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### Angaben zur Veröffentlichung / Publication details:

Colonius, Fritz. 2023. "On the global behavior of linear flows." *Proceedings of the American Mathematical Society* 151: 135–49. <https://doi.org/10.1090/proc/16189>.

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# ON THE GLOBAL BEHAVIOR OF LINEAR FLOWS

FRITZ COLONIUS

(Communicated by Wenxian Shen)

**ABSTRACT.** For linear flows on vector bundles, it is analyzed when subbundles in the Selgrade decomposition yield chain transitive subsets for the induced flow on the associated Poincaré sphere bundle.

## 1. INTRODUCTION

For linear flows on vector bundles, Selgrade's theorem describes the decomposition into subbundles obtained from the chain recurrent components of the induced flow on the projective bundle. This coincides with the finest decomposition into exponentially separated subbundles but has the advantage that it provides an intrinsic characterization using chain transitivity properties in the projective bundle. The present paper complements this point of view by considering recurrence properties of the linear flow. This is based on the compactification provided by the construction of the Poincaré sphere from the global theory of ordinary differential equations going back to Poincaré [20]; cf., e.g., Perko [19, Section 3.10].

We will consider linear flows on a vector bundle  $\pi : \mathcal{V} \rightarrow B$  over a compact chain transitive metric space  $B$ , that is, a continuous flow  $\Phi$  on  $\mathcal{V}$  preserving fibers such that the induced maps on the fibers  $\mathcal{V}_b$  are linear. Classical examples of such flows are given by linear differential equations with almost periodic coefficients and by linearized flows over a compact invariant chain transitive set. We refer to Selgrade [25], Salamon and Zehnder [23], Bronstein and Kopanskii [5], Johnson, Palmer and Sell [14], and Colonius and Kliemann [9, Chapter 5] for the theory of linear flows, and to Alves and San Martin [2] for generalizations to principal bundles. Blumenthal and Latushkin [6] generalize Selgrade's theorem to linear skew product semiflows on separable Banach bundles.

Selgrade's theorem [9, Theorem 5.2.5] states that the induced flow  $\mathbb{P}\Phi$  on the projective bundle  $\mathbb{P}\mathcal{V}$  has finitely many chain recurrent components  ${}_{\mathbb{P}}M_1, \dots, {}_{\mathbb{P}}M_l$  (this coincides with the finest Morse decomposition) and  $1 \leq l \leq d := \dim \mathcal{V}_b$ ,  $b \in B$ . Every chain recurrent component  ${}_{\mathbb{P}}M_i$  defines an invariant subbundle  $\mathcal{V}_i = \mathbb{P}^{-1}({}_{\mathbb{P}}M_i)$  of  $\mathcal{V}$  and the following decomposition into a Whitney sum holds:

$$(1.1) \quad \mathcal{V} = \mathcal{V}_1 \oplus \dots \oplus \mathcal{V}_l.$$

Note that this refines the subbundle decomposition obtained by exponential dichotomies. It is clear that in “stable” and “unstable” subbundles no recurrence properties can be expected. This is different in the nonhyperbolic case, where a “central” subbundle is present: If for an autonomous ordinary differential equation

$\dot{x} = Ax$  the matrix  $A \in \mathbb{R}^{d \times d}$  has 0 as the only eigenvalue with vanishing real part and 0 is semisimple, then the eigenspace consists of equilibria, hence it is chain transitive. Otherwise the behavior “at infinity” has to be taken into account. This can be done using the Poincaré sphere which is obtained by attaching to the sphere  $\mathbb{S}^d$  in  $\mathbb{R}^{d+1}$  a copy of  $\mathbb{R}^d$  at the north pole and by taking the central projection from the origin in  $\mathbb{R}^{d+1}$  to the northern hemisphere  $\mathbb{S}^{d,+}$  of  $\mathbb{S}^d$ .

Based on this construction, the main results of this paper are Theorem 4.3 and Theorem 4.7 showing when for linear flows subbundles  $\mathcal{V}_i$  yield chain transitive sets on the appropriately defined Poincaré sphere bundle. A sufficient condition is based on a convenient spectral concept, the Morse spectrum, which is constructed via chains in the chain recurrent components on the projective bundle  $\mathbb{P}\mathcal{V}$ . Here we instead consider chain recurrent components on the sphere bundle  $\mathbb{S}\mathcal{V}$  and an analogously defined Morse spectrum.

The contents of this paper are as follows. Section 2 recalls some notation for linear flows on vector bundles and analyzes the relation between the chain recurrent components for the induced flows on the projective bundle and the sphere bundle. Section 3 recalls the Morse spectrum and Section 4 shows the main results on the projected flow on the Poincaré sphere bundle. In Section 5 the results are illustrated for linear autonomous differential equations and for bilinear control systems (cf. Elliott [12], Jurdjevic [15]).

## 2. LINEAR FLOWS ON VECTOR BUNDLES

This section introduces some notation and describes for (continuous) linear flows on a vector bundle  $\pi : \mathcal{V} \rightarrow B$  over a metric space  $B$  the relation between the chain recurrent components on the projective bundle and the sphere bundle.

We consider vector bundles of the following form (for the following cf. Salamon and Zehnder [23, Appendix]).

**Definition 2.1.** Let  $H$  be a finite-dimensional Hilbert space over the reals  $\mathbb{R}$  and let  $B$  be a compact metric space. A vector bundle is a continuous map  $\pi : \mathcal{V} \rightarrow B$  where  $\mathcal{V}$  is a topological space with the following properties:

(i) There exist a finite cover  $\{U_a, a \in A\}$  and homeomorphisms  $\varphi_a : \pi^{-1}(U_a) \rightarrow U_a \times H$  such that  $pr_1 \circ \varphi_a = \pi$ , where  $pr_1 : U_a \times H \rightarrow U_a$  denotes the projection onto the first component.

(ii) The maps  $L_{ba} : H \rightarrow H$  defined by  $\varphi_b \circ \varphi_a^{-1}(b, h) = (b, L_{ba}h)$  are linear for every  $b \in U_a \cap U_b$ .

Without loss of generality one may assume that  $\varphi_a$  extends to a homeomorphism of  $\overline{\pi^{-1}(U_a)}$  onto  $\overline{U_a} \times H$ . The fibers  $\mathcal{V}_b$  can be given a Hilbert space structure by

$$(2.1) \quad c_1 v_1 + c_2 v_2 = \varphi_a^{-1}(b, c_1 x_1 + c_2 x_2) \text{ for } c_i \in \mathbb{R}, v_i = \varphi_a^{-1}(b, x_i) \in \mathcal{V}_b, i = 1, 2,$$

and using the continuous and positive definite linear form on  $\mathcal{V}$  defined by

$$(2.2) \quad \langle v, v' \rangle = \sum_{a \in A} d(\pi(v), B \setminus U_a) \langle v, v' \rangle_a \text{ for } \pi(v) = \pi(v').$$

For  $v = \varphi_a^{-1}(b, x)$  and  $v' = \varphi_a^{-1}(b', x')$  with  $b, b' \in U_a$  let

$$(2.3) \quad d_a(v, v') = \max \{d(b, b'), \|x - x'\|_a\}.$$

For any sequence  $v_0, \dots, v_m \in \mathcal{V}$  we define

$$\rho(v_0, \dots, v_m) = \max \left\{ \sum_{j=1}^m d_{a_j}(v_{j-1}, v_j) \mid \pi(v_{j-1}), \pi(v_j) \in U_{a_j} \text{ with } a_j \in A \right\},$$

where the maximum over the empty set is by definition  $+\infty$ . Then the distance function

$$(2.4) \quad d(v, v') = \inf \{ \rho(v_0, \dots, v_m) \mid m \in \mathbb{N}, v_j \in \mathcal{V}, v_0 = v, v_m = v' \}$$

defines a metric on  $\mathcal{V}$  that is compatible with the original topology. The corresponding projective bundle is  $\mathbb{P}\mathcal{V} = (\mathcal{V} \setminus Z) / \sim$ , where  $Z$  is the zero section and  $v \sim v'$  if  $\pi(v) = \pi(v')$ . This is a fiber bundle  $\mathbb{P}\pi : \mathbb{P}\mathcal{V} \rightarrow B$  with local trivializations  $\mathbb{P}\varphi_a : \mathbb{P}\pi^{-1}(U_a) \rightarrow U_a \times \mathbb{P}H$ . The (unit) sphere bundle is  $\mathbb{S}\pi : \mathbb{S}\mathcal{V} := \{v \in \mathcal{V} \mid \|v\| = 1\} \rightarrow B$ . The metric spaces  $\mathbb{P}\mathcal{V}$  and  $\mathbb{S}\mathcal{V}$  are compact.

