# Homogenization and parameter identification of multiscale problems of linearized elasticity

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## Abstract

In this thesis, we first consider the periodic homogenization of the linearized elasticity equation with slip-displacement conditions of a two-scale composite of two solids. The jump conditions are motivated by fibre-reinforced materials, which are often modelled by perfect bonding between fibre and matrix, which may not be true in practice. We are interested in the impact of the interface jumps in displacement on the (upscaled) partial differential equations and distinguish three different cases. While one material is connected, the other one is either disconnected, globally connected or unidirectionally connected. In all three cases, we show the existence and uniqueness of the solution and prove some general compactness and convergence results, whereby we apply the method of periodic unfolding. In the end, we derive the homogenized problem.

In the second part of the thesis, we combine the methods of homogenization and parameter identification. We consider the homogenized linear elasticity problem, whereby we assume perfect bonding on the interface of the two-scale composite, and want to deduce from measurements of the deformation on the boundary of a body the structure of the periodicity cell, which can be parametrized by finite real vector. After proving some general properties of the homogenized tensor, which describes the stiffness of the homogenized material, we show that there exists at least one solution of the minimization problem, which minimizes the $L^2$-difference of the measured deformation and the computed deformation for some given structure of the periodicity cell. Using shape optimization, in particular the Lagrangian method of Céa, we derive the Gâteaux derivative of the homogenized tensor, which we need to compute the Gâteaux derivative of the target functional. Finally, we use these results to apply gradient-based algorithms for some numerical simulations in the steady-state and time-dependent case.

# Acknowledgements

I use my mother tongue for the acknowledgements. But in short, I thank my supervisor Prof. Dr. Malte Peter, my colleagues Ursula, Ferdinand, David and Timo and my family and friends, especially my husband Christian.

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# Contents

# List of Symbols

| | | |
|---|---|---|
| $e(u)$ | linear strain tensor | 3 |
| $\sigma$ | stress tensor | 4 |
| $H^1_\Gamma(\mathcal{O})$ | space of functions in $H^1(\mathcal{O})$ with zero value on the boundary $\Gamma$ | 22 |
| $M(\alpha, \beta, \mathcal{O})$ | set of tensors of fourth order with special properties | 4 |
| $\mathcal{M}_{\mathcal{O}}(\phi)$ | mean value of function $\phi$ over the domain $\mathcal{O}$ | 10 |
| $\mathcal{T}^\varepsilon$ | periodic unfolding operator | 10 |
| $\mathcal{T}^\varepsilon_{\mathrm{b}}$ | boundary periodic unfolding operator | 13 |
| $\mathcal{T}^\varepsilon_i$ | periodic unfolding operator for transmission problems | 12 |
| $\mathcal{T}^\varepsilon_Y$ | partial periodic unfolding operator | 11 |
| $\mathcal{W}^\varepsilon_{\mathrm{c}}(\Omega)$ | solution space for the connected case | 41 |
| $\mathcal{W}^\varepsilon_{\mathrm{d}}(\Omega)$ | solution space for the disconnected case | 23 |
| $\mathcal{W}^\varepsilon_{\mathrm{m}}(\Omega)$ | solution space for the unidirectionally connected case | 52 |

# 1. Introduction

## 1.1. Motivation

Composite materials are made of two or more materials, which have different physical properties for each single component. The aim of considering such new materials is to improve certain properties such as strength, stiffness or density. For example, carbon fibres embedded in a matrix of concrete improve the flexural strength (cf. [Hambach et al., 2016, Lauff et al., 2019, Rutzen et al., 2019]) and so this composite material may be used in future for lighter and more resource-saving structures. Before the impact of length, orientation or other materials of the fibres can be investigated by simulation, we need an appropriate model describing the physical behaviour. In application, perfect bonding between fibres and matrix is assumed to simplify the problem but this is in general not true. So we are interested in the impact of interface jumps in normal and tangential direction in displacement on the weak form of the partial differential equation. Furthermore, since the fibres are often of short length compared to the whole body, we have to take the micro- and macroscale into consideration. To resolve the whole body is numerically to expensive, so we need an efficient model for simulations, which can be derived by homogenization methods.

Once we have an appropriate model, there are several things that can be investigated. In the research field of shape optimization the aim is to optimize certain quantities by finding the optimal shape of the fibres. But if the structure such as a bridge is already given, it would be of interest to know the fatigue behaviour in the interior of the structure like (microscopic) cracks or holes after it is built. Typical measurement methods without destroying the structure like computer tomography scans can not be easily applied in practice. In this case, the idea is to use measured data on the boundary of the structure to obtain by numerical calculations a better understanding of the behaviour inside. So we are interested in identifying certain parameters which describe, for example, areas of weak material.

## 1.2. Outline of the thesis

The thesis is organized as follows. In chapter 2 we introduce the equation of elasticity. In section 2.1, we formulate the linear elasticity equation, which describes the deformation of solids. In section 2.2, we state several Korn-type inequalities, which we need in the subsequent chapters to prove the existence and boundedness of the solutions. In chapter 3, we give the general idea of periodic homogenization and state briefly the different (analytical) methods

developed so far. Since we use the periodic unfolding method in this work, we give a short overview over some general known results in section 3.2. The rest of the thesis is divided into two parts, which are closely related but independent of each other.
As motivated in the beginning, Part I is concerned with the derivation of the homogenized problem of the linear elasticity equation with slip-displacement conditions, i.e. we allow jumps in displacement in normal and tangential direction at the interface of the composite. We show the existence and uniqueness of the solution, prove some compactness results and derive the upscaled problem in three different cases depending on the connectedness of materials of the composite. One material – the matrix – is connected, whereas the other material – the fibres – are disconnected, globally connected or unidirectionally connected.
In Part II we want to identify the structure of the periodicity cell of the homogenized problem of the (time-dependent) elasticity equation, where we assume perfect bonding of the composite material. After formulation of the direct problem and proving some properties of the homogenized tensor, we consider the inverse problem. We show that there exists at least one solution of the minimization problem. To be able to apply gradient-based algorithms, we derive the Gâteaux derivatives of the homogenized tensor and the target functional. At the end, we present some simulations to show the functioning of the method.

First steps towards the disconnected case in Part I can be found in my master thesis 'Homogenisierung linearer Elastizität mit Sprungbedingungen' [Wolfer, 2018] and the published paper in *Journal of Mathematical Analysis and Applications* [Lochner and Peter, 2020]. The main drawback of the first one is that we could not rewrite the homogenized problem into a cell problem and macroscopic problem. So we have chosen for the second work another solution space, where we do not allow any rigid displacement of the material in the 'holes'. This assumption might be too strong in the case that we allow jump conditions at the interface. So in this thesis we make no assumptions about the rigid displacement and prove some new compactness results. The results in the connected case have also been published in the *Journal of Mathematical Analysis and Applications* [Lochner and Peter, 2020]. The results of the parameter identification for the steady-state linear elasticity problem in chapter 9 have been published in the journal *Mathematical Methods in the Applied Sciences* [Lochner and Peter, 2022].

# 2. Equation of elasticity

The aim of the theory of elasticity is to compute the deformation of solids under forces, which is reversible when no forces are applied anymore. As we only consider small deformation, we can use the linear elasticity equation as introduced in section 2.1 to model this behaviour. In section 2.2, we state different Korn-type inequalities, which allow us to estimate the gradient of a function by its symmetric gradient. These inequalities are needed to prove the existence of solutions of the linear elasticity equation.

## 2.1. Linear elasticity

In solid mechanics, we are interested in modelling the deformation of solids under forces. If we assume that the acting forces are not so great as to cause permanent deformation or cracks and fractures and induce only small deformations, we can model the behaviour using the linear elasticity equation. Referring to [Ciarlet, 1988, Slaughter, 2002, Eck et al., 2017, Schweizer, 2018], we give a brief summary about the derivation of the linear elasticity equation.
Let the solid be of the form $\Omega \subset \mathbb{R}^3$, when no forces are applied. If forces, which satisfy the above assumptions, act on the body, the current configuration can be described by the displacement field

$$u : [0, T) \times \Omega \to \mathbb{R}^3.$$

Meaning, a material point at position $x$ in $\Omega$ is at time $t$ at position $x + u(t, x)$. A measure of how much a body is deformed is the Green strain tensor also called Green–St. Venant strain tensor

$$G \coloneqq \frac{1}{2}(\nabla u + (\nabla u)^T + (\nabla u)^T \nabla u).$$

Due the assumption of only small strains, the third summand can be neglected and we can use in the following the linearized Green strain tensor, i.e. the symmetric gradient of the displacement field

$$G \approx e(u) \coloneqq \frac{1}{2}(\nabla u + (\nabla u)^T).$$

The internal force that material particles exert on each other in the deformed configuration is described by the stress tensor $\sigma$. It is called first Piola–Kirchhoff stress tensor in Lagrangian coordinates and Cauchy stress tensor in Eulerian coordinates. Using the conservation of momentum, we get the elasticity equation

$$\partial_t(\varrho \partial_t u) - \nabla \cdot \sigma = f \quad \text{in } (0, T) \times \Omega, \tag{2.1.1}$$

where $\nabla\cdot$ is the divergence operator, $f$ is the density of volume force and $\varrho$ is the density of the material, which is time-independent since we consider the equation in the reference configuration. In the steady-state case, we can drop the first term in (2.1.1) and get

$$-\nabla \cdot \sigma = f \quad \text{in } \Omega.$$

In the linear setting, Hooke's law specifies the connection between the stress tensor $\sigma = (\sigma_{ij})_{1\leq i,j\leq 3}$ and linear strain tensor $e(u) = (e_{ij}(u))_{1\leq i,j\leq 3}$

$$\sigma_{ij} = (Ae(u))_{ij} = \sum_{k,l=1}^{3} a_{ijkl}e_{kl}(u) = \sum_{k,l=1}^{3} a_{ijkl}\frac{1}{2}\left(\partial_k u_l + \partial_l u_k\right),$$

where $A = (a_{ijkl})_{1\leq i,j,k,l\leq 3}$ is the elasticity tensor of fourth order, whose entries describe the stiffness of the material. In this thesis, we assume that the elasticity tensor $A$ is an element of the set $M(\alpha, \beta, \mathcal{O})$ for some open set $\mathcal{O} \subset \mathbb{R}^3$, which is no restriction in the application.

**Definition 2.1.1.** *Let $\alpha, \beta \in \mathbb{R}$ with $0 < \alpha < \beta$ and let $\mathcal{O}$ be an open set in $\mathbb{R}^3$. We denote by $M(\alpha, \beta, \mathcal{O})$ the set of all tensors $B = (b_{ijkl})_{1\leq i,j,k,l\leq 3}$ such that*

(i) $b_{ijkl} \in L^\infty(\mathcal{O})$ *for all* $i, j, k, l \in \{1, 2, 3\}$,

(ii) $b_{ijkl} = b_{jikl} = b_{klij}$ *for all* $i, j, k, l \in \{1, 2, 3\}$,

(iii) $\alpha|m|^2 \leq Bmm$ *for all symmetric matrices* $m$,

(iv) $|B(x)m| \leq \beta|m|$ *for all matrices* $m$

*a.e. in $\mathcal{O}$, where*

$$\begin{cases} Bm := \left((Bm)_{ij}\right)_{1\leq i,j\leq 3} = \left(\left(\sum_{k,l=1}^{3} b_{ijkl}m_{kl}\right)_{ij}\right)_{1\leq i,j\leq 3}, \\ Bm\tilde{m} := \sum_{i,j,k,l=1}^{3} b_{ijkl}m_{ij}\tilde{m}_{kl}, \\ |m| := \left(\sum_{i,j=1}^{3} m_{ij}^2\right)^{\frac{1}{2}}, \end{cases}$$

*for quadratic matrices $m = (m_{ij})_{1\leq i,j\leq 3}$ and $\tilde{m} = (\tilde{m}_{ij})_{1\leq i,j\leq 3}$.*

## 2.2. Korn-type inequalities

To prove the existence and boundedness of solutions of the linear elasticity equation, we need different Korn-type inequalities, which we introduce in this section. All the definitions and results are taken from chapters 3, 10 and 25 of [Schweizer, 2018] unless otherwise stated. As we have to evaluate Sobolev functions on the boundary, the following theorem describes in what sense these values exist.

**Theorem 2.2.1** (Trace operator). *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz-boundary. Then, the Sobolev functions $u \in H^1(\Omega)$ have boundary values in the following sense. There exists a unique continuous linear operator*

$$\tilde{\gamma} : H^1(\Omega) \to L^2(\partial\Omega),$$

*which coincides with the classical trace $u \mapsto u|_{\partial\Omega}$ for all $u \in C^1(\bar{\Omega})$.*

We define the trace operator for vector-valued functions by applying the operator from Theorem 2.2.1 to every component

$$\gamma : [H^1(\Omega)]^3 \to [L^2(\partial\Omega)]^3, \quad \gamma(u) = \begin{pmatrix} \tilde{\gamma}(u_1) \\ \tilde{\gamma}(u_2) \\ \tilde{\gamma}(u_3) \end{pmatrix}.$$

Thus, $\gamma$ is a linear continuous operator. In the following, we write $u$ instead of $\gamma(u)$ for some function $u$, if it is clear that we evaluate the function on the boundary resp. interface. The trace operator with respect to time is defined for functions in Bochner spaces.

**Theorem 2.2.2** (Trace operator in Bochner space). *Let $X$ be a Banach space, $S := (0, T)$ and $t_0 \in [0, T)$. The evaluation of $u$ at time $t_0$ is defined by the continuous linear operator*

$$\gamma_{t_0} : H^1(S; X) \to X, \quad u \mapsto -\int_{t_0}^{T} u(t)\partial_t\phi(t)\mathrm{d}t - \int_{t_0}^{T} \partial_t u(t)\phi(t)\mathrm{d}t,$$

*where $\phi \in C_c^\infty([t_0, T), \mathbb{R})$ with $\phi(t_0) = 1$ can be chosen arbitrarily. Then, there holds for a function $u \in H^1(S; X)$ the identity $u(t) = \gamma_t(u)$ for almost all $t \in [0, T]$.*

There exist different types of Korn's inequalities, all having in common that they seek to estimate the gradient by the symmetric gradient. Depending on the assumptions, different additional terms appear in the inequality. In the following theorem, often referred to as the second Korn's inequality, the extra term is the $L^2$-norm of the function itself.

**Theorem 2.2.3** (Korn's inequality). *Let $\Omega \subset \mathbb{R}^3$ a bounded domain with Lipschitz-boundary. Then, there holds for all $u \in [H^1(\Omega)]^3$*

$$\|\nabla u\|_{[L^2(\Omega)]^{3\times 3}} \leq C \left( \|e(u)\|_{[L^2(\Omega)]^{3\times 3}} + \|u\|_{[L^2(\Omega)]^3} \right)$$

*for some constant $C > 0$.*

In Corollary 5.8 from [Alessandrini et al., 2008] a similar estimate is proven with the $L^2$-norm on the boundary of the domain instead of the $L^2$-norm on the whole domain.

**Theorem 2.2.4** (Korn's inequality with control of boundary values)**.** *Let $\Omega \subset \mathbb{R}^3$ a bounded domain with Lipschitz-boundary and $\Gamma \subset \partial\Omega$ open with $|\Gamma| > 0$. Then, there holds for all $u \in [H^1(\Omega)]^3$*

$$\|u\|_{[H^1(\Omega)]^3} \leq C \left( \|e(u)\|_{[L^2(\Omega)]^{3\times 3}} + \|u\|_{[L^2(\Gamma)]^3} \right)$$

*for constants $C > 0$ depending only on $\Omega$ and $\Gamma$.*

Under additional assumption on the function space, namely zero value on part of the boundary, we can estimate the gradient by the symmetric one without any additional terms.

**Theorem 2.2.5** (Korn's inequality with zero value on part of the boundary)**.** *Let $\Omega \subset \mathbb{R}^3$ a bounded domain with Lipschitz-boundary and $\Gamma_\mathrm{D} \subset \partial\Omega$ with two-dimensional Hausdorff-measure $|\Gamma_\mathrm{D}| > 0$. Then, there holds for all $u \in H^1_{\Gamma_\mathrm{D}}(\Omega) := \{v \in [H^1(\Omega)]^3 : v = 0 \text{ on } \Gamma_\mathrm{D}\}$*

$$\|\nabla u\|_{[L^2(\Omega)]^{3\times 3}} \leq C\|e(u)\|_{[L^2(\Omega)]^{3\times 3}}$$

*for some constant $C > 0$.*

The same inequality holds for periodic function with zero mean value. Let $Y := (0, l_1) \times (0, l_2) \times (0, l_3)$ with $l_1, l_2, l_3 > 0$. An unbounded domain $\omega$ has $Y$-periodic structure, if it is invariant under shifts by $(c_1 l_1, c_2 l_2, c_3 l_3)$ with $c_1, c_2, c_3 \in \mathbb{Z}$.

**Corollary 2.2.6** (Korn's inequality for periodic functions)**.** *Let $Y := (0, l_1) \times (0, l_2) \times (0, l_3)$ with $l_1, l_2, l_3 > 0$, $\omega$ be an unbounded domain with $Y$-periodic structure and $\omega \cap Y$ a domain with Lipschitz boundary. Then there holds for all $u \in H^1_{\mathrm{per},0}(\omega) := \{v \in [H^1_{\mathrm{per}}(\omega \cap Y)]^3 : \mathcal{M}_{\omega\cap Y}(v) = 0\}$, where $\mathcal{M}_{\omega\cap Y}(v) = \frac{1}{|\omega\cap Y|}\int_{\omega\cap Y} v \,\mathrm{d}x$,*

$$\|u\|_{[H^1(\omega\cap Y)]^3} \leq C\|e(u)\|_{[L^2(\omega\cap Y)]^{3\times 3}}$$

*for constant $C > 0$ independent of $u$.*

*Proof.* We follow the proof of Theorem 2.8 from [Oleinik et al., 1992], where the same result is shown for the 1-periodic case. We notice that any $Y$-periodic rigid displacement, meaning a vector-valued function of the form $a + Ax$ with $a \in \mathbb{R}^3$ a constant vector and $A \in \mathbb{R}^{3\times 3}$ a skew-symmetric matrix, is constant. Therefore, if $u \in H^1_{\mathrm{per},0}(\omega)$ is a rigid displacement, $u \equiv 0$. Then, the result follows from Theorem 2.5 of [Oleinik et al., 1992]. □

# 3. Periodic homogenization

The concept of homogenization treats problems involving periodically oscillating coefficients on a small period or composite materials with periodic microstructure, whereby the macroscopic lengthscale is far bigger than the characteristic lengthscale of the microstructure. Resolving the microstructure would be in general numerically too costly and is therefore unfeasible for simulations. To circumvent this problem, we analytically derive an effective model describing an artificial (homogenous) material with the same macroscopic properties as the original problem. In section 3.1, we explain this process on the basis of composite materials in more detail and give a brief overview of the different methods used in literature to derive the homogenized problem. We take a closer look at the periodic unfolding method in section 3.2, which is the main method we use in this thesis.

## 3.1. General idea of homogenization

In view of applications studied in this thesis, we explain the homogenization process on the basis of composite materials and the linear elasticity equation. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $Y = (0, l_1) \times \ldots \times (0, l_n) \subset \mathbb{R}^n$ the so-called reference or periodicity cell. We define for some small scaling parameter $\varepsilon > 0$ the domain

$$\Omega^\varepsilon := \bigcup_{\xi \in \mathbb{Z}^n} \varepsilon(Y + \xi) \cap \Omega,$$

which is of $\varepsilon Y$-periodic structure (see Figure 3.1).

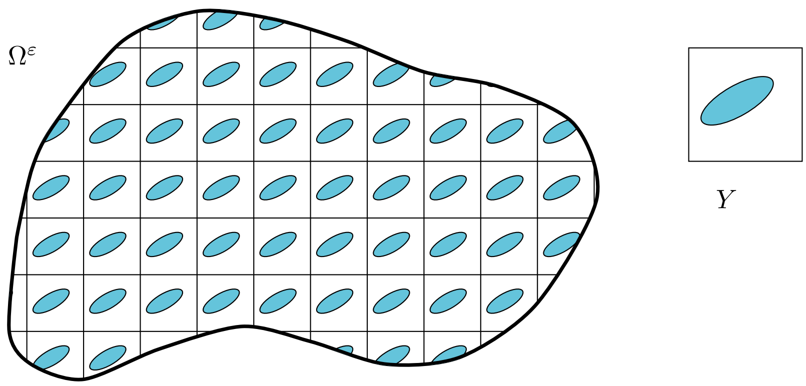


Figure 3.1.: domain $\Omega^\varepsilon$ with $\varepsilon Y$-periodic structure

For every $\varepsilon$, we consider the linear elasticity problem

$$-\nabla \cdot (A^{\varepsilon} e(u^{\varepsilon})) = f \text{ in } \Omega^{\varepsilon}$$

with appropriate boundary conditions, whereby the coefficient $A^{\varepsilon}$ describes the material properties and $f$ some volume force (for more details, see section 2.1). In periodic homogenization, we are interested in what happens passing the scaling parameter to zero. Roughly speaking with $\varepsilon$ becoming smaller the composite material is getting more and more (macroscopically) homogenous as illustrated in Figure 3.2.

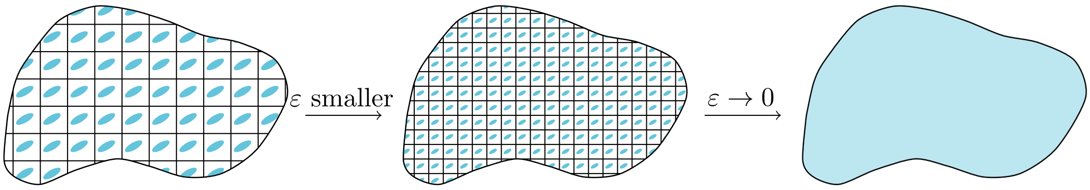


Figure 3.2.: homogenization process

Using analytical homogenization techniques, we derive an effective partial differential equation of the form

$$-\nabla \cdot (A^{\text{hom}} e(u)) = f \text{ in } \Omega$$

with some effective coefficient $A^{\text{hom}}$ describing the material property of the homogenous material, which can often be easily computed by solving the so-called cell problems. The solution $u$ has the same behaviour as $u^{\varepsilon}$ up to variations on a microscopic scale. The main advantage is that we can use the effective partial differential equation for simulations, which significantly reduces the computational effort since we do not have to resolve any microstructure.

There are several methods for obtaining the upscaled problem, briefly summarised in what follows. If we use the method of two-scale asymptotic expansion, we assume that $u^{\varepsilon}$ is of the form

$$u^{\varepsilon}(x) = u_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}\right) + \dots,$$

where $u_i(x, y)$ is $Y$-periodic in the second argument. We insert this representation in the partial differential equation and compare the terms with the same power of $\varepsilon$. This leads to equations for the $u_i$. Resolving this we get the structure of the effective partial differential equation. The main drawback is that it is just a formal ansatz, so we have to prove at the end the convergence of $u^{\varepsilon}$ to $u_0$. For more details we refer to [Cioranescu and Donato, 1999] and the references therein. A more general method is the method of oscillating test functions, also called energy method, developed by Tartar [Tartar, 1978]. Its idea is to choose special test functions built by solutions of the cell problems, which we have to guess first, and to use compactness results to pass to limit in the weak form of the partial differential equation. The method of two-scale convergence introduced by Nguetseng and Allaire [Nguetseng, 1989, Allaire, 1992] exploits the periodic structure to a greater extent, which allows us to get the homogenized problem and the convergence at once. A bounded sequence $u^\varepsilon$ in $L^2(\Omega)$ is said to two-scale converge to $u_0 \in L^2(\Omega \times Y)$ if

$$\lim_{\varepsilon \to 0} \int_\Omega u^\varepsilon(x)\psi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x = \int_\Omega \int_Y u_0(x,y)\psi(x,y)\,\mathrm{d}x\mathrm{d}y$$

for any smooth function $\psi$, which is $Y$-periodic in the second argument. We use these kind of admissible test functions in the weak form of the partial differential equation and compactness results in the setting of two-scale convergence to pass to the limit $\varepsilon \to 0$. For the periodic unfolding method, which was introducted by Cioranescu, Damlamian and Griso [Cioranescu et al., 2002], we define an operator, which seperates the micro- and macroscale by doubling the dimension. This allows us to apply well-known weak convergence results. In particular, the weak convergence of the unfolded sequence is equivalent to the two-scale convergence of the sequence (see Theorem 3.2.3). But homogenization is not restricted to the case of periodic domain or coefficients. We refer to the G-convergence introduced by Spagnolo [Spagnolo, 1968] for sequences of symmetric coefficients and to the H-convergence defined by Murat and Tartar [Murat and Tartar, 1997] for sequences of non-symmetric coefficients, whereby also the energy method is applicable in this setting. The more general convergence of functionals is the $\Gamma$-convergence in the context of calculus of variations [Braides, 2002].

## 3.2. Periodic unfolding method

The periodic unfolding method was introduced by Doina Cioranescu, Alain Damlamian and George Griso in 2002 [Cioranescu et al., 2002], formalising an idea of Todd Arbogast, Jim Douglas and Ulrich Hornung [Arbogast et al., 1990]. The main idea is to define an operator, which seperates the micro- and macroscale of a function. Although this doubles the dimension, the seperation of scales makes it easier to work with perforated domains, whose structure changes with $\varepsilon$. We refer to [Cioranescu et al., 2018] for the definitions and theorems in this section and further results in the setting of periodic unfolding, whereby this book especially includes the results of the papers [Cioranescu et al., 2008] and [Cioranescu et al., 2012]. The first paper considers a fixed domain and the second a perforated domain with connected or disconnected holes. For the first part of the thesis, we also need some results from [Donato et al., 2011], which is in the setting of a perforated domain consisting of two components.

Let $\Omega \subset \mathbb{R}^3$ be a open bounded domain with Lipschitz boundary and $Y \coloneqq (0,l_1)\times(0,l_2)\times(0,l_3)$ for some constants $l_1, l_2, l_3 > 0$. Let $x \in \mathbb{R}^3$. We denote by $[x]$ the unique linear combination of the integers $\xi_j \in \mathbb{Z}$ and the periodicity vectors $b_j \in \mathbb{R}^3$, i.e. $[x] = \sum_{j=1}^3 \xi_j b_j$, such that $\{x\} \coloneqq x - [x] \in Y$ (see Figure 3.3). In this thesis, $b_j$ is of the form $b_j = l_j e_j$, where $e_j$ is the $j$-th unit vector.

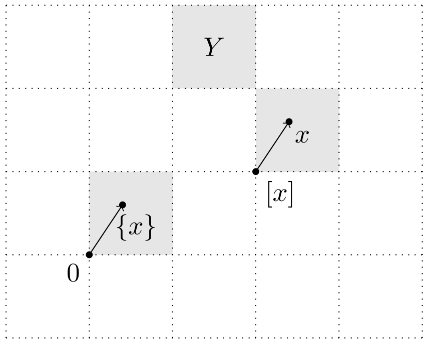


Figure 3.3.: illustration of $\{x\}$ and $[x]$

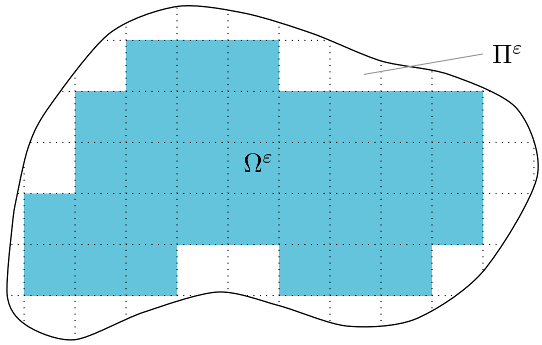


Figure 3.4.: illustration of $\Omega^\varepsilon$ and $\Pi^\varepsilon$

Therefore, we can rewrite $x \in \mathbb{R}^3$ via $x = \varepsilon\left(\left[\frac{x}{\varepsilon}\right] + \left\{\frac{x}{\varepsilon}\right\}\right)$. We split the domain $\Omega$ in two disjoint sets depending on $\varepsilon > 0$

$$\Omega^\varepsilon := \text{interior}\left(\bigcup_{\xi \in \Lambda^\varepsilon} \varepsilon(\overline{Y} + \xi)\right) \quad \text{and} \quad \Pi^\varepsilon := \Omega \backslash \Omega^\varepsilon,$$

where $\Lambda^\varepsilon := \{\xi \in \mathbb{R}^3 : \varepsilon(Y + \xi) \subset \Omega\}$ (see Figure 3.4).

**Definition 3.2.1.** *For a Lebesgue-measurable function $\phi$ on $\Omega$, the periodic unfolding operator $\mathcal{T}^\varepsilon \colon L^p(\Omega) \to L^p(\Omega \times Y)$, $p \in [1, \infty)$, is defined as*

$$\mathcal{T}^\varepsilon(\phi)(x, y) = \begin{cases} \phi(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y) & \text{for a.e. } (x, y) \in \Omega^\varepsilon \times Y, \\ 0 & \text{for a.e. } (x, y) \in \Pi^\varepsilon \times Y. \end{cases}$$

We summarise a number of well-known results for periodic unfolding, whereby we denote by $\mathcal{M}_{\mathcal{O}}(\phi) = \frac{1}{|\mathcal{O}|}\int_{\mathcal{O}} \phi \,\mathrm{d}y$ the mean value of a function $\phi$ over the domain $\mathcal{O}$ with $\mathcal{O}$ a set of finite measure.

**Proposition 3.2.2.** *Let $p \in [1, \infty)$. The operator $\mathcal{T}^\varepsilon \colon L^p(\Omega) \to L^p(\Omega \times Y)$ is linear and continuous. Furthermore, there holds*

(i) $\mathcal{T}^\varepsilon(vw) = \mathcal{T}^\varepsilon(v)\mathcal{T}^\varepsilon(w)$ *for all Lebesgue-measurable functions $v, w$ in $\Omega^\varepsilon$,*

(ii) $\displaystyle\frac{1}{|Y|}\int_{\Omega \times Y} \mathcal{T}^\varepsilon(\phi)(x, y)\,\mathrm{d}x\mathrm{d}y = \int_{\Omega^\varepsilon} \phi(x)\,\mathrm{d}x$ *for all $\phi \in L^1(\Omega^\varepsilon)$,*

(iii) $\|\mathcal{T}^\varepsilon(\phi)\|_{L^p(\Omega \times Y)} \le |Y|^{1/p}\|\phi\|_{L^p(\Omega)}$ *for all $\phi \in L^p(\Omega)$,*

(iv) $\mathcal{T}^\varepsilon(\phi) \to \phi$ *strongly in $L^p(\Omega \times Y)$ for all $\phi \in L^p(\Omega)$.*

(v) *If $\{\phi^\varepsilon\}$ is a sequence in $L^p(\Omega)$ with $\phi^\varepsilon \to \phi$ strongly in $L^p(\Omega)$, then $\mathcal{T}^\varepsilon(\phi^\varepsilon) \to \phi$ strongly in $L^p(\Omega \times Y)$.*

(vi) *If $\phi \in L^p(Y)$ $Y$-periodic and $\phi^\varepsilon(x) = \phi\left(\frac{x}{\varepsilon}\right)$, then $\mathcal{T}^\varepsilon(\phi^\varepsilon) \to \phi$ strongly in $L^p(\Omega \times Y)$.*

(vii) *If* $\phi \in W^{1,p}(\Omega)$*, then* $\nabla_y [\mathcal{T}^\varepsilon(\phi)] = \varepsilon \mathcal{T}^\varepsilon(\nabla\phi)$ *and* $\mathcal{T}^\varepsilon(\phi) \in L^p(\Omega, W^{1,p}(Y))$*.*

*Suppose* $p \in (1, \infty)$*.*

(viii) *If* $\phi^\varepsilon \in L^p(\Omega)$ *with* $\|\phi^\varepsilon\|_{L^p(\Omega)} \leq C$ *and* $\mathcal{T}^\varepsilon(\phi^\varepsilon) \rightharpoonup \phi$ *weakly in* $L^p(\Omega \times Y)$*, then* $\phi^\varepsilon \rightharpoonup \mathcal{M}_Y(\phi)$ *weakly in* $L^p(\Omega)$*.*

(ix) *If* $\phi^\varepsilon \in W^{1,p}(\Omega)$ *with* $\phi^\varepsilon \rightharpoonup \phi$ *weakly in* $W^{1,p}(\Omega)$*, then* $\mathcal{T}^\varepsilon(\phi^\varepsilon) \rightharpoonup \phi$ *weakly in* $L^p(\Omega; W^{1,p}(Y))$ *and for a subsequence, there exists* $\hat{\phi} \in L^p(\Omega; W^{1,p}_{\mathrm{per},0}(Y))$ *such that* $\mathcal{T}^\varepsilon(\nabla\phi^\varepsilon) \rightharpoonup \nabla\phi + \nabla_y\hat{\phi}$ *weakly in* $[L^p(\Omega \times Y)]^3$*.*

The next result shows the equivalence between the weak convergence of the unfolded sequence and the two-scale convergence of the sequence. A general introduction to the two-scale convergence method can be found in e.g. [Lukkassen et al., 2002].

**Theorem 3.2.3.** *Suppose* $p \in (1, \infty)$*. Let* $\{\phi^\varepsilon\}$ *be a bounded sequence in* $L^p(\Omega)$*. The following assertions are equivalent:*

(i) $\{\mathcal{T}^\varepsilon(\phi^\varepsilon)\}$ *converges weakly to* $\phi$ *in* $L^p(\Omega \times Y)$*,*

(ii) $\{\phi^\varepsilon\}$ *two-scale converges to* $\phi$*.*

This theorem allows us to apply results of two-scale convergences also in the setting of periodic unfolding. We use assertions (i) and (ii) of Theorem 3.2.3 synonymously without marking this explicitly. For time-dependent cases, we define the partial periodic unfolding operator, which is as the periodic unfolding operator from Definition 3.2.1 with the time variable considered as a parameter.

**Definition 3.2.4.** *For a Lebesgue-measurable function* $\phi$ *on* $S \times \Omega$*, the partial periodic unfolding operator* $\mathcal{T}^\varepsilon_Y \colon L^p(S \times \Omega) \to L^p(S \times \Omega \times Y)$*,* $p \in [1, \infty)$*, is defined as*

$$\mathcal{T}^\varepsilon_Y(\phi)(t, x, y) = \begin{cases} \phi(t, \varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y) & \text{for a.e. } (t, x, y) \in S \times \Omega^\varepsilon \times Y, \\ 0 & \text{for a.e. } (t, x, y) \in S \times \Pi^\varepsilon \times Y. \end{cases}$$

We summarise some results on partial periodic unfolding.

**Proposition 3.2.5.** *There holds*

(i) $\displaystyle\frac{1}{|Y|}\int_{S\times\Omega\times Y} \mathcal{T}^\varepsilon_Y(\phi)(t, x, y)\,\mathrm{d}x\mathrm{d}y\mathrm{d}t = \int_{S\times\Omega^\varepsilon} \phi(t, x)\,\mathrm{d}x\mathrm{d}t$ *for all* $\phi \in L^1(S \times \Omega)$*.*

*Suppose* $p \in [1, \infty)$*.*

(ii) $\|\mathcal{T}^\varepsilon_Y(w)\|_{L^p(S\times\Omega\times Y)} \leq |Y|^{1/p}\|w\|_{L^p(S\times\Omega)}$ *for all* $w \in L^p(S \times \Omega)$*.*

(iii) *If* $w_\varepsilon \to w$ *strongly in* $L^p(S \times \Omega)$*, then* $\mathcal{T}^\varepsilon_Y(w_\varepsilon) \to w$ *strongly in* $L^p(S \times \Omega \times Y)$*.*

(iv) *If* $w_\varepsilon \to w$ *strongly in* $L^p(\Omega; W^{1,p}(S))$*, then* $\mathcal{T}^\varepsilon_Y(w_\varepsilon) \to w$ *strongly in* $L^p(\Omega \times Y; W^{1,p}(S))$*.*

*Suppose* $p \in (1, \infty)$.

(v) *If* $\mathcal{T}_Y^\varepsilon(w_\varepsilon) \rightharpoonup \hat{w}$ *weakly in* $L^p(S \times \Omega \times Y)$*, then* $w_\varepsilon \rightharpoonup \mathcal{M}_Y(\hat{w})$ *weakly in* $L^p(S \times \Omega)$.

(vi) *If* $w_\varepsilon \rightharpoonup w$ *weakly in* $L^p(S; W^{1,p}(\Omega))$*, then, up to a subsequence, there exists some* $\hat{w} \in L^p(S \times \Omega; W^{1,p}_{\mathrm{per},0}(Y))$ *such that* $\mathcal{T}_Y^\varepsilon(\nabla_x w_\varepsilon) \rightharpoonup \nabla_x w + \nabla_y \hat{w}$ *weakly in* $[L^p(S \times \Omega \times Y)]^3$.

For transmission problems, we define the periodic unfolding operators for a two-component domain, where one component is globally connected and the other one not. We suppose that $Y_0$ and $Y_1$ are two open disjoint subsets of $Y$ such that $\overline{Y}_0 \subset Y$, $Y_1$ is connected, $Y = \overline{Y}_0 \cup Y_1$ and $\Sigma_Y := \partial Y_0$ is Lipschitz continuous. Furthermore, let

$$\Omega_0^\varepsilon := \bigcup_{\xi \in \Upsilon^\varepsilon} \varepsilon\left(Y_0 + \xi\right), \quad \Upsilon^\varepsilon := \{\xi \in \mathbb{R}^3 : \varepsilon(\overline{Y}_0 + \xi) \subset \Omega\}$$

be the $Y$-periodically extended domain $Y_0$ scaled with $\varepsilon$, $\Omega_1^\varepsilon := \Omega \setminus \overline{\Omega}_0^\varepsilon$ and

$$\Sigma^\varepsilon := \partial\Omega_0^\varepsilon = \bigcup_{\xi \in \Upsilon^\varepsilon} \varepsilon\left(\Sigma_Y + \xi\right)$$

the $Y$-periodically extended interface $\Sigma_Y$ scaled with $\varepsilon$. The following definition and proposition are from [Donato et al., 2011].

**Definition 3.2.6.** *Let* $i \in \{0, 1\}$*. For a Lebesgue-measurable function* $\phi$ *on* $\Omega_i^\varepsilon$*, the periodic unfolding operator* $\mathcal{T}_i^\varepsilon \colon L^p(\Omega_i^\varepsilon) \to L^p(\Omega \times Y_i)$*,* $p \in [1, \infty)$*, is defined as*

$$\mathcal{T}_i^\varepsilon(\phi)(x, y) := \begin{cases} \phi\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y\right) & \text{for a.e. } (x, y) \in \Omega^\varepsilon \times Y_i, \\ 0 & \text{for a.e. } (x, y) \in \Pi^\varepsilon \times Y_i. \end{cases}$$

We denote by $\tilde{f}$ the extension of the function $f$ to $\Omega$ by zero. The unfolding operator for transmission problems satisfies similar properties as in Proposition 3.2.2.

**Proposition 3.2.7.** *Let* $p \in [1, \infty)$*. The operators* $\mathcal{T}_i^\varepsilon \colon L^p(\Omega_i^\varepsilon) \to L^p(\Omega \times Y_i)$*,* $i \in \{0, 1\}$*, are linear and continuous. Furthermore,*

(i) $\mathcal{T}_i^\varepsilon(vw) = \mathcal{T}_i^\varepsilon(v)\mathcal{T}_i^\varepsilon(w)$ *for all Lebesgue-measurable functions* $v, w$ *on* $\Omega_i^\varepsilon$,

(ii) *for all* $\phi \in L^1(\Omega_i^\varepsilon)$

$$\frac{1}{|Y|}\int_{\Omega \times Y_i} \mathcal{T}_i^\varepsilon(\phi)(x, y)\,\mathrm{d}x\mathrm{d}y = \int_{\Omega^\varepsilon \cap \Omega_i^\varepsilon} \phi(x)\,\mathrm{d}x = \int_{\Omega_i^\varepsilon} \phi(x)\,\mathrm{d}x - \int_{\Pi^\varepsilon \cap \Omega_i^\varepsilon} \phi(x)\,\mathrm{d}x,$$

(iii) $\|\mathcal{T}_i^\varepsilon(\phi)\|_{L^p(\Omega \times Y_i)} \le |Y|^{1/p}\|\phi\|_{L^p(\Omega_i^\varepsilon)}$ *for all* $\phi \in L^p(\Omega_i^\varepsilon)$,

(iv) $\mathcal{T}_i^\varepsilon(\phi) \to \phi$ *strongly in* $L^p(\Omega \times Y_i)$ *for all* $\phi \in L^p(\Omega)$.

(v) *If* $\{\phi^\varepsilon\}$ *is a sequence in* $L^p(\Omega)$ *with* $\phi^\varepsilon \to \phi$ *strongly in* $L^p(\Omega)$*, then* $\mathcal{T}_i^\varepsilon(\phi^\varepsilon) \to \phi$ *strongly in* $L^p(\Omega \times Y_i)$.

(vi) *If* $\phi \in L^p(Y_i)$ $Y$*-periodic and* $\phi^\varepsilon(x) = \phi\left(\frac{x}{\varepsilon}\right)$*, then* $\mathcal{T}_i^\varepsilon(\phi^\varepsilon) \to \phi$ *strongly in* $L^p(\Omega \times Y_i)$.

(vii) *If* $\phi \in W^{1,p}(\Omega_i^\varepsilon)$*, then* $\nabla_y\left[\mathcal{T}_i^\varepsilon(\phi)\right] = \varepsilon \mathcal{T}_i^\varepsilon(\nabla\phi)$ *and* $\mathcal{T}_i^\varepsilon(\phi) \in L^p(\Omega, W^{1,p}(Y_i))$.

*Suppose* $p \in (1, \infty)$.

(viii) *If* $\phi^\varepsilon \in L^p(\Omega_i^\varepsilon)$ *with* $\|\phi^\varepsilon\|_{L^p(\Omega_i^\varepsilon)} \leq C$ *and* $\mathcal{T}_i^\varepsilon(\phi^\varepsilon) \rightharpoonup \phi$ *weakly in* $L^p(\Omega \times Y_i)$*, then* $\widetilde{\phi^\varepsilon} \rightharpoonup \frac{|Y_i|}{|Y|}\mathcal{M}_{Y_i}(\phi)$ *weakly in* $L^p(\Omega)$.

**Remark 3.2.8.** *With the same proof, the statements in Proposition 3.2.7 are also true in the case where* $Y_0$ *intersects the boundary* $\partial Y$ *in a proper sense, i.e.* $\Omega_0^\varepsilon$ *and* $\Omega_1^\varepsilon$ *have to be Lipschitz-domains.*

The unfolding operator can also be defined on the interface $\Sigma^\varepsilon$, which we will need in Part I.

**Definition 3.2.9.** *For a Lebesgue-measurable function* $\phi$ *on* $\Sigma^\varepsilon$*, the boundary unfolding operator* $\mathcal{T}_\mathrm{b}^\varepsilon \colon L^p(\Sigma^\varepsilon) \to L^p(\Omega \times \Sigma_Y)$, $p \in [1, \infty)$*, is defined as*

$$\mathcal{T}_\mathrm{b}^\varepsilon(\phi)(x,y) \coloneqq \begin{cases} \phi\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y\right) & \text{for a.e. } (x,y) \in \Omega^\varepsilon \times \Sigma_Y, \\ 0 & \text{for a.e. } (x,y) \in \Pi^\varepsilon \times \Sigma_Y. \end{cases}$$

This operator has the following properties.

**Proposition 3.2.10.** *For* $\phi \in L^1(\Sigma^\varepsilon)$*, there holds*

$$\int_{\Omega^\varepsilon \cap \Sigma^\varepsilon} \phi(x)\, \mathrm{d}S(x) = \frac{1}{\varepsilon|Y|} \int_{\Omega \times \Sigma_Y} \mathcal{T}_\mathrm{b}^\varepsilon(\phi)(x,y)\, \mathrm{d}x\mathrm{d}S(y).$$

*For* $\phi \in L^p(\Sigma^\varepsilon)$, $p \in [1, \infty)$,

$$\|\mathcal{T}_\mathrm{b}^\varepsilon(\phi)\|_{L^p(\Omega \times \Sigma_Y)} = \varepsilon^{1/p}|Y|^{1/p}\|\phi\|_{L^p(\Omega^\varepsilon \cap \Sigma^\varepsilon)}.$$

*If* $\phi$ *belongs to* $W^{1,p}(\Omega_i^\varepsilon)$, $i \in \{0, 1\}$*, then* $\mathcal{T}_b^\varepsilon(\phi) = \mathcal{T}_i^\varepsilon(\phi)|_{\Omega \times \Sigma_Y}$.

**Remark 3.2.11.** *All the results are still true for vector- and matrix-valued functions, if we use the period unfolding operator for every component and the standard inner product for vectors and the Frobenius inner product for matrices.*

In this work, we only have the case $p = 2$.

# Part I.

# Homogenization of linearized elasticity in a two-component medium with slip-displacement conditions

The deformation of a two-scale composite of two materials under forces is often modelled by the linear elasticity equation with Dirichlet and Neumann boundary conditions at separate parts of the outer boundary and perfect bonding of the two components at the internal interface. In practice the last assumption cannot always be justified. As studied in [Hambach et al., 2016], [Lauff et al., 2019] and [Rutzen et al., 2019], the flexural strength of the concrete can be improved, when it is reinforced with short carbon fibres. Although in these papers a perfect bond is assumed, the bonding of the fibres to the concrete matrix (or to other materials like ceramics) is rather weak. We address this problem by assuming general linear slip-displacement conditions in normal and tangential direction at the interface of both materials. Since the carbon fibres in this new material occur as single fibres and they are rather short (of the order of 1 cm in length), the question arise which assumptions have to be made on the connectivity in the context of homogenization. To investigate the impact of this choice, we consider three different cases. While one of the components is globally connected, the other one is either disconnected, globally connected or unidirectionally connected. At the end we derive three different upscaled problems.
The periodic homogenization of problems with imperfect internal interfaces is studied by several authors. E.g. in [Donato and Monsurrò, 2004], [Donato et al., 2011] and [Bunoiu and Timofte, 2018], thermal diffusion with homogeneous Dirichlet boundary conditions on the outer boundary is considered, where one of the subdomains is assumed to be disconnected. Nonlinear variants of these problems were treated more recently by [Donato and Nguyen, 2015], [Nguyen, 2015] and [Graf et al., 2014], whereby in the last two both components are assumed to be connected. The (vector-valued) elasticity problem with Neumann boundary condition on a part of the exterior boundary and prescribed jumps in displacements at the interface was treated in [Orlik, 2011]. A related vector-valued problem with perfect bonding and domains of tubular structure was considered by [Ptashnyk and Seguin, 2016]. The homogenization of periodic media with imperfect contacts in the time-dependent setting was for example treated in [Donato et al., 2007], where the same problem as in [Donato and Monsurrò, 2004] for the time-dependent case was studied. In [Assier et al., 2020] they examine the elastic wave propagation in one dimension with displacement and stress-discontinuity conditions at the edges of the periodicity cell in the setting of high-frequency homogenization. For the case of long wavelengths in a similar setting, we refer to [Bellis et al., 2021], where even non-linear imperfect interface conditions are allowed.
The periodic homogenization of linearized elasticity for standard (external and internal) boundary conditions is well studied, cf. e.g. [Oleinik et al., 1992] and [Cioranescu and Donato, 1999]. But since we assume slip-displacement conditions at the interface of both components, which are modelled by Robin-type interface conditions, the difficulty is to show uniform a-priori estimates of the gradient. We solve this problem in the disconnected case by neglecting rigid-body motions in an appropriate sense and by using standard extension operators. In the connected case, we apply extension operators from [Höpker, 2016], which allows us to handle the Dirichlet boundary conditions at the external boundary. In unidirectionally connected case, where we assume that the slices of the domain orthogonal to the direction of connectedness are always

the same, we cannot simply separate the relevant derivatives as is standard in a scalar-valued case since we have to take shear forces into account. Therefore, we also neglect the rigid-body motions in a proper sense.

Part I is structured as follows. In chapter 4, we introduce the general notation and assumptions used for all three kinds of microstructures and derive the weak formulation of the linear elasticity problem with slip-displacement conditions. The subsequent three chapters are structured in the same way. After proving the existence and uniform boundedness of the solution and several compactness results, we derive the homogenized problem, whereby in chapter 5 the globally disconnected case is considered, in chapter 6 the connected case and in chapter 7 the unidirectionally connected case. Chapter 8 concludes by summarising the findings of part I.

# 4. Statement of the problem for a composite with periodic microstructure

We study the upsacling of the linear elasticity equation in a composite of two materials with periodic microstructure, assuming slip-displacement conditions in normal and tangential direction at the microscopic interface. We distinguish three different microscopic structures. In the disconnected case, we assume that one material is connected, whereas the other one is not connected in any direction. In the globally connected case, the composite consists of two globally connected materials and in the unidirectionally connected case, one material is globally connected and the other one only in one direction. In this chapter, we introduce the general notation and assumptions, which are true for all three cases. The additional assumptions depending on the case are explained in more detail in the respective chapters.

Let $\Omega \subset \mathbb{R}^3$ be an open bounded connected Lipschitz-domain. We split the external boundary of $\Omega$ into two parts $\partial\Omega = \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N}}$, where $\Gamma_{\mathrm{D}}$ and $\Gamma_{\mathrm{N}}$ are disjoint sets and $\Gamma_{\mathrm{D}}$ has positive two-dimensional Hausdorff measure. To describe the periodic microstructure, we consider the reference cell $Y = (0,1)^3 \subset \mathbb{R}^3$ and two disjoint open subsets $Y_0, Y_1 \subset Y$ such that $\Sigma_Y := \overline{Y_0} \cap \overline{Y_1}$ Lipschitz-continuous and $Y = \text{interior}\,(Y_0 \cup \Sigma_Y \cup Y_1)$ (see Figure 4.1).

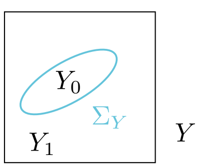

Figure 4.1.: reference cell $Y$

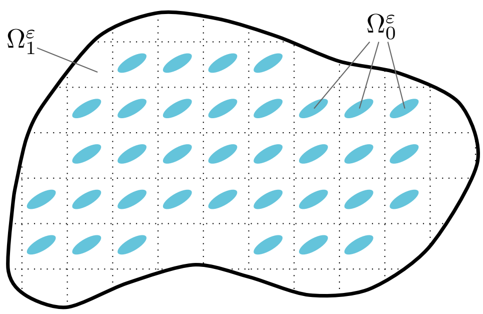

Figure 4.2.: domain $\Omega$

We define the subset of $\Omega$ of all completely contained $\varepsilon$-scaled and translated periodicity cells $Y$ as

$$\Omega^\varepsilon := \text{interior}\left( \bigcup_{\xi \in \Lambda^\varepsilon} \varepsilon(\overline{Y} + \xi) \right), \quad \Lambda^\varepsilon = \{\xi \in \mathbb{Z}^3 : \varepsilon\,(Y + \xi) \subset \Omega\},$$

and the rest of the domain as $\Pi^\varepsilon := \Omega \backslash \Omega^\varepsilon$. Furthermore, let

$$\Omega_0^\varepsilon := \text{interior}\left(\bigcup_{\xi \in \Lambda^\varepsilon} \varepsilon\left(\overline{Y_0} + \xi\right)\right)$$

the $Y$-periodically extended domain $Y_0$ scaled with $\varepsilon$ and intersected with $\Omega^\varepsilon$ (see Figure 4.2), $\Omega_1^\varepsilon := \Omega \setminus \overline{\Omega}_0^\varepsilon$ and

$$\Sigma^\varepsilon := \partial\Omega_0^\varepsilon = \bigcup_{\xi \in \Lambda^\varepsilon} \varepsilon\left(\Sigma_Y + \xi\right)$$

the $Y$-periodically extended interface $\Sigma_Y$ scaled with $\varepsilon$ and intersected with $\Omega^\varepsilon$. The idea is that $\Omega_0^\varepsilon$ represents one material in the composite and $\Omega_1^\varepsilon$ the other one.

We consider the linear elasticity equation in the steady-state case as introduced in section 2.1 with Dirichlet and Neumann boundary conditions on the outer boundary

$$\begin{cases} -\nabla \cdot \sigma^\varepsilon = f^\varepsilon & \text{in } \Omega_0^\varepsilon \cup \Omega_1^\varepsilon, \\ u^\varepsilon = 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \sigma^\varepsilon \nu = g & \text{on } \Gamma_{\mathrm{N}}, \end{cases} \tag{4.0.1}$$

where $f^\varepsilon$ is some given body force, which may depend on $\varepsilon$, $g$ some surface force and $\nu$ the outward-pointing normal to $\Gamma_{\mathrm{N}}$. The stress tensor $\sigma^\varepsilon = (\sigma_{ij}^\varepsilon)_{1 \le i,j \le 3}$ is defined by

$$\sigma_{ij}^\varepsilon = \sum_{k,l=1}^{3} a_{ijkl}^\varepsilon e_{kl}(u^\varepsilon) = \sum_{k,l=1}^{3} a_{ijkl}^\varepsilon \frac{1}{2}\left(\partial_k u_l^\varepsilon + \partial_l u_k^\varepsilon\right),$$

where $u^\varepsilon \colon \Omega^\varepsilon \to \mathbb{R}^3$ is the displacement field, $e(u^\varepsilon)$ is the linear strain tensor and $A^\varepsilon = (a_{ijkh}^\varepsilon)_{1 \le i,j,k,h \le 3}$ is a tensor of fourth order, which describes the stiffness of the materials of the solid. We assume that $A^\varepsilon$ is of the form

$$A^\varepsilon(x) = (a_{ijkh}^\varepsilon(x))_{1 \le i,j,k,h \le 3} := \left(a_{ijkh}\left(\frac{x}{\varepsilon}\right)\right)_{1 \le i,j,k,h \le 3} = A\left(\frac{x}{\varepsilon}\right),$$

where $A = (a_{ijkh})_{1 \le i,j,k,h \le 3} \in M(\alpha, \beta, Y)$ (see Definition 2.1.1) and all components $a_{ijkh}$ are $Y$-periodic for all $i, j, k, h \in \{1, 2, 3\}$. Thus, $A^\varepsilon \in M(\alpha, \beta, \Omega)$. Let $n$ be the normal to $\Sigma^\varepsilon$ with orientation from $\Omega_0^\varepsilon$ to $\Omega_1^\varepsilon$ and $\tau^1$, $\tau^2$ the tangential vectors of $\Sigma^\varepsilon$ such that $n$, $\tau^1$ are $\tau^2$ are mutually orthogonal. We define

$$u_n^\varepsilon := u^\varepsilon \cdot n \text{ resp. } u_{\tau^i}^\varepsilon := u^\varepsilon \cdot \tau^i,\ i \in \{1, 2\},$$

the projection of the displacement field in normal resp. tangential direction of the interface,

$$\sigma_n^\varepsilon := (\sigma^\varepsilon n) \cdot n \text{ resp. } \sigma_{\tau^i}^\varepsilon := (\sigma^\varepsilon n) \cdot \tau^i,\ i \in \{1, 2\},$$

the projection of the normal stress in normal resp. tangential direction of the interface and

$$[\varphi]_{\Sigma^\varepsilon} \coloneqq (\varphi_1 - \varphi_0)\,|_{\Sigma^\varepsilon}$$

the jump on the interface for some function $\varphi$, where $\varphi_\kappa \coloneqq \varphi|_{\Omega_\kappa^\varepsilon}$ is the restriction of $\varphi$ to $\Omega_\kappa^\varepsilon$ and $\varphi_\kappa|_{\Sigma^\varepsilon}$ is the trace of $\varphi_\kappa$ (we just write $\varphi_\kappa$ if it is clear) for $\kappa \in \{0,1\}$. With this notation, the slip-displacement conditions on the interface $\Sigma^\varepsilon$ are given by

$$\begin{cases} \varepsilon\,[u_n^\varepsilon]_{\Sigma^\varepsilon} = \dfrac{1}{K_\mathrm{N}}\sigma_n^{\Sigma^\varepsilon}, & \\ \varepsilon\,[u_{\tau^i}^\varepsilon]_{\Sigma^\varepsilon} = \dfrac{1}{K_\mathrm{T}}\sigma_{\tau^i}^{\Sigma^\varepsilon}, & i \in \{1,2\}, \\ [\sigma_n^\varepsilon]_{\Sigma^\varepsilon} = 0, & \\ [\sigma_{\tau^i}^\varepsilon]_{\Sigma^\varepsilon} = 0, & i \in \{1,2\}, \end{cases} \tag{4.0.2}$$

where the constants $K_\mathrm{N}, K_\mathrm{T} \geq 0$ are the normal resp. tangential stiffness and $\sigma^{\Sigma^\varepsilon}$ is the stress tensor of the interface. We refer to [Lombard and Piraux, 2006] for more details on the slip-displacement conditions.

We test problem (4.0.1) with some sufficiently smooth function $\varphi$ with $\varphi = 0$ on $\Gamma_\mathrm{D}$, use integration by parts and the Dirichlet boundary conditions to compute

$$\begin{aligned} \int_{\Omega^\varepsilon} f^\varepsilon \cdot \varphi \,\mathrm{d}x = \int_{\Omega_0^\varepsilon} \sigma_0^\varepsilon : \nabla\varphi_0 \,\mathrm{d}x + \int_{\Omega_1^\varepsilon} \sigma_1^\varepsilon : \nabla\varphi_1 \,\mathrm{d}x - \int_{\Gamma_\mathrm{N}\cap\partial\Omega_0^\varepsilon} g \cdot \varphi_0 \,\mathrm{d}S(x) \\ - \int_{\Gamma_\mathrm{N}\cap\partial\Omega_1^\varepsilon} g \cdot \varphi_1 \,\mathrm{d}S(x) + \int_{\Sigma^\varepsilon} \sigma_1^\varepsilon n \cdot \varphi_1 - \sigma_0^\varepsilon n \cdot \varphi_0 \,\mathrm{d}S(x). \end{aligned}$$

Since $\sigma^\varepsilon$ is symmetric and the jump of the normal stress on $\Sigma^\varepsilon$ is zero, meaning that $\sigma_0^\varepsilon = \sigma_1^\varepsilon$ on $\Sigma^\varepsilon$ and thus $\sigma_0^\varepsilon, \sigma_1^\varepsilon$ coincide with $\sigma^{\Sigma^\varepsilon}$ on $\Sigma^\varepsilon$, we get

$$\begin{aligned} \int_{\Omega^\varepsilon} f^\varepsilon \cdot \varphi \,\mathrm{d}x = \int_{\Omega_0^\varepsilon} \sigma_0^\varepsilon : e(\varphi_0) \,\mathrm{d}x + \int_{\Omega_1^\varepsilon} \sigma_1^\varepsilon : e(\varphi_1) \,\mathrm{d}x - \int_{\Gamma_\mathrm{N}\cap\partial\Omega_0^\varepsilon} g \cdot \varphi_0 \,\mathrm{d}S(x) \\ - \int_{\Gamma_\mathrm{N}\cap\partial\Omega_1^\varepsilon} g \cdot \varphi_1 \,\mathrm{d}S(x) + \int_{\Sigma^\varepsilon} \left(\sigma^{\Sigma^\varepsilon} n\right) \cdot (\varphi_1 - \varphi_0) \,\mathrm{d}S(x). \end{aligned}$$

We split up the normal stress of the interface in normal and tangential component and use the conditions on the interface (4.0.2) to obtain the weak formulation

$$\begin{aligned} &\int_{\Omega_0^\varepsilon} A^\varepsilon e(u_0^\varepsilon) e(\varphi_0) \mathrm{d}x + \int_{\Omega_1^\varepsilon} A^\varepsilon e(u_1^\varepsilon) e(\varphi_1) \mathrm{d}x \\ &\quad + \varepsilon \int_{\Sigma^\varepsilon} \left( K_\mathrm{N}\,[u_n^\varepsilon]_{\Sigma^\varepsilon}\, n + K_\mathrm{T} \sum_{i=1}^{2} [u_{\tau^i}^\varepsilon]_{\Sigma^\varepsilon}\, \tau^i \right) \cdot (\varphi_1 - \varphi_0) \,\mathrm{d}S(x) \\ &= \int_{\Omega_0^\varepsilon} f^\varepsilon \cdot \varphi_0 \,\mathrm{d}x + \int_{\Omega_1^\varepsilon} f^\varepsilon \cdot \varphi_1 \,\mathrm{d}x + \int_{\Gamma_\mathrm{N}\cap\partial\Omega_0^\varepsilon} g \cdot \varphi_0 \,\mathrm{d}S(x) + \int_{\Gamma_\mathrm{N}\cap\partial\Omega_1^\varepsilon} g \cdot \varphi_1 \,\mathrm{d}S(x). \end{aligned} \tag{4.0.3}$$

To complete the weak formulation, we have to define appropriate function spaces, which depend on the connectedness of $\Omega_0^\varepsilon$.

Throughout this thesis, unless otherwise stated, we denote by $C$ a constant independent of $\varepsilon$ whose value may change from line to line. We define $\mathcal{D}(\Omega) := C_c^\infty(\Omega)$ and $H^1_\Gamma(\mathcal{O}) := \{u \in [H^1(\mathcal{O})]^3 : u = 0 \text{ on } \Gamma\}$ for some open set $\mathcal{O}$ with $\Gamma \subset \partial\mathcal{O}$.

# 5. Disconnected case

In the disconnected case, we additionally require to the assumptions in chapter 4 that $Y_0 \subset\subset Y$, i.e. $Y_0$ is a relatively compact subset of $Y$. Thus, $\Omega_1^\varepsilon$ is globally connected and $\Omega_0^\varepsilon$ is the union of disconnected domains (with volume of order $\varepsilon^{-3}|\Omega|$). Since $\Gamma_\mathrm{N} \cap \Omega_0^\varepsilon = \emptyset$ and $\Gamma_\mathrm{N} \cap \Omega_1^\varepsilon = \Gamma_\mathrm{N}$, the third integral on the right-hand side in (4.0.3) vanishes.
We prove in section 5.1 the existence, uniqueness and uniform boundedness of the solution of (4.0.3). In section 5.2, we derive the homogenized problem after proving several compactness results.

## 5.1. Existence result in the disconnected case

As mentioned in the general introduction of the first part of the thesis, the difficulty is to show uniform a-priori estimates of the first derivative of the solution of (4.0.3). For the connected domain $\Omega_1^\varepsilon$, we can use well-known uniform extension operators to get $H^1$-functions on the whole domain $\Omega$. This is in general not possible for the disconnected domain $\Omega_0^\varepsilon$ but we can estimate the $\varepsilon$-scaled gradient. We define the solution space for the disconnected case as

$$\mathcal{W}_\mathrm{d}^\varepsilon(\Omega) = \{u \in \left[L^2(\Omega)\right]^3 : \; u_1 = u|_{\Omega_1^\varepsilon} \in H^1_{\Gamma_\mathrm{D}}(\Omega_1^\varepsilon), u_0 = u|_{\Omega_0^\varepsilon} \in \left[H^1(\Omega_0^\varepsilon)\right]^3\},$$

endowed with the norm

$$\|u\|^2_{\mathcal{W}_\mathrm{d}^\varepsilon(\Omega)} := \|e(u_0)\|^2_{\left[L^2(\Omega_0^\varepsilon)\right]^{3\times 3}} + \|e(u_1)\|^2_{\left[L^2(\Omega_1^\varepsilon)\right]^{3\times 3}} + \varepsilon \| \left[u\right]_{\Sigma^\varepsilon} \|^2_{[L^2(\Sigma^\varepsilon)]^3}$$

for all $u \in \mathcal{W}_\mathrm{d}^\varepsilon(\Omega)$.

**Theorem 5.1.1.** $\left(\mathcal{W}_\mathrm{d}^\varepsilon(\Omega), \|\cdot\|_{\mathcal{W}_\mathrm{d}^\varepsilon(\Omega)}\right)$ *defines a Hilbert space.*

*Proof.* Since $\|\cdot\|^2_{\mathcal{W}_\mathrm{d}^\varepsilon(\Omega)}$ is defined as a sum over norms, the subadditivity and absolute homogeneity follows directly. We obtain the positive definiteness from the fact that $u_1 = 0$ on $\Gamma_\mathrm{D}$ for $u \in \mathcal{W}_\mathrm{d}^\varepsilon(\Omega)$ and Korn's inequality from Theorem 2.2.5. Thus, $\|\cdot\|_{\mathcal{W}_\mathrm{d}^\varepsilon(\Omega)}$ defines a norm on $\mathcal{W}_\mathrm{d}^\varepsilon(\Omega)$. The trace operator and Korn's inequalities for functions with zero trace on part of the boundary for $u_1$ (cf. Theorem 2.2.5) and with control of the boundary for $u_0$ (cf.

Theorem 2.2.4) yield

$$
\begin{aligned}
c(\varepsilon)\|u\|^2_{\mathcal{W}^\varepsilon_{\mathrm{d}}(\Omega)} &\le \sum_{\xi\in\Lambda^\varepsilon} \|u_0\|^2_{[H^1(\varepsilon(Y_0+\xi))]^3} + \|u_1\|^2_{\left[H^1(\Omega^\varepsilon_1)\right]^3} \\
&\le C(\varepsilon)\left(\sum_{\xi\in\Lambda^\varepsilon} \|e(u_0)\|^2_{[L^2(\varepsilon(Y_0+\xi))]^{3\times 3}} + \|u_0\|_{[L^2(\varepsilon(\Sigma_Y+\xi))]^3} + \|e(u_1)\|^2_{\left[L^2(\Omega^\varepsilon_1)\right]^{3\times 3}}\right) \\
&\le C(\varepsilon)\left(\sum_{\xi\in\Lambda^\varepsilon} \|e(u_0)\|^2_{[L^2(\varepsilon(Y_0+\xi))]^{3\times 3}} + \|u_1-u_0\|_{[L^2(\varepsilon(\Sigma_Y+\xi))]^3}\right. \\
&\qquad \left. + \|u_1\|_{[H^1(\varepsilon(Y_1+\xi))]^3} + \|e(u_1)\|^2_{\left[L^2(\Omega^\varepsilon_1)\right]^{3\times 3}}\right) \\
&\le C(\varepsilon)\|u\|^2_{\mathcal{W}^\varepsilon_{\mathrm{d}}(\Omega)},
\end{aligned}
$$

where the constants $c$ and $C$ may depend on $\varepsilon$. Since for every $\varepsilon$ the set $\Lambda^\varepsilon$ is finite, the norms are equivalent and thus, $\mathcal{W}^\varepsilon_{\mathrm{d}}(\Omega)$ can be understood as the direct sum of Hilbert spaces

$$
\left[H^1_{\Gamma_{\mathrm{D}}\cap\partial\Omega^\varepsilon_1}(\Omega^\varepsilon_1)\right]^3 \times \prod_{\xi\in\Lambda^\varepsilon} \left[H^1(\varepsilon(Y_0+\xi))\right]^3
$$

endowed with the standard $H^1$-norms, which yields the desired result. □

We prove some uniform estimates.

**Lemma 5.1.2.** *For every $v\in\mathcal{W}^\varepsilon_{\mathrm{d}}(\Omega)$, there holds*

(i) $\varepsilon\|v_1\|^2_{[L^2(\Sigma^\varepsilon)]^3} \le C\left(\|v_1\|^2_{\left[L^2(\Omega^\varepsilon_1)\right]^3} + \varepsilon^2\|\nabla v_1\|^2_{\left[L^2(\Omega^\varepsilon_1)\right]^{3\times 3}}\right)$

(ii) $\|v_1\|^2_{\left[H^1(\Omega^\varepsilon_1)\right]^3} \le C\|e(v_1)\|^2_{\left[L^2(\Omega^\varepsilon_1)\right]^{3\times 3}}$

(iii) $\|v_0\|^2_{\left[L^2(\Omega^\varepsilon_0)\right]^3} \le C\left(\varepsilon^2\|e(v_0)\|^2_{\left[L^2(\Omega^\varepsilon_0)\right]^{3\times 3}} + \varepsilon\|v_0\|^2_{[L^2(\Sigma^\varepsilon)]^3}\right)$

*for constants $C>0$ independent of $\varepsilon$.*

*Proof.* The first estimates follows by scaling and summation together with the continuity of the trace operator

$$
\begin{aligned}
\varepsilon\|v_1\|^2_{[L^2(\Sigma^\varepsilon)]^3} &= \varepsilon^3\sum_{\xi\in\Lambda^\varepsilon}\int_{\Sigma_Y} |v_1(\varepsilon y+\varepsilon\xi)|^2 \mathrm{d}S(y) \\
&\le \varepsilon^3 C\sum_{\xi\in\Lambda^\varepsilon}\int_{Y_1} |v_1(\varepsilon y+\varepsilon\xi)|^2 \mathrm{d}y + \int_{Y_1} |\nabla_y v_1(\varepsilon y+\varepsilon\xi)|^2 \mathrm{d}y \\
&\le C\left(\|v_1\|^2_{\left[L^2(\Omega^\varepsilon_1)\right]^3} + \varepsilon^2\|\nabla v_1\|^2_{\left[L^2(\Omega^\varepsilon_1)\right]^{3\times 3}}\right).
\end{aligned}
$$

For the second inequality, we use the extension operator from Theorem 4.2 in [Oleinik et al., 1992]. This operator extents vector-valued functions defined on perforated domains, where the holes do not intersect the boundary of the reference cell or only in an appropriate way, to the whole domain and satisfies some $\varepsilon$-independent estimates. More precisely, there exists a linear extension operator $P_\varepsilon : [H^1(\Omega_1^\varepsilon)]^3 \to [H^1(\Omega)]^3$ such that

$$\begin{aligned}
&\|P_\varepsilon v\|_{[H^1(\Omega)]^3} \le C\|v\|_{\left[H^1(\Omega_1^\varepsilon)\right]^3},\\
&\|P_\varepsilon v\|_{[L^2(\Omega)]^3} + \|e(P_\varepsilon v)\|_{[L^2(\Omega)]^{3\times 3}} \le C\left(\|v\|_{\left[L^2(\Omega_1^\varepsilon)\right]^3} + \|e(v)\|_{\left[L^2(\Omega_1^\varepsilon)\right]^{3\times 3}}\right),\\
&\|e(P_\varepsilon v)\|_{[L^2(\Omega)]^{3\times 3}} \le C\|e(v)\|_{\left[L^2(\Omega_1^\varepsilon)\right]^{3\times 3}}
\end{aligned} \tag{5.1.1}$$

for all $v \in [H^1(\Omega_1^\varepsilon)]^3$. Together with Theorem 2.2.5, i.e. Korn's inequality for functions with zero trace on part of the boundary, we estimate

$$\|v_1\|^2_{\left[H^1(\Omega_1^\varepsilon)\right]^3} \le \|P_\varepsilon v_1\|^2_{[H^1(\Omega)]^3} \le C\|e(P_\varepsilon v_1)\|^2_{[L^2(\Omega)]^{3\times 3}} \le C\|e(v_1)\|^2_{\left[L^2(\Omega_1^\varepsilon)\right]^{3\times 3}}.$$

Statement (iii) follows by scaling and summation together with Korn's inequality from Theorem 2.2.4, i.e. Korn's inequality with control of boundary values,

$$\begin{aligned}
\|v_0\|^2_{\left[L^2(\Omega_0^\varepsilon)\right]^3} &= \varepsilon^3 \sum_{\xi\in\Lambda^\varepsilon} \int_{Y_0} |v_0(\varepsilon y + \varepsilon\xi)|^2 \mathrm{d}y\\
&\le \varepsilon^3 C \sum_{\xi\in\Lambda^\varepsilon} \int_{Y_0} |e_y(v_0(\varepsilon y + \varepsilon\xi))|^2 \mathrm{d}y + \int_{\Sigma_Y} |v_0(\varepsilon y + \varepsilon\xi)|^2 \mathrm{d}S(y)\\
&= C\left(\varepsilon^2 \|e(v_0)\|^2_{\left[L^2(\Omega_0^\varepsilon)\right]^{3\times 3}} + \varepsilon\|v_0\|^2_{[L^2(\Sigma^\varepsilon)]^3}\right).
\end{aligned}$$

□

There exists a unique weak solution in the space $\mathcal{W}_\mathrm{d}^\varepsilon(\Omega)$.

