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Remarks on Translation Transversal Designs

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In this paper the existence of translation transversal designs which is equivalent to the existence of certain partitions in finite groups is studied. All considerations are based on the fact that the particular component of such a partition (the component representing the point classes of the corresponding design) is a normal subgroup of the translation group.

With regard to groups admitting an (s, k, λ)-partition, on one hand the already known families of such groups are determined without using R. Baer's, O. H. Kegel's, and M. Suzuki's classification of finite groups with partition and on the other hand some new results on the structure of p-groups admitting an (s, k, λ)-partition are proved.

Furthermore, the existence of a series of nonabelian p-groups of odd order which can be represented as translation groups of certain $(s, k, 1)$-translation transversal designs is shown; moreover, the translation groups are normal subgroups of collineation groups acting regularly on the set of flags of the same designs.

1. INTRODUCTION

(1.1) DEFINITION. A finite incidence structure $\mathbb{E} := (\mathbf{P}, \mathbf{B}, \mathbf{I})$ with set of points $\mathbf{P}$, set of blocks $\mathbf{B}$, and incidence structure $\mathbf{I}$ is called a *transversal design with parameters s, k, and* λ (short: (s, k, λ)-*TD*), if the following axioms are fulfilled:

(1.1.1) There exists an equivalence relation $\sim$ on the set of points $\mathbf{P}$ of $\mathbb{E}$ satisfying

(a) $p \sim q$, if and only if $p = q$ or p and q have no block in common;

(b) each equivalence class (or point class) consists exactly of s points and $2 \leqslant s \leqslant |\mathbf{P}|$;

(c) any two points p and q from different point classes are joined by exactly $\lambda \geqslant 1$ blocks.

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(1.1.2) Each block contains exactly $k \geqslant 2$ points.

(1.1.3) Each block meets every point class (in exactly one point by (1.1.1)(a)).

Unless otherwise stated we will always assume that $\mathbb{E}$ is a *simple* incidence structure which means that different blocks are incident with different point sets. The simplicity of $\mathbb{E}$ yields $k=2$, provided that $\lambda=1$. In this case $\mathbb{E}$ is called a *complete bipartite graph.*

The monograph [4] contains many results on the existence of transversal designs. In the following we summarize some basic combinatorial properties of (s, k, λ)-TDs, which all can be proved by applying simple counting arguments.

(1.2) *Combinatorial Properties of* (s, k, λ)*-TDs.* Let $\mathbb{E} = (\mathbf{P}, \mathbf{B}, \mathbf{I})$ be an (s, k, λ)-TD with $v := |\mathbf{P}|$ points and $b := |\mathbf{B}|$ blocks.

(1.2.1) There exist exactly k point classes and $v = sk$.

(1.2.2) Each point lies on exactly $r := s\lambda$ blocks.

(1.2.3) $b = s^2\lambda$.

In many papers the existence of transversal designs $\mathbb{E}$ and their dual structures, the so-called (s, k, λ)*-nets*, is studied under the assumption that $\mathbb{E}$ admits certain *automorphism groups* (if $\lambda = 1$, also the term *collineation group* is used). In D. Jungnickel's survey [16] many further references on this topic can be found. We will deal with an important special class of transversal designs in this paper:

(1.3) DEFINITION. *A translation transversal design with parameters* s, k, *and* λ (short: (s, k, λ)*-TTD*) is an (s, k, λ)-TD admitting an automorphism group G which

(1.3.1) acts regularly (sharply transitively) on the set $\mathbf{P}$ of points and by

(1.3.2) $B \cap B^g = \{\ \}$ or $B = B^g$ for all blocks B in $\mathbf{B}$ and all g in G induces a *parallelism* on the set of blocks.

By (1.3.1) the orbit of each block leads to a partition of $\mathbf{P}$. Therefore, the parallel classes are exactly the orbits of blocks under G. The automorphism group G is called a *translation group* of $\mathbb{E}$.

Already many authors have studied intensively translation transversal designs (cf. [5, 9, 15, 21–25]). Theorem (1.5) of R. H. Schulz [23, 25] says that the existence problem of TTDs is equivalent to a combinatorial problem in group theory. (All results from group theory applied in this paper can be found in [13]).

(1.4) DEFINITION. Let G be a group of order $v \geqslant 4$. A set $\mathbb{H} := \{N\} \cup \{H_1, H_2, ..., H_r\}$ of $r+1 = s\lambda + 1$ nontrivial, pairwise different subgroups of G is called an *(s, k, λ)-partition in G*, provided that the following properties are satisfied:

(1.4.1) $|N| = s$.

(1.4.2) $|H_i| = k$ for all $i = 1, 2, ..., r$.

(1.4.3) $|N \cap H_i| = 1$ for all $i = 1, 2, ..., r$.

(1.4.4) For each x in $G - N$ there exist exactly $\lambda \geqslant 1$ subgroups in $\mathbb{H}$ containing x.

(1.4.5) $v = ks$.

The elements of $\mathbb{H}$ are called *components*. For obvious reasons we call N the *particular component* of $\mathbb{H}$.

In [23, 25], the definition of an (s, k, λ)-partition is more general. There *translation divisible designs*, a more general class of translation designs are studied. However, being interested only in *transversal* designs, in the present paper we assume that (1.4.5) holds.

(1.5) THEOREM (R. H. Schulz). *An (s, k, λ)-translation transversal design with translation group G exists, if and only if G admits an (s, k, λ)-partition.*

Sketch of Proof. If $\mathbb{H} := \{N\} \cup \{H_1, H_2, ..., H_r\}$ is an (s, k, λ)-partition in a group G, then

$$\mathbb{E}(\mathbb{H}) := (G, \{H_i x \mid x \in G,\ i = 1, 2, ..., r\}, \in) \tag{1.5.1}$$

is an (s, k, λ)-TTD with translation group G. The action of G on $\mathbb{E}(\mathbb{H})$ is induced by right-multiplication. The point classes are the right cosets of the particular component N in G.