**Definition 2.2.** A *linear flow*  $\Phi$  on a vector bundle  $\pi : \mathcal{V} \rightarrow B$  is a flow  $\Phi$  on  $\mathcal{V}$  such that for all  $\alpha \in \mathbb{R}$  and  $v, w \in \mathcal{V}$  with  $\pi(v) = \pi(w)$  and  $t \in \mathbb{R}$  one has

$$\pi(\Phi(t, v)) = \pi(\Phi(t, w)), \Phi(t, \alpha(v + w)) = \alpha\Phi(t, v) + \alpha\Phi(t, w).$$

It is convenient to denote the induced flow on the base space  $B$  by  $b \cdot t, t \in \mathbb{R}$ . The restriction of  $\Phi(t, \cdot)$  to a fiber  $\mathcal{V}_b$  is a linear map to  $\mathcal{V}_{b \cdot t}$  denoted by  $\Phi_{b \cdot t}$  with  $\|\Phi_{b \cdot t}(v)\| = \|\Phi(t, v)\|$  for  $v \in \mathcal{V}_b$ . This is an isomorphism with inverse  $\Phi_{b \cdot t, -t}$ . Furthermore,  $\Phi$  induces a flow  $\mathbb{S}\Phi$  on the sphere bundle satisfying  $\mathbb{S}\Phi(t, v) = \frac{\Phi(t, v)}{\|\Phi(t, v)\|}$  for all  $t, v$  and a flow on the projective bundle denoted by  $\mathbb{P}\Phi$ .

For  $\varepsilon, T > 0$  an  $(\varepsilon, T)$ -chain  $\zeta$  for a flow  $\Psi$  on a metric space  $X$  from  $x$  to  $y$  is given by  $n \in \mathbb{N}$ ,  $T_0, \dots, T_{n-1} \geq T$ , and  $x_0 = x, \dots, x_n = y \in X$  with  $d(\Psi(T_i, x_i), x_{i+1}) < \varepsilon$  for  $i = 0, \dots, n-1$ . For  $x \in X$  the (forward) chain limit set is

$$\Omega(x) = \{y \in X \mid \forall \varepsilon, T > 0 \exists (\varepsilon, T) \text{ chain from } x \text{ to } y\}.$$

A set is called chain transitive if  $y \in \Omega(x)$  for all  $x, y$  in this set. A chain recurrent component  $M$  is a maximal chain transitive set (on a compact metric space this is the connected components of the chain recurrent set). The monograph Alongi and Nelson [1] provides detailed proofs around chains; cf. also Robinson [21].

Next we analyze the relations between the chain recurrent components for the induced flows  $\mathbb{P}\Phi$  on the projective bundle  $\mathbb{P}\mathcal{V}$  and  $\mathbb{S}\Phi$  on the sphere bundle  $\mathbb{S}\mathcal{V}$ .

**Lemma 2.3.**

- (i) Let  $v, w \in \mathbb{S}\mathcal{V}$ . If on  $\mathbb{P}\mathcal{V}$  the point  $\mathbb{P}w$  is in  $\Omega(\mathbb{P}v)$ , then on  $\mathbb{S}\mathcal{V}$  at least one of the points  $w$  or  $-w$  is in  $\Omega(v)$ .
- (ii) Let  ${}_S M$  be a chain recurrent component on  $\mathbb{S}\mathcal{V}$ . Then the projection of  ${}_S M$  to  $\mathbb{P}\mathcal{V}$  is contained in a chain recurrent component  ${}_P M$ .
- (iii) Consider a chain recurrent component  ${}_P M$  on  $\mathbb{P}\mathcal{V}$ . Suppose that there is  $v_0 \in \mathbb{S}\mathcal{V}$  such that  $\mathbb{P}v_0 \in {}_P M$  and  $-v_0 \in \Omega(v_0)$ . Then  $\{v \in \mathbb{S}\mathcal{V} \mid \mathbb{P}v \in {}_P M\}$  is a chain recurrent component on  $\mathbb{S}\mathcal{V}$ .

*Proof.* We will frequently use the following elementary fact: If  $w \in \Omega(v)$  for the induced system on  $\mathbb{S}\mathcal{V}$ , then  $-w \in \Omega(-v)$ .

- (i) For  $\varepsilon, T > 0$ , let an  $(\varepsilon, T)$ -chain given by  $\mathbb{P}v_0 = \mathbb{P}v, \mathbb{P}v_1, \dots, \mathbb{P}v_n = \mathbb{P}w$  and  $T_0, \dots, T_{n-1} \geq T$  with  $d(\mathbb{P}\Phi(T_i, v_i), \mathbb{P}v_{i+1}) < \varepsilon$ . In order to construct an  $(\varepsilon, T)$ -chain on  $\mathbb{S}\mathcal{V}$  note that  $d(\mathbb{S}\Phi(T_0, v_0), v_1) < \varepsilon$  or  $d(\mathbb{S}\Phi(T_0, v_0), -v_1) < \varepsilon$ . In the first

case define  $v_1^1 = v_1$  and in the second case define  $v_1^1 = -v_1$ . Proceeding in this way, one finds an  $(\varepsilon, T)$ -chain  $v_0, v_1^1, \dots, v_n^1$  from  $v$  to  $w$  or  $-w$ .

Assertion (ii) is immediate from the definitions. Concerning assertion (iii) let  $v_1, v_2 \in \{v \in \mathcal{SV} \mid \mathbb{P}v \in {}_{\mathbb{P}}M\}$ . We have to show that  $v_2 \in \Omega(v_1)$ . Since  $\mathbb{P}v_1, \mathbb{P}v_2 \in {}_{\mathbb{P}}M$  it follows that  $v_2$  or  $-v_2$  is in  $\Omega(v_1)$ . In the first case we are done. In the second case it follows that  $v_2 \in \Omega(-v_1)$  and, by our assumption,  $v_0 \in \Omega(-v_0)$ . Now  $\mathbb{P}(-v_1) = \mathbb{P}v_1 \in {}_{\mathbb{P}}M$  and  $\mathbb{P}v_0 \in {}_{\mathbb{P}}M$  imply that  $\mathbb{P}v_0 \in \Omega(\mathbb{P}(-v_1))$ , hence (a)  $v_0 \in \Omega(-v_1)$  or (b)  $-v_0 \in \Omega(-v_1)$ . We claim also (b) implies that  $v_0 \in \Omega(-v_1)$ . In fact,  $-v_0 \in \Omega(-v_1)$  implies  $v_0 \in \Omega(-v_0) \subset \Omega(-v_1)$  showing the claim.

Since  $\mathbb{P}v_0, \mathbb{P}v_2 \in {}_{\mathbb{P}}M$  it follows that  $v_2 \in \Omega(v_0)$  or  $-v_2$  is in  $\Omega(v_0)$ . In the first case, one has

$$v_2 \in \Omega(v_0) \subset \Omega(-v_0) \subset \Omega(v_1)$$

and in the second case  $v_2 \in \Omega(-v_0) \subset \Omega(v_1)$ .  $\square$

We get the following result characterizing the relation between the chain recurrent components  ${}_{\mathbb{P}}M_1, \dots, {}_{\mathbb{P}}M_l, 1 \leq l \leq d$ , on the projective bundle and the chain recurrent components on the sphere bundle.

