to the infimum of $t(m)/(m-1)$. Hence we obtain

$$\frac{t(n)}{n} \leq L \leq \frac{t(m)}{m-1} \qquad \text{for all } n \geq 1,\ m \geq 2.$$

The numbers $s(n) = t(n)/L$, $n \geq 0$, form a signpost sequence. Firstly we have $s(0) = 0$. Secondly from $t(1)/1 \leq L$ we get $s(1) \in [0, 1]$. For $n = m \geq 2$ the display yields $s(n)/n \leq 1 \leq s(n)/(n-1)$, that is, $s(n) \in [n-1, n]$. Thirdly the left-right disjunction holds true. For if $s(n) = n$ and $s(m) = m-1$ then the set $D\big(n+m-1; \big(s(n), s(m)\big)\big) = \{(n, m-1), (n-1, m)\}$ contains two solutions rather than only $(n, m-1)$. Altogether the original jump point sequence is a signpost sequence if $L = 1$, and if $L \neq 1$ then it is scaled into a signpost sequence.

For the converse part let the divisor rule D have underlying signpost sequence $s(0), s(1), s(2)$ etc. Fix a seat vector $x \in \mathbb{N}^\ell(h)$. With divisor $d = 1$ the localization property yields $s(x_j) \leq x_j \leq s(x_j + 1)$ and $x \in D(h; x)$. It remains to establish uniqueness, that is, $D(h; x) = \{x\}$. (The following uniqueness proof corrects several misprints in Pukelsheim [7, page 80].)

Suppose $y \in D(h; x)$ is a second solution, $y \neq x$. If y has divisor $d(y) < 1$ then $v_j/d(y) > v_j$ implies $y_j \geq x_j$. As both vectors have component sum h they must be equal which they are not. A similar argument excludes $d(y) > 1$. Hence $d(y) = 1$, and $x_j, y_j \in R(x_j) = \{x_j - 1, x_j\}$ for all $j \leq \ell$. In view of the component sum h there are two parties $i \neq k$ with $x_i > y_i$ and $x_k < y_k$, that is, $y_i = x_i - 1$ and $y_k = x_k + 1$. This yields $x_i, x_i - 1 \in R(x_i)$, whence $x_i = s(x_i) > 0$. Similarly $x_k, x_k + 1 \in R(x_k + 1)$ necessitate $x_k = s(x_k + 1) > 0$. With $n = x_i \geq 1$ and $m = x_k + 1 \geq 2$ the fulfillments of $s(n) = n$ and $s(m) = m - 1$ violate the left-right disjunction. The supposition that besides x there exists a second solution y is untenable. That is, the solution x is unique, $D(h; x) = \{x\}$. □

For divisor rules the notions of weak-exactness and exactness coincide.

Proposition 3 *A weakly-exact divisor rule is exact.*

Proof Let $v(n)$, $n \geq 1$, be a sequence of vote vectors converging to a seat vector $x \in \mathbb{N}^{\ell}(h)$ and satisfying $v_j(n) = 0$ whenever $x_j = 0$. Since the sequence anticipates the zeros of the limit we may set aside all zero components and assume the other components to be positive, $x_j \geq 1$. We need to verify that x is the only solution in the set $A\big(h; v(n)\big)$ eventually.

Since the range $\mathbb{N}^{\ell}(h)$ is finite every sequence $y(n) \in A\big(h; v(n)\big)$, $n \geq 1$, admits convergent subsequences. Let $y(n_k)$, $k \geq 1$, be a subsequence converging to a seat vector y. Because of finiteness of $\mathbb{N}^{\ell}(h)$ there is an index k_0 such that all $k \geq k_0$ satisfy $y(n_k) = y \in A\big(h; v(n_k)\big)$. Hence there are divisors $D(n_k)$ such that $s(y_j) \leq v_j(n_k)/D(n_k) \leq s(y_j + 1)$. A transition to reciprocals shows that the divisors $D(n_k)$, $k \geq 1$, are bounded. Let $D > 0$ be an accumulation point. A passage to the limit yields $s(y_j) \leq x_j/D \leq s(y_j + 1)$, that is, $y \in A(h; x)$. Weak-exactness identifies the accumulation point to be $y = x$. As every convergent subsequence has the same limit x, so has the original sequence $y(n)$. This proves $x \in A\big(h; v(n)\big)$ eventually. □

The first five principles constitute a base catalogue for procedures to become eligible for use in electoral laws. Adherence to the principles is indicated by proper terminology.

