

# Localization by particle–hole symmetry breaking: a loop expansion

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## Abstract

Localization by a broken particle–hole (PH) symmetry in a random system of non-interacting quantum particles is studied on a  $d$ -dimensional lattice. Our approach is based on a chiral symmetry argument and the corresponding invariant measure, where the latter is described by a Grassmann functional integral. Within a loop expansion we find for small loops diffusion in the case of PH symmetry. Breaking the PH symmetry results in the creation of random dimers, which suppress diffusion and lead to localization on the scale  $\sqrt{D/|\mu|}$ , where  $D$  is the effective diffusion coefficient at PH symmetry and  $\mu$  is the parameter related to PH symmetry breaking.

Keywords: localization, particle–hole symmetry, loop expansion

(Some figures may appear in colour only in the online journal)

## 1. Introduction

There is strong evidence that disordered systems with particle–hole (PH) symmetry can avoid Anderson localization in any spatial dimension. Such a behavior was observed for one-dimensional systems at the band center of a one-dimensional tight-binding model with random hopping some time ago [1–3]. There is also numerical evidence for extended states at the band center in disordered two-dimensional lattice models [4, 5]. A renewed interest in this problem appeared with the discovery of two-dimensional Dirac-like materials, such as graphene [6–8]. Graphene is a semimetal with a very robust conductivity at the PH-symmetric Dirac point. On the other hand, after breaking the PH symmetry by doping, its conductivity changes substantially; it is either enhanced for weak disorder or reduced for strong disorder [9–12]. The effect of disorder on topological materials, based on Dirac-like models, has also been the subject of

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some recent theoretical research [13–15]. Despite of a substantial effort, there was no conclusive confirmation of localization away from the PH-symmetric Dirac point, neither from the theory side [16, 17] nor from the experiment [9–12].

While in the PH-symmetric case diffusion was identified as the dominant behavior of non-interacting Dirac particles in a random environment, the breaking of the PH symmetry was accompanied with the creation of random dimers [18]. A preliminary work, based on a perturbative renormalization group analysis, did not reveal localization, though [19]. In the following we will analyze the competition of diffusing quantum particles and randomly distributed dimers on a  $d$ -dimensional lattice without employing a perturbation theory.

Starting from a PH-symmetric random Hamiltonian  $H$ , a symmetry-breaking term is defined by a uniform shift  $\mu$  of the PH-symmetric Hamiltonian. The resulting Hamiltonian is still invariant under a chiral transformation. This invariance is relevant for the analysis of spontaneous chiral-symmetry breaking and the corresponding long-scale behavior of the model, such that we reduce our description to the corresponding invariant measure (IM). Then the IM, written in terms of a Grassmann functional integral [20], is represented by a loop expansion, which consists of graphs with four-vertices. Only the smallest loops are taken into account, which is known as the nonlinear sigma model approximation. This is used as the starting point for the analysis of the spatial correlations. As mentioned above, there is diffusion in the presence of PH symmetry with a diffusion coefficient  $D$ , which depends on the scattering rate  $\eta$  and the average Hamiltonian  $\langle H \rangle$ . After breaking the PH symmetry with  $\mu \neq 0$ , we create small loops with two sites on the lattice which represent repulsive lattice dimers. They act as obstacles for the diffusion and suppress the latter on large scales, which leads to an exponential decay of the particle correlation. Since the density of the dimers is proportional to  $|\mu|$ , this localization effect increases with this parameter. An estimation of the decay length gives  $\sqrt{D/|\mu|}$ .

## 2. Model: IM

We briefly recapitulate the main ideas which were developed in the previous work on the loop expansion and the IM for non-interacting particles in a random environment [21, 22]. For this purpose a random Hamiltonian matrix  $H$  on a lattice  $\Lambda$  is considered. It is assumed that this Hamiltonian has an internal spinor structure; its matrix elements are of the form  $H_{\mathbf{r},\mathbf{r}'}^{j,j'}$ , where  $\mathbf{r}, \mathbf{r}' \in \Lambda$  and  $j, j' = 1, 2, \dots, N$  is a spinor index, where the latter can also be a band index of a multiband Hamiltonian. We further assume that there is an  $N \times N$  unitary matrix  $U$  that (i) acts only on the spinor or band index and (ii) for which the Hamiltonian matrix obeys the relation

$$UH^T U^\dagger = -H. \quad (1)$$

This relation implies a PH transformation in the following sense: since  $H$  is Hermitian, we have  $H^T = H^*$  and the relation (1) implies for the eigenstate  $\Psi_E$  of  $H$  with the real eigenvalue  $E$

$$H^* U^\dagger \Psi_E = -U^\dagger H \Psi_E = -E U^\dagger \Psi_E.$$

Complex conjugation of this equation yields

$$H(U^\dagger \Psi_E)^* = -E(U^\dagger \Psi_E)^*,$$

such that  $\Psi_{-E} = (U^\dagger \Psi_E)^*$  is eigenstate of  $H$  with eigenvalue  $-E$ . Thus, there is a PH symmetry for  $E = 0$ . The PH symmetry of  $H$  is broken by  $\mu$  for  $H_\pm := H \pm \mu \sigma_0$ , where  $\sigma_0$  is the  $N \times N$

unit matrix. In the following we will analyze the effect of a shift of  $E = 0$  by  $\mu$ , using the nonlinear sigma model approach.