**Theorem 2.4.**

(i) *The set  $\{v \in \mathcal{SV} \mid \mathbb{P}v \in {}_{\mathbb{P}}M_i\}$  is the unique chain recurrent component on  $\mathcal{SV}$  which projects to  ${}_{\mathbb{P}}M_i$  if and only if there is  $v_0 \in \mathcal{SV}$  with  $\mathbb{P}v_0 \in {}_{\mathbb{P}}M_i$  such that  $-v_0 \in \Omega(v_0)$  for the system on the sphere bundle  $\mathcal{SV}$ .*

(ii) *For every chain recurrent component  ${}_{\mathbb{P}}M_i$  there are at most two chain recurrent components  ${}_sM^+$  and  ${}_sM^-$  on  $\mathcal{SV}$  such that*

$$(2.5) \quad \{v \in \mathcal{SV} \mid \mathbb{P}v \in {}_{\mathbb{P}}M_i\} = {}_sM^+ \cup {}_sM^-, \text{ and then } {}_sM^+ = -{}_sM^-.$$

(iii) *There are  $l_1$  chain recurrent components on  $\mathcal{SV}$  denoted by  ${}_sM_1, \dots, {}_sM_{l_1}$  with  $1 \leq l_1 \leq 2l \leq 2d$ .*

*Proof.* (i) Suppose that there is  $\mathbb{P}v_0 \in {}_{\mathbb{P}}E_i$  such that  $-v_0 \in \Omega(v_0)$ . Then Lemma 2.3(ii) shows the assertion. The converse is obvious.

(ii) Fix  $v \in \mathcal{SV}$  with  $\mathbb{P}v \in {}_{\mathbb{P}}M_i$ . For any  $w \in \mathcal{SV}$  with  $\mathbb{P}w \in {}_{\mathbb{P}}M_i$  and  $\varepsilon, T > 0$ , one finds as in Lemma 2.3(i) an  $(\varepsilon, T)$ -chain  $v_1^1, \dots, v_n^1$  on  $\mathcal{SV}$  with times  $T_0, \dots, T_{n-1} \geq T$  from  $v$  to  $w$  or  $-w$ . Define

$$A^\pm := \{v \in \mathcal{SV} \mid \mathbb{P}v \in {}_{\mathbb{P}}M_i \text{ and } \pm v \in \Omega(v_0) \text{ and } v_0 \in \Omega(\pm v)\}.$$

If  $A^\pm \neq \emptyset$  there are chain recurrent components  ${}_sM^\pm$  with  $A^\pm \subset {}_sM^\pm$ . For every point  $v$  with  $\mathbb{P}v \in {}_{\mathbb{P}}M_i$  one has  $v \in \Omega(v_0)$  or  $-v \in \Omega(v_0)$ , and  $v_0 \in \Omega(v)$  or  $v_0 \in \Omega(-v)$ . If there is  $v \in \Omega(v_0)$  and  $v_0 \in \Omega(-v)$  it follows that  $-v_0 \in \Omega(v) \subset \Omega(v_0)$ . Then (i) implies the assertion.

Hence we may assume that one of the following properties holds for every  $v \in \mathcal{SV}$  with  $\mathbb{P}v \in {}_{\mathbb{P}}M_i$ :  $v \in \Omega(v_0)$  and  $v_0 \in \Omega(v)$ , or  $-v \in \Omega(v_0)$  and  $v_0 \in \Omega(-v)$ . It follows that

$$\{v \in \mathcal{SV} \mid \mathbb{P}v \in {}_{\mathbb{P}}M_i\} \subset A^+ \cup A^- \subset {}_sM^+ \cup {}_sM^-$$

(one of the sets  ${}_sM^+$  and  ${}_sM^-$  may be void). By Lemma 2.3(ii) the projections of  ${}_sM^+$  and  ${}_sM^-$  are contained in  ${}_{\mathbb{P}}M_i$ . This proves the first equality in (2.5). The same argument with  $-v$  instead of  $v$  implies the second equality.

(iii) This is a consequence of assertion (ii).  $\square$

Next we lift chains on  $\mathcal{SV}$  to chains on  $\mathcal{V}$ .

**Definition 2.5.** Let  $\zeta$  be an  $(\varepsilon, T)$ -chain on  $\mathbb{S}\mathcal{V}$  given by  $n \in \mathbb{N}$ ,  $v_0, \dots, v_n \in \mathbb{S}\mathcal{V}$  and  $T_0, \dots, T_{n-1} \geq T$  with

$$d(\mathbb{S}\Phi(T_i, v_i), v_{i+1}) < \varepsilon \text{ for } i = 0, \dots, n-1.$$

The lift of  $\zeta$  to  $\mathcal{V}$  is the chain  $\widehat{\zeta}$  given by  $n \in \mathbb{N}$ ,  $T_0, \dots, T_{n-1} \geq T$  and

$$w_0 = v_0, w_i = \prod_{j=0}^{i-1} \|\Phi(T_j, v_j)\| v_i \text{ for } i = 1, \dots, n.$$

Furthermore,  $\alpha\widehat{\zeta}$  for  $\alpha > 0$  denotes the chain on  $\mathcal{V}$  given by  $n \in \mathbb{N}$ ,  $T_0, \dots, T_{n-1} \geq T$  and  $\alpha w_i$  for  $i = 0, \dots, n$ .

Note Lemma 2.6.

**Lemma 2.6.** Let  $\zeta$  be an  $(\varepsilon, T)$ -chain on  $\mathbb{S}\mathcal{V}$  as above and define  $C_{-1} = 1$  and  $C_i = \prod_{j=0}^i \|\Phi(T_j, v_j)\|$  for  $i = 0, \dots, n-1$ . Then for  $\varepsilon > 0$  small enough the lift  $\alpha\widehat{\zeta}$ ,  $\alpha > 0$ , to  $\mathcal{V}$  satisfies  $\|w_i\| = \alpha C_{i-1}$  and  $\Phi(T_i, \alpha w_i) = \alpha C_i \mathbb{S}\Phi(T_i, v_i)$  for all  $i$ .

*Proof.* The formula for  $\|\alpha w_i\|$  obviously holds. Together with linearity of  $\Phi$  this yields

$$\Phi(T_i, \alpha w_i) = \alpha \prod_{j=0}^i \|\Phi(T_j, v_j)\| \frac{\Phi(T_i, v_i)}{\|\Phi(T_i, v_i)\|} = \alpha C_i \mathbb{S}\Phi(T_i, v_i).$$

□

### 3. THE MORSE SPECTRUM

For linear flows on vector bundles, a number of spectral notions and their relations have been considered; cf., e.g., Sacker and Sell [22], Johnson, Palmer, and Sell [14], Grüne [13], Braga Barros and San Martin [4], Kawan and Stender [17]. An appropriate spectral notion in the present context is provided by the Morse spectrum defined as follows (cf. Colonius and Kliemann [8], and San Martin and Seco [24] and Bochi [3, Section 4] for generalizations).

For  $\varepsilon, T > 0$  let an  $(\varepsilon, T)$ -chain  $\zeta$  of  $\mathbb{P}\Phi$  be given by  $n \in \mathbb{N}$ ,  $T_0, \dots, T_{n-1} \geq T$ , and  $p_0, \dots, p_n \in \mathbb{P}\mathcal{V}$  with  $d(\mathbb{P}\Phi(T_i, p_i), p_{i+1}) < \varepsilon$  for  $i = 0, \dots, n-1$ . With total time  $\sigma = \sum_{i=0}^{n-1} T_i$  let the exponential growth rate of  $\zeta$  be

$$\lambda(\zeta) := \frac{1}{\sigma} \left( \sum_{i=0}^{n-1} \log \|\Phi(T_i, v_i)\| - \log \|v_i\| \right) \text{ with } v_i \in \mathbb{P}^{-1} p_i.$$

Define the Morse spectrum of a subbundle  $\mathcal{V}_i$  generated by  $\mathbb{P}M_i$  by

$$\Sigma_{Mo}(\mathcal{V}_i) = \left\{ \lambda \in \mathbb{R}, \text{ there are } \varepsilon^k \rightarrow 0, T^k \rightarrow \infty \text{ and } (\varepsilon^k, T^k)\text{-chains } \zeta^k \text{ in } \mathbb{P}M_i \text{ with } \lambda(\zeta^k) \rightarrow \lambda \text{ as } k \rightarrow \infty \right\}.$$

For a chain recurrent component  $\mathbb{S}M_j, j \in \{1, \dots, l_1\}$ , in  $\mathbb{S}\mathcal{V}$  let the cone bundle be

$$\mathcal{V}_j^+ := \{\alpha v \in \mathcal{V} \mid \alpha > 0 \text{ and } v \in \mathbb{S}M_j\},$$

and define the Morse spectrum  $\Sigma_{Mo}(\mathcal{V}_j^+)$  analogously via chains in  $\mathbb{S}M_j$ .

