Homogenisation of local colloid evolution induced by reaction and diffusion

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Abstract

We consider the homogenisation of a coupled reaction–diffusion process in a porous medium with evolving microstructure. A concentration-dependent reaction rate at the interface of the pores with the solid matrix induces a concentration-dependent evolution of the domain. Hence, the evolution is fully coupled with the reaction–diffusion process. In order to pass to the homogenisation limit, we employ the two-scale-transformation method. Thus, we homogenise a highly non-linear problem in a periodic and in time cylindrical domain instead. The homogenisation result is a reaction–diffusion equation, which is coupled with an internal variable, representing the local evolution of the pore structure.

Keywords: Homogenization, evolving microstructure, free boundary problem, two-scale convergence, porous medium, reaction–diffusion process
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1. Introduction

Reaction–diffusion mechanisms in porous media often induce an evolution of the solid matrix. Typical examples are reaction mechanisms producing or consuming constituents which are part of the solid matrix, e.g. in concrete carbonation (cf. [1], [2]) or crystal precipitation and dissolution (cf. [3], [4]). Similarly, if biofilms are present, these can often be viewed as a solid-matrix-type part of the porous medium on the pore scale. In this context, production of biofilm can be modelled on the microscale similarly as production of solid matrix (cf. [5], [6], [7]).

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Mathematical models for reaction and diffusion in porous media are typically obtained from upscaling processes on the pore scale by averaging or homogenisation techniques. A classic method in this context is periodic homogenisation (cf. [8], [9]), which has been extended to cope with (non-periodic) evolving microstructures (cf. [10]). The extension relies on transforming the non-periodic evolution to a periodic reference geometry, which requires modelling of this (concentration-dependent) transformation in the context of particular applications, for instance a detailed discussion for concrete carbonation can be found in [11].

$$\begin{tikzcd}
\text{evolving microproblem} \arrow[r, "\text{homogenisation on the evolving domain}"] \arrow[d, "\text{transformation}"] & \text{evolving macroproblem} \\
\text{transformed microproblem} \arrow[r, "\text{homogenisation on periodic reference domain}"] & \text{transformed macroproblem} \arrow[u, "\text{back-transformation}"']
\end{tikzcd} \tag{1}$$

The approach of transforming on a periodic reference domain has found also application in the homogenisation of thermoelasticity [12] or the homogenisation of advection–reaction–diffiusion problems in porous media (cf. [13]), where the domains evolution is a priori given. Moreover, it has been recently shown that the homogenisation of the substitute problem is equivalent to the homogenisation of the actual problem in the non-periodic mirostructure, i.e. that (1) commutes (cf. [14]). Furthermore, a new two-scale-transformation rule has been derived there, which yields a transformation-independent homogenisation result after the back-transformation.

In the present paper, we use this approach to homogenise rigorously a reaction–diffusion problem where the domain evolution is not a priori given but coupled with the solution itself. The homogenisation of problems where the evolving microstructure is coupled with the solution itself has been also considered by a level-set approach. There, the domain is described by a level-set function solving a level-set equation, which involves the other unknowns. In this framework, microscopic models for crystal precipitation and dissolution (cf. [7]) or biofilm growth in porous media (cf. [15]) have been homogenised. However, the corresponding effective macroscopic problems have been derived by formal asymptotic expansion only. Numerical simulations and analytical discussion of such type of limit models can be found in [16], [17], [18].

In this manuscript, we revisit the microscale model by [19] for one reaction–diffusion equation and derive their upscaled model by a mathematically rigorous homogenisation procedure based on the recent results of [14]. In this context, we show that such coupling of the pore structure with the solution of the reaction–diffusion equation can be handled by the two-scale-transformation method. For this purpose, we construct a concrete ε-scaled transformation for the ε-scaled domains by means of a generic parametrisable cell transformation. There, the radius of the solid obstacles becomes the parameter. By showing a certain kind of strong convergence for the radii of the ε-scaled model, we can verify the assumptions of the two-scale-transformation method. Thus, we can pass rigorously to the two-scale limit in the substitute problem. Moreover, using the two-scale-transformation rule of [14], we obtain a two-scale limit problem in the actual non-cylindrical evolving two-scale domain, which is independent of the chosen transformation. There, we split the macroscopic and microscopic variables in order to derive an effective equation. The result is a macroscopic reaction–diffusion problem coupled with an internal variable, which represents the local radius of the solid. This local radius is given by an ordinary differential equation and scales not only the time-derivative term and the reaction rate of the reaction–diffusion equation but also affects the effective diffusivity. The diffusivity is still computed by solutions of cell problems as in the case of a rigid domain. However, the domain for the cell problems is now parametrised by the internal radius and affects in this way the local effective diffusivity. A similar macroscopic model has very recently been derived in [20]. There the (slightly different) transformed microscopic model is analysed by different methods and the strong compactness results are derived by a different approach.