Definition 9 An apportionment rule that is anonymous, balanced, concordant, decent and exact is called an *apportionment method*. A divisor rule that is anonymous, balanced, concordant, decent and weakly-exact is called a *divisor method*.

A divisor method has an underlying jump point sequence which may be assumed to be a signpost sequence, by Proposition 2. For a divisor rule to advance to a divisor method only weak-exactness needs to be verified, by Proposition 3.

The five principles share a common deficiency. They are insensitive to the house size h and to the system size ℓ. They solely deal with variations in the vote vector $(v_1, \ldots, v_{\ell})$. Anonymity permutes its components, balancedness and concordance compare them by pairs, decency rescales them, and exactness addresses the exceptional circumstances when the vote vector coincides with a seat vector or converges to a seat vector.

The deficiency is overcome by the sixth principle, fairness, also known as coherence. Balinski and Young [3, page 141] treat the concept under the heading "uniformity"; Young [9, page 141] calls it "consistency". Fairness implements the idea that the whole and its parts must fit together in a coherent way. Balinski and Young [3, page 141] put it this way:

> "An inherent principle of any fair division is that *every part of a fair division should be fair.*"

The idea is the following. In a large party system $(1, \dots, L)$ with an apportionment solution $(x_1, \dots, x_L)$ for house size H, consider the subsystem $(1, \dots, \ell)$ with its induced house size $h = x_1 + \cdots + x_\ell$. In essence, fairness has two aspects. Firstly, the subvector $(x_1, \dots, x_\ell)$ of the grand solution is an apportionment solution for the subproblem. Secondly, if in the grand solution the subvector $(x_1, \dots, x_\ell)$ is replaced by a vector $(y_1, \dots, y_\ell)$ tied to it, then the resulting vector is a grand apportionment solution, too.

Of course, the ℓ subsystem parties need not be those in the initial section $(1, \dots, \ell)$, but may be extracted and arranged in an arbitrary fashion, using subscripts $(i_1, \dots, i_\ell)$.

Definition 10 An apportionment rule A is called *fair*, or *coherent*, when it satisfies coherence of partial problems as well as coherence of substituted solutions. The two properties are defined as follows:

Coherence of partial problems (PP) means that, given a grand seat vector $(x_1, \dots, x_L) \in A\big(H; (v_1, \dots, v_L)\big)$ for a grand system of L parties, the vector $(x_{i_1}, \dots, x_{i_\ell})$ is a member of the ℓ-subsystem solution set $A\big(h; (v_{i_1}, \dots, v_{i_\ell})\big)$, where $h = x_{i_1} + \cdots + x_{i_\ell}$ and $\ell \le L$.

Coherence of substituted solutions (SS) means that, given a grand seat vector $x = (x_1, \dots, x_L) \in A\big(H; (v_1, \dots, v_L)\big)$ and a ℓ-subsystem seat vector $y = (y_{i_1}, \dots, y_{i_\ell}) \in A\big(h; (v_{i_1}, \dots, v_{i_\ell})\big)$, where $h = x_{i_1} + \cdots + x_{i_\ell}$ and $\ell \le L$, substitution of y into x yields a grand solution $z \in A\big(H; (v_1, \dots, v_L)\big)$, that is, z has components

$$z_j = \begin{cases} x_j & \text{in case } j \in \{1, \dots, L\} \setminus \{i_1, \dots, i_\ell\}, \\ y_j & \text{in case } j \in \{i_1, \dots, i_\ell\}. \end{cases}$$

Coherence of partial problems is a top-down concept. It demands that a subvector that is extracted from a grand solution is a valid solution for the associated partial problem. Coherence of substituted solutions is a bottom-up requirement. Tied solutions for subproblems, when substituted into the grand solution, yield tied grand solutions.