Every Lyapunov exponent is contained in some  $\Sigma_{Mo}(\mathcal{V}_i, \Phi)$  and each  $\Sigma_{Mo}(\mathcal{V}_i, \Phi)$  is a compact interval with boundary points corresponding to Lyapunov exponents for ergodic measures. It suffices to consider periodic chains in the definition of the Morse spectrum; cf. Colonius and Kliemann [9, Chapter 5] for these claims. The

spectral intervals  $\Sigma_{Mo}(\mathcal{V}_i)$  need not be disjoint (in particular, there may exist two “central” subbundles with 0 in the Morse spectrum); cf. Salamon and Zehnder [23, Example 2.14] and also Example 5.3.

Lemma 3.1 shows that the Morse spectra on the sphere bundle coincide with the Morse spectra on the projective bundle.

**Lemma 3.1.** *If  ${}_SM_j$  is a chain recurrent component on  $SV$  that projects to a chain recurrent component  ${}_PM_i$  in  $PV$ , then the Morse spectra of the cone bundle  $\mathcal{V}_j^+$  generated by  ${}_SM_j$  and of the subbundle  $\mathcal{V}_i$  generated by  ${}_PM_i$  coincide,*

$$\Sigma_{Mo}(\mathcal{V}_j^+) = \Sigma_{Mo}(\mathcal{V}_i).$$

*Proof.* The inclusion “ $\subset$ ” holds since every  $(\varepsilon, T)$ -chain in  ${}_SM_j$  projects to an  $(\varepsilon, T)$ -chain in  ${}_PM_i$  with the same exponent. For the converse, consider an  $(\varepsilon, T)$ -chain  $\zeta$  in  ${}_PM_i$  given by  $T_0, \dots, T_{n-1} \geq T$  and points  $\mathbb{P}v_0, \dots, \mathbb{P}v_n$ . Then we can write

$$\lambda(\zeta) = \frac{1}{\sigma} \left( \sum_{i=0}^{n-1} \log \|\Phi(T_i, \pm v_i)\| - \log \|\pm v_i\| \right),$$

where, by (2.5), we may assume that  $\pm v_i \in {}_SM^+ \cup {}_SM^- \subset SV$ . Each of the chains constructed in the proof of Lemma 2.3(i) has chain exponent  $\lambda(\zeta)$ .  $\square$

Lemma 2.6 implies for the lift of a chain  $\zeta$  that  $\|w_n\| = \prod_{i=0}^{n-1} \|\Phi(T_j, v_j)\| = e^{\lambda(\zeta)\sigma}$ . Hence  $\lambda(\zeta) > 0$  and  $\lambda(\zeta) < 0$  imply  $\|w_n\| > 1$  and  $\|w_n\| < 1$ , resp.

#### 4. THE FLOW ON THE POINCARÉ SPHERE BUNDLE

This section analyzes when a subbundle  $\mathcal{V}_i$  yields a chain transitive set on the Poincaré sphere bundle. In particular, a sufficient condition is derived in terms of the Morse spectrum.

For a vector bundle  $\pi : \mathcal{V} \rightarrow B$  the projection to the Poincaré sphere can be defined in the following way (cf. Perko [19, Section 3.10] for the case in  $\mathbb{R}^d$ ). Define a Hilbert space by  $H^1 = H \times \mathbb{R}$  with inner product

$$\langle (h_1, \alpha_1), (h_2, \alpha_2) \rangle = \langle h_1, h_2 \rangle_H + \alpha_1 \alpha_2 \text{ for } (h_1, \alpha_1), (h_2, \alpha_2) \in H \times \mathbb{R},$$

and consider the corresponding vector bundle  $\pi^1 : \mathcal{V}^1 = \mathcal{V} \times \mathbb{R} \rightarrow B$  with  $\pi^1(v, r) = \pi(v)$  using the local trivializations  $\varphi_a^1 : (\pi^1)^{-1}(U_a) \rightarrow U_a \times H^1, a \in A$ , defined by

$$\varphi_a^1(v, r) = (\varphi_a(v), r) = (b, (x, r)) \text{ for } (v, r) \in \mathcal{V}^1 = \mathcal{V} \times \mathbb{R} \text{ and } (b, x) = \varphi_a(v).$$

Define a map  $\pi_P : \mathcal{V} \rightarrow S\mathcal{V}^1$  by

$$(4.1) \quad \pi_P(v) = \left( \frac{v}{\|(v, 1)\|}, \frac{1}{\|(v, 1)\|} \right) \text{ for } v \in \mathcal{V}.$$

Then  $\pi_P$  is a homeomorphism onto the open upper hemisphere of  $S\mathcal{V}^1$

$$S_P^+ \mathcal{V} := \{(v, r) \in \mathcal{V}^1 \mid \|(v, r)\| = 1 \text{ and } r > 0\} = \{\pi_P(v) \mid v \in \mathcal{V}\}.$$

We call  $S\mathcal{V}^1$  the Poincaré sphere bundle. Denote by  $S_P^0 \mathcal{V}$  the set of all  $(v, 0) \in S\mathcal{V}^1$ , the “equator” of the Poincaré sphere bundle, and define a homeomorphism

$$e : S\mathcal{V} \rightarrow S_P^0 \mathcal{V} : v \mapsto (v, 0).$$

The closed upper hemisphere bundle can be written as the disjoint union  $\overline{S_P^+ \mathcal{V}} = S_P^+ \mathcal{V} \cup S_P^0 \mathcal{V}$ .

A linear flow  $\Phi(t, v)$  on  $\mathcal{V}$  is extended to a linear flow on  $\mathcal{V}^1$  by  $\Phi^1(t, (v, r)) = (\Phi(t, v), r)$  and every trajectory  $\mathbb{S}\Phi(t, v) = \frac{\Phi(t, v)}{\|\Phi(t, v)\|}, t \in \mathbb{R}$ , on  $\mathbb{S}\mathcal{V}$  is mapped onto

$$e(\mathbb{S}\Phi(t, v)) = \mathbb{S}\Phi^1(t, (v, 0)) = (\mathbb{S}\Phi(t, v), 0), t \in \mathbb{R}.$$

**Lemma 4.1.** *The map  $e$  is a conjugacy between the flow  $\mathbb{S}\Phi$  on  $\mathbb{S}\mathcal{V}$  and the flow  $\mathbb{S}\Phi^1$  restricted to  $\mathbb{S}_P^0\mathcal{V}$ . Every chain recurrent component  ${}_sM_j$  is mapped onto a chain recurrent component  $e({}_sM_j)$  and  $\pi_P(\mathcal{V}_j^+) \cap \mathbb{S}_P^0\mathcal{V} = e({}_sM_j)$ .*

*Proof.* Obviously,  $e$  is a conjugacy mapping chain recurrent components onto chain recurrent components. The inclusion  $\pi_P(\mathcal{V}_j^+) \cap \mathbb{S}_P^0\mathcal{V} \subset e({}_sM_j)$  holds since for  $v_n \in \mathcal{V}_j^+$  with  $\|v_n\| \rightarrow \infty$  for  $n \rightarrow \infty$  one finds

$$\pi_P(v_n) = \left( \frac{v_n}{\|(v_n, 1)\|}, \frac{1}{\|(v_n, 1)\|} \right) \text{ with } \frac{1}{\|(v_n, 1)\|} \rightarrow 0$$

and every cluster point of  $\frac{v_n}{\|(v_n, 1)\|}$  is in  ${}_sM_j$ . For the converse, let  $(v, 0) \in e({}_sM_j)$ . There are  $v_n \in \mathcal{V}_j^+$  with  $\|v_n\| \rightarrow \infty$  and  $\frac{v_n}{\|(v_n, 1)\|} \rightarrow v$  for  $n \rightarrow \infty$ . Then the assertion follows from  $\pi_P(v_n) \rightarrow (v, 0)$  for  $n \rightarrow \infty$ .  $\square$