Following references [18, 19], we extend  $H$  to the random Hamiltonian matrix

$$\hat{H} = \begin{pmatrix} H_+ & 0 & 0 & 0 \\ 0 & H_- & 0 & 0 \\ 0 & 0 & H_+^T & 0 \\ 0 & 0 & 0 & H_-^T \end{pmatrix} \equiv \begin{pmatrix} \bar{H} & 0 \\ 0 & \bar{H}^T \end{pmatrix}. \quad (2)$$

At the PH symmetric point ( $\mu = 0$ ) it satisfies the relation  $\hat{H}\hat{S}_0 = -\hat{S}_0\hat{H}$  for

$$\hat{S}_0 = \begin{pmatrix} 0 & 0 & \varphi_3 U & \varphi_1 U \\ 0 & 0 & \varphi_2 U & \varphi_4 U \\ \varphi_3' U^\dagger & \varphi_2' U^\dagger & 0 & 0 \\ \varphi_1' U^\dagger & \varphi_4' U^\dagger & 0 & 0 \end{pmatrix} \quad (3)$$

with some general parameters  $\varphi_j$ . For a broken PH symmetry ( $\mu \neq 0$ ) it satisfies  $\hat{H}\hat{S} = -\hat{S}\hat{H}$  for

$$\hat{S} = \begin{pmatrix} 0 & 0 & 0 & \varphi_1 U \\ 0 & 0 & \varphi_2 U & 0 \\ 0 & \varphi_2' U^\dagger & 0 & 0 \\ \varphi_1' U^\dagger & 0 & 0 & 0 \end{pmatrix}. \quad (4)$$

The PH-symmetric case was studied previously, such that we can focus subsequently on the broken PH symmetry. Then the relation  $\hat{H}\hat{S} = -\hat{S}\hat{H}$  implies for  $\hat{H}$  the chiral symmetry  $e^{\hat{S}}\hat{H}e^{\hat{S}} = \hat{H}$  of the extended Hamiltonian defined in equation (2). The chiral symmetry reveals some interesting properties, which will be discussed next.

Some general remarks on the IM: the goal is to calculate average quantities, such as the average Green's function or the average product of two Green's functions, with respect to the random matrix elements of the Hamiltonian  $\hat{H}$  [21]. This should be seen as an alternative to studies, where the distribution of the random Hamiltonian and its spectrum is considered [23–25]. Average quantities are sufficient to discuss many physically motivated questions, such as transport [18, 22]. Although simpler than the analysis of the random distribution, the averaging with respect to the random Hamiltonian is a tedious task for a large lattice  $\Lambda$ . A common and often successful approximation to this problem is to perform a saddle-point integration (also known as the method of steepest descent) [26]. Then another problem occurs when there is no unique saddle-point solution but a manifold of saddle points due to some symmetry of the Hamiltonian. A typical example is the Hamiltonian (2) with the chiral symmetry. Since all the saddle points of the manifold are equally important, we must integrate over all of them. The resulting saddle-point integral leads to the IM. This will be discussed in detail for the specific example of  $\hat{H}$  subsequently. The saddle-point integral can be performed in this case and leads to a sum that is characterized by loops with increasing size. There is no problem with convergence on a finite lattice, since the loops are strictly repulsive and the size of the loops is restricted by the size of the lattice. The loop expansion was previously developed for the PH-symmetric case in references [21, 22] and will be adopted to the PH-symmetry broken case in the following.

First, we construct the IM that is associated with the chiral symmetry. For the matrix  $\hat{S}$  and the graded determinant (cf equation (A.1)) we get

$$\det g(e^{2\hat{S}}) = \exp[\text{Trg}(\log e^{2\hat{S}})] = \exp[2 \text{Trg}(\hat{S})] = 1,$$

since the graded trace vanishes:  $\text{Trg}(\hat{S}) = 0$ . Together with the chiral symmetry of  $\hat{H}$  this implies immediately

$$\begin{aligned} \det g(\hat{H} + i\eta) &= \det g(\hat{H} + i\eta) \det g(e^{2\hat{S}}) = \det g(e^{\hat{S}} \hat{H} e^{\hat{S}} + i\eta e^{2\hat{S}}) \\ &= \det g(\hat{H} + i\eta e^{2\hat{S}}), \end{aligned} \quad (5)$$

where  $\det g(\hat{H} + i\eta) = 1$ , according to the definition of the graded determinant. Thus,  $\det g(\hat{H} + i\eta e^{2\hat{S}})$  is invariant under the chiral transformation.

From relation (5) we can construct the IM through substituting the general parameters  $\varphi_j$  by a spatial Grassmann field  $\varphi_{\mathbf{r}j}$ . As explained in appendix A, this leads to the lattice version of the IM with

$$J_j = \det(\mathbf{1} + \varphi'_j \varphi_j - \varphi'_j h \varphi_j h^\dagger)^{-1} = \det(\mathbf{1} + \varphi_j \varphi'_j - h \varphi_j h^\dagger \varphi'_j) \quad (j = 1, 2), \quad (6)$$

where the relation is derived in (A.5), with

$$h = \mathbf{1} + 2i\eta \bar{G}_0, \quad \bar{G}_0 = (\bar{H}_0 + i\epsilon - i\eta)^{-1}. \quad (7)$$