Every divisor rule D is fair. Indeed, let $x \in D(h; v)$ be a grand solution. Then every divisor d that is feasible for x is feasible also for all partial solutions $(x_{i_1}, \dots, x_{i_\ell})$, thus establishing coherence of partial problems (PP). As for coherence of substituted solutions (SS), if the solution for the partial problem is unique, substitution of $(y_{i_1}, \dots, y_{i_\ell}) = (x_{i_1}, \dots, x_{i_\ell})$ is clearly permissible. If the partial solution is not unique, then the divisor d is unique and feasible both for the partial problem and the grand problem. Hence divisor rules are fair.

A major result of apportionment theory is the Coherence Theorem. It states that fair apportionment methods necessarily lead to divisor methods. An apportionment method A is defined to be *compatible* with a divisor method D when the inclusion $A(h; v) \subseteq D(h; v)$ holds true for every house size h and every vote vector v.

Coherence Theorem A fair apportionment method is compatible with a divisor method.

Proof See Palomares, Pukelsheim and Ramírez [6, page 17], or Pukelsheim [7, page 162]. □

Compatibility of an appportionment method A with a divisor method D implies that the methods agree whenever the solution set $D(h; v)$ is a singleton. For, if $D(h; v) = \{x\}$ then $\emptyset \neq A(h; v) \subseteq D(h; v)$ forces $A(h; v) = \{x\}$. This was meant above when saying that fair apportionment methods lead to divisor methods.

When a solution set $D(h; v)$ contains two or more seat vectors, A may differ from D. In fact, a divisor method is *complete* by offering all tied solutions possible. A fair apportionment method A may abstain from completeness by implementing a tie resolution strategy. The electoral law for the Spanish Congreso de los Diputados resolves ties by following the motto Stronger Parties First. If there are two parties whose interim quotients hit signposts, then the party with more votes is rounded upwards and the party with fewer votes is rounded downwards. Completeness is lost, yet the six principles persist.

The family of divisor methods still is huge. There are as many divisor methods as there are signpost sequences. Within this ensemble three members stand out. In the spirit of the current note we emphasize their character as a divisor method, for the alternative names and their sources see Pukelsheim [7].

The *divisor method with downward rounding* has signposts $s(n) = n$. That is, if a quotient v_j/d has a nonzero fractional part, it is truncated to its integer part. If the quotient happens to be a whole number, it stays as is or is rounded to the whole number below provided the house size h is met. Other names for this procedure are Jefferson method, or D'Hondt method, or even number method.

The *divisor method with standard rounding* has signposts $s(n) = n - 1/2$. That is, a quotient v_j/d is rounded downwards or upwards contingent on its fractional part being less than one half or greater than one half. If the fractional part is equal to one half, the quotient may be rounded either way, downwards or upwards, provided the house size h is met. Other names for this procedure are Webster method, or Sainte-Laguë method, or odd number method.

The *divisor method with upward rounding* has signposts $s(n) = n - 1$. That is, if a quotient v_j/d has a nonzero fractional part, it is rounded upwards. If the quotient happens to be a whole number, it may stay as is or it may be rounded to the whole number above provided the house size h is met. Another name for this procedure is Adams method.

4 Implied principles: anonymity and balancedness

Of the six principles, anonymity and balancedness are implied by the other four.

Proposition 4 *A fair apportionment rule is anonymous.*

Proof Anonymity follows from the special case $\ell = L$ and $h = H$ in the definition of coherence of partial problems (PP). Then any subsystem $i_1, \ldots, i_\ell$ actually is a permutation of $1, \ldots, \ell$. For $(z_1, \ldots, z_\ell) \in A\big(h; (v_1, \ldots, v_\ell)\big)$ coherence of partial problems (PP) yields $(z_{i_1}, \ldots, z_{i_\ell}) \in A\big(h; (v_{i_1}, \ldots, v_{i_\ell})\big)$. □

Proposition 4 essentially coincides with Theorem 2 of Hylland [4, page 23]. However, Hylland's approach starts from subsystems with two parties, $\ell = 2$. Therefore he needs some additional arguments for passing to subsystems $\ell \geq 2$. In constrast, our definition of fairness admits subsystems of any size $\ell \leq L$ right from the start.