Furthermore, the flow  $\Phi$  on  $\mathcal{V}$  induces a flow  $\pi_P\Phi$  on  $\mathbb{S}_P^+\mathcal{V}$  given by

$$\pi_P\Phi(t, \pi_P v) = \left( \frac{\Phi(t, v)}{\|(\Phi(t, v), 1)\|}, \frac{1}{\|(\Phi(t, v), 1)\|} \right) \text{ for } t \in \mathbb{R}, v \in \mathcal{V}.$$

**Lemma 4.2.** *Let, for  $\varepsilon > 0$  small enough,  $\zeta$  be an  $(\varepsilon, T)$ -chain on  $\mathbb{S}\mathcal{V}$  with lift  $\hat{\zeta}$  to  $\mathcal{V}$ . Then for  $\alpha > 0$  the associated chain  $\pi_P(\alpha\hat{\zeta})$  on  $\mathbb{S}_P^+\mathcal{V}$  is an  $(\varepsilon, T)$ -chain.*

*Proof.* For  $\zeta$  as in Definition 2.5, the lifted chain  $\alpha\hat{\zeta}$  is given by  $\alpha w_i = \alpha C_{i-1} v_i$  for  $i = 0, \dots, n$ . By Lemma 2.6 it follows that  $\|w_i\| = C_{i-1}$  and  $\Phi(T_i, \alpha w_i) = \alpha C_i \mathbb{S}\Phi(T_i, v_i)$ . For  $\varepsilon > 0$  small enough, there is  $a \in A$  with

$$\begin{aligned} d(\mathbb{S}\Phi(T_i, v_i), v_{i+1}) &= d_a(\mathbb{S}\Phi(T_i, v_i), v_{i+1}), \\ d(\pi_P\Phi(T_i, \alpha w_i), \pi_P(\alpha w_{i+1})) &= d_a(\pi_P\Phi(T_i, \alpha w_i), \pi_P(\alpha w_{i+1})). \end{aligned}$$

One computes

$$\begin{aligned} \pi_P\Phi(T_i, \alpha w_i) &= \left( \frac{\Phi(T_i, \alpha w_i)}{\|(\Phi(T_i, \alpha w_i), 1)\|}, \frac{1}{\|(\Phi(T_i, \alpha w_i), 1)\|} \right) \\ &= \left( \frac{\alpha C_i \mathbb{S}\Phi(T_i, v_i)}{\|(\alpha C_i \mathbb{S}\Phi(T_i, v_i), 1)\|}, \frac{1}{\|(\alpha C_i \mathbb{S}\Phi(T_i, v_i), 1)\|} \right) \\ &= \left( \frac{\alpha C_i \mathbb{S}\Phi(T_i, v_i)}{\sqrt{\alpha^2 C_i^2 + 1}}, \frac{1}{\sqrt{\alpha^2 C_i^2 + 1}} \right) \\ &= \frac{1}{\sqrt{\alpha^2 C_i^2 + 1}} (\alpha C_i \mathbb{S}\Phi(T_i, v_i), 1) \end{aligned}$$

and

$$\pi_P(\alpha w_{i+1}) = \left( \frac{\alpha w_{i+1}}{\|(\alpha w_{i+1}, 1)\|}, \frac{1}{\|(\alpha w_{i+1}, 1)\|} \right) = \left( \frac{\alpha C_i v_{i+1}}{\sqrt{\alpha^2 C_i^2 + 1}}, \frac{1}{\sqrt{\alpha^2 C_i^2 + 1}} \right).$$



The local trivializations have, with  $(b_i \cdot T_i, y_i) = \varphi_a(\mathbb{S}\Phi(T_i, v_i))$ , the form

$$\varphi_a^1(\pi_P \Phi(T_i, \alpha w_i)) = \left( b_i \cdot T_i, \frac{\alpha C_i y_i}{\sqrt{\alpha^2 C_i^2 + 1}}, \frac{1}{\sqrt{\alpha^2 C_i^2 + 1}} \right),$$

and, with  $(b_{i+1}, x_{i+1}) = \varphi_a(v_{i+1})$ ,

$$\varphi_a^1(\pi_P(\alpha w_{i+1})) = \left( b_{i+1}, \frac{\alpha C_i x_{i+1}}{\sqrt{\alpha^2 C_i^2 + 1}}, \frac{1}{\sqrt{\alpha^2 C_i^2 + 1}} \right).$$

The distance  $d_a$  in  $\mathbb{S}\mathcal{V}^1 \subset \mathcal{V}^1$  is (cf. (2.3))

$$d_a((v, r), (v', r')) = \max\{d(b, b'), \|(x, r) - (x', r')\|_a\},$$

where  $(v, r) = (\varphi_a^1)^{-1}(b, (x, r))$  and  $(v', r') = (\varphi_a^1)^{-1}(b', (x', r'))$ . It follows that

$$\begin{aligned} & d_a(\pi_P \Phi(T_i, \alpha w_i), \pi_P(\alpha w_{i+1})) \\ &= \max \left\{ d(b_i \cdot T_i, b_{i+1}), \frac{1}{\sqrt{\alpha^2 C_i^2 + 1}} \|(\alpha C_i y_i, 1) - (\alpha C_i x_{i+1}, 1)\|_a \right\} \\ &\leq \max \left\{ d(b_i \cdot T_i, b_{i+1}), \frac{\alpha C_i}{\sqrt{\alpha^2 C_i^2 + 1}} \|y_i - x_{i+1}\|_a \right\} \\ &\leq \max \{d(b_i \cdot T_i, b_{i+1}), \|y_i - x_{i+1}\|_a\} \\ &= d_a(\mathbb{S}\Phi(T_i, v_i), v_{i+1}). \end{aligned}$$

This concludes the proof.  $\square$

The following result shows that one obtains chain transitive sets on the closed upper hemisphere of the Poincaré sphere bundle from the chain recurrent components on the sphere bundle  $\mathbb{S}\mathcal{V}$ .

**Theorem 4.3.** *Let  $\mathcal{V}_j^+ = \{\alpha v \in \mathcal{V} \mid \alpha > 0, v \in {}_{\mathbb{S}}M_j\}$  be the cone bundle on  $\mathcal{V}$  generated by a chain recurrent component  ${}_{\mathbb{S}}M_j, j \in \{1, \dots, l_1\}$ , on  $\mathbb{S}\mathcal{V}$ . Then the following assertions are equivalent:*

(a) *The set  $\overline{\pi_P \mathcal{V}_j^+}$  is chain transitive on the closed upper hemisphere  $\overline{\mathbb{S}_P^+ \mathcal{V}}$  of the Poincaré sphere bundle.*

(b) *The cone bundle  $\mathcal{V}_j^+$  contains a half-line  $l = \{\alpha v_0 \mid \alpha > 0\}$  for some  $v_0 \in {}_{\mathbb{S}}M_j$  such that  $\pi_P l$  is chain transitive on  $\mathbb{S}_P^+ \mathcal{V}$ .*

*Proof.* It is clear that (a) implies (b). For the converse let  $z_1, z_2 \in \mathcal{V}_j^+$ , hence there are  $\alpha_1, \alpha_2 > 0$  and  $v_1, v_2 \in {}_{\mathbb{S}}M_j$  with  $z_1 = \alpha_1 v_1$  and  $z_2 = \alpha_2 v_2$ . Fix  $\varepsilon, T > 0$ . We construct an  $(\varepsilon, T)$ -chain from  $\pi_P z_1$  to  $\pi_P z_2$ . There is an  $(\varepsilon, T)$ -chain  $\zeta^1$  in  ${}_{\mathbb{S}}M_j$  from  $v_1$  to  $v_0$ , hence the induced chain  $\alpha_1 \zeta^1$  in  $\mathcal{V}$  starts in  $\alpha_1 v_0 = z_1$  and ends in  $\alpha_1 \gamma_1 v_0 \in l$  for some  $\gamma_1 > 0$ . Since  $v_0, v_2 \in {}_{\mathbb{S}}M_j$  one finds an  $(\varepsilon, T)$ -chain  $\zeta^2$  in  ${}_{\mathbb{S}}M_j$  from  $v_0$  to  $v_2$ . The induced chain  $\hat{\zeta}^2$  in  $\mathcal{V}$  ends in  $\gamma_2 v_2$  for some  $\gamma_2 > 0$ . This yields an  $(\varepsilon, T)$ -chain  $\frac{\alpha_2}{\gamma_2} \hat{\zeta}^2$  in  $\mathcal{V}$  from  $\frac{\alpha_2}{\gamma_2} v_0$  to  $\frac{\alpha_2}{\gamma_2} \gamma_2 v_2 = \alpha_2 v_2 = z_2$ .