This paper is organised as follows: In section 2, we derive the microscopic model (13)–(16), which consists of a reaction–diffusion problem coupled with the evolution of the domain. Then, we state the corresponding weak formulation in the evolving domain. Using a generic cell transformation, we transform the weak form to the equivalent weak form (32)–(33), (44)–(46) on the periodic substitute domain, which becomes highly non-linear. In section 3, we show the existence and uniqueness of the solution of the transformed microscopic model by a fixed point argument. There, we utilise the assumption that the radii, which define the solid domain, are a priori bounded from below and above. Moreover, we derive some ε-independent a priori estimates. In section 4, we use two-scale convergence in order to pass to the homogenisation limit. Since the coefficients in the equation depend on the solution itself, the problem becomes highly non-linear and we need a strong convergence of the solution. However, we can not derive easily a uniform bound of the time derivative of the solution of the diffusion equation. Therefore, we can not use the classical Aubin–Lions lemma. Instead, we shift the solution of the reaction–diffusion equation with respect to time and estimate the difference to the actual solution. Then, we can conclude with the Simon-Kolmogorov compactness criterion (cf. [21, Theorem 1]) the strong convergence of the concentration. Using this strong convergence, we can show a strong convergence of the radii, which allows us to apply the two-scale-transformation method. Thus, we can derive the two-scale limit problem in the cylindrical two-scale reference domain rigorously. In section 5, we transform the limit problem back and obtain the transformation-independent two-scale limit problem. Then, we split the macroscopic and microscopic variable. This gives the effective problem (152), (153) with the effective diffusion coefficient (154) and the cell problems (155), which depend on an internal variable representing the local radius.

We use the following notations. Let $f, g \in L^2(U)$ and $U \subset \mathbb{R}^m$ for $m \in \mathbb{N}_{>0}$, then we write the scalar product and the norm by: $(f, g)_U := \int_U f(x)g(x)dx, ||f||^2_U := (f, f)_U$. For $f \in H^1(U)'$ and $g \in H^1(U)$, we write the dual paring by $\langle f, g\rangle_U := \langle f, g\rangle_{H^1(U)', H^1(U)}$.

Furthermore, we use C as generic constant which is independent of ε and other variables and depends only on fixed constants. In cases, in which the generic constant can depend on other variables as for instance ε, we mark this by a subscript, e.g. we write C_ε. Moreover, let the spatial dimension be $N \in \mathbb{N}$ with $N \geq 2$.

2. The mathematical model

Let Ω be an open set in $\mathbb{R}^N$, which represents the macroscopic domain of the porous medium and let $\varepsilon = (\varepsilon_n)_{n\in\mathbb{N}}$ be a positive monotone sequence converging to zero with ε_0 sufficiently small. We assume that Ω consists of whole ε-scaled cells $Y := (0,1)^N$, i.e. $\Omega = \operatorname{int}\left(\bigcup_{k\in I_\varepsilon} \varepsilon k + \varepsilon \overline{Y}\right)$ for $I_\varepsilon := \{k \in \mathbb{Z}^N \,|\, (\varepsilon k + \varepsilon Y) \cap \Omega \neq \emptyset\}$.

We assume that the pore structure of the porous medium is given by spherical obstacles in the cells $\varepsilon k + \varepsilon Y$ for $k \in I_\varepsilon$ which can grow and shrink on the time interval $S = (0, T)$ with $0 < T < \infty$. Thus, the ε-scaled porous medium is defined by

$$\Omega_\varepsilon(t) := \Omega \setminus \bigcup_{k\in I_\varepsilon} \varepsilon \overline{B_{r_{\varepsilon,k}(t)}(k + x_M)} \tag{2}$$

where $x_M := (0.5, \dots, 0.5)^\top$ is the centre of the reference cell and $r_{\varepsilon,k}(t)$ is the ε^{-1}-scaled radius of the solid obstacle in the cell located at εk at time $t \in S$ (cf. Figure 1).

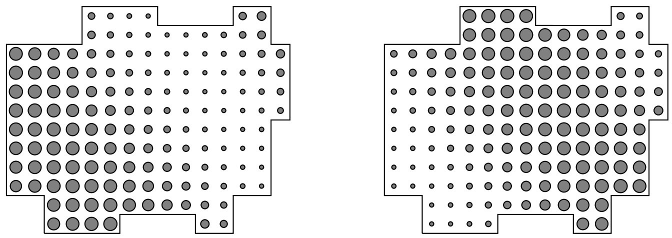

Figure 1: The domain $\Omega_\varepsilon(t)$ for $t = 0$ (left) and $t > 0$ (right)