$J_j$  is an invariant for a *global* chiral transformation, provided that  $h$  is unitary. This is the case for  $\epsilon = 0$ :  $hh^\dagger = \mathbf{1} + O(\epsilon)$  [22]. It is important to realize that the random Hamiltonian  $\hat{H}$  has been replaced by its average  $\bar{H}_0 := \langle \bar{H} \rangle$  in the IM. In references [21, 22] this IM was associated with the correlator  $K_{\mathbf{R}\mathbf{R}'}$  between the lattice sites  $\mathbf{R}$  and  $\mathbf{R}'$  through the relation

$$K_{j;\mathbf{R}\mathbf{R}'} = \frac{1}{\mathcal{N}} \int_{\Lambda} J_j \varphi_{\mathbf{R}j} \varphi'_{\mathbf{R}'j}, \quad \mathcal{N} = \int_{\Lambda} J_1 = \int_{\Lambda} J_2 \quad (8)$$

of a two-component Grassmann field  $\varphi_{\mathbf{r}j}$  ( $\mathbf{r} \in \Lambda$ ).  $\int_{\Lambda}$  is the functional integral with respect to the Grassmann field on the lattice  $\Lambda$ . This correlator indicates localization on the localization length  $\xi$  when it decays exponentially on the scale  $\xi$ . It is identical for both components of the Grassmann field  $j = 1, 2$ , such that we can drop this index subsequently. Another important feature of the Green's function  $h$  is that the relation (1) implies for the PH-symmetric case  $\mu = 0$  the relation

$$U h^\dagger U^\dagger = h, \quad (9)$$

which does not hold for  $\mu \neq 0$ .

The Grassmann field can be expressed by its Fourier components as

$$\varphi_{\mathbf{r}} = \sum_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{r}} \tilde{\varphi}_{\mathbf{k}} = \tilde{\varphi}_0 + \sum_{\mathbf{k} \neq 0} e^{-i\mathbf{k}\cdot\mathbf{r}} \tilde{\varphi}_{\mathbf{k}} \equiv \tilde{\varphi}_0 + \tau_{\mathbf{r}}, \quad (10)$$

where the zero mode  $\tilde{\varphi}_0$  does not depend on  $\mathbf{r}$ . Then the normalization becomes

$$\mathcal{N} = \int_{\Lambda} \det\{\mathbf{1} + [\tilde{\varphi}_0 + \tau - h(\tilde{\varphi}_0 + \tau)h^\dagger](\tilde{\varphi}'_0 + \tau')\}$$

and with  $hh^\dagger = \mathbf{1} - \epsilon \bar{\Gamma} + O(\epsilon^2)$  we get

$$\tilde{\varphi}_0 - h\tilde{\varphi}_0 h^\dagger = \tilde{\varphi}_0(\mathbf{1} - hh^\dagger) = \epsilon \bar{\Gamma} \tilde{\varphi}_0 + O(\epsilon^2),$$

since the space-independent  $\tilde{\varphi}_0$  commutes with  $h$ .  $\epsilon$  can be absorbed into the Grassmann integration by rescaling  $\tilde{\varphi}_0 \rightarrow \bar{\tau}/\epsilon$ . This implies for the normalization

$$\mathcal{N} = \epsilon \int_{\Lambda} \det[\mathbf{1} + (\bar{\Gamma}\bar{\tau} + \tau - h\tau h^\dagger)(\tilde{\varphi}'_0 + \tau')], \quad (11)$$

where we have neglected terms of order  $\epsilon$  inside the integrand. For the unnormalized correlator this rescaling argument for  $\tilde{\varphi}_0$  provides a constant term  $\tilde{\varphi}_0\tilde{\varphi}'_0$  from the external Grassmann variable  $\varphi_{\mathbf{R}}\varphi'_{\mathbf{R}'}$ :

$$\begin{aligned} \int_{\Lambda} J\varphi_{\mathbf{R}}\varphi'_{\mathbf{R}'} &= \int_{\Lambda} J(\tilde{\varphi}_0 + \tau_{\mathbf{R}})(\tilde{\varphi}'_0 + \tau'_{\mathbf{R}'}) \\ &= \int_{\Lambda} J\bar{\tau}(\tilde{\varphi}'_0 + \tau'_{\mathbf{R}'}) + \epsilon \int_{\Lambda} J\tau_{\mathbf{R}}(\tilde{\varphi}'_0 + \tau'_{\mathbf{R}'}), \end{aligned} \quad (12)$$

where only the first term on the right-hand side vanishes with  $\epsilon \rightarrow 0$ . This implies for the normalized correlator that the second term diverges for  $\epsilon \rightarrow 0$ , which is a consequence of the broken translational invariance due to the factor  $\tau_{\mathbf{R}}$ .

### 3. Loop expansion

The IM of equation (6) can be rewritten as

$$J = \exp\{\text{Tr}([WW^\dagger]_d\varphi\varphi')\} \det(\mathbf{1} - \varphi X^\dagger\varphi' - X\varphi\varphi' - W\varphi W^\dagger\varphi') \quad (13)$$

when we define  $h = [h]_d + W$ ,  $X = W[h]_d^\dagger$ , where  $[h]_d$  and  $[WW^\dagger]_d$  are the spatial diagonal parts of  $h$  and  $WW^\dagger$ , respectively. Here we implicitly assume the limit  $\epsilon \rightarrow 0$ . Using the determinant identity  $\det(A) = \exp[\text{Tr}(\log A)]$  this yields for the IM after expanding the logarithm

$$\log J = \text{Tr}([WW^\dagger]_d\varphi\varphi') - \sum_{l \geq 1} \frac{1}{l} \text{Tr}[(\varphi X^\dagger\varphi' + X\varphi\varphi' + W\varphi W^\dagger\varphi')^l]. \quad (14)$$

This sum terminates on a finite lattice  $\Lambda$  for  $l = |\Lambda|$  due to the Grassmann field. The trace term with the power  $l$  represents a sum of loops of length  $l$  on the lattice, such that the sum can be considered as a loop expansion of the IM [21]. Integration with respect to the Grassmann field yields graphs with four-vertices from the term  $W\varphi W^\dagger\varphi'$  because the non-zero Grassmann integral requires at each site  $\mathbf{r}$  the product  $\varphi_{\mathbf{r}}\varphi'_{\mathbf{r}}$ . The four-vertex graphs reflect the equivalence of the expression (14) with the random phase representation of the IM [22].