By chain transitivity of  $\pi_P l$  one finds an  $(\varepsilon, T)$ -chain  $\zeta^0$  in  $\mathbb{S}_P^+ \mathcal{V}$  from  $\pi_P(\alpha_1 \gamma_1 v_0) \in \pi_P l$  to  $\pi_P(\frac{\alpha_2}{\gamma_2} v_0) \in \pi_P l$ . By Lemma 4.2 the  $(\varepsilon, T)$ -chains  $\zeta^1$  and  $\zeta^2$  on  $\mathbb{S}\mathcal{V}$  induce  $(\varepsilon, T)$ -chains  $\pi_P \left( \frac{\alpha_2}{\gamma_2} \hat{\zeta}^2 \right)$  and  $\pi_P \left( \alpha_1 \hat{\zeta}^1 \right)$  on  $\mathbb{S}_P^+ \mathcal{V}$ . Then one obtains an  $(\varepsilon, T)$ -chain

from  $\pi_P z_1$  to  $\pi_P z_2$  by the concatenation

$$\pi_P \left( \frac{\alpha_2}{\gamma_2} \widehat{\zeta^2} \right) \circ \zeta^0 \circ \pi_P \left( \alpha_1 \widehat{\zeta^1} \right).$$

Thus  $\pi_P \mathcal{V}_j^+$  is chain transitive which implies that also its closure  $\overline{\pi_P \mathcal{V}_j^+}$  is chain transitive.  $\square$

Next we discuss when the existence of a half-line  $l$  as above can be guaranteed. It is clear that a necessary condition is that 0 is in the Morse spectrum. Conversely, Theorem 4.7 will show that  $l$  exists if 0 is in the interior of the Morse spectrum.

Lemma 4.4 will be needed in Step 2 of the next proof.

**Lemma 4.4.** *Let  $a, b, c$  be real numbers with  $a, b, c > 1$  and  $\frac{\log b}{\log a} \in \mathbb{R} \setminus \mathbb{Q}$ . Then for every  $\delta > 0$  there are  $k, \ell \in \mathbb{N}$  such that  $|a^k b^{-\ell} - c| < \delta$ .*

*Proof.* Since the logarithm is continuously invertible, it suffices to show that for every  $\delta > 0$  there are  $k, \ell \in \mathbb{N}$  with

$$\delta > |\log(a^k b^{-\ell}) - \log c| = |k \log a - \ell \log b - \log c|,$$

or, dividing by  $\log a > 0$ ,

$$(4.2) \quad \left| k - \ell \frac{\log b}{\log a} - \frac{\log c}{\log a} \right| < \frac{\delta}{\log a}.$$

Since  $\frac{\log b}{\log a}$  is irrational, Kronecker's theorem (cf. Cassels [7, Theorem IV, p.53]) implies that for any  $\delta > 0$  there are  $k, \ell \in \mathbb{N}$  satisfying (4.2).  $\square$

*Remark 4.5.* The proof above uses Kronecker's theorem (1884) [18]. The special case of this theorem needed here also follows from an earlier result by Tchebychev (1866) [26, Théorème, p. 679]; cf. Colonius, Santana, and Setti [11, Lemma 3.4], where Lemma 4.4 is used to prove an approximate controllability result for bilinear control systems, using a different spectral notion.

**Lemma 4.6.** *Consider a chain recurrent component  ${}_S M_j$  on the sphere bundle  $S\mathcal{V}$  and assume that there is  $\delta_0 > 0$  such that for all  $\varepsilon, T > 0$  there are periodic  $(\varepsilon, T)$ -chains  $\zeta^\pm$  in  ${}_S M_j$  with chain exponents  $\lambda(\zeta^+) > \delta_0$  and  $\lambda(\zeta^-) < -\delta_0$ . Then the cone bundle  $\mathcal{V}_j^+ := \{\alpha v \in \mathcal{V} \mid \alpha > 0, v \in {}_S M_j\}$  contains a half-line  $l = \{\alpha v \mid \alpha > 0\}$  for some  $v \in {}_S M_j$  such that  $\pi_P l$  is chain transitive.*

*Proof.*

*Step 1.* First we show that one may choose the initial points  $v^\pm$  of  $\zeta^\pm$  independently of  $\varepsilon, T$ .

Fix any point  $v \in {}_S M_j$ . Let  $\varepsilon, T > 0$  and consider for  $S \geq T$  periodic  $(\varepsilon, S)$ -chains  $\zeta^\pm$  with  $\lambda(\zeta^+) > \delta_0$  and  $\lambda(\zeta^-) < -\delta_0$ . The proof is given for  $\zeta^+$ , the proof for  $\zeta^-$  is analogous and will be omitted. It suffices to prove that for every  $\delta > 0$  there exists a periodic  $(\varepsilon, T)$ -chain  $\zeta$  through  $v$  with  $|\lambda(\zeta^+) - \lambda(\zeta)| < \delta$ .

Let  $\zeta^+$  be given by  $T_0, \dots, T_n \geq S, v_0 = v^+, v_1, \dots, v_n = v^+ \in {}_S M_j$  with total time  $\sigma$  and

$$d(\Phi(T_i, v_i), v_{i+1}) < \varepsilon \text{ for } i = 0, \dots, n-1.$$

By Colonius and Kliemann [9, Lemma B.2.23] there exists  $\bar{T}(\varepsilon, T) > 0$  such that for all  $v', v'' \in {}_S M_j$  there is an  $(\varepsilon, T)$ -chain from  $v'$  to  $v''$  with total time equal to or less than  $\bar{T}(\varepsilon, T)$ .

Choose  $(\varepsilon, T)$ -chains  $\zeta^1$  from  $v$  to  $v^+$  and  $\zeta^2$  from  $v^+$  to  $v$  on  $\mathbb{S}\mathcal{V}$ . For  $k = 1, 2$  let them be given by  $v_0^k, \dots, v_{n_k}^k$  and times  $T_0^k, \dots, T_{n_k-1}^k > T$  with total time  $\sigma_k \leq \bar{T}(\varepsilon, T)$  and chain exponent

$$\lambda(\zeta^k) = \frac{1}{\sigma_k} \sum_{i=0}^{n_k-1} \log \|\Phi(T_i^k, v_i^k)\|.$$

By compactness of the sphere bundle and continuity together with  $\sigma_k \leq \bar{T}(\varepsilon, T)$  there is a constant  $c_0 = c_0(\varepsilon, T) > 0$  such that

$$\sum_{i=0}^{n_k-1} \log \|\Phi(T_i^k, v_i^k)\| \leq c_0.$$

The concatenation  $\zeta^2 \circ \zeta^+ \circ \zeta^1$  is a periodic  $(\varepsilon, T)$ -chain through  $v$  with chain exponent

$$\begin{aligned} & \lambda(\zeta^2 \circ \zeta^+ \circ \zeta^1) \\ &= \frac{1}{\sigma + \sigma_1 + \sigma_2} \left[ \sum_{i=0}^{n-1} \log \|\Phi(T_i, v_i)\| + \sum_{i=0}^{n_1-1} \log \|\Phi(T_i^1, v_i^1)\| + \sum_{i=0}^{n_2-1} \log \|\Phi(T_i^2, v_i^2)\| \right]. \end{aligned}$$

Since  $\frac{1}{\sigma + \sigma_1 + \sigma_2} - \frac{1}{\sigma} = \frac{\sigma_1 + \sigma_2}{\sigma + \sigma_1 + \sigma_2} \cdot \frac{1}{\sigma}$  it follows that

$$\left( \frac{1}{\sigma + \sigma_1 + \sigma_2} - \frac{1}{\sigma} \right) \sum_{i=0}^{n-1} \log \|\Phi(T_i, v_i)\| = \frac{\sigma_1 + \sigma_2}{\sigma + \sigma_1 + \sigma_2} \lambda(\zeta^+).$$

Thus one may choose  $S$  and hence  $\sigma$  large enough such that as desired

$$|\lambda(\zeta^+) - \lambda(\zeta^2 \circ \zeta^+ \circ \zeta^1)| < \delta.$$

*Step 2.* For  $v \in {}_{\mathbb{S}}M_j$  the projection  $\pi_P l$  of the half-line  $l := \{\alpha v \in \mathcal{V} | \alpha > 0\}$  is chain transitive.