Balancedness is the other implied principle. Balinski and Rachev [1, page 79] and Balinski and Ramírez [2, page 112] present an indirect proof of the result. Here we argue directly. We note that those papers need to presuppose decency rather than anonymity.

Proposition 5 *A decent, weakly-exact and fair apportionment rule is balanced.*

Proof Let A be a decent, weakly-exact and fair apportionment rule awarding seat contingents z_i and z_j to two equally strong parties i and j with vote weights $v_i = v_j = v_0 > 0$. For $\delta = |z_i - z_j|$ we need to prove that $\delta = 0$ or $\delta = 1$. Due to coherence of partial problems (PP) (z_i, z_j) is a solution for the two-party problem with house size $z_i + z_j$, that is $(z_i, z_j) \in A\big(z_i + z_j; (v_0, v_0)\big)$, which we record for later usage.

Part I of the proof treats the case when the seat contingents z_i and z_j have equal parity, that is, both are even or both are odd. Then the sum $z_i + z_j$ is even and the average $z_0 = (z_i + z_j)/2$ is an integer. Decency allows a scaling of v_0 into z_0. Weak-exactness entails $A\big(z_i + z_j; (z_0, z_0)\big) = \{(z_0, z_0)\}$ and $(z_i, z_j) = (z_0, z_0)$. Hence follows $\delta = 0$ as desired.
Part II of the proof handles the case when z_i and z_j are of distinct parity. Then $\delta = |z_i - z_j|$ is odd, say $\delta = 2k + 1$ for some $k \geq 0$. The value $k = 0$ yields $\delta = 1$ as desired.

The rest of Part II perseveres in showing that a value $k \geq 1$ would lead to all sorts of contradictions and hence cannot possibly materialize. Because of distinct parity the sum $z_i + z_j$ is odd and, with $k \geq 1$, satisfies $z_i + z_j \geq 3$. The clue is a solution vector x for the apportionment of $H = 2k(z_i + z_j) \geq 6k$ seats among $4k$ parties who are equally strong,

$$x = (x_1, \ldots, x_{4k}) \in A\big(H; (v_0, \ldots, v_0)\big), \qquad H = 2k(z_i + z_j).$$

If all components of x are identical, $x_j = x_0$ say, then $4kx_0 = 2k(z_i + z_j)$ leads to the equation $2x_0 = z_i + z_j$ where the left hand side is even and the right hand side is odd. This contradiction rules out that the components of x are all equal.

We claim that the components of x take two values, $a > b$ say, one value being even, the other, odd. To this end select two components x_i and x_j that are of equal parity. Then $(x_i + x_j)/2$ is a whole number x_0, say. Due to coherence of partial problems (PP) we have $(x_i, x_j) \in A\big(x_i + x_j; (v_0, v_0)\big)$. Decency allows a scaling of v_0 into x_0. Weak-exactness entails $A\big(x_i + x_j; (x_0, x_0)\big) = \{(x_0, x_0)\}$ and $(x_i, x_j) = (x_0, x_0)$.

Since $x_i = x_0 = x_j$ all even components are the same, and all odd components are the same. The claim is proved.

Let $\alpha \in \{1, \ldots, 4k-1\}$ be the count how often a appears in the vector x. Then b has frequency $4k - \alpha$. We investigate the three cases $\alpha < 2k$, $\alpha > 2k$, and $\alpha = 2k$.