Conversely, let $\mathbb{E} = (\mathbf{P}, \mathbf{B}, \mathbf{I})$ be an (s, k, λ)-TTD with translation group G. We choose a basepoint p_0 in $\mathbf{P}$, the r blocks $B_1, B_2, ..., B_r$ incident with p_0 and the point class P_0 containing p_0. It is not difficult to see that the setwise stabilizers N of P_0 and H_i of B_i for $i = 1, 2, ..., r$ form an (s, k, λ)-partition in G. ∎

Due to the equivalence above, the structure of a translation group G of an (s, k, λ)-TTD is very restricted. The families of groups which admit an (s, k, λ)-partition, and therefore can be represented as a translation group of a translation transversal design, are known. The classification of these families was performed by R. H. Schulz and M. Biliotti/G. Micelli in the papers [5, 23, 25]. Furthermore, in [5, 9, 15, 21–25] many series of examples are constructed. This shows that the theory of translation transversal designs is well developed. The following simple, but important observation, upon which all considerations of this paper are based, however, is not proved in the above mentioned papers.[1]

(1.6) PROPOSITION. *Let G be a finite group and let $\mathbb{H} := \{N\} \cup \{H_1, H_2, ..., H_r\}$ be an (s, k, λ)-partition in G. Then the particular component N is a normal subgroup of G.*

Proof. Because of (1.4.1), (1.4.2), (1.4.3), and (1.4.5) we obtain

$$G = NH_i \qquad \text{for all} \quad i = 1, 2, ..., r. \tag{1.6.1}$$

Now let x and g be any elements of N and G, respectively. We assume that $x^g = g^{-1}xg$ does not lie in N. Let H be one of the λ components in $\mathbb{H} - \{N\}$ containing x^g. Applying (1.4.3) and (1.6.1) we can represent g uniquely as $g = nh$ whereby $n \in N$ and $h \in H$. Additionally with (1.4.3) we obtain

$$x^g \in N^g \cap H = N^{nh} \cap H = N^h \cap H = N^h \cap H^h = (N \cap H)^h = 1 \tag{1.6.2}$$

and therefore $x^g = 1$. This is a contradiction to our assumption that x^g is not an element of N, and, as x and g were chosen arbitrarily in N and G, respectively, we conclude that the particular component N is a normal subgroup of G. ▮

An immediate consequence of Proposition (1.6) is

(1.7) COROLLARY. *All components of $\mathbb{H} - \{N\}$ are pairwise isomorphic.*

Proof. As $NH = G$ for all H in $\mathbb{H} - \{N\}$, we deduce that the factor group $G/N = NH/N$ is isomorphic to $H/H \cap N$. Applying (1.4.3) we therefore obtain that G/N is isomorphic to H. Since this is true for all components H in $\mathbb{H} - \{N\}$ we obtain the aimed result. ▮

A further consequence of (1.6) is that the particular component N by rightmultiplication acts regularly on each point class and on each parallel class of blocks of the translation transversal design $\mathbb{E}(\mathbb{H})$ corresponding to $\mathbb{H}$ (cf. (1.5.1)). Therefore *all* TTDs belong to the class of the so-called *classregular transversal designs.* (Reference is made to [5, Sect. 4; 7, Sect. 5], where such designs with parameter $\lambda = 1$ are considered in connection with translation transversal designs.)

[1] In the case $\lambda = 1$, the normality of the particular component was already proved by A. Basile and P. Brutti [Struttura algebrica delle coordinate di un disegno trasversale di traslazione, *Ren. Circ. Mat. Palermo* **37** (1988), 109–119, Propositione 1.t′], but they do not study the restriction it imposes on the structure of G. The proof given there is essentially the same as the proof of (1.6) for arbitrary λ. The author thanks Professor Dr. Ralph-Hardo Schulz for this hint.

In the following section, using essentially (1.6), we want to determine the (known) families of finite groups which admit (s, k, λ)-partitions. Our proof of the classification is much more elementary than the proof given in [5, 23, 25].

In Section 3 we prove new results about the structure of p-groups with (s, k, λ)-partition. Furthermore, applying basic group theory, it is not difficult to classify all 2-groups which can be represented as translation groups of translation transversal designs. We obtain that every TTD admitting a 2-group as translation group which is not elementary abelian, is a complete bipartite graph.

Finally a concrete example will be presented in Section 4. In this connection we take up a theme of D. Jungnickel [15]: For each odd prime power q we construct a $(q^2, q, 1)$-translation transversal design $\mathbb{E}$ with nonabelian translation group G, and furthermore admitting an automorphism group K of $\mathbb{E}$ acting regularly on the set of flags (incident point-block pairs) of $\mathbb{E}$, containing G as a normal subgroup. We will also show that all TTDs with nonelementary abelian 2-group G as translation group admit a flag regular automorphism group containing G as a subgroup.

2. The Families of Groups Admitting an (s, k, λ)-Partition

(2.1) Definition. Let G be a finite group. Any nonempty set σ of nontrivial subgroups of G satisfying

$$U \cap V = 1 \qquad \text{for any two different elements } U \text{ and } V \text{ in } \sigma \quad (2.2.1)$$

and

$$\bigcup_{U \in \sigma} U = G \qquad (2.2.2)$$

is called a *partition*[2] of G. The elements of σ again are called *components of the partition.*

The following lemma of R. H. Schulz [25] shows that a finite group with (s, k, λ)-partition also admits a partition. The proof is included.

(2.2) Lemma. *Let $\mathbb{H} := \{N\} \cup \{H_1, H_2, ..., H_r\}$ be an (s, k, λ)-partition in a group G. Furthermore, for x in $G - N$ let $M_x := \bigcap_{H \in \mathbb{H},\, x \in H} H$. Then*

$$\mathbb{M} := \{N\} \cup \{M_x \mid x \in G - N\} \text{ is a partition in } G. \qquad (2.2.1)$$

[2] Sometimes the term *nontrivial partition* is used.

Proof. If y is an element of $M_x \cap (G-N)$, then by definition of M_x, any of the λ components of $\mathbb{H}$ containing x also contains y. As likewise, by definition of $\mathbb{H}$, exactly λ components of $\mathbb{H}$ contain y, we see that the set of x- and y-components in $\mathbb{H}$ coincide and we obtain $M_x = M_y$. Hence, for any pair $x, y \in G-N$ we have $M_x \cap M_y = 1$ or $M_x = M_y$. Of course, the groups M_x are nontrivial subgroups of G and have trivial intersection with N. Furthermore, by definition of $\mathbb{H}$, it is obvious that G is covered by $\mathbb{M}$ which finally shows that $\mathbb{M}$ is a partition of G. ▌

It is very important to observe that the particular component N of $\mathbb{H}$ is also a component of $\mathbb{M}$. In the case $\lambda = 1$ we have $\mathbb{M} = \mathbb{H}$. Evidently, due to Lemma (2.2), results on groups admitting a partition can be applied to study the existence of translation transversal designs. Finite groups with partition were first studied by R. Baer in [1]. In further papers, R. Baer, O. H. Kegel, and M. Suzuki [2, 3, 17, 26] were able to classify all such groups. The following main theorem of their investigations is a deep result.