#### 3.1. Nonlinear sigma model

As a special case of the loop expansion (14) only the smallest loops are considered, namely only loops that contain at most two hopping matrices, either  $X$  or  $W$ . This approximation is known as the nonlinear sigma model [27] and becomes in the present case

$$\begin{aligned} \log J_{\text{NLSM}} &= \\ &\text{Tr}([WW^\dagger]_d\varphi\varphi') - \text{Tr}(W\varphi W^\dagger\varphi') - \text{Tr}(\varphi X^\dagger\varphi' + X\varphi\varphi') \\ &\quad - \text{Tr}[(\varphi X^\dagger\varphi' + X\varphi\varphi')(\varphi X^\dagger\varphi' + X\varphi\varphi')], \end{aligned}$$

where the third term vanishes due to the trace of an off-diagonal matrix  $X$ , such that the IM reduces to

$$\begin{aligned} \log J_{\text{NLSM}} &= \text{Tr}([WW^\dagger]_d \varphi \varphi') - \text{Tr}(W \varphi W^\dagger \varphi') \\ &\quad + \text{Tr}(X^\dagger \varphi \varphi' X^\dagger \varphi \varphi' - X \varphi \varphi' X \varphi \varphi'). \end{aligned} \quad (15)$$

The first two terms represent a diffusion propagator

$$\begin{aligned} &\text{Tr}_N([WW^\dagger]_{\mathbf{r}\mathbf{r}}) \delta_{\mathbf{r}\mathbf{r}'} - \text{Tr}_N(W_{\mathbf{r}\mathbf{r}'} W_{\mathbf{r}'\mathbf{r}}^\dagger) \\ &= \sum_{\mathbf{r}''} \text{Tr}_N(W_{\mathbf{r}\mathbf{r}''} W_{\mathbf{r}''\mathbf{r}'}^\dagger) \delta_{\mathbf{r}\mathbf{r}'} - \text{Tr}_N(W_{\mathbf{r}\mathbf{r}'} W_{\mathbf{r}'\mathbf{r}}^\dagger), \end{aligned} \quad (16)$$

where  $\text{Tr}_N$  is the trace with respect to the spinor index. The quartic term reads

$$\text{Tr}(X^\dagger \varphi \varphi' X^\dagger \varphi \varphi' - X \varphi \varphi' X \varphi \varphi') = \sum_{\mathbf{r}, \mathbf{r}'} \beta_{\mathbf{r}\mathbf{r}'} \varphi_{\mathbf{r}} \varphi'_{\mathbf{r}'} \varphi_{\mathbf{r}'} \varphi'_{\mathbf{r}}.$$

$\beta$  is an imaginary matrix due to

$$\beta_{\mathbf{r}\mathbf{r}'} := \text{Tr}_N(X_{\mathbf{r}\mathbf{r}'}^\dagger X_{\mathbf{r}'\mathbf{r}}^\dagger - X_{\mathbf{r}\mathbf{r}'} X_{\mathbf{r}'\mathbf{r}}). \quad (17)$$

Relation (9) implies that  $\beta$  vanishes in the presence of the PH symmetry (i.e., for  $\mu = 0$ ). This was also observed in references [18, 19].

Next, we represent the functional integral with a quadratic form in the Grassmann field, using a coupling of the latter to a real Gaussian field  $Q_{\mathbf{r}}$ . This can be achieved by exploiting the relation

$$m_{\mathbf{r}} \bar{\beta}_{\mathbf{r}\mathbf{r}'} m_{\mathbf{r}'} - (Q_{\mathbf{r}} + V_{\mathbf{r}\mathbf{r}''} m_{\mathbf{r}''}) V_{\mathbf{r}\mathbf{r}'}^{-1} (Q_{\mathbf{r}'} + V_{\mathbf{r}'\mathbf{r}''} m_{\mathbf{r}''}) = -Q_{\mathbf{r}} V_{\mathbf{r}\mathbf{r}'}^{-1} Q_{\mathbf{r}'} - 2m_{\mathbf{r}} Q_{\mathbf{r}} \quad (18)$$

with the sum convention for paired indices and with the correlation matrix  $V = i\alpha \mathbf{1} + \bar{\beta}$  and  $\bar{\beta} = \beta/i|\mu|$ . We note that  $m_{\mathbf{r}} V_{\mathbf{r}\mathbf{r}'} m_{\mathbf{r}'} = m_{\mathbf{r}} \bar{\beta}_{\mathbf{r}\mathbf{r}'} m_{\mathbf{r}'}$  for Grassmann variables  $m_{\mathbf{r}}$ . Then we introduce the Gaussian integral

$$\langle \dots \rangle_Q := \frac{(i|\mu|/\pi)^{|\Lambda|/2}}{\sqrt{\det V}} \int_Q \exp\left(-i|\mu| \sum_{\mathbf{r}, \mathbf{r}'} Q_{\mathbf{r}} V_{\mathbf{r}\mathbf{r}'}^{-1} Q_{\mathbf{r}'}\right) \dots \prod_{\mathbf{r} \in \Lambda} dQ_{\mathbf{r}}, \quad (19)$$

where we have set the free positive parameter  $\alpha$  such that  $V$  is non-singular and its eigenvalues have positive real parts. The Grassmann integration can be performed because the argument of the exponential function is a quadratic form of the Grassmann field: "