**Theorem 5.1.3.** *Let $f^\varepsilon \in \left[L^2(\Omega)\right]^3$ and $g \in \left[L^2(\Gamma_\mathrm{N})\right]^3$. Then, there exists a unique weak solution $u \in \mathcal{W}_\mathrm{d}^\varepsilon(\Omega)$ of (4.0.3) for all $\varphi \in \mathcal{W}_\mathrm{d}^\varepsilon(\Omega)$.*

*Proof.* Our aim is to apply the Lax–Milgram theorem. Let $\varepsilon > 0$. We denote the left-hand side of (4.0.3) as a mapping $a\colon \mathcal{W}_\mathrm{d}^\varepsilon(\Omega) \times \mathcal{W}_\mathrm{d}^\varepsilon(\Omega) \to \mathbb{R}$,

$$\begin{aligned}
a(u,v) = &\int_{\Omega_0^\varepsilon} A^\varepsilon e(u_0) e(v_0)\mathrm{d}x + \int_{\Omega_1^\varepsilon} A^\varepsilon e(u_1) e(v_1)\mathrm{d}x\\
&+ \varepsilon \int_{\Sigma^\varepsilon} \left(K_\mathrm{N} \left[u_n\right]_{\Sigma^\varepsilon} n + K_\mathrm{T} \sum_{i=1}^{2} \left[u_{\tau^i}\right]_{\Sigma^\varepsilon} \tau^i\right) \cdot (v_1 - v_0)\,\mathrm{d}S(x)
\end{aligned} \tag{5.1.2}$$

and the right-hand side of (4.0.3) as a mapping $F\colon \mathcal{W}_{\mathrm{d}}^{\varepsilon}(\Omega) \to \mathbb{R}$,

$$F(v) = \int_{\Omega} f^{\varepsilon} \cdot v \,\mathrm{d}x + \int_{\Gamma_{\mathrm{N}}} g \cdot v_1 \,\mathrm{d}S(x),$$

whereby we merged the first two terms of (4.0.3). This is feasible, since $\Sigma^{\varepsilon}$ is a Lebesgue null set with respect to the three dimensional Lebesgue measure and $\mathcal{W}_{\mathrm{d}}^{\varepsilon}(\Omega) \subset [L^2(\Omega)]^3$. We prove that $a$ is a continuous coercive bilinear form. The linearity in both components follows directly from the linearity of the integrals. Let $u, v \in \mathcal{W}_{\mathrm{d}}^{\varepsilon}(\Omega)$. With the properties of $A^{\varepsilon} \in M(\alpha, \beta, \Omega)$ and the splitting of $u_0$ and $u_1$ in normal and tangential part, we get

$$\begin{aligned} a(u,u) &\geq \alpha \int_{\Omega_0^{\varepsilon}} |e(u_0)|^2 \mathrm{d}x + \alpha \int_{\Omega_1^{\varepsilon}} |e(u_1)|^2 \mathrm{d}x + \varepsilon \int_{\Sigma^{\varepsilon}} K_{\mathrm{N}} \, [u_n]_{\Sigma^{\varepsilon}}^2 + K_{\mathrm{T}} \sum_{i=1}^{2} [u_{\tau^i}]_{\Sigma^{\varepsilon}}^2 \,\mathrm{d}S(x) \\ &\geq \alpha \|e(u_0)\|^2_{\left[L^2(\Omega_0^{\varepsilon})\right]^{3\times 3}} + \alpha \|e(u_1)\|^2_{\left[L^2(\Omega_1^{\varepsilon})\right]^{3\times 3}} + \min\{K_{\mathrm{N}}, K_{\mathrm{T}}\} \varepsilon \| \, [u]_{\Sigma^{\varepsilon}} \, \|^2_{[L^2(\Sigma^{\varepsilon})]^3} \\ &\geq \min\{\alpha, K_{\mathrm{N}}, K_{\mathrm{T}}\} \|u\|^2_{\mathcal{W}_{\mathrm{d}}^{\varepsilon}(\Omega)}, \end{aligned}$$

which shows that $a$ is coercive. Using the boundedness of $A^{\varepsilon}$ and Hölder's inequality, we receive the continuity of $a$

$$\begin{aligned} |a(u,v)| &\leq C \left( \|e(u_0)\|_{\left[L^2(\Omega_0^{\varepsilon})\right]^{3\times 3}} \|e(v_0)\|_{\left[L^2(\Omega_0^{\varepsilon})\right]^{3\times 3}} + \|e(u_1)\|_{\left[L^2(\Omega_1^{\varepsilon})\right]^{3\times 3}} \|e(v_1)\|_{\left[L^2(\Omega_1^{\varepsilon})\right]^{3\times 3}} \right) \\ &\quad + \max\{K_{\mathrm{N}}, K_{\mathrm{T}}\} \sqrt{\varepsilon} \|[u]_{\Sigma^{\varepsilon}}\|_{[L^2(\Sigma^{\varepsilon})]^3} \sqrt{\varepsilon} \|[v]_{\Sigma^{\varepsilon}}\|_{[L^2(\Sigma^{\varepsilon})]^3} \\ &\leq C \|u\|_{\mathcal{W}_{\mathrm{d}}^{\varepsilon}(\Omega)} \|v\|_{\mathcal{W}_{\mathrm{d}}^{\varepsilon}(\Omega)}. \end{aligned}$$

It remains to prove that $F$ is linear and continuous. The linearity is clear. Let $v \in \mathcal{W}_{\mathrm{d}}^{\varepsilon}(\Omega)$. By Hölder's inequality

$$|F(v)| \leq C \|f^{\varepsilon}\|_{[L^2(\Omega)]^3} \|v\|_{[L^2(\Omega)]^3} + \|g\|_{[L^2(\Gamma_{\mathrm{N}})]^3} \|v_1\|_{[L^2(\Gamma_{\mathrm{N}})]^3}.$$

We have to estimate the terms $\|v\|_{[L^2(\Omega)]^3}$ and $\|v_1\|_{[L^2(\Gamma_{\mathrm{N}})]^3}$ by the norm $\|v\|_{\mathcal{W}_{\mathrm{d}}^{\varepsilon}(\Omega)}$. Using Lemma 5.1.2 (i)–(iii),

$$\|v_0\|^2_{\left[L^2(\Omega_0^{\varepsilon})\right]^3} \leq C \left( \varepsilon^2 \|e(v_0)\|^2_{\left[L^2(\Omega_0^{\varepsilon})\right]^{3\times 3}} + \varepsilon \|[v]_{\Sigma^{\varepsilon}}\|^2_{[L^2(\Sigma^{\varepsilon})]^3} + \varepsilon \|v_1\|^2_{[L^2(\Sigma^{\varepsilon})]^3} \right) \leq C \|v\|^2_{\mathcal{W}_{\mathrm{d}}^{\varepsilon}(\Omega)} \tag{5.1.3}$$

and

$$\|v_1\|_{\left[L^2(\Omega_1^{\varepsilon})\right]^3} \leq C \|e(v_1)\|_{\left[L^2(\Omega_1^{\varepsilon})\right]^{3\times 3}} \leq C \|v\|_{\mathcal{W}_{\mathrm{d}}^{\varepsilon}(\Omega)}. \tag{5.1.4}$$

Again by the extension operator $P_{\varepsilon}$ from (5.1.1), the trace operator and Lemma 5.1.2 (ii), we receive

$$\|v_1\|_{[L^2(\Gamma_{\mathrm{N}})]^3} \leq C \|P_{\varepsilon} v_1\|_{[H^1(\Omega)]^3} \leq C \|v_1\|_{\left[H^1(\Omega_1^{\varepsilon})\right]^3} \leq C \|e(v_1)\|_{\left[L^2(\Omega_1^{\varepsilon})\right]^{3\times 3}} \leq C \|v\|_{\mathcal{W}_{\mathrm{d}}^{\varepsilon}(\Omega)}.$$

Thus,

$$|F(v)| \le C\left(\|f^\varepsilon\|_{[L^2(\Omega)]^3} + \|g\|_{[L^2(\Gamma_\mathrm{N})]^3}\right)\|v\|_{\mathcal{W}_\mathrm{d}^\varepsilon(\Omega)}.$$

So all assumptions of the Lax–Milgram theorem are fulfilled and we get the existence and uniqueness of the solution. □

In [Donato and Monsurrò, 2004] and [Monsurrò, 2003], a similar approach to the proof of existence of solutions was chosen in the scalar case, where Poincaré's inequality instead of Korn's inequality is used. Under the additional assumption of uniform boundedness of $f^\varepsilon$ the weak solution $u^\varepsilon$ is uniformly bounded.

**Theorem 5.1.4.** *Let $u^\varepsilon \in \mathcal{W}_\mathrm{d}^\varepsilon(\Omega)$ be the weak solution of* (4.0.3) *and $f^\varepsilon$ bounded independent of $\varepsilon$ in $\left[L^2(\Omega)\right]^3$. Then, there exists an $\varepsilon$-independent constant $C$ with*

$$\|u^\varepsilon\|_{\mathcal{W}_\mathrm{d}^\varepsilon(\Omega)} \le C.$$

*Proof.* Using the same estimates as in the proof of Theorem 5.1.3 with $v = u^\varepsilon$, we get

$$\min\{\alpha, K_\mathrm{N}, K_\mathrm{T}\}\|u^\varepsilon\|^2_{\mathcal{W}_\mathrm{d}^\varepsilon(\Omega)} \le a(u^\varepsilon, u^\varepsilon) = F(u^\varepsilon) \le C\left(\|f^\varepsilon\|_{[L^2(\Omega)]^3} + \|g\|_{[L^2(\Gamma_\mathrm{N})]^3}\right)\|u^\varepsilon\|_{\mathcal{W}_\mathrm{d}^\varepsilon(\Omega)},$$

which shows the uniform boundedness of $\|u^\varepsilon\|_{\mathcal{W}_\mathrm{d}^\varepsilon(\Omega)}$. □

In scalar-valued cases the uniform boundedness of the gradient is directly obtained but in our case we first can only estimate the symmetric gradient uniformly. Therefore, we have the study the rigid-body motions more closely. For $u_1^\varepsilon$ we can use the fact that $u_1^\varepsilon = 0$ on $\Gamma_\mathrm{D}$ to estimate the full gradient. Since the $\varepsilon$-scaled jumps at the interface are bounded uniformly, i.e. we have a weak bonding of the materials, $u_0^\varepsilon$ satisfies a (weaker) boundary condition at each connected subset of $\Omega_0^\varepsilon$, which allows us to estimate the $\varepsilon$-scaled gradient of $u_0^\varepsilon$ uniformly.

**Theorem 5.1.5.** *Let $u^\varepsilon \in \mathcal{W}_\mathrm{d}^\varepsilon(\Omega)$ with $\|u^\varepsilon\|_{\mathcal{W}_\mathrm{d}^\varepsilon(\Omega)} \le C$ for an $\varepsilon$-independent constant $C$. Then, the following quantities are bounded uniformly in $\varepsilon$*

$$\|u^\varepsilon\|_{[L^2(\Omega)]^3}, \quad \varepsilon\|\nabla u_0^\varepsilon\|_{\left[L^2(\Omega_0^\varepsilon)\right]^{3\times3}}, \quad \|\nabla u_1^\varepsilon\|_{\left[L^2(\Omega_1^\varepsilon)\right]^{3\times3}}.$$

*Proof.* If we choose $v = u^\varepsilon$ in the estimates (5.1.3) and (5.1.4) and note that the constants there are independent of $\varepsilon$, we get, together with the uniform boundedness of $u^\varepsilon$ in $\mathcal{W}_\mathrm{d}^\varepsilon(\Omega)$, that

$$\|u^\varepsilon\|_{[L^2(\Omega)]^3} \le C.$$

The uniform boundedness of $\|\nabla u_1^\varepsilon\|_{\left[L^2(\Omega_1^\varepsilon)\right]^{3\times3}}$ follows directly from Lemma 5.1.2 (ii). As in the proof of Lemma 5.1.2 (iii) we obtain by scaling and summation together with Korn's inequality from Theorem 2.2.4

$$\begin{aligned}
\varepsilon^2\|\nabla u_0^\varepsilon\|^2_{[L^2(\Omega_0^\varepsilon)]^{3\times 3}} &= \varepsilon^5 \sum_{\xi\in\Lambda^\varepsilon}\int_{Y_0} |\nabla u_0^\varepsilon(\varepsilon y+\varepsilon\xi)|^2 \mathrm{d}y \\
&\leq \varepsilon^3 C \sum_{\xi\in\Lambda^\varepsilon}\int_{Y_0} |e_y(u_0^\varepsilon(\varepsilon y+\varepsilon\xi))|^2 \mathrm{d}y + \int_{\Sigma_Y} |u_0^\varepsilon(\varepsilon y+\varepsilon\xi)|^2 \mathrm{d}S(y) \\
&= C\left(\varepsilon^2\|e(u_0^\varepsilon)\|^2_{[L^2(\Omega_0^\varepsilon)]^{3\times 3}} + \varepsilon\|u_0^\varepsilon\|^2_{[L^2(\Sigma^\varepsilon)]^3}\right) \\
&= C\left(\varepsilon^2\|e(u_0^\varepsilon)\|^2_{[L^2(\Omega_0^\varepsilon)]^{3\times 3}} + \varepsilon\|[u]_{\Sigma^\varepsilon}\|^2_{[L^2(\Sigma^\varepsilon)]^3} + \varepsilon\|u_1\|^2_{[L^2(\Sigma^\varepsilon)]^3}\right),
\end{aligned}$$

which together with Lemma 5.1.2 (i) yields the uniform boundedness. □

## 5.2. Homogenization in the disconnected case

First, we prove in subsection 5.2.1 some general compactness results via the periodic unfolding method, which we apply subsequently in subsection 5.2.2 to derive the homogenized problem. Due to the structure of the domain, we need the periodic unfolding operator for imperfect transmission problems stated in Definition 3.2.6.

### 5.2.1. Compactness results in the disconnected case

Since $\Omega_1^\varepsilon$ can be seen as a perforated domain, the weak convergence of the unfolded sequence $\{\mathcal{T}_1^\varepsilon(u_1^\varepsilon)\}$ with $u_1^\varepsilon \in H^1_{\Gamma_\mathrm{D}}(\Omega_1^\varepsilon)$ can be easily proven by using well-known convergence result for perforated domains. We define the Hilbert space

$$\left[L^2(\Omega, H^1_{\mathrm{per},0}(Y_1))\right]^3 \coloneqq \{u \in \left[L^2(\Omega, H^1_{\mathrm{per}}(Y_1))\right]^3 : \mathcal{M}_{Y_1}(u) = 0\}.$$

**Theorem 5.2.1.** *Let $\{u_1^\varepsilon\}$ be a sequence with $u_1^\varepsilon \in H^1_{\Gamma_\mathrm{D}}(\Omega_1^\varepsilon)$ and*

$$\|u_1^\varepsilon\|_{[L^2(\Omega_1^\varepsilon)]^3} + \|e(u_1^\varepsilon)\|_{[L^2(\Omega_1^\varepsilon)]^{3\times 3}} \leq C \tag{5.2.1}$$

*for a constant $C$ independent of $\varepsilon$. Then, there exists a subsequence (again denoted by $\{u_1^\varepsilon\}$), $u_1 \in H^1_{\Gamma_\mathrm{D}}(\Omega)$ and $\hat{u}_1 \in \left[L^2(\Omega, H^1_{\mathrm{per},0}(Y_1))\right]^3$ such that*

$$\begin{aligned}
\mathcal{T}_1^\varepsilon(u_1^\varepsilon) &\rightharpoonup u_1 \ \textit{weakly in} \ \left[L^2(\Omega, H^1(Y_1))\right]^3, \\
\mathcal{T}_1^\varepsilon(e(u_1^\varepsilon)) &\rightharpoonup e(u_1) + e_y(\hat{u}_1) \ \textit{weakly in} \ \left[L^2(\Omega\times Y_1)\right]^{3\times 3}.
\end{aligned}$$

*Proof.* Let $\{u_1^\varepsilon\}$ be a sequence with $u_1^\varepsilon \in H^1_{\Gamma_\mathrm{D}}(\Omega_1^\varepsilon)$, for which (5.2.1) holds. By Theorem 5.1.5, every function $u_1^\varepsilon$ is bounded independent of $\varepsilon$ in $\left[H^1(\Omega_1^\varepsilon)\right]^3$. Since the domain $\Omega_1^\varepsilon$ is connected, we can apply Theorem 4.43 from [Cioranescu et al., 2018] to obtain, up to a subsequence,

$$\mathcal{T}_1^\varepsilon(u_1^\varepsilon) \rightharpoonup u_1 \text{ weakly in } \left[L^2(\Omega, H^1(Y_1))\right]^3,$$
$$\mathcal{T}_1^\varepsilon(\nabla u_1^\varepsilon) \rightharpoonup \nabla u_1 + \nabla_y \hat{u}_1 \text{ weakly in } \left[L^2(\Omega \times Y_1)\right]^{3\times 3}$$

for some $u_1 \in H^1_{\Gamma_{\mathrm{D}}}(\Omega)$ and $\hat{u}_1 \in \left[L^2(\Omega, H^1_{\mathrm{per},0}(Y_1))\right]^3$. The linearity of $\mathcal{T}_1^\varepsilon$ and the definition of the linear strain tensor $e(u_1^\varepsilon) = \frac{1}{2}\left(\nabla u_1^\varepsilon + (\nabla u_1^\varepsilon)^T\right)$ directly yield

$$\mathcal{T}_1^\varepsilon(e(u_1^\varepsilon)) \rightharpoonup e(u_1) + e_y(\hat{u}_1) \text{ weakly in } \left[L^2(\Omega \times Y_1)\right]^{3\times 3}.$$

□

The next lemma states the connection between the symmetric gradient of a function $\phi$ and of the unfolded function $\mathcal{T}_\kappa^\varepsilon(\phi)$.

**Lemma 5.2.2.** *Let $\kappa \in \{0,1\}$ and $\phi \in \left[H^1(\Omega_\kappa^\varepsilon)\right]^3$. Then, there holds*

$$e_y(\mathcal{T}_\kappa^\varepsilon(\phi)) = \varepsilon \mathcal{T}_\kappa^\varepsilon(e(\phi)).$$

*Proof.* Let $\kappa \in \{0,1\}$ and $\phi \in \left[H^1(\Omega_\kappa^\varepsilon)\right]^3$. Using Proposition 3.2.7 (vii) and the linearity of $\mathcal{T}_\kappa^\varepsilon$, we compute

$$[e_y(\mathcal{T}_\kappa^\varepsilon(\phi))]_{ij} = \frac{1}{2}\left[\partial_{y_i}\mathcal{T}_\kappa^\varepsilon(\phi_j) + \partial_{y_j}\mathcal{T}_\kappa^\varepsilon(\phi_i)\right] = \frac{1}{2}\varepsilon\left[\mathcal{T}_\kappa^\varepsilon(\partial_{x_i}\phi_j) + \mathcal{T}_\kappa^\varepsilon(\partial_{x_j}\phi_i)\right] = \varepsilon\mathcal{T}_\kappa^\varepsilon(e_{ij}(\phi))$$

for $i, j \in \{1,2,3\}$. □

The ansatz of perforated domain can not be applied to $\Omega_0^\varepsilon$ since it consists of a union of disconnected domains. So we have to prove further compactness results.

**Theorem 5.2.3.** *Let $\{u_0^\varepsilon\}$ be a sequence with $u_0^\varepsilon \in \left[H^1(\Omega_0^\varepsilon)\right]^3$ satisfying*

$$\|u_0^\varepsilon\|_{[L^2(\Omega_0^\varepsilon)]^3} + \varepsilon\|\nabla u_0^\varepsilon\|_{[L^2(\Omega_0^\varepsilon)]^{3\times 3}} \le C \quad \text{and} \quad \|e(u_0^\varepsilon)\|_{[L^2(\Omega_0^\varepsilon)]^{3\times 3}} \le C \tag{5.2.2}$$

*for a constant $C$ independent of $\varepsilon$. Then, there exists a subsequence (again denoted by $\{u_0^\varepsilon\}$) and $u_0 \in \left[L^2(\Omega, H^1(Y_0))\right]^3$ such that*

$$\mathcal{T}_0^\varepsilon(u_0^\varepsilon) \rightharpoonup u_0 \textit{ weakly in } \left[L^2(\Omega, H^1(Y_0))\right]^3,$$
$$\varepsilon\mathcal{T}_0^\varepsilon(\nabla u_0^\varepsilon) \rightharpoonup \nabla_y u_0 \textit{ weakly in } \left[L^2(\Omega \times Y_0)\right]^{3\times 3},$$
$$\varepsilon\mathcal{T}_0^\varepsilon(e(u_0^\varepsilon)) \to 0 \textit{ strongly in } \left[L^2(\Omega \times Y_0)\right]^{3\times 3}.$$

*Furthermore, the limit function satisfies $u_0 = B(x)y + c(x)$ for some skew-symmetric matrix $B(x) \in \mathbb{R}^{3\times 3}$ and some appropriate function $c$.*

*Proof.* Let $\{u_0^\varepsilon\}$ be a sequence satisfying $u_0^\varepsilon \in \left[H^1(\Omega_0^\varepsilon)\right]^3$ and (5.2.2). We estimate with Proposition 3.2.7 (iii) and (vii)

$$\begin{aligned}
&\|\mathcal{T}_0^\varepsilon(u_0^\varepsilon)\|_{[L^2(\Omega\times Y_0)]^3} \leq |Y|^{\frac{1}{2}}\|u_0^\varepsilon\|_{\left[L^2(\Omega_0^\varepsilon)\right]^3} \leq C,\\
&\|\nabla_y\mathcal{T}_0^\varepsilon(u_0^\varepsilon)\|_{[L^2(\Omega\times Y_0)]^{3\times 3}} = \varepsilon\|\mathcal{T}_0^\varepsilon(\nabla u_0^\varepsilon)\|_{[L^2(\Omega\times Y_0)]^{3\times 3}} \leq \varepsilon|Y|^{\frac{1}{2}}\|\nabla u_0^\varepsilon\|_{\left[L^2(\Omega_0^\varepsilon)\right]^{3\times 3}} \leq C,
\end{aligned}$$

which shows that $\mathcal{T}_0^\varepsilon(u_0^\varepsilon)$ is bounded in $\left[L^2(\Omega, H^1(Y_0))\right]^3$. Since $\left[L^2(\Omega, H^1(Y_0))\right]^3$ is a Hilbert space, there exists a subsequence (again denoted by $\{u_0^\varepsilon\}$) and $u_0 \in \left[L^2(\Omega, H^1(Y_0)\right]^3$ such that

$$\begin{aligned}
\mathcal{T}_0^\varepsilon(u_0^\varepsilon) &\rightharpoonup u_0 \text{ weakly in } \left[L^2(\Omega\times Y_0)\right]^3,\\
\nabla_y(\mathcal{T}_0^\varepsilon(u_0^\varepsilon)) &\rightharpoonup \nabla_y u_0 \text{ weakly in } \left[L^2(\Omega\times Y_0)\right]^{3\times 3}.
\end{aligned}$$

Because

$$\|e_y(\mathcal{T}_0^\varepsilon(u_0^\varepsilon))\|_{[L^2(\Omega\times Y_0)]^{3\times 3}} = \varepsilon\|\mathcal{T}_0^\varepsilon(e(u_0^\varepsilon))\|_{[L^2(\Omega\times Y_0)]^{3\times 3}} \leq \varepsilon|Y|^{\frac{1}{2}}\|e(u_0^\varepsilon)\|_{\left[L^2(\Omega_0^\varepsilon)\right]^{3\times 3}} \leq \varepsilon C$$

the symmetric gradient of $u_0$ is zero, i.e. $e_y(u_0) = 0$. Thus, $u_0$ only allows rigid-body motions with respect to $y$, i.e.

$$u_0 = B(x)y + c(x)$$

for some skew-symmetric matrix $B(x) \in \mathbb{R}^{3\times 3}$ and some appropriate function $c$. Clearly, $B \in \left[L^2(\Omega)\right]^{3\times 3}$ and $c \in \left[L^2(\Omega)\right]^3$. □

To prove the next theorem, we define an appropriate sequence neglecting rigid-body motions to obtain the weak convergence of the unfolded symmetric gradient of $u_0^\varepsilon$.

**Theorem 5.2.4.** *Let $\{u_0^\varepsilon\}$ be a sequence with $u_0^\varepsilon \in \left[H^1(\Omega_0^\varepsilon)\right]^3$ and*

$$\|u_0^\varepsilon\|_{\left[L^2(\Omega_0^\varepsilon)\right]^3} + \|e(u_0^\varepsilon)\|_{\left[L^2(\Omega_0^\varepsilon)\right]^{3\times 3}} \leq C$$

*for a constant $C$ independent of $\varepsilon$. Then, there exists a function $\hat{u}_0 \in \left[L^2(\Omega, H^1(Y_0))\right]^3$ such that up to a subsequence*

$$\begin{aligned}
Z_0^\varepsilon \coloneqq \frac{1}{\varepsilon}\left[\mathcal{T}_0^\varepsilon(u_0^\varepsilon) - r^\varepsilon\right] &\rightharpoonup \hat{u}_0 \text{ weakly in } \left[L^2(\Omega, H^1(Y_0))\right]^3,\\
\mathcal{T}_0^\varepsilon(e(u_0^\varepsilon)) &\rightharpoonup e_y(\hat{u}_0) \text{ weakly in } \left[L^2(\Omega\times Y_0)\right]^{3\times 3},
\end{aligned}$$

*where $r^\varepsilon(x,y) = R^\varepsilon(x)y + c^\varepsilon(x)$ with skew-symmetric matrix*

$$R^\varepsilon(x) = \mathcal{M}_{Y_0}(\nabla_y\mathcal{T}_0^\varepsilon(u_0^\varepsilon)(x,y) - e_y(\mathcal{T}_0^\varepsilon(u_0^\varepsilon)(x,y)))$$

*and*

$$c^\varepsilon(x) = \mathcal{M}_{Y_0}(\mathcal{T}_0^\varepsilon(u_0^\varepsilon)(x,y) - R^\varepsilon(x)y).$$

*for* $(x,y) \in \Omega \times Y_0$*. Furthermore, there holds* $\mathcal{M}_{Y_0}(\hat{u}_0) = 0$.

*Proof.* Let $\{u_0^\varepsilon\}$ be a bounded sequence as in the assumption. By definition $r^\varepsilon$ is piecewise constant in $x$. If we apply the Poincaré–Wirtinger inequality in $H^1(Y_0)$, Korn's inequality (cf. Theorem 25.4 from [Schweizer, 2018]) and Proposition 3.2.7 (iii) and (vii), we receive

$$\begin{aligned}\|Z_0^\varepsilon\|_{[L^2(\Omega,H^1(Y_0))]^3} &\leq \frac{C}{\varepsilon}\|\nabla_y \mathcal{T}_0^\varepsilon(u_0^\varepsilon) - R^\varepsilon\|_{[L^2(\Omega\times Y_0)]^{3\times 3}} \leq \frac{C}{\varepsilon}\|e_y(\mathcal{T}_0^\varepsilon(u_0^\varepsilon))\|_{[L^2(\Omega\times Y_0)]^{3\times 3}} \\ &= C\|\mathcal{T}_0^\varepsilon(e(u_0^\varepsilon))\|_{[L^2(\Omega\times Y_0)]^{3\times 3}} \leq C|Y|^{\frac{1}{2}}\|e(u_0^\varepsilon)\|_{\left[L^2(\Omega_0^\varepsilon)\right]^{3\times 3}} \leq C\end{aligned}$$

Thus, the sequence $\{Z_0^\varepsilon\}$ is bounded in the Hilbert space $\left[L^2(\Omega, H^1(Y_0))\right]^3$ and, therefore, there exists a function $\hat{u}_0 \in \left[L^2(\Omega, H^1(Y_0))\right]^3$ such that up to a subsequence

$$Z_0^\varepsilon \rightharpoonup \hat{u}_0 \text{ weakly in } \left[L^2(\Omega, H^1(Y_0))\right]^3.$$

Using the skew-symmetry of $R^\varepsilon$,

$$\mathcal{T}_0^\varepsilon(e(u_0^\varepsilon)) = \mathcal{T}_0^\varepsilon(e(u_0^\varepsilon)) - \frac{1}{2\varepsilon}(R^\varepsilon + (R^\varepsilon)^T) = e_y(Z_0^\varepsilon) \rightharpoonup e_y(\hat{u}_0) \text{ weakly in } \left[L^2(\Omega\times Y_0)\right]^{3\times 3}.$$

Since

$$\mathcal{M}_{Y_0}(Z_0^\varepsilon) = \frac{1}{\varepsilon}\left[\mathcal{M}_{Y_0}(\mathcal{T}_0^\varepsilon(u_0^\varepsilon)) - \mathcal{M}_{Y_0}(r^\varepsilon)\right] = 0$$

for all $\varepsilon$, we receive $\mathcal{M}_{Y_0}(\hat{u}_0) = 0$. □

Due to Proposition 3.2.7 (vii), there exists the trace of the unfolding operator with respect to $y$ and we can prove the following result.

**Theorem 5.2.5.** *Let* $u^\varepsilon, \varphi \in \mathcal{W}_\mathrm{d}^\varepsilon(\Omega)$*. Then, there holds*

$$\begin{aligned}\frac{1}{\varepsilon|Y|}\int_\Omega\int_{\Sigma_Y} (\mathcal{T}_1^\varepsilon(u_1^\varepsilon) - \mathcal{T}_0^\varepsilon(u_0^\varepsilon))\cdot(\mathcal{T}_1^\varepsilon(\varphi_1) - \mathcal{T}_0^\varepsilon(\varphi_0))\,\mathrm{d}S(y)\,\mathrm{d}x \\ = \int_{\Sigma^\varepsilon}(u_1^\varepsilon - u_0^\varepsilon)\cdot(\varphi_1 - \varphi_0)\,\mathrm{d}S(x).\end{aligned}$$

*Proof.* Since $(u_1^\varepsilon - u_0^\varepsilon)\cdot(\varphi_1 - \varphi_0) \in L^1(\Sigma^\varepsilon)$, we can use Proposition 3.2.10 to obtain

$$\begin{aligned}&\int_{\Sigma^\varepsilon\cap\Omega^\varepsilon}(u_1^\varepsilon - u_0^\varepsilon)\cdot(\varphi_1 - \varphi_0)\,\mathrm{d}S(x) \\ &\quad= \frac{1}{\varepsilon|Y|}\int_\Omega\int_{\Sigma_Y}\mathcal{T}_\mathrm{b}^\varepsilon\left((u_1^\varepsilon - u_0^\varepsilon)\cdot(\varphi_1 - \varphi_0)\right)\mathrm{d}S(y)\,\mathrm{d}x \\ &\quad= \frac{1}{\varepsilon|Y|}\int_\Omega\int_{\Sigma_Y}(\mathcal{T}_1^\varepsilon(u_1^\varepsilon) - \mathcal{T}_0^\varepsilon(u_0^\varepsilon))\cdot(\mathcal{T}_1^\varepsilon(\varphi_1) - \mathcal{T}_0^\varepsilon(\varphi_0))\,\mathrm{d}S(y)\,\mathrm{d}x.\end{aligned}$$

This shows the result because $\Sigma^\varepsilon \cap \Omega^\varepsilon = \Sigma^\varepsilon$. □

If we choose $\varphi = u^\varepsilon$ in Theorem 5.2.5, we get the following result.

**Corollary 5.2.6.** *Let $u^\varepsilon \in \mathcal{W}_{\mathrm{d}}^\varepsilon(\Omega)$. Then, there holds*

$$\frac{1}{\varepsilon|Y|}\int_\Omega\int_{\Sigma_Y}|\mathcal{T}_1^\varepsilon(u_1^\varepsilon)-\mathcal{T}_0^\varepsilon(u_0^\varepsilon)|^2\mathrm{d}S(y)\,\mathrm{d}x=\int_{\Sigma^\varepsilon}|u_1^\varepsilon-u_0^\varepsilon|^2\mathrm{d}S(x).$$

### 5.2.2. Passage to the limit in the disconnected case

With the compactness results from the subsection before, we derive the homogenized problem, which we rewrite afterwards with the help of cell problems as a microscopic and macroscopic problem.

**Theorem 5.2.7.** *Let $\{u^\varepsilon\}$ be a sequence of weak solutions of the problem* (4.0.3) *with $u^\varepsilon \in \mathcal{W}_{\mathrm{d}}^\varepsilon(\Omega)$ and $\{f^\varepsilon\}$ a bounded sequence in $\left[L^2(\Omega)\right]^3$ such that*

$$\mathcal{T}_\kappa^\varepsilon(f^\varepsilon)\rightharpoonup f|_{\Omega\times Y_\kappa}\ \textit{weakly in}\ \left[L^2(\Omega\times Y_\kappa)\right]^3$$

*for some $f\in\left[L^2(\Omega\times Y)\right]^3$ and $\kappa\in\{0,1\}$. Then,*

$$\begin{cases}\mathcal{T}_1^\varepsilon(u_1^\varepsilon)\rightharpoonup u_1\ \textit{weakly in}\ \left[L^2(\Omega,H^1(Y_1))\right]^3,\\ \mathcal{T}_1^\varepsilon(e(u_1^\varepsilon))\rightharpoonup e(u_1)+e_y(\hat{u}_1)\ \textit{weakly in}\ \left[L^2(\Omega\times Y_1)\right]^{3\times3},\\ \mathcal{T}_0^\varepsilon(u_0^\varepsilon)\rightharpoonup u_0\ \textit{weakly in}\ \left[L^2(\Omega,H^1(Y_0))\right]^3,\\ \mathcal{T}_0^\varepsilon(e(u_0^\varepsilon))\rightharpoonup e_y(\hat{u}_0)\ \textit{weakly in}\ \left[L^2(\Omega\times Y_0)\right]^{3\times3},\end{cases}\tag{5.2.3}$$

*with*

$$u=(u_1,\hat{u}_1,u_0)\in H^1_{\Gamma_{\mathrm{D}}}(\Omega)\times\left[L^2(\Omega,H^1_{\mathrm{per},0}(Y_1))\right]^3\times\left[L^2(\Omega,H^1(Y_0))\right]^3,$$

*where*

$$u_0=B(x)y+c(x)$$

*with $B\in[L^2(\Omega)]^{3\times3}$ skew-symmetric and $c\in[L^2(\Omega)]^3$, and $\hat{u}_0\in\left[L^2(\Omega,H^1(Y_0))\right]^3$ with $\mathcal{M}_{Y_0}(\hat{u}_0)=0$. Furthermore, $u$ is the solution of the problem*

$$\begin{aligned}&\int_\Omega\int_{Y_1}A(y)(e(u_1)+e_y(\hat{u}_1))(e(v_1)+e_y(\hat{v}_1))\,\mathrm{d}y\mathrm{d}x\\&\quad+\int_\Omega\int_{\Sigma_Y}\left(K_{\mathrm{N}}\left[u_1\cdot n-u_0\cdot n\right]n+K_{\mathrm{T}}\sum_{i=1}^2\left[u_1\cdot\tau^i-u_0\cdot\tau^i\right]\tau^i\right)\cdot(v_1-v_0)\,\mathrm{d}S(y)\,\mathrm{d}x\\&=\int_\Omega\int_{Y_1}f\,\mathrm{d}y\cdot v_1\,\mathrm{d}x+\int_\Omega\int_{Y_0}f\cdot v_0\,\mathrm{d}y\mathrm{d}x+\int_{\Gamma_{\mathrm{N}}}g\cdot v_1\,\mathrm{d}S(x)\end{aligned}\tag{5.2.4}$$

*for all $v=(v_1,\hat{v}_1,v_0)\in H^1_{\Gamma_{\mathrm{D}}}(\Omega)\times\left[L^2(\Omega,H^1_{\mathrm{per},0}(Y_1))\right]^3\times\left[L^2(\Omega,H^1(Y_0))\right]^3$ with $v_0=\tilde{B}(x)y+\tilde{c}(x)$, $\tilde{B}\in[L^2(\Omega)]^{3\times3}$ skew-symmetric and $\tilde{c}\in[L^2(\Omega)]^3$.*

*Proof.* Let $\{u^\varepsilon\}$ be a sequence of weak solutions of problem (4.0.3) with $u^\varepsilon\in\mathcal{W}_{\mathrm{d}}^\varepsilon(\Omega)$. From

Theorem 5.1.4 and Theorem 5.1.5, we get the uniform boundedness of the following quantities

$$\|u^\varepsilon\|_{\mathcal{W}_{\mathrm{d}}^\varepsilon(\Omega)}, \quad \|u^\varepsilon\|_{[L^2(\Omega)]^3}, \quad \varepsilon\|\nabla u_0^\varepsilon\|_{\left[L^2(\Omega_0^\varepsilon)\right]^{3\times 3}}, \quad \|\nabla u_1^\varepsilon\|_{\left[L^2(\Omega_1^\varepsilon)\right]^{3\times 3}}.$$

Then, the convergences (5.2.3) follow directly from Theorems 5.2.1, 5.2.3 and 5.2.4. We rewrite the weak formulation of problem (4.0.3) using Theorem 5.2.5 and Proposition 3.2.7 (i) and (ii) to receive the unfolded problem

$$\begin{aligned}
&\int_\Omega\int_{Y_1} \mathcal{T}_1^\varepsilon(A^\varepsilon)\mathcal{T}_1^\varepsilon(e(u_1^\varepsilon))\mathcal{T}_1^\varepsilon(e(\varphi_1))\,\mathrm{d}y\mathrm{d}x + \int_{\Pi^\varepsilon\cap\Omega_1^\varepsilon} A^\varepsilon e(u_1^\varepsilon)e(\varphi_1)\mathrm{d}x \\
&\quad + \int_\Omega\int_{Y_0} \mathcal{T}_0^\varepsilon(A^\varepsilon)\mathcal{T}_0^\varepsilon(e(u_0^\varepsilon))\mathcal{T}_0^\varepsilon(e(\varphi_0))\,\mathrm{d}y\mathrm{d}x \\
&\quad + \int_\Omega\int_{\Sigma_Y} \left(K_{\mathrm{N}}\left[\mathcal{T}_1^\varepsilon(u_1^\varepsilon\cdot n) - \mathcal{T}_0^\varepsilon(u_0^\varepsilon\cdot n)\right]n\right)\cdot\left(\mathcal{T}_1^\varepsilon(\varphi_1) - \mathcal{T}_0^\varepsilon(\varphi_0)\right)\mathrm{d}S(y)\,\mathrm{d}x \\
&\quad + \int_\Omega\int_{\Sigma_Y} \left(K_{\mathrm{T}}\sum_{i=1}^{2}\left[\mathcal{T}_1^\varepsilon(u_1^\varepsilon\cdot\tau^i) - \mathcal{T}_0^\varepsilon(u_0^\varepsilon\cdot\tau^i)\right]\tau^i\right)\cdot\left(\mathcal{T}_1^\varepsilon(\varphi_1) - \mathcal{T}_0^\varepsilon(\varphi_0)\right)\mathrm{d}S(y)\,\mathrm{d}x \\
&= \int_\Omega\int_{Y_1} \mathcal{T}_1^\varepsilon(f^\varepsilon)\cdot\mathcal{T}_1^\varepsilon(\varphi_1)\,\mathrm{d}y\mathrm{d}x + \int_{\Pi^\varepsilon\cap\Omega_1^\varepsilon} f^\varepsilon\cdot\varphi_1\,\mathrm{d}x + \int_\Omega\int_{Y_0} \mathcal{T}_0^\varepsilon(f^\varepsilon)\cdot\mathcal{T}_0^\varepsilon(\varphi_0)\,\mathrm{d}y\mathrm{d}x \\
&\quad + \int_{\Gamma_{\mathrm{N}}} g\cdot\varphi_1\,\mathrm{d}S(x).
\end{aligned} \tag{5.2.5}$$

Let $v_1$ be an element of

$$\mathcal{D}_{\Gamma_{\mathrm{D}}}(\overline{\Omega}) := \{\phi \in [C^\infty(\Omega)]^3 : v \text{ is equal to 0 in a neighbourhood of } \Gamma_{\mathrm{D}}\},$$

$v_0, w_0, w_1 \in [\mathcal{D}(\Omega)]^3$ and

$$\psi_0^\varepsilon(x) := \psi_0\left(\frac{x}{\varepsilon}\right) \quad \text{and} \quad \psi_1^\varepsilon(x) := \psi_1\left(\frac{x}{\varepsilon}\right)$$

with $\psi_0 \in \left[H^1(Y_0)\right]^3$ and $\psi_1 \in \left[H^1_{\mathrm{per},0}(Y_1)\right]^3$ $Y$-periodically extended. We define the test functions as

$$\varphi_0 = \varphi_0^\varepsilon := v_0 + \varepsilon\hat{v}_0^\varepsilon \quad \text{and} \quad \varphi_1 = \varphi_1^\varepsilon := v_1 + \varepsilon\hat{v}_1^\varepsilon$$

with $\hat{v}_0^\varepsilon(x) = \hat{v}_0(x, \frac{x}{\varepsilon})$ and $\hat{v}_1^\varepsilon(x) = \hat{v}_1(x, \frac{x}{\varepsilon})$, where

$$\hat{v}_0(x,y) = ((w_0)_i(x)(\psi_0)_i(y))_{1\le i\le 3} \quad \text{and} \quad \hat{v}_1(x,y) = ((w_1)_i(x)(\psi_1)_i(y))_{1\le i\le 3}.$$

Then, $\varphi^\varepsilon \in \mathcal{W}_{\mathrm{d}}^\varepsilon(\Omega)$,

$$\mathcal{T}_0^\varepsilon(\varphi_0^\varepsilon) \in \left[L^2(\Omega, H^1(Y_0))\right]^3 \text{ and } \mathcal{T}_1^\varepsilon(\varphi_1^\varepsilon) \in \left[L^2(\Omega, H^1(Y_1))\right]^3.$$

Let $\kappa \in \{0,1\}$ and $i \in \{1,2,3\}$. With Proposition 3.2.7 (i) we estimate

$$\begin{aligned}\| (\mathcal{T}_\kappa^\varepsilon(\hat{v}_\kappa^\varepsilon) - \hat{v}_\kappa)_i \|_{L^2(\Omega\times Y_\kappa)} &= \|\mathcal{T}_\kappa^\varepsilon(w_\kappa)_i \mathcal{T}_\kappa^\varepsilon(\psi_\kappa^\varepsilon)_i - (w_\kappa)_i(\psi_\kappa)_i\|_{L^2(\Omega\times Y_\kappa)} \\ &\leq \|\mathcal{T}_\kappa^\varepsilon(w_\kappa)_i \left(\mathcal{T}_\kappa^\varepsilon(\psi_\kappa^\varepsilon)_i - (\psi_\kappa)_i\right)\|_{L^2(\Omega\times Y_\kappa)} + \|\left(\mathcal{T}_\kappa^\varepsilon(w_\kappa)_i - (w_\kappa)_i\right)(\psi_\kappa)_i\|_{L^2(\Omega\times Y_\kappa)} \\ &\leq C\|\mathcal{T}_\kappa^\varepsilon(\psi_\kappa^\varepsilon)_i - (\psi_\kappa)_i\|_{L^2(\Omega\times Y_\kappa)} + C\|\mathcal{T}_\kappa^\varepsilon(w_\kappa)_i - (w_\kappa)_i\|_{L^\infty(\Omega\times Y_\kappa)}.\end{aligned}$$

Because of the compact support of $w_\kappa$, $\mathcal{T}_\kappa^\varepsilon(w_\kappa)$ and $w_\kappa$ vanish in $\Pi^\varepsilon \times Y_\kappa$ for $\varepsilon$ small enough, wherefore

$$\|\mathcal{T}_\kappa^\varepsilon(w_\kappa)_i - (w_\kappa)_i\|_{L^\infty(\Omega\times Y_\kappa)} \leq C\varepsilon \text{diameter}(Y_\kappa)$$

for $\varepsilon$ small enough. From Proposition 3.2.7 (vi), we obtain that

$$\mathcal{T}_\kappa^\varepsilon(\psi_\kappa^\varepsilon) \to \psi_\kappa \quad \text{strongly in } \left[L^2(\Omega\times Y_\kappa)\right]^3.$$

Summing up, the right-hand side of the above inequality converges to zero as $\varepsilon \to 0$, i.e.

$$\mathcal{T}_\kappa^\varepsilon(\hat{v}_\kappa^\varepsilon) \to \hat{v}_\kappa \text{ strongly in } \left[L^2(\Omega\times Y_\kappa)\right]^3.$$

By Proposition 3.2.7 (iv),

$$\mathcal{T}_\kappa^\varepsilon(v_\kappa) \to v_\kappa \quad \text{strongly in } \left[L^2(\Omega\times Y_\kappa)\right]^3.$$

Thus, there holds

$$\mathcal{T}_\kappa^\varepsilon(\varphi_\kappa^\varepsilon) \to v_\kappa \text{ strongly in } \left[L^2(\Omega\times Y_\kappa)\right]^3.$$

Every component of the symmetric gradient of $\varphi_\kappa^\varepsilon$ satisfies

$$\begin{aligned}e_{ij}(\varphi_\kappa^\varepsilon)(x) = e_{ij}(v_\kappa)(x) + \frac{1}{2}\Big[&\varepsilon\partial_{x_i}(w_\kappa)_j(x)(\psi_\kappa)_j\left(\frac{x}{\varepsilon}\right) + (w_\kappa)_j(x)\partial_{y_i}(\psi_\kappa)_j\left(\frac{x}{\varepsilon}\right) \\ &+\varepsilon\partial_{x_j}(w_\kappa)_i(x)(\psi_\kappa)_i\left(\frac{x}{\varepsilon}\right) + (w_\kappa)_i(x)\partial_{y_j}(\psi_\kappa)_i\left(\frac{x}{\varepsilon}\right)\Big],\end{aligned}$$

$i,j \in \{1,2,3\}$. If we apply the periodic unfolding operator to $e_{ij}(\varphi_\kappa^\varepsilon)$ and use the properties from Proposition 3.2.7,

$$\mathcal{T}_\kappa^\varepsilon(e_{ij}(\varphi_\kappa^\varepsilon)) \to e_{ij}(v_\kappa) + \frac{1}{2}\left[(w_\kappa)_j\partial_{y_i}(\psi_\kappa)_j + (w_\kappa)_i\partial_{y_j}(\psi_\kappa)_i\right] = e_{ij}(v_\kappa) + \left(e_y(\hat{v}_\kappa)\right)_{ij}$$

strongly in $L^2(\Omega\times Y_\kappa)$. Thus,

$$\mathcal{T}_\kappa^\varepsilon(e(\varphi_\kappa^\varepsilon)) \to e(v_\kappa) + e_y(\hat{v}_\kappa) \text{ strongly in } \left[L^2(\Omega\times Y_\kappa)\right]^{3\times 3}.$$

The integrals

$$\int_{\Pi^\varepsilon\cap\Omega_1^\varepsilon} A^\varepsilon e(u_1^\varepsilon)e(\varphi_1^\varepsilon)\mathrm{d}x \quad \text{and} \quad \int_{\Pi^\varepsilon\cap\Omega_1^\varepsilon} f^\varepsilon\cdot\varphi_1^\varepsilon\,\mathrm{d}x$$

vanish for $\varepsilon$ small enough because $\hat{v}_1^\varepsilon$ has compact support in $\Omega$ and for the terms with $v_1$ we can estimate the integrals by

$$C\beta|\Pi^\varepsilon \cap \Omega_1^\varepsilon| \|e(u_1^\varepsilon)\|^2_{[L^2(\Omega_1^\varepsilon)]^{3\times 3}} \quad \text{resp.} \quad C|\Pi^\varepsilon \cap \Omega_1^\varepsilon| \|f^\varepsilon\|_{[L^2(\Omega)]^3},$$

which converge to zero as $\varepsilon \to 0$. We estimate the interface term with respect to the normal direction using Hölder's inequality, Corollary 5.2.6 and boundedness of the solution

$$\begin{aligned}
&\int_\Omega \int_{\Sigma_Y} \left(K_\mathrm{N} \left[\mathcal{T}_1^\varepsilon(u_1^\varepsilon \cdot n) - \mathcal{T}_0^\varepsilon(u_0^\varepsilon \cdot n)\right] n\right) \cdot \mathcal{T}_\kappa^\varepsilon(\varepsilon \hat{v}_\kappa^\varepsilon)\, \mathrm{d}S(y)\, \mathrm{d}x \\
&\quad \le K_N \left(\int_\Omega \int_{\Sigma_Y} |\left[\mathcal{T}_1^\varepsilon(u_1^\varepsilon) - \mathcal{T}_0^\varepsilon(u_0^\varepsilon)\right] \cdot n|^2 \mathrm{d}S(y)\, \mathrm{d}x\right)^{\frac{1}{2}} \|\mathcal{T}_\kappa^\varepsilon(\varepsilon \hat{v}_\kappa^\varepsilon)\|_{[L^2(\Omega\times\Sigma_Y)]^3} \\
&\quad \le C\varepsilon^{\frac{1}{2}} \|[u^\varepsilon]_{\Sigma^\varepsilon}\|_{[L^2(\Sigma^\varepsilon)]^3}\, \varepsilon \|(\mathcal{T}_\kappa^\varepsilon((w_\kappa)_i)(\psi_\kappa)_i)_{1\le i\le 3}\|_{[L^2(\Omega\times\Sigma_Y)]^3} \\
&\quad \le C\varepsilon.
\end{aligned}$$

Therefore, this integral converges to zero. The analogous result holds for the terms with $\tau_i$, $i \in \{1, 2\}$, instead of $n$. Due to Proposition 3.2.7 (iv)

$$\begin{aligned}
\mathcal{T}_\kappa^\varepsilon(v_\kappa) &\to v_\kappa \text{ strongly in } \left[L^2(\Omega \times Y_\kappa)\right]^3, \\
\mathcal{T}_\kappa^\varepsilon(\nabla v_\kappa) &\to \nabla v_\kappa \text{ strongly in } \left[L^2(\Omega \times Y_\kappa)\right]^{3\times 3},
\end{aligned}$$

which yields that $\nabla_y(\mathcal{T}_0^\varepsilon(v_\kappa)) \to 0$ strongly in $\left[L^2(\Omega \times Y_\kappa)\right]^{3\times 3}$ and thus

$$\mathcal{T}_\kappa^\varepsilon(v_\kappa) \to v_\kappa \text{ strongly in } \left[L^2(\Omega, H^1(Y_\kappa))\right]^3.$$

So we even get the convergence of the traces of $\mathcal{T}_\kappa^\varepsilon(v_\kappa)$ with respect to $y$. If we plug in the test function in (5.2.5) and pass to the limit, we get

$$\begin{aligned}
&\int_\Omega \int_{Y_1} A(y)(e(u_1) + e_y(\hat{u}_1))(e(v_1) + e_y(\hat{v}_1))\, \mathrm{d}y \mathrm{d}x \\
&\quad + \int_\Omega \int_{Y_0} A(y) e_y(\hat{u}_0)(e(v_0) + e_y(\hat{v}_0))\, \mathrm{d}y \mathrm{d}x \\
&\quad + \int_\Omega \int_{\Sigma_Y} \left(K_\mathrm{N} \left[u_1 \cdot n - u_0 \cdot n\right] n + K_\mathrm{T} \sum_{i=1}^2 \left[u_1 \cdot \tau^i - u_0 \cdot \tau^i\right] \tau^i\right) \cdot (v_1 - v_0)\, \mathrm{d}S(y)\, \mathrm{d}x \\
&= \int_\Omega \int_{Y_1} f\, \mathrm{d}y \cdot v_1\, \mathrm{d}x + \int_\Omega \int_{Y_0} f\, \mathrm{d}y \cdot v_0\, \mathrm{d}x + \int_{\Gamma_\mathrm{N}} g \cdot v_1\, \mathrm{d}S(x).
\end{aligned} \tag{5.2.6}$$

Since $\mathcal{D}(\Omega) \times H^1(Y_0)$ is dense in $L^2(\Omega, H^1(Y_0))$, we can choose as a test function $v_1 = \hat{v}_1 = v_0 = 0$ and $\hat{v}_0 = \hat{u}_0$ to obtain

$$\int_\Omega \int_{Y_0} A(y) e_y(\hat{u}_0) e_y(\hat{u}_0)\, \mathrm{d}y \mathrm{d}x = 0.$$

Using the coercivity of $A$ we get that $e_y(\hat{u}_0) \equiv 0$ and the second integral in (5.2.6) vanishes. Since $\mathcal{D}_{\Gamma_\mathrm{D}}(\bar{\Omega})$ is dense in $H^1_{\Gamma_\mathrm{D}}(\Omega)$ (cf. Theorem 3.1 from [Bernard, 2011]), $\mathcal{D}(\Omega) \times H^1_{\mathrm{per},0}(Y_1)$ is dense in $L^2(\Omega, H^1_{\mathrm{per},0}(Y_1))$ and $\mathcal{D}(\Omega)$ is dense in $L^2(\Omega)$, the homogenized problem (5.2.6) is true for all $v_1 \in H^1_{\Gamma_\mathrm{D}}(\Omega)$, $\hat{v}_1 \in \left[L^2(\Omega, H^1_{\mathrm{per},0}(Y_1))\right]^3$ and $v_0 \in [L^2(\Omega)]^3$. We now consider test functions $\varphi_1^\varepsilon = 0$ and $\varphi_0^\varepsilon(x) = r_0(x, \frac{x}{\varepsilon})$, where $r_0(x,y) = \tilde{B}(x)y$ is $Y$-periodic extended with $\tilde{B} \in [C_c^\infty(\Omega)]^{3\times 3}$ skew-symmetric. Thus, $\varphi_0^\varepsilon \in \left[H^1(\Omega_0^\varepsilon)\right]^3$ and

$$\mathcal{T}_0^\varepsilon(\varphi_0^\varepsilon) \to r_0 \text{ strongly in } \left[L^2(\Omega \times Y_0)\right]^3$$

with the same proof as above. For $\varepsilon$ small enough, using the skew-symmetry of $\tilde{B}$

$$\begin{aligned}
\mathcal{T}_0^\varepsilon(e_{11}(\varphi_0^\varepsilon)) &= \mathcal{T}_0^\varepsilon(\partial_{x_1}\tilde{b}_{12})y_2 + \mathcal{T}_0^\varepsilon(\partial_{x_1}\tilde{b}_{13})y_3,\\
\mathcal{T}_0^\varepsilon(e_{12}(\varphi_0^\varepsilon)) &= \frac{1}{2}\Big(-\mathcal{T}_0^\varepsilon(\partial_{x_1}\tilde{b}_{12})y_1 - \mathcal{T}_0^\varepsilon(\tilde{b}_{12})\frac{1}{\varepsilon} + \mathcal{T}_0^\varepsilon(\partial_{x_1}\tilde{b}_{23})y_3 + \mathcal{T}_0^\varepsilon(\partial_{x_2}\tilde{b}_{12})y_2\\
&\qquad + \mathcal{T}_0^\varepsilon(\tilde{b}_{12})\frac{1}{\varepsilon} + \mathcal{T}_0^\varepsilon(\partial_{x_2}\tilde{b}_{13})y_3\Big)\\
&= \frac{1}{2}\left(-\mathcal{T}_0^\varepsilon(\partial_{x_1}\tilde{b}_{12})y_1 + \mathcal{T}_0^\varepsilon(\partial_{x_1}\tilde{b}_{23})y_3 + \mathcal{T}_0^\varepsilon(\partial_{x_2}\tilde{b}_{12})y_2 + \mathcal{T}_0^\varepsilon(\partial_{x_2}\tilde{b}_{13})y_3\right),\\
\mathcal{T}_0^\varepsilon(e_{13}(\varphi_0^\varepsilon)) &= \frac{1}{2}\left(-\mathcal{T}_0^\varepsilon(\partial_{x_1}\tilde{b}_{13})y_1 - \mathcal{T}_0^\varepsilon(\partial_{x_1}\tilde{b}_{23})y_2 + \mathcal{T}_0^\varepsilon(\partial_{x_3}\tilde{b}_{12})y_2 + \mathcal{T}_0^\varepsilon(\partial_{x_3}\tilde{b}_{13})y_3\right),\\
\mathcal{T}_0^\varepsilon(e_{22}(\varphi_0^\varepsilon)) &= -\mathcal{T}_0^\varepsilon(\partial_{x_2}\tilde{b}_{12})y_1 + \mathcal{T}_0^\varepsilon(\partial_{x_2}\tilde{b}_{23})y_3,\\
\mathcal{T}_0^\varepsilon(e_{23}(\varphi_0^\varepsilon)) &= \frac{1}{2}\left(-\mathcal{T}_0^\varepsilon(\partial_{x_2}\tilde{b}_{13})y_1 - \mathcal{T}_0^\varepsilon(\partial_{x_2}\tilde{b}_{23})y_2 - \mathcal{T}_0^\varepsilon(\partial_{x_3}\tilde{b}_{12})y_1 + \mathcal{T}_0^\varepsilon(\partial_{x_3}\tilde{b}_{23})y_3\right),\\
\mathcal{T}_0^\varepsilon(e_{33}(\varphi_0^\varepsilon)) &= -\mathcal{T}_0^\varepsilon(\partial_{x_3}\tilde{b}_{13})y_1 - \mathcal{T}_0^\varepsilon(\partial_{x_3}\tilde{b}_{23})y_2.
\end{aligned}$$

With the same arguments as before

$$\mathcal{T}_0^\varepsilon(e(\varphi_0^\varepsilon)) \to e(r_0) \text{ strongly in } \left[L^2(\Omega \times Y_0)\right]^{3\times 3}.$$

Passing to the limit in (5.2.5) and use the fact that $e_y(\hat{u}_0) \equiv 0$, we obtain

$$\begin{aligned}
-\int_\Omega \int_{\Sigma_Y} &\left(K_\mathrm{N}\left[u_1 \cdot n - u_0 \cdot n\right] n + K_\mathrm{T} \sum_{i=1}^{2} \left[u_1 \cdot \tau^i - u_0 \cdot \tau^i\right] \tau^i\right) \cdot r_0 \,\mathrm{d}S(y)\,\mathrm{d}x\\
&= \int_\Omega \int_{Y_0} f \cdot r_0 \,\mathrm{d}y\,\mathrm{d}x.
\end{aligned}$$

Since $[C_c^\infty(\Omega)]^{3\times 3}$ is dense in $\left[L^2(\Omega)\right]^{3\times 3}$, the equality is true for all $r_0(x,y) = \tilde{B}(x)y$ with $\tilde{B} \in \left[L^2(\Omega)\right]^{3\times 3}$. Summing up, we get the desired result. Using the uniqueness of the solution, which we prove in Theorem 5.2.9, all the convergences above hold true for the whole sequence. $\square$

In the next theorem, we rewrite the homogenized problem from Theorem 5.2.7 with the help of auxiliary cell problems as a macroscopic problem. The steps of the proof of the next theorem follow [Höpker, 2016].

**Theorem 5.2.8.** *Let the sequence $\{u^\varepsilon\}$ be as in Theorem 5.2.7. We can reformulate the homogenized problem (5.2.4) as follows: Find $u_1 \in H^1_{\Gamma_\mathrm{D}}(\Omega)$, $u_0 \in \left[L^2(\Omega, H^1(Y_0))\right]^3$, where $u_0 = B(x)y + c(x)$ with $B \in [L^2(\Omega)]^{3\times 3}$ skew-symmetric and $c \in [L^2(\Omega)]^3$, such that*

$$\begin{aligned} &\int_\Omega A_1^{\mathrm{hom}} e(u_1) e(v_1)\,\mathrm{d}x \\ &\quad + \int_\Omega \int_{\Sigma_Y} \left( K_\mathrm{N}(u_1\cdot n - u_0 \cdot n)n + K_\mathrm{T} \sum_{i=1}^{2} (u_1 \cdot \tau^i - u_0 \cdot \tau^i)\tau^i \right) \cdot (v_1 - v_0)\,\mathrm{d}S(y)\mathrm{d}x \\ &= \int_\Omega \int_{Y_1} f\,\mathrm{d}y \cdot v_1\,\mathrm{d}x + \int_\Omega \int_{Y_0} f \cdot v_0\,\mathrm{d}y\mathrm{d}x + \int_{\Gamma_\mathrm{N}} g\cdot v_1\,\mathrm{d}S(x) \end{aligned} \tag{5.2.7}$$

*where*

$$(A_1^{\mathrm{hom}})_{ijkh} = \int_{Y_1} a_{ijkh}(y) - \sum_{l,m=1}^{3} a_{ijlm} \left(e_y(\chi_1^{kh})\right)_{lm} \mathrm{d}y$$

*and $\chi_1^{lm} \in \left[H^1_{\mathrm{per},0}(Y_1)\right]^3$, $l,m \in \{1,2,3\}$, is the unique solution of*

$$\begin{cases} \left( -\displaystyle\sum_{j=1}^{3} \frac{\partial}{\partial y_j} \left[ \left(Ae_y(\chi_1^{lm})\right)_{ij} - a_{ijlm} \right] \right)_{1\le i\le 3} = 0 & \text{in } Y_1, \\ \left( -\displaystyle\sum_{j=1}^{3} \left[ \left(Ae_y(\chi_1^{lm})\right)_{ij} - a_{ijlm} \right] n_j \right)_{1\le i\le 3} = 0 & \text{on } \Sigma_Y. \end{cases} \tag{5.2.8}$$

*Proof.* Choosing $v_0 = v_1 = 0$ in (5.2.4), we get for all $\hat{v}_1 \in \left[L^2(\Omega, H^1_{\mathrm{per},0}(Y_1))\right]^3$

$$\int_\Omega \int_{Y_1} A\left(e(u_1) + e_y(\hat{u}_1)\right) e_y(\hat{v}_1)\,\mathrm{d}y\mathrm{d}x = 0.$$

Thus,

$$\int_{Y_1} A(y)\left(e(u_1)(x) + e_y(\hat{u}_1)(x,y)\right) e_y(\hat{v}_1)(y)\,\mathrm{d}y = 0$$

for a.e. $x \in \Omega$ and $\hat{v}_1 \in \left[H^1_{\mathrm{per},0}(Y_1)\right]^3$. Due to Korn's inequality for periodic functions with mean value zero, there exists a unique solution $\varphi \in \left[H^1_{\mathrm{per},0}(Y_1)\right]^3$ of

$$\int_{Y_1} A(y)\left(e(u_1)(x) + e_y(\varphi)(y)\right) e_y(\hat{v}_1)(y)\,\mathrm{d}y = 0 \tag{5.2.9}$$

for all $\hat{v}_1 \in \left[H^1_{\mathrm{per},0}(Y)\right]^3$ and a.e. $x \in \Omega$. We consider the cell problems: Find the weak solution $\chi_1^{lm} \in \left[H^1_{\mathrm{per},0}(Y_1)\right]^3$, $l,m \in \{1,2,3\}$, of the problem (5.2.8). Using the symmetry of $A$, the weak formulation is

$$\int_{Y_1} Ae_y(\chi_1^{lm})e_y(\hat{v}_1) - (Ae_y(\hat{v}_1))_{lm}\mathrm{d}y = 0$$

for all $\hat{v}_1 \in \left[H^1_{\mathrm{per},0}(Y_1)\right]^3$. Using Korn's inequality for periodic functions with zero mean value, it follows from Lax–Milgram's theorem that there exists a unique solution $\chi_1^{lm}$ of the cell problem. If we plug in $-\sum_{l,m=1}^{3} e_{lm}(u_1)(x)\chi_1^{lm}(y)$ for $\varphi$ in (5.2.9), we receive

$$\begin{aligned}\int_{Y_1} A(y)\left[e(u_1)+e_y(\varphi)\right]e_y(\hat{v}_1)\mathrm{d}y &= \int_{Y_1}\left[A(y)e(u_1)-\sum_{l,m=1}^{3} e_{lm}(u_1)A(y)e_y(\chi_1^{lm})\right]e_y(\hat{v}_1)\mathrm{d}y\\ &= \int_{Y_1} A(y)e(u_1)e_y(\hat{v}_1)-A(y)e(u_1)e_y(\hat{v}_1)\mathrm{d}y=0.\end{aligned}$$

Hence $-\sum_{l,m=1}^{3} e_{lm}(u_1)\chi_1^{lm}$ is the unique solution of the problem (5.2.9) and thus,

$$\hat{u}_1(x,y) = -\sum_{l,m=1}^{3} e_{lm}(u_1)(x)\chi_1^{lm}(y).$$

Using this equality in (5.2.4), we receive

$$\begin{aligned}&\int_\Omega\int_{Y_1} A(y)\left[e(u_1)+e_y(\hat{u}_1)\right]\left[e(v_1)+e_y(\hat{v}_1)\right]\mathrm{d}y\mathrm{d}x\\ &\quad=\int_\Omega\int_{Y_1} A(y)\left[e(u_1)-\sum_{l,m=1}^{3} e_{lm}(u_1)e_y(\chi_1^{lm})\right]e(v_1)\mathrm{d}y\mathrm{d}x\\ &\quad=\int_\Omega\sum_{i,j,k,h=1}^{3}\left(\int_{Y_1} a_{ijkh}(y)\mathrm{d}y\right)e_{kh}(u_1)e_{ij}(v_1)\\ &\qquad\quad-\sum_{i,j,l,m=1}^{3}\left(\int_{Y_1}\sum_{k,h=1}^{3} a_{ijkh}(y)\left(e_y(\chi_1^{lm})\right)_{kh}\mathrm{d}y\right)e_{lm}(u_1)e_{ij}(v_1)\mathrm{d}x.\end{aligned}$$

So the homogenized tensor $A_1^{\mathrm{hom}}$ on $\Omega\times Y_1$ is given by

$$(A_1^{\mathrm{hom}})_{ijkh} = \int_{Y_1} a_{ijkh}(y)-\sum_{l,m=1}^{3} a_{ijlm}\left(e_y(\chi_1^{kh})\right)_{lm}\mathrm{d}y,$$

and the homogenized problem (5.2.4) can be reformulated as the macroscopic problem (5.2.7). □

Similar as in Theorem II.1.1 from [Oleinik et al., 1992], it can be proven that there exist constants $\alpha^{\mathrm{hom}},\beta^{\mathrm{hom}}\in\mathbb{R}$ with $0<\alpha^{\mathrm{hom}}<\beta^{\mathrm{hom}}$ such that $A_1^{\mathrm{hom}}\in M(\alpha^{\mathrm{hom}},\beta^{\mathrm{hom}},\Omega)$. We use this property to prove the uniqueness of the solution of (5.2.7).

**Theorem 5.2.9.** *The solutions $u_1\in H^1_{\Gamma_{\mathrm{D}}}(\Omega)$ and $u_0\in\left[L^2(\Omega,H^1(Y_0))\right]^3$, where $u_0=B(x)y+c(x)$ with $B\in[L^2(\Omega)]^{3\times3}$ skew-symmetric and $c\in[L^2(\Omega)]^3$, of the macroscopic problem* (5.2.7) *are unique.*

*Proof.* Let $u=(u_1,u_0), w=(w_1,w_0)\in H^1_{\Gamma_{\mathrm{D}}}(\Omega)\times\left[L^2(\Omega,H^1(Y_0))\right]^3$ be two solutions of the problem (5.2.7), where $u_0 = B(x)y + c(x)$ and $w_0 = \tilde{B}(x)y + \tilde{c}(x)$ with $B, \tilde{B} \in [L^2(\Omega)]^{3\times 3}$ skew-symmetric and $c, \tilde{c} \in [L^2(\Omega)]^3$. Then there holds

$$
\begin{aligned}
0 = \int_\Omega A_1^{\mathrm{hom}} e(u_1 - w_1) e(v_1)\,\mathrm{d}x + \int_\Omega \int_{\Sigma_Y} \Big( & K_{\mathrm{N}}[u_1 - u_0 - w_1 + w_0] \cdot nn \\
& + K_{\mathrm{T}} \sum_{i=1}^{2} [u_1 - u_0 - w_1 + w_0] \cdot \tau^i \tau^i \Big) \cdot (v_1 - v_0)\,\mathrm{d}S(y)\mathrm{d}x.
\end{aligned}
$$

If we choose as test function the difference of both solutions, i.e. $v_1 := u_1 - w_1$ and $v_0 := u_0 - w_0$, we can estimate, using that $A_1^{\mathrm{hom}}$ is coercive and Korn's inequality for functions with zero on part of the boundary,

$$
\begin{aligned}
0 &= \int_\Omega A_1^{\mathrm{hom}} e(u_1 - w_1) e(u_1 - w_1)\,\mathrm{d}x + \int_\Omega \int_{\Sigma_Y} \Big( K_{\mathrm{N}}\,[u_1 - u_0 - w_1 + w_0] \cdot nn \\
&\quad + K_{\mathrm{T}} \sum_{i=1}^{2} [u_1 - u_0 - w_1 + w_0] \cdot \tau^i \tau^i \Big) \cdot (u_1 - w_1 - u_0 + w_0)\,\mathrm{d}S(y)\,\mathrm{d}x \\
&\geq \alpha^{\mathrm{hom}} C \|u_1 - w_1\|^2_{[H^1(\Omega)]^3} + \min\{K_{\mathrm{N}}, K_{\mathrm{T}}\} \|u_1 - u_0 - w_1 + w_0\|^2_{[L^2(\Omega\times\Sigma_Y)]^3}.
\end{aligned}
$$

Thus, $u_1 = w_1$, which yields together with $\|u_1 - u_0 - w_1 + w_0\|^2_{[L^2(\Omega\times\Sigma_Y)]^3} = 0$ that $u_0 = w_0$, i.e.

$$
(B(x) - \tilde{B}(x))y + (c(x) - \tilde{c}(x)) = 0
$$

for a.e. $(x, y) \in \Omega \times \Sigma_Y$. Varying along $y \in \Sigma_Y$ for fixed $x \in \Omega$, we obtain $B = \tilde{B}$ and $c = \tilde{c}$. □

A similar result holds for the case of a more general elasticity tensor $A^\varepsilon$.