(2.3) THEOREM (R. Baer, O. H. Kegel, and M. Suzuki). *Let G be a finite group admitting a partition. Then G belongs to one of the following families of groups*:

(2.3.1) *G is a p-group, $|G| \geqslant p^2$ and $H_p(G) := \langle x \in G \mid x^p \neq 1 \rangle$, the H_p-subgroup of G, is a proper subgroup of G.*

(2.3.2) *G is a* Frobenius group, *i.e., G has a proper subgroup H satisfying $H^g \cap H = 1$ for any g in $G - H$.*

(2.3.3) *G is a* Hughes–Thompson group *(short: HT-group), i.e., G is neither a p-group nor a Frobenius group and there exists a prime factor p of $|G|$ such that the H_p-subgroup of G has index p in G.*

(2.3.4) *G is isomorphic to Σ_4, the* symmetric group of degree 4.

(2.3.5) *G is isomorphic to $PGL(2, p^f)$, the* projective linear group of degree 2 over *$GF(p^f)$ for some prime power $p^f \geqslant 4$.*

(2.3.6) *G is isomorphic to $PSL(2, p^f)$, the* special linear projective group of degree 2 over *$GF(p^f)$ for some prime power $p^f \geqslant 4$.*

(2.3.7) *G is isomorphic to $Sz(q)$, the* Suzuki group with parameter q, where $q = 2^{2m+1} \geqslant 8$.

This theorem is applied in [5, 23, 25] to classify all families of groups admitting an (s, k, λ)-partition:

(2.4) THEOREM (R. H. Schulz, M. Biliotti and G. Micelli). *The groups of the families* (2.3.1), (2.3.2), *and* (2.3.3) *are exactly the finite groups admitting an (s, k, λ)-partition for suitable parameters s, k, and λ.*

The proof of Theorem (2.4) given in [5, 23, 25] essentially proceeds as follows: The known partitions of the groups in (2.3.4)–(2.3.7) are investigated, and, as the cardinalities of the components of these partitions do not fit with the parameters of an (s, k, λ)-partition, these groups can be excluded.

Hence, besides the classification theorem (2.3), further properties of the groups in (2.3.4)–(2.3.7) are used. (In [18] a list of all partitions of finite groups with nontrivial Fitting subgroup can be found.) Applying Proposition (1.6), we are able to exclude these groups only using information about their nontrivial normal subgroups. For example, the groups in (2.3.6) and (2.3.7) do not even have to be considered as automorphism groups of translation transversal designs, as it is well known that these families consist of simple groups.

It seems that the families of groups admitting an (s, k, λ)-partition can be classified without using the deep classification theorem of R. Baer, O. H. Kegel, and M. Suzuki. Indeed, in the following we will show that in order to prove Theorem (2.4) it is sufficient to apply some results from R. Baer [1, p. 333–359].

(2.5) DEFINITION. A partition σ in G is called *normal*, if

$$H^g \in \sigma \qquad \text{for all} \quad H \text{ in } \sigma \text{ and all } g \text{ in } G. \tag{2.5.1}$$

Consequently, in normal partitions, we have

$$H = H^g \text{ or } H \cap H^g = 1 \qquad \text{for all} \quad H \text{ in } \sigma \text{ and all } g \text{ in } G. \tag{2.5.2}$$

As most of the results in [1] are formulated for normal partitions, yet we are not able to apply them to the partition $\mathbb{M}$ induced by an (s, k, λ)-partition $\mathbb{H}$ (cf. (2.2.1)), since in general $\mathbb{M}$ is not normal. The following method of O. H. Kegel shows how we can use $\mathbb{M}$ to construct a normal partition in G. In particular, any group G with partition likewise admits a normal partition (cf. also [1, the remark subsequent to the proof of Satz 4.7]).

(2.6) LEMMA. *Let G be a group with (s, k, λ)-partition $\mathbb{H}$ and particular component N. Then G also admits a normal partition containing N as component.*

Proof. Analogous to (2.2.1) let $\mathbb{M}$ be the partition of G which is induced by $\mathbb{H}$. For any automorphism α of G, the set $\mathbb{M}^\alpha := \{U^\alpha \mid U \in \mathbb{M}\}$ likewise is a partition of G. In particular, for any g in G, $\mathbb{M}^g = \{U^g \mid U \in \mathbb{M}\}$ is a partition in G. Now let

$$\mathbf{M} := \bigcup_{g \in G} \mathbb{M}^g \tag{2.6.1}$$

and for a in $G-\{1\}$ let

$$U_a := \bigcap_{\substack{Y\in \mathbf{M}\\ a\in Y}} Y. \tag{2.6.2}$$

We consider

$$\sigma(\mathbb{H}) := \{U_a \mid a\in G-\{1\}\} \tag{2.6.3}$$

and show that $\sigma(\mathbb{H})$ is a normal partition of G which contains the particular component N, as N is a normal subgroup of G: Now let U_a be any subgroup in $\sigma(\mathbb{H})$. We assume that $1\neq x$ is an element of U_a. By definition of U_a we obtain

$$U_a = \bigcap_{\substack{Y\in \mathbf{M}\\ a\in Y}} Y \geqslant \bigcap_{\substack{Y\in \mathbf{M}\\ x\in Y}} = U_x \tag{2.6.4}$$

(observe that x lies in any subgroup in $\mathbf{M}$ containing a). Furthermore, as $\mathbb{M}^g$ for any g in G is a partition of G, we see that the components containing x are exactly the components containing a. Hence we obtain $U_a = U_x$. In particular, for any a, b in $G-\{1\}$, we have $U_a = U_b$ or $U_a \cap U_b = 1$. Therefore $\sigma(\mathbb{H})$ is a partition in G. The normality of $\sigma(\mathbb{H})$ now follows immediately from the fact that

$$U_a^g = \bigcap_{\substack{Y\in \mathbf{M}\\ a\in Y}} Y^g = \bigcap_{\substack{Y\in \mathbf{M}\\ a^g\in Y}} Y = U_{a^g} \tag{2.6.5}$$

holds for all U_a in $\sigma(\mathbb{H})$ and all g in G. ∎

Before applying the results of [1] to $\sigma(\mathbb{H})$, we have to mention that trivially

(2.7) the particular component N is a so-called $\sigma(\mathbb{H})$-*admissible* normal subgroup of G.