$$\begin{aligned} K_{\mathbf{R}\mathbf{R}'} &= \mathcal{N} \left\langle \int_{\Lambda} \exp \left\{ \tilde{\gamma}_0 \sum_{\mathbf{r}} \varphi_{\mathbf{r}} \varphi'_{\mathbf{r}} - \sum_{\mathbf{r}, \mathbf{r}'} \gamma_{\mathbf{r}\mathbf{r}'} \varphi_{\mathbf{r}} \varphi'_{\mathbf{r}'} \right. \right. \\ &\quad \left. \left. - 2i|\mu| \sum_{\mathbf{r} \in \Lambda} Q_{\mathbf{r}} \varphi_{\mathbf{r}} \varphi'_{\mathbf{r}} \right\} \varphi_{\mathbf{R}} \varphi'_{\mathbf{R}'} \right\rangle_Q = \mathcal{N} \langle \text{adj}_{\mathbf{R}\mathbf{R}'}(\tilde{\gamma}_0 - \gamma - 2i|\mu|Q) \rangle_Q \end{aligned} \quad (20)$$

with  $\gamma_{\mathbf{r}\mathbf{r}'} = \text{Tr}_N(W_{\mathbf{r}\mathbf{r}'} W_{\mathbf{r}'\mathbf{r}}^\dagger)$  and  $\tilde{\gamma}_0 = \sum_{\mathbf{r}'} \gamma_{\mathbf{r}\mathbf{r}'}$ . The adjugate matrix can be expressed by the determinant as

$$\text{adj}_{\mathbf{R}\mathbf{R}'}(\tilde{\gamma}_0 - \gamma - 2i|\mu|Q) = \det(\tilde{\gamma}_0 - \gamma - 2i|\mu|Q) (\tilde{\gamma}_0 - \gamma - 2i|\mu|Q)_{\mathbf{R}\mathbf{R}'}^{-1}, \quad (21)$$

provided that  $\tilde{\gamma}_0 - \gamma + 2i|\mu|Q$  is not singular. The latter can always be arranged by a deformation of the  $Q_r$  path of integration in the complex plane. The deformation moves the poles of the inverse matrix away from the real axis, which results in an exponential decay with respect to  $|\mathbf{R} - \mathbf{R}'|$ .

### 3.2. Estimation of the localization length

To analyze the spatial behavior of the inverse matrix we use the plane wave eigenvector  $\Phi_{\mathbf{k}} = (\exp[i\mathbf{k} \cdot \mathbf{r}])$  of the translational invariant matrix  $\gamma$  with eigenvalue  $\tilde{\gamma}_{\mathbf{k}}$ :

$$(\tilde{\gamma}_0 - \gamma - 2i|\mu|Q)\Phi_{\mathbf{k}} = (\tilde{\gamma}_0 - \tilde{\gamma}_{\mathbf{k}} - 2i|\mu|Q)\Phi_{\mathbf{k}} \quad (22)$$

with a diagonal matrix on the right-hand side. Since we have

$$\begin{aligned} \Phi_{\mathbf{k}} &= (\tilde{\gamma}_0 - \gamma - 2i|\mu|Q)^{-1}(\tilde{\gamma}_0 - \gamma - 2i|\mu|Q)\Phi_{\mathbf{k}} \\ &= (\tilde{\gamma}_0 - \gamma - 2i|\mu|Q)^{-1}(\tilde{\gamma}_0 - \tilde{\gamma}_{\mathbf{k}} - 2i|\mu|Q)\Phi_{\mathbf{k}}, \end{aligned} \quad (23)$$

we get for  $\Phi'_{\mathbf{k}} := (\tilde{\gamma}_0 - \tilde{\gamma}_{\mathbf{k}} - 2i|\mu|Q)\Phi_{\mathbf{k}}$  the equation

$$(\tilde{\gamma}_0 - \gamma - 2i|\mu|Q)^{-1}\Phi'_{\mathbf{k}} = \Phi_{\mathbf{k}} = (\tilde{\gamma}_0 - \tilde{\gamma}_{\mathbf{k}} - 2i|\mu|Q)^{-1}\Phi'_{\mathbf{k}}. \quad (24)$$

Then we consider the basis  $\{\Phi'_{\mathbf{k}}\}$  and define the vector  $\phi'_{\mathbf{R}} := \int_{\mathbf{k}} \phi_{\mathbf{k},\mathbf{R}}\Phi'_{\mathbf{k}}$ . This gives with equation (24)

$$(\tilde{\gamma}_0 - \gamma - 2i|\mu|Q)^{-1}\phi'_{\mathbf{R}} = \int_{\mathbf{k}} \phi_{\mathbf{k},\mathbf{R}}(\tilde{\gamma}_0 - \tilde{\gamma}_{\mathbf{k}} - 2i|\mu|Q)^{-1}\Phi'_{\mathbf{k}} = \int_{\mathbf{k}} \phi_{\mathbf{k},\mathbf{R}}\Phi_{\mathbf{k}}. \quad (25)$$

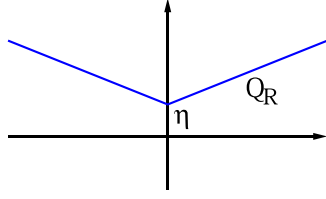
Using the special local vector  $\phi_{\mathbf{R}'} = (\delta_{\mathbf{R}'\mathbf{r}})$ , we get

$$\begin{aligned} \phi_{\mathbf{R}'} \cdot (\tilde{\gamma}_0 - \gamma - 2i|\mu|Q)^{-1}\phi'_{\mathbf{R}} &= \int_{\mathbf{k}} \phi_{\mathbf{k},\mathbf{R}}(\phi_{\mathbf{R}'} \cdot (\tilde{\gamma}_0 - \tilde{\gamma}_{\mathbf{k}} - 2i|\mu|Q)^{-1}\Phi'_{\mathbf{k}}) \\ &= \int_{\mathbf{k}} \phi_{\mathbf{k},\mathbf{R}}(\phi_{\mathbf{R}'} \cdot \Phi_{\mathbf{k}}) = \int_{\mathbf{k}} \phi_{\mathbf{k},\mathbf{R}}e^{i\mathbf{k} \cdot \mathbf{R}'}. \end{aligned} \quad (26)$$