We construct for all  $\alpha_0 v, \alpha_1 v \in l$  and every  $\varepsilon, T > 0$  an  $(\varepsilon, T)$ -chain from  $\pi_P(\alpha_0 v)$  to  $\pi_P(\alpha_1 v)$ . The strategy for the proof is to go  $k$  times through the periodic chain  $\zeta^+$  and  $\ell$  times through the periodic chain  $\zeta^-$ , and to adjust the numbers  $k, \ell \in \mathbb{N}$  such that in  $\mathcal{V}$  one approaches  $\alpha_1 v$ . Fix  $\varepsilon, T > 0$ , add a superscript  $\pm$  to the components of the chain  $\zeta^\pm$ , and abbreviate

$$(4.3) \quad \beta^\pm = \prod_{i=0}^{n^\pm-1} \|\Phi(T_i^\pm, v_i^\pm)\|.$$

Define for  $k, \ell \in \mathbb{N}$  a periodic  $(\varepsilon, T)$ -chain  $\zeta^{k,\ell}$  through  $v$  on  $\mathbb{S}\mathcal{V}$  by the concatenation

$$\zeta^{k,\ell} = (\zeta^-)^\ell \circ (\zeta^+)^k.$$

The endpoint of the lift  $\widehat{\zeta^{k,\ell}}$  of  $\zeta^{k,\ell}$  to  $\mathcal{V}$  is given by  $(\beta^-)^\ell (\beta^+)^k v$ . The chain  $\alpha_0 \widehat{\zeta^{k,\ell}}$  starts in  $\alpha_0 v$  and ends in  $\alpha_0 w_n = (\beta^+)^k (\beta^-)^\ell \alpha_0 v$ .

Lemma 4.4 implies the following: If one can choose  $\beta^+$  and  $\beta^-$  such that  $\frac{\log \beta^-}{\log \beta^+}$  is irrational then for every  $\delta > 0$  there are  $k, \ell \in \mathbb{N}$  with

$$\left| (\beta^+)^k (\beta^-)^\ell - \frac{\alpha_1}{\alpha_0} \right| < \delta.$$

Hence one can choose  $k, \ell$  such that

$$\left\| (\beta^-)^\ell (\beta^+)^k \alpha_0 v - \alpha_1 v \right\| < \alpha_0 \delta < \varepsilon.$$

This yields a  $(2\varepsilon, T)$ -chain  $\zeta^{k,\ell}$  on  $\mathbb{S}\mathcal{V}$  from  $\alpha_0 v$  to  $\alpha_1 v$  such that, by Lemma 4.2,  $\pi_P(\alpha_0 \widehat{\zeta^{k,\ell}})$  is a  $(2\varepsilon, T)$ -chain from  $\pi_P(\alpha_0 v)$  to  $\pi_P(\alpha_1 v)$ .

Since  $\varepsilon, T > 0$  are arbitrary, this completes Step 2 once we have justified the assumption that we can choose  $\beta^+$  and  $\beta^-$  as in (4.3) such that  $\frac{\log \beta^-}{\log \beta^+}$  is irrational. If this is not the case, we will adjust  $v_1^+$  appropriately. First note that we may assume for the length of the periodic chain  $\zeta^+$  that  $n^+ > 1$  (otherwise we replace it by  $\zeta^+ \circ \zeta^+$ ). The map  $\Phi(T_1^+, \cdot)$  restricted to the fiber  $\mathcal{V}_{\pi(v)}$  with range  $\mathcal{V}_{\pi(v) \cdot T_1^+}$  is a linear isomorphism. It maps any neighborhood of  $v_1^+$  onto a neighborhood of  $\Phi(T_1^+, v_1^+)$ . Thus one finds arbitrarily close to  $v_1^+$  points  $v_1 \in \mathcal{V}_{\pi(v)}$  such that for

$$\beta_0 := \|\Phi(T_1, v_1)\| \prod_{i=0, i \neq 1}^{n^+-1} \|\Phi(T_i^+, v_i^+)\|,$$

the quotient  $\frac{\log \beta^-}{\log \beta_0}$  is irrational. Choosing the neighborhood  $N$  small enough we may replace  $v_1^+$  by  $v_1$  without destroying the property of a  $(2\varepsilon, T)$ -chain from  $\alpha_0 v$  to  $\alpha_1 v$ .

□

Theorem 4.7 provides a sufficient condition for the property that the cone bundle induced by a central subbundle in the Selgrade decomposition yields a chain transitive set on the upper hemisphere of the Poincaré sphere bundle.

**Theorem 4.7.** *If a cone bundle  $\mathcal{V}_j^+$  generated by a chain recurrent component  ${}_s M_j, j \in \{1, \dots, l_1\}$ , on  $\mathbb{S}\mathcal{V}$  satisfies  $0 \in \text{int}(\Sigma_{Mo}(\mathcal{V}_j^+))$ , then the set  $\overline{\pi_P \mathcal{V}_j^+}$  is chain transitive on the closed upper hemisphere  $\mathbb{S}_P^+ \mathcal{V}$  of the Poincaré sphere bundle.*

*Proof.* The Morse spectrum over a chain recurrent component in the projective bundle is a bounded interval, cf. [9, Proposition 6.2.14]. By Lemma 3.1 the same is valid for the Morse spectrum  $\Sigma_{Mo}({}_s M_i)$ . If  $0 \in \text{int}(\Sigma_{Mo}({}_s M_i))$ , it follows that there is  $\delta_0 > 0$  such that for all  $\varepsilon, T > 0$  there are periodic  $(\varepsilon, T)$ -chains  $\zeta^+, \zeta^-$  on  ${}_s M_i$ , with  $\lambda(\zeta^+) > \delta_0$  and  $\lambda(\zeta^-) < -\delta_0$  for some  $\delta_0 > 0$ . Thus Lemma 4.6 implies that  $\mathcal{V}_j^+$  contains a half-line  $l = \{\alpha v \mid \alpha > 0\}$  for some  $v \in {}_s M_j$  such that  $\pi_P l$  is chain transitive on  $\mathbb{S}_P^+ \mathcal{V}$ . Now the assertion follows from Theorem 4.3. □

## 5. EXAMPLES

In this section the results above are illustrated by autonomous linear differential equations and by bilinear control systems.

**Example 5.1.** Consider

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Here  ${}_s M^\pm = \{(0, \pm 1)\} \subset \mathbb{S}^1$  are chain recurrent components generating the cones  $V^\pm = \{(0, \pm \alpha) \mid \alpha > 0\}$ . The solutions in  $\mathbb{R}^2$  in the eigenspace  $\{0\} \times \mathbb{R}$  for the eigenvalue 0 are equilibria, hence  $V^\pm$  form chain transitive sets in  $\mathbb{R}^2$ . The Morse spectrum is given by  $\Sigma_{Mo}(V^\pm) = \{0\}$  and on the closed upper hemisphere  $\mathbb{S}^{2,+}$  one

obtains the chain transitive sets

$$\overline{\pi_P V^+} = \{(0, s_2, s_3) \in \mathbb{S}^2 \mid s_2 \geq 0, s_3 \geq 0\},$$

$$\pi_P V^- = \{(0, s_2, s_3) \in \mathbb{S}^2 \mid s_2 \leq 0, s_3 \geq 0\}.$$

The intersection with the equator  $\mathbb{S}^{2,0} = \{(s_1, s_2, s_3) \in \mathbb{S}^2 \mid s_3 = 0\}$  is given by  $\overline{\pi_P V^\pm} \cap \mathbb{S}^{2,0} = (0, \pm 1, 0) = e((0, \pm 1))$ .