(If σ is a normal partition of G, then a nontrivial subgroup K of G is called σ-*admissible*, if for any component U of σ we have $U \leqslant K$ whenever $U\cap K > 1$).

Now two cases remain to be considered:

(1) $\sigma(\mathbb{H})$ contains a component X which is equal to its own normalizer in G, i.e., $N_G(X) := \{g\in G \mid X^g = X\} = X$.

(2) The normalizer of any component X of $\sigma(\mathbb{H})$ contains X properly.

(2.8) *Case* 1. Assume that there exists a component X in $\sigma(\mathbb{H})$ satisfying $N_G(X)=X$. Of course X is different from the particular component N of $\mathbb{H}$. By the normality of $\sigma(\mathbb{H})$ we have that $X \cap X^g = 1$ for all g in $G-X$. Hence G is a Frobenius group and belongs to family (2.3.2). By the famous theorem of Frobenius (cf. [13, Chap. V, 7.6]), the set

$$K := G - \left(\bigcup_{g \in G} X^g - \{1\} \right)$$

is a normal subgroup of G. K is called the *Frobenius kernel* of G and X is called a *Frobenius complement* of K in G. Furthermore, having $N \cap X^g = 1$ for all g in G, the particular component N is a subgroup of the Frobenius kernel K.

(2.9) *Case* 2. Assume that the normalizer of every component X of $\sigma(\mathbb{H})$ contains X properly. An application of [1, Satz 5.1] implies the following facts:

— All elements x in $G-N$ have the same prime order, say p. Consequently, the H_p-subgroup of G is a proper subgroup of G and a subgroup of the particular component N.

— G is either a p-group with $|G| \geqslant p^2$ or a HT-group and therefore belongs to one of the families in (2.3.1) or (2.3.3). In the latter case one furthermore obtains $[G : N] = p$ and $N = H_p(G)$, i.e., the particular component N is equal to the H_p-subgroup of G and therefore, by definition, is generated by all elements in G having order different from p.

Besides the basic results developed in [1], the proof of Satz 5.1 [1] essentially proceeds using the Theorem of Frobenius and the Theorem of Hughes/Thompson [12] concerning the structure of H_p-groups. In any case, by the Theorem of Hughes/Thompson and Kegel [12, 19], we obtain that the particular component N of an (s, k, λ)-partition always is nilpotent. (As mentioned in [1, 19], the proof of Satz 5.1 [1] can be simplified using this result.)

In order to complete the proof of Theorem (2.4) we finally have to show that the groups in (2.3.1)–(2.3.3) really do admit (s, k, λ)-partitions for certain $\lambda \geqslant 1$:

Considering (2.9) it is not difficult to see that $\sigma(\mathbb{H})$ is equal to $\mathbb{H}$ provided that G is a Hughes–Thompson group. In this case all components of $\sigma(\mathbb{H}) - \{N\}$ are cyclic of order p whence $\sigma(\mathbb{H})$ is an $(|N|, p, 1)$-partition in G. The results in [1] furthermore imply that $\{H_p(G)\} \cup \{\langle x \rangle \mid x \in G - H_p(G)\}$ is the only partition in a HT-group G. Hence the parameter λ necessarily is equal to 1. Examples of HT-groups and further results on TTDs admitting such a translation group can be found in [22].

Let G be a p-group admitting a partition. By (2.2.1) we obtain that $|G| \geqslant p^2$ and that $H_p(G)$ is a proper subgroup of G. If we choose a maximal subgroup N of G containing $H_p(G)$, then, by definition of $H_p(G)$, we see that $\mathbb{H} := \{N\} \cup \{\langle x \rangle \mid x \in G - N\}$ is an $(|N|, p, 1)$-partition in G.

In the following section we will continue studying the structure of p-groups with (s, k, λ)-partition and in Section 4 we will give examples in nonabelian p-groups of odd order, where the particular component does not have index p in G.

Finally, let G be a Frobenius group with Frobenius kernel K and a Frobenius complement X. Then $\sigma := \{K\} \cup \{X^g \mid g \in G\}$ is a $(|K|, |X|, 1)$-partition in G. This partition is called the *Frobenius partition*[3] of G. (A Frobenius group has a unique Frobenius partition, cf. [13, Chap. V, 8.17].)

This finally concludes the proof of Theorem (2.4).

It is an interesting question to deal with, whether a Frobenius group admits (s, k, λ)-partitions different from the Frobenius partition. This problem is considered by R. H. Schulz, A. Herzer, M. Biliotti, and G. Micelli in [5, 9, 21, 24].

(2.10) THEOREM (R. H. Schulz). *Let $\mathbb{H}$ be an (s, k, λ)-partition in a Frobenius group G with particular component N. If N is a proper subgroup of the Frobenius kernel K, then K is a p-group.*

We have already seen in (2.8) that the particular component N of an (s, k, λ)-partition $\mathbb{H}$ in a Frobenius group is a subgroup of the Frobenius kernel K. Evidently, if $N = K$, then $\mathbb{H}$ is the Frobenius partition, and necessarily the parameter λ is equal to 1. In order to prove Theorem (2.10), R. H. Schulz applies a basic, but very important argument from R. Baer [1] regarding the commutator of two elements lying in different components of a partition in a group G. (A similar argument is used to prove Lemma (3.1) in the following section.)

In [5, 21] some series of $(s, k, 1)$-partitions in Frobenius groups with K properly containing N are constructed. Examples, where $\lambda > 1$ can be found in [9, 24]. In some of the series the Frobenius kernel is elementary abelian, but there are also other examples (cf. [5, 9]). The methods of all these constructions are based on an idea of A. Herzer [8].

The possible parameter pairs (s, k) of a Frobenius partition are determined in D. Jungnickel [15] and the possible parameter pairs (s, k) for TTDs with $\lambda = 1$ can be found in D. Jungnickel [16, 6.3.4].

[3] In [1] σ is called *minimal Frobenius partition*. Here we use the same terminology as in [13].

3. On the Structure of p-Groups Admitting an (s, k, λ)-Partition

In this section we continue investigating the structure of finite p-groups G admitting an (s, k, λ)-partition $\mathbb{H}$. Let N be the particular component of $\mathbb{H}$. Applying R. Baer's Satz 5.1 from [1] in (2.9), we have seen that

(3.1) $H_p(G) = \langle x \in G \mid x^p \neq 1 \rangle$ is a proper subgroup of N.