Finally, we define the expansion coefficients as  $\phi_{\mathbf{k},\mathbf{R}} = e^{-i\mathbf{k} \cdot \mathbf{R}}/(\tilde{\gamma}_0 - \tilde{\gamma}_{\mathbf{k}} - 2i|\mu|Q_{\mathbf{R}})$  and obtain

$$\begin{aligned} \phi_{\mathbf{R}'} \cdot (\tilde{\gamma}_0 - \gamma - 2i|\mu|Q)^{-1}\phi'_{\mathbf{R}} &= \int_{\mathbf{k}} \frac{e^{i\mathbf{k} \cdot (\mathbf{R}' - \mathbf{R})}}{\tilde{\gamma}_0 - \tilde{\gamma}_{\mathbf{k}} - 2i|\mu|Q_{\mathbf{R}}}, \\ \phi'_{\mathbf{R}} &= \int_{\mathbf{k}} \frac{e^{-i\mathbf{k} \cdot \mathbf{R}}}{\tilde{\gamma}_0 - \tilde{\gamma}_{\mathbf{k}} - 2i|\mu|Q_{\mathbf{R}}} (\tilde{\gamma}_0 - \tilde{\gamma}_{\mathbf{k}} - 2i|\mu|Q)\Phi_{\mathbf{k}}. \end{aligned} \quad (27)$$

The decay of this expression with respect to  $|\mathbf{R} - \mathbf{R}'|$  characterizes the localization even before averaging, i.e., for any realization of  $Q_{\mathbf{R}}$ . To calculate the decay we choose the contour  $\Gamma$  of the  $Q_{\mathbf{R}}$  integration as  $Q_{\mathbf{R}} = e^{i \operatorname{sgn}(Q)\zeta} \bar{Q} + i\eta$  with real  $-\infty < \bar{Q} < \infty$ ,  $\eta > 0$  and  $0 < \zeta < \pi/4$ , as visualized in figure 1. Moreover, we assume that  $\tilde{\gamma}_0 - \tilde{\gamma}_{\mathbf{k}} = Dk^2$ , which is the inverse diffusion



**Figure 1.** Contour  $\Gamma$  of the  $Q_{\mathbf{R}}$  integration in the complex plane. The angle with respect to the abscissa is  $\pm\zeta$  ( $0 < \zeta < \pi/4$ ).

propagator with diffusion coefficient  $D$ . Then the  $k_1$  integration can be carried out for the decay along the one-direction of the lattice as

$$\begin{aligned} \int_{\mathbf{k}} \tilde{\gamma}_0 - \tilde{\gamma}_{\mathbf{k}} - 2i|\mu|Q_{\mathbf{R}} &= \int_{\mathbf{k}_{\perp}} \int_{-\infty}^{\infty} \frac{e^{ik_1(R'_1 - R_1)}}{Dk_1^2 + Dk_{\perp}^2 - 2i|\mu|Q_{\mathbf{R}}} dk_1 \\ &= \pi \int_{\mathbf{k}_{\perp}} \frac{e^{-|R'_1 - R_1| \sqrt{k_{\perp}^2 - 2i|\mu|(e^{i \operatorname{sgn}(\bar{Q})\zeta} \bar{Q} + i\eta)}/D}}{D \sqrt{k_{\perp}^2 - 2i|\mu|(e^{i \operatorname{sgn}(\bar{Q})\zeta} \bar{Q} + i\eta)}/D}}, \end{aligned} \quad (28)$$

where  $\mathbf{k}_{\perp}$  is the  $\mathbf{k}$  vector perpendicular to the one-direction. From the decay of the exponential function in this expression we extract a localization length as

$$\xi = \frac{1}{\operatorname{Re} \left( \sqrt{k_{\perp}^2 - 2i|\mu|(e^{i \operatorname{sgn}(\bar{Q})\zeta} \bar{Q} + i\eta)}/D \right)} \leq c(\eta, |\bar{Q}|, \zeta) \sqrt{D/|\mu|}, \quad (29)$$

where the coefficient  $c(\eta, |\bar{Q}|, \zeta)$  of the upper bound is of order 1.

### 3.3. Higher order terms of the loop expansion

The contribution of large loops can be analyzed for the case when we neglect the string term  $W\varphi W^{\dagger}\varphi'$  inside the trace of equation (14), which provides the IM without any string contribution as

$$J_{\text{loops}} = \det(\mathbf{1} - X\varphi\varphi') \det(\mathbf{1} + X^{\dagger}\varphi\varphi')^{-1}. \quad (30)$$

The correlator  $\int_{\Lambda} J_{\text{loops}} \varphi_{\mathbf{R}} \varphi_{\mathbf{R}'}$  vanishes for  $\mathbf{R}' \neq \mathbf{R}$ , since only a string formed by  $W\varphi W^{\dagger}\varphi'$  can connect  $\varphi_{\mathbf{R}}$  and  $\varphi_{\mathbf{R}'}$ . On the other hand, in the presence of a string, these loops are obstacles for the string in the full IM  $J$  and resemble the situation of classical percolation by a geometric restriction of the string formation. Reducing the loop distribution to small loops, we have found in section 3.2 that this leads to localization. Therefore, a similar localization effect is anticipated also for a distribution of larger loops, the possibility of percolating strings for the full distribution of loops cannot be ruled out though.