**Remark 5.2.10.** *We consider the general elasticity tensor $A^\varepsilon \in M(\alpha, \beta, \Omega)$ instead of $A^\varepsilon = A(\frac{\cdot}{\varepsilon}) \in M(\alpha, \beta, \Omega)$. If we additional assume that $\mathcal{T}^\varepsilon(A^\varepsilon) \to C$ a.e. in $\Omega \times Y$, then, $C \in M(\alpha, \beta, \Omega \times Y)$, which can be shown as in the beginning of the proof of Theorem 9.1.2, and the homogenized problem is of the form: Find the unique weak solutions $u_1 \in H^1_{\Gamma_{\mathrm{D}}}(\Omega)$, $u_0 \in \left[L^2(\Omega, H^1(Y_0))\right]^3$, where $u_0 = B(x)y + c(x)$ with $B \in [L^2(\Omega)]^{3\times 3}$ skew-symmetric and $c \in [L^2(\Omega)]^3$, such that*

$$
\begin{aligned}
&\int_\Omega C_1^{\mathrm{hom}} e(u_1) e(v_1)\,\mathrm{d}x \\
&\quad + \int_\Omega \int_{\Sigma_Y} \left( K_{\mathrm{N}}(u_1 - u_0) \cdot nn + K_{\mathrm{T}} \sum_{i=1}^{2} (u_1 - u_0) \cdot \tau^i \tau^i \right) \cdot (v_1 - v_0)\,\mathrm{d}S(y)\mathrm{d}x \\
&= \int_\Omega \int_{Y_1} f\,\mathrm{d}y \cdot v_1\,\mathrm{d}x + \int_\Omega \int_{Y_0} f \cdot v_0\,\mathrm{d}y\,\mathrm{d}x + \int_{\Gamma_{\mathrm{N}}} g \cdot v_1\,\mathrm{d}S(x)
\end{aligned}
$$

*with*

$$
(C_1^{\mathrm{hom}})_{ijkh}(x) = \int_{Y_1} c_{ijkh}(x, y) - \sum_{l,m=1}^{3} c_{ijlm}(x, y) \left(e_y(\chi_1^{kh})\right)_{lm} (x, y)\mathrm{d}y
$$

*and* $\chi_1^{lm} \in \left[L^\infty(\Omega, H^1_{\text{per},0}(Y_1))\right]^3$, $l, m \in \{1,2,3\}$, *is the unique solution of*

$$\int_{Y_1} C(x,y)e_y(\chi_1^{lm})(x,y)e_y(\hat{v}_1)(y) - (C(x,y)e_y(\hat{v}_1)(y))_{lm}\mathrm{d}y = 0$$

*for all* $\hat{v}_1 \in \left[H^1_{\text{per},0}(Y_1)\right]^3$ *and a.e.* $x \in \Omega$.

If we assume that the normal and tangential stiffness $K_\mathrm{N}$ and $K_\mathrm{T}$ are the same, the problem (5.2.7) can be simplified.

**Remark 5.2.11.** *In the case where* $K_\mathrm{N} = K_\mathrm{T}$ *the solution* $u_0$ *can be expressed by* $u_1$ *and the volume force* $f$*, more precisely*

$$\int_{\Sigma_Y} u_0 \,\mathrm{d}S(y) = |\Sigma_Y| u_1 + |Y_0| \mathcal{M}_{Y_0}(f),$$

*i.e. the larger the Hausdorff measure of the interface* $\Sigma_Y$ *is, the value of* $u_0$ *is greater. Then, problem* (5.2.7) *is of the form*

$$\int_\Omega A_1^{\text{hom}} e(u_1)e(v_1)\,\mathrm{d}x = \int_\Omega |Y_1| \mathcal{M}_{Y_1}(f)\cdot v_1 \,\mathrm{d}x + \int_{\Gamma_\mathrm{N}} g\cdot v_1 \,\mathrm{d}S(x),$$

*which is the same weak form as in the case, where* $Y_0$ *describes holes.*

# 6. Globally connected case

In the globally connected case, we additionally require to the assumptions in chapter 4 that $\Omega$ can be represented as a union of axis-parallel cuboids with corner coordinates in $\mathbb{Q}^3$, which is not a restriction since every Lipschitz-domain can be approximated by a domain of cuboidal structure. Furthermore, we assume that the boundary $\partial Y_0 \cap \partial Y$ resp. $\partial Y_1 \cap \partial Y$ on opposite faces of the periodicity cell $Y$ is the same and $\Omega_0^\varepsilon$ and $\Omega_1^\varepsilon$ are two globally connected sets. We allow the scaling factor $\varepsilon$ to be only such that $\varepsilon^{-1}\Omega$ can be represented as a finite union of axis-parallel cuboids with corner coordinates in $\mathbb{Z}^3$. Thus, we can ensure that the domain $\Omega$ can completely be filled up with scaled reference cells (see Figure 6.1).

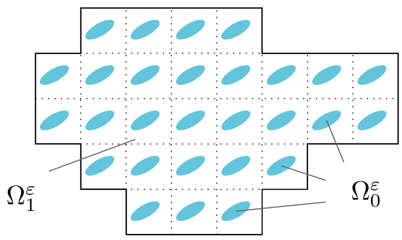


Figure 6.1.: domain $\Omega$

This condition can be relaxed, but we impose it in what follows to avoid well-known technicalities induced by otherwise non-matching boundaries of $\Omega$ and its $\varepsilon$-periodic approximation. Due to the choice of $\varepsilon$, there holds $\Omega^\varepsilon = \Omega$.
We prove the existence, uniqueness and uniform boundedness of the solution of (4.0.3) in section 6.1 and derive the homogenized problem after proving a compactness result in section 6.2. The results in this chapter have been published in the *Journal of Mathematical Analysis and Applications* [Lochner and Peter, 2020].

## 6.1. Existence result in the connected case

Since $\Gamma_{\mathrm{D}} \cap \partial\Omega_0^\varepsilon \neq \emptyset$ and $\Gamma_{\mathrm{D}} \cap \partial\Omega_1^\varepsilon \neq \emptyset$, we can choose in the connected case the solution space as the set of piecewise $H^1$-functions with zero value on part of the boundary, i.e.

$$\mathcal{W}_{\mathrm{c}}^\varepsilon(\Omega) = \{u \in \left[L^2(\Omega)\right]^3 : \; u_0 \in H^1_{\Gamma_{\mathrm{D}} \cap \partial\Omega_0^\varepsilon}(\Omega_0^\varepsilon), u_1 \in H^1_{\Gamma_{\mathrm{D}} \cap \partial\Omega_1^\varepsilon}(\Omega_1^\varepsilon)\},$$

endowed with the same norm as in the disconnected case, i.e.

$$\|u\|^2_{\mathcal{W}^\varepsilon_{\mathrm{c}}(\Omega)} := \|e(u_0)\|^2_{[L^2(\Omega^\varepsilon_0)]^{3\times 3}} + \|e(u_1)\|^2_{[L^2(\Omega^\varepsilon_1)]^{3\times 3}} + \varepsilon\|[u]_{\Sigma^\varepsilon}\|^2_{[L^2(\Sigma^\varepsilon)]^3}.$$

By the trace theorem and Korn's inequality from Theorem 4.4 in [Höpker, 2016]

$$c\left(\|u_0\|^2_{[H^1(\Omega^\varepsilon_0)]^3} + \|u_1\|^2_{[H^1(\Omega^\varepsilon_1)]^3}\right) \le \|u\|^2_{\mathcal{W}^\varepsilon_{\mathrm{d}}(\Omega)} \le C\left(\|u_0\|^2_{[H^1(\Omega^\varepsilon_0)]^3} + \|u_1\|^2_{[H^1(\Omega^\varepsilon_1)]^3}\right)$$

with constants $c, C > 0$ independent of $\varepsilon$. The equivalence of the norms implies that the space $\mathcal{W}^\varepsilon_{\mathrm{c}}(\Omega)$ defines a Hilbert space.

**Theorem 6.1.1.** *Let $f^\varepsilon \in \left[L^2(\Omega)\right]^3$ and $g \in \left[L^2(\Gamma_{\mathrm{N}})\right]^3$. Then, there exists a unique weak solution $u^\varepsilon \in \mathcal{W}^\varepsilon_{\mathrm{c}}(\Omega)$ of* (4.0.3) *for all $\varphi \in \mathcal{W}^\varepsilon_{\mathrm{c}}(\Omega)$ and all admissible* $0 < \varepsilon \le 1$.

*Proof.* We prove this statement via Lax–Milgram theorem similar as in Theorem 5.1.3. Let $\varepsilon > 0$. Since the left-hand side of (4.0.3) is of the same form as in the disconnected case and the norms on $\mathcal{W}^\varepsilon_{\mathrm{d}}(\Omega)$ and $\mathcal{W}^\varepsilon_{\mathrm{c}}(\Omega)$ are identical, we obtain with the same estimates as in the proof of Theorem 5.1.3 that mapping $a : \mathcal{W}^\varepsilon_{\mathrm{c}}(\Omega) \times \mathcal{W}^\varepsilon_{\mathrm{c}}(\Omega) \to \mathbb{R}$, defined as in (5.1.2), is a coercive continuous bilinear form. We denote the right-hand side of (4.0.3) as the mapping $F\colon \mathcal{W}^\varepsilon_{\mathrm{c}}(\Omega) \to \mathbb{R}$,

$$F(v) := \int_\Omega f^\varepsilon \cdot v \,\mathrm{d}x + \int_{\Gamma_{\mathrm{N}}\cap\partial\Omega^\varepsilon_0} g\cdot v_0 \,\mathrm{d}S(x) + \int_{\Gamma_{\mathrm{N}}\cap\partial\Omega^\varepsilon_1} g\cdot v_1 \,\mathrm{d}S(x), \tag{6.1.1}$$

whereby we merged the first two terms of (4.0.3). This is feasible, since $\Sigma^\varepsilon$ is a Lebesgue null set with respect to the three dimensional Lebesgue measure and $\mathcal{W}^\varepsilon_{\mathrm{c}}(\Omega) \subset [L^2(\Omega)]^3$. We have to prove that $F$ is linear and continuous, i.e. $F \in \mathcal{W}^\varepsilon_{\mathrm{c}}(\Omega)'$. The linearity follows directly. Let $v \in \mathcal{W}^\varepsilon_{\mathrm{c}}(\Omega)$. Using Hölder's inequality and Korn's inequality from Theorem 4.4 in [Höpker, 2016] applied on the two connected domains $\Omega^\varepsilon_0$ resp. $\Omega^\varepsilon_1$, we obtain

$$\begin{aligned}\left|\int_\Omega f^\varepsilon \cdot v \,\mathrm{d}x\right| &\le C\|f^\varepsilon\|_{[L^2(\Omega)]^3}\left(\|e(v_0)\|^2_{[L^2(\Omega^\varepsilon_0)]^{3\times 3}} + \|e(v_1)\|^2_{[L^2(\Omega^\varepsilon_1)]^{3\times 3}}\right)^{\frac{1}{2}}\\ &\le C\|f^\varepsilon\|_{[L^2(\Omega)]^3}\|v\|_{\mathcal{W}^\varepsilon_{\mathrm{c}}(\Omega)}.\end{aligned}$$

Let $\kappa \in \{0, 1\}$. By Hölder's inequality we receive

$$\left|\int_{\Gamma_{\mathrm{N}}\cap\partial\Omega^\varepsilon_\kappa} g\cdot v_\kappa \,\mathrm{d}S(x)\right| \le \|g\|_{[L^2(\Gamma_{\mathrm{N}})]^3}\|v_\kappa\|_{[L^2(\Gamma_{\mathrm{N}}\cap\partial\Omega^\varepsilon_\kappa)]^3}.$$

In order to estimate the term $\|v_\kappa\|_{[L^2(\Gamma_{\mathrm{N}}\cap\partial\Omega^\varepsilon_\kappa)]^3}$ by $\|v\|_{\mathcal{W}^\varepsilon_{\mathrm{c}}(\Omega)}$, we consider the extension operator $L^\varepsilon_\kappa : \Omega^\varepsilon_\kappa \to \Omega$ defined in Theorem 3.4 in [Höpker, 2016]. We apply the trace operator on $L^\varepsilon_\kappa v_\kappa$, use the estimates of the trace operator and Korn's inequality from Theorem 4.4. in [Höpker, 2016] to estimate

$$\|v_\kappa\|_{[L^2(\Gamma_\mathrm{N}\cap\partial\Omega^\varepsilon_\kappa)]^3} \leq C\|L^\varepsilon_\kappa(v_\kappa)\|_{[H^1(\Omega)]^3} \leq C\|v_\kappa\|_{[H^1(\Omega^\varepsilon_\kappa)]^3} \leq C\|e(v_\kappa)\|_{[L^2(\Omega^\varepsilon_\kappa]^{3\times 3}}. \tag{6.1.2}$$

Summarising the previous estimates, we receive the continuity of the mapping $F$

$$|F(v)| \leq C\left(\|f^\varepsilon\|_{[L^2(\Omega)]^3} + \|g\|_{[L^2(\Gamma_\mathrm{N})]^3}\right)\|v\|_{\mathcal{W}^\varepsilon_\mathrm{c}(\Omega)}.$$

So all assumptions of the Lax–Milgram theorem are fulfilled, which guarantees the existence and uniqueness of the solution. □

Under the additional assumption of uniform boundedness of $f^\varepsilon$ the weak solutions $u^\varepsilon$ are $\varepsilon$-independent bounded.

**Theorem 6.1.2.** *Let $u^\varepsilon \in \mathcal{W}^\varepsilon_\mathrm{c}(\Omega)$ be the weak solutions of* (4.0.3) *and $f^\varepsilon$ bounded independent of $\varepsilon$ in $\left[L^2(\Omega)\right]^3$. Then, there exists a constant $C$ independent of $\varepsilon$ such that*

$$\|u^\varepsilon\|_{\mathcal{W}^\varepsilon_\mathrm{c}(\Omega)} \leq C$$

*for all admissible* $0 < \varepsilon \leq 1$.

*Proof.* Since all the constants in the estimates of the proof of Theorem 6.1.1 are independent of $\varepsilon$, we get by choosing $v = u^\varepsilon$

$$\min\{\alpha, K_\mathrm{N}, K_\mathrm{T}\}\|u^\varepsilon\|^2_{\mathcal{W}^\varepsilon_\mathrm{c}(\Omega)} \leq a(u^\varepsilon, u^\varepsilon) = F(u^\varepsilon) \leq C\left(\|f^\varepsilon\|_{[L^2(\Omega)]^3} + \|g\|_{[L^2(\Gamma_\mathrm{N})]^3}\right)\|u^\varepsilon\|_{\mathcal{W}^\varepsilon_\mathrm{c}(\Omega)},$$

which shows the uniform boundedness of $\|u^\varepsilon\|_{\mathcal{W}^\varepsilon_\mathrm{c}(\Omega)}$. □

## 6.2. Homogenization in the connected case

First, we prove in subsection 6.2.1 a compactness result, for which we need the extension operator from [Höpker, 2016] to treat the boundary conditions on the exterior boundary $\Gamma_\mathrm{D}$. We apply this result afterwards in subsection 6.2.2 to derive the homogenized problem in the connected case.

### 6.2.1. Compactness result in the connected case

The following theorem provides us with information about the weak convergence of the unfolded extended solutions $\mathcal{T}^\varepsilon(\tilde{u}^\varepsilon_\kappa)$, $\kappa \in \{0,1\}$. We define the Hilbert space

$$\left[L^2(\Omega, H^1_{\mathrm{per},0}(Y))\right]^3 := \{u \in \left[L^2(\Omega, H^1_\mathrm{per}(Y))\right]^3 : \mathcal{M}_Y(u) = 0\}.$$

**Theorem 6.2.1.** *For $\kappa \in \{0,1\}$, let $\{u^\varepsilon_\kappa\}$ be a sequence with $u^\varepsilon_\kappa \in H^1_{\Gamma_\mathrm{D}\cap\partial\Omega^\varepsilon_\kappa}(\Omega^\varepsilon_\kappa)$ and*

$$\|e(u^\varepsilon_\kappa)\|_{[L^2(\Omega^\varepsilon_\kappa)]^{3\times 3}} \leq C \tag{6.2.1}$$

*for a constant $C$ independent of $\varepsilon$. Then, there exists, up to a subsequence, $u_\kappa \in H^1_{\Gamma_\mathrm{D}}(\Omega)$ and $\bar{u}_\kappa \in \left[L^2(\Omega, H^1_{\mathrm{per},0}(Y))\right]^3$, such that*

$$\mathcal{T}^\varepsilon(\tilde{u}^\varepsilon_\kappa) \rightharpoonup u_\kappa \textit{ weakly in } \left[L^2(\Omega, H^1(Y))\right]^3,$$
$$\mathcal{T}^\varepsilon\left(e(\tilde{u}^\varepsilon_\kappa)\right) \rightharpoonup e(u_\kappa) + e_y(\bar{u}_\kappa) \textit{ weakly in } \left[L^2(\Omega \times Y)\right]^{3\times 3},$$

*where $\tilde{\cdot}$ is the extension to $\Omega$ defined in Theorem 3.5 in [Höpker, 2016] and $\mathcal{T}^\varepsilon$ is the unfolding operator from Definition 3.2.1.*

*Proof.* Let $\{u^\varepsilon_\kappa\}$ be a bounded sequence as in (6.2.1) with $u^\varepsilon_\kappa \in H^1_{\Gamma_\mathrm{D}\cap\partial\Omega^\varepsilon_\kappa}(\Omega^\varepsilon_\kappa)$. The extension operator $L_\varepsilon \colon H^1_{\Gamma_\mathrm{D}\cap\partial\Omega^\varepsilon_\kappa}(\Omega^\varepsilon_\kappa) \to H^1_{\Gamma_\mathrm{D}}(\Omega)$, $u \mapsto \tilde{u}$ from Theorem 3.5 in [Höpker, 2016] satisfies

$$\|\tilde{u}^\varepsilon_\kappa\|_{[H^1(\Omega)]^3} \leq C\|u^\varepsilon_\kappa\|_{[H^1(\Omega^\varepsilon_\kappa)]^3}.$$

By Korn's inequality from Theorem 4.4 in [Höpker, 2016], we estimate

$$\|u^\varepsilon_\kappa\|_{[L^2(\Omega^\varepsilon_\kappa)]^3} + \|\nabla u^\varepsilon_\kappa\|_{[L^2(\Omega^\varepsilon_\kappa)]^{3\times 3}} \leq C\|e(u^\varepsilon_\kappa)\|_{[L^2(\Omega^\varepsilon_\kappa]^{3\times 3}} \leq C. \tag{6.2.2}$$

Thus, the sequence of extended functions $\{\tilde{u}^\varepsilon_\kappa\}$ is bounded in $H^1_{\Gamma_\mathrm{D}}(\Omega)$. Since $H^1_{\Gamma_\mathrm{D}}(\Omega)$ is a Hilbert space, there exists, up to a subsequence, $u_\kappa \in H^1_{\Gamma_\mathrm{D}}(\Omega)$ with

$$\tilde{u}^\varepsilon_\kappa \rightharpoonup u_\kappa \text{ weakly in } H^1_{\Gamma_\mathrm{D}}(\Omega).$$

So we can apply Proposition 3.2.2 (ix) to get

$$\mathcal{T}^\varepsilon(\tilde{u}^\varepsilon_\kappa) \rightharpoonup u_\kappa \text{ weakly in } \left[L^2(\Omega, H^1(Y))\right]^3$$

and

$$\mathcal{T}^\varepsilon\left(\nabla\tilde{u}^\varepsilon_\kappa\right) \rightharpoonup \nabla u_\kappa + \nabla_y \bar{u}_\kappa \text{ weakly in } \left[L^2(\Omega \times Y)\right]^{3\times 3}.$$

for some $\bar{u}_\kappa \in \left[L^2(\Omega, H^1_{\mathrm{per},0}(Y))\right]^3$. The linearity of $\mathcal{T}^\varepsilon$ and the definition of the linear strain tensor $e(\tilde{u}^\varepsilon_\kappa) = \frac{1}{2}\left(\nabla\tilde{u}^\varepsilon_\kappa + (\nabla\tilde{u}^\varepsilon_\kappa)^T\right)$ directly yield

$$\mathcal{T}^\varepsilon\left(e(\tilde{u}^\varepsilon_\kappa)\right) \rightharpoonup e(u_\kappa) + e_y(\bar{u}_\kappa) \text{ weakly in } \left[L^2(\Omega \times Y)\right]^{3\times 3}.$$

□

If we restrict the unfolded sequence to $\Omega \times Y_\kappa$, $\kappa \in \{0, 1\}$, we obtain the following lemma.

**Theorem 6.2.2.** *For $\kappa \in \{0, 1\}$, let $\{u^\varepsilon_\kappa\}$ be a sequence with $u^\varepsilon_\kappa \in H^1_{\Gamma_\mathrm{D}\cap\partial\Omega^\varepsilon_\kappa}(\Omega^\varepsilon_\kappa)$ and*

$$\|e(u^\varepsilon_\kappa)\|_{[L^2(\Omega^\varepsilon_\kappa)]^{3\times 3}} \leq C$$

*for a constant $C$ independent of $\varepsilon$. Then, there exists, up to a subsequence, $u_\kappa \in H^1_{\Gamma_\mathrm{D}}(\Omega)$ and*

$\hat{u}_\kappa \in \left[L^2(\Omega, H^1_{\mathrm{per},0}(Y_\kappa))\right]^3$, *such that*

$$\begin{aligned} \mathcal{T}^\varepsilon_\kappa(u^\varepsilon_\kappa) &\rightharpoonup u_\kappa \text{ weakly in } \left[L^2(\Omega, H^1(Y_\kappa))\right]^3, \\ \mathcal{T}^\varepsilon_\kappa\left(e(u^\varepsilon_\kappa)\right) &\rightharpoonup e(u_\kappa) + e_y(\hat{u}_\kappa) \text{ weakly in } \left[L^2(\Omega \times Y_\kappa)\right]^{3\times 3}. \end{aligned} \tag{6.2.3}$$

*Proof.* Let $\kappa \in \{0,1\}$ and $\{u^\varepsilon_\kappa\}$ be as in the assumptions. By restricting the unfolded extended sequence to $\Omega \times Y_\kappa$ in (the proof of) Theorem 6.2.1 we obtain

$$\begin{aligned} \mathcal{T}^\varepsilon(\tilde{u}^\varepsilon_\kappa)|_{\Omega\times Y_\kappa} &\rightharpoonup u_\kappa \text{ weakly in } \left[L^2(\Omega, H^1(Y_\kappa))\right]^3, \\ \mathcal{T}^\varepsilon(\nabla\tilde{u}^\varepsilon_\kappa)|_{\Omega\times Y_\kappa} &\rightharpoonup \nabla u_\kappa + \nabla_y \bar{u}_\kappa \text{ weakly in } \left[L^2(\Omega \times Y_\kappa)\right]^{3\times 3}. \end{aligned} \tag{6.2.4}$$

for $u_\kappa \in H^1_{\Gamma_\mathrm{D}}(\Omega)$ and $\bar{u}_\kappa \in \left[L^2(\Omega, H^1_{\mathrm{per},0}(Y))\right]^3$. There holds

$$\mathcal{T}^\varepsilon(\tilde{u}^\varepsilon_\kappa)|_{\Omega\times Y_\kappa} = \mathcal{T}^\varepsilon_\kappa(\tilde{u}^\varepsilon_\kappa) = \mathcal{T}^\varepsilon_\kappa(u^\varepsilon_\kappa) \quad \text{and} \quad \mathcal{T}^\varepsilon(\nabla\tilde{u}^\varepsilon_\kappa)|_{\Omega\times Y_\kappa} = \mathcal{T}^\varepsilon_\kappa(\nabla\tilde{u}^\varepsilon_\kappa) = \mathcal{T}^\varepsilon_\kappa(\nabla u^\varepsilon_\kappa),$$

wherefore the first convergence in (6.2.3) follows directly. To prove the second convergence result, we define

$$Z^\varepsilon := \frac{1}{\varepsilon}(\mathcal{T}^\varepsilon_\kappa(u^\varepsilon_\kappa) - \mathcal{M}_{Y_\kappa}(u^\varepsilon_\kappa)) - \nabla u_\kappa(y - \mathcal{M}_{Y_\kappa}(y)).$$

Since by construction, $\mathcal{M}_{Y_\kappa}(Z^\varepsilon) = 0$, we can apply Poincaré–Wirtinger inequality to estimate

$$\begin{aligned} \|Z^\varepsilon\|_{[L^2(\Omega\times Y_\kappa)]^3} &\le C\|\nabla_y Z^\varepsilon\|_{[L^2(\Omega\times Y_\kappa)]^{3\times3}} \\ &\le C\left(\|\mathcal{T}^\varepsilon_\kappa(\nabla u^\varepsilon_\kappa)\|_{[L^2(\Omega\times Y_\kappa)]^{3\times3}} + \|\nabla u_\kappa\|_{[L^2(\Omega\times Y_\kappa)]^{3\times3}}\right), \end{aligned}$$

which is uniformly bounded using (6.2.2). Consequently, there exists $\hat{u}_\kappa$ such that

$$Z^\varepsilon \rightharpoonup \hat{u}_\kappa \text{ weakly in } \left[L^2(\Omega, H^1(Y_\kappa))\right]^3.$$

Since $\mathcal{M}_{Y_\kappa}(Z^\varepsilon) = 0$, one has $\mathcal{M}_{Y_\kappa}(\hat{u}_\kappa) = 0$. From (6.2.4)

$$\nabla_y Z^\varepsilon = \mathcal{T}^\varepsilon_\kappa(\nabla u^\varepsilon_\kappa) - \nabla u_\kappa \rightharpoonup \nabla_y \bar{u}_\kappa \text{ weakly in } \left[L^2(\Omega \times Y_\kappa)\right]^{3\times3},$$

which implies that $\nabla_y \hat{u}_\kappa = \nabla_y \bar{u}_\kappa$. So $\hat{u}_\kappa$ is also $Y$-periodic, which ends up the proof. □

### 6.2.2. Passage to the limit in the connected case

We apply the compactness result from the last subsection to derive the homogenized problem, which we subsequently write as a microscopic and macroscopic problem.

**Theorem 6.2.3.** *Let $\{u^\varepsilon\}$ be a sequence of weak solutions of the problem (4.0.3) with $u^\varepsilon \in \mathcal{W}^\varepsilon_\mathrm{c}(\Omega)$ and $\{f^\varepsilon\}$ a bounded sequence in $\left[L^2(\Omega)\right]^3$ such that*

$$\mathcal{T}^\varepsilon_\kappa(f^\varepsilon) \rightharpoonup f|_{\Omega\times Y_\kappa} \text{ weakly in } \left[L^2(\Omega \times Y_\kappa)\right]^3$$

*for some* $f \in \left[L^2(\Omega \times Y)\right]^3$ *and* $\kappa \in \{0,1\}$. *Then, there exist* $u_\kappa \in H^1_{\Gamma_\mathrm{D}}(\Omega)$ *and* $\hat{u}_\kappa \in \left[L^2(\Omega, H^1_{\mathrm{per},0}(Y_\kappa))\right]^3$, $\kappa \in \{0,1\}$, *such that*

$$\begin{cases} \mathcal{T}^\varepsilon(u^\varepsilon_\kappa) \rightharpoonup u_\kappa(x) \text{ weakly in } \left[L^2(\Omega, H^1(Y_\kappa))\right]^3, \\ \mathcal{T}^\varepsilon\left(e(u^\varepsilon_\kappa)\right) \rightharpoonup e(u_\kappa) + e_y(\hat{u}_\kappa) \text{ weakly in } \left[L^2(\Omega \times Y_\kappa)\right]^{3\times 3}. \end{cases} \tag{6.2.5}$$

*Furthermore,*

$$u = (u_1, \hat{u}_1, u_0, \hat{u}_0) \in H^1_{\Gamma_\mathrm{D}}(\Omega) \times \left[L^2(\Omega, H^1_{\mathrm{per},0}(Y_1))\right]^3 \times H^1_{\Gamma_\mathrm{D}}(\Omega) \times \left[L^2(\Omega, H^1_{\mathrm{per},0}(Y_0))\right]^3$$

*is the solution of the problem*

$$\begin{aligned} &\int_\Omega \int_{Y_1} A\left(e(u_1) + e_y(\hat{u}_1)\right)\left(e(v_1) + e_y(\hat{v}_1)\right) \mathrm{d}y\mathrm{d}x \\ &\quad + \int_\Omega \int_{Y_0} A\left(e(u_0) + e_y(\hat{u}_0)\right)\left(e(v_0) + e_y(\hat{v}_0)\right) \mathrm{d}y\mathrm{d}x \\ &\quad + \int_\Omega \int_{\Sigma_Y} \left(K_\mathrm{N}(u_1 \cdot n - u_0 \cdot n)n + K_\mathrm{T} \sum_{i=1}^2 (u_1 \cdot \tau^i - u_0 \cdot \tau^i)\tau^i\right) \cdot (v_1 - v_0)\, \mathrm{d}S(y)\mathrm{d}x \\ &= \int_\Omega \int_{Y_1} f\, \mathrm{d}y \cdot v_1\, \mathrm{d}x + \int_\Omega \int_{Y_0} f\, \mathrm{d}y \cdot v_0\, \mathrm{d}x + \int_{\Gamma_\mathrm{N}} g \cdot h_1 v_1\, \mathrm{d}S(x) + \int_{\Gamma_\mathrm{N}} g \cdot h_0 v_0\, \mathrm{d}S(x) \end{aligned} \tag{6.2.6}$$

*for all* $v = (v_1, \hat{v}_1, v_0, \hat{v}_0) \in H^1_{\Gamma_\mathrm{D}}(\Omega) \times \left[L^2(\Omega, H^1_{\mathrm{per},0}(Y_1))\right]^3 \times H^1_{\Gamma_\mathrm{D}}(\Omega) \times \left[L^2(\Omega, H^1_{\mathrm{per},0}(Y_0))\right]^3$.

*Proof.* Let $\{u^\varepsilon\}$ be the sequence of weak solutions of problem (4.0.3). It is by Theorem 6.1.1 and Theorem 6.1.2 unique and uniformly bounded, i.e.

$$\|u^\varepsilon\|_{\mathcal{W}^\varepsilon_c(\Omega)} \leq C.$$

Then, the convergences (6.2.5) follow directly from Theorem 6.2.2. We rewrite problem (4.0.3) using Proposition 3.2.10, Proposition 3.2.7 (i) and (ii) and the fact that $\Pi^\varepsilon = \emptyset$

$$\begin{aligned} &\int_\Omega \int_{Y_1} \mathcal{T}^\varepsilon_1(A^\varepsilon)\mathcal{T}^\varepsilon_1(e(u^\varepsilon_1))\mathcal{T}^\varepsilon_1(e(\varphi_1))\, \mathrm{d}y\mathrm{d}x + \int_\Omega \int_{Y_0} \mathcal{T}^\varepsilon_0(A^\varepsilon)\mathcal{T}^\varepsilon_0(e(u^\varepsilon_0))\mathcal{T}^\varepsilon_0(e(\varphi_0))\, \mathrm{d}y\mathrm{d}x \\ &\quad + \int_\Omega \int_{\Sigma_Y} \left(K_\mathrm{N}\left[\mathcal{T}^\varepsilon_1(u^\varepsilon_1 \cdot n) - \mathcal{T}^\varepsilon_0(u^\varepsilon_0 \cdot n)\right] n\right) \cdot \left(\mathcal{T}^\varepsilon_1(\varphi_1) - \mathcal{T}^\varepsilon_0(\varphi_0)\right) \mathrm{d}S(y)\, \mathrm{d}x \\ &\quad + \int_\Omega \int_{\Sigma_Y} \left(K_\mathrm{T} \sum_{i=1}^2 \left[\mathcal{T}^\varepsilon_1(u^\varepsilon_1 \cdot \tau^i) - \mathcal{T}^\varepsilon_0(u^\varepsilon_0 \cdot \tau^i)\right]\tau^i\right) \cdot \left(\mathcal{T}^\varepsilon_1(\varphi_1) - \mathcal{T}^\varepsilon_0(\varphi_0)\right) \mathrm{d}S(y)\, \mathrm{d}x \\ &= \int_\Omega \int_{Y_1} \mathcal{T}^\varepsilon_1(f^\varepsilon) \cdot \mathcal{T}^\varepsilon_1(\varphi_1)\, \mathrm{d}y\mathrm{d}x + \int_\Omega \int_{Y_0} \mathcal{T}^\varepsilon_0(f^\varepsilon) \cdot \mathcal{T}^\varepsilon_0(\varphi_0)\, \mathrm{d}y\mathrm{d}x + \int_{\Gamma_\mathrm{N} \cap \partial\Omega^\varepsilon_1} g \cdot \varphi_1\, \mathrm{d}S(x) \\ &\quad + \int_{\Gamma_\mathrm{N} \cap \partial\Omega^\varepsilon_0} g \cdot \varphi_0\, \mathrm{d}S(x). \end{aligned} \tag{6.2.7}$$

Let $v_0, v_1$ be elements of

$$\mathcal{D}_{\Gamma_\mathrm{D}}(\overline{\Omega}) \coloneqq \{\phi \in [C^\infty(\Omega)]^3 : v \text{ is equal to 0 in a neighbourhood of } \Gamma_\mathrm{D}\},$$

$w_0, w_1 \in [\mathcal{D}(\Omega)]^3$ and

$$\psi_0^\varepsilon(x) \coloneqq \psi_0\left(\frac{x}{\varepsilon}\right) \quad \text{and} \quad \psi_1^\varepsilon(x) \coloneqq \psi_1\left(\frac{x}{\varepsilon}\right)$$

with $\psi_0 \in \left[H^1_{\mathrm{per},0}(Y_0)\right]^3$ and $\psi_1 \in \left[H^1_{\mathrm{per},0}(Y_1)\right]^3$ $Y$-periodically extended. We define the test functions as

$$\varphi_0 = \varphi_0^\varepsilon \coloneqq v_0 + \varepsilon \hat{v}_0^\varepsilon \quad \text{and} \quad \varphi_1 = \varphi_1^\varepsilon \coloneqq v_1 + \varepsilon \hat{v}_0^\varepsilon$$

with $\hat{v}_0^\varepsilon(x) = \hat{v}_0(x, \frac{x}{\varepsilon})$ and $\hat{v}_1^\varepsilon(x) = \hat{v}_1(x, \frac{x}{\varepsilon})$, where

$$\hat{v}_0(x,y) = ((w_0)_i(x)(\psi_0)_i(y))_{1\le i\le 3} \quad \text{and} \quad \hat{v}_1(x,y) = ((w_1)_i(x)(\psi_1)_i(y))_{1\le i\le 3}\,.$$

Since $\varphi_0^\varepsilon|_{\Gamma_\mathrm{D}} = 0$ and $\varphi_1^\varepsilon|_{\Gamma_\mathrm{D}} = 0$, they satisfy $\varphi^\varepsilon \in \mathcal{W}_\mathrm{c}^\varepsilon(\Omega)$,

$$\mathcal{T}_0^\varepsilon(\varphi_0^\varepsilon) \in \left[L^2(\Omega, H^1(Y_0))\right]^3 \text{ and } \mathcal{T}_1^\varepsilon(\varphi_1^\varepsilon) \in \left[L^2(\Omega, H^1(Y_1))\right]^3.$$

With the same arguments as in the proof of Theorem 5.2.7

$$\begin{aligned}\mathcal{T}_\kappa^\varepsilon(\varphi_\kappa^\varepsilon) &\to v_\kappa \text{ strongly in } \left[L^2(\Omega\times Y_\kappa)\right]^3,\\ \mathcal{T}_\kappa^\varepsilon(e(\varphi_\kappa^\varepsilon)) &\to e(v_\kappa) + e_y(\hat{v}_\kappa) \text{ strongly in } \left[L^2(\Omega\times Y_\kappa)\right]^{3\times 3}\end{aligned}$$

for $\kappa \in \{0,1\}$ and

$$\int_\Omega \int_{\Sigma_Y} \left(K_\mathrm{N}\left[\mathcal{T}_1^\varepsilon(u_1^\varepsilon\cdot n) - \mathcal{T}_0^\varepsilon(u_0^\varepsilon\cdot n)\right] n\right)\cdot \mathcal{T}_\kappa^\varepsilon(\varepsilon\hat{v}_\kappa^\varepsilon)\,\mathrm{d}S(y)\,\mathrm{d}x \to 0$$

resp. for $\tau_i$, $i \in \{1,2\}$ instead of $n$. Since $w_\kappa$, $\kappa \in \{0,1\}$, has compact support in $\Omega$ we can rewrite the boundary integral

$$\int_{\Gamma_\mathrm{N}\cap\partial\Omega_\kappa^\varepsilon} g\cdot\varphi_\kappa^\varepsilon\,\mathrm{d}S(x) = \int_{\Gamma_\mathrm{N}} g\cdot v_\kappa \chi_{\Gamma_\mathrm{N}\cap\partial\Omega_\kappa^\varepsilon}\,\mathrm{d}S(x)$$

for $\varepsilon$ small enough. We follow the approach of section 7.6.2 in [Höpker, 2016] to prove the convergence. Since $\chi_{\Gamma_\mathrm{N}\cap\partial\Omega_\kappa^\varepsilon}$ is bounded in $L^2(\Gamma_\mathrm{N})$, there exists a subsequence (again denoted by $\varepsilon$) and a function $h_\kappa \in L^2(\Gamma_\mathrm{N})$ with

$$\chi_{\Gamma_\mathrm{N}\cap\partial\Omega_\kappa^\varepsilon} \rightharpoonup h_\kappa \text{ weakly in } L^2(\Gamma_\mathrm{N}).$$

The set $\{u \in L^2(\Gamma_\mathrm{N}) : 0 \le u \le 1\}$ is closed and convex and therefore weakly closed, so $0 \le h_\kappa \le 1$. The function $g\cdot v_\kappa$ is independent of $\varepsilon$, so it converges strongly to $g\cdot v_\kappa$ in $L^2(\Gamma_\mathrm{N})$.

Summing up,

$$\int_{\Gamma_\mathrm{N}\cap\partial\Omega_\kappa^\varepsilon} g\cdot\varphi_\kappa^\varepsilon \,\mathrm{d}S(x) \to \int_{\Gamma_\mathrm{N}} g\cdot h_\kappa v_\kappa \,\mathrm{d}S(x).$$

Under additional assumption on the boundary $\partial\Omega$ and the exterior boundaries of $Y_0$ and $Y_1$ the limit functions $h_0$ and $h_1$ can be formulated explicitly (cf. Theorem 7.17 in [Höpker, 2016]). If we plug in the test functions in (6.2.7) and pass to the limit, we obtain (6.2.6). Since $\mathcal{D}_{\Gamma_\mathrm{D}}(\overline{\Omega})$ is dense in $H^1_{\Gamma_\mathrm{D}}(\Omega)$ (cf. Theorem 3.1 from [Bernard, 2011]) and $\mathcal{D}(\Omega)\times H^1_{\mathrm{per},0}(Y_\kappa)$ is dense in $L^2(\Omega, H^1_{\mathrm{per},0}(Y_\kappa))$, the homogenized problem is true for all $v_0, v_1 \in H^1_{\Gamma_\mathrm{D}}(\Omega)$, $\hat{v}_0 \in \left[L^2(\Omega, H^1_{\mathrm{per},0}(Y_0))\right]^3$ and $\hat{v}_1 \in \left[L^2(\Omega, H^1_{\mathrm{per},0}(Y_1))\right]^3$. Due to the uniqueness of the solution, which we prove below, all the convergences above hold true for the whole sequence. □

In the next theorem we want to split the problem (6.2.6) into a micro- and macroscopic problem.

**Theorem 6.2.4.** *Let $\{u^\varepsilon\}$ be as in Theorem 6.2.3. We can reformulate the homogenized problem* (6.2.6)*: Find $u_0, u_1 \in H^1_{\Gamma_\mathrm{D}}(\Omega)$ with*

$$\begin{aligned}
&\int_\Omega A_1^{\mathrm{hom}} e(u_1)e(v_1)\,\mathrm{d}x + \int_\Omega A_0^{\mathrm{hom}} e(u_0)e(v_0)\,\mathrm{d}x \\
&\quad + \int_\Omega\int_{\Sigma_Y} \left( K_\mathrm{N}(u_1\cdot n - u_0\cdot n)n + K_\mathrm{T}\sum_{i=1}^{2}(u_1\cdot\tau^i - u_0\cdot\tau^i)\tau^i \right)\cdot(v_1 - v_0)\,\mathrm{d}S(y)\mathrm{d}x \\
&\quad = \int_\Omega\int_{Y_1} f\,\mathrm{d}y\cdot v_1\,\mathrm{d}x + \int_\Omega\int_{Y_0} f\,\mathrm{d}y\cdot v_0\,\mathrm{d}x + \int_{\Gamma_\mathrm{N}} g\cdot h_1 v_1\,\mathrm{d}S(x) + \int_{\Gamma_\mathrm{N}} g\cdot h_0 v_0\,\mathrm{d}S(x)
\end{aligned} \tag{6.2.8}$$

*whereby*

$$(A_\kappa^{\mathrm{hom}})_{ijkh} = \int_{Y_\kappa} a_{ijkh}(y) - \sum_{l,m=1}^{3} a_{ijlm}\left(e_y(\chi_\kappa^{kh})\right)_{lm}\mathrm{d}y$$

*and $\chi_\kappa^{lm} \in \left[H^1_{\mathrm{per},0}(Y_\kappa)\right]^3$, $l, m \in \{1,2,3\}$, is the unique solution of*

$$\begin{cases}
\left(-\sum_{j=1}^{3}\frac{\partial}{\partial y_j}\left[\left(Ae_y(\chi_\kappa^{lm})\right)_{ij} - a_{ijlm}\right]\right)_{1\le i\le 3} = 0 & \text{in } Y_\kappa, \\
\left(-\sum_{j=1}^{3}\left[\left(Ae_y(\chi_\kappa^{lm})\right)_{ij} - a_{ijlm}\right]n_j\right)_{1\le i\le 3} = 0 & \text{on } \Sigma_Y
\end{cases}$$

*for $\kappa \in \{0,1\}$.*

*Proof.* This result can be shown in the same way as in the proof of Theorem 5.2.8. □

Similar as in Theorem II.1.1 from [Oleinik et al., 1992], it can be proven that there exist $\alpha_\kappa^{\mathrm{hom}}$, $\beta_\kappa^{\mathrm{hom}} \in \mathbb{R}$ with $0 < \alpha_\kappa^{\mathrm{hom}} < \beta_\kappa^{\mathrm{hom}}$, $\kappa \in \{0,1\}$, such that $A_\kappa^{\mathrm{hom}} \in M(\alpha_\kappa^{\mathrm{hom}}, \beta_\kappa^{\mathrm{hom}}, \Omega)$. With this fact we are able to prove the uniqueness of the solution of (6.2.8).

**Theorem 6.2.5.** *There exist unique solutions $u_0, u_1 \in H^1_{\Gamma_\mathrm{D}}(\Omega)$ of the macroscopic problem (6.2.8).*

*Proof.* Let $u = (u_1, u_0), w = (w_1, w_0) \in H^1_{\Gamma_\mathrm{D}}(\Omega) \times H^1_{\Gamma_\mathrm{D}}(\Omega)$ be two solutions of the problem (6.2.8). If we choose as test function the difference of both solutions, i.e. $v_1 \coloneqq u_1 - w_1$ and $v_0 \coloneqq u_0 - w_0$, we can estimate, using the coercivity of $A^{\mathrm{hom}}_\kappa$, $\kappa \in \{0,1\}$, and Korn's inequality for functions with zero on part of the boundary,

$$\begin{aligned}
0 &= \int_\Omega A_0^{\mathrm{hom}} e(u_0 - w_0) e(u_0 - w_0)\,\mathrm{d}x + \int_\Omega A_1^{\mathrm{hom}} e(u_1 - w_1) e(u_1 - w_1)\,\mathrm{d}x \\
&\quad + \int_\Omega \int_{\Sigma_Y} \Big( K_\mathrm{N} [u_1 - u_0 - w_1 + w_0] \cdot nn + K_\mathrm{T} \sum_{i=1}^{2} [u_1 - u_0 - w_1 + w_0] \cdot \tau^i \tau^i \Big) \\
&\qquad\qquad \cdot (u_1 - w_1 - u_0 + w_0)\,\mathrm{d}S(y)\mathrm{d}x \\
&\geq \alpha_0^{\mathrm{hom}} C \|u_0 - w_0\|^2_{[H^1(\Omega)]^3} + \alpha_1^{\mathrm{hom}} C \|u_1 - w_1\|^2_{[H^1(\Omega)]^3} \\
&\quad + \min\{K_\mathrm{N}, K_\mathrm{T}\} |\Sigma_Y| \|u_1 - u_0 - w_1 + w_0\|^2_{[L^2(\Omega)]^3}.
\end{aligned}$$

Thus, $u_0 = w_0$ and $u_1 = w_1$. $\square$

A similar result holds for the case of a more general elasticity tensor $A^\varepsilon$.

**Remark 6.2.6.** *As in the disconnected case (cf. Remark 5.2.10), we assume that $A^\varepsilon \in M(\alpha, \beta, \Omega)$ and $\mathcal{T}^\varepsilon(A^\varepsilon) \to C$ a.e. in $\Omega \times Y$. The homogenized problem is of the same form as (6.2.8) but the homogenized tensors $A^{\mathrm{hom}}_\kappa$, $\kappa \in \{0,1\}$, satisfy*

$$(A_\kappa^{\mathrm{hom}})_{ijkh}(x) = \int_{Y_\kappa} c_{ijkh}(x,y) - \sum_{l,m=1}^{3} c_{ijlm}(x,y) \left( e_y(\chi_\kappa^{kh}) \right)_{lm} (x,y)\mathrm{d}y$$

*and $\chi_\kappa^{lm} \in \left[L^\infty(\Omega, H^1_{\mathrm{per},0}(Y_\kappa))\right]^3$, $l, m \in \{1,2,3\}$, is the unique solution of*

$$\int_{Y_\kappa} C(x,y) e_y(\chi_\kappa^{lm})(x,y) e_y(\hat{v}_\kappa)(y) - (C(x,y) e_y(\hat{v}_\kappa)(y))_{lm} \mathrm{d}y = 0$$

*for all $\hat{v}_\kappa \in \left[H^1_{\mathrm{per},0}(Y_\kappa)\right]^3$ and for a.e. $x \in \Omega$.*

# 7. Unidirectionally connected case

In this case, we are interested in domains, where one component represents fibers of tubular structure, which are connected in one direction. We require additionally to the assumptions in chapter 4 that the slices of $\Omega$ parallel to one of the $x_i$–$x_j$-planes with $i \neq j \in \{1, 2, 3\}$ are equal. Without restriction, we choose $i = 1$ and $j = 2$ (see Figure 7.1). So, in the notation of chapter 4,

$$\Omega = \Omega' \times (0, L_3)$$

with $0 < L_3$, where $\Omega' \subset \mathbb{R}^2$ is an open bounded connected Lipschitz-domain. Let $\Omega' \times \{0\} \subset \Gamma_{\mathrm{D}}$ and $g = 0$ on $\Omega' \times \{L_3\} \subset \Gamma_{\mathrm{N}}$, where $g$ is the boundary load in (4.0.1).

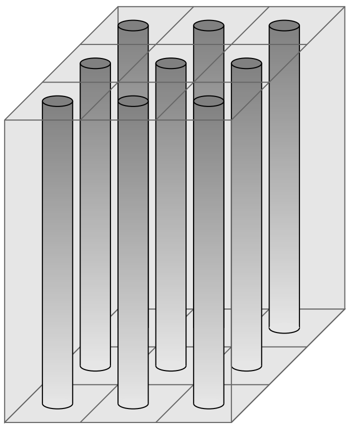

Figure 7.1.: domain $\Omega$

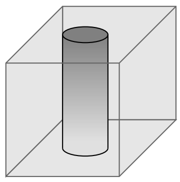

Figure 7.2.: reference cell $Y$

The reference cell $Y = \text{interior}\,(Y_0 \cup \Sigma_Y \cup Y_1)$ is of the form

$$Y_0 = Y_0' \times (0,1), \quad Y_1 = Y_1' \times (0,1), \quad \Sigma_Y = \Sigma_{Y'} \times (0,1),$$

where $Y_0', Y_1'$ are two open subsets of $Y' = (0,1)^2 \subset \mathbb{R}^2$ such that $Y_0'$ is a relatively compact subset of $Y'$, $Y_0' \cap Y_1' = \emptyset$, $\Sigma_{Y'} := \overline{Y_0'} \cap \overline{Y_1'}$ Lipschitz-continuous and $Y' = \text{interior}\,(Y_0' \cup \Sigma_{Y'} \cup Y_1')$ (see Figure 7.2). The stiffness tensor $A^\varepsilon$ only depends on $\Omega'$, i.e.

$$A^\varepsilon(x') = (a^\varepsilon_{ijkh}(x'))_{1\leq i,j,k,h\leq 3} := \left(a_{ijkh}\left(\frac{x'}{\varepsilon}\right)\right)_{1\leq i,j,k,h\leq 3} = A\left(\frac{x'}{\varepsilon}\right)$$

for $x' \in \Omega'$, where $A(\,\cdot\,) = (a_{ijkh}(\,\cdot\,))_{1\leq i,j,k,h\leq 3} \in M(\alpha, \beta, Y')$ and all components $a_{ijkh}$ are

$Y'$-periodic for all $i, j, k, h \in \{1, 2, 3\}$. Thus, $A^\varepsilon \in M(\alpha, \beta, \Omega')$.
We prove the existence, uniqueness and uniform boundedness of the solution of (4.0.3) in section 7.1 and derive the homogenized problem after proving some compactness results in section 7.2.

## 7.1. Existence result in the unidirectionally connected case

Due to the assumption on the domain, $\Gamma_{\mathrm{D}} \cap \partial\Omega_0^\varepsilon \neq \emptyset$ and $\Gamma_{\mathrm{D}} \cap \partial\Omega_1^\varepsilon \neq \emptyset$. So we define the solution space as the set of piecewise $H^1$-functions with zero value on part of the boundary, i.e.

$$\mathcal{W}_{\mathrm{m}}^\varepsilon(\Omega) = \{u \in \left[L^2(\Omega)\right]^3 : \ u_0 \in H^1_{\Gamma_{\mathrm{D}} \cap \partial\Omega_0^\varepsilon}(\Omega_0^\varepsilon), u_1 \in H^1_{\Gamma_{\mathrm{D}} \cap \partial\Omega_1^\varepsilon}(\Omega_1^\varepsilon)\},$$

endowed with the norm

$$\|u\|^2_{\mathcal{W}_{\mathrm{m}}^\varepsilon(\Omega)} \coloneqq \|e(u_0)\|^2_{\left[L^2(\Omega_0^\varepsilon)\right]^{3\times 3}} + \|e(u_1)\|^2_{\left[L^2(\Omega_1^\varepsilon)\right]^{3\times 3}} + \varepsilon \| \left[u\right]_{\Sigma^\varepsilon} \|^2_{[L^2(\Sigma^\varepsilon)]^3}.$$

**Theorem 7.1.1.** $\left(\mathcal{W}_{\mathrm{m}}^\varepsilon(\Omega), \|\cdot\|_{\mathcal{W}_{\mathrm{m}}^\varepsilon(\Omega)}\right)$ *defines a Hilbert space.*

*Proof.* It can be shown as in Theorem 5.1.1 that $\|\cdot\|_{\mathcal{W}_{\mathrm{m}}^\varepsilon(\Omega)}$ defines a norm. The trace operator and Korn's inequality for functions with zero trace on part of the boundary (cf. Theorem 2.2.5) yield

$$\begin{aligned} c(\varepsilon)\|u\|^2_{\mathcal{W}_{\mathrm{m}}^\varepsilon(\Omega)} &\leq \sum_{\xi \in \tilde{\Lambda}^\varepsilon} \|u_0\|^2_{\left[H^1(\Omega_\xi^\varepsilon)\right]^3} + \|u_1\|^2_{\left[H^1(\Omega_1^\varepsilon)\right]^3} \\ &\leq C(\varepsilon) \left( \sum_{\xi \in \tilde{\Lambda}^\varepsilon} \|e(u_0)\|^2_{\left[L^2(\Omega_\xi^\varepsilon)\right]^{3\times 3}} + \|e(u_1)\|^2_{\left[L^2(\Omega_1^\varepsilon)\right]^{3\times 3}} \right) \leq C(\varepsilon)\|u\|^2_{\mathcal{W}_{\mathrm{m}}^\varepsilon(\Omega)}, \end{aligned}$$

where $\tilde{\Lambda}^\varepsilon = \{\xi \in \mathbb{Z}^2 : \varepsilon\,(Y' + \xi) \subset \Omega'\}$, $\Omega_\xi^\varepsilon \coloneqq \text{interior}\left(\varepsilon\left(\overline{Y_0'} + \xi\right)\right) \times (0, L_3) \cap \Omega_0^\varepsilon$ and the constants $c$ and $C$ may depend on $\varepsilon$. Since for every $\varepsilon$ the set $\tilde{\Lambda}^\varepsilon$ is finite, the norms are equivalent and thus, $\mathcal{W}_{\mathrm{m}}^\varepsilon(\Omega)$ can be seen as the direct sum of Hilbert spaces

$$\left[H^1_{\Gamma_{\mathrm{D}} \cap \partial\Omega_1^\varepsilon}(\Omega_1^\varepsilon)\right]^3 \times \prod_{\xi \in \tilde{\Lambda}^\varepsilon} \left[H^1_{\Gamma_{\mathrm{D}} \cap \partial\Omega_\xi^\varepsilon}(\Omega_\xi^\varepsilon)\right]^3$$

endowed with the standard $H^1$-norms, which yields the desired result. ☐

We need the following estimates to prove the existence and uniform boundedness of the solution.

**Lemma 7.1.2.** *For every* $v \in \mathcal{W}_{\mathrm{m}}^\varepsilon(\Omega)$*, there holds*

(i) $\varepsilon\|v_1\|^2_{[L^2(\Sigma^\varepsilon)]^3} \leq C \left( \|v_1\|^2_{\left[L^2(\Omega_1^\varepsilon)\right]^3} + \varepsilon^2 \|\nabla v_1\|^2_{\left[L^2(\Omega_1^\varepsilon)\right]^{3\times 3}} \right)$

(ii) $\|v_0\|^2_{[L^2(\Omega_0^\varepsilon)]^3} \leq C\left(\varepsilon^2\|e(v_0)\|^2_{[L^2(\Omega_0^\varepsilon)]^{3\times 3}} + \varepsilon\|v_0\|^2_{[L^2(\Sigma^\varepsilon)]^3}\right)$

*for constants $C > 0$ independent of $\varepsilon$.*

*Proof.* The proof is as in Lemma 5.1.2 with $Y_\kappa := Y'_\kappa \times (0,1)$, $\kappa \in \{0,1\}$, and $\Lambda^\varepsilon := \{\xi \in \mathbb{Z}^3 : \varepsilon\,(Y' \times (0,1) + \xi) \subset \Omega\}$. □

There exists a unique weak solution in the space $\mathcal{W}^\varepsilon_{\mathrm{m}}(\Omega)$.

**Theorem 7.1.3.** *Let $f^\varepsilon \in \left[L^2(\Omega)\right]^3$ and $g \in \left[L^2(\Gamma_\mathrm{N})\right]^3$. Then, there exists a unique weak solution $u \in \mathcal{W}^\varepsilon_{\mathrm{m}}(\Omega)$ of (4.0.3) for all $\varphi \in \mathcal{W}^\varepsilon_{\mathrm{m}}(\Omega)$.*

*Proof.* We prove the result via combination of the proofs of Theorem 5.1.3 and Theorem 6.1.1. We denote the left-hand side of (4.0.3) as a mapping $a\colon \mathcal{W}^\varepsilon_{\mathrm{m}}(\Omega) \times \mathcal{W}^\varepsilon_{\mathrm{m}}(\Omega) \to \mathbb{R}$,

$$\begin{aligned} a(u,v) = &\int_{\Omega_0^\varepsilon} A^\varepsilon e(u_0)e(v_0)\mathrm{d}x + \int_{\Omega_1^\varepsilon} A^\varepsilon e(u_1)e(v_1)\mathrm{d}x \\ &+ \varepsilon\int_{\Sigma^\varepsilon}\left(K_\mathrm{N}\,[u_n]_{\Sigma^\varepsilon}\,n + K_\mathrm{T}\sum_{i=1}^{2}[u_{\tau^i}]_{\Sigma^\varepsilon}\,\tau^i\right)\cdot(v_1 - v_0)\,\mathrm{d}S(x) \end{aligned}$$

and the right-hand side of (4.0.3) as a mapping $F\colon \mathcal{W}^\varepsilon_{\mathrm{m}}(\Omega) \to \mathbb{R}$,

$$F(v) = \int_\Omega f^\varepsilon \cdot v\,\mathrm{d}x + \int_{\Gamma_\mathrm{N}\cap\partial\Omega_1^\varepsilon} g\cdot v_1\,\mathrm{d}S(x),$$

The integral over $\Gamma_\mathrm{N} \cap \partial\Omega_0^\varepsilon$ vanishes since $g = 0$ on $\Omega' \times \{L_3\} \supset \Gamma_\mathrm{N} \cap \partial\Omega_0^\varepsilon$. As the norms on $\mathcal{W}^\varepsilon_{\mathrm{d}}(\Omega)$ and $\mathcal{W}^\varepsilon_{\mathrm{m}}(\Omega)$ are identical, the same proof as in Theorem 5.1.3 shows that $a$ is a coercive continuous bilinear form. It remains to prove that $F$ is linear and continuous. The linearity is clear. Let $v \in \mathcal{W}^\varepsilon_{\mathrm{m}}(\Omega)$. Using Korn's inequality from Theorem 4.4 in [Höpker, 2016] as in the connected case for $v_1$ (cf. Theorem 6.1.1), we obtain

$$\|v_1\|^2_{[L^2(\Omega_1^\varepsilon)]^3} \leq C\|e(v_1)\|^2_{[L^2(\Omega_1^\varepsilon)]^{3\times 3}} \leq C\|v\|^2_{\mathcal{W}^\varepsilon_{\mathrm{m}}(\Omega)} \tag{7.1.1}$$

and with Lemma 7.1.2

$$\begin{aligned} \|v_0\|^2_{[L^2(\Omega_0^\varepsilon)]^3} &\leq C\left(\varepsilon^2\|e(v_0)\|^2_{[L^2(\Omega_0^\varepsilon)]^{3\times 3}} + \varepsilon\|v_0\|^2_{[L^2(\Sigma^\varepsilon)]^3}\right) \\ &\leq C\left(\varepsilon^2\|e(v_0)\|^2_{[L^2(\Omega_0^\varepsilon)]^{3\times 3}} + \varepsilon\|[v]_{\Sigma^\varepsilon}\|^2_{[L^2(\Sigma^\varepsilon)]^3} + \|v_1\|^2_{[L^2(\Omega_1^\varepsilon)]^3} + \varepsilon^2\|\nabla v_1\|^2_{[L^2(\Omega_1^\varepsilon)]^{3\times 3}}\right) \\ &\leq C\|v\|^2_{\mathcal{W}^\varepsilon_{\mathrm{m}}(\Omega)}. \end{aligned} \tag{7.1.2}$$

Thus, Hölder's inequality leads to

$$|F(v)| \leq C\|f^\varepsilon\|_{[L^2(\Omega)]^3}\|v\|_{\mathcal{W}^\varepsilon_{\mathrm{m}}(\Omega)} + \|g\|_{[L^2(\Gamma_\mathrm{N})]^3}\|v_1\|_{[L^2(\Gamma_\mathrm{N}\cap\partial\Omega_1^\varepsilon)]^3}.$$

Since estimate (6.1.2) is also true for $v_1$, we receive

$$|F(v)| \leq C \left( \|f^\varepsilon\|_{[L^2(\Omega)]^3} + \|g\|_{[L^2(\Gamma_{\mathrm{N}})]^3} \right) \|v\|_{\mathcal{W}^\varepsilon_{\mathrm{m}}(\Omega)}.$$

Therefore, all assumptions of the Lax–Milgram theorem are fulfilled and we get the existence and uniqueness of the solution. □

If $f^\varepsilon$ is bounded independently of $\varepsilon$, the weak solution $u^\varepsilon$ is uniformly bounded.

**Theorem 7.1.4.** *Let $u^\varepsilon \in \mathcal{W}^\varepsilon_{\mathrm{m}}(\Omega)$ be the weak solution of* (4.0.3) *and $f^\varepsilon$ bounded independently of $\varepsilon$ in $\left[L^2(\Omega)\right]^3$. Then, there exists an $\varepsilon$-independent constant $C$ with*

$$\|u^\varepsilon\|_{\mathcal{W}^\varepsilon_{\mathrm{m}}(\Omega)} \leq C.$$

*Proof.* Using the estimates in the proof of Theorem 7.1.3 with $v = u^\varepsilon$, we get

$$\min\{\alpha, K_{\mathrm{N}}, K_{\mathrm{T}}\} \|u^\varepsilon\|^2_{\mathcal{W}^\varepsilon_{\mathrm{m}}(\Omega)} \leq a(u^\varepsilon, u^\varepsilon) = F(u^\varepsilon) \leq C \left( \|f^\varepsilon\|_{[L^2(\Omega)]^3} + \|g\|_{[L^2(\Gamma_{\mathrm{N}})]^3} \right) \|u^\varepsilon\|_{\mathcal{W}^\varepsilon_{\mathrm{m}}(\Omega)},$$

which shows the uniform boundedness of $\|u^\varepsilon\|_{\mathcal{W}^\varepsilon_{\mathrm{m}}(\Omega)}$. □

**Lemma 7.1.5.** *With the same assumption as in Theorem 7.1.4, there holds*

$$\|u^\varepsilon\|^2_{[L^2(\Omega)]^3} + \|e(u^\varepsilon_0)\|^2_{\left[L^2(\Omega^\varepsilon_0)\right]^{3\times 3}} + \|e(u^\varepsilon_1)\|^2_{\left[L^2(\Omega^\varepsilon_1)\right]^{3\times 3}} \leq C$$

*for a constant $C > 0$ independent of $\varepsilon$.*

*Proof.* If we choose $v = u^\varepsilon$ in the estimates (7.1.1) and (7.1.2) and note that the constants there are independent of $\varepsilon$, we get, together with the uniform boundedness from Theorem 7.1.4, the desired result. □

# 7.2. Homogenization results in the unidirectionally connected case

We prove in subsection 7.2.1 some compactness results via the periodic unfolding method, which we apply in subsection 7.2.2 to derive the homogenized problem.

## 7.2.1. Compactness result in the unidirectionally connected case

Since $A$ only depends on $Y'$, we are, in addition to the standard weak convergences of the unfolded sequence, also interested in the convergence of the mean value over $(0,1)$ of the unfolded sequence. Therefore, we define the linear and continuous operator $\mathcal{M}^1_{(0,1)} : L^2(\Omega \times Y_1) \to L^2(\Omega \times Y'_1)$,

$$\mathcal{M}^1_{(0,1)}(u)(x, y') := \int_0^1 u(x, y', y_3)\,\mathrm{d}y_3$$

and the symmetric gradient with respect to $y'$

$$e_{y'}(w) := e_{(y_1,y_2)}(w) := \begin{pmatrix} \partial_{y_1} w_1 & \frac{1}{2}(\partial_{y_1} w_2 + \partial_{y_2} w_1) & \frac{1}{2}\partial_{y_1} w_3 \\ \frac{1}{2}(\partial_{y_1} w_2 + \partial_{y_2} w_1) & \partial_{y_2} w_2 & \frac{1}{2}\partial_{y_2} w_3 \\ \frac{1}{2}\partial_{y_1} w_3 & \frac{1}{2}\partial_{y_2} w_3 & 0 \end{pmatrix}.$$

**Theorem 7.2.1.** *Let $\{u_1^\varepsilon\}$ be a sequence with $u_1^\varepsilon \in H^1_{\Gamma_\mathrm{D}\cap\partial\Omega_1^\varepsilon}(\Omega_1^\varepsilon)$ and*

$$\|e(u_1^\varepsilon)\|_{\left[L^2(\Omega_1^\varepsilon)\right]^{3\times 3}} \leq C$$

*for a constant $C$ independent of $\varepsilon$. Then, there exists, up to a subsequence, $u_1 \in H^1_{\Gamma_\mathrm{D}}(\Omega)$ and $\hat{u}_1 \in \left[L^2(\Omega, H^1_{\mathrm{per},0}(Y_1))\right]^3$, such that*

$$\begin{aligned} \mathcal{T}_1^\varepsilon(u_1^\varepsilon) &\rightharpoonup u_1 \text{ weakly in } \left[L^2(\Omega, H^1(Y_1))\right]^3, \\ \mathcal{T}_1^\varepsilon\left(e(u_1^\varepsilon)\right) &\rightharpoonup e(u_1) + e_y(\hat{u}_1) \text{ weakly in } \left[L^2(\Omega\times Y_1)\right]^{3\times 3}. \end{aligned} \tag{7.2.1}$$

*Furthermore,*

$$\begin{aligned} \mathcal{M}^1_{(0,1)}(\mathcal{T}_1^\varepsilon(u_1^\varepsilon)) &\rightharpoonup u_1 \text{ weakly in } \left[L^2(\Omega, H^1(Y_1'))\right]^3, \\ \mathcal{M}^1_{(0,1)}(\mathcal{T}_1^\varepsilon(e(u_1^\varepsilon))) &\rightharpoonup e(u_1) + e_{y'}(\bar{u}_1) \text{ weakly in } \left[L^2(\Omega\times Y_1')\right]^{3\times 3}, \end{aligned}$$

*where the mean value operator $\mathcal{M}^1_{(0,1)}$ is applied to every component and $\bar{u}_1 = \mathcal{M}^1_{(0,1)}(\hat{u}_1)$.*

*Proof.* Let $\{u_1^\varepsilon\}$ be a sequence with $u_1^\varepsilon \in H^1_{\Gamma_\mathrm{D}\cap\partial\Omega_1^\varepsilon}(\Omega_1^\varepsilon)$ and $\|e(u_1^\varepsilon)\|_{\left[L^2(\Omega_1^\varepsilon)\right]^{3\times 3}} \leq C$. The domain $\Omega_1^\varepsilon$ is of the same structure as in the connected case, so we can directly apply Theorem 6.2.2 to obtain the weak convergences (7.2.1). Thus, for all $v \in \left[L^2(\Omega\times Y_1')\right]^3$

$$\begin{aligned} \int_\Omega\int_{Y_1'} \mathcal{M}^1_{(0,1)}(\mathcal{T}_1^\varepsilon(u_1^\varepsilon))(x,y')\cdot v(x,y')\,\mathrm{d}y'\mathrm{d}x &= \int_\Omega\int_{Y_1} \mathcal{T}_1^\varepsilon(u_1^\varepsilon)(x,y)\cdot v(x,y')\,\mathrm{d}y\mathrm{d}x \\ \to \int_\Omega\int_{Y_1} u_1(x)\cdot v(x,y')\,\mathrm{d}y\mathrm{d}x &= \int_\Omega\int_{Y_1'} u_1(x)\cdot v(x,y')\,\mathrm{d}y'\mathrm{d}x \end{aligned}$$

and with Theorem 7.2.2

$$\int_\Omega\int_{Y_1'} \partial_{y_i}\mathcal{M}^1_{(0,1)}(\mathcal{T}_1^\varepsilon(u_1^\varepsilon))(x,y')\cdot v(x,y')\,\mathrm{d}y'\mathrm{d}x = \int_\Omega\int_{Y_1} \partial_{y_i}\mathcal{T}_1^\varepsilon(u_1^\varepsilon)(x,y)\cdot v(x,y')\,\mathrm{d}y\mathrm{d}x \to 0,$$

for $i \in \{1,2\}$. Analogously, for all $v \in \left[L^2(\Omega\times Y_1')\right]^{3\times 3}$

$$\begin{aligned} &\int_\Omega\int_{Y_1'} \mathcal{M}^1_{(0,1)}(\mathcal{T}_1^\varepsilon(e(u_1^\varepsilon)))(x,y') : v(x,y')\,\mathrm{d}y'\mathrm{d}x \\ &\qquad \to \int_\Omega\int_{Y_1'} \left(e(u_1)(x) + \int_0^1 e_y(\hat{u}_1)(x,y',y_3)\mathrm{d}y_3\right) : v(x,y')\,\mathrm{d}y'\mathrm{d}x. \end{aligned}$$

Using that $\hat{u}_1$ is 1-periodic with respect to $y_3$ and Theorem 7.2.2, there holds

$$\int_0^1 e_y(\hat{u}_1)(x, y', y_3)\mathrm{d}y_3 = e_{y'}(\bar{u}_1)(x, y')$$

with $\bar{u}_1(x, y') = \mathcal{M}^1_{(0,1)}(\hat{u}_1)$. □

The following technical theorem derives the weak derivative of the mean value of a function with respect to one variable.