The proof of this fact contains an important basic idea and is therefore presented in the following:

As N is a normal subgroup of G (see Proposition (1.6)), there exists an element $x \neq 1$ in $N \cap \Omega_1(Z(G))$, where $\Omega_1(Z(G)) = \langle x \in Z(G) \mid x^p = 1 \rangle$ is the largest elementary abelian subgroup of $Z(G)$. Analogous to (2.2.1) let $\mathbb{M}$ denote the partition in G induced by $\mathbb{H}$. Let h be any element in $G - N$. Furthermore, let H and K be the components in $\mathbb{M}$ containing h and hx, respectively. Due to the choice of x and h, we obtain that N, H and K are three different components of $\mathbb{M}$. Furthermore, as $x \in \Omega_1(Z(G))$, we see that $(xh)^p = x^p h^p = h^p$. Therefore, the element h^p lies in $K \cap H = 1$ whence we obtain $h^p = 1$. As h was arbitrarily chosen in $G - N$, this holds for any h in $G - N$ and shows the validity of (3.1). ∎

Taking advantage of the idea presented in the proof of (3.1), it is not difficult to see that:

(3.2) If $H_p(G) \neq 1$, then $Z(G) \leqslant N$.

Therefore, we have:

(3.3) If there exists a component H in $\mathbb{H} - \{N\}$ with $H \cap Z(G) \neq 1$, then G is of exponent p.

Statement (3.1) in particular implies that all components in $\mathbb{H} - \{N\}$ have exponent p. In the special case where $p = 2$ this shows that each component in $\mathbb{H} - \{N\}$ is elementary abelian (since each 2-group of exponent 2 is elementary abelian). Using once more the normality of N in G and some elementary facts on p-groups, we are able to show that this holds in any case:

(3.4) Theorem. *Let $\mathbb{H}$ be an (s, k, λ)-partition in a p-group G with particular component N. Then the Frattini subgroup $\Phi(G)$ is a subgroup of N and each component in $\mathbb{H} - \{N\}$ is elementary abelian.*

Proof. Assume that there exists a component H in $\mathbb{H} - \{N\}$ which is not elementary abelian. Then, with Corollary (1.7) we obtain that no component in $\mathbb{H} - \{N\}$ is elementary abelian. Therefore, the Frattini subgroup $\Phi(H)$ of H is nontrivial for any H in $\mathbb{H} - \{N\}$. Let $|\Phi(H)| = p^m$ with $m \geqslant 1$ (observe that m is constant for any H in $\mathbb{H} - \{N\}$ by (1.7)).

As $\Phi(U) \leqslant \Phi(G)$ for any subgroup U of G, using (1.4.3) and (1.4.4), we obtain the following lower bound for the cardinality of $\Phi(G)$:

$$(|\Phi(G)| - 1)\,\lambda \geqslant r(p^m - 1), \qquad \text{where} \quad r = |N|\,\lambda. \tag{3.4.1}$$

Now let $|N| = p^n$ and $|H| = p^h$ for $H \in \mathbb{H} - \{N\}$. Then (1.4.3) and (1.4.5) imply $|G| = p^{n+h}$ and with (3.4.1) we have

$$|\Phi(G)| \geqslant p^n(p^m - 1) + 1 = p^{n+m} - p^n + 1. \tag{3.4.2}$$

As $m \geqslant 1$ by assumption and as $|\Phi(G)|$ is a power of p, we even obtain

$$|\Phi(G)| \geqslant p^{n+m}. \tag{3.4.3}$$

Next, as N is a normal subgroup of G, we may apply [13, Chap. III, 3.14.c] to obtain $\Phi(G/N) = \Phi(G)N/N$. Moreover, as G/N is isomorphic to H and $\Phi(G)N/N$ is isomorphic to $\Phi(G)/(\Phi(G) \cap N)$, we obtain

$$\begin{aligned} |\Phi(G)| &= |\Phi(G) \cap N|\,[\Phi(G) : \Phi(G) \cap N] \\ &= |\Phi(G) \cap N|\,|\Phi(H)| = |\Phi(G) \cap N|\,p^m. \end{aligned} \tag{3.4.4}$$

Using (3.4.3), we now see that $|\Phi(G) \cap N| \geqslant p^n = |N|$. Hence the component N is a subgroup of $\Phi(G)$ and due to $NH = G$ for all H in $\mathbb{H} - \{N\}$, in particular we have $H\Phi(G) = G$ for any H in $\mathbb{H} - \{N\}$. But then [13, Chap. III, 3.2] implies that $G = H$, a contradiction.

We therefore conclude that $m = 0$, i.e., $\Phi(H) = 1$ for any H in $\mathbb{H} - \{N\}$. Consequently all components of $\mathbb{H} - \{N\}$ are elementary abelian. Furthermore, as H is isomorphic to G/N and as $\Phi(G)$ is the smallest subgroup of G such that $G/\Phi(G)$ is elementary abelian, we finally obtain that $\Phi(G)$ is a subgroup of N. ∎

Studying p-groups G with $1 \neq H_p(G) \neq G$ one is confronted with a very difficult problem in group theory, namely *Hughes' H_p-problem.* D. R. Hughes conjectured in [11] that $[G : H_p(G)] = p$ always holds provided that $1 \neq H_p(G) \neq G$. But an example of G. E. Wall [27] disproves the conjecture (the existence of a nonabelian 5-group G with $H_5(G) \neq 1$ and $[G : H_5(G)] = 25$ is shown in [27]). We do not go into further detail here and refer to T. Meixner [20], where *generalized Hughes subgroups* of p-groups are studied and further results and references on the H_p-problem are stated.