To understand the interaction between loops and a string, where the latter is created by  $W\varphi W^{\dagger}\varphi'$ , we consider the spatial off-diagonal elements of the average Hamiltonian  $\langle H \rangle$  (i.e., the hopping terms of  $\langle H \rangle$ ). In order to vary the hopping rate we introduce the parameter  $s$  ( $0 < s < 1$ ) by the rescaling transformation  $\langle H \rangle \rightarrow s\langle H \rangle$ . For simplicity, we assume that  $\langle H \rangle$



consists only of nearest-neighbor hopping terms. Then a reduction of  $s$  means a reduction of the hopping probability. Expanding the effective Green's function  $h$  of equation (7) in powers of  $s$  gives

$$h = \begin{pmatrix} h_+ & 0 \\ 0 & h_- \end{pmatrix}, \quad h_{\pm} = e^{\pm i\phi\mu} \mathbf{1} - \frac{2i\eta}{(\pm\mu - i\eta)^2} s \langle H \rangle + O(s^2)$$

$$e^{\pm i\phi\mu} := \frac{\pm\mu + i\eta}{\pm\mu - i\eta}. \quad (31)$$

In the PH-symmetric case  $\mu = 0$  the term linear in the Hermitian matrix  $\langle H \rangle$  changes its sign under Hermitian conjugation (i.e., it is anti-Hermitian). Moreover, we write  $X = sX_1 + O(s^2)$  and  $W = sW_1 + O(s^2)$ . This enables us to extract a scaling factor  $s$  in (14) and, after rescaling the Grassmann field  $\varphi$  by  $s$ , we obtain

$$\log J = \text{Tr}(s[W_1 W_1^\dagger]_d \varphi \varphi') - \sum_{l \geq 1} \frac{1}{l} \text{Tr} \left[ \left( \varphi X_1^\dagger \varphi' + X_1 \varphi \varphi' + s W_1 \varphi W_1^\dagger \varphi' + O(s^2) \right)^l \right]. \quad (32)$$

Then we treat  $s$  as an expansion parameter to separate the terms in  $\log J$  according to their scaling dimension with respect to  $s$  as

$$\log J = \text{Tr} \log(\mathbf{1} - X_1 \varphi \varphi') - \text{Tr} \log(\mathbf{1} + X_1^\dagger \varphi \varphi') + s \left\{ \text{Tr}([W_1 W_1^\dagger]_d \varphi \varphi') - \text{Tr}(W_1 \varphi W_1^\dagger \varphi') \right\} + O(s^2). \quad (33)$$

This means that on a scale larger than  $1/s$  we can neglect terms of  $O(s^2)$ . Moreover, since  $X_1^\dagger = -X_1$  in the PH-symmetric case, the term without  $s$  vanishes, and the leading term on large scales  $1/s$  is just the diffusion propagator linear in  $s$ . The situation is different when  $X_1^\dagger \neq -X_1$  in the PH-symmetry broken case. Then the term without  $s$  survives and can dominate the diffusion term.

#### 4. Discussion and conclusion

The effect of PH-symmetry breaking is characterized by the appearance of small repulsive dimers in a lattice system that is diffusive at the PH-symmetric point. This means that diffusion, a classical process, is disturbed by a random distribution of small obstacles. These obstacles are complex (i.e., they have a phase factor) due to the imaginary matrix  $\beta$  in equation (17). This effect can be seen as a quantum effect, since for classical diffusion in a random environment there are only real obstacles. The dimers are represented by a Gaussian field with a complex correlation matrix in the Grassmann functional integral, as given in equations (19) and (20). This reduces the original problem of the IM in equations (6) and (8) to the rather elementary case of diffusion in the presence of random obstacles. The effect of the latter has been estimated and gives a localization length, whose upper bound is  $\sqrt{D/|\mu|}$  with the diffusion coefficient  $D$  and the PH-symmetry breaking parameter  $\mu$ . This surprisingly elementary result, which depends only on the ratio of the two model parameters, reflects the competition between diffusion and PH-symmetry breaking. The diffusion coefficient, which

is determined through the expression  $\tilde{\gamma}_0 - \tilde{\gamma}_k = Dk^2$ , depends on the average Hamiltonian  $H_0$ , the scattering rate  $\eta$  and  $\mu$  according to the expressions in equation (7). On the other hand, the localization effect due to PH-symmetry breaking does not agree with the conventional picture of a mobility edge somewhere in the band of the random Hamiltonian and the related second order phase transition for dimensionality  $d > 2$  [28, 29]. Moreover, our result is different from the self-consistent approach to Anderson localization by Vollhardt and Wölfle [30, 31], who found that the diffusion coefficient  $D$  vanishes at the transition to Anderson localization but leaves a pole of the effective propagator on the real axis instead of moving it away into the complex plane. These differences might be related to the fact that we have considered a special class of Hamiltonians, based on the property (1) and the chiral symmetry. Moreover, the type of PH-symmetry breaking in equation (2) is special. With our results we cannot rule out that there are other types of PH-symmetry breaking which lead to the conventional Anderson transition.

The type of Hamiltonian obeying (1) is known in the form of the Dirac Hamiltonian, which is realized for low-energy quasi particles in graphene. As mentioned in the introduction, this material has been the subject of intense experimental as well as theoretical research for a number of years. Our result of a finite localization length in the case of a broken PH symmetry might be useful for the characterization of transport in doped graphene. For this system the diffusion coefficient of the two-dimensional Dirac fermions with finite momentum cut-off reads  $D = \frac{(\hbar v_F)^2}{\eta} \gamma_D$  [18], where the scalar  $\gamma_D$  depends on the momentum cut-off. The Fermi velocity in graphene is typically  $v_F \approx 10^6$  m s<sup>-1</sup> and a typical Fermi energy  $\mu$  is up to  $|\mu| \approx 0.5$  eV. Moreover, the typical scattering rate is  $\eta \approx 0.02$  eV. Together with the Planck constant  $\hbar \approx 6.6 \cdot 10^{-16}$  eV s we get for the localization length  $\xi \sim \sqrt{D/|\mu|} \approx 10^{-8}$  m. One should keep in mind that a finite system size prevents us to distinguish extended states from localized states whose localization length is larger than the system size. This could be important for experiments with graphene flakes and for numerical simulations of the localization effect away from the Dirac point, especially for a small PH-symmetry breaking parameter  $\mu$ . In those cases the localization effect should be observable for localization lengths smaller than the system size.