**Theorem 7.2.2.** *Let $\kappa \in \{0, 1\}$ and $u \in [L^2(\Omega, H^1(Y_\kappa))]^3$. Then,*

$$\int_0^1 u(\cdot, \cdot, y_3)\, \mathrm{d}y_3 \in [L^2(\Omega, H^1(Y'_\kappa))]^3$$

*with weak derivative*

$$\int_0^1 \partial_{y_i} u(x, y', y_3)\, \mathrm{d}y_3$$

*for $i \in \{1, 2\}$.*

*Proof.* By Hölder's inequality

$$\int_\Omega \int_{Y'_0} \Big| \int_0^1 u(x, y', y_3)\, \mathrm{d}y_3 \Big|^2 \mathrm{d}y'\mathrm{d}x \leq \|u\|^2_{[L^2(\Omega\times Y_0)]^3},$$

which shows that $\int_0^1 u(\cdot, \cdot, y_3)\, \mathrm{d}y_3 \in [L^2(\Omega \times Y'_\kappa)]^3$. Let $\varphi \in [\mathcal{D}(Y'_\kappa)]^3$ and $i \in \{1, 2\}$. Since $\varphi$ is independent of $y_3$ we get

$$\int_{Y'_\kappa} \int_0^1 u(x, y', y_3)\, \mathrm{d}y_3 \cdot \partial_{y_i}\varphi(y')\, \mathrm{d}y' = \int_{Y'_\kappa} \int_0^1 u(x, y', y_3) \cdot \partial_{y_i}\varphi(y')\, \mathrm{d}y_3\mathrm{d}y'. \tag{7.2.2}$$

By applying Hölder's inequality twice

$$\begin{aligned}
&\int_0^1 \int_{Y'_\kappa} |u(x, y', y_3) \cdot \partial_{y_i}\varphi(y')|\, \mathrm{d}y'\mathrm{d}y_3 \\
&\qquad \leq \int_0^1 \left( \int_{Y'_\kappa} |u(x, y', y_3)|^2\, \mathrm{d}y' \right)^{1/2} \left( \int_{Y'_\kappa} |\partial_{y_i}\varphi(y')|^2\, \mathrm{d}y' \right)^{1/2} \mathrm{d}y_3 \\
&\qquad \leq \|u(x, \cdot)\|_{L^2(Y_0)} \|\partial_{y_i}\varphi\|_{L^\infty(Y'_\kappa)} |Y_\kappa|^{1/2}.
\end{aligned}$$

Thus, by Tonelli's theorem we can interchange the integrals in (7.2.2) and use the weak differentiablity of $\hat{u}_\kappa$ with respect to $y$ to obtain

$$\begin{aligned}\int_{Y'_\kappa}\int_0^1 u(x,y',y_3)\cdot\partial_{y_i}\varphi(y')\,\mathrm{d}y_3\mathrm{d}y' &= \int_0^1\int_{Y'_\kappa} u(x,y',y_3)\cdot\partial_{y_i}\varphi(y')\,\mathrm{d}y'\mathrm{d}y_3\\ &= -\int_0^1\int_{Y'_\kappa}\partial_{y_i}u(x,y',y_3)\cdot\varphi(y')\,\mathrm{d}y'\mathrm{d}y_3.\end{aligned}$$

By applying Hölder's inequality twice

$$\begin{aligned}&\int_0^1\int_{Y'_\kappa}|\partial_{y_i}u(x,y',y_3)\cdot\varphi(y')|\,\mathrm{d}y'\mathrm{d}y_3\\ &\qquad\leq \int_0^1\left(\int_{Y'_\kappa}|\partial_{y_i}u(x,y',y_3)|^2\,\mathrm{d}y'\right)^{1/2}\left(\int_{Y'_\kappa}|\varphi(y')|^2\,\mathrm{d}y'\right)^{1/2}\mathrm{d}y_3\\ &\qquad\leq \|\partial_{y_i}u(x,\cdot)\|_{L^2(Y_\kappa)}\|\varphi\|_{L^\infty(Y'_\kappa)}|Y_\kappa|^{1/2}.\end{aligned}$$

So again by Tonelli's theorem, we obtain for almost all $x\in\Omega$

$$\int_{Y'_\kappa}\int_0^1 u(x,y',y_3)\,\mathrm{d}y_3\cdot\partial_{y_i}\varphi(y')\,\mathrm{d}y' = -\int_{Y'_\kappa}\int_0^1\partial_{y_i}u(x,y',y_3)\,\mathrm{d}y_3\cdot\varphi(y')\,\mathrm{d}y'.$$

Since

$$\int_\Omega\int_{Y'_\kappa}\left|\int_0^1\partial_{y_i}u(x,y',y_3)\,\mathrm{d}y_3\right|^2\mathrm{d}y'\mathrm{d}x\leq\|\partial_{y_i}u\|^2_{[L^2(\Omega\times Y_\kappa)]^3}$$

$\int_0^1\partial_{y_i}u(x,y',y_3)\,\mathrm{d}y_3\in[L^2(\Omega\times Y'_\kappa)]^3$ and as $\varphi$ was arbitrary, we get the desired result. □

As in the disconnected case we have in general only a uniform estimate of the symmetric gradient of $u_0^\varepsilon$. But since we have weak bonding of the materials and every connected subset of $\Omega_0^\varepsilon$ is fixed at one part of the outer boundary we can prove some general compactness results.

**Theorem 7.2.3.** *Let $\{u^\varepsilon\}$ be a sequence with $u^\varepsilon\in\mathcal{W}^\varepsilon_{\mathrm{m}}(\Omega)$ and*

$$\|u^\varepsilon\|_{\mathcal{W}^\varepsilon_{\mathrm{m}}(\Omega)}\leq C$$

*for a constant $C$ independent of $\varepsilon$. Then, there exists, up to a subsequence, $u_0\in[L^2(\Omega,H^1(Y_0))]^3$ such that*

$$\begin{aligned}\mathcal{T}_0^\varepsilon(u_0^\varepsilon)&\rightharpoonup u_0 \text{ weakly in } \left[L^2(\Omega,H^1(Y_0))\right]^3,\\ \nabla_y\mathcal{T}_0^\varepsilon(u_0^\varepsilon)&\rightharpoonup\nabla_y u_0 \text{ weakly in } \left[L^2(\Omega\times Y_0)\right]^{3\times3},\\ e_y(\mathcal{T}_0^\varepsilon(u_0^\varepsilon))&\to 0 \text{ strongly in } \left[L^2(\Omega\times Y_0)\right]^{3\times3}.\end{aligned}$$

*Moreover, $u_0$ is of the form $u_0(x,y)=B(x)y+c(x)$ for some skew-symmetric matrix $B\in\left[L^2(\Omega)\right]^{3\times3}$ and appropriate function $c\in\left[L^2(\Omega)\right]^3$.*

*Proof.* Let $\{u^\varepsilon\}$ be as in the assumption. Then, there holds due Proposition 3.2.7 (iii) and (7.1.2)

$$\|\mathcal{T}_0^\varepsilon(u_0^\varepsilon)\|_{[L^2(\Omega\times Y_0)]^3} \le |Y|^{\frac{1}{2}}\|u_0^\varepsilon\|_{\left[L^2(\Omega_0^\varepsilon)\right]^3} \le C\|u^\varepsilon\|_{\mathcal{W}_{\mathrm{m}}^\varepsilon(\Omega)} \le C$$

and if we apply Korn's inequality (see Theorem 2.2.3) with respect to $Y_0$ and Lemma 5.2.2, we get

$$\begin{aligned}
\|\nabla_y\mathcal{T}_0^\varepsilon(u_0^\varepsilon)\|^2_{[L^2(\Omega\times Y_0)]^{3\times3}} &\le C\left(\|e_y(\mathcal{T}_0^\varepsilon(u_0^\varepsilon))\|^2_{[L^2(\Omega\times Y_0)]^{3\times3}} + \|\mathcal{T}_0^\varepsilon(u_0^\varepsilon)\|^2_{[L^2(\Omega\times Y_0)]^3}\right)\\
&\le C\left(\varepsilon^2\|\mathcal{T}_0^\varepsilon(e(u_0^\varepsilon))\|^2_{[L^2(\Omega\times Y_0)]^{3\times3}} + \|\mathcal{T}_0^\varepsilon(u_0^\varepsilon)\|^2_{[L^2(\Omega\times Y_0)]^3}\right)\\
&\le C,
\end{aligned}$$

which shows that $\{\mathcal{T}_0^\varepsilon(u_0^\varepsilon)\}$ is a bounded sequence in $[L^2(\Omega, H^1(Y_0))]^3$. Thus, there exists a subsequence (again denoted by $\varepsilon$) such that

$$\begin{aligned}
\mathcal{T}_0^\varepsilon(u_0^\varepsilon) &\rightharpoonup u_0 \text{ weakly in } \left[L^2(\Omega\times Y_0)\right]^3,\\
\nabla_y\mathcal{T}_0^\varepsilon(u_0^\varepsilon) &\rightharpoonup \nabla_y u_0 \text{ weakly in } \left[L^2(\Omega\times Y_0)\right]^{3\times3}.
\end{aligned}$$

Since

$$\|e_y(\mathcal{T}_0^\varepsilon(u_0^\varepsilon))\|_{[L^2(\Omega\times Y_0)]^{3\times3}} \le \varepsilon\|\mathcal{T}_0^\varepsilon(e(u_0^\varepsilon))\|_{[L^2(\Omega\times Y_0)]^{3\times3}} \le \varepsilon|Y|^{\frac{1}{2}}\|e(u_0^\varepsilon)\|_{\left[L^2(\Omega_0^\varepsilon)\right]^{3\times3}} \le \varepsilon C,$$

we receive the strong convergence

$$e_y(\mathcal{T}_0^\varepsilon(u_0^\varepsilon)) \to 0 \text{ strongly in } \left[L^2(\Omega\times Y_0)\right]^{3\times3},$$

resulting in $e_y(u_0)(x,y) = 0$ for almost every $(x,y)\in\Omega\times Y_0$. So there are only rigid-body motions with respect to $Y_0$ possible, i.e. $u_0(x,y) = B(x)y + c(x)$ with $B(x)\in\mathbb{R}^{3\times3}$ skew-symmetric. Clearly, $B\in\left[L^2(\Omega)\right]^{3\times3}$ and $c\in\left[L^2(\Omega)\right]^3$. □

The limit function $u_0$ satisfies some more properties.

**Theorem 7.2.4.** *The limit function $u_0\in[L^2(\Omega,H^1(Y_0))]^3$ from Theorem 7.2.3 is of the form $u_0(x,y) = b(x)(y_2,-y_1,0)^T + c(x)$ with $b\in L^2(\Omega)$.*

*Proof.* Since $\mathcal{T}_0^\varepsilon(u_0^\varepsilon)\in[L^2(\Omega,H^1(Y_0))]^3$, the traces with respect to $y_3$ exist. Following the steps of the proof of Proposition 3.1 in [Cioranescu et al., 2008], we compute for all $\varphi\in\left[\mathcal{D}(\Omega\times Y_0')\right]^3$

by an obvious change of variable

$$\begin{aligned}
&\int_{\Omega\times Y_0'} (\mathcal{T}_0^\varepsilon(u_0^\varepsilon)(x,(y',1)) - \mathcal{T}_0^\varepsilon(u_0^\varepsilon)(x,(y',0))) \cdot \varphi(x,y')\,\mathrm{d}x\mathrm{d}y' \\
&\quad = \int_{\Omega\times Y_0'} \left(u_0^\varepsilon\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon(y',1)\right) - u_0^\varepsilon\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon(y',0)\right)\right) \cdot \varphi(x,y')\,\mathrm{d}x\mathrm{d}y' \\
&\quad = \int_{\Omega\times Y_0'} u_0^\varepsilon\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon(y',0)\right) \cdot (\varphi(x-\varepsilon e_3,y') - \varphi(x,y'))\,\mathrm{d}x\mathrm{d}y' \\
&\quad = \int_{\Omega\times Y_0'} \mathcal{T}_0^\varepsilon(u_0^\varepsilon)(x,(y',0)) \cdot (\varphi(x-\varepsilon e_3,y') - \varphi(x,y'))\,\mathrm{d}x\mathrm{d}y',
\end{aligned}$$

which converges to zero, using the weak convergence from Theorem 7.2.3. Thus, $u_0$ is $y_3$-periodic and we obtain

$$0 = u_0(x,(y',1)) - u_0(x,(y',0)) = B(x)(y',1) - B(x)(y',0) = (b_{13}(x), b_{23}(x), 0)$$

for almost every $x \in \Omega$. Since $B$ is skew-symmetric, $u_0$ is of the form

$$u_0(x,y) = B(x)y + c(x) = (b_{12}(x)y_2, b_{21}(x)y_1, 0)^T + c(x) = b_{12}(x)(y_2, -y_1, 0)^T + c(x).$$

The result follows by setting $b(x) \coloneqq b_{12}(x)$. $\square$

From Theorem 7.2.3 and Theorem 7.2.4, we directly obtain the following weak convergences.

**Lemma 7.2.5.** *There holds*

$$\begin{aligned}
\partial_{y_3}\mathcal{T}_0^\varepsilon(u_0^\varepsilon) &\rightharpoonup 0 \text{ weakly in } \left[L^2(\Omega\times Y_0)\right]^3, \\
\partial_{y_1}\mathcal{T}_0^\varepsilon(u_0^\varepsilon)_3 &\rightharpoonup 0 \text{ weakly in } L^2(\Omega\times Y_0), \\
\partial_{y_2}\mathcal{T}_0^\varepsilon(u_0^\varepsilon)_3 &\rightharpoonup 0 \text{ weakly in } L^2(\Omega\times Y_0).
\end{aligned}$$

The third component of $u_0$ is weak differentiable in $e_3$-direction.

**Theorem 7.2.6.** *The function $c(x)$ from Theorem 7.2.4 satisfies $\partial_3 c_3(x) \in L^2(\Omega)$ and $c_3|_{\Gamma_\mathrm{D}} = 0$.*

*Proof.* The proof follows the idea of the proof of Lemma 2.3 from [Allaire and Murat, 1993] applied to the unfolded sequence. For the readability, we define $\tilde{u}_0^\varepsilon \coloneqq (u_0^\varepsilon)_3$. We consider the unfolded mean value over $Y_0$ of the third component of $u_0^\varepsilon$, i.e.

$$\mathcal{M}_{Y_0}^\varepsilon(\tilde{u}_0^\varepsilon)(x) \coloneqq \frac{1}{|Y_0|}\int_{Y_0} \mathcal{T}_0^\varepsilon(\tilde{u}_0^\varepsilon)(x,y)\,\mathrm{d}y.$$

Since

$$\|\mathcal{M}_{Y_0}^\varepsilon(\tilde{u}_0^\varepsilon)\|_{L^2(\Omega)} \le \left(\frac{|Y|}{|Y_0|}\right)^{\frac{1}{2}} \|\tilde{u}_0^\varepsilon\|_{L^2(\Omega_0^\varepsilon)} \le C,$$

there exists a function $\tilde{u}_0 \in L^2(\Omega)$ such that

$$\mathcal{M}^\varepsilon_{Y_0}(\tilde{u}^\varepsilon_0) \rightharpoonup \tilde{u}_0 \text{ weakly in } L^2(\Omega).$$

We estimate the difference of the mean values of adjacent cells in $e_3$-direction by using the change of variable theorem and Lemma 2.2(2) from [Allaire and Murat, 1993]

$$\begin{aligned}
&|\mathcal{M}^\varepsilon_{Y_0}(\tilde{u}^\varepsilon_0)(x) - \mathcal{M}^\varepsilon_{Y_0}(\tilde{u}^\varepsilon_0)(x+\varepsilon e_3)| \\
&\qquad = \left| \frac{1}{|Y_0|} \int_{Y_0} \mathcal{T}^\varepsilon_0(\tilde{u}^\varepsilon_0)(x,y)\,\mathrm{d}y - \frac{1}{|Y_0|} \int_{Y_0} \mathcal{T}^\varepsilon_0(\tilde{u}^\varepsilon_0)(x,e_3+y)\,\mathrm{d}y \right| \\
&\qquad = \left| \frac{1}{|Y_0|} \int_{Y_0'} \left( \int_0^1 \mathcal{T}^\varepsilon_0(\tilde{u}^\varepsilon_0)(x,y',y_3)\,\mathrm{d}y_3 - \int_1^2 \mathcal{T}^\varepsilon_0(\tilde{u}^\varepsilon_0)(x,y',y_3)\,\mathrm{d}y_3 \right) \mathrm{d}y' \right| \\
&\qquad \le \frac{C}{|Y_0|} \int_{Y_0'} \left( \int_0^2 |\partial_{y_3} \mathcal{T}^\varepsilon_0(\tilde{u}^\varepsilon_0)(x,y',y_3)|^2 \mathrm{d}y_3 \right)^{\frac{1}{2}} \mathrm{d}y' \\
&\qquad \le C \frac{|Y_0'|^{\frac{1}{2}}}{|Y_0|} \|\partial_{y_3} \mathcal{T}^\varepsilon_0(\tilde{u}^\varepsilon_0)(x,\cdot)\|_{L^2(Z_0)} \\
&\qquad \le C\varepsilon \|\mathcal{T}^\varepsilon_0(\partial_3 \tilde{u}^\varepsilon_0)(x,\cdot)\|_{L^2(Z_0)} \\
&\qquad = C\varepsilon \left( \int_{Z_0} \left| \partial_3 \tilde{u}^\varepsilon_0 \left( \varepsilon \left[ \frac{x}{\varepsilon} \right] + \varepsilon y \right) \right|^2 \mathrm{d}y \right)^{\frac{1}{2}} \\
&\qquad = C\varepsilon^{-\frac{1}{2}} \|\partial_3 \tilde{u}^\varepsilon_0\|_{L^2(\varepsilon[\frac{x}{\varepsilon}]+\varepsilon Z_0)}
\end{aligned}$$

with $Z_0 := Y_0 \cup (Y_0 + e_3) \cup (\partial Y_0 \cap \partial(Y_0 + e_3))$. Let $\omega \subset \Omega$ convex such that $\overline{\omega} \subset \Omega$ and $\omega \subset \Omega^\varepsilon$ for $\varepsilon$ small enough. Let $h > 0$ sufficiently small such that $x + he_3 \in \Omega$ for all $x \in \omega$. If $h \le \varepsilon$, we have to distinguish two cases. If $[\frac{x+he_3}{\varepsilon}] = [\frac{x}{\varepsilon}]$, then

$$|\mathcal{M}^\varepsilon_{Y_0}(\tilde{u}^\varepsilon_0)(x) - \mathcal{M}^\varepsilon_{Y_0}(\tilde{u}^\varepsilon_0)(x+he_3)| = 0,$$

and if $[\frac{x+he_3}{\varepsilon}] = [\frac{x}{\varepsilon}] + e_3$, we use the previous estimate to get

$$|\mathcal{M}^\varepsilon_{Y_0}(\tilde{u}^\varepsilon_0)(x) - \mathcal{M}^\varepsilon_{Y_0}(\tilde{u}^\varepsilon_0)(x+he_3)| \le C\varepsilon^{-\frac{1}{2}} \|\partial_3 \tilde{u}^\varepsilon_0\|_{L^2(\varepsilon[\frac{x}{\varepsilon}]+\varepsilon Z_0)}.$$

Thus,

$$\begin{aligned}
&\int_\omega |\mathcal{M}^\varepsilon_{Y_0}(\tilde{u}^\varepsilon_0)(x) - \mathcal{M}^\varepsilon_{Y_0}(\tilde{u}^\varepsilon_0)(x+he_3)|^2 \,\mathrm{d}x \\
&\qquad \le \sum_{\xi \in \Lambda_\varepsilon} \int_{\varepsilon(\xi+Y_0)} |\mathcal{M}^\varepsilon_{Y_0}(\tilde{u}^\varepsilon_0)(x) - \mathcal{M}^\varepsilon_{Y_0}(\tilde{u}^\varepsilon_0)(x+he_3)|^2 \,\mathrm{d}x \\
&\qquad \le \sum_{\xi \in \Lambda_\varepsilon} C\varepsilon^{-1} \|\partial_3 \tilde{u}^\varepsilon_0\|^2_{L^2(\varepsilon\xi+\varepsilon Z_0)} (\varepsilon - 0)(\varepsilon - 0)(\varepsilon - (\varepsilon - h)) \\
&\qquad \le 2C\varepsilon h \|\partial_3 \tilde{u}^\varepsilon_0\|^2_{L^2(\Omega^\varepsilon_0)} \\
&\qquad \le \varepsilon h C.
\end{aligned}$$

If $h > \varepsilon$ there exists an $n \in \mathbb{N}$ and $\tilde{h} < \varepsilon$ with $h = n\varepsilon + \tilde{h}$. Since $\omega$ is convex, we can split the intervall $(x, x + he_3)$ into the intervalls

$$(x + j\varepsilon e_3, x + (j+1)\varepsilon e_3),\ j \in \{0, \dots, n-1\}, \quad \text{and} \quad (x + n\varepsilon e_3, x + (n\varepsilon + \tilde{h})e_3).$$

With the results from the case $h < \varepsilon$, we estimate

$$\begin{aligned}
&\|\mathcal{M}^{\varepsilon}_{Y_0}(\tilde{u}^{\varepsilon}_0)(x) - \mathcal{M}^{\varepsilon}_{Y_0}(\tilde{u}^{\varepsilon}_0)(x + he_3)\|_{L^2(\omega)} \\
&\quad\le \sum_{j=0}^{n-1} \|\mathcal{M}^{\varepsilon}_{Y_0}(\tilde{u}^{\varepsilon}_0)(x + j\varepsilon e_3) - \mathcal{M}^{\varepsilon}_{Y_0}(\tilde{u}^{\varepsilon}_0)(x + (j+1)\varepsilon e_3)\|_{L^2(\omega)} \\
&\qquad\quad + \|\mathcal{M}^{\varepsilon}_{Y_0}(\tilde{u}^{\varepsilon}_0)(x + n\varepsilon e_3) - \mathcal{M}^{\varepsilon}_{Y_0}(\tilde{u}^{\varepsilon}_0)(x + (n\varepsilon + \tilde{h})e_3)\|_{L^2(\omega)} \\
&\quad\le C(n\varepsilon + (\varepsilon\tilde{h})^{\frac{1}{2}}) \\
&\quad\le Ch.
\end{aligned}$$

Since $\mathcal{M}^{\varepsilon}_{Y_0}(\tilde{u}^{\varepsilon}_0)$ converges weakly to $\tilde{u}_0$ in $L^2(\Omega)$,

$$\mathcal{M}^{\varepsilon}_{Y_0}(\tilde{u}^{\varepsilon}_0)(\cdot) - \mathcal{M}^{\varepsilon}_{Y_0}(\tilde{u}^{\varepsilon}_0)(\cdot + he_3) \rightharpoonup \tilde{u}_0(\cdot) - \tilde{u}_0(\cdot + he_3)$$

weakly in $L^2(\omega)$. Thus,

$$\|\tilde{u}_0(\cdot) - \tilde{u}_0(\cdot + he_3)\|_{L^2(\omega)} \le \liminf_{\varepsilon \to 0} \|\mathcal{M}^{\varepsilon}_{Y_0}(\tilde{u}^{\varepsilon}_0)(\cdot) - \mathcal{M}^{\varepsilon}_{Y_0}(\tilde{u}^{\varepsilon}_0)(\cdot + he_3)\|_{L^2(\omega)} \le Ch,$$

and since $\omega$ was arbitrary, this shows that $\partial_3 \tilde{u}_0 \in L^2(\Omega)$. Furthermore, we know from Theorem 7.2.4 that

$$\mathcal{T}^{\varepsilon}_0(\tilde{u}^{\varepsilon}_0) \rightharpoonup c_3 \text{ weakly in } L^2(\Omega, H^1(Y_0)).$$

Since the mean-value operator is linear and continuous, we also get that

$$\mathcal{M}^{\varepsilon}_{Y_0}(\tilde{u}^{\varepsilon}_0) = \mathcal{M}_{Y_0}(\mathcal{T}^{\varepsilon}_0(\tilde{u}^{\varepsilon}_0)) \rightharpoonup \mathcal{M}_{Y_0}(c_3) = c_3 \text{ weakly in } L^2(\Omega),$$

which yields that $c_3 = \tilde{u}_0 \in L^2(\Omega)$. It can be shown that $c_3|_{\Gamma_{\mathrm{D}}} = 0$ as in Lemma 4.41 from [Cioranescu et al., 2018] by extending the function by zero outside of the domain $\Omega$. $\square$

In the next theorem, we prove the weak convergence of a sequence, where the symmetric gradient with respect to $y$ of this sequence coincides with the unfolded symmetric gradient of $u^{\varepsilon}_0$. The idea is to neglect the rigid-body motions on the microscopic scale because they do not induce forces.

**Theorem 7.2.7.** *Let $\{u^{\varepsilon}\}$ be a sequence with $u^{\varepsilon} \in \mathcal{W}^{\varepsilon}_{\mathrm{m}}(\Omega)$ and*

$$\|u^{\varepsilon}\|_{\mathcal{W}^{\varepsilon}_{\mathrm{m}}(\Omega)} \le C$$

*for a constant $C$ independent of $\varepsilon$. Then, there exists a function $\hat{u}_0 \in \left[L^2(\Omega, H^1(Y_0))\right]^3$ such* *that up to a subsequence*

$$Z_0^\varepsilon \coloneqq \frac{1}{\varepsilon}\left[\mathcal{T}_0^\varepsilon(u_0^\varepsilon) - r^\varepsilon\right] \rightharpoonup \hat{u}_0 \text{ weakly in } \left[L^2(\Omega, H^1(Y_0))\right]^3,$$
$$\mathcal{T}_0^\varepsilon(e(u_0^\varepsilon)) \rightharpoonup e_y(\hat{u}_0) \text{ weakly in } \left[L^2(\Omega \times Y_0)\right]^{3\times 3},$$

*where* $r^\varepsilon(x,y) = B^\varepsilon(x)y + c^\varepsilon(x)$ *with skew-symmetric matrix*

$$B^\varepsilon(x) \coloneqq \mathcal{M}_{Y_0}(\nabla_y \mathcal{T}_0^\varepsilon(u_0^\varepsilon)(x,y) - e_y(\mathcal{T}_0^\varepsilon(u_0^\varepsilon))(x,y))$$

*and*

$$c^\varepsilon(x) = \mathcal{M}_{Y_0}(\mathcal{T}_0^\varepsilon(u_0^\varepsilon)(x,y) - B^\varepsilon y)$$

*for a.e.* $(x,y) \in \Omega \times Y_0$*. Furthermore, there holds* $\mathcal{M}_{Y_0}(\hat{u}_0) = 0$ *and* $\mathcal{M}_{Y_0}(\nabla \times \hat{u}_0) = 0$*.*

*Proof.* Let $\{u^\varepsilon\}$ be a bounded sequence as in the assumption. The first part can be proven as in Theorem 5.2.4. Since

$$\begin{aligned}\mathcal{M}_{Y_0}(\partial_{y_i}(Z_0^\varepsilon)_j - \partial_{y_j}(Z_0^\varepsilon)_i) &= \frac{1}{\varepsilon}\mathcal{M}_{Y_0}(\partial_{y_i}\mathcal{T}_0^\varepsilon(u_0^\varepsilon)_j - b_{ji}^\varepsilon - \partial_{y_j}\mathcal{T}_0^\varepsilon(u_0^\varepsilon)_i + b_{ij}^\varepsilon)\\ &= \frac{1}{\varepsilon}(\mathcal{M}_{Y_0}(\partial_{y_i}\mathcal{T}_0^\varepsilon(u_0^\varepsilon)_j - \partial_{y_j}\mathcal{T}_0^\varepsilon(u_0^\varepsilon)_i) + 2b_{ij}^\varepsilon) = 0\end{aligned}$$

for $i,j \in \{1,2,3\}$ with $i \neq j$ and for all $\varepsilon$, we receive $\mathcal{M}_{Y_0}(\nabla \times \hat{u}_0) = 0$. □

We can split the weak limit of the third component of the sequence $\{Z_0^\varepsilon\}$ into a non-periodic and a 1-periodic part with respect to $y_3$.

**Lemma 7.2.8.** *With the same assumption as in Theorem 7.2.7, the third component of the sequence* $\{Z_0^\varepsilon\}$ *satisfies*

$$(Z_0^\varepsilon)_3 \rightharpoonup \tilde{u}_0 + \partial_3 c_3(x)\left(y_3 - \frac{1}{2}\right) \text{ weakly in } L^2(\Omega, H^1(Y_0))$$

*with* $\tilde{u}_0 \in L^2(\Omega, H^1(Y_0))$ *1-periodic with respect to* $y_3$*.*

*Proof.* We define the sequence $F_0^\varepsilon \coloneqq (Z_0^\varepsilon)_3 - \partial_3 c_3(x)(y_3 - \frac{1}{2})$, which is bounded in $L^2(\Omega, H^1(Y_0))$ due to Theorem 7.2.6 and Theorem 7.2.7. Thus, there exists a $\tilde{u}_0 \in L^2(\Omega, H^1(Y_0))$ such that

$$F_0^\varepsilon \rightharpoonup \tilde{u}_0 \text{ weakly in } L^2(\Omega, H^1(Y_0)).$$

We prove that $\tilde{u}_0$ is 1-periodic with respect to $y_3$. Let $\varphi \in \mathcal{D}(\Omega \times Y_0')$, then by change of variable

$$
\begin{aligned}
\int_{\Omega\times Y_0'} &\left(F_0^\varepsilon(x,(y',1)) - F_0^\varepsilon(x,(y',0))\right)\varphi(x,y')\,\mathrm{d}x\mathrm{d}y' \\
&= \int_{\Omega\times Y_0'} (\mathcal{T}_0^\varepsilon(u_0^\varepsilon))_3(x,(y',0))\frac{1}{\varepsilon}(\varphi(x-\varepsilon e_3,y') - \varphi(x,y'))\,\mathrm{d}x\mathrm{d}y' \\
&\quad - \int_{\Omega\times Y_0'} \frac{1}{\varepsilon}\left(B^\varepsilon(x)(y',1)^T - B^\varepsilon(x)(y',0)^T\right)_3 \varphi(x,y')\,\mathrm{d}x\mathrm{d}y' \\
&\quad - \int_{\Omega\times Y_0'} \partial_3 c_3(x)\varphi(x,y')\,\mathrm{d}x\mathrm{d}y'
\end{aligned}
$$

which converges to

$$
\begin{aligned}
\int_{\Omega\times Y_0'} &\left(\tilde{u}_0(x,(y',1)) - \tilde{u}_0(x,(y',0))\right)\varphi(x,y')\,\mathrm{d}x\mathrm{d}y' \\
&= -\int_{\Omega\times Y_0'} c_3(x)\partial_3\varphi(x,y')\,\mathrm{d}x\mathrm{d}y' - \int_{\Omega\times Y_0'} \partial_3 c_3(x)\varphi(x,y')\,\mathrm{d}x\mathrm{d}y' = 0,
\end{aligned}
$$

since $c_3$ is weakly differentiable with respect to $x_3$ and

$$
\left(B^\varepsilon(x)(y',1)^T - B^\varepsilon(x)(y',0)^T\right)_3 = \left(B^\varepsilon(x)(0,0,1)^T\right)_3 = 0
$$

due to the skew-symmety of $B^\varepsilon$. From the definition of $F_0^\varepsilon$ there follows

$$
(Z_0^\varepsilon)_3 \rightharpoonup \tilde{u}_0 + \partial_3 c_3(x)\left(y_3 - \frac{1}{2}\right) \text{ weakly in } L^2(\Omega, H^1(Y_0))
$$

with $\tilde{u}_0 \in L^2(\Omega, H^1(Y_0))$ 1-periodic with respect to $y_3$. □

We define the linear and continuous operator $\mathcal{M}^0_{(0,1)} : L^2(\Omega\times Y_0) \to L^2(\Omega\times Y_0')$,

$$
\mathcal{M}^0_{(0,1)}(u)(x,y') := \int_0^1 u(x,y',y_3)\,\mathrm{d}y_3.
$$

**Corollary 7.2.9.** *Let $\{u^\varepsilon\}$ be a sequence with $u^\varepsilon \in \mathcal{W}^\varepsilon_{\mathrm{m}}(\Omega)$ and*

$$
\|u^\varepsilon\|_{\mathcal{W}^\varepsilon_{\mathrm{m}}(\Omega)} \le C
$$

*for a constant $C$ independent of $\varepsilon$. Then, there exists, up to a subsequence, functions $b \in L^2(\Omega)$, $c \in \left[L^2(\Omega)\right]^3$ with $\partial_3 c_3 \in L^2(\Omega)$, $\hat{u}_0 \in \left[L^2(\Omega, H^1(Y_0))\right]^2$ and $\tilde{u}_0 \in L^2(\Omega, H^1(Y_0))$ 1-periodic with respect to $y_3$ such that*

$$
\mathcal{M}^0_{(0,1)}(\mathcal{T}_0^\varepsilon(u_0^\varepsilon)) \rightharpoonup b(x)(y_2,-y_1,0)^T + c(x)
$$

*weakly in* $\left[L^2(\Omega, H^1(Y_0'))\right]^3$ *and*

$$\mathcal{M}^0_{(0,1)}(\mathcal{T}^\varepsilon_0(e(u^\varepsilon_0))) \rightharpoonup \begin{pmatrix} \partial_{y_1}(\bar{u}_0)_1 & \frac{1}{2}(\partial_{y_1}(\bar{u}_0)_2 + \partial_{y_2}(\bar{u}_0)_1) & \frac{1}{2}(\partial_{y_1}(\bar{u}_0)_3 + \int_0^1 \partial_{y_3}(\hat{u}_0)_1 \,\mathrm{d}y_3) \\ * & \partial_{y_2}(\bar{u}_0)_2 & \frac{1}{2}(\partial_{y_2}(\bar{u}_0)_3 + \int_0^1 \partial_{y_3}(\hat{u}_0)_2 \,\mathrm{d}y_3) \\ sym & * & \partial_3 c_3 \end{pmatrix}$$

*weakly in* $\left[L^2(\Omega \times Y_0')\right]^{3\times 3}$*, whereby* $\mathcal{M}^0_{(0,1)}$ *is applied to every component and*

$$\bar{u}_0 \coloneqq \int_0^1 ((\hat{u}_0)_1, (\hat{u}_0)_2, \tilde{u}_0)^T \mathrm{d}y_3.$$

*Furthermore,* $\mathcal{M}_{Y_0'}(\bar{u}_0)(x) = 0$ *and* $\mathcal{M}_{Y_0'}(\partial_{y_1}(\bar{u}_0)_2 - \partial_{y_2}(\bar{u}_0)_1)(x) = 0$ *for almost all* $x \in \Omega$.

*Proof.* We apply the continuous operator $\mathcal{M}^0_{(0,1)}$ to the weak convergent sequences in Theorem 7.2.4 and Theorem 7.2.7, wherefore with Theorem 7.2.2 as in Theorem 7.2.1

$$\mathcal{M}^0_{(0,1)}(\mathcal{T}^\varepsilon_0(u^\varepsilon_0)) \rightharpoonup b(x)(y_2, -y_1, 0)^T + c(x)$$

weakly in $\left[L^2(\Omega, H^1(Y_0'))\right]^3$ and

$$\mathcal{M}^0_{(0,1)}(\mathcal{T}^\varepsilon_0(e(u^\varepsilon_0))) \rightharpoonup \mathcal{M}^0_{(0,1)}(e_y(\hat{u}_0))$$

weakly in $\left[L^2(\Omega \times Y_0')\right]^{3\times 3}$. By Lemma 7.2.8 the symmetric gradient of $\hat{u}_0$ is of the form

$$e_y(\hat{u}_0) = \begin{pmatrix} \partial_{y_1}(\hat{u}_0)_1 & \frac{1}{2}(\partial_{y_1}(\hat{u}_0)_2 + \partial_{y_2}(\hat{u}_0)_1) & \frac{1}{2}(\partial_{y_1}\tilde{u}_0 + \partial_{y_3}(\hat{u}_0)_1) \\ * & \partial_{y_2}(\hat{u}_0)_2 & \frac{1}{2}(\partial_{y_2}\tilde{u}_0 + \partial_{y_3}(\hat{u}_0)_2) \\ \text{sym} & * & \partial_{y_3}\tilde{u}_0 + \partial_3 c_3(x) \end{pmatrix}.$$

Using Theorem 7.2.2 and the 1-periodicity of $\tilde{u}_0$ with respect to $y_3$ we obtain

$$\mathcal{M}^0_{(0,1)}(e_y(\hat{u}_0)) = \begin{pmatrix} \partial_{y_1}(\bar{u}_0)_1 & \frac{1}{2}(\partial_{y_1}(\bar{u}_0)_2 + \partial_{y_2}(\bar{u}_0)_1) & \frac{1}{2}(\partial_{y_1}(\bar{u}_0)_3 + \int_0^1 \partial_{y_3}(\hat{u}_0)_1 \,\mathrm{d}y_3) \\ * & \partial_{y_2}(\bar{u}_0)_2 & \frac{1}{2}(\partial_{y_2}(\bar{u}_0)_3 + \int_0^1 \partial_{y_3}(\hat{u}_0)_2 \,\mathrm{d}y_3) \\ \text{sym} & * & \partial_3 c_3(x) \end{pmatrix}$$

with $\bar{u}_0 \coloneqq \int_0^1 ((\hat{u}_0)_1, (\hat{u}_0)_2, \tilde{u}_0)^T \mathrm{d}y_3$. Since

$$\int_0^1 (\hat{u}_0)_3 \,\mathrm{d}y_3 = \int_0^1 \tilde{u}_0 + \partial_3 c_3 \left(y_3 - \frac{1}{2}\right) \mathrm{d}y_3 = \int_0^1 \tilde{u}_0 \,\mathrm{d}y_3$$

there follows

$$0 = \mathcal{M}^0_{Y_0}(\hat{u}_0)(x) = \mathcal{M}^0_{Y_0'}(\bar{u}_0)(x)$$

for almost all $x \in \Omega$. Using again Theorem 7.2.2

$$0 = \mathcal{M}^0_{Y_0}(\partial_{y_1}(\hat{u}_0)_2 - \partial_{y_2}(\hat{u}_0)_1)(x) = \mathcal{M}^0_{Y_0'}(\partial_{y_1}(\bar{u}_0)_2 - \partial_{y_2}(\bar{u}_0)_1)(x)$$

for almost all $x \in \Omega$. □

### 7.2.2. Passage to the limit in the unidirectionally connected case

We apply the compactness results from the subsection above to derive the homogenized problem.

**Theorem 7.2.10.** *Let $\{u^\varepsilon\}$ be a sequence of weak solutions of the problems (4.0.3) with $u^\varepsilon \in \mathcal{W}^\varepsilon_{\mathrm{m}}(\Omega)$ and $\{f^\varepsilon\}$ a bounded sequence in $\left[L^2(\Omega)\right]^3$ such that*

$$\mathcal{T}^\varepsilon_\kappa(f^\varepsilon) \rightharpoonup f|_{\Omega\times Y_\kappa} \text{ weakly in } \left[L^2(\Omega\times Y_\kappa)\right]^3$$

*for some $f \in \left[L^2(\Omega\times Y)\right]^3$ and $\kappa \in \{0,1\}$. Then, there exist functions*

$$(u_1, \bar{u}_1, u_0) \in H^1_{\Gamma_{\mathrm{D}}}(\Omega) \times \left[L^2(\Omega, H^1_{\mathrm{per},0}(Y_1'))\right]^3 \times \left[L^2(\Omega, H^1(Y_0'))\right]^3,$$

*where*

$$u_0(x,y) = b(x)(y_2, -y_1, 0)^T + c(x)$$

*with $b \in L^2(\Omega)$ and $c \in \left[L^2(\Omega)\right]^3$ with $\partial_3 c_3 \in L^2(\Omega)$ and $c_3|_{\Gamma_{\mathrm{D}}} = 0$, and*

$$(\hat{u}_0, \tilde{u}_0) \in \left[L^2(\Omega, H^1(Y_0))\right]^2 \times L^2(\Omega, H^1(Y_0))$$

*with $\tilde{u}_0$ 1-periodic with respect to $y_3$, which satisfy, up to a subsequence,*

$$\begin{cases} \mathcal{M}^1_{(0,1)}(\mathcal{T}^\varepsilon_1(u^\varepsilon_1)) \rightharpoonup u_1 \text{ weakly in } \left[L^2(\Omega, H^1(Y_1'))\right]^3, \\ \mathcal{M}^1_{(0,1)}(\mathcal{T}^\varepsilon_1(e(u^\varepsilon_1))) \rightharpoonup e(u_1) + e_{y'}(\bar{u}_1) \text{ weakly in } \left[L^2(\Omega\times Y_1')\right]^{3\times 3}, \\ \mathcal{M}^0_{(0,1)}(\mathcal{T}^\varepsilon_0(u^\varepsilon_0)) \rightharpoonup u_0 \text{ weakly in } \left[L^2(\Omega, H^1(Y_0'))\right]^3, \end{cases} \tag{7.2.3}$$

*and*

$$\mathcal{M}^0_{(0,1)}(\mathcal{T}^\varepsilon_0(e(u^\varepsilon_0))) \rightharpoonup \begin{pmatrix} \partial_{y_1}(\bar{u}_0)_1 & \frac{1}{2}(\partial_{y_1}(\bar{u}_0)_2 + \partial_{y_2}(\bar{u}_0)_1) & \frac{1}{2}(\partial_{y_1}(\bar{u}_0)_3 + \int_0^1 \partial_{y_3}(\hat{u}_0)_1 \,\mathrm{d}y_3) \\ * & \partial_{y_2}(\bar{u}_0)_2 & \frac{1}{2}(\partial_{y_2}(\bar{u}_0)_3 + \int_0^1 \partial_{y_3}(\hat{u}_0)_2 \,\mathrm{d}y_3) \\ sym & * & \partial_3 c_3 \end{pmatrix}$$

*weakly in $\left[L^2(\Omega\times Y_0')\right]^{3\times 3}$, where $\bar{u}_0 \coloneqq \int_0^1((\hat{u}_0)_1, (\hat{u}_0)_2, \tilde{u}_0)^T \mathrm{d}y_3$ with $\mathcal{M}_{Y_0'}(\bar{u}_0)(x) = 0$ and $\mathcal{M}_{Y_0'}(\partial_{y_1}(\bar{u}_0)_2 - \partial_{y_2}(\bar{u}_0)_1)(x) = 0$ for almost all $x \in \Omega$.*

*Furthermore, $u = (u_1, \bar{u}_1, u_0, \hat{u}_0, \tilde{u}_0)$ is the solution of the problem*

$$\begin{aligned}
&\int_\Omega \int_{Y_1'} A(y')(e(u_1) + e_{y'}(\bar{u}_1))(e(v_1) + e_{y'}(\hat{v}_1))\,\mathrm{d}y'\mathrm{d}x \\
&\quad + \int_\Omega \int_{Y_0'} A(y') \left( \begin{pmatrix} \partial_{y_1}(\bar{u}_0)_1 & \frac{1}{2}(\partial_{y_1}(\bar{u}_0)_2 + \partial_{y_2}(\bar{u}_0)_1) & \frac{1}{2}(\partial_{y_1}(\bar{u}_0)_3 + \int_0^1 \partial_{y_3}(\hat{u}_0)_1\,\mathrm{d}y_3) \\ * & \partial_{y_2}(\bar{u}_0)_2 & \frac{1}{2}(\partial_{y_2}(\bar{u}_0)_3 + \int_0^1 \partial_{y_3}(\hat{u}_0)_2\,\mathrm{d}y_3) \\ sym & * & 0 \end{pmatrix} \right. \\
&\qquad \left. + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \partial_3 c_3 \end{pmatrix} \right) (e(v_0) + e_{y'}(\hat{v}_0))\,\mathrm{d}y'\mathrm{d}x \\
&\quad + \int_\Omega \begin{pmatrix} \int_{\Sigma_{Y'}} (K_\mathrm{N}\,[u_1' \cdot n' - u_0' \cdot n']\,n_1' + K_\mathrm{T}\,[u_1' \cdot \tau' - u_0' \cdot \tau']\,\tau_1')\,\mathrm{d}S(y') \\ \int_{\Sigma_{Y'}} (K_\mathrm{N}\,[u_1' \cdot n' - u_0' \cdot n']\,n_2' + K_\mathrm{T}\,[u_1' \cdot \tau' - u_0' \cdot \tau']\,\tau_2')\,\mathrm{d}S(y') \\ |\Sigma_{Y'}| K_\mathrm{T}((u_1)_3 - c_3) \end{pmatrix} \cdot (v_1 - v_0)\,\mathrm{d}x \\
&= \int_\Omega \int_{Y_1'} \bar{f}\,\mathrm{d}y' \cdot v_1\,\mathrm{d}x + \int_\Omega \int_{Y_0'} \bar{f}\,\mathrm{d}y' \cdot v_0\,\mathrm{d}x + \int_{\Gamma_\mathrm{N}} g \cdot v_1\,\mathrm{d}S(x)
\end{aligned} \tag{7.2.4}$$

*for all $v_0, v_1 \in H^1_{\Gamma_\mathrm{D}}(\Omega)$, $\hat{v}_1 \in \left[L^2(\Omega, H^1_{\mathrm{per},0}(Y_1'))\right]^3$ and $\hat{v}_0 \in \left[L^2(\Omega, H^1(Y_0'))\right]^3$, where $\bar{f} = \int_0^1 f \mathrm{d}y_3$, $u_1' = ((u_1)_1, (u_1)_2)$, $u_0' = b(x)(y_2, -y_1) + (c_1, c_2)$, $n'$ is the normal vector and $\tau'$ is the tangential vector of $\Sigma_{Y'}$.*

*Proof.* Let $\{u^\varepsilon\}$ be a sequence of weak solutions of problem (4.0.3) with $u^\varepsilon \in \mathcal{W}^\varepsilon_\mathrm{m}(\Omega)$. From Theorem 7.1.4 and Lemma 7.1.5 we get the uniform boundedness

$$\|u^\varepsilon\|^2_{[L^2(\Omega)]^3} + \|e(u_0^\varepsilon)\|^2_{\left[L^2(\Omega_0^\varepsilon)\right]^{3\times 3}} + \|e(u_1^\varepsilon)\|^2_{\left[L^2(\Omega_1^\varepsilon)\right]^{3\times 3}} \leq C.$$

Then, the convergences (7.2.3) follow directly from Theorem 7.2.1 and Corollary 7.2.9. We rewrite the weak formulation of problem (4.0.3) using Proposition 3.2.10 and Proposition 3.2.7 (i) and (ii) to receive the unfolded problem

$$\begin{aligned}
&\int_\Omega \int_{Y_1} \mathcal{T}_1^\varepsilon(A^\varepsilon)\mathcal{T}_1^\varepsilon(e(u_1^\varepsilon))\mathcal{T}_1^\varepsilon(e(\varphi_1))\,\mathrm{d}y\mathrm{d}x + \int_{\Pi^\varepsilon \cap \Omega_1^\varepsilon} A^\varepsilon e(u_1^\varepsilon) e(\varphi_1)\mathrm{d}x \\
&\quad + \int_\Omega \int_{Y_0} \mathcal{T}_0^\varepsilon(A^\varepsilon)\mathcal{T}_0^\varepsilon(e(u_0^\varepsilon))\mathcal{T}_0^\varepsilon(e(\varphi_0))\,\mathrm{d}y\mathrm{d}x \\
&\quad + \int_\Omega \int_{\Sigma_Y} (K_\mathrm{N}\,[\mathcal{T}_1^\varepsilon(u_1^\varepsilon \cdot n) - \mathcal{T}_0^\varepsilon(u_0^\varepsilon \cdot n)]\,n) \cdot (\mathcal{T}_1^\varepsilon(\varphi_1) - \mathcal{T}_0^\varepsilon(\varphi_0))\,\mathrm{d}S(y)\,\mathrm{d}x \\
&\quad + \int_\Omega \int_{\Sigma_Y} \left( K_\mathrm{T} \sum_{i=1}^2 \left[\mathcal{T}_1^\varepsilon(u_1^\varepsilon \cdot \tau^i) - \mathcal{T}_0^\varepsilon(u_0^\varepsilon \cdot \tau^i)\right] \tau^i \right) \cdot (\mathcal{T}_1^\varepsilon(\varphi_1) - \mathcal{T}_0^\varepsilon(\varphi_0))\,\mathrm{d}S(y)\,\mathrm{d}x \\
&= \int_\Omega \int_{Y_1} \mathcal{T}_1^\varepsilon(f^\varepsilon) \cdot \mathcal{T}_1^\varepsilon(\varphi_1)\,\mathrm{d}y\mathrm{d}x + \int_{\Pi^\varepsilon \cap \Omega_1^\varepsilon} f^\varepsilon \cdot \varphi_1\,\mathrm{d}x + \int_\Omega \int_{Y_0} \mathcal{T}_0^\varepsilon(f^\varepsilon) \cdot \mathcal{T}_0^\varepsilon(\varphi_0)\,\mathrm{d}y\mathrm{d}x \\
&\quad + \int_{\Gamma_\mathrm{N}} g \cdot \varphi_1\,\mathrm{d}S(x).
\end{aligned} \tag{7.2.5}$$

We define

$$\mathcal{D}_{\Gamma_{\mathrm{D}}}(\overline{\Omega}) := \{\phi \in C^\infty(\Omega) : v \text{ is equal to 0 in a neighbourhood of } \Gamma_{\mathrm{D}}\}.$$

Let $v_0, v_1 \in \left[\mathcal{D}_{\Gamma_{\mathrm{D}}}(\overline{\Omega})\right]^3$, $w_0, w_1 \in [\mathcal{D}(\Omega)]^3$ and

$$\psi_0^\varepsilon(x') := \psi_0\left(\frac{x'}{\varepsilon}\right) := \psi_0\left(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\right) \quad \text{and} \quad \psi_1^\varepsilon(x') := \psi_1\left(\frac{x'}{\varepsilon}\right) := \psi_1\left(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\right)$$

with $\psi_0 \in \left[H^1(Y_0')\right]^3$ and $\psi_1 \in \left[H^1_{\mathrm{per},0}(Y_1')\right]^3$ $Y'$-periodically extended. We choose as test functions

$$\varphi_0 = \varphi_0^\varepsilon := v_0 + \varepsilon \hat{v}_0^\varepsilon \quad \text{and} \quad \varphi_1 = \varphi_1^\varepsilon := v_1 + \varepsilon \hat{v}_1^\varepsilon$$

with $\hat{v}_0^\varepsilon(x) = \hat{v}_0(x, (\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}))$ and $\hat{v}_1^\varepsilon(x) = \hat{v}_1(x, (\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}))$, where

$$\hat{v}_0(x, y') = ((w_0)_i(x)(\psi_0)_i(y'))_{1\le i\le 3} \quad \text{and} \quad \hat{v}_1(x, y') = ((w_1)_i(x)(\psi_1)_i(y'))_{1\le i\le 3}.$$

Then, $\varphi^\varepsilon \in \mathcal{W}^\varepsilon_{\mathrm{m}}(\Omega)$,

$$\mathcal{T}_0^\varepsilon(\varphi_0^\varepsilon) \in \left[L^2(\Omega, H^1(Y_0))\right]^3 \text{ and } \mathcal{T}_1^\varepsilon(\varphi_1^\varepsilon) \in \left[L^2(\Omega, H^1(Y_1))\right]^3.$$

With the same arguments as in the proof of Theorem 5.2.7

$$\begin{aligned} \mathcal{T}_\kappa^\varepsilon(\varphi_\kappa^\varepsilon) &\to v_\kappa \text{ strongly in } \left[L^2(\Omega \times Y_\kappa)\right]^3, \\ \mathcal{T}_\kappa^\varepsilon(v_\kappa) &\to v_\kappa \text{ strongly in } \left[L^2(\Omega, H^1(Y_\kappa))\right]^3 \end{aligned}$$

for $\kappa \in \{0, 1\}$,

$$\int_{\Pi^\varepsilon \cap \Omega_1^\varepsilon} A^\varepsilon e(u_1^\varepsilon) e(\varphi_1^\varepsilon) \mathrm{d}x \to 0 \quad \text{and} \quad \int_{\Pi^\varepsilon \cap \Omega_1^\varepsilon} f^\varepsilon \cdot \varphi_1^\varepsilon \, \mathrm{d}x \to 0$$

as $\varepsilon \to 0$ and

$$\int_\Omega \int_{\Sigma_Y} \left(K_{\mathrm{N}} \left[\mathcal{T}_1^\varepsilon(u_1^\varepsilon \cdot n) - \mathcal{T}_0^\varepsilon(u_0^\varepsilon \cdot n)\right] n\right) \cdot \mathcal{T}_\kappa^\varepsilon(\varepsilon \hat{v}_\kappa^\varepsilon) \, \mathrm{d}S(y) \, \mathrm{d}x \to 0$$

resp. for $\tau_i$, $i \in \{1, 2\}$ instead of $n$. Every component of the symmetric gradient of $\varphi_\kappa^\varepsilon$ satisfies

$$\begin{aligned} e_{ij}(\varphi_\kappa^\varepsilon) = e_{ij}(v_\kappa) + \frac{1}{2}\Bigg[&\varepsilon \partial_{x_i}(w_\kappa)_j(x)(\psi_\kappa)_j\left(\frac{x'}{\varepsilon}\right) + (w_\kappa)_j(x)\partial_{y_i}(\psi_\kappa)_j\left(\frac{x'}{\varepsilon}\right) \\ &+ \varepsilon \partial_{x_j}(w_\kappa)_i(x)(\psi_\kappa)_i\left(\frac{x'}{\varepsilon}\right) + (w_\kappa)_i(x)\partial_{y_j}(\psi_\kappa)_i\left(\frac{x'}{\varepsilon}\right)\Bigg] \end{aligned}$$

for $i,j\in\{1,2\}$,

$$e_{i3}(\varphi_\kappa^\varepsilon)=e_{i3}(v_\kappa)+\frac{1}{2}\left[\varepsilon\partial_{x_i}(w_\kappa)_3(x)(\psi_\kappa)_3\left(\frac{x'}{\varepsilon}\right)+(w_\kappa)_3(x)\partial_{y_i}(\psi_\kappa)_3\left(\frac{x'}{\varepsilon}\right)\right.$$
$$\left.+\varepsilon\partial_{x_3}(w_\kappa)_i(x)(\psi_\kappa)_i\left(\frac{x'}{\varepsilon}\right)\right]$$

for $i\in\{1,2\}$ and

$$e_{33}(\varphi_\kappa^\varepsilon)=e_{33}(v_\kappa)+\varepsilon\partial_{x_3}(w_\kappa)_3(x)(\psi_\kappa)_3\left(\frac{x'}{\varepsilon}\right).$$

If we apply the periodic unfolding operator to $e_{ij}(\varphi_\kappa^\varepsilon)$ and use the properties from Proposition 3.2.7, we get

$$\mathcal{T}_\kappa^\varepsilon(e_{ij}(\varphi_\kappa^\varepsilon))\to e_{ij}(v_\kappa)+\left(e_{y'}(\hat{v}_\kappa)\right)_{ij}\ \text{ strongly in } L^2(\Omega\times Y_\kappa).$$

Thus,

$$\mathcal{T}_\kappa^\varepsilon(e(\varphi_\kappa^\varepsilon))\to e(v_\kappa)+e_{y'}(\hat{v}_\kappa)\text{ strongly in }\left[L^2(\Omega\times Y_\kappa)\right]^{3\times 3}.$$

If we plug in the test function in (7.2.5), use the weak convergence results from subsection 7.2.1 and pass to the limit, we get

$$\begin{aligned}
&\int_\Omega\int_{Y_1}A(y')(e(u_1)+e_y(\hat{u}_1))(e(v_1)+e_{y'}(\hat{v}_1))\,\mathrm{d}y\mathrm{d}x\\
&\quad+\int_\Omega\int_{Y_0}A(y')\left(\begin{pmatrix}\partial_{y_1}(\hat{u}_0)_1 & \frac{1}{2}(\partial_{y_1}(\hat{u}_0)_2+\partial_{y_2}(\hat{u}_0)_1) & \frac{1}{2}(\partial_{y_1}\tilde{u}_0+\partial_{y_3}(\hat{u}_0)_1)\\ * & \partial_{y_2}(\hat{u}_0)_2 & \frac{1}{2}(\partial_{y_2}\tilde{u}_0+\partial_{y_3}(\hat{u}_0)_2)\\ \text{sym} & * & \partial_{y_3}\tilde{u}_0\end{pmatrix}\right.\\
&\qquad\left.+\begin{pmatrix}0&0&0\\0&0&0\\0&0&\partial_3c_3\end{pmatrix}\right)(e(v_0)+e_{y'}(\hat{v}_0))\,\mathrm{d}y\mathrm{d}x\\
&\quad+\int_\Omega\int_{\Sigma_Y}\Big(K_\mathrm{N}\left[u_1\cdot n-(b(x)(y_2,-y_1,0)^T+c(x))\cdot n\right]n\\
&\qquad+K_\mathrm{T}\sum_{i=1}^{2}\left[u_1\cdot\tau^i-(b(x)(y_2,-y_1,0)^T+c(x))\cdot\tau^i\right]\tau^i\Big)\cdot(v_1-v_0)\,\mathrm{d}S(y)\,\mathrm{d}x\\
&=\int_\Omega\int_{Y_1}f\,\mathrm{d}y\cdot v_1\,\mathrm{d}x+\int_\Omega\int_{Y_0}f\,\mathrm{d}y\cdot v_0\,\mathrm{d}x+\int_{\Gamma_\mathrm{N}}g\cdot v_1\,\mathrm{d}S(x).
\end{aligned}\tag{7.2.6}$$

Due to the structure of $\Sigma_Y$ the normal vector and one of tangent vectors can be chosen independent of $y_3$, i.e. $n=(n_1',n_2',0)$, $\tau^1=(\tau_1',\tau_2',0)$ and $\tau^2=(0,0,1)$ with $n'=(n_1',n_2')$ and

$\tau' = (\tau'_1, \tau'_2)$ the normal and tangential vector of $\Sigma_{Y'}$. So the interface term is of the form

$$
\begin{aligned}
&\int_\Omega \int_{\Sigma_Y} \Big( K_\mathrm{N} \left[ u_1 \cdot n - (b(x)(y_2, -y_1, 0)^T + c(x)) \cdot n \right] n \\
&\qquad + K_\mathrm{T} \sum_{i=1}^{2} \left[ u_1 \cdot \tau^i - (b(x)(y_2, -y_1, 0)^T + c(x)) \cdot \tau^i \right] \tau^i \Big) \cdot (v_1 - v_0) \,\mathrm{d}S(y) \,\mathrm{d}x \\
&= \int_\Omega \begin{pmatrix} \int_{\Sigma_{Y'}} \left( K_\mathrm{N} \left[ u'_1 \cdot n' - u'_0 \cdot n' \right] n'_1 + K_\mathrm{T} \left[ u'_1 \cdot \tau' - u'_0 \cdot \tau' \right] \tau'_1 \right) \mathrm{d}S(y') \\ \int_{\Sigma_{Y'}} \left( K_\mathrm{N} \left[ u'_1 \cdot n' - u'_0 \cdot n' \right] n'_2 + K_\mathrm{T} \left[ u'_1 \cdot \tau' - u'_0 \cdot \tau' \right] \tau'_2 \right) \mathrm{d}S(y') \\ |\Sigma_{Y'}| K_\mathrm{T}((u_1)_3 - c_3) \end{pmatrix} \cdot (v_1 - v_0) \,\mathrm{d}x.
\end{aligned}
$$

Since most of the functions in (7.2.6) are independent of $y_3$, we can rewrite the problem to obtain (7.2.4) for all $v_0, v_1 \in H^1_{\Gamma_\mathrm{D}}(\Omega)$, $\hat{v}_1 \in \left[ L^2(\Omega, H^1_{\mathrm{per},0}(Y'_1)) \right]^3$ and $\hat{v}_0 \in \left[ L^2(\Omega, H^1(Y'_0)) \right]^3$, where we have used that $\left[ \mathcal{D}_{\Gamma_\mathrm{D}}(\bar{\Omega}) \right]^3$ is dense in $H^1_{\Gamma_\mathrm{D}}(\Omega)$ (by Theorem 3.1 from [Bernard, 2011]), $\mathcal{D}(\Omega) \times H^1_{\mathrm{per}}(Y'_1)$ is dense in $L^2(\Omega, H^1_{\mathrm{per}}(Y'_1))$ and $\mathcal{D}(\Omega) \times H^1(Y'_0)$ is dense in $L^2(\Omega, H^1(Y'_0))$. $\square$

If we assume that $A$ is isotropic, we can simplify the problem.

**Theorem 7.2.11.** *Additional to the assumption of Theorem 7.2.10, let A be isotropic. Then, $u = (u_1, u_0)$ is the solution of the problem*

$$
\begin{aligned}
&\int_\Omega A_1^{\mathrm{hom}} e(u_1) e(v_1) \,\mathrm{d}x + \int_\Omega A_0^{\mathrm{hom}} \partial_3 c_3 \partial_3 (v_0)_3 \,\mathrm{d}x \\
&\quad + \int_\Omega \begin{pmatrix} \int_{\Sigma_{Y'}} ((u_1)_1 - b y_2 - c_1) \zeta_1(n') + ((u_1)_2 + b y_1 - c_2) \zeta_2(n') \,\mathrm{d}S(y') \\ \int_{\Sigma_{Y'}} ((u_1)_1 - b y_2 - c_1) \zeta_2(n') + ((u_1)_2 + b y_1 - c_2) \zeta_3(n') \,\mathrm{d}S(y') \\ |\Sigma_{Y'}| K_\mathrm{T}((u_1)_3 - c_3) \end{pmatrix} \cdot (v_1 - v_0) \,\mathrm{d}x \\
&= \int_\Omega \int_{Y'_1} \bar{f} \,\mathrm{d}y' \cdot v_1 \,\mathrm{d}x + \int_\Omega \int_{Y'_0} \bar{f} \,\mathrm{d}y' \cdot v_0 \,\mathrm{d}x + \int_{\Gamma_\mathrm{N}} g \cdot v_1 \,\mathrm{d}S(x), \qquad (7.2.7)
\end{aligned}
$$

*for all $v_1 \in H^1_{\Gamma_\mathrm{D}}(\Omega)$, $v_0 \in \left[ L^2(\Omega) \right]^3$ with $\partial_3 (v_0)_3 \in L^2(\Omega)$ and $(v_0)_3|_{\Gamma_\mathrm{D}} = 0$, where*

$$
\begin{aligned}
\zeta_1(n') &= K_\mathrm{N}(n'_1)^2 + K_\mathrm{T}(n'_2)^2 = (K_\mathrm{N} - K_\mathrm{T})(n'_1)^2 + K_\mathrm{T}, \\
\zeta_2(n') &= (K_\mathrm{N} - K_\mathrm{T}) n'_1 n'_2, \\
\zeta_3(n') &= K_\mathrm{N}(n'_2)^2 + K_\mathrm{T}(n'_1)^2 = (K_\mathrm{N} - K_\mathrm{T})(n'_2)^2 + K_\mathrm{T}.
\end{aligned}
$$

*Furthermore,*

$$
A_0^{\mathrm{hom}} := \int_{Y'_0} a_{3333} - \frac{a_{1133}^2 a_{2222} + a_{1111} a_{2233}^2 - 2 a_{1122} a_{1133} a_{2233}}{a_{1111} a_{2222} - (a_{1122})^2} \,\mathrm{d}y'
$$

*and*

$$
(A_1^{\mathrm{hom}})_{ijkh} = \int_{Y'_1} a_{ijkh}(y') - \sum_{l,m=1}^{3} a_{ijlm}(y') \left( e_{y'}(\chi_1^{kh}) \right)_{lm} \mathrm{d}y',
$$

*where* $\chi_1^{lm} \in \left[H^1_{\mathrm{per},0}(Y_1')\right]^3$, $l,m \in \{1,2,3\}$, *is the weak solution of*

$$\int_{Y_1'} A(y')e_{y'}(\chi_1^{lm})e_{y'}(\hat{v}_1) - (Ae_{y'}(\hat{v}_1))_{lm}\,\mathrm{d}y' = 0$$

*for all* $\hat{v}_1 \in \left[H^1_{\mathrm{per},0}(Y_1')\right]^3$.