Consider the case $\alpha < 2k$. We omit $2k$ components with value b to pass from $x \in \mathbb{N}^{4k}(H)$ to the subvector $y \in \mathbb{N}^{2k}(h)$. The components of y add to $h = H - 2kb = 2k(z_i + z_j - b)$. Coherence of partial problems (PP) implies $y \in A\big(h; (v_0, \ldots, v_0)\big)$. Decency allows to scale v_0 into $z_i + z_j - b$. For $\widetilde{x} = (z_i + z_j - b, \ldots, z_i + z_j - b) \in \mathbb{N}^{2k}(h)$ weak-exactness entails $A(h; \widetilde{x}) = \{\widetilde{x}\}$. Now $\widetilde{x} \neq y$ shows that the case $\alpha < 2k$ is infeasible. The case $\alpha > 2k$ is excluded by a similar argument.

This leaves the case $\alpha = 2k$. Since x satisfies $\alpha a + (4k - \alpha)b = H = 2k(z_i + z_j)$ we get $a + b = z_i + z_j$. Coherence of partial problems (PP) implies $(a, b) \in A\big(z_i + z_j; (v_0, v_0)\big)$. As recorded in the beginning the set also contains (z_i, z_j). Coherence of substituted solutions (SS) is applied $2k$ times to replace (a, b) in the vector x by (z_i, z_j). The resulting vector z has $2k$ components equal to z_i, and $2k$ components equal to z_j, like $z = (z_i, \ldots, z_i, z_j, \ldots, z_j) \in A\big(H; (v_0, \ldots, v_0)\big)$. We may assume $z_j = z_i + \delta$. By omitting $2k - 1$ components with value z_j, we pass from the vector $z \in \mathbb{N}^{4k}(H)$ to the subvector $y = (z_i, \ldots, z_i, z_i + \delta) \in \mathbb{N}^{2k+1}(h)$. Insertion of $\delta = 2k+1$ yields $h = (2k+1)z_i + \delta = (2k+1)(z_i + 1)$. Coherence of partial problems (PP) implies $y \in A\big(h; (v_0, \ldots, v_0)\big)$. By decency we may scale v_0 into $z_i + 1$. For $\widetilde{x} = (z_i + 1, \ldots, z_i + 1) \in \mathbb{N}^{2k+1}(h)$ weak-exactness entails $A(h; \widetilde{x}) = \{\widetilde{x}\}$. Now $\widetilde{x} \neq y$ shows that the case $\alpha = 2k$ is infeasible either.

Since all three cases $\alpha < 2k$, $\alpha > 2k$, and $\alpha = 2k$ are infeasible a value $k \geq 1$ cannot materialize. The proof is complete. □

Although anonymity and balancedness are logically dispensable we prefer to list them explicitly. They count among the pertinent principles that aid in the identification of concrete apportionment methods among all abstract apportionment rules.

5 Independent principles: concordance, decency, exactness, fairness

The remaining four principles are concordance, decency, exactness, and fairness. They turn out to be logically independent, none of them is implied by the other three. We will show this by constructing appropriate examples.

Proposition 6 *There exists an apportionment rule violating concordance, but satisfying decency, exactness and fairness as well as anonymity and balancedness.*

Proof The following example is inspired by the apportionment rule ψ of Balinski and Ramírez [2, page 114]. Define a rounding rule R by expanding the jump points $n - 1/2$ of standard rounding into regions $(n - 3/4, n - 1/4)$, according to

$$R\left(\frac{v_j}{d}\right) = \begin{cases} \{n\} & \text{in case } \dfrac{v_j}{d} \in \left[n-\dfrac{1}{4}, n+\dfrac{1}{4}\right], \\ \{n-1, n\} & \text{in case } \dfrac{v_j}{d} \in \left(n-\dfrac{3}{4}, n-\dfrac{1}{4}\right), \\ \{0\} & \text{in case } \dfrac{v_j}{d} \in \left[0, \dfrac{1}{4}\right]. \end{cases}$$

The accompanying divisor rule A fails to be concordant.