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## Data availability statement

No new data were created or analysed in this study.

## Appendix A. The IM

Since  $\hat{S}^k = 0$  for  $k > 2$  we can write for the matrix of the IM

$$\begin{aligned} \hat{H}_0 + i\epsilon + i\eta e^{2\hat{S}} &= \hat{H}_0 + i\epsilon + i\eta(1 + 2\hat{S} + 2\hat{S}^2) = \hat{H}_0 + i\epsilon - i\eta + 2i\eta(\mathbf{1} - \hat{S})^{-1} \\ &= (\mathbf{1} - \hat{S})^{-1} \hat{G}^{-1} (\mathbf{1} - \hat{G}\hat{S}) \hat{G}_0^{-1} \end{aligned}$$

with

$$\hat{G}_0^{-1} = \hat{H}_0 + i\epsilon - i\eta, \quad \hat{G} = \begin{pmatrix} h & 0 \\ 0 & h^T \end{pmatrix}.$$

The definition of graded trace is

$$\text{Tr} g \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \text{Tr} A - \text{Tr} D$$

and of the graded determinant is

$$\det g \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{\det A}{\det D} \det(\mathbf{1} - BD^{-1}CA^{-1}), \quad (\text{A.1})$$

where the latter implies

$$\text{det}g(\hat{G}) = \text{det}g(\hat{G}_0) = 1.$$

This gives for the IM

$$J = \text{det}g(\mathbf{1} - \hat{S})^{-1} \text{det}g(\mathbf{1} - \hat{G}\hat{S}).$$

Moreover, using

$$\hat{S} = \begin{pmatrix} 0 & \bar{S} \\ \bar{S}' & 0 \end{pmatrix}, \quad \bar{S} = \begin{pmatrix} 0 & \varphi_1 U \\ \varphi_2 U & 0 \end{pmatrix} = \begin{pmatrix} \varphi_1 U & 0 \\ 0 & \varphi_2 U \end{pmatrix} \sigma_1,$$

$$\bar{S}' = \begin{pmatrix} 0 & \varphi_2' U^\dagger \\ \varphi_1' U^\dagger & 0 \end{pmatrix} = \sigma_1 \begin{pmatrix} \varphi_1' U^\dagger & 0 \\ 0 & \varphi_2' U^\dagger \end{pmatrix},$$

we can express the IM via (A.1) in terms of determinants as

$$J = \det(\mathbf{1} + \bar{S}\bar{S}' - h\bar{S}h^T\bar{S}'). \quad (\text{A.2})$$

With  $H_0 := \langle H \rangle$  and

$$h = \begin{pmatrix} (H_0 + \mu + i\epsilon - i\eta)(H_0 + \mu + i\epsilon + i\eta)^{-1} & 0 \\ 0 & (H_0 - \mu + i\epsilon - i\eta)(H_0 - \mu + i\epsilon + i\eta)^{-1} \end{pmatrix}$$

we have  $hh^\dagger = \mathbf{1} + O(\epsilon)$  and

$$\sigma_1 U h^T U^\dagger \sigma_1 = \begin{pmatrix} (H_0 + \mu - i\epsilon + i\eta)(H_0 + \mu - i\epsilon - i\eta)^{-1} & 0 \\ 0 & (H_0 - \mu - i\epsilon + i\eta)(H_0 - \mu - i\epsilon - i\eta)^{-1} \end{pmatrix} = h^\dagger.$$

This implies for equation (A.2)

$$J = \det(\mathbf{1} + \varphi_1 \varphi_1' - h \varphi_1 h^\dagger \varphi_1') \det(\mathbf{1} + \varphi_2 \varphi_2' - h \varphi_2 h^\dagger \varphi_2'). \quad (\text{A.3})$$

Finally, we apply the determinant identity to write

$$\det(\mathbf{1} + \varphi_j \varphi_j' - h \varphi_j h^\dagger \varphi_j') = \exp \left\{ - \sum_{l \geq 1} \frac{1}{l} \text{Tr} [(-\varphi_j \varphi_j' + h \varphi_j h^\dagger \varphi_j')^l] \right\}. \quad (\text{A.4})$$

The properties of the Grassmann variables imply

$$\begin{aligned}\sum_{l \geq 1} \frac{1}{l} \text{Tr}[(-\varphi_j \varphi'_j + h \varphi_j h^\dagger \varphi'_j)^l] &= \text{Tr}(\varphi'_j \varphi_j) - \sum_{l \geq 1} \frac{1}{l} \text{Tr}[(\varphi'_j h \varphi_j h^\dagger)^l] \\ &= -\sum_{l \geq 1} \frac{1}{l} \text{Tr}[(-\varphi'_j \varphi_j + \varphi'_j h \varphi_j h^\dagger)^l]\end{aligned}$$

such that

$$\det(\mathbf{1} + \varphi_j \varphi'_j - h \varphi_j h^\dagger \varphi'_j) = \det(\mathbf{1} + \varphi'_j \varphi_j - \varphi'_j h \varphi_j h^\dagger)^{-1}. \quad (\text{A.5})$$

This is the relation in equation (6).

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