*Proof.* Since the normal and tangent vector of $\Sigma_{Y'}$ are orthogonal, there holds $n' = (n_1', n_2')$ and $\tau' = (n_2', -n_1')$. So the interface term is of the form

$$\begin{aligned}
&\int_\Omega \begin{pmatrix} \int_{\Sigma_{Y'}} \left(K_\mathrm{N}\left[u_1'\cdot n' - u_0'\cdot n'\right] n_1' + K_\mathrm{T}\left[u_1'\cdot\tau' - u_0'\cdot\tau'\right]\tau_1'\right)\,\mathrm{d}S(y') \\ \int_{\Sigma_{Y'}} \left(K_\mathrm{N}\left[u_1'\cdot n' - u_0'\cdot n'\right] n_2' + K_\mathrm{T}\left[u_1'\cdot\tau' - u_0'\cdot\tau'\right]\tau_2'\right)\,\mathrm{d}S(y') \\ |\Sigma_{Y'}|K_\mathrm{T}((u_1)_3 - c_3) \end{pmatrix} \cdot (v_1 - v_0)\,\mathrm{d}x \\
&= \int_\Omega \begin{pmatrix} \int_{\Sigma_{Y'}} ((u_1)_1 - by_2 - c_1)\zeta_1(n') + ((u_1)_2 + by_1 - c_2)\zeta_2(n')\,\mathrm{d}S(y') \\ \int_{\Sigma_{Y'}} ((u_1)_1 - by_2 - c_1)\zeta_2(n') + ((u_1)_2 + by_1 - c_2)\zeta_3(n')\,\mathrm{d}S(y') \\ |\Sigma_{Y'}|K_\mathrm{T}((u_1)_3 - c_3) \end{pmatrix} \cdot (v_1 - v_0)\,\mathrm{d}x
\end{aligned}$$

with

$$\begin{aligned}
\zeta_1(n') &= K_\mathrm{N}(n_1')^2 + K_\mathrm{T}(n_2')^2 = (K_\mathrm{N} - K_\mathrm{T})(n_1')^2 + K_\mathrm{T}, \\
\zeta_2(n') &= (K_\mathrm{N} - K_\mathrm{T})n_1'n_2', \\
\zeta_3(n') &= K_\mathrm{N}(n_2')^2 + K_\mathrm{T}(n_1')^2 = (K_\mathrm{N} - K_\mathrm{T})(n_2')^2 + K_\mathrm{T}.
\end{aligned}$$

If we choose $v_0 = v_1 = \hat{v}_1 = 0$ in (7.2.4), then

$$\begin{aligned}
\int_\Omega\int_{Y_0'} A(y') &\left( \begin{pmatrix} \partial_{y_1}(\bar{u}_0)_1 & \frac{1}{2}(\partial_{y_1}(\bar{u}_0)_2 + \partial_{y_2}(\bar{u}_0)_1) & \frac{1}{2}(\partial_{y_1}(\bar{u}_0)_3 + \int_0^1 \partial_{y_3}(\hat{u}_0)_1\,\mathrm{d}y_3) \\ * & \partial_{y_2}(\bar{u}_0)_2 & \frac{1}{2}(\partial_{y_2}(\bar{u}_0)_3 + \int_0^1 \partial_{y_3}(\hat{u}_0)_2\,\mathrm{d}y_3) \\ \mathrm{sym} & * & 0 \end{pmatrix} \right. \\
&\left. + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \partial_3 c_3 \end{pmatrix} \right) e_{y'}(\hat{v}_0)\,\mathrm{d}y'\mathrm{d}x = 0
\end{aligned}$$

for all $\hat{v}_0 \in [L^2(\Omega, H^1(Y_0'))]^3$. Using the isotropy of $A$, we can simplify the limit problem to

$$\begin{aligned}
\int_\Omega \int_{Y_0'} & (a_{1111}(y')\partial_{y_1}(\bar{u}_0)_1 + a_{1122}(y')\partial_{y_2}(\bar{u}_0)_2)\partial_{y_1}(\hat{v}_0)_1 \\
& + a_{1212}(y')(\partial_{y_1}(\bar{u}_0)_2 + \partial_{y_2}(\bar{u}_0)_1)(\partial_{y_1}(\hat{v}_0)_2 + \partial_{y_2}(\hat{v}_0)_1) \\
& + a_{1313}(y')\left(\partial_{y_1}(\bar{u}_0)_3 + \int_0^1 \partial_{y_3}(\hat{u}_0)_1 \,\mathrm{d}y_3\right)\partial_{y_1}(\hat{v}_0)_3 \\
& + (a_{1122}(y')\partial_{y_1}(\bar{u}_0)_1 + a_{2222}(y')\partial_{y_2}(\bar{u}_0)_2)\partial_{y_2}(\hat{v}_0)_2 \\
& + a_{2323}(y')\left(\partial_{y_2}(\bar{u}_0)_3 + \int_0^1 \partial_{y_3}(\hat{u}_0)_2 \,\mathrm{d}y_3\right)\partial_{y_2}(\hat{v}_0)_3 \,\mathrm{d}y'\mathrm{d}x \\
& = -\int_\Omega \int_{Y_0'} a_{1133}(y')\partial_3 c_3 \partial_{y_1}(\hat{v}_0)_1 + a_{2233}(y')\partial_3 c_3 \partial_{y_2}(\hat{v}_0)_2 \,\mathrm{d}y'\mathrm{d}x.
\end{aligned} \tag{7.2.8}$$

In this equation, only the symmetric gradient of the test functions, i.e. $e_{y'}(\hat{v}_0)$, is of interest. So we can restrict the test function space to $Z(\Omega, Y_0') := Z_1(\Omega, Y_0') \times Z_2(\Omega, Y_0')$ defined by

$$Z_1(\Omega, Y_0') := \{w \in [L^2(\Omega, H^1(Y_0'))]^2 : \mathcal{M}_{Y_0'}(w) = 0, \mathcal{M}_{Y_0'}(\partial_{y_1} w_2 - \partial_{y_2} w_1) = 0 \text{ f.a.a. } x \in \Omega\}$$

and

$$Z_2(\Omega, Y_0') := \{w \in L^2(\Omega, H^1(Y_0')) : \mathcal{M}_{Y_0'}(w) = 0 \text{ f.a.a. } x \in \Omega\}$$

equipped with the standard norms. $Z_1(Y_0')$ and $Z_2(Y_0')$ are as closed subspaces of $[L^2(\Omega, H^1(Y_0'))]^2$ resp. $L^2(\Omega, H^1(Y_0'))$ again Hilbert spaces. Clearly, functions, which only depend on $\Omega$ have no impact on the symmetric gradient with respect to $y$, so we can postulate that $\mathcal{M}_{Y_0'}(\hat{v}_0) = 0$. Assuming that there exists a function $v \in [L^2(\Omega, H^1(Y_0'))]^3$ with $\mathcal{M}_{Y_0'}(v) = 0$ and $e_{y'}(v) \neq e_{y'}(w)$ for all $w \in Z(\Omega, Y_0')$, then,

$$\mathcal{M}_{Y_0'}(\partial_{y_1} v_2 - \partial_{y_2} v_1) = \gamma(x)$$

for some function $\gamma \in L^2(\Omega) \setminus \{0\}$. Otherwise $v \in Z(\Omega, Y_0')$. If we define

$$w(x, y') := \begin{pmatrix} v_1(x, y') + \frac{\gamma(x)}{2}(y_2 - \mathcal{M}_{Y_0'}(y_2)) \\ v_2(x, y') - \frac{\gamma(x)}{2}(y_1 - \mathcal{M}_{Y_0'}(y_1)) \\ v_3(x, y') \end{pmatrix}$$

there holds $\mathcal{M}_{Y_0'}(w) = 0$ and

$$\mathcal{M}_{Y_0'}(\partial_{y_1} w_2 - \partial_{y_2} w_1) = \mathcal{M}_{Y_0'}\left(\partial_{y_1} v_2 - \frac{\gamma(x)}{2} - \left(\partial_{y_2} v_1 + \frac{\gamma(x)}{2}\right)\right) = 0,$$

i.e. $w \in Z(\Omega, Y_0')$. In addition $e_{y'}(w) = e_{y'}(v)$, which is a contradiction to the assumption. Furthermore, there holds $Z_1(\Omega, Y_0') \cap \mathcal{R} = \{0\}$ with

$$\mathcal{R} := \{A(x)y' + c(x) \in [L^2(\Omega, H^1(Y_0')]^2 : A \in [L^2(\Omega)]^{2\times 2} \text{ skew-symmetric}, c \in [L^2(\Omega)]^2\}$$

the space of rigid displacements with respect to $y'$. So by Theorem 2.5 from [Oleinik et al., 1992] there holds Korn's inequality in two dimensions, i.e.

$$\|v(x,\cdot)\|_{[H^1(Y_0')]^2} \leq C \left\| \begin{pmatrix} e_{11}^y(v)(x,\cdot) & e_{12}^y(v)(x,\cdot) \\ e_{12}^y(v)(x,\cdot) & e_{22}^y(v)(x,\cdot) \end{pmatrix} \right\|_{[L^2(Y_0')]^{2\times 2}}$$

for all $v \in Z_1(\Omega, Y_0')$ and almost all $x \in \Omega$ with $e_{ij}^y = \frac{1}{2}(\partial_{y_i} v_j + \partial_{y_j} v_i)$, $i,j \in \{1,2\}$, wherefore

$$\|v\|_{[L^2(\Omega,H^1(Y_0'))]^2} \leq C \left\| \begin{pmatrix} e_{11}^y(v) & e_{12}^y(v) \\ e_{12}^y(v) & e_{22}^y(v) \end{pmatrix} \right\|_{[L^2(\Omega\times Y_0')]^{2\times 2}}. \tag{7.2.9}$$

If we choose $(\hat{v}_0)_3 = 0$ in (7.2.8) the problem simplifies to

$$\begin{aligned} \int_\Omega \int_{Y_0'} &(a_{1111}(y')\partial_{y_1}(\bar{u}_0)_1 + a_{1122}(y')\partial_{y_2}(\bar{u}_0)_2)\partial_{y_1}(\hat{v}_0)_1 \\ &+ a_{1212}(y')(\partial_{y_1}(\bar{u}_0)_2 + \partial_{y_2}(\bar{u}_0)_1)(\partial_{y_1}(\hat{v}_0)_2 + \partial_{y_2}(\hat{v}_0)_1) \\ &+ (a_{1122}(y')\partial_{y_1}(\bar{u}_0)_1 + a_{2222}(y')\partial_{y_2}(\bar{u}_0)_2)\partial_{y_2}(\hat{v}_0)_2 \,\mathrm{d}y'\mathrm{d}x \\ &= -\int_\Omega \int_{Y_0'} a_{1133}(y')\partial_3 c_3(x)\partial_{y_1}(\hat{v}_0)_1 + a_{2233}(y')\partial_3 c_3(x)\partial_{y_2}(\hat{v}_0)_2 \,\mathrm{d}y'\mathrm{d}x \end{aligned}$$

for all $((\hat{v}_0)_1, (\hat{v}_0)_2) \in Z_1(\Omega, Y_0')$. We define this equation as

$$\tilde{a}(w,v) = \tilde{F}(v) \tag{7.2.10}$$

with $\tilde{a} : Z_1(\Omega, Y_0') \times Z_1(\Omega, Y_0') \to \mathbb{R}$,

$$\tilde{a}(w,v) = \int_\Omega \int_{Y_0'} A(y') \begin{pmatrix} e_{11}^y(w) & e_{12}^y(w) & 0 \\ e_{12}^y(w) & e_{22}^y(w) & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_{11}^y(v) & e_{12}^y(v) & 0 \\ e_{12}^y(v) & e_{22}^y(v) & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathrm{d}y'\mathrm{d}x$$

and $\tilde{F} : Z_1(\Omega, Y_0') \to \mathbb{R}$,

$$\tilde{F}(v) = -\int_\Omega \int_{Y_0'} a_{1133}(y')\partial_3 c_3(x)\partial_{y_1} v_1 + a_{2233}(y')\partial_3 c_3(x)\partial_{y_2} v_2 \,\mathrm{d}y'\mathrm{d}x.$$

Since $A \in M(\alpha, \beta, Y')$ we receive by (7.2.9)

$$\begin{aligned} \tilde{a}(w,w) &= \int_\Omega \int_{Y_0'} A(y') \begin{pmatrix} e_{11}^y(w) & e_{12}^y(w) & 0 \\ e_{12}^y(w) & e_{22}^y(w) & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_{11}^y(w) & e_{12}^y(w) & 0 \\ e_{12}^y(w) & e_{22}^y(w) & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathrm{d}y'\mathrm{d}x \\ &\geq \alpha \int_\Omega \int_{Y_0'} \sum_{i,j=1}^2 (e_{ij}^y(w))^2 \,\mathrm{d}y'\mathrm{d}x \geq C\|w\|^2_{[L^2(\Omega,H^1(Y_0'))]^2} \end{aligned}$$

and

$$\tilde{a}(w,v) \leq C\|w\|_{[L^2(\Omega,H^1(Y_0'))]^2}\|v\|_{[L^2(\Omega,H^1(Y_0'))]^2},$$

which shows that $\tilde{a}$ is a continuous and coercive bilinear form. Since $a_{ijkl} \in L^\infty(Y')$

$$\tilde{F}(v) \leq C\beta\|\partial_3 c_3\|_{L^2(\Omega)}(\|\partial_{y_1} v_1\|_{L^2(\Omega\times Y_0')} + \|\partial_{y_2} v_2\|_{L^2(\Omega\times Y_0')}) \leq C\|v\|_{[L^2(\Omega,H^1(Y_0'))]^2},$$

which yields that $\tilde{F}$ is linear and continuous. Due the theorem of Lax–Milgram there exists a unique solution $w \in Z_1(\Omega, Y_0')$ of (7.2.10). Since $\bar{u}_0$ also solves this equation there must hold $w = ((\bar{u}_0)_1, (\bar{u}_0)_2)^T$. It follows that

$$\begin{aligned}\begin{pmatrix}(\bar{u}_0)_1\\(\bar{u}_0)_2\end{pmatrix}(x,y') = &- \frac{\partial_3 c_3(x)}{a_{1111}a_{2222}-(a_{1122})^2}\begin{pmatrix}(a_{1133}a_{2222}-a_{1122}a_{2233})y_1\\(a_{1111}a_{2233}-a_{1122}a_{1133})y_2\end{pmatrix}\\&+ \mathcal{M}_{Y_0'}\left(\frac{\partial_3 c_3(x)}{a_{1111}a_{2222}-(a_{1122})^2}\begin{pmatrix}(a_{1133}a_{2222}-a_{1122}a_{2233})y_1\\(a_{1111}a_{2233}-a_{1122}a_{1133})y_2\end{pmatrix}\right).\end{aligned}$$

If we choose $(\hat{v}_0)_1 = (\hat{v}_0)_2 = 0$ in (7.2.8) the problem simplifies to

$$\begin{aligned}0 = \int_\Omega\int_{Y_0'} &a_{1313}(y')\left(\partial_{y_1}(\bar{u}_0)_3 + \int_0^1 \partial_{y_3}(\hat{u}_0)_1\,\mathrm{d}y_3\right)\partial_{y_1}(\hat{v}_0)_3\\&+ a_{2323}(y')\left(\partial_{y_2}(\bar{u}_0)_3 + \int_0^1 \partial_{y_3}(\hat{u}_0)_2\,\mathrm{d}y_3\right)\partial_{y_2}(\hat{v}_0)_3\,\mathrm{d}y'\mathrm{d}x\end{aligned} \tag{7.2.11}$$

for all $(\hat{v}_0)_3 \in Z_2(\Omega, Y_0')$. Let $(\hat{v}_0)_3(x,y') = \tilde{v}_0(x)(y_1 - \mathcal{M}_{Y_0'}(y_1))$ with $\tilde{v}_0 \in L^2(\Omega)$. Then (7.2.11) is of the form

$$0 = \int_\Omega\int_{Y_0'} a_{1313}(y')\left(\partial_{y_1}(\bar{u}_0)_3 + \int_0^1 \partial_{y_3}(\hat{u}_0)_1\,\mathrm{d}y_3\right)\mathrm{d}y'\,\tilde{v}_0\,\mathrm{d}x.$$

By the fundamental lemma of the calculus of variations

$$\int_{Y_0'} a_{1313}(y')\left(\partial_{y_1}(\bar{u}_0)_3 + \int_0^1 \partial_{y_3}(\hat{u}_0)_1\,\mathrm{d}y_3\right)\mathrm{d}y' = 0 \tag{7.2.12}$$

for almost every $x \in \Omega$. Analogously,

$$\int_{Y_0'} a_{2323}(y')\left(\partial_{y_2}(\bar{u}_0)_3 + \int_0^1 \partial_{y_3}(\hat{u}_0)_2\,\mathrm{d}y_3\right)\mathrm{d}y' = 0 \tag{7.2.13}$$

for almost every $x \in \Omega$. We plug in the representation of the solution $((\bar{u}_0)_1, (\bar{u}_0)_2)^T$ in the second integral of (7.2.4) and use (7.2.11), (7.2.12), (7.2.13)

$$
\begin{aligned}
&\int_\Omega \int_{Y_0'} -\partial_3 c_3(x) a_{1133}(y')(\partial_1 (v_0)_1 + \partial_{y_1}(\hat v_0)_1) - \partial_3 c_3(x) a_{2233}(y')(\partial_2 (v_0)_2 + \partial_{y_2}(\hat v_0)_2)\\
&\qquad + a_{1313}(y')\left(\partial_{y_1}(\bar u_0)_3 + \int_0^1 \partial_{y_3}(\hat u_0)_1 \,\mathrm{d}y_3\right)(\partial_1 (v_0)_3 + \partial_3 (v_0)_1 + \partial_{y_1}(\hat v_0)_3)\\
&\qquad + a_{2323}(y')\left(\partial_{y_2}(\bar u_0)_3 + \int_0^1 \partial_{y_3}(\hat u_0)_2 \,\mathrm{d}y_3\right)(\partial_2 (v_0)_3 + \partial_3 (v_0)_2 + \partial_{y_2}(\hat v_0)_3)\\
&\qquad + (a_{1133}(y')\partial_{y_1}(\bar u_0)_1 + a_{2233}(y')\partial_{y_2}(\bar u_0)_2)\partial_3 (v_0)_3\\
&\qquad + a_{1133}(y')\partial_3 c_3(\partial_1 (v_0)_1 + \partial_{y_1}(\hat v_0)_1) + a_{2233}(y')\partial_3 c_3(\partial_2 (v_0)_2 + \partial_{y_2}(\hat v_0)_2)\\
&\qquad + a_{3333}(y')\partial_3 c_3 \partial_3 (v_0)_3 \,\mathrm{d}y'\mathrm{d}x\\
&= \int_\Omega \int_{Y_0'} a_{1313}(y')\left(\partial_{y_1}(\bar u_0)_3 + \int_0^1 \partial_{y_3}(\hat u_0)_1 \,\mathrm{d}y_3\right)(\partial_1 (v_0)_3 + \partial_3 (v_0)_1 + \partial_{y_1}(\hat v_0)_3)\\
&\qquad + a_{2323}(y')\left(\partial_{y_2}(\bar u_0)_3 + \int_0^1 \partial_{y_3}(\hat u_0)_2 \,\mathrm{d}y_3\right)(\partial_2 (v_0)_3 + \partial_3 (v_0)_2 + \partial_{y_2}(\hat v_0)_3)\\
&\qquad - \frac{a_{1133}^2 a_{2222} + a_{1111} a_{2233}^2 - 2 a_{1122} a_{1133} a_{2233}}{a_{1111} a_{2222} - (a_{1122})^2} \partial_3 c_3 \partial_3 (v_0)_3\\
&\qquad + a_{3333}(y')\partial_3 c_3 \partial_3 (v_0)_3 \,\mathrm{d}y'\mathrm{d}x\\
&= \int_\Omega \int_{Y_0'} a_{1313}(y')\left(\partial_{y_1}(\bar u_0)_3 + \int_0^1 \partial_{y_3}(\hat u_0)_1 \,\mathrm{d}y_3\right)\partial_{y_1}(\hat v_0)_3\\
&\qquad + a_{2323}(y')\left(\partial_{y_2}(\bar u_0)_3 + \int_0^1 \partial_{y_3}(\hat u_0)_2 \,\mathrm{d}y_3\right)\partial_{y_2}(\hat v_0)_3 \,\mathrm{d}y'\mathrm{d}x + \int_\Omega A_0^{\mathrm{hom}}(y')\partial_3 c_3 \partial_3 (v_0)_3 \,\mathrm{d}x\\
&= \int_\Omega A_0^{\mathrm{hom}}(y')\partial_3 c_3 \partial_3 (v_0)_3 \,\mathrm{d}x,
\end{aligned}
$$

whereby

$$
A_0^{\mathrm{hom}} := \int_{Y_0'} a_{3333} - \frac{a_{1133}^2 a_{2222} + a_{1111} a_{2233}^2 - 2 a_{1122} a_{1133} a_{2233}}{a_{1111} a_{2222} - (a_{1122})^2} \,\mathrm{d}y'.
$$

Summing up, (7.2.4) can be rewritten as

$$
\begin{aligned}
&\int_\Omega \int_{Y_1'} A(y')(e(u_1) + e_{y'}(\bar u_1))(e(v_1) + e_{y'}(\hat v_1)) \,\mathrm{d}y'\mathrm{d}x + \int_\Omega A_0^{\mathrm{hom}} \partial_3 c_3 \partial_3 (v_0)_3 \,\mathrm{d}x\\
&\qquad + \int_\Omega \begin{pmatrix} \int_{\Sigma_{Y'}} ((u_1)_1 - b y_2 - c_1)\zeta_1(n) + ((u_1)_2 + b y_1 - c_2)\zeta_2(n) \,\mathrm{d}S(y') \\ \int_{\Sigma_{Y'}} ((u_1)_1 - b y_2 - c_1)\zeta_2(n) + ((u_1)_2 + b y_1 - c_2)\zeta_3(n) \,\mathrm{d}S(y') \\ |\Sigma_{Y'}| K_{\mathrm{T}}((u_1)_3 - c_3) \end{pmatrix} \cdot (v_1 - v_0) \,\mathrm{d}x\\
&= \int_\Omega \int_{Y_1'} \bar f \,\mathrm{d}y' \cdot v_1 \,\mathrm{d}x + \int_\Omega \int_{Y_0'} \bar f \,\mathrm{d}y' \cdot v_0 \,\mathrm{d}x + \int_{\Gamma_{\mathrm{N}}} g \cdot v_1 \,\mathrm{d}S(x)
\end{aligned}
$$

for all $v_1 \in H^1_{\Gamma_{\mathrm{D}}}(\Omega)$, $(v_0)_1, (v_0)_2 \in L^2(\Omega)$ and $(v_0)_3 \in H^1(\Omega)$ with $(v_0)_3|_{\Gamma_{\mathrm{D}}} = 0$ and $\hat v_1 \in$

$\left[L^2(\Omega, H^1_{\mathrm{per},0}(Y_1'))\right]^3$. Choosing $v_0 = v_1 = 0$, we get for all $\hat{v}_1 \in \left[L^2(\Omega, H^1_{\mathrm{per},0}(Y_1'))\right]^3$

$$\int_\Omega \int_{Y_1'} A(y')(e(u_1) + e_{y'}(\bar{u}_1))e_{y'}(\hat{v}_1)\,\mathrm{d}y'\mathrm{d}x = 0.$$

Thus,

$$\int_{Y_1'} A(y')\,(e(u_1)(x) + e_{y'}(\bar{u}_1)(x,y'))\,e_{y'}(\hat{v}_1)(y')\,\mathrm{d}y' = 0$$

for a.e. $x \in \Omega$ and $\hat{v}_1 \in \left[H^1_{\mathrm{per},0}(Y_1')\right]^3$. Due to Korn's inequality for periodic functions with mean value zero, the same estimate as in (7.2.9) is true. Thus, for all $v \in \left[H^1_{\mathrm{per},0}(Y_1')\right]^3$

$$\int_{Y_1'} A(y')e_{y'}(v)e_{y'}(v)\,\mathrm{d}y' \geq \alpha \int_{Y_1'} |e_{y'}(v)|^2\,\mathrm{d}y' \geq C\|\nabla_{y'}v\|^2_{[L^2(\Omega\times Y_1'))]^{3\times 2}} \geq C\|v\|^2_{[L^2(\Omega,H^1(Y_1'))]^3},$$

whereby we have used Poincaré inequality for the last estimate. Moreover,

$$\begin{aligned}\int_{Y_1'} A(y')e_{y'}(v)e_{y'}(w)\,\mathrm{d}y' &\leq \|A(y')e_{y'}(v)\|_{[L^2(Y_1')]^{3\times 3}}\|e_{y'}(w)\|_{[L^2(Y_1')]^{3\times 3}} \\ &\leq C\|v\|_{[L^2(\Omega,H^1(Y_1'))]^3}\|w\|_{[L^2(\Omega,H^1(Y_1'))]^3}\end{aligned}$$

for all $v, w \in \left[H^1_{\mathrm{per},0}(Y_1')\right]^3$ and

$$\begin{aligned}\int_{Y_1'} A(y')e(u_1)(x)e_{y'}(v)\,\mathrm{d}y' &\leq C|u_1(x)|\|A(y')e_{y'}(v)\|_{[L^2(Y_1')]^{3\times 3}} \\ &\leq C|u_1(x)|\|v\|_{[L^2(\Omega,H^1(Y_1'))]^3}.\end{aligned}$$

for all $v \in \left[H^1_{\mathrm{per},0}(Y_1')\right]^3$. So by the theorem of Lax–Milgram there exists a unique solution $\varphi \in \left[H^1_{\mathrm{per},0}(Y_1')\right]^3$ of

$$\int_{Y_1'} A(y')\,(e(u_1)(x) + e_{y'}(\varphi)(y'))\,e_{y'}(\hat{v}_1)(y')\,\mathrm{d}y' = 0 \tag{7.2.14}$$

for all $\hat{v}_1 \in \left[H^1_{\mathrm{per},0}(Y_1')\right]^3$ and a.e. $x \in \Omega$. We consider the cell problems: Find $\chi_1^{lm} \in \left[H^1_{\mathrm{per},0}(Y_1')\right]^3$, $l, m \in \{1,2,3\}$, such that

$$\int_{Y_1'} A(y')e_{y'}(\chi_1^{lm})e_{y'}(\hat{v}_1) - (Ae_{y'}(\hat{v}_1))_{lm}\,\mathrm{d}y' = 0$$

for all $\hat{v}_1 \in \left[H^1_{\mathrm{per},0}(Y_1')\right]^3$. With a similar proof as before we can apply the theorem of Lax–Milgram to obtain that there exists a unique solution $\chi_1^{lm}$ of the cell problem. If we plug in

$-\sum_{l,m=1}^{3} e_{lm}(u_1)(x)\chi_1^{lm}(y)$ for $\varphi$ in (7.2.14), we receive

$$\begin{aligned}\int_{Y_1'} A(y')\left[e(u_1)+e_{y'}(\varphi)\right] e_{y'}(\hat{v}_1)\mathrm{d}y \\ &= \int_{Y_1'}\left[A(y')e(u_1)-\sum_{l,m=1}^{3} e_{lm}(u_1)A(y')e_{y'}(\chi_1^{lm})\right] e_{y'}(\hat{v}_1)\,\mathrm{d}y' \\ &= \int_{Y_1'} A(y')e(u_1)e_{y'}(\hat{v}_1)-A(y')e(u_1)e_{y'}(\hat{v}_1)\,\mathrm{d}y'=0.\end{aligned}$$

Hence,

$$\bar{u}_1(x,y')=-\sum_{l,m=1}^{3} e_{lm}(u_1)(x)\chi_1^{lm}(y').$$

Using this equality we receive

$$\begin{aligned}\int_\Omega\int_{Y_1'} A(y')\left[e(u_1)+e_{y'}(\bar{u}_1)\right]\left[e(v_1)+e_{y'}(\hat{v}_1)\right]\mathrm{d}y'\mathrm{d}x \\ &= \int_\Omega\int_{Y_1'} A(y')\left[e(u_1)-\sum_{l,m=1}^{3} e_{lm}(u_1)e_{y'}(\chi_1^{lm})\right] e(v_1)\,\mathrm{d}y\mathrm{d}x \\ &= \int_\Omega \sum_{i,j,k,h=1}^{3}\left(\int_{Y_1'} a_{ijkh}(y')\,\mathrm{d}y'\right) e_{kh}(u_1)e_{ij}(v_1) \\ &\quad -\sum_{i,j,l,m=1}^{3}\left(\int_{Y_1'}\sum_{k,h=1}^{3} a_{ijkh}(y')\left(e_{y'}(\chi_1^{lm})\right)_{kh}\mathrm{d}y'\right) e_{lm}(u_1)e_{ij}(v_1)\mathrm{d}x.\end{aligned}$$

So the homogenized tensor $A_1^{\mathrm{hom}}$ is given by

$$(A_1^{\mathrm{hom}})_{ijkh}=\int_{Y_1'} a_{ijkh}(y')-\left(A(y')e_{y'}(\chi_1^{kh})\right)_{ij}\mathrm{d}y'$$

and the homogenized problem can be reformulated as the macroscopic problem

$$\begin{aligned}&\int_\Omega A_1^{\mathrm{hom}}e(u_1)e(v_1)\,\mathrm{d}y'\mathrm{d}x+\int_\Omega A_0^{\mathrm{hom}}\partial_3c_3\partial_3(v_0)_3\,\mathrm{d}x \\ &\quad+\int_\Omega\begin{pmatrix}\int_{\Sigma_{Y'}}((u_1)_1-by_2-c_1)\zeta_1(n')+((u_1)_2+by_1-c_2)\zeta_2(n')\,\mathrm{d}S(y') \\ \int_{\Sigma_{Y'}}((u_1)_1-by_2-c_1)\zeta_2(n')+((u_1)_2+by_1-c_2)\zeta_3(n')\,\mathrm{d}S(y') \\ |\Sigma_{Y'}|K_{\mathrm{T}}((u_1)_3-c_3)\end{pmatrix}\cdot(v_1-v_0)\,\mathrm{d}x \\ &=\int_\Omega\int_{Y_1'}\bar{f}\,\mathrm{d}y'\cdot v_1\,\mathrm{d}x+\int_\Omega\int_{Y_0'}\bar{f}\,\mathrm{d}y'\cdot v_0\,\mathrm{d}x+\int_{\Gamma_{\mathrm{N}}} g\cdot v_1\,\mathrm{d}S(x)\end{aligned}$$

for all $v_1\in H^1_{\Gamma_{\mathrm{D}}}(\Omega)$ and $v_0\in\left[L^2(\Omega)\right]^3$ with $\partial_3(v_0)_3\in L^2(\Omega)$ and $(v_0)_3|_{\Gamma_{\mathrm{D}}}=0$. $\square$

As in the disconnected case, a similar proof as in Theorem II.1.1 from [Oleinik et al., 1992] shows that there exist constants $\alpha^{\mathrm{hom}}, \beta^{\mathrm{hom}} \in \mathbb{R}$ with $0 < \alpha^{\mathrm{hom}} < \beta^{\mathrm{hom}}$ such that $A_1^{\mathrm{hom}} \in M(\alpha^{\mathrm{hom}}, \beta^{\mathrm{hom}}, \Omega)$.

Until now, we have all the results only up to a subsequence. We can prove the uniqueness of the solutions of the homogenized problem, if we have more information as for example in the case, where $\Sigma_{Y'}$ is an ellipse. Let $\Sigma_{Y'}$ be an ellipse with center $(z_1, z_2) \in (0,1)^2$ and half-axes $p, q \in (0, \frac{1}{2})$ such that the ellipse is completely contained in the unit cell $(0,1)^2$. We can parametrise the submanifold by

$$\Phi : [0, 2\pi) \to \Sigma_{Y'}, \quad \varphi \mapsto (p\cos\varphi + z_1, q\sin\varphi + z_2)$$

The normal vector is of the form

$$n' = \frac{1}{\sqrt{q^2\cos^2\varphi + p^2\sin^2\varphi}} \begin{pmatrix} q\cos\varphi \\ p\sin\varphi \end{pmatrix}.$$

By the integration rule for submanifolds,

$$\int_{\Sigma_{Y'}} h(y')\mathrm{d}S(y') = \int_0^{2\pi} h(\Phi(\varphi))\sqrt{\det(D\Phi^T D\Phi)}\,\mathrm{d}\varphi = \int_0^{2\pi} h(\Phi(\varphi))\sqrt{p^2\sin^2\varphi + q^2\cos^2\varphi}\,\mathrm{d}\varphi$$

for some function $h$, it follows that

$$\int_{\Sigma_{Y'}} n_1'n_2'\mathrm{d}S(y') = 0, \quad \int_{\Sigma_{Y'}} y_1 n_1'n_2'\mathrm{d}S(y') = 0, \quad \int_{\Sigma_{Y'}} y_2 n_1'n_2'\mathrm{d}S(y') = 0$$

and

$$\int_{\Sigma_{Y'}} y_i(n_1')^2\mathrm{d}S(y') = z_i\int_{\Sigma_{Y'}} (n_1')^2\mathrm{d}S(y'), \quad \int_{\Sigma_{Y'}} y_i(n_2')^2\mathrm{d}S(y') = z_i\int_{\Sigma_{Y'}} (n_2')^2\mathrm{d}S(y')$$

for $i \in \{1,2\}$. So the homogenized problem (7.2.7) simplifies to

$$\begin{aligned}
&\int_\Omega A_1^{\mathrm{hom}} e(u_1)e(v_1)\,\mathrm{d}y'\mathrm{d}x + \int_\Omega A_0^{\mathrm{hom}}\partial_3 c_3\partial_3(v_0)_3\,\mathrm{d}x \\
&\quad + \int_\Omega \begin{pmatrix} \left(K_\mathrm{N}\int_{\Sigma_{Y'}}(n_1')^2\mathrm{d}S(y') + K_\mathrm{T}\int_{\Sigma_{Y'}}(n_2')^2\mathrm{d}S(y')\right)((u_1)_1 - bz_2 - c_1) \\ \left(K_\mathrm{N}\int_{\Sigma_{Y'}}(n_2')^2\mathrm{d}S(y') + K_\mathrm{T}\int_{\Sigma_{Y'}}(n_1')^2\mathrm{d}S(y')\right)((u_1)_2 + bz_1 - c_2) \\ |\Sigma_{Y'}|K_\mathrm{T}((u_1)_3 - c_3) \end{pmatrix} \cdot (v_1 - v_0)\,\mathrm{d}x \\
&= \int_\Omega\int_{Y_1'} \bar{f}\,\mathrm{d}y' \cdot v_1\,\mathrm{d}x + \int_\Omega\int_{Y_0'} \bar{f}\,\mathrm{d}y' \cdot v_0\,\mathrm{d}x + \int_{\Gamma_\mathrm{N}} g\cdot v_1\,\mathrm{d}S(x). \qquad (7.2.15)
\end{aligned}$$

If we define $u_0 := (bz_2 + c_1, -bz_1 + c_2, c_3)$, we can show that the solution $u_0$ and $u_1$ are unique, but not the uniqueness of $b$ and $c$.

**Theorem 7.2.12.** *There exist unique solutions $u_0 \in \left[L^2(\Omega)\right]^3$ with $\partial_3(u_0)_3 \in L^2(\Omega)$ and*

$(u_0)_3|_{\Gamma_\mathrm{D}} = 0$ *and* $u_1 \in H^1_{\Gamma_\mathrm{D}}(\Omega)$ *of the problem* (7.2.15).

*Proof.* Let $u = (u_1, u_0), w = (w_1, w_0)$ be two solutions of the problem (7.2.15). We choose as test function the difference of both solutions, i.e. $v_1 \coloneqq u_1 - w_1$ and $v_0 \coloneqq u_0 - w_0$. Thus, we can estimate, using the coercivity of $A_1^{\mathrm{hom}}$, the fact that $A_0^{\mathrm{hom}} > 0$ and all constants of the interface term are greater than zero, and Korn's inequality for functions with zero on part of the boundary,

$$\begin{aligned}
0 =& \int_\Omega A_1^{\mathrm{hom}} e(u_1 - w_1) e(u_1 - w_1)\,\mathrm{d}x + \int_\Omega A_0^{\mathrm{hom}} \partial_3 (u_0 - w_0)_3 \partial_3 (u_0 - w_0)_3 \,\mathrm{d}x \\
&+ \int_\Omega \begin{pmatrix} \left(K_\mathrm{N} \int_{\Sigma_{Y'}} (n_1')^2 \mathrm{d}S(y') + K_\mathrm{T} \int_{\Sigma_{Y'}} (n_2')^2 \mathrm{d}S(y')\right) ((u_1 - w_1 - u_0 + w_0)_1) \\ \left(K_\mathrm{N} \int_{\Sigma_{Y'}} (n_2')^2 \mathrm{d}S(y') + K_\mathrm{T} \int_{\Sigma_{Y'}} (n_1')^2 \mathrm{d}S(y')\right) ((u_1 - w_1 - u_0 + w_0)_2) \\ |\Sigma_{Y'}| K_\mathrm{T}((u_1 - w_1 - u_0 + w_0)_3) \end{pmatrix} \\
&\qquad \cdot (u_1 - w_1 - u_0 + w_0)\,\mathrm{d}x \\
\geq& \alpha_1^{\mathrm{hom}} C \|u_1 - w_1\|^2_{[H^1(\Omega)]^3} + A_0^{\mathrm{hom}} \|\partial_3 (u_0 - w_0)_3\|^2_{L^2(\Omega)} + c \|u_1 - u_0 - w_1 + w_0\|^2_{[L^2(\Omega)]^3}.
\end{aligned}$$

Thus, $u_1 = w_1$ and $u_0 = w_0$. □

We get a similar result as in Theorem 7.2.11 if we use a more general elasticity tensor $A^\varepsilon$.

**Remark 7.2.13.** *As in the disconnected case (cf. Remark 5.2.10), we assume that* $A^\varepsilon \in M(\alpha, \beta, \Omega)$ *and* $\mathcal{T}^\varepsilon(A^\varepsilon) \to C$ *a.e. in* $\Omega \times Y$ *with* $C$ *independent of* $y_3$*. Then, the homogenized problem is of the same form as* (7.2.7) *but the homogenized tensors* $A_\kappa^{\mathrm{hom}}$*,* $\kappa \in \{0, 1\}$*, satisfy*

$$\begin{aligned}
A_0^{\mathrm{hom}}(x) = \int_{Y_0'} & c_{3333}(x, y') - \frac{1}{c_{1111}(x, y') c_{2222}(x, y') - (c_{1122}(x, y'))^2} \left(c_{1133}^2(x, y') c_{2222}(x, y')\right. \\
& \left. + c_{1111}(x, y') c_{2233}^2(x, y') - 2 c_{1122}(x, y') c_{1133}(x, y') c_{2233}(x, y')\right) \mathrm{d}y'
\end{aligned}$$

*and*

$$(A_1^{\mathrm{hom}})_{ijkh}(x) = \int_{Y_1'} c_{ijkh}(x, y') - \sum_{l,m=1}^{3} c_{ijlm}(x, y') \left(e_{y'}(\chi_1^{kh})\right)_{lm} (x, y')\,\mathrm{d}y'$$

*with* $\chi_\kappa^{lm} \in \left[L^\infty(\Omega, H^1_{\mathrm{per},0}(Y_1'))\right]^3$*,* $l, m \in \{1, 2, 3\}$*, is the unique weak solution of*

$$\int_{Y_1'} C(x, y') e_{y'}(\chi_1^{lm})(x, y') e_{y'}(\hat{v}_1)(y') - (C(x, y) e_{y'}(\hat{v}_1)(y'))_{lm}\,\mathrm{d}y' = 0$$

*for all* $\hat{v}_1 \in \left[H^1_{\mathrm{per},0}(Y_1')\right]^3$ *and a.e.* $x \in \Omega$.

# 8. Conclusion and outlook

After deriving the homogenized problems (5.2.7), (6.2.8), (7.2.7) in the disconnected (chapter 5), globally connected (chapter 6) and unidirectionally connected case (chapter 7), we are interested in the differences and similarities.
Since $\Omega_1^\varepsilon$ is globally connected for all $\varepsilon$ in all three cases, we were able to use the well-known extension operators to define functions on the whole domain $\Omega$. This allowed us to prove uniform boundedness and compactness results. Due to the similar structure of the domains the homogenized tensors $A_1^{\text{hom}}$ are of the same form in all three cases, namely

$$(A_1^{\text{hom}})_{ijkh} = \int_{Y_1} a_{ijkh}(y) - \sum_{l,m=1}^{3} a_{ijlm} \left(e_y(\chi_1^{kh})\right)_{lm} \mathrm{d}y$$

resp. as an integral over $Y_1'$ with $\chi_1^{lm}$, $l, m \in \{1, 2, 3\}$, the solutions of the associated cell problems in $Y_1$ resp. $Y_1'$. The differences in the upscaled problem arise due to the connectedness conditions of $\Omega_0^\varepsilon$. Although we could not estimate the gradient uniformly in $\Omega_0^\varepsilon$ in the disconnected and unidirectionally connected case, we were able to prove some compactness results using the weak boundedness of the material and that $\Omega_0^\varepsilon$ is fixed at one part of the outer boundary in the unidirectionally connected case. In the homogenized problem of the disconnected case, there is no contribution of any homogenized tensor $A_0^{\text{hom}}$ since there is no stress transmitted globally by the material of the disconnected domain. In contrast, in the connected domain the homogenized tensor $A_0^{\text{hom}}$ is of the same structure as $A_1^{\text{hom}}$, i.e. the stress is globally transmitted. The unidirectionally connected case can be seen as a mixture of the other two. So we have as in the connected case some macroscopic stress transmission but only in one direction. However, there are no further contributions as in the disconnected case, i.e. there is no macroscopic contribution of shear stresses and normal stresses in the disconnected direction. All three cases have in common that $u_0$ and $u_1$ are connected through the interface term. Although there are differences in the homogenized tensor $A_0^{\text{hom}}$ the density of the material does always play a role in all three cases as $f$ is a force density w.r.t. volume. If the normal and tangential stiffness $K_{\mathrm{N}}$ and $K_{\mathrm{T}}$ are the same, we obtain in the disconnected case the same homogenized problem as if we would assume that $\Omega_0^\varepsilon$ are holes without material. This rises from the fact that we allow jumps in deformations in normal and tangential directions at the interface. The different conditions at the exterior boundary follow from the different periodic structures of the domains.
In terms of the application to concretes reinforced with short carbon fibres, it is unlikely that the disconnected or connected case are useful models. As future work, we could study the more

general case of connectedness in one direction, where the slices in every plane are not equal, i.e. we allow $Y_0 \neq Y_0' \times (0,1)$. Furthermore, it may be of interest to compare simulation results of the derived homogenized problems with experimental data.

# Part II.

# Parameter identification for the linearized elasticity problem

To simulate the deformation of a two-scale composite of two materials under forces, the solution of the homogenized linear elasticity problem is often needed. If the material parameters, the microstructure and the volume and boundary forces are known, the computation of the direct problem is a classical result. It is of the form of the linear elasticity equation with an effective elasticity tensor, which is computed from solutions of auxiliary problems defined in the periodicity cell and may depend on the macroscopic variable. Another interesting aspect for application is to obtain information about the interior structure of a body without destroying it or using other tools like scanning electron microscope, which are often not feasible in practice. As in the direct problem, we consider a composite of two solids with periodic microstructure. If measured data of the deformation of the composite's exterior boundary under known forces is given, we prove results, using the methods of periodic homogenization and parameter identification, which allow us to compute the parameters characterizing the microscopic structure. The comparison with the original microstructure of the material may help to detect any changes in material, e.g. the extent of the (microscopic) damage (size of weak domains). Generally, computing the direct problem, i.e. finding the displacement field when the microstructure is known, is much easier (and less numerically costly) than the inverse problem, i.e. deducing the structure of the periodicity cell from the measurements of deformations on the boundary.
In this work, we consider a composite of two solids, which are perfectly bonded at the internal interface of the periodicity cell, whereby one component is completely contained in the cell and its geometry can be modelled by some finite vector of real parameters $\tau$. We derive results on the inverse problem to investigate the minimization problems

$$\operatorname*{arg\,min}_{\tau \in I_\eta} \frac{1}{2} \|u[\tau] - u_m\|^2_{[L^2(\partial\Omega)]^3} \quad \text{resp.} \quad \operatorname*{arg\,min}_{\tau \in I_\eta} \frac{1}{2} \|u[\tau] - u_m\|^2_{[L^2(S \times \partial\Omega)]^3}$$

for the stationary resp. time-dependent case, where the parameter $\tau \in I_\eta$, $I_\eta$ a compact set, describes the geometry of the inclusions in the periodicity cell, $u[\tau] : \Omega \to \mathbb{R}^3$ resp. $u[\tau] : S \times \Omega \to \mathbb{R}^3$ is the displacement field for some given $\tau$ and $u_m$ is the measured displacement.
Parameter identification problems in the context of shape optimization and homogenization are studied by several authors. In [Allaire et al., 2018], topology optimization in connection with homogenization of the linearized elasticity in two dimension is considered, where the microstructure consists of a cell with a rectangular central hole. The aim is to find the optimal length, width and rotation of the rectangle. In [Allaire et al., 2011], they investigate in a non-multiscale setting the damage evolution in linear elasticity via shape optimization, wherefore they have to compute the shape derivative. They handle the difficulty that the boundary of the damaged region moves and the full strain and stress tensors are not continuous through the interface. In [Michailidis, 2014, chapter 6.7], they study the linear elasticity equation together with some thermal stress tensor in the setting of inverse homogenization. They use the method of Céa in connection with a smoothed-interface instead of a sharp interface as we con-

sider here. A more application-oriented work is [Orlik et al., 2016], where textile-materials are optimized via homogenization and beam approximation. They prove results under the assumption that the homogenized tensor is constant and under different assumptions on the elasticity tensor. Apart from identifying the shape from measurements on the boundary, many authors are interested in finding the (microscopic) material parameters as e.g. in [Hartmann et al., 2021] and [Schmidt et al., 2015]. Apart from elasticity, there are several other applications, where parameter identification problems arise. For example in [Hintermueller and Laurain, 2008] the Electrical Impedance Tomography is considered, where the aim is to find the electrical conductivity and permittivity under special structural assumptions. Some general results in shape optimization by homogenization method can be found in [Allaire, 2002] and [Delfour and Zolésio, 2011] and in the theory of inverse problems in e.g. [Isakov, 1998] and [Kirsch, 2011].

Part II is structured as follows. In chapter 9 we first formulate the direct problem of the stationary linear elasticity equation (with perfect bond at the internal interface) and state the well-known existence result and the homogenized problem, whereby we prove some properties of the homogenized tensor. In the second part of this chaper, we study the inverse problem. We show that there exists at least one solution of the inverse problem and derive the Gâteaux derivative of the target functional to be able to apply gradient-based algorithm. In chapter 10, we consider the direct and inverse problem in the time-dependent case. After proving the existence of solution, we derive the homogenized problem via the periodic unfolding method. We use some results from the steady-state case to prove the existence of at least one solution of the inverse problem and to gain the Gâteaux derivation of the target functional. In chapter 11, we present some numerical experiments to showcast the functioning of the method. Chapter 12 summarises the results of part II.

Throughout this part, we denote by $C$ a constant independent of $\varepsilon$ whose value may change from line to line.

# 9. Parameter identification for the steady-state linearized elasticity problem

We are interested in identifying parameters describing the periodic microstructure of a two-scale composite of two solids. If measured data of the deformation on the exterior boundary under known forces and the materials of the composite are given, we prove that there exists at least one solution of minimization problem, which is a finite real vector describing the shape of the periodicity cell.

Concretely, in section 9.1 we consider the direct problem, i.e. we derive the homogenized problem of the steady-state linear elasticity problem by using the periodic unfolding method, and prove some properties of the homogenized tensor. We need these subsequently in section 9.2 to prove the existence of a solution of the inverse problem. Afterwards, we derive the Gâteaux derivative of the target function to be able to apply gradient-based optimization algorithm.

The results in this chapter have been published in the journal *Mathematical Methods in the Applied Sciences* [Lochner and Peter, 2022].

## 9.1. Statement of the direct problem

Let $\Omega$ be an open bounded connected Lipschitz-domain in $\mathbb{R}^3$, $\Gamma_{\mathrm{D}} \subset \partial\Omega$ closed with positive two-dimensional Hausdorff measure and $\Gamma_{\mathrm{N}} \coloneqq \partial\Omega \setminus \Gamma_{\mathrm{D}}$. Let $\nu$ be the outward-pointing normal to $\Gamma_{\mathrm{N}}$. We assume that the periodic microstructure can be described by the (scaled) reference cell $Y = (0, l_1) \times (0, l_2) \times (0, l_3) \subset \mathbb{R}^3$ with $l_1, l_2, l_3 > 0$.

### 9.1.1. Periodic and homogenized problem

We consider the linear elasticity equation in the steady-state case as introduced in section 2.1 to model the deformation of the domain $\Omega$ under body load $f$ and boundary force $g$

$$\begin{cases} -\nabla \cdot (A^\varepsilon e(u^\varepsilon)) = f & \text{in } \Omega, \\ u^\varepsilon = 0 & \text{on } \Gamma_{\mathrm{D}}, \\ (A^\varepsilon e(u^\varepsilon))\nu = g & \text{on } \Gamma_{\mathrm{N}}, \end{cases} \tag{9.1.1}$$

where $u^\varepsilon : \Omega \to \mathbb{R}^3$ is the displacement field and $A^\varepsilon$ the tensor of fourth order, which describes the stiffness of the material of the solid. The associated weak formulation is: Find $u^\varepsilon \in H^1_{\Gamma_\mathrm{D}}(\Omega) := \{u \in \left[H^1(\Omega)\right]^3 : u = 0 \text{ on } \Gamma_\mathrm{D}\}$ such that

$$\int_\Omega A^\varepsilon e(u^\varepsilon) e(v)\,\mathrm{d}x = \int_\Omega f \cdot v\,\mathrm{d}x + \int_{\Gamma_\mathrm{N}} g \cdot v\,\mathrm{d}S(x) \tag{9.1.2}$$

for all $v \in H^1_{\Gamma_\mathrm{D}}(\Omega)$. There exists a unique weak solution of this problem for every $\varepsilon$.

**Theorem 9.1.1.** *Let $A^\varepsilon \in M(\alpha, \beta, \Omega)$ (see Definition 2.1.1), $f \in \left[L^2(\Omega)\right]^3$ and $g \in \left[L^2(\Gamma_\mathrm{N})\right]^3$. Then, there exists a unique weak solution $u^\varepsilon \in H^1_{\Gamma_\mathrm{D}}(\Omega)$ of the problem* (9.1.2)*. Moreover, $u^\varepsilon$ is bounded in $H^1_{\Gamma_\mathrm{D}}(\Omega)$*

$$\|u^\varepsilon\|_{[H^1(\Omega)]^3} \leq \frac{C}{\alpha}\left(\|f\|_{[L^2(\Omega)]^3} + \|g\|_{[L^2(\Gamma_\mathrm{N})]^3}\right),$$

*where $C$ is a constant only depending on $\Omega$.*

*Proof.* The proof can be found in Theorem 10.6. in [Cioranescu and Donato, 1999], whereby we use additionally Korn's inequality to estimate the gradient by the symmetric gradient. □

We use the periodic unfolding operator from Definition 3.2.1 and the properties stated in Proposition 3.2.2 to derive the homogenized problem, whereby we assume that the unfolded tensor $\mathcal{T}^\varepsilon(A^\varepsilon)$ converges pointwise almost everywhere as $\varepsilon \to 0$.

**Theorem 9.1.2.** *Let $f \in \left[L^2(\Omega)\right]^3$, $g \in \left[L^2(\Gamma_\mathrm{N})\right]^3$, $\{A^\varepsilon\}$ a sequence of tensors in $M(\alpha, \beta, \Omega)$ and $\{u^\varepsilon\}$ the associated sequence of weak solutions of* (9.1.2)*. Suppose that*

$$B^\varepsilon := \mathcal{T}^\varepsilon(A^\varepsilon) \to B \text{ a.e. in } \Omega \times Y.$$

*Then, $B \in M(\alpha, \beta, \Omega \times Y)$ and there exists $u \in H^1_{\Gamma_\mathrm{D}}(\Omega)$ and $\hat{u} \in \left[L^2(\Omega, H^1_{\mathrm{per},0}(Y))\right]^3$ such that*

$$u^\varepsilon \to u \text{ strongly in } \left[L^2(\Omega)\right]^3, \tag{9.1.3}$$

$$\mathcal{T}^\varepsilon(u^\varepsilon) \rightharpoonup u \text{ weakly in } \left[L^2(\Omega, H^1(Y))\right]^3, \tag{9.1.4}$$

$$\mathcal{T}^\varepsilon(\nabla u^\varepsilon) \rightharpoonup \nabla u + \nabla_y \hat{u} \text{ weakly in } \left[L^2(\Omega \times Y)\right]^{3\times 3}. \tag{9.1.5}$$

*Furthermore, $(u, \hat{u})$ is the unique solution of*

$$\begin{aligned}\frac{1}{|Y|}\int_{\Omega\times Y} B(x,y)(e(u)(x) + e_y(\hat{u})(x,y))(e(v)(x) + e_y(\hat{v})(x,y))\,\mathrm{d}x\mathrm{d}y \\ = \int_\Omega f \cdot v\,\mathrm{d}x + \int_{\Gamma_\mathrm{N}} g \cdot v\,\mathrm{d}S(x)\end{aligned} \tag{9.1.6}$$

*for all $v \in H^1_{\Gamma_\mathrm{D}}(\Omega)$ and $\hat{v} \in \left[L^2(\Omega, H^1_{\mathrm{per},0}(Y))\right]^3$.*

*Proof.* From Satz IV.4.4 in [Werner, 2009] we get the measurability of $B$. We know that $A^\varepsilon \in M(\alpha, \beta, \Omega)$ for all $\varepsilon > 0$ and

$$\mathcal{T}^\varepsilon(A^\varepsilon)(x,y) = \begin{cases} A^\varepsilon(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y) & \text{for a.e. } (x,y) \in \Omega^\varepsilon \times Y, \\ 0 & \text{for a.e. } (x,y) \in \Pi^\varepsilon \times Y. \end{cases}$$

For a.e. $(x,y) \in \Omega \times Y$ there holds

$$\begin{aligned} b_{ijkh}(x,y) = \lim_{\varepsilon\to 0} b^\varepsilon_{ijkh}(x,y) &= \lim_{\varepsilon\to 0} \begin{cases} a^\varepsilon_{ijkh}(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y) & \text{if } (x,y) \in \Omega^\varepsilon \times Y \\ 0 & \text{if } (x,y) \in \Pi^\varepsilon \times Y \end{cases} \\ &= \lim_{\varepsilon\to 0} \begin{cases} a^\varepsilon_{jikh}(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y) & \text{if } (x,y) \in \Omega^\varepsilon \times Y \\ 0 & \text{if } (x,y) \in \Pi^\varepsilon \times Y \end{cases} \\ &= \lim_{\varepsilon\to 0} b^\varepsilon_{jikh}(x,y) = b_{jikh}(x,y). \end{aligned}$$

Analogously, $b_{ijkh} = b_{khij}$. Since $|\Pi^\varepsilon| \to 0$, there exists some $\varepsilon_0 > 0$ such that for all $\varepsilon \leq \varepsilon_0$ and all symmetric matrices $m$

$$\alpha|m|^2 \leq A^\varepsilon(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y)mm = \mathcal{T}^\varepsilon(A^\varepsilon)(x,y)mm \to B(x,y)mm,$$

which shows that $B$ is coercive. Furthermore, we get the boundedness

$$\beta|m| \geq |A^\varepsilon(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y)m| = |\mathcal{T}^\varepsilon(A^\varepsilon)(x,y)m| \to |B(x,y)m|$$

for all matrices $m$. Thus all entries of $B$ are in $L^\infty(\Omega \times Y)$. Summing up all results, we obtain that $B \in M(\alpha, \beta, \Omega \times Y)$. Due to Theorem 9.1.1 the solutions $u^\varepsilon$ are uniformly bounded in $\left[H^1(\Omega)\right]^3$ and so we can apply Proposition 3.2.2 (ix) to get the convergences (9.1.3)–(9.1.5). Choosing standard test functions in (9.1.2) and passing to the limit, we receive (9.1.6). Due to the $Y$-periodicity of $\hat{v} \in \left[L^2(\Omega, H^1_{\mathrm{per},0}(Y))\right]^3$ and Korn's inequality for functions with zero value on part of the boundary (see Theorem 2.2.5) and for periodic functions with zero mean value (see Corollary 2.2.6), we estimate for all $(v, \hat{v}) \in H^1_{\Gamma_\mathrm{D}}(\Omega) \times \left[L^2(\Omega, H^1_{\mathrm{per},0}(Y))\right]^3$

$$\begin{aligned} \|e(v) + e_y(\hat{v})\|^2_{[L^2(\Omega\times Y)]^{3\times 3}} &= \int_{\Omega\times Y} |e(v)|^2 + 2e(v) : e_y(\hat{v}) + |e_y(\hat{v})|^2 \mathrm{d}y\mathrm{d}x \\ &= |Y|\|e(v)\|^2_{[L^2(\Omega)]^{3\times 3}} + \|e_y(\hat{v})\|^2_{[L^2(\Omega\times Y)]^{3\times 3}} \\ &\geq C\left(|Y|\|v\|^2_{[H^1(\Omega)]^3} + \|\hat{v}\|^2_{[L^2(\Omega, H^1(Y))]^3}\right). \end{aligned}$$

Using this inequality and the fact that $B \in M(\alpha, \beta, \Omega\times Y)$, we get by Lax–Milgram theorem the existence and uniqueness of the weak solution of (9.1.6) in $H^1_{\Gamma_\mathrm{D}}(\Omega) \times \left[L^2(\Omega, H^1_{\mathrm{per},0}(Y))\right]^3$. □

In the next step, we want to split the homogenized problem into a macroscopic and a cell problem.

**Theorem 9.1.3.** *The homogenized problem* (9.1.6) *is of the form: Find* $u \in H^1_{\Gamma_{\mathrm{D}}}(\Omega)$ *such that*

$$\int_\Omega A^{\mathrm{hom}} e(u) e(v)\,\mathrm{d}x = \int_\Omega f \cdot v\,\mathrm{d}x + \int_{\Gamma_{\mathrm{N}}} g \cdot v\,\mathrm{d}S(x) \tag{9.1.7}$$

*for all* $v \in H^1_{\Gamma_{\mathrm{D}}}(\Omega)$*, where* $A^{\mathrm{hom}} = (a^{\mathrm{hom}}_{ijkl})_{1\le i,j,k,l\le 3}$ *with*

$$a^{\mathrm{hom}}_{ijkl}(x) = \frac{1}{|Y|}\int_Y B(x,y) e_{ij}(e_{kl} - e_y(w^{kl})(x,y))\,\mathrm{d}y \tag{9.1.8}$$

*for a.e.* $x \in \Omega$ *and* $w^{kl}$*,* $k,l \in \{1,2,3\}$*, is the unique solution in* $\left[L^\infty(\Omega, H^1_{\mathrm{per},0}(Y))\right]^3$ *of the cell problem*

$$\int_Y B(x,y)\left(e_y(w^{kl})(x,y) - e_{kl}\right) e(\varphi)\,\mathrm{d}y = 0 \tag{9.1.9}$$

*for all* $\varphi \in [H^1_{\mathrm{per},0}(Y)]^3$.

*Proof.* We consider the problem of finding $w^{kl} \in \left[L^2(\Omega, H^1_{\mathrm{per},0}(Y))\right]^3$ such that

$$\int_{\Omega\times Y} B(x,y) e_y(w^{kl})(x,y) e_y(v)(x,y)\,\mathrm{d}y\mathrm{d}x = \int_{\Omega\times Y} B(x,y) e_{kl} e_y(v)(x,y)\,\mathrm{d}y\mathrm{d}x$$

for all $v \in \left[L^2(\Omega, H^1_{\mathrm{per},0}(Y))\right]^3$. Since $B \in M(\alpha,\beta,\Omega\times Y)$ and by Korn's inequality for periodic functions with mean value zero, the left-hand side is a continuous coercive bilinear form. The right-hand side is a linear continuous functional, so we can apply the theorem of Lax–Milgram to receive the existence and uniqueness of the solution $w^{kl} \in \left[L^2(\Omega, H^1_{\mathrm{per},0}(Y))\right]^3$ of this problem. Furthermore, due to the fundamental lemma of the calculus of variations the solution has to satisfy

$$\int_Y B(x,y) e_y(w^{kl})(x,y) e_y(v)(y)\mathrm{d}y = \int_Y B(x,y) e_{kl} e_y(v)(y)\mathrm{d}y$$

for a.e. $x \in \Omega$ and $v \in \left[H^1_{\mathrm{per},0}(Y)\right]^3$. Using again that $B \in M(\alpha,\beta,\Omega\times Y)$ and Korn's inequality for periodic functions with mean value zero, we get that

$$\|\nabla_y(w^{kl})(x,\cdot)\|_{[L^2(Y)]^{3\times 3}} \le C$$

for a constant $C$ independent of $x$. Thus, $w^{kl} \in [L^\infty(\Omega, H^1_{\mathrm{per},0}(Y))]^3$. Choosing $v = 0$ in (9.1.6) yields

$$\frac{1}{|Y|}\int_{\Omega\times Y} B(x,y)(e(u)(x) + e_y(\hat{u})(x,y)) e_y(\hat{v})(x,y)\,\mathrm{d}x\mathrm{d}y = 0$$

for all $\hat{v} \in [L^2(\Omega; H^1_{\mathrm{per},0}(Y))]^3$. If we plug in $\hat{u} = -\sum_{l,m=1}^3 e_{lm}(u)(x)w^{lm}(x,y)$, we receive

$$\begin{aligned}
&\frac{1}{|Y|}\int_{\Omega\times Y}\left[B(x,y)e(u) - \sum_{l,m=1}^{3} e_{lm}(u)B(x,y)e_y(w^{lm})\right] e_y(\hat{v})\,\mathrm{d}y\mathrm{d}x \\
&\qquad = \frac{1}{|Y|}\int_{\Omega\times Y} B(x,y)e(u)e_y(\hat{v}) - B(x,y)e(u)e_y(\hat{v})\,\mathrm{d}y\mathrm{d}x = 0
\end{aligned}$$

because $w^{lm}$ solves the cell problem. Since $\hat{u}$ is unique, we can rewrite the limit problem (9.1.6)

$$\begin{aligned}
&\frac{1}{|Y|}\int_\Omega\int_Y B(x,y)\left[e(u) + e_y(\hat{u})\right]\left[e(v) + e_y(\hat{v})\right]\mathrm{d}y\mathrm{d}x \\
&= \int_\Omega \frac{1}{|Y|}\int_Y B(x,y)\left[e(u) - \sum_{l,m=1}^{3} e_{lm}(u)e_y(w^{lm})\right] e(v)\,\mathrm{d}y\mathrm{d}x \\
&= \int_\Omega \sum_{i,j,k,h=1}^{3}\left(\frac{1}{|Y|}\int_Y b_{ijkh}(x,y)\mathrm{d}y\right) e_{kh}(u)e_{ij}(v) \\
&\qquad - \sum_{i,j,l,m=1}^{3}\left(\frac{1}{|Y|}\int_Y \sum_{k,h=1}^{3} b_{ijkh}(y)\left(e_y(w^{lm})\right)_{kh}\mathrm{d}y\right) e_{lm}(u)e_{ij}(v)\,\mathrm{d}x.
\end{aligned}$$

So the homogenized tensor $A^{\mathrm{hom}}$ is given by (9.1.8) and the homogenized problem (9.1.6) can be reformulated into the macroscopic problem (9.1.7). $\square$

**Remark 9.1.4.** *The strong formulation of* (9.1.7) *is*

$$\begin{cases}
-\nabla\cdot(A^{\mathrm{hom}}e(u)) = f & \text{in } \Omega, \\
u = 0 & \text{on } \Gamma_{\mathrm{D}}, \\
(A^{\mathrm{hom}}e(u))\nu = g & \text{on } \Gamma_{\mathrm{N}}.
\end{cases}$$

*If $B$ is $Y$-periodic, we can write the cell problem* (9.1.9) *in the strong form*

$$\begin{cases}
-\nabla_y\cdot(B(x,\cdot)(e_y(w^{kl})(x,\cdot) - e_{kl})) = 0 \text{ in } Y, \\
w^{kl}(x,\cdot)\, Y\text{-periodic with } \mathcal{M}_Y(w^{kl}(x,\cdot)) = 0
\end{cases}$$

*for a.e. $x \in \Omega$.*

### 9.1.2. Properties of the homogenized tensor and homogenized problem

We want to show that the homogenized tensor $A^{\mathrm{hom}}$ (see (9.1.8)) is under an additional assumption in the set $M(\alpha, \frac{\beta^2}{\alpha}, \Omega)$. First, we compute the boundedness of the cell solutions.

**Lemma 9.1.5.** *The solution $w^{kh} \in \left[L^\infty(\Omega, H^1_{\mathrm{per},0}(Y))\right]^3$ of the cell problem* (9.1.9) *is bounded* *as follows*

$$\|e_y(w^{kh})\|_{[L^\infty(\Omega,L^2(Y))]^{3\times 3}} \leq \frac{\beta|Y|^{1/2}}{\alpha}.$$

*Proof.* We use that $B(x,y) \in M(\alpha,\beta,\Omega\times Y)$ to estimate

$$\begin{aligned}
\alpha\|e_y(w^{kh}(x,\cdot))\|^2_{[L^2(Y)]^{3\times 3}} &\leq \int_Y B(x,y)e_y(w^{kh})e_y(w^{kh})\,\mathrm{d}y = \int_Y B(x,y)e_{kh}e_y(w^{kh})\,\mathrm{d}y \\
&\leq |Y|^{1/2}\left(\int_Y |\sum_{i,j=1}^{3} b_{ijkh}e^y_{ij}(w^{kh})|^2\mathrm{d}y\right)^{1/2} \\
&\leq |Y|^{1/2}\left(\int_Y |B(x,y)e_y(w^{kh})|^2\mathrm{d}y\right)^{1/2} \\
&\leq |Y|^{1/2}\beta\|e_y(w^{kh})(x,\cdot)\|_{[L^2(Y)]^{3\times 3}}
\end{aligned}$$

for a.e. $x\in\Omega$. □

We need some auxiliary lemmas.

**Lemma 9.1.6.** *Let $v \in \left[H^1_{\mathrm{per},0}(Y)\right]^3$. Then, $v$ can be extended $Y$-periodically to an element of $\left[H^1_{\mathrm{loc}}(\mathbb{R}^3)\right]^3$.*

*Proof.* Let $v \in \left[H^1_{\mathrm{per},0}(Y)\right]^3$. Clearly, we can extend the function $Y$-periodically, which we denote by $\tilde{v}$. It remains to prove that $v \in [H^1_{\mathrm{loc}}(\mathbb{R}^3)]^3$. Therefore, let $K$ be a compact subset of $\mathbb{R}^3$. We define the sets

$$\mathcal{Z}(K) := \{\xi \in \mathbb{R}^3 : \xi = (l_1\tilde{\xi}_1, l_2\tilde{\xi}_2, l_3\tilde{\xi}_3) \text{ for some } \tilde{\xi}\in\mathbb{Z}^3, K\cap(Y+\xi)\neq\emptyset\}$$

and

$$\tilde{K} := \bigcup_{\xi\in\mathcal{Z}}(Y+\xi).$$

Thus, $\mathcal{Z}(K)$ consists of finitely many elements and $K\subset\tilde{K}$. Then, using the transformation formula

$$\begin{aligned}
\|\tilde{v}\|^2_{[L^2(K)]^3} &= \int_K |\tilde{v}(y)|^2\mathrm{d}y \leq \int_{\tilde{K}} |\tilde{v}(y)|^2\mathrm{d}y = \sum_{\xi\in\mathcal{Z}(K)}\int_{Y+\xi}|\tilde{v}(y)|^2\mathrm{d}y = \sum_{\xi\in\mathcal{Z}(K)}\int_Y |\tilde{v}(y+\xi)|^2\mathrm{d}y \\
&= \sum_{\xi\in\mathcal{Z}(K)}\int_Y |v(y)|^2\mathrm{d}y = \|v\|^2_{[L^2(Y)]^3}\sum_{\xi\in\mathcal{Z}(K)}\frac{1}{|Y|}\int_Y \mathrm{d}y = \frac{|\tilde{K}|}{|Y|}\|v\|^2_{[L^2(Y)]^3}
\end{aligned}$$

and, analogously,

$$\|\nabla\tilde{v}\|^2_{[L^2(K)]^{3\times 3}} = \int_K |\nabla\tilde{v}(y)|^2\mathrm{d}y \leq \int_{\tilde{K}} |\nabla\tilde{v}(y)|^2\mathrm{d}y = \frac{|\tilde{K}|}{|Y|}\|\nabla v\|^2_{[L^2(Y)]^{3\times 3}}.$$

Since $K$ was arbitrary, we get the desired result. □

**Lemma 9.1.7.** *Let* $v \in \left[L^2(Y)\right]^{3\times 3}$ *with*

$$\int_Y v : \nabla\varphi \,\mathrm{d}y = 0$$

*for all* $\varphi \in \left[H^1_{\mathrm{per},0}(Y)\right]^3$*. Then,* $v$ *can be extended* $Y$*-periodically to an element of* $\left[L^2_{\mathrm{loc}}(\mathbb{R}^3)\right]^3$*, denoted again by* $v$*, such that* $-\nabla\cdot v = 0$ *in* $\left[\mathcal{D}'(\mathbb{R}^3)\right]^3$*.*

*Proof.* Let $v \in \left[L^2(Y)\right]^{3\times 3}$ satisfy $\int_Y v : \nabla\varphi \,\mathrm{d}y = 0$ for all $\varphi \in [H^1_{\mathrm{per},0}(Y)]^3$. Then, in the sense of distributions

$$-\int_Y (\nabla\cdot v)\cdot\varphi \,\mathrm{d}y = \int_Y v : \nabla\varphi \,\mathrm{d}y = 0 = \int_Y 0\cdot\varphi \,\mathrm{d}y$$

for all $\varphi \in [C_c^\infty(Y)]^3$, wherefore the distributional derivate fulfills $-\nabla\cdot v = 0 \in [L^2(Y)]^3$. So $v$ is an element of the space $H(Y,\mathrm{div}) \coloneqq \{w \in \left[L^2(Y)\right]^{3\times 3} : \nabla\cdot w \in \left[L^2(Y)\right]^3\}$. Using Prop. 3.47 (ii) from [Cioranescu and Donato, 1999], there holds for all $\phi \in \left[H^1(Y)\right]^3$

$$-\int_Y (\nabla\cdot v)\cdot\phi \,\mathrm{d}y = \int_Y v : \nabla\phi \,\mathrm{d}y + \langle vn, \phi\rangle_{\left[H^{-1/2}(\partial Y)\right]^3, \left[H^{1/2}(\partial Y)\right]^3}$$

where $n$ is the normal of $\partial Y$. With the results from above, we get for all $\varphi \in \left[H^1_{\mathrm{per},0}(Y)\right]^3$

$$0 = \langle vn, \varphi\rangle_{\left[H^{-1/2}(\partial Y)\right]^3, \left[H^{1/2}(\partial Y)\right]^3},$$

which proves that $v$ is $Y$-periodic. With Lemma 9.1.6 we can extend $v$ $Y$-periodically, again denoted by $v$, such that $v \in \left[L^2_{\mathrm{loc}}(\mathbb{R}^3)\right]^3$. It remains to prove that $-\nabla\cdot v = 0$ in $\left[\mathcal{D}'(\mathbb{R}^3)\right]^3$. Let $\varphi \in \left[C_c^\infty(\mathbb{R}^3)\right]^3$. Due to the compact support, there exists a bounded set $K \subset \mathbb{R}^3$ such that $\varphi \in [C_0^\infty(K)]^3$ and $K = \bigcup_{\xi\in\mathcal{Z}}(Y+\xi)$ for some set $\mathcal{Z} \subset \mathbb{R}^3$ with finitely many elements, which satisfy $(Y+\xi)\cap(Y+\tilde{\xi}) = \emptyset$ for all $\xi \neq \tilde{\xi} \in \mathcal{Z}$. So we get

$$\begin{aligned}
-\int_{\mathbb{R}^3} (\nabla\cdot v)\cdot\varphi \,\mathrm{d}x &= \int_K v : \nabla\varphi \,\mathrm{d}x = \sum_{\xi\in\mathcal{Z}} \int_{Y+\xi} v : \nabla\varphi \,\mathrm{d}x = \sum_{\xi\in\mathcal{Z}} \int_Y v(y) : \nabla\varphi(y+\xi) \,\mathrm{d}y \\
&= \sum_{\xi\in\mathcal{Z}} \int_Y -(\nabla\cdot v(y))\cdot\varphi(y+\xi)\,\mathrm{d}y + \langle v(\cdot)n, \varphi(\cdot+\xi)\rangle_{\left[H^{-1/2}(\partial Y)\right]^3, \left[H^{1/2}(\partial Y)\right]^3} \\
&= \sum_{\xi\in\mathcal{Z}} \langle v(\cdot)n, \varphi(\cdot+\xi)\rangle_{\left[H^{-1/2}(\partial Y)\right]^3, \left[H^{1/2}(\partial Y)\right]^3}.
\end{aligned}$$

The sum disappears since $v$ is periodic and either $\varphi$ is continuous on $\partial Y + \xi$ or already zero. $\square$

Let $A \in M(\alpha, \beta, Y)$ and $m \in \mathbb{R}^{3\times 3}$ be a symmetric matrix. We use the Voigt notation to rewrite the tensor of fourth order as a $6\times 6$ matrix and the symmetric matrix as a vector of

$\mathbb{R}^6$, i.e.

$$A^{\mathrm{V}} = \begin{pmatrix} a_{1111} & a_{1122} & a_{1133} & a_{1123} & a_{1113} & a_{1112} \\ a_{1122} & a_{2222} & a_{2233} & a_{2223} & a_{2213} & a_{2212} \\ a_{1133} & a_{2233} & a_{3333} & a_{3323} & a_{3313} & a_{3312} \\ a_{1123} & a_{2223} & a_{3323} & a_{2323} & a_{2313} & a_{2312} \\ a_{1113} & a_{2213} & a_{3313} & a_{2313} & a_{1313} & a_{1312} \\ a_{1112} & a_{2212} & a_{3312} & a_{2312} & a_{1312} & a_{1212} \end{pmatrix}, \quad m^{\mathrm{V}} = \begin{pmatrix} m_{11} \\ m_{22} \\ m_{33} \\ 2m_{23} \\ 2m_{13} \\ 2m_{12} \end{pmatrix}.$$

**Lemma 9.1.8.** *Let $A \in M(\alpha, \beta, Y)$. Then, the inverse of $A^{\mathrm{V}}$ exists and is symmetric. Furthermore, there holds*

$$A^{-1}w : w \geq \frac{\alpha}{\beta^2}|w|^2 \tag{9.1.10}$$

*for all symmetric matrices $w \in \mathbb{R}^{3\times 3}$, where $A^{-1}w$ is defined by*

$$\begin{pmatrix} ((A^{\mathrm{V}})^{-1}w^{\mathrm{V}})_1 & \frac{1}{2}((A^{\mathrm{V}})^{-1}w^{\mathrm{V}})_5 & \frac{1}{2}((A^{\mathrm{V}})^{-1}w^{\mathrm{V}})_6 \\ \frac{1}{2}((A^{\mathrm{V}})^{-1}w^{\mathrm{V}})_5 & ((A^{\mathrm{V}})^{-1}w^{\mathrm{V}})_2 & \frac{1}{2}((A^{\mathrm{V}})^{-1}w^{\mathrm{V}})_4 \\ \frac{1}{2}((A^{\mathrm{V}})^{-1}w^{\mathrm{V}})_6 & \frac{1}{2}((A^{\mathrm{V}})^{-1}w^{\mathrm{V}})_4 & ((A^{\mathrm{V}})^{-1}w^{\mathrm{V}})_3 \end{pmatrix}. \tag{9.1.11}$$

*Proof.* Let $A \in M(\alpha, \beta, Y)$ and $\lambda^{\mathrm{V}} \in \mathbb{R}^6$. Then, the associated matrix

$$\lambda := \begin{pmatrix} \lambda_1^{\mathrm{V}} & \frac{1}{2}\lambda_5^{\mathrm{V}} & \frac{1}{2}\lambda_6^{\mathrm{V}} \\ \frac{1}{2}\lambda_5^{\mathrm{V}} & \lambda_2^{\mathrm{V}} & \frac{1}{2}\lambda_4^{\mathrm{V}} \\ \frac{1}{2}\lambda_6^{\mathrm{V}} & \frac{1}{2}\lambda_4^{\mathrm{V}} & \lambda_3^{\mathrm{V}} \end{pmatrix}.$$

is symmetric and we can use that $A \in M(\alpha, \beta, Y)$ to estimate

$$(A^{\mathrm{V}}\lambda^{\mathrm{V}}, \lambda^{\mathrm{V}}) = A\lambda\lambda \geq \alpha|\lambda|^2 \geq \frac{\alpha}{2}|\lambda^{\mathrm{V}}|^2,$$

where $(\cdot,\cdot)$ denotes the standard scalar product. This shows that $A^{\mathrm{V}}$ is positive definite and due to the definition symmetric. Thus, the inverse of $A^{\mathrm{V}}$ exists and is symmetric. To prove the inequality (9.1.10), we follow the proof of Proposition 8.3 in [Cioranescu and Donato, 1999], where the same result is shown for tensors of second order. Let $w \in \mathbb{R}^{3\times 3}$ be a symmetric matrix and $m = A^{-1}w$ defined as in (9.1.11). Then, $Am = w$ and because $A \in M(\alpha, \beta, Y)$

$$A^{-1}w : w = Amm \geq \alpha|m|^2 = \alpha|A^{-1}w|^2. \tag{9.1.12}$$

Since $A$ is linear operator, we can estimate the operator norm

$$\|A\| = \sup_{\mathbb{R}^{3\times 3} \ni u \neq 0} \frac{|Au|}{|u|} \leq \frac{\beta|u|}{|u|} = \beta.$$

Thus,

$$|w| = |Am| \leq |m|\|A\| \leq \beta|A^{-1}w|.$$

Together with estimate (9.1.12), we get

$$A^{-1}w : w \geq \alpha |A^{-1}w|^2 \geq \frac{\alpha}{\beta^2}|w|^2.$$

□

We now have all the results to show that $A^{\text{hom}} \in M(\alpha, \frac{\beta^2}{\alpha}, \Omega)$.