For the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

the unit sphere  $S^1$  is a chain recurrent component. The solutions in  $\mathbb{R}^2$  are unbounded, hence the system on  $\mathbb{R}^2$  has no recurrence properties. The compactification provided by the Poincaré sphere yields the chain transitive set

$$\overline{\pi_P(\mathbb{R}^2)} = \{(s_1, s_2, s_3) \in \mathbb{S}^2 \mid s_3 \geq 0\}.$$

For general autonomous linear differential equations the assumption in Theorem 4.7 that 0 is in the interior of the Morse spectrum is never satisfied, since the Morse spectrum reduces to the real parts of the eigenvalues. A direct application of Theorem 4.3 yields the following result.

**Corollary 5.2.** *For an autonomous linear differential equation given by  $\dot{x} = Ax$  with  $A \in \mathbb{R}^{d \times d}$ , suppose that the subspace*

$$V := \bigoplus \{E(\mu) \mid \mu \in \text{spec}(A) : \text{Re } \mu = 0\}$$

*is nontrivial; here  $E(\mu)$  is the (real) generalized eigenspace for the eigenvalue  $\mu$ .*

*Then the projection of  $V$  to projective space  $\mathbb{P}^{d-1}$  is a chain recurrent component  ${}_{\mathbb{P}}M$  and there are one or two cones  $V^+$  and  $V^-$  such that  $V^\pm \cap \mathbb{S}^{d-1}$  are chain recurrent components on the unit sphere  $\mathbb{S}^{d-1}$  which project onto  ${}_{\mathbb{P}}M$ . The closure of the projection  $\pi_P V^\pm$  is chain transitive for the induced flow on the closed upper hemisphere  $\overline{\mathbb{S}_P^+ \mathbb{R}^d} = \overline{\mathbb{S}^{d,+}}$  of the Poincaré sphere.*

*Proof.* It is well known that the projection of  $V$  is a chain recurrent component  ${}_{\mathbb{P}}M$  in projective space  $\mathbb{P}^{d-1}$  (cf., e.g., Colonius and Kliemann [10, Theorem 4.1.3] for a proof). Then it easily follows that there are one or two chain recurrent components on the unit sphere which project onto  ${}_{\mathbb{P}}M$ . The last assertion of the corollary follows from Theorem 4.3, since for an eigenvalue  $\mu$  with  $\text{Re } \mu = 0$  the (real) eigenspace  $E(\mu)$  consists of a continuum of periodic solutions and hence contains a chain transitive half-line. Thus also the projection to  $\mathbb{S}_P^+ \mathbb{R}^d$  is chain transitive.  $\square$

Another class of linear flows is given by homogeneous control systems of the form

$$(5.1) \quad \dot{x}(t) = A_0 x(t) + \sum_{i=1}^m u_i(t) A_i x(t), \quad u(t) \in \Omega,$$

where  $A_0, A_1, \dots, A_m \in \mathbb{R}^{d \times d}$ . The control functions  $u = (u_1, \dots, u_m)$  have values in a convex and compact subset  $\Omega \subset \mathbb{R}^m$  and the set of admissible controls is  $\mathcal{U} = \{u \in L^\infty(\mathbb{R}, \mathbb{R}^m) \mid u(t) \in \Omega \text{ for almost all } t\}$ . Denote the solutions of (5.1) with initial condition  $x(0) = x$  by  $\varphi(t, x, u)$ ,  $t \in \mathbb{R}$ .

A control system of this form defines a linear flow, the control flow, on the vector bundle  $\mathcal{V} = \mathcal{U} \times \mathbb{R}^d$  given by  $\Phi : \mathbb{R} \times \mathcal{U} \times \mathbb{R}^d \rightarrow \mathcal{U} \times \mathbb{R}^d$ ,  $(t, u, x) \mapsto (u(t + \cdot), \varphi(t, x, u))$  where  $u(t + \cdot)(s) := u(t + s)$ ,  $s \in \mathbb{R}$ , is the right shift and  $\mathcal{U} \subset L^\infty(\mathbb{R}, \mathbb{R}^m)$  is

considered in a metric for the weak\* topology. Then  $\mathcal{U}$  is compact and chain transitive; cf. Colonius and Kliemann [9, Chapter 4] or Kawan [16, Section 1.4].

The following two-dimensional example is a minor modification of [9, Example 5.5.12]. Here four chain recurrent components on the sphere bundle are present, and the corresponding cone bundles contain 0 in the interior of the Morse spectrum.

**Example 5.3.** Consider the control system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{4} \\ -\frac{1}{4} & 0 \end{bmatrix} + u_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + u_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + u_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

with

$$u(t) = (u_1(t), u_2(t), u_3(t)) \in \Omega = [-1, 1] \times [-1/4, 1/4] \times [-1/4, 1/4] \subset \mathbb{R}^3.$$

This defines a linear flow  $\Phi$  on the vector bundle  $\mathcal{U} \times \mathbb{R}^2$  given by  $\Phi(t, u, x) = (u(t + \cdot), \varphi(t, x, u))$ . With

$$A_1 := \left\{ (x_1, x_2) \in \mathbb{R}^2 \mid x_2 = \alpha x_1, \alpha \in [-\sqrt{2}, -1/\sqrt{2}] \right\},$$

$$A_2 := \left\{ (x_1, x_2) \in \mathbb{R}^2 \mid x_2 = \alpha x_1, \alpha \in [1/\sqrt{2}, \sqrt{2}] \right\},$$

the two chain recurrent components of  $\mathbb{P}\Phi$  on the projective bundle  $\mathcal{U} \times \mathbb{P}^1$  are

$$\mathbb{P}M_i = \left\{ (u, p) \in \mathcal{U} \times \mathbb{P}^1 \mid \mathbb{P}\varphi(t, p, u) \in \pi_{\mathbb{P}} A_i \text{ for } t \in \mathbb{R} \right\}, \quad i = 1, 2.$$

With

$$A_i^+ = A_i \cap ((0, \infty) \times \mathbb{R}), \quad A_i^- = A_i \cap ((-\infty, 0) \times \mathbb{R}),$$

the four chain recurrent components for  $\mathbb{S}\Phi$  on the sphere bundle  $\mathbb{S}(\mathcal{U} \times \mathbb{R}^2) = \mathcal{U} \times \mathbb{S}^1$  are given by

$$\mathbb{S}M_i^\pm = \left\{ (u, s) \in \mathcal{U} \times \mathbb{S}^1 \mid \mathbb{S}\varphi(t, s, u) \in \pi_{\mathbb{S}} A_i^\pm \text{ for } t \in \mathbb{R} \right\}, \quad i = 1, 2.$$

One computes the Morse spectra of the generated cone bundles  $\mathcal{V}_i^\pm$  as (here they are determined by the eigenvalues for constant controls)

$$\Sigma_{Mo}(\mathcal{V}_1^\pm) = [-2, 1/2] \quad \text{and} \quad \Sigma_{Mo}(\mathcal{V}_2^\pm) = [-1/2, 2].$$

Hence for each of them the Morse spectrum contains 0 in the interior. By Theorem 4.7 the projections  $\pi_P \mathcal{V}_i^\pm$  are chain transitive on the upper hemisphere  $\mathbb{S}_P^+(\mathcal{U} \times \mathbb{R}^2) = \mathcal{U} \times \mathbb{S}^{2,+}$  of the Poincaré sphere bundle.

#### ACKNOWLEDGMENTS

I am indebted to Alexandre Santana for helpful discussions concerning the Poincaré sphere and to Mauro Patrão for a hint to Kronecker's theorem.

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