The reason is that the rounding rule R is not set-monotonic. For instance, $q = 3/8$ may be rounded to $n = 1 \in R(q) = \{0, 1\}$, and $Q = 5/8$ to $N = 0 \in R(Q) = \{0, 1\}$. Now $q < Q$ and $n > N$ invalidate set-monotonicity and concordance. Clearly the apportionment rule A is decent, exact and fair. It is also anonymous and balanced, by Propositions 4 and 5. □

Proposition 7 *There exists an apportionment rule violating decency, but satisfying concordance, exactness and fairness as well as anonymity and balancedness.*

Proof An example is obtained when the vote counts of party j and the jump points of party j are linked in a particular way. In a two-party system, we use $s(n) = n-1+1/v_1$ for the first party, and $t(n) = n - 1 + 1/v_2$ for the second party. Any solution $(x_1, x_2) \in A\big(h; (v_1, v_2)\big)$ then is accompanied by a divisor $d > 0$ such that

$$s(x_1) \leq \frac{v_1}{d} \leq s(x_1+1), \qquad t(x_2) \leq \frac{v_2}{d} \leq t(x_2+1).$$

It is easily verified that $A\big(5; (10, 1)\big) = \{(5, 0)\}$, with divisor $d = 2$, while $A\big(5; (100, 10)\big) = \{(4, 1)\}$, with divisor $d = 25$. The votes are scaled by the factor ten, but the solutions differ. Hence the rule fails to be decent. Yet concordance, exactness and fairness hold true, as do anonymity and balancedness. □

Proposition 8 *There exists an apportionment rule violating exactness, but satisfying concordance, decency, and fairness as well as anonymity and balancedness.*

Proof Consider the *round-robin rule* A whose solution sets $A\big(h; (v_1, \ldots, v_\ell)\big)$ are composed as follows. With parties ordered by decreasing strength, $v_{i_1} \geq v_{i_2} \geq \cdots \geq v_{i_\ell}$, the seats are dealt out in a round-robin fashion in the order $i_1, \ldots, i_\ell$, one by one until no seats are left. Equivalently, write the house size as a multiple of the system size, $h = m\ell + r$, with a residual $r < \ell$. Then allocate to every party m seats and assign the r residual seats to the r strongest parties. Ties may occur when several parties are equally strong. The round-robin rule takes into account not the numerical size of the vote counts, but only their ordering. It is evident that the rule is not weakly-exact and hence not exact. Yet it is concordant, decent and fair as well as anonymous (Proposition 4) and balanced. □

A whole family of non-exact apportionment rules emerges when the original vote counts v_j are power-weighted according to v_j^e, with some exponent $0 < e \neq 1$:

$$A(h; v) = \Big\{ x \in \mathbb{N}^\ell(h) \,\Big|\, s(x_1) \leq \frac{v_1^e}{d} \leq s(x_1+1), \ldots, s(x_\ell) \leq \frac{v_\ell^e}{d} \leq s(x_\ell+1) \text{ for some } d > 0 \Big\}.$$

When $s(n) = n$ the inequalities $x_j \leq v_j^e/d \leq x_j + 1$ turn into $x_j^{1/e} \leq v_j/d^{1/e} \leq (x_j+1)^{1/e}$, the latter conforming to the divisor rule with jump point sequence $t(n) = n^{1/e}$. The rule is used in Estonia with exponent $e = 10/9$, that is, with jump point sequence $t(n) = n^{0.9}$, as mentioned by Janson [5, pages 271].

With exponents $e < 1$ the power-weighted rules are of interest in the European Parliament for a degressive representation of the Member States, see Pukelsheim and Grimmett [8]. We note that there is a subtle distinction whether the vote vectors $v = (v_1, v_2, \ldots)$ are considered to be given whence the rule to be investigated is $A(h; v)$, or whether the power vectors $v^e = (v_1^e, v_2^e, \ldots)$ are the given quantities whence the rule to be investigated would be $A(h; v^e)$.

Proposition 9 *There exists an apportionment rule violating fairness, but satisfying concordance, decency, and exactness as well as anonymity and balancedness.*

Proof The most prominent example is the quota method with residual fit by largest remainders, also known as Hamilton method, or method of largest remainders. It is anonymous, balanced, concordant, decent and exact, but fails to satisfy fairness. □

Violation of the fairness principle is treated in the literature under the heading of the new states paradox, see Balinski and Young [3, page 44] or Pukelsheim [7, page 177].