**Theorem 9.1.9.** *If $B$ is additionally $Y$-periodic in the second argument, there holds $A^{\text{hom}} \in M(\alpha, \frac{\beta^2}{\alpha}, \Omega)$.*

*Proof.* We prove the Theorem for a.e. $\hat{x} \in \Omega$. Since $B \in M(\alpha, \beta, \Omega \times Y)$ and Lemma 9.1.5 holds, we get

$$\begin{aligned}
|a_{ijkl}^{\text{hom}}(\hat{x})| &\leq \frac{1}{|Y|}\int_Y |B(\hat{x}, y)e_{ij}(e_{kl} - e_y(w^{kl})(\hat{x}, y))|\mathrm{d}y \\
&\leq \|b_{ijkl}\|_{L^\infty(\Omega\times Y)} + \frac{\beta}{|Y|^{1/2}}\|e_y(w^{kl})\|_{[L^\infty(\Omega, L^2(Y))]^{3\times 3}} \leq C
\end{aligned}$$

for a constant $C$ independent of $\hat{x}$, which proves that $a_{ijkl}^{\text{hom}} \in L^\infty(\Omega)$. Using $w^{kl}$ as a test function in (9.1.9) for the cell problem of $w^{ij}$ (and the other way round) and the symmetry of $B$ we receive

$$\begin{aligned}
a_{ijkl}^{\text{hom}}(\hat{x}) &= \frac{1}{|Y|}\int_Y B(\hat{x}, y)e_{ij}e_{kl}\,\mathrm{d}y - \frac{1}{|Y|}\int_Y B(\hat{x}, y)e_{ij}e_y(w^{kl})(\hat{x}, y)\,\mathrm{d}y \\
&= \frac{1}{|Y|}\int_Y B(\hat{x}, y)e_{ij}e_{kl}\,\mathrm{d}y - \frac{1}{|Y|}\int_Y B(\hat{x}, y)e_y(w^{ij})(\hat{x}, y)e_y(w^{kl})(\hat{x}, y)\,\mathrm{d}y \\
&= \frac{1}{|Y|}\int_Y B(\hat{x}, y)e_{kl}e_{ij}\,\mathrm{d}y - \frac{1}{|Y|}\int_Y B(\hat{x}, y)e_{kl}e_y(w^{ij})(\hat{x}, y)\,\mathrm{d}y = a_{klij}^{\text{hom}}(\hat{x})
\end{aligned}$$

and

$$\begin{aligned}
a_{ijkl}^{\text{hom}}(\hat{x}) &= \frac{1}{|Y|}\int_Y B(\hat{x}, y)e_{ij}(e_{kl} - e(w^{kl})(\hat{x}, y))\,\mathrm{d}y \\
&= \frac{1}{|Y|}\int_Y B(\hat{x}, y)e_{ji}(e_{kl} - e(w^{kl})(\hat{x}, y))\,\mathrm{d}y = a_{jikl}^{\text{hom}}(\hat{x}),
\end{aligned}$$

which shows the symmetry of $A^{\text{hom}}$. In the next step, we prove the coercivity of $A^{\text{hom}}$ with the coercivity constant $\alpha$. We extend $B$ $Y$-periodically in the second argument to obtain a tensor $B^\varepsilon(x) \coloneqq B(\hat{x}, \frac{x}{\varepsilon})$, which is well-defined for $x \in \Omega$ and $\varepsilon > 0$ small enough. Clearly, $B^\varepsilon \in M(\alpha, \beta, \Omega)$ for every $\varepsilon$. Let $m = (m_{kh})_{1\leq k,h\leq 3} \in \mathbb{R}^{3\times 3}$ be a symmetric matrix and

$$v^\varepsilon(x) \coloneqq \sum_{k,h=1}^{3} m_{kh}w_\varepsilon^{kh}(x) \coloneqq \sum_{k,h=1}^{3} m_{kh}\left((x_h\delta_{ik})_{1\leq i\leq 3} - \varepsilon w^{kh}\left(\hat{x}, \frac{x}{\varepsilon}\right)\right)$$

with $w^{kh}(\hat{x},\cdot)$ $Y$-periodically extended as in Lemma 9.1.6. Then, $v^\varepsilon \in \left[H^1(\Omega)\right]^3$,

$$v^\varepsilon \to \sum_{k,h=1}^{3} (m_{kh}x_h\delta_{ik})_{1\le i\le 3} = \sum_{h=1}^{3}(m_{ih}x_h)_{1\le i\le 3}$$

strongly in $\left[L^2(\Omega)\right]^3$ and, since $\nabla_y w^{kh}\left(\hat{x},\frac{x}{\varepsilon}\right) \rightharpoonup \mathcal{M}_Y(\nabla_y w^{kh}(\hat{x},y)) = 0$ weakly in $\left[L^2(\Omega)\right]^3$,

$$\partial_j v^\varepsilon(x) = \sum_{h=1}^{3}(m_{ih}\partial_j x_h)_{1\le i\le 3} - \sum_{k,h=1}^{3} m_{kh}\partial_{y_j} w^{kh}\left(\hat{x},\frac{x}{\varepsilon}\right) \rightharpoonup (m_{ij})_{1\le i\le 3}$$

weakly in $\left[L^2(\Omega)\right]^3$ for $j\in\{1,2,3\}$. Furthermore,

$$\begin{aligned}
(B^\varepsilon e(v^\varepsilon))_{ij}(x) &= \sum_{k,h=1}^{3} b_{ijkh}\left(\hat{x},\frac{x}{\varepsilon}\right)\left(m_{kh} - \sum_{p,q=1}^{3} m_{pq}(e_y(w^{pq}))_{kh}\left(\hat{x},\frac{x}{\varepsilon}\right)\right)\\
&\rightharpoonup \frac{1}{|Y|}\int_Y \sum_{k,h=1}^{3} b_{ijkh}(\hat{x},y)\left(m_{kh} - \sum_{p,q=1}^{3} m_{pq}(e_y(w^{pq}))_{kh}(\hat{x},y)\right)\mathrm{d}y\\
&= \sum_{p,q=1}^{3} m_{pq}\frac{1}{|Y|}\int_Y \sum_{k,h=1}^{3} b_{ijkh}(\hat{x},y)\left(\delta_{kp}\delta_{hq} - (e_y(w^{pq}))_{kh}(\hat{x},y)\right)\mathrm{d}y\\
&= (A^{\mathrm{hom}}(\hat{x})m)_{ij}
\end{aligned}$$

weakly in $L^2(\Omega)$ for all $i,j\in\{1,2,3\}$. Next, we prove that

$$\int_\Omega B^\varepsilon e(v^\varepsilon)e(v^\varepsilon)\varphi(x)\,\mathrm{d}x \to \int_\Omega A^{\mathrm{hom}}(\hat{x})mm\varphi(x)\,\mathrm{d}x.$$

for all $\varphi\in C_c^\infty(\Omega)$. We compute

$$e_{ij}(\varphi v^\varepsilon) = \varphi e_{ij}(v^\varepsilon) + \frac{1}{2}\partial_i\varphi v_j^\varepsilon + \frac{1}{2}\partial_j\varphi v_i^\varepsilon$$

and for $\tilde{x} \coloneqq \sum_{h=1}^{3}(m_{ih}x_h)_{1\le i\le 3}$

$$\begin{aligned}
e_{ij}^x(\varphi\tilde{x}) &= \frac{1}{2}(\partial_{x_i}\varphi\tilde{x}_j + \varphi\partial_{x_i}\tilde{x}_j + \partial_{x_j}\varphi\tilde{x}_i + \varphi\partial_{x_j}\tilde{x}_i)\\
&= \frac{1}{2}\sum_{h=1}^{3}\left(\partial_{x_i}\varphi m_{jh}x_h + \partial_{x_j}\varphi m_{ih}x_h\right) + \varphi\frac{1}{2}(m_{ji}+m_{ij})\\
&= \frac{1}{2}\sum_{h=1}^{3}\left(\partial_{x_i}\varphi m_{jh}x_h + \partial_{x_j}\varphi m_{ih}x_h\right) + \varphi m_{ij}
\end{aligned}$$

by the symmetry of $m$. Using the symmetry of $B(\hat{x},\cdot)$ we can apply Lemma 9.1.7 to $v \coloneqq$

$B(\hat{x},\cdot)\left(e_y(w^{kl})(\hat{x},\cdot)-e_{kl}\right)$. Thus,

$$-\nabla_y\cdot(B(\hat{x},\cdot)(e(w^{pq})(\hat{x},\cdot)-e_{pq})=0\quad\text{in }[\mathcal{D}'(\mathbb{R}^3)]^3$$

and

$$\int_\Omega B^\varepsilon e(v^\varepsilon)e(\varphi v^\varepsilon)\,\mathrm{d}x=\sum_{p,q=1}^3 m_{pq}\left(\int_\Omega B\left(\hat{x},\frac{x}{\varepsilon}\right)\left(e_{pq}-e_y(w^{pq})\left(\hat{x},\frac{x}{\varepsilon}\right)\right)e(\varphi v^\varepsilon)\mathrm{d}x\right)=0,$$

since we can approximate $v^\varepsilon$ by $C^\infty(\Omega)$ functions. Using this result and the strong and weak convergences from above

$$\int_\Omega B^\varepsilon e(v^\varepsilon)e(v^\varepsilon)\varphi\,\mathrm{d}x=\int_\Omega B^\varepsilon e(v^\varepsilon)e(\varphi v^\varepsilon)\mathrm{d}x-\int_\Omega\sum_{i,j=1}^3(B^\varepsilon e(v^\varepsilon))_{ij}\frac{1}{2}\left(\partial_i\varphi v_j^\varepsilon+\partial_j\varphi v_i^\varepsilon\right)\mathrm{d}x$$
$$\to-\int_\Omega\sum_{i,j=1}^3(A^{\mathrm{hom}}(\hat{x})m)_{ij}\frac{1}{2}\sum_{h=1}^3(\partial_i\varphi m_{jh}x_h+\partial_j\varphi m_{ih}x_h)\,\mathrm{d}x=\int_\Omega A^{\mathrm{hom}}(\hat{x})mm\varphi\,\mathrm{d}x.$$

The last equation holds since

$$-\int_\Omega A^{\mathrm{hom}}(\hat{x})me\left(\varphi\tilde{x}\right)\mathrm{d}x=-A^{\mathrm{hom}}(\hat{x})m\int_\Omega e\left(\varphi\tilde{x}\right)\mathrm{d}x=0.$$

The coercivity of $B^\varepsilon$ together with the weak lower semicontinuity of the $L^2$-norm yields for $\varepsilon\to0$ and for all $\varphi\in C_c^\infty(\Omega)$ with $\varphi\geq0$

$$\int_\Omega A^{\mathrm{hom}}(\hat{x})mm\varphi\,\mathrm{d}x=\lim_{\varepsilon\to0}\int_\Omega B^\varepsilon e(v^\varepsilon)e(v^\varepsilon)\varphi\,\mathrm{d}x\geq\liminf_{\varepsilon\to0}\alpha\int_\Omega|e(v^\varepsilon)|^2\varphi\,\mathrm{d}x\geq\int_\Omega\alpha|m|^2\varphi\,\mathrm{d}x.$$

Rearranged,
$$\int_\Omega\left(A^{\mathrm{hom}}(\hat{x})mm-\alpha|m|^2\right)\varphi\,\mathrm{d}x\geq0,$$

which shows that $A^{\mathrm{hom}}(\hat{x})mm\geq\alpha|m|^2$. It remains to prove the last property of $A^{\mathrm{hom}}$, namely, $|A^{\mathrm{hom}}(\hat{x})m|\leq\frac{\beta^2}{\alpha}|m|$ for all matrices $m$. Let $m\in\mathbb{R}^{3\times3}$. We define as before

$$v^\varepsilon(x)\coloneqq\sum_{k,h=1}^3 m_{kh}w_\varepsilon^{kh}(x)\coloneqq\sum_{k,h=1}^3 m_{kh}\left((x_h\delta_{ik})_{1\leq i\leq3}-\varepsilon w^{kh}\left(\hat{x},\frac{x}{\varepsilon}\right)\right)$$

If we apply Lemma 9.1.8 on $w=B^\varepsilon e(v^\varepsilon)$, whereby $(B^\varepsilon)^{-1}w$ is defined as in (9.1.11), we get for all $\varphi\in C_c^\infty(\Omega)$ with $\varphi\geq0$

$$\int_\Omega B^\varepsilon e(v^\varepsilon)e(v^\varepsilon)\varphi\,\mathrm{d}x=\int_\Omega(B^\varepsilon)^{-1}w:w\varphi\,\mathrm{d}x\geq\frac{\alpha}{\beta^2}\int_\Omega|w|^2\varphi\,\mathrm{d}x=\frac{\alpha}{\beta^2}\int_\Omega|B^\varepsilon e(v^\varepsilon)|^2\varphi\,\mathrm{d}x.$$

The same convergences as before are true, since only the symmetry of $B^\varepsilon$ and $A^{\mathrm{hom}}(\hat{x})$ was needed. We have used the symmetry of the matrix only to get the weak convergence of $e(v^\varepsilon)$ to $m$. Thus, passing to the limit yields

$$\int_\Omega A^{\mathrm{hom}}(\hat{x})mm\varphi \,\mathrm{d}x \geq \frac{\alpha}{\beta^2}\int_\Omega |A^{\mathrm{hom}}(\hat{x})m|^2\varphi \,\mathrm{d}x.$$

Since $\varphi \geq 0$ was arbitrary and we can apply Cauchy–Schwarz inequality, we get

$$|A^{\mathrm{hom}}(\hat{x})m|^2 \leq \frac{\beta^2}{\alpha}A^{\mathrm{hom}}(\hat{x})mm \leq \frac{\beta^2}{\alpha}|A^{\mathrm{hom}}(\hat{x})m||m|.$$

Thus,

$$|A^{\mathrm{hom}}(\hat{x})m| \leq \frac{\beta^2}{\alpha}|m|.$$

□

**Remark 9.1.10.** *Although we have already proven in Theorem 9.1.2 the existence and uniqueness of the solution, we can use the last result to show this directly by applying the Lax–Milgram theorem to the problem* (9.1.7)*. As a consequence we get*

$$\|u\|_{[H^1(\Omega)]^3} \leq C\left(\|f\|_{[L^2(\Omega)]^3} + \|g\|_{[L^2(\Gamma_{\mathrm{N}})]^3}\right) \tag{9.1.13}$$

*for a constant $C$ independent of the structure of the cell $Y$.*

## 9.2. Inverse problem

In the previous section, we were in the setting that if volume and boundary forces $f$ and $g$ are given we can easily compute the displacement field $u$, since the microstructure was known. From now on, we only know the volume and boundary forces $f$ and $g$ and some measured displacement $u_m$ on the exterior boundary. With this information we want to deduce the microstructure of the reference cell $Y$, which can be described by a finite vector of real parameters $\tau$. Concretely, we consider $Y$ consisting of two parts, where one is a connected Lipschitz domain $Y_0$ completely contained in $Y$, whose geometry is uniquely given by some finite real vector.

Although the following results are true in this general setting, we restrict us to case, where $Y_0$ is an ellipsoid as illustrated in Figure 9.1, for explicit examples and computations. Let $Y_0[\tau]$ be an open ellipsoid centered in the middle of the reference cell $Y$ with axis orientation in direction of the standard unit vectors and axis lengths $\tau = (\tau_1, \tau_2, \tau_2) \in [\eta, l_1-\eta]\times[\eta, l_2-\eta]\times[\eta, l_3-\eta] =: I_\eta$ for some small $\eta$ (see Figure 9.2). Furthermore, we define $Y_1[\tau] := Y\backslash\overline{Y_0[\tau]}$ and $\Sigma_Y[\tau] := \partial Y_0[\tau]$. Thus, $Y = Y_0 \cup Y_1 \cup \Sigma_Y$.

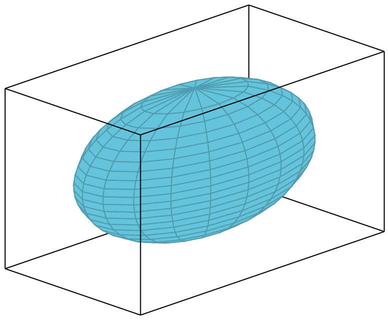

Figure 9.1.: ellipsoid $Y_0$ in the cuboid $Y$

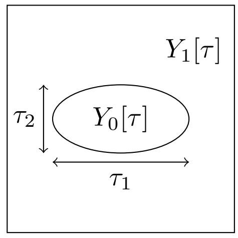


Figure 9.2.: periodicity cell $Y$ in 2D

As mentioned in the beginning of part II, we consider a perfectly bonded composite of two materials. So we can define the elasticity tensor $A^\varepsilon[\tau]$ as

$$A^\varepsilon[\tau](x) = A^0(x)\chi_{Y_0[\tau]}\left(\frac{x}{\varepsilon}\right) + A^1(x)\chi_{Y_1[\tau]}\left(\frac{x}{\varepsilon}\right)$$

with $\chi_{Y_0[\tau]}$ resp. $\chi_{Y_1[\tau]}$ the characteristic function of the $Y$-periodic extended domain $Y_0[\tau]$ resp. $Y_1[\tau]$ and some fourth order tensors $A^0, A^1 \in M(\alpha, \beta, \Omega)$ such that

$$\begin{aligned}
&\mathcal{T}^\varepsilon(A^\varepsilon[\tau])(x,y) \\
&\qquad = \begin{cases} A^0(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y)\chi_{Y_0[\tau]}(y) + A^1(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y)\chi_{Y_1[\tau]}(y) & \text{for a.e. } (x,y) \in \Omega^\varepsilon \times Y, \\ 0 & \text{for a.e. } (x,y) \in \Pi^\varepsilon \times Y \end{cases} \\
&\qquad \to A^0(x)\chi_{Y_0[\tau]}(y) + A^1(x)\chi_{Y_1[\tau]}(y) =: B[\tau](x,y)
\end{aligned}$$

for a.e. $(x,y) \in \Omega \times Y$. In this case, $A^{\mathrm{hom}} = (a^{\mathrm{hom}}_{ijkl})_{1 \le i,j,k,l \le 3}$ with

$$\begin{aligned}
a^{\mathrm{hom}}_{ijkl}[\tau](x) =& \frac{1}{|Y|}\int_{Y_0[\tau]} A^0(x)e_{ij}(e_{kl} - e_y(w^{kl})(x,y))\,\mathrm{d}y \\
&+ \frac{1}{|Y|}\int_{Y_1[\tau]} A^1(x)e_{ij}(e_{kl} - e_y(w^{kl})(x,y))\,\mathrm{d}y
\end{aligned}$$

and since $B[\tau]$ is $Y$-periodic, $A^{\mathrm{hom}}[\tau] \in M(\alpha, \frac{\beta^2}{\alpha}, \Omega)$ by Theorem 9.1.9. Because $B[\tau](x,\cdot)$ is piecewise smooth in $Y$, we even get that restricted cell solution $w^{kl}(x,\cdot)$ to $Y_0[\tau]$ resp. $Y_1[\tau]$ belongs to $[C^\infty(Y_0[\tau])]^3$ and $[C^\infty(Y_1[\tau])]^3$ due to Theorem 6.2 in Chapter I of [Oleinik et al., 1992]. We define the input–output operator:

**Definition 9.2.1** (input–output operator)**.**

$$\mathcal{L}_\tau : \left[L^2(\Omega)\right]^3 \times \left[L^2(\Gamma_{\mathrm{N}})\right]^3 \to [L^2(\partial\Omega)]^3, \quad (f,g) \mapsto u[\tau]|_{\partial\Omega},$$

*where $u[\tau]$ is the solution of the homogenized problem* (9.1.7) *for given $\tau$.*

Using the properties of the trace operator and (9.1.13) the continuity of the linear operator $\mathcal{L}_\tau$

follows directly. We study the following inverse problem.

**Definition 9.2.2** (Inverse Problem)**.** *Let $0 < \eta < \min\{l_1, l_2, l_3\}$. Find $\tau \in I_\eta$ such that for given measured data $u_m \in \left[L^2(\partial\Omega)\right]^3$, when forces $(f, g)$ are applied, $\tau$ is the solution of the minimization problem*

$$\underset{\tau \in I_\eta}{\arg\min}\, \mathcal{J}(\tau) \coloneqq \underset{\tau \in I_\eta}{\arg\min}\, \frac{1}{2}\|\mathcal{L}_\tau(f, g) - u_m\|^2_{[L^2(\partial\Omega)]^3}. \tag{9.2.1}$$

Clearly, other functionals than $\mathcal{J}$ could be used.

### 9.2.1. Existence result

In this section, we prove that there exists at least one solution of the inverse problem (9.2.1). We decompose the mapping $\mathcal{Z} : I_\eta \to [L^2(\partial\Omega)]^3$, $\tau \mapsto \mathcal{L}_\tau(f, g)$ for given $(f, g)$ into the continuous trace operator $\mathcal{T} : \left[H^1(\Omega)\right]^3 \to [L^2(\partial\Omega)]^3$ and the operator $\mathcal{H}_{f,g} : I_\eta \to H^1_{\Gamma_\mathrm{D}}(\Omega)$, $\tau \mapsto u[\tau]$, i.e. $\mathcal{Z}(\tau) = \mathcal{T} \circ \mathcal{H}_{f,g}(\tau)$. To prove the continuity of $\mathcal{Z}$ we have to show that $\mathcal{H}_{f,g}$ is continuous.

**Theorem 9.2.3.** *The operator $\mathcal{H}_{f,g}$ is continuous.*

*Proof.* Let $\tau_n, \hat{\tau} \in I_\eta$ with $\tau_n \to \hat{\tau}$ for $n \to \infty$ and $u[\tau_n]$, $u[\hat{\tau}]$ the corresponding weak solutions of the homogenized problem (9.1.7). Then, they satisfy for all $\varphi \in H^1_{\Gamma_\mathrm{D}}(\Omega)$

$$a(u[\tau_n], \varphi; \tau_n) = F(\varphi), \quad a(u[\hat{\tau}], \varphi; \hat{\tau}) = F(\varphi),$$

where $a : H^1_{\Gamma_\mathrm{D}}(\Omega) \times H^1_{\Gamma_\mathrm{D}}(\Omega) \to \mathbb{R}$ is the bilinear form of the left-hand side of (9.1.7), i.e.

$$a(v, \varphi; \tau) = \int_\Omega A^{\mathrm{hom}}[\tau] e(v) e(\varphi)\, \mathrm{d}x,$$

and $F : H^1_{\Gamma_\mathrm{D}}(\Omega) \to \mathbb{R}$ is the $\tau$-independent functional of the right-hand side of (9.1.7), i.e.

$$F(\varphi) = \int_\Omega f \cdot \varphi\, \mathrm{d}x + \int_{\Gamma_\mathrm{N}} g \cdot \varphi\, \mathrm{d}S(x).$$

The third index of $a$ emphasizes that the bilinear form depends on the given $\tau$ through the homogenized tensor $A^{\mathrm{hom}}[\tau]$. Taking the difference of both equations yield

$$\int_\Omega A^{\mathrm{hom}}[\tau_n] e(u[\tau_n] - u[\hat{\tau}]) e(\varphi) \mathrm{d}x = \int_\Omega (A^{\mathrm{hom}}[\hat{\tau}] - A^{\mathrm{hom}}[\tau_n]) e(u[\hat{\tau}]) e(\varphi) \mathrm{d}x.$$

We choose as test function $\varphi = u[\tau_n] - u[\hat{\tau}]$ and use that $A^{\mathrm{hom}}[\tau_n]$ is coercive to estimate

$$\begin{aligned}
\alpha \|e(u[\tau_n] - u[\hat{\tau}])\|^2_{[L^2(\Omega)]^{3\times 3}} &\\
\leq \int_\Omega (A^{\mathrm{hom}}[\hat{\tau}] - A^{\mathrm{hom}}[\tau_n]) e(u[\hat{\tau}]) e(u[\tau_n] - u[\hat{\tau}]) \mathrm{d}x &\\
\leq \|(A^{\mathrm{hom}}[\hat{\tau}] - A^{\mathrm{hom}}[\tau_n]) e(u[\hat{\tau}])\|_{[L^2(\Omega)]^{3\times 3}} \|e(u[\tau_n] - u[\hat{\tau}])\|_{[L^2(\Omega)]^{3\times 3}}. &
\end{aligned}$$

We obtain by Korn's inequality for functions with zero value on part of the boundary (see Theorem 2.2.5)

$$c\|\mathcal{H}_{f,g}(\tau_n) - \mathcal{H}_{f,g}(\hat{\tau})\|_{[H^1(\Omega)]^3} \leq \|(A^{\text{hom}}[\hat{\tau}] - A^{\text{hom}}[\tau_n])e(u[\hat{\tau}])\|_{[L^2(\Omega)]^{3\times 3}} \tag{9.2.2}$$

for some constant $c > 0$ independent of $\tau$. $a^{\text{hom}}_{ijkl}[\tau]$, $i, j, k, l \in \{1, 2, 3\}$, is bounded in $L^\infty(\Omega)$ independent of $\tau \in I_\eta$ since we can estimate, using that $B[\tau] \in M(\alpha, \beta, \Omega \times Y)$ and Lemma 9.1.5,

$$\begin{aligned}|a^{\text{hom}}_{ijkl}[\tau](x)| &\leq \frac{1}{|Y|}\|B[\tau](x,\cdot)e_{ij}\|_{[L^2(Y)]^{3\times 3}}\|e_{kl} - e(w^{kl}[\tau](x,\cdot))\|_{[L^2(Y)]^{3\times 3}} \\ &\leq \beta C\|e(w^{kl}[\tau](x,\cdot))\|_{[L^2(Y)]^{3\times 3}} \leq C\end{aligned}$$

for a.e. $x \in \Omega$ and for some constant $C > 0$ independent of $\tau$ and $x$. Thus,

$$|(A^{\text{hom}}[\hat{\tau}](x) - A^{\text{hom}}[\tau_n](x))e(u[\hat{\tau}](x))|^2 \leq C|e(u[\hat{\tau}](x))|^2 \in L^1(\Omega)$$

for a.e. $x \in \Omega$ and $C > 0$ independent of $x$. By Theorem 9.2.4, what we show next,

$$|(a^{\text{hom}}_{ijkl}[\tau_n](x) - a^{\text{hom}}_{ijkl}[\hat{\tau}](x))e(u[\hat{\tau}](x))|^2 \to 0$$

pointwise for a.e. $x \in \Omega$ as $n \to \infty$. So by the dominated convergence theorem the right-hand side of (9.2.2) converges to 0 as $n \to \infty$, which proves that $\mathcal{H}_{f,g}$ is continuous. □

We show the pointwise convergence of $A^{\text{hom}}[\tau_n](x)$ to $A^{\text{hom}}[\tau](x)$ as $\tau_n \to \tau$, which was used to prove the last Theorem.

**Theorem 9.2.4.** *For a.e. $x \in \Omega$, $a^{\text{hom}}_{ijkl}[\tau_n](x)$ converges to $a^{\text{hom}}_{ijkl}[\hat{\tau}](x)$ for $\tau_n \to \hat{\tau}$ in $I_\eta$ and every $i, j, k, l \in \{1, 2, 3\}$.*

*Proof.* Let $x \in \Omega$ and $\tau_n \in I_\eta$ with $\tau_n \to \hat{\tau}$. Since $I_\eta$ is compact, $\hat{\tau} \in I_\eta$. We use the definition of $A^{\text{hom}}$ to estimate

$$\begin{aligned}&|a^{\text{hom}}_{ijkl}[\tau_n](x) - a^{\text{hom}}_{ijkl}[\hat{\tau}](x)| \\ &= \left|\frac{1}{|Y|}\int_Y B[\tau_n](x,y)e_{ij}(e_{kl} - e(w^{kl}[\tau_n]))\,\mathrm{d}y - \frac{1}{|Y|}\int_Y B[\hat{\tau}](x,y)e_{ij}(e_{kl} - e(w^{kl}[\hat{\tau}]))\,\mathrm{d}y\right| \\ &\leq \frac{1}{|Y|}\int_Y \left|(B[\tau_n](x,y) - B[\hat{\tau}](x,y))\,e_{ij}\left(e_{kl} - e(w^{kl}[\hat{\tau}])\right)\right|\,\mathrm{d}y \\ &\quad + \frac{1}{|Y|}\left|\int_Y B[\tau_n](x,y)e_{ij}\left(e(w^{kl}[\tau_n]) - e(w^{kl}[\hat{\tau}])\right)\,\mathrm{d}y\right| \\ &=: \mathcal{I}^1_n(x) + \mathcal{I}^2_n(x).\end{aligned}$$

We prove that both terms $\mathcal{I}_n^1$ and $\mathcal{I}_n^2$ converge to zero. Starting with $\mathcal{I}_n^1$,

$$\begin{aligned}\mathcal{I}_n^1(x) &\leq \frac{1}{|Y|}\| \left(B[\tau_n](x,\cdot) - B[\hat{\tau}](x,\cdot)\right) e_{ij}\|_{[L^2(Y)]^{3\times 3}}\|e_{kl} - e(w^{kl}[\hat{\tau}](x,\cdot))\|_{[L^2(Y)]^{3\times 3}} \\ &\leq \frac{1}{|Y|^{1/2}}\left(1+\frac{\beta}{\alpha}\right)\sum_{h,r=1}^{3}\|b_{ijhr}[\tau_n](x,\cdot) - b_{ijhr}[\hat{\tau}](x,\cdot)\|_{L^2(Y)},\end{aligned}$$

where we have applied Lemma 9.1.5. Since $b_{ijhr}[\tau_n] \in L^\infty(\Omega\times Y)$ and $Y$ is bounded, we obtain that $b_{ijhr}[\tau_n](x,\cdot) \in L^2(Y)$ for a.e. $x \in \Omega$ and

$$\begin{aligned}&\|b_{ijhr}[\tau_n](x,\cdot) - b_{ijhr}[\hat{\tau}](x,\cdot)\|_{L^2(Y)} \\ &\quad\leq \|a^0_{ijhr}(x)(\chi_{Y_0[\tau_n]}(\cdot) - \chi_{Y_0[\hat{\tau}]}(\cdot))\|_{L^2(Y)} + \|a^1_{ijhr}(x)(\chi_{Y_1[\tau_n]}(\cdot) - \chi_{Y_1[\hat{\tau}]}(\cdot))\|_{L^2(Y)} \\ &\quad\leq |a^0_{ijhr}(x)|\|\chi_{Y_0[\tau_n]} - \chi_{Y_0[\hat{\tau}]}\|_{L^2(Y)} + |a^1_{ijhr}(x)|\|\chi_{Y_1[\tau_n]} - \chi_{Y_1[\hat{\tau}]}\|_{L^2(Y)} \\ &\quad\to 0,\end{aligned}$$

because $A^0, A^1 \in L^\infty(\Omega)$. The second term

$$\mathcal{I}_n^2(x) = \frac{1}{|Y|}\left|\sum_{h,r=1}^{3}\int_Y b_{ijhr}[\tau_n](x,y)\left(e_{hr}(w^{kl}[\tau_n]) - e_{hr}(w^{kl}[\hat{\tau}])\right)\,\mathrm{d}y\right|$$

converges to zero, if we show that

$$e_{hr}(w^{kl}[\tau_n])(x,\cdot) \rightharpoonup e_{hr}(w^{kl}[\hat{\tau}])(x,\cdot) \text{ weakly in } L^2(Y)$$

for $n\to\infty$ and $h,r \in \{1,2,3\}$, because we already know the strong convergence

$$b_{ijhr}[\tau_n](x,\cdot) \to b_{ijhr}[\hat{\tau}](x,\cdot)$$

in $L^2(Y)$ from above. Due to Lemma 9.1.5 the sequence of solutions of the cell problems $\{w^{kl}[\tau_n]\}$ is uniformly bounded in $[L^\infty(\Omega, H^1_{\mathrm{per},0}(Y))]^3$. Thus, for a.e. $x \in \Omega$ there exists a subsequence (again denoted by $\tau_n$) and a function $\tilde{w} \in [H^1_{\mathrm{per},0}(Y)]^3$ such that

$$w^{kl}[\tau_n](x,\cdot) \rightharpoonup \tilde{w}(\cdot) \text{ weakly in } [H^1_{\mathrm{per},0}(Y)]^3.$$

We equate the cell problems (9.1.9) for the weak solution $w^{kl}[\tau_n]$ and $w^{kl}[\hat{\tau}]$ and pass to the limit

$$\begin{aligned}\int_Y B[\hat{\tau}](x,y)(e_{kl} - e(\tilde{w}))e(\varphi)\,\mathrm{d}y &= \lim_{n\to\infty}\int_Y B[\tau_n](x,y)(e_{kl} - e(w^{kl}[\tau_n]))e(\varphi)\,\mathrm{d}y = 0 \\ &= \int_Y B[\hat{\tau}](x,y)(e_{kl} - e(w^{kl}[\hat{\tau}]))e(\varphi)\,\mathrm{d}y\end{aligned}$$

for all $\varphi \in [C^\infty_{\mathrm{per}}(Y)]^3$. Taking as a test function $\varphi(y) = w^{kl}[\hat{\tau}](x,y) - \tilde{w}(y)$, the coercivity of

$B$ and Korn's inequality for periodic functions (see Corollary 2.2.6) lead to

$$
\begin{aligned}
c\|w^{kl}[\hat{\tau}]-\tilde{w}\|_{[H^1(Y)]^3} &\leq \alpha\|e(w^{kl}[\hat{\tau}]-\tilde{w})\|_{[L^2(Y)]^{3\times 3}} \\
&\leq \int_Y B[\hat{\tau}](x,y)(e(w^{kl}[\hat{\tau}])-e(\tilde{w}))(e(w^{kl}[\hat{\tau}])-e(\tilde{w}))\mathrm{d}y = 0.
\end{aligned}
$$

Thus, $\tilde{w}$ coincides with $w^{kl}[\hat{\tau}](x,\cdot)$ and since the subsequence was chosen arbitrary and the limit function is unique, we get the convergence of the whole sequence

$$
w^{kl}[\tau_n](x,\cdot) \rightharpoonup w^{kl}[\hat{\tau}](x,\cdot) \text{ weakly in } [H^1_{\mathrm{per},0}(Y)]^3
$$

for a.e. $x \in \Omega$. ☐

Now, we can prove that there exists at least one solution of the inverse problem (9.2.1).

**Theorem 9.2.5.** *There exists at least one solution of the minimization problem* (9.2.1).

*Proof.* The operator $\mathcal{Z}(\tau) = \mathcal{T}\circ\mathcal{H}_{f,g}(\tau)$ is continuous since it is a composition of the continuous trace operator $\mathcal{T}$ and by Theorem 9.2.3 of the continuous operator $\mathcal{H}_{f,g}$. Since the set $I_\eta$ is compact, we can apply the extreme value theorem to guarantee that there exists at least one $\hat{\tau} \in I_\eta$ such that

$$
\hat{\tau} = \arg\min_{\tau\in I_\eta} \mathcal{J}(\tau).
$$

☐

The solution space is compact.

**Lemma 9.2.6.** *The solution space of the homogenized problem* (9.1.7)

$$
\mathbb{L}_{f,g} \coloneqq \left\{u \in H^1_{\Gamma_{\mathrm{D}}}(\Omega) : u \textit{ solution of } (9.1.7) \textit{ for some } \tau \in I_\eta\right\}
$$

*for fixed* $(f,g)$ *is compact.*

*Proof.* Let $\{u_n\} \subset \mathbb{L}_{f,g}$ be a sequence of solutions. Then, there exists for every $n \in \mathbb{N}$ a vector $\tau_n \in I_\eta$ such that $u_n = u[\tau_n]$. Because $I_\eta$ is a compact set there exists a subsequence of $\{\tau_n\}$, again denoted by $n$, such that $\tau_n$ converges to some $\hat{\tau} \in I_\eta$. Since by Theorem 9.2.3 $\mathcal{H}_{f,g}$ is continuous, we receive the convergence of $u_n$ to $u[\hat{\tau}]$ in $H^1_{\Gamma_{\mathrm{D}}}(\Omega)$. ☐

### 9.2.2. Gâteaux derivative of $A^{\mathrm{hom}}$

In the following we compute the Gâteaux derivative of $A^{\mathrm{hom}}$ by applying the concept of shape derivatives, which we need in the next section to derive the Gâteaux derivative of $\mathcal{J}$ to facilitate the use of gradient-based optimization algorithms.
Concretely, we use the Lagrangian method of Céa following the idea of [Allaire et al., 2011] to compute the Gâteaux derivative of $A^{\mathrm{hom}}$. For this, we define for all $x \in \Omega$ a Lagrangian

function $\mathfrak{L}^x_{ijkl}$ which coincides with

$$\mathfrak{J}^x_{ijkl}(Y_0) \coloneqq \int_Y B[Y_0](x,y) e_{ij}(e_{kl} - e(w_{kl})(x,y)) \,\mathrm{d}y$$

in some special points, where $B[Y_0](x,y) \coloneqq A^0(x)\chi_{Y_0}(y) + A^1(x)(1-\chi_{Y_0}(y))$ with $A^0, A^1 \in M(\alpha,\beta,\Omega)$ and $w_{kl} \in \left[L^\infty(\Omega, H^1_{\mathrm{per}}(Y))\right]^3$ is the weak solution of

$$\begin{cases} -\nabla_y \cdot (B[Y_0](x,\cdot)(e_y(w_{kl}) - e_{kl})) = 0 & \text{in } Y, \\ \mathcal{M}_Y(w_{kl}) = 0. \end{cases} \tag{9.2.3}$$

The reason for considering of $\mathfrak{L}^x_{ijkl}$ instead of $\mathfrak{J}^x_{ijkl}$ is that we can apply standard shape derivative results, which is not possible for the second function since the solutions $w_{kl}$ of the cell problem also depend on $Y_0$. For the readability, we omit the index $y$ in the divergence $\nabla_y\cdot$ and in the symmetric gradient $e_y(\cdot)$ because all the computations in this subsection are done for some fixed $x \in \Omega$. Since the spatial derivative of $w_{kl}$ may be discontinuous at the interface $\Sigma_Y$, we rewrite the problem as a transmission problem: For a.e. $x \in \Omega$ find $(w^{x,1}_{kl}, w^{x,0}_{kl}) \in V$ with

$$V \coloneqq \left\{(u^1,u^0) \in \left[H^1(Y_1)\right]^3 \times \left[H^1(Y_0)\right]^3 : u^1 \text{ is Y-periodic}, \mathcal{M}_Y(u^1\chi_{Y_1} + u^0\chi_{Y_0}) = 0\right\}$$

such that

$$\begin{cases} -\nabla \cdot (A^1_x(e(w^{x,1}_{kl}) - e_{kl})) = 0 & \text{in } Y_1, \\ w^{x,1}_{kl} = w^{x,0}_{kl} & \text{on } \Sigma_Y, \\ A^1_x(e(w^{x,1}_{kl}) - e_{kl})n^1 + A^0_x(e(w^{x,0}_{kl}) - e_{kl})n^0 = 0 & \text{on } \Sigma_Y \end{cases} \tag{9.2.4}$$

and

$$\begin{cases} -\nabla \cdot (A^0_x(e(w^{x,0}_{kl}) - e_{kl})) = 0 & \text{in } Y_0, \\ w^{x,0}_{kl} = w^{x,1}_{kl} & \text{on } \Sigma_Y, \\ A^1_x(e(w^{x,1}_{kl}) - e_{kl})n^1 + A^0_x(e(w^{x,0}_{kl}) - e_{kl})n^0 = 0 & \text{on } \Sigma_Y, \end{cases} \tag{9.2.5}$$

where $A^1_x \coloneqq A^1(x)$, $A^0_x \coloneqq A^0(x)$ and $n = n^0 = -n^1$ is the outward unit normal vector of the interface $\Sigma_Y$ with direction from $Y_0$ to $Y_1$. The transmission problem is equivalent to (9.2.3), since

$$w^x_{kl}(y) \coloneqq w^{x,1}_{kl}(y)\chi_{Y_1}(y) + w^{x,0}_{kl}(y)\chi_{Y_0}(y) \in \left[H^1_{\mathrm{per},0}(Y)\right]^3$$

due to the assumptions on the interface and for all $\varphi \in [H^1_{\mathrm{per},0}(Y)]^3$

$$\begin{aligned} 0 &= -\int_{Y_1} \nabla \cdot (A^1_x(e(w^{x,1}_{kl}) - e_{kl})) \cdot \varphi \,\mathrm{d}y - \int_{Y_0} \nabla \cdot (A^0_x(e(w^{x,0}_{kl}) - e_{kl})) \cdot \varphi \,\mathrm{d}y \\ &= -\int_{\partial Y} A^1_x(e(w^{x,1}_{kl}) - e_{kl})n^1\varphi \,\mathrm{d}S(y) - \int_{\Sigma_Y} A^1_x(e(w^{x,1}_{kl}) - e_{kl})n^1\varphi \,\mathrm{d}S(y) \\ &\quad + \int_{Y_1} A^1_x(e(w^{x,1}_{kl}) - e_{kl})e(\varphi)\mathrm{d}y - \int_{\Sigma_Y} A^0_x(e(w^{x,0}_{kl}) - e_{kl})n^0\varphi \,\mathrm{d}S(y) \end{aligned}$$

$$
\begin{aligned}
&+\int_{Y_0} A_x^0(e(w_{kl}^{x,0}) - e_{kl})e(\varphi)\mathrm{d}y \\
&= -\int_{\Sigma_Y} \left[A_x^1(e(w_{kl}^{x,1}) - e_{kl})n^1 + A_x^0(e(w_{kl}^{x,0}) - e_{kl})n^0\right] \varphi \,\mathrm{d}S(y) \\
&\quad + \int_Y B[Y_0](x,y)(e(w_{kl}^x)(y) - e_{kl})e(\varphi)(y)\mathrm{d}y \\
&= \int_Y B[Y_0](x,y)(e(w_{kl}^x)(y) - e_{kl})e(\varphi)(y)\mathrm{d}y.
\end{aligned}
$$

Clearly, the restricted solution $w_{kl}(x,\cdot)$ of (9.2.3) to the domain $Y_0$ resp. $Y_1$ solves the transmission problem, i.e. $w_{kl}(x,\cdot) = w_{kl}^{x,1}$ in $Y_1$ and $w_{kl}(x,\cdot) = w_{kl}^{x,0}$ in $Y_0$. We define the general Lagrangian function $\mathfrak{L}_{ijkl}^x$, where $q^1, q^0$ play the role of Lagrange multiplier,

$$
\begin{aligned}
&\mathfrak{L}_{ijkl}^x(v^0, v^1, q^0, q^1, Y_0) \\
&\quad := \int_{Y_0} A_x^0 e_{ij}(e_{kl} - e(v^0))\,\mathrm{d}y + \int_{Y_1} A_x^1 e_{ij}(e_{kl} - e(v^1))\,\mathrm{d}y \\
&\quad\quad - \int_{Y_0} A_x^0(e(v^0) - e_{kl})e(q^0)\,\mathrm{d}y - \int_{Y_1} A_x^1(e(v^1) - e_{kl})e(q^1)\,\mathrm{d}y \\
&\quad\quad - \frac{1}{2}\int_{\Sigma_Y} (A_x^1(e(v^1) - e_{kl}) + A_x^0(e(v^0) - e_{kl}))n \cdot (q^1 - q^0)\,\mathrm{d}S(y) \\
&\quad\quad - \frac{1}{2}\int_{\Sigma_Y} (A_x^1(e(q^1) + e_{ij}) + A_x^0(e(q^0) + e_{ij}))n \cdot (v^1 - v^0)\,\mathrm{d}S(y) \\
&\quad = -\int_{Y_0} A_x^0(e(q^0) + e_{ij})(e(v^0) - e_{kl})\,\mathrm{d}y - \int_{Y_1} A_x^1(e(q^1) + e_{ij})(e(v^1) - e_{kl})\,\mathrm{d}y \\
&\quad\quad - \frac{1}{2}\int_{\Sigma_Y} (A_x^1(e(v^1) - e_{kl}) + A_x^0(e(v^0) - e_{kl}))n \cdot (q^1 - q^0)\,\mathrm{d}S(y) \\
&\quad\quad - \frac{1}{2}\int_{\Sigma_Y} (A_x^1(e(q^1) + e_{ij}) + A_x^0(e(q^0) + e_{ij}))n \cdot (v^1 - v^0)\,\mathrm{d}S(y)
\end{aligned}
$$

for $v^0, v^1, q^0, q^1 \in \left[H^1_{\mathrm{per},0}(Y)\right]^3$. Since only the values of $v^\alpha$ and $q^\alpha$ in $Y_\alpha$ and on $\Sigma_Y$, $\alpha \in \{0,1\}$, are relevant for computation of the integral, we sometimes plug in functions, which are elements of $\left[H^1(Y_\alpha)\right]^3$. This is no problem, since they can be easily extended to functions of the whole domain and the extension does not change the computation of $\mathfrak{L}_{ijkl}^x$. In the next two lemmas we compute some conditions for optimal points.

**Lemma 9.2.7.** *The solutions $u^1$ and $u^0$ of* (9.2.4) *resp.* (9.2.5) *satisfy the optimality condition*

$$
0 = \frac{\partial \mathfrak{L}_{ijkl}^x}{\partial q^1}(u^0, u^1, p^0, p^1, Y_0)(\phi) = \frac{\partial \mathfrak{L}_{ijkl}^x}{\partial q^0}(u^1, u^0, p^0, p^1, Y_0)(\phi) \tag{9.2.6}
$$

*for all $\phi \in \left[H^1_{\mathrm{per},0}(Y)\right]^3$. Therefore, the solution $w_{kl}^x := w_{kl}(x,\cdot)$ of* (9.2.3) *fulfils the condition*

$$
0 = \frac{\partial \mathfrak{L}_{ijkl}^x}{\partial q^1}(w_{kl}^x, w_{kl}^x, p^0, p^1, Y_0)(\phi) = \frac{\partial \mathfrak{L}_{ijkl}^x}{\partial q^0}(w_{kl}^x, w_{kl}^x, p^0, p^1, Y_0)(\phi)
$$

*for all* $\phi \in \left[H^1_{\mathrm{per},0}(Y)\right]^3$ *in particular.*

*Proof.* Let $\phi \in \left[H^1_{\mathrm{per},0}(Y)\right]^3$. We compute the directional derivatives

$$
\begin{aligned}
&\frac{\partial \mathfrak{L}^x_{ijkl}}{\partial q^1}(v^0, v^1, q^0, q^1, Y_0)(\phi) \\
&\quad = -\int_{Y_1} A^1_x e(\phi)(e(v^1) - e_{kl})\,\mathrm{d}y - \frac{1}{2}\int_{\Sigma_Y} (A^1_x(e(v^1) - e_{kl}) + A^0(e(v^0) - e_{kl}))n \cdot \phi \,\mathrm{d}S(y) \\
&\qquad - \frac{1}{2}\int_{\Sigma_Y} A^1_x e(\phi) n \cdot (v^1 - v^0)\,\mathrm{d}S(y) \\
&\quad = \int_{Y_1} \nabla \cdot (A^1_x(e(v^1) - e_{kl}) \cdot \phi \,\mathrm{d}y - \int_{\Sigma_Y} A^1_x(e(v^1) - e_{kl})n^1 \cdot \phi \,\mathrm{d}S(y) \\
&\qquad - \frac{1}{2}\int_{\Sigma_Y} (A^1_x(e(v^1) - e_{kl}) + A^0_x(e(v^0) - e_{kl}))n \cdot \phi \,\mathrm{d}S(y) \\
&\qquad - \frac{1}{2}\int_{\Sigma_Y} A^1_x e(\phi) n \cdot (v^1 - v^0)\,\mathrm{d}S(y) \\
&\quad = \int_{Y_1} \nabla \cdot (A^1_x(e(v^1) - e_{kl}) \cdot \phi \,\mathrm{d}y + \frac{1}{2}\int_{\Sigma_Y} (A^1_x(e(v^1) - e_{kl}) - A^0_x(e(v^0) - e_{kl}))n \cdot \phi \,\mathrm{d}S(y) \\
&\qquad - \frac{1}{2}\int_{\Sigma_Y} A^1_x e(\phi) n \cdot (v^1 - v^0)\,\mathrm{d}S(y)
\end{aligned}
$$

and analogously,

$$
\begin{aligned}
&\frac{\partial \mathfrak{L}^x_{ijkl}}{\partial q^0}(v^0, v^1, q^0, q^1, Y_0)(\phi) \\
&\quad = \int_{Y_0} \nabla \cdot (A^0_x(e(v^0) - e_{kl}) \cdot \phi \,\mathrm{d}y + \frac{1}{2}\int_{\Sigma_Y} (A^1_x(e(v^1) - e_{kl}) - A^0_x(e(v^0) - e_{kl}))n \cdot \phi \,\mathrm{d}S(y) \\
&\qquad - \frac{1}{2}\int_{\Sigma_Y} A^0_x e(\phi) n \cdot (v^1 - v^0)\,\mathrm{d}S(y).
\end{aligned}
$$

Choosing $v^1 = u^1$ and $v^0 = u^0$ yields that all the integrals vanish. Thus, (9.2.6) holds. The second statement of the lemma follows directly since $w^x_{kl}$ solves the transmission problem as mentioned above. $\square$

We define the adjoint transmission problem: Find $(p^1, p^0) \in V$ such that

$$
\begin{cases}
-\nabla \cdot (A^\alpha_x(e(p^\alpha) + e_{ij})) = 0 & \text{in } Y_\alpha, \\
p^1 = p^0 & \text{on } \Sigma_Y, \\
A^1_x(e(p^1) + e_{ij})n^1 + A^0_x(e(p^0) + e_{ij})n^0 = 0 & \text{on } \Sigma_Y
\end{cases}
\tag{9.2.7}
$$

for $\alpha \in \{0, 1\}$, which is equivalent to the problem

$$
\begin{cases}
-\nabla \cdot (B[Y_0](x, \cdot)(e(p) + e_{ij})) = 0 & \text{in } Y, \\
\mathcal{M}_Y(p) = 0.
\end{cases}
\tag{9.2.8}
$$

An analogous computation as before shows the equivalance of both problems.

**Lemma 9.2.8.** *The solution* $(p^0, p^1)$ *of the adjoint transmission problem* (9.2.7) *satisfies the optimality condition*

$$0 = \frac{\partial \mathfrak{L}^x_{ijkl}}{\partial v^1}(u^0, u^1, p^0, p^1, Y_0)(\phi) = \frac{\partial \mathfrak{L}^x_{ijkl}}{\partial v^0}(u^0, u^1, p^0, p^1, Y_0)(\phi)$$

*for all* $\phi \in \left[H^1_{\mathrm{per},0}(Y)\right]^3$. *Furthermore, the solution* $w^x_{ij} := w_{ij}(x, \cdot)$ *of* (9.2.3) *for* $k = i$ *and* $l = j$ *fulfils the condition*

$$0 = \frac{\partial \mathfrak{L}^x_{ijkl}}{\partial v^1}(u^0, u^1, -w^x_{ij}, -w^x_{ij}, Y_0)(\phi) = \frac{\partial \mathfrak{L}^x_{ijkl}}{\partial v^0}(u^0, u^1, -w^x_{ij}, -w^x_{ij}, Y_0)(\phi)$$

*for all* $\phi \in \left[H^1_{\mathrm{per},0}(Y)\right]^3$.

*Proof.* Let $\phi \in \left[H^1_{\mathrm{per},0}(Y)\right]^3$. We compute the directional derivatives

$$\begin{aligned}
&\frac{\partial \mathfrak{L}^x_{ijkl}}{\partial v^1}(v^0, v^1, q^0, q^1, Y_0)(\phi) \\
&\quad = -\int_{Y_1} A^1_x(e(q^1) + e_{ij})e(\phi)\,\mathrm{d}y - \frac{1}{2}\int_{\Sigma_Y} A^1_x e(\phi)n \cdot (q^1 - q^0)\,\mathrm{d}S(y) \\
&\qquad - \frac{1}{2}\int_{\Sigma_Y} (A^1_x(e(q^1) + e_{ij}) + A^0_x(e(q^0) + e_{ij}))n \cdot \phi\,\mathrm{d}S(y) \\
&\quad = \int_{Y_1} \nabla \cdot (A^1_x(e(q^1) + e_{ij})) \cdot \phi\,\mathrm{d}y - \int_{\Sigma_Y} A^1_x(e(q^1) + e_{ij})n^1 \cdot \phi\,\mathrm{d}S(y) \\
&\qquad - \frac{1}{2}\int_{\Sigma_Y} A^1_x e(\phi)n \cdot (q^1 - q^0)\,\mathrm{d}S(y) - \frac{1}{2}\int_{\Sigma_Y} (A^1_x(e(q^1) + e_{ij}) + A^0_x(e(q^0) + e_{ij}))n \cdot \phi\,\mathrm{d}S(y) \\
&\quad = \int_{Y_1} \nabla \cdot (A^1_x(e(q^1) + e_{ij})) \cdot \phi\,\mathrm{d}y - \frac{1}{2}\int_{\Sigma_Y} A^1_x e(\phi)n \cdot (q^1 - q^0)\,\mathrm{d}S(y) \\
&\qquad + \frac{1}{2}\int_{\Sigma_Y} (A^1_x(e(q^1) + e_{ij}) - A^0_x(e(q^0) + e_{ij}))n \cdot \phi\,\mathrm{d}S(y)
\end{aligned}$$

and analogously,

$$\begin{aligned}
\frac{\partial \mathfrak{L}^x_{ijkl}}{\partial v^0}(v^0, v^1, q^0, q^1, Y_0)(\phi) &= \int_{Y_0} \nabla \cdot (A^0_x(e(q^0) + e_{ij})) \cdot \phi\,\mathrm{d}y - \frac{1}{2}\int_{\Sigma_Y} A^0_x e(\phi)n \cdot (q^1 - q^0)\,\mathrm{d}S(y) \\
&\quad + \frac{1}{2}\int_{\Sigma_Y} (A^1_x(e(q^1) + e_{ij}) - A^0_x(e(q^0) + e_{ij}))n \cdot \phi\,\mathrm{d}S(y).
\end{aligned}$$

Choosing $q^1 = p^1$ and $q^0 = p^0$ yields that all the integrals vanish, wherefore the first statement follows. The function $-w^x_{ij} := -w_{ij}(x, \cdot)$, where $w_{ij}$ is the solution of (9.2.3) for $k = i, l = j$, is the solution of (9.2.8). Thus, the second statement of the lemma follows directly. □

We introduce the shape differentation, whereby we refer to [Michailidis, 2014] for the following definitions and propositions and for further details.

**Definition 9.2.9.** *Let $\Omega_0$ be a reference domain, $\Omega = \{x + \theta(x) : x \in \Omega_0\} =: (Id + \theta)(\Omega_0)$ for some vector field $\theta : \mathbb{R}^3 \to \mathbb{R}^3$. A functional $\mathcal{F} : \Omega \to \mathbb{R}$ is said to be shape differentiable at $\Omega_0$ if the application $\theta \mapsto \mathcal{F}((Id + \theta)(\Omega_0))$ is Fréchet differentiable at 0 in the Banach space $\left[W^{1,\infty}(\mathbb{R}^3)\right]^3$. Then, the following asymptotic expansion holds in the vicinity of 0:*

$$\mathcal{F}((Id + \theta)(\Omega)) = \mathcal{F}(\Omega) + \mathcal{F}'(\Omega)(\theta) + o(\theta) \quad \text{with } \lim_{\theta \to 0} \frac{|o(\theta)|}{\|\theta\|} = 0,$$

*where $\mathcal{F}'(\Omega)$ is a continuous linear form on $[W^{1,\infty}(\mathbb{R}^3)]^3$.*

As in the standard differentiation of functions we can also define directional derivatives.

**Definition 9.2.10.** *The directional derivative of a functional $\mathcal{F} : \Omega \to \mathbb{R}$ at $\Omega$ in the direction $\theta \in \left[W^{1,\infty}(\mathbb{R}^3)\right]^3$ is defined by (if it exists)*

$$\mathcal{F}'(\Omega)(\theta) = \lim_{\delta \to 0} \frac{\mathcal{F}((Id + \delta\theta)\Omega) - \mathcal{F}(\Omega)}{\delta}.$$

If the function $\mathcal{F}$ is defined as an integral, where the integrand does not depend on the domain, there exists an explicit formula for the shape derivative.

**Proposition 9.2.11.** *Let $\Omega_0 \subset \mathbb{R}^3$ a smooth bounded open set. If $f \in W^{1,1}(\mathbb{R}^3)$ and $\mathcal{F} : \mathbb{C}(\Omega_0) \to \mathbb{R}$ is defined by $\mathcal{F}(\Omega) = \int_\Omega f(x)\mathrm{d}x$, where $\mathbb{C}(\Omega_0) \coloneqq \{\Omega = (Id + \theta)(\Omega_0)$ with $\theta \in [W^{1,\infty}(\mathbb{R}^3)]^3\}$, then $\mathcal{F}$ is differentiable at $\Omega_0$ and*

$$\mathcal{F}'(\Omega_0)(\theta) = \int_{\Omega_0} \nabla \cdot (\theta(x) f(x))\mathrm{d}x = \int_{\partial\Omega_0} \theta(x) \cdot n(x) f(x)\mathrm{d}S(x)$$

*for all $\theta \in \left[W^{1,\infty}(\mathbb{R}^3)\right]^3$.*

The Proposition is still true if $\Omega_0$ is regular enough to apply the transformation formula and Gauß's theorem.

**Proposition 9.2.12.** *Let $\Omega_0 \subset \mathbb{R}^3$ a smooth bounded open set. If $f \in W^{2,1}(\mathbb{R}^3)$ and $\mathcal{F} : \mathbb{C}(\Omega_0) \to \mathbb{R}$ is defined by $\mathcal{F}(\Omega) = \int_{\partial\Omega} f(x)\mathrm{d}S(x)$, where $\mathbb{C}(\Omega_0) \coloneqq \{\Omega = (Id + \theta)(\Omega_0)$ with $\theta \in \left[W^{1,\infty}(\mathbb{R}^3)\right]^3\}$, then $\mathcal{F}$ is differentiable at $\Omega_0$ and for all $\theta \in \left[W^{1,\infty}(\mathbb{R}^3)\right]^3$*

$$\mathcal{F}'(\Omega_0)(\theta) = \int_{\partial\Omega_0} \nabla f \cdot \theta + f(\nabla \cdot \theta - \nabla\theta n \cdot n)\mathrm{d}S(x) = \int_{\partial\Omega_0} \left(\frac{\partial f}{\partial n} + Hf\right) \theta \cdot n\mathrm{d}S(x),$$

*where $H = \nabla \cdot n$ is the mean curvature of $\partial\Omega_0$.*

The weak solution $w_{kl}$ of the cell problem (9.2.3) is not shape differentiable. But we can show that, upon a suitable extension outside of $Y_\alpha$, $\alpha \in \{0, 1\}$, the solutions $w_{kl}^{x,0}$ and $w_{kl}^{x,1}$ of the transmission problem are shape differentiable.

**Lemma 9.2.13.** *The solutions $w_{kl}^{x,1}$ of (9.2.4) and $w_{kl}^{x,0}$ of (9.2.5) are shape differentiable for a.e. $x \in \Omega$ and $\theta \in \left[W_0^{1,\infty}(Y)\right]^3$.*

*Proof.* The Lemma can be shown as in the proof of Theorem 5.3.2 from [Henrot and Pierre, 2005]. We summarise the main ideas. We consider the cell problem (9.2.3) on the transformed domain $Y_\theta := (Id + \theta)(Y)$ for some $\theta \in [W_0^{1,\infty}(Y)]^3$, which guarantees that we still have a domain which can be extended periodically. Using the change of variable theorem, the weak formulation can be rewritten as an integral over the reference cell $Y$. Since this integrand is of class $C^1$ with respect to $\theta$ and $v \in [H^1_{\mathrm{per},0}(Y)]^3$, we can apply the implicit function theorem to get the desired result after suitable extension of the solutions $w_{kl}^{x,\alpha}$ outside of $Y_\alpha$, $\alpha \in \{0,1\}$. $\square$

We are now able to prove that the Lagrangian $\mathfrak{L}^x_{ijkl}$ is equal to the functional $\mathfrak{J}^x_{ijkl}(Y_0)$ in the optimal point $(w_{kl}^{x,0}, w_{kl}^{x,1}, -w_{ij}^{x,0}, -w_{ij}^{x,1}, Y_0)$.

**Lemma 9.2.14.** *The shape derivative of the objective function $\mathfrak{J}^x_{ijkl}(Y_0)$ exists and is given by*

$$(\mathfrak{J}^x_{ijkl})'(Y_0)(\theta) = \frac{\partial \mathfrak{L}^x_{ijkl}}{\partial Y_0}(w_{kl}^{x,0}, w_{kl}^{x,1}, -w_{ij}^{x,0}, -w_{ij}^{x,1}, Y_0)(\theta),$$

*for all $\theta \in [W_0^{1,\infty}(Y)]^3$.*

*Proof.* The following identity holds

$$\begin{aligned}
&\mathfrak{L}^x_{ijkl}(w_{kl}^{x,0}, w_{kl}^{x,1}, q^0, q^1, Y_0) \\
&\quad= \int_{Y_0} A^0_x e_{ij}(e_{kl} - e(w_{kl}^{x,0}))\,\mathrm{d}y + \int_{Y_1} A^1_x e_{ij}(e_{kl} - e(w_{kl}^{x,1}))\,\mathrm{d}y \\
&\quad\quad - \int_{Y_0} A^0_x (e(w_{kl}^{x,0}) - e_{kl}) e(q^0)\,\mathrm{d}y - \int_{Y_1} A^1_x (e(w_{kl}^{x,1}) - e_{kl}) e(q^1)\,\mathrm{d}y \\
&\quad\quad - \frac{1}{2}\int_{\Sigma_Y} (A^1_x(e(w_{kl}^{x,1}) - e_{kl}) + A^0_x(e(w_{kl}^{x,0}) - e_{kl}))n \cdot (q^1 - q^0)\,\mathrm{d}S(y) \\
&\quad\quad - \frac{1}{2}\int_{\Sigma_Y} (A^1_x(e(q^1) + e_{ij}) + A^0_x(e(q^0) + e_{ij}))n \cdot (w_{kl}^{x,1} - w_{kl}^{x,0})\,\mathrm{d}S(y) \\
&\quad= \mathfrak{J}^x_{ijkl}(Y_0) - \int_{Y_0} A^0_x (e(w_{kl}^{x,0}) - e_{kl}) e(q^0)\,\mathrm{d}y - \int_{Y_1} A^1_x (e(w_{kl}^{x,1}) - e_{kl}) e(q^1)\,\mathrm{d}y \\
&\quad\quad + \int_{\Sigma_Y} (A^1_x(e(w_{kl}^{x,1}) - e_{kl})n^1 \cdot q^1 + A^0_x(e(w_{kl}^{x,0}) - e_{kl})n^0 \cdot q^0\,\mathrm{d}S(y) \\
&\quad= \mathfrak{J}^x_{ijkl}(Y_0) + \int_{Y_0} \nabla \cdot (A^0_x(e(w_{kl}^{x,0}) - e_{kl}))q^0\,\mathrm{d}y + \int_{Y_1} \nabla \cdot (A^1_x(e(w_{kl}^{x,1}) - e_{kl}))q^1\,\mathrm{d}y \\
&\quad= \mathfrak{J}^x_{ijkl}(Y_0),
\end{aligned}$$

using the properties of $w_{kl}^{x,1}$ and $w_{kl}^{x,0}$. Since $\mathfrak{L}^x_{ijkl}$ and $w_{kl}^{x,\kappa}$, $\kappa \in \{0,1\}$, are shape differentiable, we get the identity

$$(\mathfrak{J}^x_{ijkl})'(Y_0)(\theta) = \frac{\partial \mathfrak{L}^x_{ijkl}}{\partial Y_0}(w_{kl}^{x,0}, w_{kl}^{x,1}, q^0, q^1, Y_0)(\theta) + \sum_{\kappa=0}^{1} \frac{\partial \mathfrak{L}^x_{ijkl}}{\partial v^\kappa}(w_{kl}^{x,0}, w_{kl}^{x,1}, q^0, q^1, Y_0)\frac{\partial w_{kl}^{x,\kappa}}{\partial Y_0}(\theta).$$

In the special case where $q^0 = -w_{ij}^{x,0}$ and $q^1 = -w_{ij}^{x,1}$ the last two terms disappear. $\square$

With this lemma we can compute the shape derivative of $\mathfrak{L}^x_{ijkl}$ instead of $\mathfrak{J}^x_{ijkl}(Y_0)$, which is much easier. Since $v^0, v^1, q^0, q^0$ do not depend on the structure of $Y_0$, we can apply Proposition 9.2.11 and Proposition 9.2.12 to compute the shape derivative of the Lagrangian $\mathfrak{L}^x_{ijkl}$.

$$
\begin{aligned}
&\frac{\partial \mathfrak{L}^x_{ijkl}}{\partial Y_0}(w^{x,0}_{kl}, w^{x,1}_{kl}, -w^{x,0}_{ij}, -w^{x,1}_{ij}, Y_0)(\theta)\\
&= \int_{\Sigma_Y} A^0_x e_{ij}(e_{kl} - e(w^{x,0}_{kl}))\theta \cdot n^0 \mathrm{d}S(y) + \int_{\Sigma_Y} A^1_x e_{ij}(e_{kl} - e(w^{x,1}_{kl}))\theta \cdot n^1 \mathrm{d}S(y)\\
&\quad - \int_{\Sigma_Y} A^0_x(e(w^{x,0}_{kl}) - e_{kl})e(-w^{x,0}_{ij})\theta \cdot n^0 \mathrm{d}S(y) - \int_{\Sigma_Y} A^1_x(e(w^{x,1}_{kl}) - e_{kl})e(-w^{x,1}_{ij})\theta \cdot n^1 \mathrm{d}S(y)\\
&\quad - \frac{1}{2}\int_{\Sigma_Y}\left(\frac{\partial}{\partial n} + H\right)\left[(A^1_x(e(w^{x,1}_{kl}) - e_{kl}) + A^0_x(e(w^{x,0}_{kl}) - e_{kl}))n \cdot (-w^{x,1}_{ij} + w^{x,0}_{ij})\right]\theta \cdot n^0\, \mathrm{d}S(y)\\
&\quad - \frac{1}{2}\int_{\Sigma_Y}\left(\frac{\partial}{\partial n} + H\right)\left[(A^1_x(e(-w^{x,1}_{ij}) + e_{ij}) + A^0_x(e(-w^{x,0}_{ij}) + e_{ij}))n \cdot (w^{x,1}_{kl} - w^{x,0}_{kl})\right]\theta \cdot n^0\, \mathrm{d}S(y),
\end{aligned}
$$

where $H$ is the mean curvature. Because $w^{x,1}_{kl} = w^{x,0}_{kl}$ and $w^{x,1}_{ij} = w^{x,0}_{ij}$ on $\Sigma_Y$, the integrals involving $H$ vanish on $\Sigma_Y$, i.e.

$$
\begin{aligned}
&\frac{\partial \mathfrak{L}^x_{ijkl}}{\partial Y_0}(w^{x,0}_{kl}, w^{x,1}_{kl}, -w^{x,0}_{ij}, -w^{x,1}_{ij}, Y_0)(\theta)\\
&= \int_{\Sigma_Y} A^0_x e_{ij}(e_{kl} - e(w^{x,0}_{kl}))\theta \cdot n\, \mathrm{d}S(y) - \int_{\Sigma_Y} A^1_x e_{ij}(e_{kl} - e(w^{x,1}_{kl}))\theta \cdot n\, \mathrm{d}S(y)\\
&\quad - \int_{\Sigma_Y} A^0_x(e(w^{x,0}_{kl}) - e_{kl})e(-w^{x,0}_{ij})\theta \cdot n\, \mathrm{d}S(y) + \int_{\Sigma_Y} A^1_x(e(w^{x,1}_{kl}) - e_{kl})e(-w^{x,1}_{ij})\theta \cdot n\, \mathrm{d}S(y)\\
&\quad - \frac{1}{2}\int_{\Sigma_Y}\frac{\partial}{\partial n}\left[(A^1_x(e(w^{x,1}_{kl}) - e_{kl}) + A^0_x(e(w^{x,0}_{kl}) - e_{kl}))n \cdot (-w^{x,1}_{ij} + w^{x,0}_{ij})\right]\theta \cdot n\, \mathrm{d}S(y)\\
&\quad - \frac{1}{2}\int_{\Sigma_Y}\frac{\partial}{\partial n}\left[(A^1_x(e(-w^{x,1}_{ij}) + e_{ij}) + A^0_x(e(-w^{x,0}_{ij}) + e_{ij}))n \cdot (w^{x,1}_{kl} - w^{x,0}_{kl})\right]\theta \cdot n\, \mathrm{d}S(y).
\end{aligned}
$$

Using the same argument and the fact that $A^1_x(e(w^{x,1}_{kl}) - e_{kl})n = A^0_x(e(w^{x,0}_{kl}) - e_{kl})n$ on $\Sigma_Y$ leads to

$$
\begin{aligned}
&\frac{\partial \mathfrak{L}^x_{ijkl}}{\partial Y_0}(w^{x,0}_{kl}, w^{x,1}_{kl}, -w^{x,0}_{ij}, -w^{x,1}_{ij}, Y_0)(\theta)\\
&\quad = \int_{\Sigma_Y} A^0_x(e(w^{x,0}_{kl}) - e_{kl})(e(w^{x,0}_{ij}) - e_{ij})\theta \cdot n\, \mathrm{d}S(y)\\
&\qquad - \int_{\Sigma_Y} A^1_x(e(w^{x,1}_{kl}) - e_{kl})(e(w^{x,1}_{ij}) - e_{ij})\theta \cdot n\, \mathrm{d}S(y)\\
&\qquad + \int_{\Sigma_Y} A_x(e(w^x_{kl}) - e_{kl})n \cdot \frac{\partial(w^{x,1}_{ij} - w^{x,0}_{ij})}{\partial n}\theta \cdot n\, \mathrm{d}S(y)\\
&\qquad + \int_{\Sigma_Y} A_x(e(w^x_{ij}) - e_{ij})n \cdot \frac{\partial(w^{x,1}_{kl} - w^{x,0}_{kl})}{\partial n}\theta \cdot n\, \mathrm{d}S(y),
\end{aligned} \tag{9.2.9}
$$

whereby we denote by $A_x(e(w^x_{kl}) - e_{kl})n$ and $A_x(e(w^x_{ij}) - e_{ij})n$ the continuous quantities through the interface.
This formula can be simplified under additional assumptions. We assume that the material is isotropic, i.e. $A^\kappa_x$, $\kappa \in \{0,1\}$, is a tensor of the form

$$A^\kappa_x = 2\mu^\kappa_x I_4 + \lambda^\kappa_x I_2 \otimes I_2,$$

where $\lambda^\kappa \in L^\infty(\Omega)$ and $\mu^\kappa \in L^\infty(\Omega)$ are the Lamé parameters depending on the macrovariable and $I_2$ and $I_4$ are the identity tensors of second and fourth order. At each point of $\Sigma_Y$, we define the local orthonormal basis $(t, n)$ with unit normal vector $n$ and both unit tangential vectors as a collection $t$. A $3 \times 3$-matrix in local basis can be written as

$$m = \begin{pmatrix} m_{tt} & m_{tn} \\ m_{nt} & m_{nn} \end{pmatrix} \text{ with } m_{tt} \in \mathbb{R}^{2\times 2}, m_{tn} \in \mathbb{R}^2, m_{nn} \in \mathbb{R}$$

**Lemma 9.2.15.** *$A_x(e(w^x_{kl}) - e_{kl})n$ and $e(w^x_{kl})_{tt}$ are continuous across the interface $\Sigma_Y$. All other components have jumps on the interface given by*

(i) $[(e(w^x_{kl}) - e_{kl})_{nn}] = \left[\frac{1}{2\mu_x+\lambda_x}\right] (A_x(e(w^x_{kl}) - e_{kl}))_{nn} - \left[\frac{\lambda_x}{2\mu_x+\lambda_x}\right] \mathrm{tr}(e(w^x_{kl}) - e_{kl})_{tt}$,