However, Hughes' conjecture is true if $p = 2$ (see [10]). We will use this result to prove Theorem (3.5), where we are going to show that the finite 2-groups admitting a partition can completely be characterized:

(3.5) THEOREM. *Let σ be a partition in a finite 2-group G. Then one of the following cases holds*:

(3.5.1) *G is elementary abelian.*

(3.5.2) *σ is an $(s, 2, 1)$-partition. In this case the particular component N is a maximal subgroup of G and equal to $H_2(G)$. The translation transversal design corresponding to σ is a complete bipartite graph. Furthermore, if $H = \{1, h\}$ is a complement of N in G, then $n^h = n^{-1}$ for any n in N. In particular, N is abelian.*

The Proof is Simple. Assume that G is not elementary abelian. Then the exponent of G is greater than 2 whence $H_2(G)$ is different from the trivial subgroup 1. Let N be a component of σ containing a nontrivial element x of $\Omega_1(Z(G))$. The same argument used in the proof of (3.1) shows that $H_2(G)$ is a subgroup of N. Hence $1 \neq H_2(G) \neq G$. Now [10, Lemma 4] (which is also simple to prove) says that $[G : H_2(G)] = 2$. Therefore $N = H_2(G)$ is a maximal subgroup of G and each component H in $\sigma - \{N\}$ is cyclic of order 2. In particular, σ is an $(|N|, 2, 1)$-partition in G. This proves the first part of (3.5.2).

Let x and n be any elements of $G - N$ and N, respectively. Then nx lies in $G - N$ and consequently has order 2. We obtain $1 = (nx)^2 = nxnx = nn^x$ and therefore $n^x = n^{-1}$. Hence the inversion of elements in N is an automorphism of N. This finally shows that N is abelian. ▮

We continue with some remarks:

(3.6) *Remarks.* (3.6.1) If $\mathbb{H}$ is an (s, k, λ)-partition in a 2-group G which is not elementary abelian, then $\lambda = 1$ and $k = 2$. This follows immediately by applying (3.5.2) to the partition $\mathbb{M}$ in G induced by $\mathbb{H}$.

(3.6.2) Let K be an abelian group and let G be the generalized dihedral group belonging to K, i.e., G is the semidirect product of $\langle \alpha \rangle$ with K, where α is the inversion of elements of K. If K is finite, then G admits a $(|K|, 2, 1)$-partition with particular component N which is isomorphic to K.

As $k^\alpha = k^{-1}$ for any k in K, the order of any nontrivial fixed point x of α has order 2. Hence, if $|K| = |N|$ is odd, then G is a Frobenius group with Frobenius kernel N. If the order of K is even, then, provided that K is not a 2-group, $N = H_2(G)$ and G is a HT-group.

(3.6.3) The case $p = 2$ shows that the index $[G : N]$ of the particular component N in an (s, k, λ)-partition in G is equal to 2 provided that G is not elementary abelian. In the next section, we will give an example to illustrate that this does not hold in p-groups of odd order: Any finite elementary abelian group of odd order can be represented as a complement of a particular component in a suitable $(s, k, 1)$-partition in a certain p-group which is not elementary abelian.

An application of Theorem (3.4) and Theorem (3.5) to the situation exposed in Theorem (2.10) allows the following extension of that result:

(3.7) THEOREM. *Let $\mathbb{H}$ be an (s, k, λ)-partiton in a Frobenius group G and assume that the particular component N is a proper subgroup of the Frobenius kernel K of G.*

Then the factor group K/N is elementary abelian. Moreover, if K is a 2-group, then K is elementary abelian.

Proof. The first assertion follows immediately from (2.10) in combination with (3.2). Therefore, let K be a 2-group. It is easy to see that the set $\{N\} \cup \{H \cap K \mid H \in \mathbb{H} - \{N\}\}$ forms an (s, l, λ)-partition in K, where $l = |H \cap K|$ and H is any component of $\mathbb{H} - \{N\}$. Assume that K is not elementary abelian. Then by Theorem (3.5) we obtain that $l = 2$. As G is a Frobenius group, each component H in $\mathbb{H} - \{N\}$ likewise belongs to this family; its Frobenius kernel is $H \cap K$. Due to properties of Frobenius groups each complement of $H \cap K$ in H by conjugation acts fixed-point-freely on $H \cap K$. Because of $|H \cap K| = 2$ the automorphism group of $H \cap K$ is trivial. Hence, we obtain $H = H \cap K$ and therefore $G = K$. But this is a contradiction. ∎

We are now going to state a basic, but useful induction argument:

(3.8) LEMMA AND DEFINITION. *Let G be a finite group and let $\mathbb{H} = \{N\} \cup \{H_1, H_2, ..., H_r\}$ be an (s, k, λ)-partition in G with particular component N. If X is a proper subgroup of N and normal in G, then, as a multiset, $\mathbb{H}_X := \{N/X\} \cup \{UX/X \mid U \in \mathbb{H} - \{N\}\}$ satisfies the properties (1.4.1)–(1.4.5) with parameter triple $(s|X|^{-1}, k, \lambda|X|)$.*

As UX/X and VX/X might be equal for even different components U and V of $\mathbb{H} - \{N\}$, $\mathbb{H}_X$ in general is not an $(s|X|^{-1}, k, \lambda|X|)$-partition in G/X.[4]

Therefore $\mathbb{H}_X$ is called a generalized $(s|X|^{-1}, k, \lambda|X|)$-partition. *Again, N/X is called the particular component of $\mathbb{H}_X$.*

Proof. It is simple to see that the properties (1.4.1), (1.4.2), (1.4.3), and (1.4.5) are satisfied. In order to verify (1.4.4), we have to show that each element aX of $G/X - N/X$ is contained in exactly $\lambda|X|$ members of $\{UX/X \mid U \in \mathbb{H}\}$. Now $aX \in UX/X$, if and only if the intersection $U \cap aX$ is not empty. Furthermore, for all x in X the element ax lies in exactly λ components of $\mathbb{H}$. Now, if $x \neq y$, the elements ax and ay do not lie in a common component of $\mathbb{H}$. (Otherwise we would obtain $1 \neq (ax)^{-1}(ay) = x^{-1}y \in X \cap H$ for

[4] The author thanks Professor Dr. Ralph-Hardo Schulz for this hint.

one component H in $\mathbb{H}-\{N\}$. As X is a subgroup of N, that would lead to a contradiction to (1.4.3).) This yields that the number of components U in $\mathbb{H}$ satisfying $U \cap aX \neq \{\ \}$ is exactly $\lambda|X|$ and therefore finally proves (3.8). ∎

The classification of groups admitting an (s, k, λ)-partition shows that the particular component N is always nilpotent (compare~~d~~ with the remarks subsequently given to (2.9)). Therefore, if N is not elementary abelian, N always contains a nontrivial characteristic subgroup X satisfying the assumptions of (3.8).