Another example, somewhat playful, is the apportionment rule that would apply the three divisor methods with downward, standard, and upward rounding cyclically to house sizes 1, 2, 3, then 4, 5, 6, next 7, 8, 9 etc. Since the three divisor methods are distinct, solutions on different levels of the house size generally cannot satisfy the coherence principle.

6 An illustrative example

Finally we illustrate by example that the generality of arbitrary divisor rules may lead to results which appear preposterous. The following exhibit uses the data of the 2017 election to the German Bundestag. Seven parties participated in the process for the apportionment of the 598 nominal seats.

Column Ideal Share shows the parties' seat fractions as promised by ideal proportionality. A party's ideal share of seats is the product of its vote share and the total number of available seats.

Column DivStd displays the seat contingents obtained from the divisor method with standard rounding. The select divisor is $d = 73{,}900$, that is, every 73,900 votes justify roughly one seat. Ideal seat shares and actual seat contingents conform extremely well.

The last three columns exhibit apportionment results of divisor rules which do not qualify for divisor methods. The divisor rule with jump points $t(n) = 2^{n-1}$ is used in Macau and the divisor rule with jump points $t(n) = n^{0.9}$ in Estonia, see Janson [5, pages 266, 271]. The divisor rule with jump points $t(n) = 2 - 1/n$ is adjoined for curiosity, as an example of a jump point sequence that is bounded. The three divisor rules fail the exactness principle since the limit of $t(n)/n$ equals zero or infinity, rather than being unity.

Political party	Vote count	Ideal share	DivStd $n-1/2$	Macau 2^{n-1}	Estonia $n^{0.9}$	Bounded $2-1/n$
"CDU"	12,447,656	168.45	168	87	179	596
"SPD"	9,539,381	129.09	129	86	133	2
"AfD"	5,878,115	79.55	80	86	78	0
"FDP"	4,999,449	67.66	68	85	65	0
"LINKE"	4,297,270	58.15	58	85	55	0
"GRÜNE"	4,158,400	56.27	56	85	53	0
"CSU"	2,869,688	38.83	39	84	35	0
Sum	44,189,959	598.00	598	598	598	598

In column Macau the inequality to check is $2^{x_j-1} = t(x_j) \leq v_j/d \leq t(x_j+1) = 2^{x_j}$. This turns into $x_j - 1 \leq a + \log(v_j)/\log(2) \leq x_j$, where $a = -\log(d)/\log(2)$. Hence the seat contingent of party j is given by

$$x_j = \left\lceil a + \frac{\log(v_j)}{\log(2)} \right\rceil, \qquad a = 62.53.$$

That is, all parties share a common base of 62.53 seat fractions. Thereafter the vote counts are added with a logarithmic transformation. Being concave the transformation has a degressive effect, favoring weaker parties at the expense of stronger parties. The Macau rule results in almost uniform seat contingents, ranging from 87 seats for the strongest party to 84 seats for the weakest party.

In column Estonia the critical inequality is $x_j^{9/10} \leq v_j/d \leq (x_j+1)^{9/10}$. This turns into $x_j \leq v_j^{10/9}/b \leq x_j + 1$, where $b = d^{10/9}$. Thus the seat contingent of party j satisfies

$$x_j = \left\lfloor \frac{v_j^{10/9}}{b} \right\rfloor, \qquad b = 425{,}000.$$

Here the vote counts undergo a power transformation. With exponent $10/9 > 1$ the transformation is convex. The effect is progressive, favoring stronger parties at the expense of weaker parties. The spread of the seat contingents is wider than in column DivStd.

In column Bounded the formula for the seat contingents looks strange,

$$x_j = \left\lfloor \frac{1}{2 - v_j/c} \right\rfloor, \qquad c = 6{,}229{,}050.$$

Almost all of the seats are apportioned to the strongest party, just two seats are left for the second-strongest party. The 22,202,922 voters who supported the five weakest parties—more than half of the electorate—are denied representation in parliament, which would be a preposterous outcome of an election.

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