(ii) $[(e(w^x_{kl}) - e_{kl})_{tn}] = \left[\frac{1}{2\mu_x}\right] (A_x(e(w^x_{kl}) - e_{kl}))_{tn}$,

(iii)
$$\begin{aligned}[(A_x(e(w^x_{kl}) - e_{kl}))_{tt}] = [2\mu_x](e(w^x_{kl}) - e_{kl})_{tt} + \Bigg(\left[\frac{2\mu_x\lambda_x}{2\mu_x+\lambda_x}\right] \mathrm{tr}(e(w^x_{kl}) - e_{kl})_{tt} \\ + \left[\frac{\lambda_x}{2\mu_x+\lambda_x}\right] (A_x(e(w^x_{kl}) - e_{kl}))_{nn}\Bigg) I_2,\end{aligned}$$

*where the square brackets denote the jump on the interface, i.e. $[f] = f^1 - f^0$.*

*Proof.* Let $\kappa \in \{0,1\}$. Using $I_4 = (\delta_{il}\delta_{jk})_{1\le i,j,k,l\le 3}$ and $I_2 \otimes I_2 = (\delta_{ij}\delta_{kl})_{1\le i,j,k,l\le 3}$ and the similarity invariance of the trace, we compute

$$\begin{aligned}
&\frac{1}{2\mu^\kappa_x + \lambda^\kappa_x}(A^\kappa_x(e(w^{x,\kappa}_{kl}) - e_{kl}))_{nn} - \frac{\lambda^\kappa_x}{2\mu^\kappa_x + \lambda^\kappa_x}\mathrm{tr}(e(w^{x,\kappa}_{kl}) - e_{kl})_{tt} \\
&\quad= \frac{1}{2\mu^\kappa_x + \lambda^\kappa_x}\left(\left(2\mu^\kappa_x(\delta_{ip}\delta_{jm})^3_{i,j,m,p=1} + \lambda^\kappa_x(\delta_{ij}\delta_{mp})^3_{i,j,m,p=1}\right)(e(w^{x,\kappa}_{kl}) - e_{kl})\right)_{nn} \\
&\quad\quad - \frac{\lambda^\kappa_x}{2\mu^\kappa_x + \lambda^\kappa_x}\mathrm{tr}(e(w^{x,\kappa}_{kl}) - e_{kl})_{tt} \\
&\quad= \frac{1}{2\mu^\kappa_x + \lambda^\kappa_x}(2\mu^\kappa_x(e(w^{x,\kappa}_{kl}) - e_{kl})_{nn} + \lambda^\kappa_x\mathrm{tr}(e(w^{x,\kappa}_{kl}) - e_{kl})((\delta_{mp})^3_{m,p=1})_{nn}) \\
&\quad\quad - \frac{\lambda^\kappa_x}{2\mu^\kappa_x + \lambda^\kappa_x}\mathrm{tr}(e(w^{x,\kappa}_{kl}) - e_{kl}) + \frac{\lambda^\kappa_x}{2\mu^\kappa_x + \lambda^\kappa_x}(e(w^{x,\kappa}_{kl}) - e_{kl})_{nn} \\
&\quad= \frac{1}{2\mu^\kappa_x + \lambda^\kappa_x}(2\mu^\kappa_x(e(w^{x,\kappa}_{kl}) - e_{kl})_{nn} + \lambda^\kappa_x(e(w^{x,\kappa}_{kl}) - e_{kl})_{nn}) \\
&\quad= (e(w^{x,\kappa}_{kl}) - e_{kl})_{nn}
\end{aligned}$$

and

$$\frac{1}{2\mu_x^\kappa}(A_x^\kappa(e(w_{kl}^{x,\kappa}) - e_{kl}))_{tn} = \frac{1}{2\mu_x^\kappa}\left(2\mu_x^\kappa(e(w_{kl}^{x,\kappa}) - e_{kl})_{tn} + \lambda_x^\kappa \mathrm{tr}(e(w_{kl}^{x,\kappa}) - e_{kl})((\delta_{mp})_{m,p=1}^3)_{tn}\right)$$
$$= (e(w_{kl}^{x,\kappa}) - e_{kl})_{tn}$$

and

$$\begin{aligned}
&2\mu_x^\kappa(e(w_{kl}^{x,\kappa}) - e_{kl})_{tt} + \frac{2\mu_x^\kappa\lambda_x^\kappa}{2\mu_x^\kappa + \lambda_x^\kappa}\mathrm{tr}(e(w_{kl}^{x,\kappa}) - e_{kl})_{tt}I_2 + \frac{\lambda_x^\kappa}{2\mu_x^\kappa + \lambda_x^\kappa}(A_x^\kappa(e(w_{kl}^{x,\kappa}) - e_{kl}))_{nn}I_2\\
&=2\mu_x^\kappa(e(w_{kl}^{x,\kappa}) - e_{kl})_{tt} + \frac{2\mu_x^\kappa\lambda_x^\kappa}{2\mu_x^\kappa + \lambda_x^\kappa}\mathrm{tr}(e(w_{kl}^{x,\kappa}) - e_{kl})_{tt}I_2\\
&\quad+\frac{\lambda_x^\kappa}{2\mu_x^\kappa + \lambda_x^\kappa}(2\mu_x^\kappa(e(w_{kl}^{x,\kappa}) - e_{kl})_{nn} + \lambda_x^\kappa\mathrm{tr}(e(w_{kl}^{x,\kappa}) - e_{kl})((\delta_{mp})_{m,p=1}^3)_{nn})I_2\\
&=2\mu_x^\kappa(e(w_{kl}^{x,\kappa}) - e_{kl})_{tt} + \frac{\lambda_x^\kappa}{2\mu_x^\kappa + \lambda_x^\kappa}(2\mu_x^\kappa\mathrm{tr}(e(w_{kl}^{x,\kappa}) - e_{kl}) + \lambda_x^\kappa\mathrm{tr}(e(w_{kl}^{x,\kappa}) - e_{kl}))I_2\\
&=2\mu_x^\kappa(e(w_{kl}^{x,\kappa}) - e_{kl})_{tt} + \lambda_x^\kappa\mathrm{tr}(e(w_{kl}^{x,\kappa}) - e_{kl})(I_3)_{tt}\\
&=(A_x^\kappa(e(w_{kl}^{x,\kappa}) - e_{kl}))_{tt}.
\end{aligned}$$

The statements (i)–(iii) follow directly by taking the difference. $\square$

With this lemma we can rewrite the first two integrands of $\frac{\partial \mathfrak{L}_{ijkl}^x}{\partial Y_0}$

$$\begin{aligned}
&A_x^1(e(w_{kl}^{x,1}) - e_{kl})(e(w_{ij}^{x,1}) - e_{ij})\\
&=(A_x^1(e(w_{kl}^{x,1}) - e_{kl}))_{tt}(e(w_{ij}^{x,1}) - e_{ij})_{tt} + 2(A_x^1(e(w_{kl}^{x,1}) - e_{kl}))_{tn}(e(w_{ij}^{x,1}) - e_{ij})_{tn}\\
&\quad+(A_x^1(e(w_{kl}^{x,1}) - e_{kl}))_{nn}(e(w_{ij}^{x,1}) - e_{ij})_{nn}\\
&=2\mu_x^1(e(w_{kl}^{x,1}) - e_{kl})_{tt}(e(w_{ij}^{x,1}) - e_{ij})_{tt} + \frac{2\mu_x^1\lambda_x^1}{2\mu_x^1 + \lambda_x^1}\mathrm{tr}(e(w_{kl}^{x,1}) - e_{kl})_{tt}I_2(e(w_{ij}^{x,1}) - e_{ij})_{tt}\\
&\quad+\frac{\lambda_x^1}{2\mu_x^1 + \lambda_x^1}(A_x^1(e(w_{kl}^{x,1}) - e_{kl}))_{nn}I_2(e(w_{ij}^{x,1}) - e_{ij})_{tt}\\
&\quad+2(A_x^1(e(w_{kl}^{x,1}) - e_{kl}))_{tn}\frac{1}{2\mu_x^1}(A_x^1(e(w_{ij}^{x,1}) - e_{ij}))_{tn}\\
&\quad+(A_x^1(e(w_{kl}^{x,1}) - e_{kl}))_{nn}\frac{1}{2\mu_x^1 + \lambda_x^1}(A_x^1(e(w_{ij}^{x,1}) - e_{ij}))_{nn}\\
&\quad-(A_x^1(e(w_{kl}^{x,1}) - e_{kl}))_{nn}\frac{\lambda_x^1}{2\mu_x^1 + \lambda_x^1}\mathrm{tr}(e(w_{ij}^{x,1}) - e_{ij})_{tt}\\
&=2\mu_x^1(e(w_{kl}^{x,1}) - e_{kl})_{tt}(e(w_{ij}^{x,1}) - e_{ij})_{tt} + \frac{2\mu_x^1\lambda_x^1}{2\mu_x^1 + \lambda_x^1}\mathrm{tr}(e(w_{kl}^{x,1}) - e_{kl})_{tt}\mathrm{tr}(e(w_{ij}^{x,1}) - e_{ij})_{tt}\\
&\quad+\frac{1}{\mu_x^1}(A_x^1(e(w_{kl}^{x,1}) - e_{kl}))_{tn}(A_x^1(e(w_{ij}^{x,1}) - e_{ij}))_{tn}\\
&\quad+\frac{1}{2\mu_x^1 + \lambda_x^1}(A_x^1(e(w_{kl}^{x,1}) - e_{kl}))_{nn}(A_x^1(e(w_{ij}^{x,1}) - e_{ij}))_{nn}
\end{aligned}$$

and analogously for the term $A_x^0(e(w_{kl}^{x,0}) - e_{kl})(e(w_{ij}^{x,0}) - e_{ij})$. Thus,

$$
\begin{aligned}
&A_x^1(e(w_{kl}^{x,1}) - e_{kl})(e(w_{ij}^{x,1}) - e_{ij}) - A_x^0(e(w_{kl}^{x,0}) - e_{kl})(e(w_{ij}^{x,0}) - e_{ij}) \\
=&[2\mu_x](e(w_{kl}^x) - e_{kl})_{tt}(e(w_{ij}^x) - e_{ij})_{tt} + \left[\frac{2\mu_x\lambda_x}{2\mu_x + \lambda_x}\right] \mathrm{tr}(e(w_{kl}^x) - e_{kl})_{tt}\mathrm{tr}(e(w_{ij}^x) - e_{ij})_{tt} \\
&+ \left[\frac{1}{\mu_x}\right] (A_x(e(w_{kl}^x) - e_{kl}))_{tn}(A_x(e(w_{ij}^x) - e_{ij}))_{tn} \\
&+ \left[\frac{1}{2\mu_x + \lambda_x}\right] (A_x(e(w_{kl}^x) - e_{kl}))_{nn}(A_x(e(w_{ij}^x) - e_{ij}))_{nn},
\end{aligned}
$$

which is an expression of only continuous functions at the interface.

**Lemma 9.2.16.** *For two displacements $w_{kl}^x$ and $q$ there holds: if $q = 0$ on $\Sigma_Y$, then*

$$
A_x(e(w_{kl}^x) - e_{kl})n \cdot \frac{\partial q}{\partial n} = 2(A_x(e(w_{kl}^x) - e_{kl})n) \cdot (e(q)n) - (A_x(e(w_{kl}^x) - e_{kl}))_{nn}e(q)_{nn}
$$

*on $\Sigma_Y$.*

*Proof.* Since $q = 0$ on $\Sigma_Y$, there holds $\nabla q\, t = 0$ for all tangential vectors $t$. Thus, if $(t_1, t_2, n)$ is an orthonormal basis,

$$
\begin{aligned}
(A_x(e(w_{kl}^x) - e_{kl}))_{nn}e(q)_{nn} =&\frac{1}{2}n^T(A_x(e(w_{kl}^x) - e_{kl}))nn^T(\nabla q + (\nabla q)^T)n \\
=&\frac{1}{2}n^T(A_xe(w_{kl}^x) - e_{kl}))n(n^T\nabla qn + n^T(\nabla q)^Tn) \\
=&n^T(A_x(e(w_{kl}^x) - e_{kl}))(nn^T + t_1t_1^T + t_2t_2^T)(\nabla q)^Tn \\
=&(A_x(e(w_{kl}^x) - e_{kl}))n \cdot (\nabla q)^Tn,
\end{aligned}
$$

where we have used the fact that $nn^T + t_1t_1^T + t_2t_2^T = (t_1, t_2, n)(t_1, t_2, n)^T = I_3$. So we can prove the equality in the lemma

$$
\begin{aligned}
&2(A_x(e(w_{kl}^x) - e_{kl})n) \cdot (e(q)n) - (A_x(e(w_{kl}^x) - e_{kl}))_{nn}e(q)_{nn} \\
&= (A_x(e(w_{kl}^x) - e_{kl})n) \cdot (\nabla qn + (\nabla q)^Tn) - (A_x(e(w_{kl}^x) - e_{kl}))_{nn}e(q)_{nn} \\
&= (A_x(e(w_{kl}^x) - e_{kl})n) \cdot \frac{\partial q}{\partial n} + (A_x(e(w_{kl}^x) - e_{kl})n) \cdot ((\nabla q)^Tn) - (A_x(e(w_{kl}^x) - e_{kl}))_{nn}e(q)_{nn} \\
&= (A_x(e(w_{kl}^x) - e_{kl})n) \cdot \frac{\partial q}{\partial n}.
\end{aligned}
$$

□

We apply this to the third integrand of $\frac{\partial \mathfrak{L}^x_{ijkl}}{\partial Y_0}$

$$
\begin{aligned}
&A_x(e(w^x_{kl}) - e_{kl})n \cdot \frac{\partial(w^{x,1}_{ij} - w^{x,0}_{ij})}{\partial n} \\
=&2A_x(e(w^x_{kl}) - e_{kl})n \cdot (e(w^{x,1}_{ij}) - e(w^{x,0}_{ij}))n - (A_x(e(w^x_{kl}) - e_{kl}))_{nn}(e(w^{x,1}_{ij}) - e(w^{x,0}_{ij}))_{nn} \\
=&\left[\frac{2}{2\mu_x + \lambda_x}\right](A_x(e(w^x_{kl}) - e_{kl}))_{nn}(A_x(e(w^x_{ij}) - e_{ij}))_{nn} \\
&- \left[\frac{2\lambda_x}{2\mu_x + \lambda_x}\right](A_x(e(w_{kl}) - e_{kl}))_{nn}\mathrm{tr}(e(w^x_{ij}) - e_{ij})_{tt} \\
&+ \left[\frac{1}{\mu_x}\right](A_x(e(w^x_{kl}) - e_{kl}))_{tn}(A_x(e(w^x_{ij}) - e_{ij}))_{tn} \\
&- \left[\frac{1}{2\mu_x + \lambda_x}\right](A_x(e(w^x_{kl}) - e_{kl}))_{nn}(A_x(e(w^x_{ij}) - e_{ij}))_{nn} \\
&+ \left[\frac{\lambda_x}{2\mu_x + \lambda_x}\right](A_x(e(w^x_{kl}) - e_{kl}))_{nn}\mathrm{tr}(e(w^x_{ij}) - e_{ij})_{tt} \\
=&\left[\frac{1}{2\mu_x + \lambda_x}\right](A_x(e(w^x_{kl}) - e_{kl}))_{nn}(A_x(e(w^x_{ij}) - e_{ij}))_{nn} \\
&- \left[\frac{\lambda_x}{2\mu_x + \lambda_x}\right](A_x(e(w^x_{kl}) - e_{kl}))_{nn}\mathrm{tr}(e(w^x_{ij}) - e_{ij})_{tt} \\
&+ \left[\frac{1}{\mu_x}\right](A_x(e(w^x_{kl}) - e_{kl}))_{tn}(A_x(e(w^x_{ij}) - e_{ij}))_{tn}
\end{aligned}
$$

and analogously for the fourth integrand $A_x(e(w^x_{ij}) - e_{ij})n \cdot \frac{\partial(w^{x,1}_{kl} - w^{x,0}_{kl})}{\partial n}$. Summing up all the results, we obtain for the shape derivative of $\mathfrak{L}^x_{ijkl}$

$$
\begin{aligned}
&\frac{\partial \mathfrak{L}^x_{ijkl}}{\partial Y_0}(w^{x,0}_{kl}, w^{x,1}_{kl}, -w^{x,0}_{ij}, -w^{x,1}_{ij}, Y_0)(\theta) \\
&\quad = - \int_{\Sigma_Y} \Big([2\mu_x](e(w^x_{kl}) - e_{kl})_{tt}(e(w^x_{ij}) - e_{ij})_{tt} \\
&\qquad + \left[\frac{2\mu_x\lambda_x}{2\mu_x + \lambda_x}\right]\mathrm{tr}(e(w^x_{kl}) - e_{kl})_{tt}\mathrm{tr}(e(w^x_{ij}) - e_{ij})_{tt} \\
&\qquad + \left[\frac{1}{\mu_x}\right](A_x(e(w^x_{kl}) - e_{kl}))_{tn}(A_x(e(w^x_{ij}) - e_{ij}))_{tn} \\
&\qquad + \left[\frac{1}{2\mu_x + \lambda_x}\right](A_x(e(w^x_{kl}) - e_{kl}))_{nn}(A_x(e(w^x_{ij}) - e_{ij}))_{nn}\Big)\,\theta \cdot n \,\mathrm{d}S(y) \\
&\quad + \int_{\Sigma_Y} \Big(\left[\frac{1}{2\mu_x + \lambda_x}\right](A_x(e(w^x_{kl}) - e_{kl}))_{nn}(A_x(e(w^x_{ij}) - e_{ij}))_{nn} \\
&\qquad - \left[\frac{\lambda_x}{2\mu_x + \lambda_x}\right](A_x(e(w^x_{kl}) - e_{kl}))_{nn}\mathrm{tr}(e(w^x_{ij}) - e_{ij})_{tt} \\
&\qquad + \left[\frac{1}{\mu_x}\right](A_x(e(w^x_{kl}) - e_{kl}))_{tn}(A_x(e(w^x_{ij}) - e_{ij}))_{tn}\Big)\,\theta \cdot n \,\mathrm{d}S(y)
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_{\Sigma_Y} \Bigg( \left[ \frac{1}{2\mu_x + \lambda_x} \right] (A_x(e(w_{ij}^x) - e_{ij}))_{nn} (A_x(e(w_{kl}^x) - e_{kl}))_{nn} \\
&- \left[ \frac{\lambda_x}{2\mu_x + \lambda_x} \right] (A_x(e(w_{ij}^x) - e_{ij}))_{nn} \mathrm{tr}(e(w_{kl}^x) - e_{kl})_{tt} \\
&+ \left[ \frac{1}{\mu_x} \right] (A_x(e(w_{ij}^x) - e_{ij}))_{tn} (A_x(e(w_{kl}^x) - e_{kl}))_{tn} \Bigg) \theta \cdot n \, \mathrm{d}S(y).
\end{aligned}
$$

Rewriting this as one integral proves the following result.

**Theorem 9.2.17.** *The shape derivative of the Lagrangian $\mathfrak{L}_{ijkl}^x(Y_0)$ is of the form*

$$
\begin{aligned}
&\frac{\partial \mathfrak{L}_{ijkl}^x}{\partial Y_0}(w_{kl}^{x,0}, w_{kl}^{x,1}, -w_{ij}^{x,0}, -w_{ij}^{x,1}, Y_0)(\theta) \\
&\quad = \int_{\Sigma_Y} \Bigg( -[2\mu_x](e(w_{kl}^x) - e_{kl})_{tt} (e(w_{ij}^x) - e_{ij})_{tt} \\
&\qquad - \left[ \frac{2\mu_x \lambda_x}{2\mu_x + \lambda_x} \right] \mathrm{tr}(e(w_{kl}^x) - e_{kl})_{tt} \mathrm{tr}(e(w_{ij}^x) - e_{ij})_{tt} \\
&\qquad + \left[ \frac{1}{\mu_x} \right] (A_x(e(w_{kl}^x) - e_{kl}))_{tn} (A_x(e(w_{ij}^x) - e_{ij}))_{tn} \\
&\qquad + \left[ \frac{1}{2\mu_x + \lambda_x} \right] (A_x(e(w_{kl}^x) - e_{kl}))_{nn} (A_x(e(w_{ij}^x) - e_{ij}))_{nn} \\
&\qquad - \left[ \frac{\lambda_x}{2\mu_x + \lambda_x} \right] (A_x(e(w_{kl}^x) - e_{kl}))_{nn} \mathrm{tr}(e(w_{ij}^x) - e_{ij})_{tt} \\
&\qquad - \left[ \frac{\lambda_x}{2\mu_x + \lambda_x} \right] (A_x(e(w_{ij}^x) - e_{ij}))_{nn} \mathrm{tr}(e(w_{kl}^x) - e_{kl})_{tt} \Bigg) \theta \cdot n \, \mathrm{d}S(y)
\end{aligned}
$$

*for all $\theta \in [W_0^{1,\infty}(Y)]^3$.*

We want to derive some $Y$-periodic vector fields $\Theta_1, \Theta_2, \Theta_3 \in [W_0^{1,\infty}(Y)]^3$ such that

$$
Y_0[\tau_1 + \delta\tau_1, \tau_2 + \delta\tau_2, \tau_3 + \delta\tau_3] = (Id + \delta\tau_1\Theta_1 + \delta\tau_2\Theta_2 + \delta\tau_3\Theta_3)(Y_0[\tau])
$$

and

$$
Y_1[\tau_1 + \delta\tau_1, \tau_2 + \delta\tau_2, \tau_3 + \delta\tau_3] = (Id + \delta\tau_1\Theta_1 + \delta\tau_2\Theta_2 + \delta\tau_3\Theta_3)(Y_1[\tau])
$$

with small increment $(\delta\tau_1, \delta\tau_2, \delta\tau_3)$. Then, Lemma 9.2.14 yields

$$
\frac{\partial a_{ijkl}^{\mathrm{hom}}}{\partial \tau_h}[\tau](x) = \frac{1}{|Y|}(\mathfrak{J}_{ijkl}^x)'(\Theta_h) = \frac{1}{|Y|}\frac{\partial \mathfrak{L}_{ijkl}^x}{\partial Y_0}(\Theta_h) \tag{9.2.10}
$$

for $h \in \{1, 2, 3\}$, where the last term can be easily computed in the isotropic case by Theorem 9.2.17.

Now, we derive $\Theta_i$ explicitly in the case of ellipsoids, which are given by the implicit equation

$$
El(\tau_1, \tau_2, \tau_3) : \frac{1}{\tau_1^2}\Big(y_1 - \frac{l_1}{2}\Big)^2 + \frac{1}{\tau_2^2}\Big(y_2 - \frac{l_1}{2}\Big)^2 + \frac{1}{\tau_3^2}\Big(y_3 - \frac{l_3}{2}\Big)^2 = \frac{1}{4}
$$

for some $\tau \in I_\eta$. The idea is to define appropriate functions $\Theta_i$, which describe the displacement of the points in direction $e_i$. Let $i = 1$, the other functions follow analogously. We consider the displacement in the case, where we change $\tau_1$ to $\tau_1 + \delta\tau_1$ for some small $\delta\tau_1$ and the other parameters $\tau_2, \tau_3$ remain the same. Let $\hat{y}_2 \in (\frac{l_2}{2} - \frac{\tau_2}{2}, \frac{l_2}{2} + \frac{\tau_2}{2})$. Then, choose $, \hat{y}_3 \in \Big(\frac{l_3}{2} - \tau_3\sqrt{\frac{1}{4} - \frac{1}{\tau_2^2}(\hat{y}_2 - \frac{l_2}{2})^2}, \frac{l_3}{2} + \tau_3\sqrt{\frac{1}{4} - \frac{1}{\tau_2^2}(\hat{y}_2 - \frac{l_2}{2})^2}\Big)$. Thus, there exists $\hat{y}_1 \in (0, \frac{l_1}{2})$ with $\hat{y} = (\hat{y}_1, \hat{y}_2, \hat{y}_3) \in El(\tau_1, \tau_2, \tau_3)$. We demand that $\Theta_1$ fulfils

- $(\hat{y}_1, \hat{y}_2, \hat{y}_3) + \delta\tau_1\Theta_1(\hat{y}) = (\tilde{y}_1, \hat{y}_2, \hat{y}_3) \in El(\tau_1 + \delta\tau_1, \tau_2, \tau_3)$
- $\Theta_1(0, \hat{y}_2, \hat{y}_3) = \Theta_1(\frac{l_1}{2}, \hat{y}_2, \hat{y}_3) = \Theta_1(l_1, \hat{y}_2, \hat{y}_3) = 0.$

The first condition leads to

$$\begin{aligned}
\Theta_1(\hat{y}) =& \frac{1}{\delta\tau_1}(\tilde{y}_1 - \hat{y}_1)e_1 \\
=& \frac{1}{\delta\tau_1}\Bigg(\frac{l_1}{2} \pm (\tau_1 + \delta\tau_1)\sqrt{\frac{1}{4} - \frac{1}{\tau_2^2}\Big(\hat{y}_2 - \frac{l_2}{2}\Big)^2 - \frac{1}{\tau_3^2}\Big(\hat{y}_3 - \frac{l_3}{2}\Big)^2} \\
& - \frac{l_1}{2} \mp \tau_1\sqrt{\frac{1}{4} - \frac{1}{\tau_2^2}\Big(\hat{y}_2 - \frac{l_2}{2}\Big)^2 - \frac{1}{\tau_3^2}\Big(\hat{y}_3 - \frac{l_3}{2}\Big)^2}\Bigg) e_1 \\
=& \pm\sqrt{\frac{1}{4} - \frac{1}{\tau_2^2}\Big(\hat{y}_2 - \frac{l_2}{2}\Big)^2 - \frac{1}{\tau_3^2}\Big(\hat{y}_3 - \frac{l_3}{2}\Big)^2}\, e_1,
\end{aligned} \tag{9.2.11}$$

where we used the ellipsoid equation to find the representation of $\hat{y}_1$ and $\tilde{y}_1$. We make as an ansatz $\Theta_1(y_1, \hat{y}_2, \hat{y}_3) = a^{(l_1,l_2,l_3)}_{(\tau_1,\tau_2,\tau_3)}(\hat{y}_2, \hat{y}_3)\sin(by_1)e_1$ for some function $a$ and some constant $b$. Due to the second condition we choose $b = \frac{2\pi}{l_1}$ and due to (9.2.11) we receive

$$a^{(l_1,l_2,l_3)}_{(\tau_1,\tau_2,\tau_3)}(\hat{y}_2, \hat{y}_3) = -\frac{\sqrt{\frac{1}{4} - \frac{1}{\tau_2^2}(\hat{y}_2 - \frac{l_2}{2})^2 - \frac{1}{\tau_3^2}(\hat{y}_3 - \frac{l_3}{2})^2}}{\sin\Big(\frac{2\pi}{l_1}\tau_1\sqrt{\frac{1}{4} - \frac{1}{\tau_2^2}(\hat{y}_2 - \frac{l_2}{2})^2 - \frac{1}{\tau_3^2}(\hat{y}_3 - \frac{l_3}{2})^2}\Big)}.$$

Until now the function $\Theta_1$ is only defined for $y_1 \in [0, l_1]$ and

$$(y_2, y_3) \in \mathcal{A}_{\tau_2,\tau_3} := \left\{(y_2, y_3) \in \mathbb{R}^2 : \frac{1}{\tau_2^2}\Big(y_2 - \frac{l_2}{2}\Big)^2 + \frac{1}{\tau_3^2}\Big(y_3 - \frac{l_3}{2}\Big)^2 < \frac{1}{4}\right\}.$$

The next step is to find a continuous extension on $Y$. Let $y_1$ be fixed. We choose a sequence $(y_2, y_3) \subset \mathcal{A}_{\tau_2,\tau_3}$ such that

$$(y_2, y_3) \to (\tilde{y}_2, \tilde{y}_3) := \left(\tilde{y}_2, \frac{l_3}{2} \pm \tau_3\sqrt{\frac{1}{4} - \frac{1}{\tau_2^2}\Big(\tilde{y}_2 - \frac{l_2}{2}\Big)^2}\right) \in \partial\mathcal{A}_{\tau_2,\tau_3}$$

for some $\tilde{y}_2 \in \left(\frac{l_2}{2} - \frac{\tau_2}{2}, \frac{l_2}{2} + \frac{\tau_2}{2}\right)$. Therefore, we get the limit

$$\lim_{(y_2,y_3)\to(\tilde{y}_2,\tilde{y}_3)} \Theta_1(y) = \lim_{t\to 0, t>0} -\frac{t}{\sin(\frac{2\pi}{l_1}\tau_1 t)} \sin\left(\frac{2\pi}{l_1} y_1\right) e_1 = -\frac{l_1}{2\pi\tau_1} \sin\left(\frac{2\pi}{l_1} y_1\right) e_1.$$

Now we can define $\Theta_1$ on the whole domain $Y$

$$\Theta_1(y) = \begin{cases} a^{(l_1,l_2,l_3)}_{(\tau_1,\tau_2,\tau_3)}(y_2,y_3)\sin(\frac{2\pi}{l_1}y_1)e_1 & ,\text{if } \frac{1}{\tau_2^2}(y_2-\frac{l_2}{2})^2 + \frac{1}{\tau_3^2}(y_3-\frac{l_3}{2})^2 < \frac{1}{4} \\ -\frac{l_1}{2\pi\tau_1}\sin(\frac{2\pi}{l_1}y_1)e_1 & ,\text{if } \frac{1}{\tau_2^2}(y_2-\frac{l_2}{2})^2 + \frac{1}{\tau_3^2}(y_3-\frac{l_3}{2})^2 \geq \frac{1}{4} \end{cases} \tag{9.2.12}$$

and analogously

$$\Theta_2(y) = \begin{cases} a^{(l_2,l_1,l_3)}_{(\tau_2,\tau_1,\tau_3)}(y_1,y_3)\sin(\frac{2\pi}{l_2}y_2)e_2 & ,\text{if } \frac{1}{\tau_1^2}(y_1-\frac{l_1}{2})^2 + \frac{1}{\tau_3^2}(y_3-\frac{l_3}{2})^2 < \frac{1}{4} \\ -\frac{l_2}{2\pi\tau_2}\sin(\frac{2\pi}{l_2}y_2)e_2 & ,\text{if } \frac{1}{\tau_1^2}(y_1-\frac{l_1}{2})^2 + \frac{1}{\tau_3^2}(y_3-\frac{l_3}{2})^2 \geq \frac{1}{4} \end{cases} \tag{9.2.13}$$

and

$$\Theta_3(y) = \begin{cases} a^{(l_3,l_1,l_2)}_{(\tau_3,\tau_1,\tau_2)}(y_1,y_2)\sin(\frac{2\pi}{l_3}y_3)e_3 & ,\text{if } \frac{1}{\tau_1^2}(y_1-\frac{l_1}{2})^2 + \frac{1}{\tau_2^2}(y_2-\frac{l_2}{2})^2 < \frac{1}{4} \\ -\frac{l_3}{2\pi\tau_3}\sin(\frac{2\pi}{l_3}y_3)e_3 & ,\text{if } \frac{1}{\tau_1^2}(y_1-\frac{l_1}{2})^2 + \frac{1}{\tau_2^2}(y_2-\frac{l_2}{2})^2 \geq \frac{1}{4}. \end{cases} \tag{9.2.14}$$

It remains to prove that $\Theta_i \in [W^{1,\infty}_0(Y)]^3$ for $i \in \{1,2,3\}$. Clearly, $\Theta_1$ is continuous and piecewise continuous differentiable. It remains to prove that the boundary terms of the partial derivative exists, if we compute the limit to the boundary starting from the inner of the ellipse. We compute

$$\frac{\partial}{\partial y_1}(\Theta_1)_1(y) = \frac{\partial}{\partial y_1}\left(a^{(l_1,l_2,l_3)}_{(\tau_1,\tau_2,\tau_3)}(y_2,y_3)\sin\left(\frac{2\pi}{l_1}y_1\right)\right) = \frac{2\pi}{l_1} a^{(l_1,l_2,l_3)}_{(\tau_1,\tau_2,\tau_3)}(y_2,y_3)\cos\left(\frac{2\pi}{l_1}y_1\right)$$

and if we define $s(y_2,y_3) \coloneqq \sqrt{\frac{1}{4} - \frac{1}{\tau_2^2}(y_2-\frac{l_2}{2})^2 - \frac{1}{\tau_3^2}(y_3-\frac{l_3}{2})^2}$

$$\begin{aligned} \frac{\partial}{\partial y_2}(\Theta_1)_1(y) &= \frac{\partial}{\partial y_2}\left(a^{(l_1,l_2,l_3)}_{(\tau_1,\tau_2,\tau_3)}(y_2,y_3)\sin\left(\frac{2\pi}{l_1}y_1\right)\right) \\ &= \frac{\frac{1}{\tau_2^2}(y_2-\frac{l_2}{2})\left[\sin(\frac{2\pi}{l_1}\tau_1 s(y_2,y_3)) - \frac{2\pi}{l_1}\tau_1 s(y_2,y_3)\cos(\frac{2\pi}{l_1}\tau_1 s(y_2,y_3))\right]}{s(y_2,y_3)\sin^2(\frac{2\pi}{l_1}\tau_1 s(y_2,y_3))}\sin\left(\frac{2\pi}{l_1}y_1\right). \end{aligned}$$

We want to pass this term to the limit $(y_2,y_3) \to (\tilde{y}_2,\tilde{y}_3) \in \partial\mathcal{A}_{\tau_2,\tau_3}$. Therefore we use the

Taylor series of Sine and Cosine at 0

$$
\begin{aligned}
\sin\Big(\frac{2\pi}{l_1}\tau_1 s(y_2,y_3)\Big) =& \frac{2\pi}{l_1}\tau_1 s(y_2,y_3) - \frac{1}{6}\Big(\frac{2\pi}{l_1}\tau_1 s(y_2,y_3)\Big)^3 + \mathcal{O}(s(y_2,y_3)^5)\\
\cos\Big(\frac{2\pi}{l_1}\tau_1 s(y_2,y_3)\Big) =& 1 - \frac{1}{2}\Big(\frac{2\pi}{l_1}\tau_1 s(y_2,y_3)\Big)^2 + \mathcal{O}(s(y_2,y_3)^4)\\
\sin^2\Big(\frac{2\pi}{l_1}\tau_1 s(y_2,y_3)\Big) =& \Big(\frac{2\pi}{l_1}\tau_1 s(y_2,y_3)\Big)^2 + \mathcal{O}(s(y_2,y_3)^4).
\end{aligned}
$$

to rewrite the partial derivative

$$
\frac{\partial}{\partial y_2}(\Theta_1)_1(y) = \frac{1}{\tau_2^2}\Big(y_2 - \frac{l_2}{2}\Big)\frac{\big(-\frac{1}{6}+\frac{1}{2}\big)\,(\frac{2\pi}{l_1}\tau_1 s(y_2,y_3))^3 + \mathcal{O}(s(y_2,y_3)^5)}{(\frac{2\pi}{l_1}\tau_1)^2 s(y_2,y_3)^3 + \mathcal{O}(s(y_2,y_3)^5)}\sin\Big(\frac{2\pi}{l_1}y_1\Big).
$$

Passing to the limit and using the fact that $s(y_2,y_3)\to 0$ for $(y_2,y_3)\to(\tilde{y}_2,\tilde{y}_3)$, results in

$$
\lim_{(y_2,y_3)\to(\tilde{y}_2,\tilde{y}_3)}\frac{\partial}{\partial y_2}(\Theta_1)_1(y) = \frac{2\pi\tau_1}{3l_1\tau_2^2}\Big(\tilde{y}_2 - \frac{l_2}{2}\Big)\sin\Big(\frac{2\pi}{l_1}y_1\Big).
$$

Due to symmetry an analogous result holds for $\frac{\partial}{\partial y_3}(\Theta_1)_1(y)$. Summing up, this shows that $\Theta_1 \in [W_0^{1,\infty}(Y)]^3$. Due to the results above, we also obtain that the mapping $(Id + \delta\tau_1\Theta_1 + \delta\tau_2\Theta_2 + \delta\tau_3\Theta_3)(y)$ is strictly monotonically increasing for $\delta\tau_1, \delta\tau_2, \delta\tau_3$ small enough. Thus, $(Id + \delta\tau_1\Theta_1 + \delta\tau_2\Theta_2 + \delta\tau_3\Theta_3)(Y[\tau]) = Y[\tau_1+\delta\tau_1, \tau_2+\delta\tau_2, \tau_3+\delta\tau_3]$.

### 9.2.3. Gâteaux derivative of $\mathcal{J}$

Finally, we can derive the Gâteaux derivative of (9.2.1), namely of

$$
\mathcal{J}(\tau) \coloneqq \frac{1}{2}\int_{\partial\Omega}|\mathcal{L}_\tau(f,g) - u_m|^2 \mathrm{d}S(x).
$$

We know that $u[\tau]$ is the weak solution of

$$
\int_\Omega A^{\mathrm{hom}}[\tau]e(u[\tau])e(v)\,\mathrm{d}x = \int_\Omega f\cdot v\,\mathrm{d}x + \int_{\Gamma_{\mathrm{N}}} g\cdot v\,\mathrm{d}S(x)
$$

and $u[\tau+\varepsilon\tilde{\tau}]$ of

$$
\int_\Omega A^{\mathrm{hom}}[\tau+\varepsilon\tilde{\tau}]e(u[\tau+\varepsilon\tilde{\tau}])e(v)\,\mathrm{d}x = \int_\Omega f\cdot v\,\mathrm{d}x + \int_{\Gamma_{\mathrm{N}}} g\cdot v\,\mathrm{d}S(x)
$$

for all $v \in H^1_{\Gamma_\mathrm{D}}(\Omega)$. Taking the difference of both equations, dividing by $\varepsilon$ and passing to the limit yields

$$\begin{aligned} 0 &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_\Omega A^{\mathrm{hom}}[\tau]e(u[\tau+\varepsilon\tilde\tau]-u[\tau])e(v)\,\mathrm{d}x \\ &\quad+\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_\Omega (A^{\mathrm{hom}}[\tau+\varepsilon\tilde\tau]-A^{\mathrm{hom}}[\tau])e(u[\tau+\varepsilon\tilde\tau])e(v)\,\mathrm{d}x \\ &= \int_\Omega A^{\mathrm{hom}}[\tau]e(\delta u(\tau,\tilde\tau))e(v)\,\mathrm{d}x + \int_\Omega \delta A^{\mathrm{hom}}(\tau,\tilde\tau)e(u[\tau])e(v)\,\mathrm{d}x. \end{aligned} \tag{9.2.15}$$

From the last subsection, we know that every element of $\delta A^{\mathrm{hom}}$ satisfies

$$\delta a^{\mathrm{hom}}_{ijkl}(\tau,\tilde\tau)(x) = \sum_{h=1}^{3}\frac{\partial a^{\mathrm{hom}}_{ijkl}}{\partial\tau_h}[\tau](x)\tilde\tau_h. \tag{9.2.16}$$

Furthermore, the first term of $\frac{\partial a^{\mathrm{hom}}_{ijkl}}{\partial\tau_h}[\tau]$ is an element of $L^\infty(\Omega)$

$$\begin{aligned} &\left|\int_{\Sigma_Y} -[2\mu_x](e(w^x_{kl})-e_{kl})_{tt}(e(w^x_{ij})-e_{ij})_{tt}\theta\cdot n\mathrm{d}S(y)\right| \\ &\le |[2\mu_x]|\|\theta\cdot n\|_{L^\infty(\Sigma_Y)}\int_{\Sigma_Y}|(e(w^x_{kl})-e_{kl})_{tt}(e(w^x_{ij})-e_{ij})_{tt}|\mathrm{d}S(y) \\ &\le 2\|\mu\|_{L^\infty(\Omega)}\|\theta\|_{L^\infty(\Sigma_Y)^3}\|(e(w^x_{kl})-e_{kl})_{tt}\|_{L^2(\Sigma_Y)}\|(e(w^x_{ij})-e_{ij})_{tt}\|_{L^2(\Sigma_Y)} \\ &\le C\left(\|(e(w^x_{kl}))_{tt}\|_{L^2(\Sigma_Y)}+\|(e_{kl})_{tt}\|_{L^2(\Sigma_Y)}\right)\left(\|(e(w^x_{ij}))_{tt}\|_{L^2(\Sigma_Y)}+\|(e_{ij})_{tt}\|_{L^2(\Sigma_Y)}\right) \\ &\le C\left(\|\sum_{i,j=1}^{3}t_ie_{ij}(w^x_{kl})t_j\|_{L^2(\Sigma_Y)}+\|2t_kt_l\|_{L^2(\Sigma_Y)}\right) \\ &\qquad\qquad\left(\|\sum_{k,l=1}^{3}t_ke_{kl}(w^x_{ij})t_l\|_{L^2(\Sigma_Y)}+\|2t_it_j\|_{L^2(\Sigma_Y)}\right) \\ &\le C\left(\|e(w^x_{kl})\|_{L^2(\Sigma_Y)^{3\times 3}}+2|\Sigma_Y|\right)\left(\|e(w^x_{ij})\|_{L^2(\Sigma_Y)^{3\times 3}}+2|\Sigma_Y|\right) \\ &\le C\left(\|w_{kl}\|_{L^\infty(\Omega,H^1(Y))^3}+2|\Sigma_Y|\right)\left(\|w_{ij}\|_{L^\infty(\Omega,H^1(Y))^3}+2|\Sigma_Y|\right) \\ &\le C. \end{aligned}$$

The proof for the other terms follow analogously. Thus, $\frac{\partial a^{\mathrm{hom}}_{ijkl}}{\partial\tau_h}[\tau]\in L^\infty(\Omega)$, and since $u[\tau]\in H^1_{\Gamma_\mathrm{D}}(\Omega)$ is some known quantity we can apply the theorem of Lax–Milgram to get the existence and uniqueness of the solution $\delta u(\tau,\tilde\tau)\in H^1_{\Gamma_\mathrm{D}}(\Omega)$ of (9.2.15). Using (9.2.16), we can rewrite the problem: Find for $h\in\{1,2,3\}$ the functions $\frac{\partial u}{\partial\tau_h}\in H^1_{\Gamma_\mathrm{D}}(\Omega)$ such that

$$\int_\Omega A^{\mathrm{hom}}[\tau]e\left(\frac{\partial u}{\partial\tau_h}\right)e(v)\,\mathrm{d}x = -\int_\Omega\frac{\partial A^{\mathrm{hom}}}{\partial\tau_h}[\tau]e(u[\tau])e(v)\,\mathrm{d}x.$$

Then, due to the uniqueness of the solutions,

$$\nabla u[\tau] \cdot \tilde{\tau} \coloneqq \sum_{h=1}^{3} \frac{\partial u}{\partial \tau_h} \tilde{\tau}_h = \delta u(\tau, \tilde{\tau}). \tag{9.2.17}$$

We derive the first variation of $\mathcal{J}$

$$\begin{aligned}
\delta \mathcal{J}(\tau, \tilde{\tau}) &= \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{\partial\Omega} |u[\tau + \varepsilon\tilde{\tau}] - u_m|^2 - |u[\tau] - u_m|^2 \mathrm{d}S(x) \\
&= \lim_{\varepsilon \to 0} \int_{\partial\Omega} \frac{1}{2}(u[\tau + \varepsilon\tilde{\tau}] + u[\tau] - 2u_m) \cdot \frac{1}{\varepsilon}(u[\tau + \varepsilon\tilde{\tau}] - u[\tau])\mathrm{d}S(x) \\
&= \int_{\partial\Omega} (u[\tau] - u_m) \cdot \delta u(\tau, \tilde{\tau})\mathrm{d}S(x).
\end{aligned}$$

Using (9.2.17), we can determine the Gâteaux derivative of the objective function

$$\begin{aligned}
\delta \mathcal{J}(\tau, \tilde{\tau}) &= \int_{\partial\Omega} (u[\tau] - u_m) \cdot \delta u(\tau, \tilde{\tau})\mathrm{d}S(x) \\
&= \int_{\partial\Omega} (u[\tau] - u_m) \cdot \nabla u[\tau]\mathrm{d}S(x)\tilde{\tau} =: \nabla \mathcal{J}(\tau) \cdot \tilde{\tau}.
\end{aligned} \tag{9.2.18}$$

# 10. Parameter identification for the time-dependent linearized elasticity problem

As in the steady-state case (see chapter 9), we want to identify the microstructure of a two-scale composite of two solids. But now we consider the time-dependent linearized elasticity equation. Therefore, we show that if measured data of the deformation on the exterior boundary over some time interval are given, we get at least one solution of the minimization problem, which is to identify a finite vector of real parameters describing the shape of the microstructure.
In section 10.1, we consider the direct problem, i.e. we prove the existence of the solution of the time-dependent linear elasticity problem and derive the homogenized problem by using the periodic unfolding method. In section 10.2, we show that there exists a solution of the inverse problem and derive the Gâteaux derivative of the target function by using the Gâteaux derivative of homogenized tensor from subsection 9.2.2.

## 10.1. Statement of the direct problem

Let $S = (0, T)$ with $0 < T < \infty$, $\Omega \subset \mathbb{R}^3$ be a open bounded connected Lipschitz-domain, $\Gamma_{\mathrm{D}} \subset \partial\Omega$ closed with positive two-dimensional Hausdorff measure and $\Gamma_{\mathrm{N}} := \partial\Omega \setminus \Gamma_{\mathrm{D}}$. Let $\nu$ be the outward-pointing normal to $\Gamma_{\mathrm{N}}$. We define the Banach spaces

$$H^1_{\Gamma_{\mathrm{D}}}(\Omega) := \left\{ u \in \left[H^1(\Omega)\right]^3 | u = 0 \text{ on } \Gamma_{\mathrm{D}} \right\} \quad \text{and} \quad L^2_{\varrho}(\Omega) := \left[L^2(\Omega)\right]^3$$

equipped with norms

$$\|u\|_{H^1_{\Gamma_{\mathrm{D}}}(\Omega)} = \|e(u)\|_{[L^2(\Omega)]^{3\times 3}} \quad \text{and} \quad \|u\|_{\varrho} = \sqrt{\langle u, u\rangle_{\varrho}},$$

where $\langle \cdot, \cdot \rangle_{\varrho}$ is the weighted scalar product

$$\langle u, v\rangle_{\varrho} := \int_{\Omega} \varrho(x) u(x) v(x) \mathrm{d}x$$

on the space $L^2_{\varrho}(\Omega) \times L^2_{\varrho}(\Omega)$ with $0 < \varrho \in L^\infty(\Omega)$. $\|\cdot\|_{H^1_{\Gamma_{\mathrm{D}}}(\Omega)}$ is a norm on $H^1_{\Gamma_{\mathrm{D}}}(\Omega)$ because of Korn's inequality for functions with zero value on part of the boundary (cf. Theorem 2.2.5).

Then, there holds
$$H^1_{\Gamma_{\mathrm{D}}}(\Omega) \subset L^2_\varrho(\Omega) = \left(L^2_\varrho(\Omega)\right)^* \subset (H^1_{\Gamma_{\mathrm{D}}}(\Omega))^*,$$
where $H^1_{\Gamma_{\mathrm{D}}}(\Omega)$ is a separable Hilbert space. If we write $L^2(\mathcal{O})$ for some open set $\mathcal{O}$, we equip this Hilbert space with the standard norm $\|u\|^2_{L^2(\mathcal{O})} = \int_{\mathcal{O}} |u|^2 \mathrm{d}x$.

### 10.1.1. General existence result

We consider the time-dependent linear elasticity equation as introduced in section 2.1 to modell the deformation of the domain under body load $f$ (given as a density per mass) and boundary force $g$ during time interval $S$

$$\begin{cases} \partial_t(\varrho\partial_t u) - \nabla\cdot(Ae(u)) &= \varrho f \quad & \text{in } S\times\Omega, \\ u &= 0 & \text{on } S\times\Gamma_{\mathrm{D}}, \\ (Ae(u))\nu &= g & \text{on } S\times\Gamma_{\mathrm{N}}, \\ u(0,x) &= u_0(x) & \text{a.e. in } \Omega, \\ \partial_t u(0,x) &= u_1(x) & \text{a.e. in } \Omega, \end{cases} \tag{10.1.1}$$

where $u_0$ and $u_1$ are the initial values of $u$ and $\partial_t u$ at time $t=0$. There exists a unique weak solution of this problem.

**Theorem 10.1.1.** *Let $\varrho_0 \in \mathbb{R}$, $\varrho \in L^\infty(\Omega)$ with $0 < \varrho_0 < \varrho(x)$ for a.e. $x\in\Omega$, $A \in M(\alpha,\beta,\Omega)$ (see Definition 2.1.1), $f \in L^2(S; L^2_\varrho(\Omega))$, $u_0 \in H^1_{\Gamma_{\mathrm{D}}}(\Omega)$, $u_1 \in L^2_\varrho(\Omega)$, $g \in H^1\left(S; \left[L^2(\Gamma_{\mathrm{N}})\right]^3\right)$. Then, there exists a unique weak solution $u \in L^2(S; H^1_{\Gamma_{\mathrm{D}}}(\Omega))$ with $u\in L^\infty(S;H^1_{\Gamma_{\mathrm{D}}}(\Omega))$, $\partial_t u \in L^\infty(S; L^2_\varrho(\Omega))$ and $\partial_t(\varrho\partial_t u) \in L^2(S; (H^1_{\Gamma_{\mathrm{D}}}(\Omega))^*)$ in the sense of distributions, as well as $u \in C^0(\bar{S}; L^2_\varrho(\Omega))$, of the problem (10.1.1), i.e. for all $v \in L^2(S; H^1_{\Gamma_{\mathrm{D}}}(\Omega))$ with $\partial_t v \in L^2(S; L^2_\varrho(\Omega))$ and $v(T) = 0$ there holds*

$$\begin{aligned} &-\int_0^T \int_\Omega \varrho\partial_t u\cdot\partial_t v \,\mathrm{d}x\mathrm{d}t + \int_0^T\int_\Omega Ae(u)e(v)\,\mathrm{d}x\mathrm{d}t \\ &\qquad = \int_0^T\int_\Omega \varrho f\cdot v\,\mathrm{d}x\mathrm{d}t + \int_0^T\int_{\Gamma_{\mathrm{N}}} g\cdot v\,\mathrm{d}S(x)\mathrm{d}t + \int_\Omega \varrho u_1\cdot v(0)\,\mathrm{d}x \end{aligned} \tag{10.1.2}$$

*and $u(0) = u_0$.*

*Proof.* We prove this theorem with the help of the Galerkin method and similar to the proof of Theorem 12.4 from [Schweizer, 2018].

**(i) Existence of Galerkin-solutions:** Since $H^1_{\Gamma_{\mathrm{D}}}(\Omega)$ is separable, we find $n$-dimensional subspaces $H^{1,n}_{\Gamma_{\mathrm{D}}}(\Omega) := \operatorname{span}\{w_1,\dots,w_n\} \subset H^1_{\Gamma_{\mathrm{D}}}(\Omega)$ such that for every $v\in H^1_{\Gamma_{\mathrm{D}}}(\Omega)$ there exists a sequence $\{v_n\}$ with $v_n \in H^{1,n}_{\Gamma_{\mathrm{D}}}(\Omega)$ such that $v_n$ converges strongly to $v$ in $H^1_{\Gamma_{\mathrm{D}}}(\Omega)$. Furthermore, let $\{u_{0n}\}$ and $\{u_{1n}\}$ sequences in $H^1_{\Gamma_{\mathrm{D}}}(\Omega)$ with $u_{0n}, u_{1n} \in H^{1,n}_{\Gamma_{\mathrm{D}}}(\Omega)$ and

$$u_{0n} \to u_0 \text{ strongly in } H^1_{\Gamma_{\mathrm{D}}}(\Omega) \quad\text{and}\quad u_{1n}\to u_1 \text{ strongly in } L^2_\varrho(\Omega). \tag{10.1.3}$$

We want to find a weak solution of problem (10.1.1) projected on the space $H^{1,n}_{\Gamma_\mathrm{D}}(\Omega)$. Meaning, we are searching for some function

$$u_n : S \to H^{1,n}_{\Gamma_\mathrm{D}}(\Omega), \quad u_n(t) = \sum_{j=1}^{n} z_{nj}(t) w_j \tag{10.1.4}$$

with $u_n(0) = u_{0n}$ and $\partial_t u_n(0) = u_{1n}$, such that, using $\partial_t(\varrho \partial_t u_n) = \varrho \partial_t^2(u_n)$,

$$\langle \partial_t^2 u_n(t), w_l \rangle_\varrho + a(u_n(t), w_l) = \langle f(t), w_l \rangle_\varrho + \langle g(t), w_l \rangle_{L^2(\Gamma_\mathrm{N})} \tag{10.1.5}$$

for every $l \in \{1, \ldots, n\}$ and a.e. $t \in S$, where

$$a(u, v) := \int_\Omega Ae(u)e(v)\,\mathrm{d}x.$$

If we use (10.1.4), we can rewrite (10.1.5)

$$\sum_{j=1}^{n} \partial_t^2 z_{nj}(t) \langle w_j, w_l \rangle_\varrho + \sum_{j=1}^{n} z_{nj}(t) a(w_j, w_l) = \langle f(t), w_l \rangle_\varrho + \langle g(t), w_l \rangle_{L^2(\Gamma_\mathrm{N})}. \tag{10.1.6}$$

Since the scalar product is coercive, the matrix $W \in \mathbb{R}^{n \times n}$ with $W_{jl} = \langle w_j, w_l \rangle_\varrho$ satisfies

$$z^T W z = z^T \begin{pmatrix} \langle w_1, w_1 \rangle_\varrho & \cdots & \langle w_1, w_n \rangle_\varrho \\ \vdots & \ddots & \vdots \\ \langle w_n, w_1 \rangle_\varrho & \cdots & \langle w_n, w_n \rangle_\varrho \end{pmatrix} z = \langle \sum_{i=1}^{n} z_i w_i, \sum_{j=1}^{n} z_j w_j \rangle_\varrho > 0$$

for all $z \neq 0 \in \mathbb{R}^n$. Thus, $W$ is positive definite and therefore invertible. So (10.1.6) is a linear ordinary differential equation (ODE) of second order. After reformulation as a system of ODEs of first order and using the initial conditions, we can apply Carathéodory theory, which guarantees the existence of a unique solution $u_n$ on the interval $\bar{S}$.

**(ii) A priori estimates:** We multiply equation (10.1.6) with $\partial_t z_{nl}$, summarise over $l \leq n$ and integrate from 0 to $t_1$ with $0 < t_1 \leq T$. We receive

$$\begin{aligned} \int_0^{t_1} \langle \partial_t^2 u_n(t), &\partial_t u_n(t) \rangle_\varrho \,\mathrm{d}t + \int_0^{t_1} a(u_n(t), \partial_t u_n(t))\,\mathrm{d}t \\ &= \int_0^{t_1} \langle f(t), \partial_t u_n(t) \rangle_\varrho \,\mathrm{d}t + \int_0^{t_1} \langle g(t), \partial_t u_n(t) \rangle_{L^2(\Gamma_\mathrm{N})}\,\mathrm{d}t. \end{aligned}$$

Since $A$ is symmetric and independent of $t$ and we can apply integration by parts formula in the last integral, we get

$$\frac{1}{2}\left[\|\partial_t u_n(t)\|_\varrho^2 + a(u_n(t), u_n(t))\right]_0^{t_1} = \int_0^{t_1} \langle f(t), \partial_t u_n(t)\rangle_\varrho \mathrm{d}t$$
$$+ \langle g(t_1), u_n(t_1)\rangle_{L^2(\Gamma_\mathrm{N})} - \langle g(0), u_{0n}\rangle_{L^2(\Gamma_\mathrm{N})} - \int_0^{t_1} \langle \partial_t g(t), u_n(t)\rangle_{L^2(\Gamma_\mathrm{N})}\mathrm{d}t.$$

We can estimate the left-hand side using the coercivity and boundedness of $A$

$$\frac{1}{2}\|\partial_t u_n(t_1)\|_\varrho^2 + \frac{1}{2}a(u_n(t_1), u_n(t_1)) - \frac{1}{2}\|u_{1n}\|_\varrho^2 - \frac{1}{2}a(u_{0n}, u_{0n})$$
$$\geq \frac{1}{2}\|\partial_t u_n(t_1)\|_\varrho^2 + \frac{\alpha}{2}\|u_n(t_1)\|^2_{H^1_{\Gamma_\mathrm{D}}(\Omega)} - \frac{1}{2}\|u_{1n}\|_\varrho^2 - \frac{\beta}{2}\|u_{0n}\|^2_{H^1_{\Gamma_\mathrm{D}}(\Omega)}$$

and the right-hand side using Hölder's and Young's inequality and the estimate of the trace operator

$$\int_0^{t_1} \langle f(t), \partial_t u_n(t)\rangle_\varrho \mathrm{d}t + \langle g(t_1), u_n(t_1)\rangle_{L^2(\Gamma_\mathrm{N})} - \langle g(0), u_{0n}\rangle_{L^2(\Gamma_\mathrm{N})} - \int_0^{t_1} \langle \partial_t g(t), u_n(t)\rangle_{L^2(\Gamma_\mathrm{N})}\mathrm{d}t$$
$$\leq \int_0^{t_1} \|f(t)\|_\varrho \|\partial_t u_n(t)\|_\varrho \mathrm{d}t + \|g(t_1)\|_{[L^2(\Gamma_\mathrm{N})]^3}\|u_n(t_1)\|_{[L^2(\Gamma_\mathrm{N})]^3}$$
$$+ \|g(0)\|_{[L^2(\Gamma_\mathrm{N})]^3}\|u_{0n}\|_{[L^2(\Gamma_\mathrm{N})]^3} + \int_0^{t_1} \|\partial_t g(t)\|_{[L^2(\Gamma_\mathrm{N})]^3}\|u_n(t)\|_{[L^2(\Gamma_\mathrm{N})]^3}\mathrm{d}t$$
$$\leq \frac{1}{2}\int_0^{t_1} \|f(t)\|_\varrho^2 \mathrm{d}t + \frac{1}{2}\int_0^{t_1} \|\partial_t u_n(t)\|_\varrho^2 \mathrm{d}t + \frac{1}{2\varepsilon}\|g(t_1)\|^2_{[L^2(\Gamma_\mathrm{N})]^3} + \frac{\varepsilon}{2}C_{trace}\|u_n(t_1)\|^2_{H^1_{\Gamma_\mathrm{D}}(\Omega)}$$
$$+ \frac{1}{2}\|g(0)\|^2_{[L^2(\Gamma_\mathrm{N})]^3} + \frac{1}{2}C_{trace}\|u_{0n}\|^2_{H^1_{\Gamma_\mathrm{D}}(\Omega)} + \frac{1}{2}\int_0^{t_1} \|\partial_t g(t)\|^2_{[L^2(\Gamma_\mathrm{N})]^3}\mathrm{d}t$$
$$+ \frac{1}{2}C_{trace}\int_0^{t_1} \|u_n(t)\|^2_{H^1_{\Gamma_\mathrm{D}}(\Omega)}\mathrm{d}t.$$

Choosing $\varepsilon = \frac{\alpha}{2C_{trace}}$ and using the fact that $g$ is continuous in $t$, we receive

$$\|\partial_t u_n(t_1)\|_\varrho^2 + \frac{\alpha}{2}\|u_n(t_1)\|^2_{H^1_{\Gamma_\mathrm{D}}(\Omega)}$$
$$\leq \|u_{1n}\|_\varrho^2 + (\beta + C_{trace})\|u_{0n}\|^2_{H^1_{\Gamma_\mathrm{D}}(\Omega)} + \int_0^{t_1} C_{trace}\|u_n(t)\|^2_{H^1_{\Gamma_\mathrm{D}}(\Omega)} + \|\partial_t u_n(t)\|_\varrho^2 \mathrm{d}t$$
$$+ \int_0^{t_1} \|f(t)\|_\varrho^2 + \frac{2C^T_{trace}C_{trace}}{\alpha}\|g(t)\|^2_{[L^2(\Gamma_\mathrm{N})]^3} + C^T_{trace}\|g(t)\|^2_{[L^2(\Gamma_\mathrm{N})]^3} + \|\partial_t g(t)\|^2_{[L^2(\Gamma_\mathrm{N})]^3}\mathrm{d}t,$$

where $C^T_{trace}$ is the continuity constant of the trace operator of Bochner spaces (cf. Theorem 2.2.2). Gronwall's Lemma yields

$$\|\partial_t u_n(t_1)\|_\varrho^2 + \|u_n(t_1)\|^2_{H^1_{\Gamma_\mathrm{D}}(\Omega)}$$
$$\leq c_1 e^{c_2 t_1}\left(\|u_{1n}\|_\varrho^2 + \|u_{0n}\|^2_{H^1_{\Gamma_\mathrm{D}}(\Omega)} + \|f\|^2_{L^2(S;L^2_\varrho)} + \|g\|^2_{H^1(S;[L^2(\Gamma_\mathrm{N})]^3)}\right)$$

with constants $c_1, c_2$ only depending on $\alpha$, $\beta$. Since every convergent sequence is bounded, we get (for $n$ big enough) the a priori estimates

$$\begin{aligned}
\|\partial_t u_n\|^2_{L^\infty(S;L^2_\varrho(\Omega))} &\le c_3 e^{c_2 T}\left(\|u_1\|^2_\varrho + \|u_0\|^2_{H^1_{\Gamma_\mathrm{D}}(\Omega)} + \|f\|^2_{L^2(S;L^2_\varrho)} + \|g\|^2_{H^1(S;[L^2(\Gamma_\mathrm{N})]^3)}\right),\\
\|u_n\|^2_{L^\infty(S;H^1_{\Gamma_\mathrm{D}}(\Omega))} &\le c_3 e^{c_2 T}\left(\|u_1\|^2_\varrho + \|u_0\|^2_{H^1_{\Gamma_\mathrm{D}}(\Omega)} + \|f\|^2_{L^2(S;L^2_\varrho)} + \|g\|^2_{H^1(S;[L^2(\Gamma_\mathrm{N})]^3)}\right).
\end{aligned} \tag{10.1.7}$$

**(iii) Passing to the limit $n \to \infty$:** Due to the estimate (10.1.7), the sequence $\{u_n\}$ (resp. $\{\partial_t u_n\}$) is uniformly bounded in $L^\infty(S; H^1_{\Gamma_\mathrm{D}}(\Omega))$ (resp. $L^\infty(S; L^2_\varrho(\Omega))$). Therefore, there exists a weakly-* convergent subsequence such that

$$\begin{aligned}
u_n &\overset{*}{\rightharpoonup} u \text{ weakly-* in } L^\infty(S; H^1_{\Gamma_\mathrm{D}}(\Omega)),\\
\partial_t u_n &\overset{*}{\rightharpoonup} \partial_t u \text{ weakly-* in } L^\infty(S; L^2_\varrho(\Omega)).
\end{aligned} \tag{10.1.8}$$

We multiply equation (10.1.5) with some function $\varphi \in C^1(\bar{S})$ with $\varphi(T) = 0$, integrate over $S$ and integrate by parts in the first integral

$$\begin{aligned}
-\int_0^T \langle \partial_t u_n(t), \partial_t\varphi(t) w_l\rangle_\varrho \mathrm{d}t &+ \int_0^T a(u_n(t), \varphi(t) w_l)\mathrm{d}t\\
&= \int_0^T \langle f(t), \varphi(t) w_l\rangle_\varrho \mathrm{d}t + \int_0^T \langle g(t), \varphi(t) w_l\rangle_{L^2(\Gamma_\mathrm{N})}\mathrm{d}t + \langle u_{1n}, \partial_t\varphi(0) w_l\rangle_\varrho.
\end{aligned}$$

Due to the weak-* convergences (10.1.8), we can pass to the limit

$$\begin{aligned}
-\int_0^T \langle \partial_t u(t), \partial_t\varphi(t) w_l\rangle_\varrho \mathrm{d}t &+ \int_0^T a(u(t), \varphi(t) w_l)\mathrm{d}t\\
&= \int_0^T \langle f(t), \varphi(t) w_l\rangle_\varrho \mathrm{d}t + \int_0^T \langle g(t), \varphi(t) w_l\rangle_{L^2(\Gamma_\mathrm{N})}\mathrm{d}t + \langle u_1, \partial_t\varphi(0) w_l\rangle_\varrho.
\end{aligned}$$

The linear span of the function $\varphi w_l$, $l \in \mathbb{N}$, is dense in $L^2(S; H^1_{\Gamma_\mathrm{D}}(\Omega))$ (more details can be found in the proof of Theorem 12.4 in [Schweizer, 2018]). Clearly,

$$\begin{aligned}
\int_0^T \langle \partial_t u_n(t), \varphi(t) w_l\rangle_\varrho \mathrm{d}t &= -\int_0^T \langle u_n(t), \partial_t\varphi(t) w_l\rangle_\varrho \mathrm{d}t - \langle u_{0n}, \varphi(0) w_l\rangle_\varrho,\\
\int_0^T \langle \partial_t u(t), \varphi(t) w_l\rangle_\varrho \mathrm{d}t &= -\int_0^T \langle u(t), \partial_t\varphi(t) w_l\rangle_\varrho \mathrm{d}t - \langle u(0), \varphi(0) w_l\rangle_\varrho.
\end{aligned}$$

If we pass to limit $n \to \infty$, we get

$$\langle u_{0n}, \varphi(0) w_l\rangle_\varrho \to \langle u(0), \varphi(0) w_l\rangle_\varrho,$$

which shows that $u_{0n} \rightharpoonup u(0)$ in $L^2_\varrho(\Omega)$. Together with (10.1.3) it follows that $u(0) = u_0$. So we have found a weak solution $u$ of (10.1.1) with $u \in L^\infty(S; H^1_{\Gamma_\mathrm{D}}(\Omega))$, $\partial_t u \in L^\infty(S; L^2_\varrho(\Omega))$ and $\partial_t(\varrho\partial_t u) \in L^2(S; (H^1_{\Gamma_\mathrm{D}}(\Omega))^*)$ in the sense of distributions. Thus, $u \in L^2(S; H^1_{\Gamma_\mathrm{D}}(\Omega))$,

$\partial_t u \in L^2(S; L^2_\varrho(\Omega))$ and due to Theorem 10.9 from [Schweizer, 2018] $u \in C^0(\bar{S}; L^2_\varrho(\Omega))$.
**(iv) Uniqueness of the solution:** We assume that there exists two weak solutions $u_a$ and $u_b$. Due to the linearity, there holds for $u \coloneqq u_a - u_b$

$$u(0) = 0, \quad \partial_t u(0) = 0$$

and

$$-\int_0^T \langle \partial_t u(t), \partial_t v(t) \rangle_\varrho \, \mathrm{d}t + \int_0^T a(u(t), v(t)) \, \mathrm{d}t = 0$$

for all $v \in L^2(S; H^1_{\Gamma_\mathrm{D}}(\Omega))$ with $\partial_t v \in L^2(S; L^2_\varrho(\Omega))$ and $v(T) = 0$. Let $0 \le s \le T$, $\chi_s$ the characteristic function of the interval $[0, s]$ and

$$v(t) \coloneqq \int_0^t \chi_s(\tau) u(\tau) \, \mathrm{d}\tau - \int_0^T \chi_s(\tau) u(\tau) \, \mathrm{d}\tau.$$

Then, $v$ is an admissable test function, $v(t) = v(T) = 0$ for $s \le t \le T$, $v$ is absolute continuous in $[0, T]$ and $\partial_t v(t) = \chi_s(t) u(t)$ a.e. in $S$. So we get for the first integral

$$-\int_0^T \langle \partial_t u(t), \partial_t v(t) \rangle_\varrho \, \mathrm{d}t = -\int_0^s \langle \partial_t u(t), u(t) \rangle_\varrho \, \mathrm{d}t = -\frac{1}{2} \|u(s)\|_\varrho^2.$$

Similarly, we receive

$$\int_0^T a(u(t), v(t)) \, \mathrm{d}t = \int_0^s a(\partial_t v(t), v(t)) \, \mathrm{d}t = -\frac{1}{2} a(v(0), v(0)).$$

Summing up,

$$0 = \frac{1}{2} \|u(s)\|_\varrho^2 + \frac{1}{2} a(v(0), v(0)) \ge \frac{1}{2} \|u(s)\|_\varrho^2 + \frac{\alpha}{2} \|v(0)\|^2_{H^1_{\Gamma_\mathrm{D}}(\Omega)}$$

for a.e. $s \in S$. Therefore, $u = 0$ a.e. □

As a simple consequence we receive a linear continuous operator, which maps the initial values and given forces to the solution of (10.1.2).

**Corollary 10.1.2.** *Under the same assumption as in Theorem 10.1.1, we get the linear and continuous operator*

$$\mathcal{L} : H^1_{\Gamma_\mathrm{D}}(\Omega) \times L^2_\varrho(\Omega) \times L^2(S; L^2_\varrho(\Omega)) \times H^1(S; \left[L^2(\Gamma_\mathrm{N})\right]^3) \to L^\infty(S; H^1_{\Gamma_\mathrm{D}}(\Omega)) \times L^\infty(S; L^2_\varrho(\Omega))$$

*with*

$$\mathcal{L}(u_0, u_1, f, g) = (u, \partial_t u),$$

*where $u$ is the weak solution of* (10.1.1)*. Furthermore, there exists a constant $C$ independent* *of $u_0, u_1, f, g$, such that*

$$\|u\|^2_{L^\infty(S;H^1_{\Gamma_\mathrm{D}}(\Omega))} + \|\partial_t u\|^2_{L^\infty(S;L^2_\varrho(\Omega))} \leq C\left(\|u_1\|^2_\varrho + \|u_0\|^2_{H^1_{\Gamma_\mathrm{D}}(\Omega)} + \|f\|^2_{L^2(S;L^2_\varrho(\Omega))} + \|g\|^2_{H^1(S;[L^2(\Gamma_\mathrm{N})]^3)}\right)$$

*Proof.* This result follows directly from the proof of Theorem 10.1.1 using (10.1.7) and the weak lower semicontinuity of the norm. □

### 10.1.2. Periodic and homogenized problem

We define the reference cell $Y = (0, l_1) \times (0, l_2) \times (0, l_3) \subset \mathbb{R}^3$ with $l_1, l_2, l_3 > 0$. We consider a bounded sequences $\{(A^\varepsilon, \varrho^\varepsilon)\}$ in $M(\alpha, \beta, \Omega) \times L^\infty(\Omega)$, where $A^\varepsilon$ is a tensor of fourth order and $\varrho^\varepsilon$ a mass density. Additionally, all $\varrho^\varepsilon$ should satisfy $0 < \varrho_0 < \varrho^\varepsilon(x) < \varrho_1$ for some $\varrho_0, \varrho_1 \in \mathbb{R}$ and for a.e. $x \in \Omega$. The norms $\|\cdot\|_{L^2_{\varrho^\varepsilon}}$ and $\|\cdot\|_{L^2(\Omega)}$ are equivalent since

$$\sqrt{\varrho_0}\|u\|_{[L^2(\Omega)]^3} \leq \|u\|_{\varrho^\varepsilon} \leq \sqrt{\varrho_1}\|u\|_{[L^2(\Omega)]^3}. \tag{10.1.9}$$

We define for every $\varepsilon$ the time-dependent linear elasticity problem

$$\begin{cases} \partial_t(\varrho^\varepsilon \partial_t u^\varepsilon) - \nabla\cdot(A^\varepsilon e(u^\varepsilon)) = f & \text{in } S\times\Omega, \\ u^\varepsilon = 0 & \text{on } S\times\Gamma_\mathrm{D}, \\ A^\varepsilon e(u^\varepsilon)\nu = g & \text{on } S\times\Gamma_\mathrm{N}, \\ u^\varepsilon(0,x) = u_0(x) & \text{a.e. in } \Omega, \\ \partial_t u^\varepsilon(0,x) = u_1(x) & \text{a.e. in } \Omega. \end{cases} \tag{10.1.10}$$

We assume that $f$ is given as a volume force. It can be rewritten as $\varrho^\varepsilon \frac{f}{\varrho^\varepsilon}$ with $\frac{f}{\varrho^\varepsilon}$ force per mass, which is well-defined since $0 < \varrho_0 < \varrho^\varepsilon$. Thus, we can apply Theorem 10.1.1 to get the existence and uniqueness of the weak solution.