It is remarkable that the results (1.6), (1.7), (3.1)–(3.4) remain valid for generalized (s, k, λ)-partitions (this can be seen immediately by considering once more the proofs; the essential fact is that the underlying group G is covered by partitions). A generalized (s, k, λ)-partition corresponds to a translation transversal design which is not necessarily simple.

We conclude this section with some further remarks on the structure of p-groups admitting an (s, k, λ)-partition. As the case $p=2$ is completely settled, from now on we assume that p is odd.

(3.9) *Remarks.* Let G be a p-group of odd order and let $\mathbb{H}$ be an (s, k, λ)-partition in G with particular component N.

(3.9.1) The center of G is elementary abelian.

(3.9.2) The commutator group G' is equal to the Frattini group $\Phi(G)$.

Proof. (3.9.1) follows immediately from R. Baer [1, Lemma 2.1] and is moreover valid for all p-groups admitting a partition (the proof proceeds using the idea already presented in the proof of (3.1)).

If G is abelian, the statement (3.9.2) follows from (3.9.1). Assume therefore that G is nonabelian. In p-groups the commutator group is always a subgroup of the Frattini group. Hence, by Theorem (3.4), we have that G' is a subgroup of the particular component N. Applying (3.8) to $X=G'$ we obtain that $\{UX/X \mid U \in \mathbb{H}\}$ is a generalized $(s|X|^{-1}, k, \lambda|X|)$-partition in G/G'. As G/G' is abelian, from (3.9.1) we furthermore obtain that this factor group is even elementary abelian (observe that (3.9.1) remains valid for generalized (s, k, λ)-partitions). Hence $\Phi(G)$ is a subgroup of G' and we obtain the aimed result. ∎

Next, a sufficient condition for G having exponent p is proved.

(3.9.3) If G has class at most 2, then the exponent of G is equal to p.

Proof. If G is abelian, the assertion follows from (3.9.1). We may therefore assume that G has class 2. Then G' is a subgroup of the center $Z(G)$ of G and by (3.9.1) is elementary abelian. By (3.1) it is enough to show that all nontrivial elements of the particular component N have order p. Let therefore n and x be any elements of N and $G-N$, respectively. We apply the commutator formulas from [13, Chap. III, Sect. 1]) to this special case and obtain

$$n^p = (nxx^{-1})^p = (nx)^p x^{-p}[x^{-1}, nx]^{\binom{p}{2}}.$$

As nx is not an element of N and as p is a divisor of $\binom{p}{2}$, we obtain $n^p = 1$, the desired result. ∎

We continue with a sufficient condition for $Z(G)$ being a subgroup of the particular component N:

(3.9.4) If N is abelian, then G is elementary abelian or $Z(G) \leqslant N$.

Proof. Let z be an element of $Z(G)$ and let H be any component of $\mathbb{H} - \{N\}$. By (1.4.3) and (1.4.5) there exist uniquely determined elements n of N and h of H such that $z = nh$. As H is abelian, we see that $n = zh^{-1}$ lies in the centralizer of H in G. Due to the assumption that N is abelian, we have that n centralizes $NH = G$ and therefore n lies in the center of G. The same argument shows that likewise h lies in $Z(G)$. Altogether, we may conclude that $Z(G) = (N \cap Z(G))(H \cap Z(G))$. In particular $|H \cap Z(G)|$ is constant for all H in $\mathbb{H} - \{N\}$.

Now let $|Z(G)| = p^a$ and $|N \cap Z(G)| = p^b$ and therefore $|H \cap Z(G)| = p^{a-b}$ for all H in $\mathbb{H} - \{N\}$. We assume that $Z(G)$ is not a subgroup of N. Using (1.4.3) and (1.4.4) we obtain

$$\begin{aligned} \lambda p^b(p^{a-b} - 1) &= \lambda |Z(G) - (N \cap Z(G))| \\ &= r(p^{a-b} - 1) = \lambda |N| \, (p^{a-b} - 1) \end{aligned}$$

and as $a > b$ by assumption, we see that $p^b = |N|$. But then N is a proper subgroup of $Z(G)$. Therefore N centralizes the component H whence H, because of $NH = G$, is a normal subgroup of G. This implies that G is isomorphic to $N \times H$. Hence G is abelian and therefore by (3.9.1) elementary abelian. ∎

Induction and a combination of (3.8) with (3.9.4) show:

(3.9.5) If N is abelian, then every proper subgroup of the lower central series of G is a subgroup of N.

Proof. Trivially the assertion is true, if G is abelian. If G is nonabelian, then by (3.9.4) $Z(G) \leqslant N$. Now we use (3.8) with $X := Z(G)$ and consider the generalized $(s|Z(G)|^{-1}, k, \lambda|Z(G)|)$-partition $\{UZ(G)/Z(G) \mid U \in \mathbb{H}\}$ in $G/Z(G)$. As (3.9.4) is also applicable to generalized (s, k, λ)-partitions and as G is nilpotent, the procedure is finished after a finite number of steps and the statement follows by induction. ∎

We finally mention that in [25] a construction of (s, k, λ)-partitions in elementary abelian groups is given. A very large spectrum of parameters is covered with this construction.

4. Translation Transversal Designs with Flag Regular Automorphism Groups

In this section we assume that the parameter λ is equal to 1. Let $\mathbb{H} := \{N\} \cup \{H_x \mid x \in N\}$ be an $(s, k, 1)$-partition in a group G and analogous to (1.5.1) let $\mathbb{E}(\mathbb{H}) = (G, \{Hg \mid H \in \mathbb{H} - \{N\},\ g \in G\}, \in)$ be the translation transversal design corresponding to $\mathbb{H}$.

As all components in $\mathbb{H} - \{N\}$ by Corollary (1.7) are pairwise isomorphic, it is suggestive to deal with the following question:

(4.1) *Problem.* Does there exist a subgroup Γ of the automorphism group of G satisfying the following properties?

(4.1.1) $N^\tau = N$ for any τ in Γ.

(4.1.2) Γ acts regularly on the set $\mathbb{H} - \{N\}$.

Assuming that G admits such an automorphism group Γ, it is not difficult to see that Γ induces a collineation group on $\mathbb{E}(\mathbb{H})$. We even have:

(4.2.) PROPOSITION. *The collineation group D of $\mathbb{E}(\mathbb{H})$ which is generated by G and Γ acts as a flag regular automorphism group on $\mathbb{E}(\mathbb{H})$, i.e., for any two incident point-block-pairs (p_1, B_1) and (p_2, B_2) there exists a unique element δ in D satisfying $(p_1, B_1)^\delta := (p_1^\delta, B_1^\delta) = (p_2, B_2)$. Moreover, G is a normal subgroup of D and D is isomorphic to a semidirect product of G with Γ.*

Proof. For any x in G the collineation induced by x on $\mathbb{E}(\mathbb{H})$ is also denoted with x. In defining

$$g^{(x,\tau)} := (gx)^\tau \qquad \text{for all } g \text{ in } G \tag{4.2.1}$$

and

$$(Hg)^{(x,\tau)} := H^\tau(gx)^\tau \text{ for all } H \text{ in } \mathbb{H} - \{N\} \text{ and all } g \text{ in } G,$$

the set $D := G \times \Gamma$ with multiplication

$$(x, \tau)(y, \rho) := (xy^{\tau^{-1}}, \tau\rho) \tag{4.2.2}$$

becomes a collineation group of $\mathbb{E}(\mathbb{H})$. Using (4.2.2) it is not difficult to see that G is a normal subgroup of D.

Now let (x, Hx) and (y, Ky) be any two flags of $\mathbb{E}(\mathbb{H})$. Furthermore, let τ be the unique element in Γ satisfying $H^\tau = K$. A simple calculation shows that the collineation $(x^{-1}y^\tau, \tau)$ maps the flag (x, Hx) onto the flag (y, Ky). Hence D acts transitively on the set of flags of $\mathbb{E}(\mathbb{H})$. Moreover, as by (1.2) the number of flags of $\mathbb{E}(\mathbb{H})$ is equal to

$$rv = s\lambda v = |\mathbb{H} - \{N\}|\ |G| = |G \times \Gamma|,$$

we obtain likewise the regularity of the action of D. ▮

From this aspect, Frobenius groups in connection with translation transversal designs were studied first by D. Jungnickel in [15]:

(4.3) PROPOSITION. *If G is a Frobenius group and $\mathbb{H}$ is the Frobenius partition of G, then the inner automorphism group of G induced by the Frobenius kernel N just leads to an action on $\mathbb{H}$ satisfying* (4.1.1) *and* (4.1.2). *Hence any translation transversal design induced by the Frobenius partition of a Frobenius group admits a flag regular collineation group.*

In the following example we construct a series of nonabelian p-groups of odd order which can be represented as translation groups of translation transversal designs admitting a flag regular collineation group. For that purpose we use a method of [6].

(4.4) EXAMPLE. Let q be an odd prime power and let $GF(q)$ denote the Galois field of order q. We introduce a multiplication on the set $G := \{(a, b, c) \mid a, b, c \in GF(q)\}$ of triples over $GF(q)$:

$$(a_1, b_1, c_1)(a_2, b_2, c_2) := (a_1 + a_2, b_1 + b_2, c_1 + c_2 - a_2 b_1). \quad (4.4.1)$$

With help of the formulas

$$(a, b, c)^{-1} = (-a, -b, -c - ab) \quad (4.4.2)$$

and

$$[(a_1, b_1, c_1), (a_2, b_2, c_2)] = (0, 0, a_1 b_2 - a_2 b_1), \quad (4.4.3)$$

it is simple to show that G is a nonabelian group of order q^3 and class 2.

Now let $N := \{(0, b, c) \mid b, c \in GF(q)\}$. By definition of the multiplication in G, we have that N is a subgroup of G. Furthermore, (4.4.3) shows that N contains the commutator subgroup G' of G. Hence N is even a normal subgroup of G.

(4.4.4) For ρ and μ in $GF(q)$ let

$$f_{\rho,\mu} : G \to G,$$
$$(a, b, c) \to (a, \rho a + b, -2^{-1}\rho a(a-1) + \mu a + c).$$

Using again the multiplication in G, it is not difficult to see that $f_{\rho,\mu}$ is an automorphism of G. Moreover, because of $f_{\rho,\mu}f_{\tau,\nu}=f_{\rho+\tau,\mu+\nu}$, we have that

(4.4.5) $\Gamma:=\{f_{\rho,\mu}\mid \rho,\mu\in GF(q)\}$ is a subgroup of the automorphism group of G.

Furthermore, N is fixed elementwise by Γ. Now we define

$$H:=\{(a,0,0)\mid a\in GF(q)\} \tag{4.4.6}$$

and

$$H_{\rho,\mu}:=f_{\rho,\mu}(H)=\{(a,\rho a,-2^{-1}\rho a(a-1)+\mu a)\mid a\in GF(q)\}. \tag{4.4.7}$$

A simple calculation shows that

$$\mathbb{H}:=\{N\}\cup\{H_{\rho,\mu}\mid \rho,\mu\in GF(q)\} \text{ is a } (q^2,q,1)\text{-partition in } G. \tag{4.4.8}$$

Let $\mathbf{T}:=\mathbb{E}(\mathbb{H})$ be the translation transversal design corresponding to $\mathbb{H}$. By (4.4.7) and Proposition (4.2) the group of collineations of $\mathbf{T}$ generated by G and Γ acts regularly on the set of flags of $\mathbf{T}$.

We conclude our investigations with a further example concerning groups of even order:

(4.5) Example. Let G be a finite group with abelian subgroup N of index 2 in G. Assume that there exists a complement $X:=\{1,x\}$ of N in G. For all n in N let $n^x=n^{-1}$; hence G is a generalized dihedral group. As already remarked in (3.6.2), the set of complements of N in G together with N form an $(|N|,2,1)$-partition $\mathbb{H}$ in G where N is the particular component. For any a in N, we define the following mapping:

$$\tau_a\colon G\to G,\ g\to\begin{cases} g, & \text{if } g\in N\\ xab, & \text{if } g=xb\in G-N.\end{cases} \tag{4.5.1}$$

Once more, a simple calculation shows that τ_a is an automorphism of G. Moreover, by the fact that $\tau_a\tau_b$ is equal to τ_{ab}, the set $\Gamma:=\{\tau_a\mid a\in N\}$ is a subgroup of the automorphism group of G and isomorphic to N. The definition of these automorphisms shows that N is fixed elementwise by Γ and that Γ acts regularly on the set of complements of N in G.

Hence, by (4.2) the incidence structure $\mathbb{E}(\mathbb{H})$ is a complete bipartite graph admitting both, a collineation group acting regularly on its flags and a translation group acting in the manner of (1.3). We finally mention that any partition in a 2-group which is not elementary abelian by Theorem (3.5) leads to such a complete bipartite graph.

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