**Theorem 10.1.3.** *Let $(A^\varepsilon, \varrho^\varepsilon)$ be defined as above, $u_0 \in H^1_{\Gamma_\mathrm{D}}(\Omega)$, $u_1 \in [L^2(\Omega)]^3$, $f \in \left[L^2(S\times\Omega)\right]^3$ and $g \in H^1(S;[L^2(\Gamma_\mathrm{N})]^3)$. Then, there exists a unique weak solution $u^\varepsilon \in L^2(S;H^1_{\Gamma_\mathrm{D}}(\Omega))$ of* (10.1.10) *with $u^\varepsilon \in L^\infty(S;H^1_{\Gamma_\mathrm{D}}(\Omega))$, $\partial_t u^\varepsilon \in L^\infty(S;[L^2(\Omega)]^3)$ and $\partial_t(\varrho^\varepsilon\partial_t u^\varepsilon) \in L^2(S;(H^1_{\Gamma_\mathrm{D}}(\Omega))^*)$ in the sense of distributions, as well as $u^\varepsilon \in C^0(\bar{S};[L^2(\Omega)]^3)$, i.e. for all $v \in L^2(S;H^1_{\Gamma_\mathrm{D}}(\Omega))$ with $\partial_t v \in L^2(S;L^2_{\varrho^\varepsilon}(\Omega))$ and $v(T) = 0$ there holds*

$$\begin{aligned} -\int_0^T\int_\Omega \varrho^\varepsilon\partial_t u^\varepsilon\cdot\partial_t v\,\mathrm{d}x\mathrm{d}t + \int_0^T\int_\Omega A^\varepsilon e(u^\varepsilon)e(v)\,\mathrm{d}x\mathrm{d}t \\ = \int_0^T\int_\Omega f\cdot v\,\mathrm{d}x\mathrm{d}t + \int_0^T\int_{\Gamma_\mathrm{N}} g\cdot v\,\mathrm{d}S(x)\mathrm{d}t + \int_\Omega \varrho^\varepsilon u_1\cdot v(0)\,\mathrm{d}x \end{aligned} \tag{10.1.11}$$

*and* $u^\varepsilon(0) = u_0$. *Furthermore,*

$$\begin{aligned}\|u^\varepsilon\|^2_{L^\infty(S;H^1_{\Gamma_\mathrm{D}}(\Omega))} &+ \|\partial_t u^\varepsilon\|^2_{L^\infty(S;[L^2(\Omega)]^3)} \\ &\leq C\left(\|u_1\|^2_{[L^2(\Omega)]^3} + \|u_0\|^2_{H^1_{\Gamma_\mathrm{D}}(\Omega)} + \|f\|^2_{L^2(S;[L^2(\Omega)]^3)} + \|g\|^2_{H^1(S;[L^2(\Gamma_\mathrm{N})]^3)}\right)\end{aligned} \tag{10.1.12}$$

*for some constant* $C$ *independent of* $\varepsilon$.

*Proof.* The result follows directly from Theorem 10.1.1 and Corollary 10.1.2 and the equivalence of the norms (10.1.9). □

Under the assumptions of Theorem 10.1.3

$$u^\varepsilon \overset{*}{\rightharpoonup} u \text{ weakly-* in } L^\infty(S;H^1_{\Gamma_\mathrm{D}}(\Omega)) \quad \text{and} \quad \partial_t u^\varepsilon \overset{*}{\rightharpoonup} \partial_t u \text{ weakly-* in } L^\infty(S;[L^2(\Omega)]^3).$$

Since $L^\infty(S;H^1_{\Gamma_\mathrm{D}}(\Omega)) \subset L^2(S;H^1_{\Gamma_\mathrm{D}}(\Omega))$ and $L^\infty(S;[L^2(\Omega)]^3) \subset \left[L^2(S\times\Omega)\right]^3$, we even know that

$$u^\varepsilon \rightharpoonup u \text{ weakly in } L^2(S;H^1_{\Gamma_\mathrm{D}}(\Omega)) \quad \text{and} \quad \partial_t u^\varepsilon \rightharpoonup \partial_t u \text{ weakly in } \left[L^2(S\times\Omega)\right]^3. \tag{10.1.13}$$

We want to pass to the limit in (10.1.11). Therefore, we need the partial periodic unfolding operator $\mathcal{T}^\varepsilon_Y$ from Definition 3.2.4 and the properties stated in Proposition 3.2.5. For functions independent of time, we can use the standard periodic unfolding operator $\mathcal{T}^\varepsilon$ defined in Definition 3.2.1.

**Theorem 10.1.4.** *Let* $\{u^\varepsilon\}$ *be a sequence with* $u^\varepsilon \in L^\infty(S;H^1_{\Gamma_\mathrm{D}}(\Omega))$, $\partial_t u^\varepsilon \in L^\infty(S;[L^2(\Omega)]^3)$, $u^\varepsilon(0) = u_0$ *and*

$$\|u^\varepsilon\|^2_{L^2(S;H^1_{\Gamma_\mathrm{D}}(\Omega))} + \|\partial_t u^\varepsilon\|^2_{[L^2(S\times\Omega)]^3} \leq C$$

*for a constant* $C$ *independent of* $\varepsilon$. *Then, there exists a subsequence (again denoted by* $\{u^\varepsilon\}$*),* $u \in L^2(S;H^1_{\Gamma_\mathrm{D}}(\Omega)) \cap H^1(S;[L^2(\Omega)]^3)$ *with* $u(0) = u_0$ *and* $\hat{u} \in L^2(S\times\Omega;[H^1_{\mathrm{per},0}(Y)]^3)$ *such that*

$$\mathcal{T}^\varepsilon_Y(u^\varepsilon) \rightharpoonup u \textit{ weakly in } [L^2(S\times\Omega\times Y)]^3, \tag{10.1.14}$$

$$\mathcal{T}^\varepsilon_Y(\partial_t u^\varepsilon) \rightharpoonup \partial_t u \textit{ weakly in } [L^2(S\times\Omega\times Y)]^3, \tag{10.1.15}$$

$$\mathcal{T}^\varepsilon_Y(\nabla_x u^\varepsilon) \rightharpoonup \nabla u + \nabla_y \hat{u} \textit{ weakly in } [L^2(S\times\Omega\times Y)]^{3\times 3}. \tag{10.1.16}$$

*Proof.* Due to the uniform boundedness of $u^\varepsilon$, we can estimate by using Proposition 3.2.5 (ii)

$$\begin{aligned}&\|\mathcal{T}^\varepsilon_Y(u^\varepsilon)\|_{[L^2(S\times\Omega\times Y)]^3} \leq |Y|^{1/2}\|u^\varepsilon\|_{[L^2(S\times\Omega)]^3} \leq C,\\ &\|\partial_t\mathcal{T}^\varepsilon_Y(u^\varepsilon)\|_{[L^2(S\times\Omega\times Y)]^3} = \|\mathcal{T}^\varepsilon_Y(\partial_t u^\varepsilon)\|_{[L^2(S\times\Omega\times Y)]^3} \leq |Y|^{1/2}\|\partial_t u^\varepsilon\|_{[L^2(S\times\Omega)]^3} \leq C,\\ &\|\nabla_y\mathcal{T}^\varepsilon_Y(u^\varepsilon)\|_{[L^2(S\times\Omega\times Y)]^{3\times3}} = \|\varepsilon\mathcal{T}^\varepsilon_Y(\nabla u^\varepsilon)\|_{[L^2(S\times\Omega\times Y)]^{3\times3}} \leq \varepsilon|Y|^{1/2}\|\nabla u^\varepsilon\|_{[L^2(S\times\Omega)]^{3\times3}} \leq \varepsilon C.\end{aligned}$$

Thus,

$$\|\mathcal{T}^\varepsilon_Y(u^\varepsilon)\|_{H^1(S;[L^2(\Omega\times Y)]^3)} \leq C \quad \text{and} \quad \|\mathcal{T}^\varepsilon_Y(u^\varepsilon)\|_{L^2(S\times\Omega;[H^1(Y)]^3)} \leq C.$$

So there exist functions $u \in H^1(S; [L^2(\Omega \times Y)]^3)$ with

$$\mathcal{T}_Y^\varepsilon(u^\varepsilon) \rightharpoonup u \text{ weakly in } [L^2(S \times \Omega \times Y)]^3,$$
$$\partial_t \mathcal{T}_Y^\varepsilon(u^\varepsilon) \rightharpoonup \partial_t u \text{ weakly in } [L^2(S \times \Omega \times Y)]^3$$

and $\hat{u} \in L^2(S \times \Omega; [H^1(Y)]^3)$ with

$$\mathcal{T}_Y^\varepsilon(u^\varepsilon) \rightharpoonup \hat{u} \text{ weakly in } [L^2(S \times \Omega \times Y)]^3,$$
$$\nabla_y \mathcal{T}_Y^\varepsilon(u^\varepsilon) \to 0 \text{ weakly in } [L^2(S \times \Omega \times Y)]^{3\times 3}.$$

Therefore $\hat{u}$ is independent of $y$ and since the weak limit is unique, there holds $\hat{u} = u$. The condition $u(0) = u_0$ follows directly from $u^\varepsilon(0) = u_0$ for all $\varepsilon > 0$. Proposition 3.2.5 (vi) shows (10.1.16). □

Because of Proposition 3.2.5 (v), we even get that $u^\varepsilon \rightharpoonup \mathcal{M}_Y(u) = u$ weakly in $\left[L^2(S \times \Omega)\right]^3$ in Theorem 10.1.4. This yields that the limit function $u$ coincides with $u$ from (10.1.13). In the next step, we want to pass to the limit $\varepsilon \to 0$ in (10.1.11).

**Theorem 10.1.5.** *Let $\{(A^\varepsilon, \varrho^\varepsilon)\}$ be defined as above, $f \in \left[L^2(S \times \Omega)\right]^3$, $g \in H^1(S; [L^2(\Gamma_\mathrm{N})]^3)$, $u_0 \in H^1_{\Gamma_\mathrm{D}}(\Omega)$, $u_1 \in [L^2(\Omega)]^3$ and $\{u^\varepsilon\}$ the associated sequence of weak solutions of* (10.1.11). *Then, the weak convergences* (10.1.13),(10.1.14),(10.1.15) *and* (10.1.16) *hold. Suppose that*

$$B^\varepsilon = \mathcal{T}^\varepsilon(A^\varepsilon) \to B \textit{ a.e. in } \Omega \times Y$$

*and*

$$\mathcal{T}^\varepsilon(\varrho^\varepsilon) \to \varrho \textit{ a.e. in } \Omega \times Y.$$

*Then, $B \in M(\alpha, \beta, \Omega \times Y)$, $0 < \varrho_0 \leq \varrho(x) \leq \varrho_1$ for a.e. $x \in \Omega$ and*

$$(u, \hat{u}) \in L^2(S; H^1_{\Gamma_\mathrm{D}}(\Omega)) \times L^2(S \times \Omega; [H^1_{\mathrm{per},0}(Y)]^3)$$

*with $\partial_t u \in [L^2(S \times \Omega)]^3$ and $u(0) = u_0$ is the weak solution of*

$$\begin{aligned}
&-\int_0^T \int_\Omega \frac{1}{|Y|} \int_Y \varrho(x,y)\mathrm{d}y\, \partial_t u(t,x) \cdot \partial_t v(t,x)\, \mathrm{d}x\mathrm{d}t \\
&\quad + \frac{1}{|Y|} \int_0^T \int_{\Omega \times Y} B(x,y)(e(u)(t,x) + e_y(\hat{u})(t,x,y))(e(v)(t,x) + e_y(\hat{v})(t,x,y))\, \mathrm{d}x\mathrm{d}y\mathrm{d}t \\
&= \int_0^T \int_\Omega f(t,x) \cdot v(t,x)\, \mathrm{d}x\mathrm{d}t + \int_0^T \int_{\Gamma_\mathrm{N}} g \cdot v(t,x)\, \mathrm{d}S(x)\mathrm{d}t \\
&\quad + \int_\Omega \frac{1}{|Y|} \int_Y \varrho(x,y)\, \mathrm{d}y\, u_1(x) \cdot v(0,x)\, \mathrm{d}x
\end{aligned} \tag{10.1.17}$$

*for all $v \in L^2(S; H^1_{\Gamma_\mathrm{D}}(\Omega))$ with $\partial_t v \in [L^2(S \times \Omega)]^3$, $v(T) = 0$ and $\hat{v} \in L^2(S \times \Omega; [H^1_{\mathrm{per},0}(Y)]^3)$.*

*Proof.* The weak convergences (10.1.13)–(10.1.16) follow directly from (10.1.12) and Theorem 10.1.4. We have already proven in Theorem 9.1.2 that $B \in M(\alpha, \beta, \Omega \times Y)$. Since $0 < \varrho_0 < \varrho^\varepsilon(x) < \varrho_1$ for a.e. $x \in \Omega$ clearly $0 < \varrho_0 \leq \varrho(x,y) \leq \varrho_1$ for a.e. $(x,y) \in \Omega \times Y$. We rewrite the weak formulation (10.1.11) using the partial periodic unfolding method

$$\begin{aligned}
&-\frac{1}{|Y|}\int_0^T \int_{\Omega \times Y} \mathcal{T}^\varepsilon(\varrho^\varepsilon)\mathcal{T}_Y^\varepsilon(\partial_t u^\varepsilon) \cdot \mathcal{T}_Y^\varepsilon(\partial_t v)\,\mathrm{d}x\mathrm{d}y\mathrm{d}t \\
&\quad+\frac{1}{|Y|}\int_0^T \int_{\Omega \times Y} \mathcal{T}^\varepsilon(A^\varepsilon)\mathcal{T}_Y^\varepsilon(e(u^\varepsilon))\mathcal{T}_Y^\varepsilon(e(v))\,\mathrm{d}x\mathrm{d}y\mathrm{d}t + \mathcal{I}_1 \\
&=\frac{1}{|Y|}\int_0^T \int_{\Omega \times Y} \mathcal{T}_Y^\varepsilon(f) \cdot \mathcal{T}_Y^\varepsilon(v)\,\mathrm{d}x\mathrm{d}y\mathrm{d}t + \int_0^T \int_{\Gamma_\mathrm{N}} g \cdot v\,\mathrm{d}S(x)\mathrm{d}t \\
&\quad+\frac{1}{|Y|}\int_{\Omega \times Y} \mathcal{T}^\varepsilon(\varrho^\varepsilon)\mathcal{T}^\varepsilon(u_1) \cdot \mathcal{T}^\varepsilon(v(0))\,\mathrm{d}x\mathrm{d}y + \mathcal{I}_2,
\end{aligned} \tag{10.1.18}$$

where

$$\begin{aligned}
\mathcal{I}_1 &= -\int_0^T \int_{\Pi^\varepsilon} \varrho^\varepsilon \partial_t u^\varepsilon \cdot \partial_t v\,\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\Pi^\varepsilon} A^\varepsilon e(u^\varepsilon)e(v)\,\mathrm{d}x\mathrm{d}t \\
\mathcal{I}_2 &= \int_0^T \int_{\Pi^\varepsilon} f \cdot v\,\mathrm{d}x\mathrm{d}t + \int_{\Pi^\varepsilon} \varrho^\varepsilon u_1 \cdot v(0)\,\mathrm{d}x
\end{aligned}$$

We choose as test functions $v(t,x) = \varphi(t)w(x)$ with $\varphi \in C_c^1([0,T))$ and $w \in \mathcal{D}_{\Gamma_\mathrm{D}}(\overline{\Omega})$, where

$$\mathcal{D}_{\Gamma_\mathrm{D}}(\overline{\Omega}) \coloneqq \{\phi \in [C^\infty(\Omega)]^3 : v \text{ is equal to } 0 \text{ in a neighbourhood of } \Gamma_\mathrm{D}\}.$$

Then, it follows from Proposition 3.2.5 (iii)

$$\begin{aligned}
&\mathcal{T}_Y^\varepsilon(v) \to \varphi w \text{ strongly in } [L^2(S \times \Omega \times Y)]^3, \\
&\mathcal{T}_Y^\varepsilon(\partial_t v) \to \partial_t \varphi w \text{ strongly in } [L^2(S \times \Omega \times Y)]^3, \\
&\mathcal{T}_Y^\varepsilon(e(v)) \to \varphi e(w) \text{ strongly in } [L^2(S \times \Omega \times Y)]^{3\times 3}.
\end{aligned}$$

So passing to the limit in (10.1.18) and using the almost everywhere convergence of $\mathcal{T}^\varepsilon(\varrho^\varepsilon)(x,y)$ and $\mathcal{T}^\varepsilon(A^\varepsilon)(x,y)$, yields

$$\begin{aligned}
&-\frac{1}{|Y|}\int_0^T \int_{\Omega \times Y} \varrho(x,y)\partial_t u(t,x) \cdot \partial_t \varphi(t) w(x)\,\mathrm{d}x\mathrm{d}y\mathrm{d}t \\
&\qquad+\frac{1}{|Y|}\int_0^T \int_{\Omega \times Y} B(x,y)(e(u)(t,x) + e_y(\hat{u})(t,x,y))\varphi(t)e(w)(x)\,\mathrm{d}x\mathrm{d}y\mathrm{d}t \\
&=\frac{1}{|Y|}\int_0^T \int_{\Omega \times Y} f(t,x) \cdot \varphi(t)w(x)\,\mathrm{d}x\mathrm{d}y\mathrm{d}t + \int_0^T \int_{\Gamma_\mathrm{N}} g \cdot \varphi(t)w(x)\,\mathrm{d}S(x)\mathrm{d}t \\
&\qquad+\frac{1}{|Y|}\int_{\Omega \times Y} \varrho(x,y)u_1(x) \cdot \varphi(0)w(x)\,\mathrm{d}x\mathrm{d}y.
\end{aligned} \tag{10.1.19}$$

The terms $\mathcal{I}_1$ and $\mathcal{I}_2$ vanish because of Hölder's inequality and the fact that

$$\int_{\Pi^\varepsilon} |v|^2 \mathrm{d}x, \int_{\Pi^\varepsilon} |\partial_t v|^2 \mathrm{d}x, \int_{\Pi^\varepsilon} |e(v)|^2 \mathrm{d}x, \int_{\Pi^\varepsilon} |v(0)|^2 \mathrm{d}x \to 0.$$

In the next step, we choose as a testfunctions $v(t,x) = \varepsilon\varphi(t)\hat{w}^\varepsilon(x)$ with $\hat{w}^\varepsilon(x) = \hat{w}(x, \frac{x}{\varepsilon})$, where

$$\hat{w}(x,y) = (\psi_i(x)\eta_i(y))_{1\le i\le 3}$$

and $\varphi \in C_c^\infty(S)$, $\psi \in \mathcal{D}(\Omega)$ and $\eta \in \left[H^1_{\mathrm{per},0}(Y)\right]^3$. Similar as in the proof of Theorem 5.2.7, we get the convergences

$$\begin{aligned}
&\mathcal{T}^\varepsilon_Y(v) = \varepsilon\varphi\mathcal{T}^\varepsilon(\hat{w}^\varepsilon) \to 0 \text{ strongly in } [L^2(S\times\Omega\times Y)]^3,\\
&\mathcal{T}^\varepsilon_Y(\partial_t v) = \varepsilon\partial_t\varphi\mathcal{T}^\varepsilon(\hat{w}^\varepsilon) \to 0 \text{ strongly in } [L^2(S\times\Omega\times Y)]^3,\\
&\mathcal{T}^\varepsilon_Y(e(v)) = \varphi\mathcal{T}^\varepsilon(\varepsilon e(\hat{w}^\varepsilon)) \to \varphi e_y(\hat{w}) \text{ strongly in } [L^2(S\times\Omega\times Y)]^{3\times 3}.
\end{aligned}$$

So passing to the limit in (10.1.18) yields

$$\frac{1}{|Y|}\int_0^T \int_{\Omega\times Y} B(x,y)(e(u)(t,x) + e_y(\hat{u})(t,x,y))\varphi(t) e_y(\hat{w})(x,y)\,\mathrm{d}x\mathrm{d}y\mathrm{d}t = 0. \qquad (10.1.20)$$

If we add (10.1.19) and (10.1.20) and use the fact that $C^1_c([0,T)) \otimes \mathcal{D}_{\Gamma_\mathrm{D}}(\Omega)$ is dense in $L^2(S; H^1_{\Gamma_\mathrm{D}}(\Omega))$ (cf. Theorem 3.1 from [Bernard, 2011]) and $C_c^\infty(S) \otimes (\mathcal{D}(\Omega) \otimes H^1_{\mathrm{per},0}(Y))^3$ dense in $L^2(S\times\Omega; [H^1_{\mathrm{per},0}(Y)]^3)$, we obtain (10.1.17). □

We can reformulate the homogenized problem (10.1.17).

**Theorem 10.1.6.** *Find $u \in L^2(S; H^1_{\Gamma_\mathrm{D}}(\Omega))$ with $\partial_t u \in [L^2(S\times\Omega)]^3$ and $u(0) = u_0$ such that*

$$\begin{aligned}
&-\int_0^T\int_\Omega \mathcal{M}_Y(\varrho(x,\cdot))\partial_t u(t,x)\cdot\partial_t v(t,x)\,\mathrm{d}x\mathrm{d}t + \int_0^T\int_\Omega A^{\mathrm{hom}}(x)e(u)(t,x)e(v)(t,x)\,\mathrm{d}x\mathrm{d}t\\
&= \int_0^T\int_\Omega f(t,x)\cdot v(t,x)\,\mathrm{d}x\mathrm{d}t + \int_0^T\int_{\Gamma_\mathrm{N}} g\cdot v(t,x)\,\mathrm{d}S(x)\mathrm{d}t + \int_\Omega \mathcal{M}_Y(\varrho(x,\cdot))u_1(x)\cdot v(0,x)\,\mathrm{d}x
\end{aligned} \qquad (10.1.21)$$

*for all $v \in L^2(S; H^1_{\Gamma_\mathrm{D}}(\Omega))$ with $\partial_t v \in [L^2(S\times\Omega)]^3$ and $v(T) = 0$, where $A^{\mathrm{hom}} = (a^{\mathrm{hom}}_{ijkl})_{1\le i,j,k,l\le 3}$ with*

$$a^{\mathrm{hom}}_{ijkl}(x) = \frac{1}{|Y|}\int_Y B(x,y)e_{ij}(e_{kl} - e_y(w^{kl})(x,y))\mathrm{d}y$$

*for a.e. $x \in \Omega$ and $w^{kl} \in [L^\infty(\Omega, H^1_{\mathrm{per},0}(Y))]^3$, $k,l \in \{1,2,3\}$, is the unique solution of the cell problem*

$$\int_Y B(x,y)\left(e_y(w^{kl})(\cdot,y) - e_{kl}\right)e_y(v)(y)\mathrm{d}y = 0 \qquad (10.1.22)$$

*for all $v \in \left[H^1_{\mathrm{per},0}(Y)\right]^3$.*

*Proof.* We have already proven in Theorem 9.1.3 that there exists a unique solution $w^{kl} \in$

$[L^\infty(\Omega, H^1_{\mathrm{per},0}(Y))]^3$ of (10.1.22). Choosing $v = 0$ in (10.1.17) yields

$$\frac{1}{|Y|}\int_0^T \int_{\Omega\times Y} B(x,y)(e(u)(t,x) + e_y(\hat{u})(t,x,y))e_y(\hat{v})(t,x,y)\,\mathrm{d}x\mathrm{d}y\mathrm{d}t = 0$$

for all $\hat{v} \in L^2(S\times\Omega; [H^1_{\mathrm{per},0}(Y)]^3)$. Using the fundamental lemma of calculus of variations we receive

$$\frac{1}{|Y|}\int_Y B(x,y)(e(u)(t,x) + e_y(\hat{u})(t,x,y))e_y(\hat{v})(y)\,\mathrm{d}y = 0$$

for a.e. $t \in S$ and $x \in \Omega$ and all $\hat{v} \in \left[H^1_{\mathrm{per},0}(Y)\right]^3$. Due to Korn's inequality for periodic functions with mean value zero (see Corollary 2.2.6), there exists for a.e. $t \in S$ and $x \in \Omega$ a unique solution $\varphi \in \left[H^1_{\mathrm{per},0}(Y)\right]^3$ of

$$\frac{1}{|Y|}\int_Y B(x,y)(e(u)(t,x) + e_y(\varphi)(y))e_y(\hat{v})(y)\,\mathrm{d}y = 0 \tag{10.1.23}$$

for all $\hat{v} \in [H^1_{\mathrm{per},0}(Y)]^3$. With the same computation as in the proof of Theorem 9.1.3, we get that $\varphi = -\sum_{l,m=1}^3 e_{lm}(u)(t,x)w^{lm}(x,y)$ is a solution of (10.1.23). Due to uniqueness and since $(t,x)$ was arbitrary, we obtain that

$$\hat{u}(t,x,y) = -\sum_{l,m=1}^{3} e_{lm}(u)(t,x)w^{lm}(x,y).$$

So again with the same computation as in the proof of Theorem 9.1.3 we get the macroscopic problem (10.1.21). $\square$

We know from Theorem 9.1.9 that $A^{\mathrm{hom}} \in M(\alpha, \frac{\beta^2}{2}, \Omega)$. With this fact we can prove the uniqueness of the solution of the homogenized problem.

**Theorem 10.1.7.** *There exists a unique solution $u \in L^2(S; H^1_{\Gamma_\mathrm{D}}(\Omega))$ with $\partial_t u \in [L^2(S\times\Omega)]^3$ and $u(0) = u_0$ of problem (10.1.21).*

*Proof.* We assume that there exists two weak solutions $u_a$ and $u_b$. Due to the linearity, there holds $u(0) = 0$ for $u := u_a - u_b$ and

$$-\int_0^T\int_\Omega \mathcal{M}_Y(\varrho(x,\cdot))\partial_t u(t,x)\cdot\partial_t v(t,x)\,\mathrm{d}x\mathrm{d}t + \int_0^T\int_\Omega A^{\mathrm{hom}}(x)e(u)(t,x)e(v)(t,x)\,\mathrm{d}x\mathrm{d}t = 0$$

for all $v \in L^2(S; H^1_{\Gamma_\mathrm{D}}(\Omega))$ with $\partial_t v \in [L^2(S\times\Omega)]^3$ and $v(T) = 0$. Let $0 \le s \le T$, $\chi_s$ the characteristic function of the interval $[0,s]$ and

$$v(t,x) := \int_0^t \chi_s(\tau)u(\tau,x)\,\mathrm{d}\tau - \int_0^T \chi_s(\tau)u(\tau,x)\,\mathrm{d}\tau.$$

Then, $v$ is an admissable test function, $v(t,\cdot) = v(T,\cdot) = 0$ for $s \le t \le T$, $v$ is absolutely continuous in $[0,T]$ and $\partial_t v(t.x) = \chi_s(t)u(t,x)$ a.e. in $S\times\Omega$. So we can compute

$$\begin{aligned}
0 &= -\int_0^T\int_\Omega \mathcal{M}_Y(\varrho(x,\cdot))\partial_t u(t,x)\cdot\partial_t v(t,x)\,\mathrm{d}x\mathrm{d}t + \int_0^T\int_\Omega A^{\mathrm{hom}}(x)e(u)(t,x)e(v)(t,x)\,\mathrm{d}x\mathrm{d}t\\
&= -\int_0^s\int_\Omega \mathcal{M}_Y(\varrho(x,\cdot))\partial_t u(t,x)\cdot u(t,x)\,\mathrm{d}x\mathrm{d}t + \int_0^s\int_\Omega A^{\mathrm{hom}}(x)e(\partial_t v)(t,x)e(v)(t,x)\,\mathrm{d}x\mathrm{d}t\\
&= -\int_0^s\int_\Omega \mathcal{M}_Y(\varrho(x,\cdot))\partial_t u(t,x)\cdot u(t,x)\,\mathrm{d}x\mathrm{d}t - \frac{1}{2}\int_\Omega A^{\mathrm{hom}}(x)e(v)(0,x)e(v)(0,x)\,\mathrm{d}x,
\end{aligned}$$

where we have used the symmetry of $A^{\mathrm{hom}}$. The coercivity of $A^{\mathrm{hom}}$ yields

$$\begin{aligned}
0 &= \int_0^s\int_\Omega \mathcal{M}_Y(\varrho(x,\cdot))\partial_t u(t,x)\cdot u(t,x)\,\mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_\Omega A^{\mathrm{hom}}(x)e(v)(0,x)e(v)(0,x)\,\mathrm{d}x\\
&\geq \frac{1}{2}\varrho_0\|u(s,\cdot)\|^2_{[L^2(\Omega)]^3} + \frac{\alpha}{2}\|v(0,\cdot)\|^2_{H^1_{\Gamma_{\mathrm{D}}}(\Omega)}
\end{aligned}$$

for a.e. $s\in S$. Thus, $u=0$ a.e. $\square$

**Remark 10.1.8.** *The strong formulation of* (10.1.21) *is*

$$\begin{cases}
\partial_t\left(\mathcal{M}_Y(\varrho)\partial_t u\right) - \nabla\cdot\left(A^{\mathrm{hom}}e(u)\right) &= f \quad \text{in } S\times\Omega,\\
u &= 0 \quad \text{on } S\times\Gamma_{\mathrm{D}},\\
\left(A^{\mathrm{hom}}e(u)\right)\cdot\nu &= g \quad \text{on } S\times\Gamma_{\mathrm{N}},\\
u(0,x) &= u_0(x) \quad \text{a.e. in } \Omega,\\
\partial_t u(0,x) &= u_1(x) \quad \text{a.e. in } \Omega.
\end{cases}$$

*If $B(x,\cdot)$ is $Y$-periodic, we can rewrite the cell problem* (10.1.22) *in the strong form*

$$\begin{cases}
-\nabla_y\cdot\left(B(x,y)(e_y(w^{kl})(x,y)-e_{kl})\right) = 0 \text{ in } Y,\\
w^{kl}(x,\cdot)\, Y\text{-periodic with } \mathcal{M}_Y(w^{kl}(x,\cdot)) = 0.
\end{cases}$$

*for a.e. $x\in\Omega$, which is the same cell problem as in the steady-state case.*

## 10.2. Inverse problem

With the results from the previous section, we are able to compute the displacement field $u$ of the homogenized problem if body and boundary forces are given and the sequence of elasticity tensors $\{A^\varepsilon\}$ satisfies some properties. A classical example for $A^\varepsilon$ is a tensors of the form $A^\varepsilon(x) = A\left(\frac{x}{\varepsilon}\right)$ with $A$ $Y$-periodic, i.e. the material properties only depend on the microstructure. So in this case, the microstructure is known. From now on, we want to derive the microstructure of $Y$, when measured data on the exterior boundary under known volume and boundary forces $f$ and $g$ over some time interval $S$ are given.
We make the same structural assumptions about the reference cell $Y$ and choose the same sequence $\{A^\varepsilon\}$ of tensors of fourth order as stated in the beginning of section 9.2 in the steady-

state case. So $A^{\mathrm{hom}} = (a^{\mathrm{hom}}_{ijkl})_{1\le i,j,k,l\le 3} \in M(\alpha, \frac{\beta^2}{\alpha}, \Omega)$ with

$$\begin{aligned} a^{\mathrm{hom}}_{ijkl}[\tau](x) =& \frac{1}{|Y|}\int_{Y_0[\tau]} A^0(x) e_{ij}(e_{kl} - e_y(w^{kl})(x,y))\,\mathrm{d}y \\ &+ \frac{1}{|Y|}\int_{Y_1[\tau]} A^1(x) e_{ij}(e_{kl} - e_y(w^{kl})(x,y))\,\mathrm{d}y. \end{aligned}$$

Furthermore, we assume that $\varrho^\varepsilon[\tau]$ is of the form

$$\varrho^\varepsilon[\tau](x) = \varrho_0(x)\chi_{Y_0[\tau]}\left(\frac{x}{\varepsilon}\right) + \varrho_1(x)\chi_{Y_1[\tau]}\left(\frac{x}{\varepsilon}\right)$$

for some $0 < \varrho_0, \varrho_1 \in L^\infty(\Omega)$ such that

$$\begin{aligned} \mathcal{T}^\varepsilon(\varrho^\varepsilon[\tau])(x,y) &= \begin{cases} \varrho_0(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y)\chi_{Y_0[\tau]}(y) + \varrho_1(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y)\chi_{Y_1[\tau]}(y) & \text{for a.e. } (x,y) \in \Omega^\varepsilon \times Y, \\ 0 & \text{for a.e. } (x,y) \in \Pi^\varepsilon \times Y \end{cases} \\ &\to \varrho_0(x)\chi_{Y_0[\tau]}(y) + \varrho_1(x)\chi_{Y_1[\tau]}(y) =: \varrho[\tau](x,y) \end{aligned}$$

for a.e. $(x,y) \in \Omega \times Y$. We define the input–output operator, which maps the body and boundary forces and the initial values to the solution of the homogenized problem (10.1.21).

**Definition 10.2.1** (input–output operator)**.**

$$\mathcal{L}_\tau : [L^2(S\times\Omega)]^3 \times H^1(S;[L^2(\Gamma_{\mathrm{N}})]^3) \times H^1_{\Gamma_{\mathrm{D}}}(\Omega) \times [L^2(\Omega)]^3 \to \left[L^2(S\times\partial\Omega)\right]^3$$

*with*

$$(f, g, u_0, u_1) \mapsto u[\tau]|_{\partial\Omega},$$

*where* $u[\tau] \in L^2(S; H^1_{\Gamma_{\mathrm{D}}}(\Omega)) \cap H^1(S;[L^2(\Omega)]^3)$ *is the solution of the homogenized problem* (10.1.21) *for given* $\tau$.

This operator satisfies some properties.

**Theorem 10.2.2.** *The operator* $\mathcal{L}_\tau$ *is linear and continuous. Furthermore,* $u[\tau]$ *satisfies*

$$\|u[\tau]\|^2_{[L^2(S\times\partial\Omega)]^3} \le C\left(\|u_1\|^2_{[L^2(\Omega)]^3} + \|u_0\|^2_{H^1_{\Gamma_{\mathrm{D}}}(\Omega)} + \|f\|^2_{L^2(S;[L^2(\Omega)]^3)} + \|g\|^2_{H^1(S;[L^2(\Gamma_{\mathrm{N}})]^3)}\right)$$

*for some constant* $C$ *independent of the structure of the reference cell* $Y$.

*Proof.* Since $A^{\mathrm{hom}} \in M(\alpha, \frac{\beta^2}{\alpha}, \Omega)$, it follows directly from Theorem 10.1.1 and Corollary 10.1.2 that

$$\begin{aligned} &\|u[\tau]\|^2_{L^\infty(S;H^1_{\Gamma_{\mathrm{D}}}(\Omega))} + \|\partial_t u[\tau]\|^2_{L^\infty(S;[L^2(\Omega)]^3)} \\ &\qquad \le C\left(\|u_1\|^2_{[L^2(\Omega)]^3} + \|u_0\|^2_{H^1_{\Gamma_{\mathrm{D}}}(\Omega)} + \|f\|^2_{L^2(S;[L^2(\Omega)]^3)} + \|g\|^2_{H^1(S;[L^2(\Gamma_{\mathrm{N}})]^3)}\right) \end{aligned}$$

for some constant $C$ depending only on $\alpha$, $\beta$ but not on the structure of the periodicity cell $Y$. The continuity of the trace operator and $L^\infty \subset L^2$ yields the desired result. $\square$

We want to study the following inverse problem.

**Definition 10.2.3** (Inverse Problem)**.** *Let* $0 < \eta < \min\{l_1, l_2, l_3\}$*. Find* $\tau \in I_\eta$ *such that for given measured data* $u_m \in \left[L^2(S \times \partial\Omega)\right]^3$*, when forces* $(f, g)$ *are applied and initial conditions* $u_0, u_1$ *are given,* $\tau$ *is the solution of the minimization problem*

$$\operatorname*{arg\,min}_{\tau \in I_\eta} \mathcal{J}(\tau) \coloneqq \operatorname*{arg\,min}_{\tau \in I_\eta} \frac{1}{2} \|\mathcal{L}_\tau(f, g, u_0, u_1) - u_m\|^2_{[L^2(S\times\partial\Omega)]^3}. \tag{10.2.1}$$

## 10.2.1. Existence result

We prove that there exists at least one solution of the inverse problem (10.2.1).

**Theorem 10.2.4.** *The inverse problem* (10.2.1) *has at least one optimal solution* $\tau^* \in I_\eta$*.*

*Proof.* Let $\{\tau_n\}$ be a minimizing sequence in $I_\eta$ such that

$$\lim_{n\to\infty} \mathcal{J}(\tau_n) = \inf\{\mathcal{J}(\tau) : \tau \in I_\eta\} \geq 0.$$

Obviously, the sequence $\{\tau_n\}$ is bounded. Since $I_\eta$ is a compact set in $\mathbb{R}^3$, there exists a subsequence (again denoted by $\{\tau_n\}$) such that $\tau_n \to \tau^*$ for some $\tau^* \in I_\eta$ as $n \to \infty$. Let $\{u[\tau_n]\}$ be the associated sequence of weak solutions of the homogenized problem (10.1.21). We receive as in the proof of Theorem 10.2.2 the uniform boundedness of $\{u[\tau_n]\}$ in $L^2(S; H^1_{\Gamma_\mathrm{D}}(\Omega)) \cap H^1(S; [L^2(\Omega)]^3)$. Thus, there exists a subsequence of $\{\tau_n\}$ – again denoted by $\{\tau_n\}$ – such that

$$u[\tau_n] \rightharpoonup \tilde{u} \text{ weakly in } L^2(S; H^1_{\Gamma_\mathrm{D}}(\Omega)) \cap H^1(S; [L^2(\Omega)]^3). \tag{10.2.2}$$

In the next step we prove that $\tilde{u} = u[\tau^*]$. For every $\tau_n$ the function $u[\tau_n]$ is the solution of

$$a(u[\tau_n], v; \tau_n) = F(v; \tau_n)$$

for all $v \in L^2(S; H^1_{\Gamma_\mathrm{D}}(\Omega)) \cap H^1(S; [L^2(\Omega)]^3)$ with $v(T) = 0$, where

$$a : L^2(S; H^1_{\Gamma_\mathrm{D}}(\Omega)) \cap H^1(S; [L^2(\Omega)]^3) \times L^2(S; H^1_{\Gamma_\mathrm{D}}(\Omega)) \cap H^1(S; [L^2(\Omega)]^3) \to \mathbb{R}$$

is the bilinear form of the left-hand side of (10.1.21), i.e.

$$\begin{aligned} a(w, v; \hat{\tau}) = &- \int_0^T \int_\Omega \mathcal{M}_Y(\varrho[\hat{\tau}](x, \cdot))\partial_t w(t, x) \cdot \partial_t v(t, x) \,\mathrm{d}x\mathrm{d}t \\ &+ \int_0^T \int_\Omega A^{\mathrm{hom}}[\hat{\tau}](x) e(w)(t, x) e(v)(t, x) \,\mathrm{d}x\mathrm{d}t, \end{aligned}$$

and

$$F : L^2(S; H^1_{\Gamma_\mathrm{D}}(\Omega)) \cap H^1(S; [L^2(\Omega)]^3) \to \mathbb{R}$$

is the linear functional of the right-hand side, i.e.

$$\begin{aligned} F(v;\hat{\tau}) = & \int_0^T \int_\Omega f(t,x) \cdot v(t,x) \,\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\Gamma_\mathrm{N}} g \cdot v(t,x) \,\mathrm{d}S(x)\mathrm{d}t \\ & + \int_\Omega \mathcal{M}_Y(\varrho[\hat{\tau}](x,\cdot)) u_1(x) \cdot v(0,x) \,\mathrm{d}x. \end{aligned}$$

The index $\hat{\tau}$ emphasizes that the bilinear and linear form depend on the parameter $\hat{\tau}$ through $A^{\mathrm{hom}}[\hat{\tau}]$ and $\varrho[\hat{\tau}]$. For the readability, we omit the arguments $(t,x)$ of the functions. In the first step we show that

$$|a(u[\tau_n], v; \tau_n) - a(\tilde{u}, v; \tau^*)| \to 0 \tag{10.2.3}$$

for $n \to \infty$. We rewrite the difference and use Hölder's inequality

$$\begin{aligned} & |a(u[\tau_n], v; \tau_n) - a(\tilde{u}, v; \tau^*)| \\ = & \Bigg| -\int_0^T \int_\Omega \left(\mathcal{M}_Y(\varrho[\tau_n](x,\cdot)) - \mathcal{M}_Y(\varrho[\tau^*](x,\cdot))\right) \partial_t u[\tau_n] \cdot \partial_t v \,\mathrm{d}x\mathrm{d}t \\ & - \int_0^T \int_\Omega \mathcal{M}_Y(\varrho[\tau^*](x,\cdot))(\partial_t u[\tau_n] - \partial_t \tilde{u}) \cdot \partial_t v \,\mathrm{d}x\mathrm{d}t \\ & + \int_0^T \int_\Omega \left(A^{\mathrm{hom}}[\tau_n](x) - A^{\mathrm{hom}}[\tau^*](x)\right) e(\tilde{u}) e(v) \,\mathrm{d}x\mathrm{d}t \\ & + \int_0^T \int_\Omega A^{\mathrm{hom}}[\tau_n](x)(e(u[\tau_n]) - e(\tilde{u})) e(v) \,\mathrm{d}x\mathrm{d}t \Bigg| \\ \leq & \|\mathcal{M}_Y(\varrho[\tau_n] - \varrho[\tau^*])\|_{L^\infty(\Omega)} \|\partial_t u[\tau_n]\|_{[L^2(S\times\Omega)]^3} \|\partial_t v\|_{[L^2(S\times\Omega)]^3} \\ & + \|\mathcal{M}_Y(\varrho[\tau^*])\|_{L^\infty(\Omega)} \left| \int_0^T \int_\Omega \partial_t(u[\tau_n] - \tilde{u}) \cdot \partial_t v \,\mathrm{d}x\mathrm{d}t \right| \\ & + \|(A^{\mathrm{hom}}[\tau_n] - A^{\mathrm{hom}}[\tau^*]) e(\tilde{u})\|_{[L^2(S\times\Omega)]^{3\times 3}} \|e(v)\|_{[L^2(S\times\Omega)]^{3\times 3}} \\ & + \left| \int_0^T \int_\Omega \left( \sum_{k,l=1}^3 a^{\mathrm{hom}}_{ijkl}[\tau_n] e_{kl}(v) \right)_{i,j=1,2,3} : (e(u[\tau_n] - \tilde{u})) \,\mathrm{d}x\mathrm{d}t \right| . \end{aligned}$$

Since $0 < \varrho_0, \varrho_1 \in L^\infty(\Omega)$, we receive for a.e. $x \in \Omega$

$$\begin{aligned}
&|\mathcal{M}_Y(\varrho[\tau_n](x) - \varrho[\tau^*](x))| \\
&= \frac{1}{|Y|}\left|\int_Y (\varrho_0(x)(\chi_{Y_0[\tau_n]}(y) - \chi_{Y_0[\tau^*]}(y)) + \varrho_1(x)(\chi_{Y_1[\tau_n]}(y) - \chi_{Y_1[\tau^*]}(y))\mathrm{d}y\right| \\
&\leq \frac{1}{|Y|}\left(\varrho_0(x)|Y_0[\tau_n]\backslash Y_0[\tau^*] \cup Y_0[\tau^*]\backslash Y_0[\tau_n]| + \varrho_1(x)|Y_1[\tau_n]\backslash Y_1[\tau^*] \cup Y_1[\tau^*]\backslash Y_1[\tau_n]|\right) \\
&\leq \frac{C}{|Y|}\left(|Y_0[\tau_n]\backslash Y_0[\tau^*] \cup Y_0[\tau^*]\backslash Y_0[\tau_n]| + |Y_1[\tau_n]\backslash Y_1[\tau^*] \cup Y_1[\tau^*]\backslash Y_1[\tau_n]|\right) \to 0.
\end{aligned} \tag{10.2.4}$$

Since the last term in the inequality is independent of $x$, we even get the convergence in $L^\infty(\Omega)$. The same proof as in the proof of Theorem 9.2.3 shows that

$$\|(A^{\hom}[\tau_n] - A^{\hom}[\tau^*])e(\tilde{u})\|_{[L^2(S\times\Omega)]^{3\times 3}} \to 0.$$

Using the pointwise convergence of $A^{\hom}[\tau_n]$ to $A^{\hom}[\tau^*]$ (cf. Theorem 9.2.4) and the weak convergence (10.2.2) this shows (10.2.3). In the second step, we prove

$$|F(v,\tau_n) - F(v,\tau^*)| \to 0 \tag{10.2.5}$$

for $n \to \infty$. We estimate

$$\begin{aligned}
|F(v;\tau_n) - F(v;\tau^*)| \leq &\|\mathcal{M}_Y(\varrho[\tau_n] - \varrho[\tau^*])\|_{L^\infty(\Omega)}\left|\int_\Omega u_1(x)\cdot v(0,x)\,\mathrm{d}x\right| \\
\leq &C\|\mathcal{M}_Y(\varrho[\tau_n] - \varrho[\tau^*])\|_{L^\infty(\Omega)}.
\end{aligned}$$

Thus, (10.2.5) follows directly from (10.2.4). So we can conclude from the first and second step that

$$a(\tilde{u}, v;\tau^*) = \lim_{n\to\infty} a(u[\tau_n], v;\tau_n) = \lim_{n\to\infty} F(v;\tau_n) = F(v;\tau^*). \tag{10.2.6}$$

Since $u[\tau_n], \tilde{u} \in L^2(S; H^1_{\Gamma_D}) \cap H^1(S;[L^2(\Omega)]^3)$ we can apply Theorem 10.9 from [Schweizer, 2018] to get that $u_n, \tilde{u} \in C^0(\bar{S};[L^2(\Omega)]^3)$. Thus, using Proposition 10.8 from [Schweizer, 2018]

$$u[\tau_n](t) = \gamma_t(u[\tau_n]), \quad \tilde{u}(t) = \gamma_t(\tilde{u}) \quad \text{for all } t \in \bar{S},$$

where the trace operator $\gamma_t$ for Bochner spaces is defined in Theorem 2.2.2. The definition of the trace operator and the weak convergence of the solutions in $H^1(S;[L^2(\Omega)]^3)$ yield for all $\phi \in C_c^\infty([0,T))$ with $\phi(0) = 1$

$$\begin{aligned}
u_0(x) = u[\tau_n](0,x) = \gamma_0(u[\tau_n]) = &-\int_0^T u[\tau_n](t)\partial_t\phi(t)\mathrm{d}t - \int_0^T \partial_t u[\tau_n](t)\phi(t)\mathrm{d}t \\
&\to -\int_0^T \tilde{u}(t)\partial_t\phi(t)\mathrm{d}t - \int_0^T \partial_t\tilde{u}(t)\phi(t)\mathrm{d}t = \gamma_0(\tilde{u}) = \tilde{u}(0,x),
\end{aligned}$$

which shows that

$$\tilde{u}(0,x) = u_0(x).$$

Summing up all these results, we have shown that $\tilde{u} = u[\tau^*]$ due to the uniqueness of the solution $u[\tau^*]$ of (10.1.21). The functional $\mathcal{F} : \left[L^2(S \times \partial\Omega)\right]^3 \to \mathbb{R}, v \mapsto \|v - u_m\|^2_{L^2(S\times\partial\Omega)}$ is continuous and convex, since for all $\lambda \in (0,1)$ and $v, w \in \left[L^2(S \times \partial\Omega)\right]^3$

$$\begin{aligned}
\mathcal{F}(\lambda v + (1-\lambda)w) &= \|\lambda(v - u_m) + (1-\lambda)(w - u_m)\|^2_{[L^2(S\times\partial\Omega)]^3} \\
&\leq \left(\lambda\|v - u_m\|_{[L^2(S\times\partial\Omega)]^3} + (1-\lambda)\|w - u_m\|_{[L^2(S\times\partial\Omega)]^3}\right)^2 \\
&\leq \lambda F(v) + (1-\lambda)F(w).
\end{aligned}$$

By Theorem 13.8 from [Schweizer, 2018] we obtain that $\mathcal{F}$ is weakly lower semi-continuous. Thus, we can conclude that

$$\mathcal{J}(\tau^*) \leq \liminf_{n\to\infty} \mathcal{J}(\tau_n) = \lim_{n\to\infty} \mathcal{J}(\tau_n) = \inf\{\mathcal{J}(\tau) : \tau \in I_\eta\} \leq \mathcal{J}(\tau^*),$$

which shows that $\tau^*$ is a solution of the inverse problem (10.2.1). □

### 10.2.2. Gâteaux derivative of $\mathcal{J}$

In this section, we compute the Gâteaux derivative of the functional of the minimization problem (10.2.1), namely of

$$\mathcal{J}(\tau) := \frac{1}{2}\int_S \int_{\partial\Omega} |\mathcal{L}_\tau(f,g) - u_m|^2 \mathrm{d}S(x)\mathrm{d}t.$$

Apart from the Gâteaux derivative of $A^{\mathrm{hom}}$, which we have already computed in section 9.2.2, we need the Gâteaux derivative of $\mathcal{M}_Y(\varrho[\tau])$. Therefore we define

$$\mathcal{F}(Y) := \int_{Y_0} \varrho_0(x)\mathrm{d}y + \int_{Y_1} \varrho_1(x)\mathrm{d}y =: \mathcal{F}_1(Y_0) + \mathcal{F}_1(Y_1)$$

for a.e. $x \in \Omega$. We compute the directional derivative

$$\begin{aligned}
\mathcal{F}'(Y)(\theta) &= \lim_{\delta\to 0} \frac{\mathcal{F}((Id + \delta\theta)(Y)) - \mathcal{F}(Y)}{\delta} \\
&= \lim_{\delta\to 0} \frac{\mathcal{F}_1((Id+\delta\theta)(Y_0)) + \mathcal{F}_2((Id+\delta\theta)(Y_1)) - \mathcal{F}_1(Y_0) - \mathcal{F}_2(Y_1)}{\delta} \\
&= \mathcal{F}_1'(Y_0)(\theta) + \mathcal{F}_2'(Y_1)(\theta)
\end{aligned}$$

for all $\theta \in [W^{1,\infty}(\mathbb{R}^3)]^3$. Due to Proposition 9.2.11, the last two term exist and are of the form

$$\begin{aligned}\mathcal{F}'(Y)(\theta) =& \mathcal{F}_1'(Y_0)(\theta) + \mathcal{F}_2'(Y_1)(\theta)\\ =& \int_{\partial Y_0} \varrho_0(x)\theta \cdot n \mathrm{d}S(y) + \int_{\partial Y} \varrho_1 \theta \cdot n \mathrm{d}S(y) + \int_{\partial Y_0} \varrho_1(x)\theta \cdot (-n) \mathrm{d}S(y).\end{aligned}$$

In our setting, we are only interested in $\Theta_i \in [W_0^{1,\infty}(Y)]^3$, $i \in \{1,2,3\}$, chosen as in (9.2.12), (9.2.13), (9.2.14), which guarantees that

$$(Id + \delta\tau_1\Theta_1 + \delta\tau_2\Theta_2 + \delta\tau_3\Theta_3)(Y[\tau]) = Y[\tau_1 + \delta\tau_1, \tau_2 + \delta\tau_2, \tau_3 + \delta\tau_3]$$

and the structural assumption is maintained, i.e. the transformed $Y_0$ is still an ellipsoid. Since $\frac{1}{|Y|}\mathcal{F}(Y[\tau]) = \mathcal{M}_Y(\varrho[\tau])$, we obtain

$$\begin{aligned}\delta\mathcal{M}_Y(\varrho[\tau],\hat{\tau})(x) =& \sum_{i=1}^{3} \frac{1}{|Y|}\mathcal{F}'(Y[\tau])(\Theta_i)\hat{\tau}_i\\ =& \frac{1}{|Y|}\int_{\partial Y_0[\tau]} (\varrho_0(x) - \varrho_1(x))(\hat{\tau}_1\Theta_1 + \hat{\tau}_2\Theta_2 + \hat{\tau}_3\Theta_3) \cdot n \,\mathrm{d}S(y).\end{aligned}$$

The integrals $\int_{\partial Y} \varrho_1(x)\Theta_i \cdot n\mathrm{d}S(y)$, $i \in \{1,2,3\}$, vanish due to the definition of $\Theta_i$. Now we are able to compute the Gâteaux derivative of (10.2.1). Let $u[\tau]$ and $u[\tau + \varepsilon\hat{\tau}]$ be the weak solutions of (10.1.21) for given $\tau$ resp. $\tau + \varepsilon\hat{\tau}$. We take the difference of both equations and divide by $\varepsilon$

$$\begin{aligned}&-\frac{1}{\varepsilon}\int_0^T\int_\Omega \left(\mathcal{M}_Y(\varrho[\tau+\varepsilon\hat{\tau}](x,\cdot)) - \mathcal{M}_Y(\varrho[\tau](x,\cdot))\right)\partial_t u[\tau+\varepsilon\hat{\tau}] \cdot \partial_t v \,\mathrm{d}x\mathrm{d}t\\ &-\frac{1}{\varepsilon}\int_0^T\int_\Omega \mathcal{M}_Y(\varrho[\tau](x,\cdot))(\partial_t u[\tau+\varepsilon\hat{\tau}] - \partial_t\tilde{u}) \cdot \partial_t v \,\mathrm{d}x\mathrm{d}t\\ &+\frac{1}{\varepsilon}\int_0^T\int_\Omega \left(A^{\mathrm{hom}}[\tau+\varepsilon\hat{\tau}](x) - A^{\mathrm{hom}}[\tau](x)\right) e(u[\tau+\varepsilon\hat{\tau}])e(v)\,\mathrm{d}x\mathrm{d}t\\ &+\frac{1}{\varepsilon}\int_0^T\int_\Omega A^{\mathrm{hom}}(x)[\tau](e(u[\tau+\varepsilon\hat{\tau}]) - e(\tilde{u}))e(v)\,\mathrm{d}x\mathrm{d}t\\ =&\frac{1}{\varepsilon}\int_\Omega \mathcal{M}_Y(\varrho[\tau+\varepsilon\hat{\tau}](x,\cdot) - \varrho[\tau](x,\cdot))u_1(x) \cdot v(0,x)\,\mathrm{d}x.\end{aligned}$$

We pass to limit to get the first variation

$$\begin{aligned}&-\int_0^T\int_\Omega \delta\mathcal{M}_Y(\varrho[\tau],\hat{\tau})(x)\partial_t u[\tau] \cdot \partial_t v \,\mathrm{d}x\mathrm{d}t - \int_0^T\int_\Omega \mathcal{M}_Y(\varrho[\tau](x,\cdot))\partial_t(\delta u(\tau,\hat{\tau})) \cdot \partial_t v\,\mathrm{d}x\mathrm{d}t\\ &+\int_0^T\int_\Omega \delta A^{\mathrm{hom}}(\tau,\hat{\tau})e(u[\tau])e(v)\,\mathrm{d}x\mathrm{d}t + \int_0^T\int_\Omega A^{\mathrm{hom}}[\tau]e(\delta u(\tau,\hat{\tau}))e(v)\,\mathrm{d}x\mathrm{d}t \qquad (10.2.7)\\ =&\int_\Omega \delta\mathcal{M}_Y(\varrho[\tau],\hat{\tau})(x)u_1(x) \cdot v(0,x)\,\mathrm{d}x.\end{aligned}$$

From the steady-state case we already know that

$$\delta a_{ijkl}^{\mathrm{hom}}(\tau,\hat{\tau})(x) = \sum_{h=1}^{3} \frac{\partial a_{ijkl}^{\mathrm{hom}}}{\partial \tau_h}[\tau](x)\hat{\tau}_h \in L^\infty(\Omega).$$

In the case where $\rho_0$ and $\rho_1$ are independent of $x$, we get that $\partial_t(\delta\mathcal{M}_Y(\varrho[\tau],\hat{\tau})\partial_t u[\tau]) \in L^2(S,(H^1_{\Gamma_\mathrm{D}}(\Omega))^*)$ and so we can apply Satz 1.1 from [Gajewski et al., 1974] to get the existence and uniqueness of the solution $\delta u(\tau,\hat{\tau}) \in L^2(S;H^1_{\Gamma_\mathrm{D}}(\Omega)) \cap H^1(S;[L^2(\Omega)]^3)$ of (10.2.7) with initial condition $\delta u(\tau,\hat{\tau})(0,x) = 0$. We could also use more general $\rho_0$ and $\rho_1$ as long as $\partial_t(\delta\mathcal{M}_Y(\varrho[\tau],\hat{\tau})\partial_t u[\tau]) \in L^2(S,(H^1_{\Gamma_\mathrm{D}}(\Omega))^*)$. We rewrite the problem: Let $h \in \{1,2,3\}$. Find the function $\frac{\partial u}{\partial \tau_h}$ such that

$$\begin{aligned}
&-\int_0^T \int_\Omega \mathcal{M}_Y(\varrho[\tau](x,\cdot))\partial_t\left(\frac{\partial u}{\partial \tau_h}\right)\cdot \partial_t v \,\mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega A^{\mathrm{hom}}[\tau]e\left(\frac{\partial u}{\partial \tau_h}\right)e(v)\,\mathrm{d}x\mathrm{d}t \\
&= \int_\Omega \frac{\partial \mathcal{M}_Y}{\partial \tau_h}(\varrho[\tau])(x)u_1(x)\cdot v(0,x)\,\mathrm{d}x + \int_0^T \int_\Omega \frac{\partial \mathcal{M}_Y}{\partial \tau_h}(\varrho[\tau])(x)\partial_t u[\tau]\cdot \partial_t v\,\mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^T \int_\Omega \frac{\partial A^{\mathrm{hom}}}{\partial \tau_h}[\tau]e(u[\tau])e(v)\,\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

where

$$\frac{\partial \mathcal{M}_Y}{\partial \tau_h}(\varrho[\tau])(x) = \frac{1}{|Y|}\int_{\partial Y_0[\tau]} (\varrho_0(x)-\varrho_1(x))\Theta_h(y)\cdot n\,\mathrm{d}S(y). \tag{10.2.8}$$

Then, due to uniqueness

$$\nabla u[\tau]\cdot\hat{\tau} := \sum_{h=1}^{3}\frac{\partial u}{\partial \tau_h}\hat{\tau}_h = \delta u(\tau,\hat{\tau}).$$

We compute the first variation of $\mathcal{J}$

$$\begin{aligned}
\delta\mathcal{J}(\tau,\hat{\tau}) &= \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_S\int_{\partial\Omega} |u[\tau+\varepsilon\tilde{\tau}]-u_m|^2 - |u[\tau]-u_m|^2\,\mathrm{d}S(x)\mathrm{d}t \\
&= \int_S\int_{\partial\Omega}(u[\tau]-u_m)\delta u(\tau,\hat{\tau})\,\mathrm{d}S(x)\mathrm{d}t.
\end{aligned}$$

Using (10.2.8), we get the Gâteaux derivative of the target functional

$$\delta\mathcal{J}(\tau,\hat{\tau}) = \int_S\int_{\partial\Omega}(u[\tau]-u_m)\nabla u[\tau]\mathrm{d}S(x)\mathrm{d}t\cdot\hat{\tau} =: \nabla\mathcal{J}(\tau)\cdot\hat{\tau}. \tag{10.2.9}$$

# 11. Numerical simulations

In this section, we present some numerical simulations to showcast that we can derive the ellipsoidal microstructure from measurements of the deformation on the boundary of a beam, whereby we use a generally known gradient-based algorithm. The material of the beam $\Omega$ with volume 120 cm $\times$ 40 cm $\times$ 40 cm is a composite of carbon and concrete, whereby the carbon fibres with ellipsoidal structure are distributed periodically on a microscopic scale. The material properties for the concrete are Young's modulus $E_{\text{con}} = 20$ GPa, Poisson's ratio $\nu_{\text{con}} = 0.2$ and density $\varrho_{\text{con}} = 2300\,\text{kg/m}^3$ and for the carbon Young's modulus $E_{\text{car}} = 230$ GPa, Poisson's ratio $\nu_{\text{car}} = 0.2$ and density $\varrho_{\text{car}} = 1800\,\text{kg/m}^3$. To handle the different scales, we non-dimensionalize the cell problem. We assume that reference cell has sidelengths $2 \times 1 \times 1$ and the fibre is an ellipsoid centred in the middle of the cuboid with axis lengths $\tau = (\tau_1, \tau_2, \tau_3) \in [0.12, 1.88] \times [0.12, 0.88]^2$. We fix the beam on one of the small faces and assume no volume forces and zero initial values for the deformation. The boundary load on the upper part of the surface is given by $g \equiv -2e_3$ [GPa] in the steady-state case and by $g(t) = -0.8te_3$ [GPa] in the time-dependent case, where $t$ is the time variable. In both cases there is zero boundary load on the rest of the surface. In this setup, we consider the inverse problems of the form

$$\operatorname*{arg\,min}_{\tau \in I_\eta} \mathcal{J}(\tau) := \operatorname*{arg\,min}_{\tau \in I_\eta} \frac{1}{2|\partial\Omega|^2} \int_{\partial\Omega} |u[\tau] - u_m|^2 \mathrm{d}S(x)$$

for the steady-state case and

$$\operatorname*{arg\,min}_{\tau \in I_\eta} \mathcal{J}(\tau) := \operatorname*{arg\,min}_{\tau \in I_\eta} \frac{1}{2|\partial\Omega|^2} \int_S \int_{\partial\Omega} |u[\tau] - u_m|^2 \mathrm{d}S(x)\mathrm{d}t,$$

for the time-dependent case, where $I_\eta := [0.12, 1.88] \times [0.12, 0.88]^2$, $u_m$ is the deformation of the beam for the target value $\tau^{\text{target}} = (1.5, 0.6, 0.4)$, $u[\tau]$ is the solution of homogenized problem (9.1.7) resp. (10.1.21) for given $\tau$ and $S = [0, 3]s$ is the time interval. The functional $\mathcal{J}$ slightly differs from the definitions in the sections above by the constant $\frac{1}{|\partial\Omega|^2}$, which has no impact on the analytical results up to a scaling factor.

To solve the minimization problem numerically, we use MATLAB® (version R2020a) and the finite element simulation software COMSOL Multiphysics® [Com, 2020], which can be connected by COMSOL LiveLink$^{\text{TM}}$ for MATLAB®. In a first step, we compute in COMSOL the solutions of the cell problems (9.1.9), which we use to calculate the homogenized tensor (9.1.8), the mean value of the density (in the time-dependent case) and their Gâteaux deriva-

tives with the formula from Theorem 9.2.17 and (10.2.8). We save these values in MATLAB to pass them as parameters to another COMSOL model to solve the homogenized problem (9.1.7) resp. (10.1.21) (in the time-dependent case). We use the solution to compute the target functional $\mathcal{J}$ and its Gâteaux derivative (9.2.18) resp. (10.2.9) (in the time-dependent case). These values are needed to apply the MATLAB function `fmincon` with the algorithm 'sqp-legacy', which is a gradient-based alogrithm solving the minimization problem.
In both cases – the stationary and time-dependent – we start the iteration with the same boundary value of $I_\eta$ as the initial guess, namely $\tau^{\text{start}} = (0.12, 0.12, 0.12)$. In the steady-state case, the values of $\tau_1, \tau_2$ and $\tau_3$ in every iteration step are plotted in Figure 11.1, whereby the constant functions show the values of $\tau_1^{\text{target}}, \tau_2^{\text{target}}$ and $\tau_3^{\text{target}}$. In Figure 11.2 the associated values of $\mathcal{J}$ in every iteration step are plotted. After 32 steps the algorithm terminates since the relative first-order optimality is less than the optimality tolerance of $10^{-6}$. We obtain the value $\tau^{\text{end}} = (1.483, 0.624, 0.396)$.

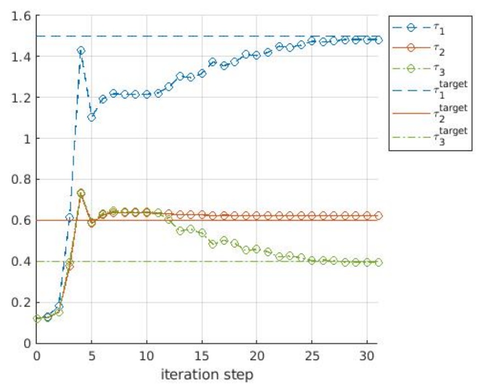


Figure 11.1.: values of $\tau$ in each iteration step in the steady-state case

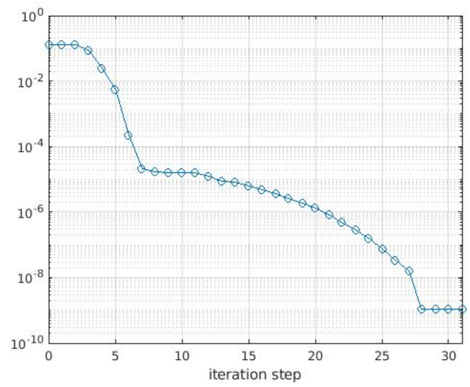


Figure 11.2.: target functional on a logarithmic scale in each iteration step

In the time-dependent case, the values of $\tau_1, \tau_2$ and $\tau_3$ in every iteration step are plotted in Figure 11.3 and the corresponding values of $\mathcal{J}$ in Figure 11.4. The algorithm terminates after 32 steps when the relative first-order optimality is less than the optimality tolerance of $10^{-6}$. We obtain the value $\tau^{\text{end}} = (1.507, 0.591, 0.402)$.

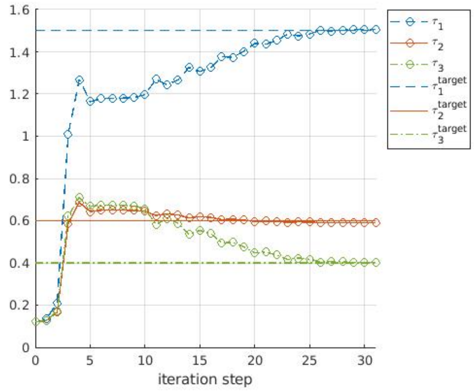


Figure 11.3.: values of $\tau$ in each iteration step in the time-dependent case

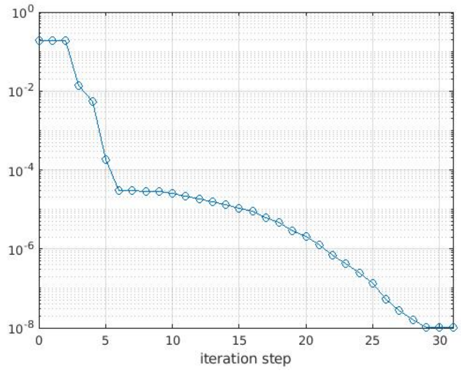


Figure 11.4.: target functional on a logarithmic scale in each iteration step

The results of the simulations show that we get a good approximation of the target value $\tau^{\text{target}} = (1.5, 0.6, 0.4)$ at the end, which demonstrates the functioning of the method. In this example, we have assumed that we know the exact deformation on the boundary of the beam in every mesh point, which is not possible in practice. So to quantify this properly, further research has to be done such as a stability and sensitivity analysis, but which is beyond the scope of this thesis. Furthermore, we could investigate, especially in the time-dependent case, the impact of different test loadings on the perfomance of the optimization algorithm.

# 12. Conclusion and outlook

The question of identifying the microscopic structure of the composite material was the motivation for the second part of this thesis. We considered resp. derived the homogenized problem of the stationary resp. time-dependent linear elasticity, where we have assumed perfect bonding of fibres and matrix. In both cases, we can split the problem into a microscopic and macroscopic part, which are connected through the effective elasticity tensor, whose elements are based on the solutions of the microscopic problem. Only this elliptic cell problem takes the representative structure of the composite into account. In the corresponding inverse problem, we were interested in identifying the parameters describing the structure of this representative cell from macroscopic measurements on the boundary. We proved that there exists at least one solution of the minimization problem

$$\operatorname*{arg\,min}_{\tau \in I_\eta} \mathcal{J}(\tau) \coloneqq \operatorname*{arg\,min}_{\tau \in I_\eta} \frac{1}{2} \|\mathcal{L}_\tau(f,g) - u_m\|^2_{[L^2(\partial\Omega)]^3}$$

resp.

$$\operatorname*{arg\,min}_{\tau \in I_\eta} \mathcal{J}(\tau) \coloneqq \operatorname*{arg\,min}_{\tau \in I_\eta} \frac{1}{2} \|\mathcal{L}_\tau(f,g,u_0,u_1) - u_m\|^2_{[L^2(S \times \partial\Omega)]^3}$$

under given volume and boundary forces $f$ and $g$ and measured data $u_m$, where $\tau$ is a finite vector of real paramaters describing the structure of the periodicity cell. To be able to solve this problem numerically by generally known gradient-based algorithm, we derived the formula (9.2.18) for the Gâteaux derivative of $\mathcal{J}$. For this we have to compute the shape derivative of the homogenized tensor and solve several partial differential equations. Numerical simulations for an ellipsoidal microstructure showed that we can reconstruct the size of the axes of the ellipsoids. However, to quantify this would require a proper stability and sensitivity analysis, which is beyond the scope of this thesis. Although we have only considered isotropic materials with special microstructure at the end, the results are still true for the case of more general microstructure as long as there holds (9.2.10) for some appropriate $\Theta$ or if we use formula (9.2.9) instead of Theorem 9.2.17 for anisotropic materials.

In addition to a more detailed study of the numerics, there arise several interesting aspects in this context, but which go beyond the scope of this work. One could consider more advanced models like the linear elasticity equation with slip displacement conditions as studied in the first part of the thesis. Furthermore, the microscopic structure could depend on the macroscopic variable $x$, i.e. the representative cell is not same everywhere in the domain. Thus, the parameter $\tau$ is a function of $x$ and not a finite real vector anymore. This is a classical generalization of the direct problem but requires now for the inverse problem to find a function in an infinite-dimensional space instead of a finite one.

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