We assume that the size of the obstacles $\varepsilon B_{r_{\varepsilon,k}(t)}(k + x_M)$ is affected by reactions on their surfaces $\Gamma_{\varepsilon,k}(t) := \partial \varepsilon B_{r_{\varepsilon,k}(t)}(k + x_M)$. The reactions rate $\varepsilon f(u_\varepsilon(t,x), r_{\varepsilon,k}(t))$ depends on the concentration rate u_ε and on the radius $r_{\varepsilon,k}$ of $S_{\varepsilon,k}$. Because the reaction rate depends on the radius, we can ensure $r_{\min} \leq r_{\varepsilon,k}(t) \leq r_{\max}$ for every $k \in I_\varepsilon$ and every $t \in S$ for constants $0 < r_{\min} < r_{\max} < 0.5$ by the assumptions:

$$f(\cdot, r) \geq 0 \text{ for } r \leq r_{\min}, \tag{3}$$

$$f(\cdot, r) \leq 0 \text{ for } r \geq r_{\max}. \tag{4}$$

Moreover, we assume that f is uniformly Lipschitz continuous and bounded, i.e. there exists a constant C such that

$$f(u_1, r_1) - f(u_2, r_2) \leq C(|u_1 - u_2| + |r_1 - r_2|), \tag{5}$$

$$|f(u_1, r_1)| \leq C_f \tag{6}$$

for $u_1, u_2 \in \mathbb{R}$ and $r_1, r_2 \in \mathbb{R}$.

We consider the case that the formed or vanishing solid has a constant concentration density c_s. Thus, the conservation of mass yields

$$\frac{d}{dt} |\varepsilon B_{r_{\varepsilon,k}(t)}(k + x_M)| c_s = \int_{\Gamma_{\varepsilon,k}(t)} j_\varepsilon(t,x) \cdot n(t,x) \, d\sigma_x \tag{7}$$

where $j_\varepsilon(t,x)$ is the flux through $\Gamma_{\varepsilon,k}(t)$ and n is the inner unit normal of $\varepsilon B_{r_{\varepsilon,k}(t)}(k + x_M)$. We note that this flux consists of the diffusive flux and a flux which is induced by the evolution of the domain. We model the diffusive flux $j_{D,\varepsilon} = -D\nabla u_\varepsilon(t,x)$ by Fick's law with a diffusion coefficient D. The second flux, which is induced by the evolution of the domain, can be understood in the following sense: when the carrier medium becomes solid any excess dissolved concentration separates from the carrier medium and is pushed away, i.e. $j_{\Gamma_\varepsilon}(t,x) = -v_{\Gamma_{\varepsilon,k}(t,x)}u(t,x)$, where $v_{\Gamma_{\varepsilon,k}}$ is the velocity of the boundary deformation. We note that $v_{\Gamma_{\varepsilon,k}}$ can be formulated explicitly by $v_{\Gamma_{\varepsilon,k}}(t,x) = -\varepsilon\partial_t r_{\varepsilon,k}(t)n(t,x)$. Thus, the total flux on the boundary is

$$j_\varepsilon(t,x) = j_{D,\varepsilon}(t,x) + j_{\Gamma_\varepsilon}(t,x) = -D\nabla u_\varepsilon(t,x) - v_{\Gamma_{\varepsilon,k}}(t,x)u_\varepsilon(t,x) \qquad (8)$$

for $t\in S$ and $x\in\Gamma_{\varepsilon,k}(t)$. On the other hand, the flux at the boundary in the normal direction, $j_\varepsilon(t,x)\cdot n(t,x)$, represent the consumption or gain of concentration due to the reactions on $\Gamma_{\varepsilon,k}(t)$, which yields

$$(-D\nabla u_\varepsilon(t,x) - v_{\Gamma_{\varepsilon,k}}(t,x)u_\varepsilon(t,x))\cdot n(t,x) = j_\varepsilon(t,x)\cdot n(t,x) = \varepsilon f(u_\varepsilon(t,x), r_{\varepsilon,k}(t)) \qquad (9)$$

and equivalently

$$-D\nabla u_\varepsilon(t,x)\cdot n(t,x) + \varepsilon\partial_t r_{\varepsilon,k}(t)u_\varepsilon(t,x) = \varepsilon f(u_\varepsilon(t,x), r_{\varepsilon,k}(t)). \qquad (10)$$

Inserting (9) in (7) yields

$$\frac{d}{dt}|\varepsilon B_{r_{\varepsilon,k}(t)}(k+x_M)|c_s = \int_{\Gamma_{\varepsilon,k}(t)} \varepsilon f(u_\varepsilon(t,x), r_{\varepsilon,k}(t))d\sigma_x \qquad (11)$$

and elementary calculus implies

$$\frac{d}{dt}|\varepsilon B_{r_{\varepsilon,k}(t)}(k+x_M)| = \varepsilon^N\frac{d}{dt}V_N(r_{\varepsilon,k}(t)) = \varepsilon^N S_{N-1}(r_{\varepsilon,k}(t))\partial_t r_{\varepsilon,k}(t),$$

where $V_N(r)$ denotes the volume of the N-ball with radius r and $S_N(r)$ denotes the surface of the N-sphere with radius r. Thus, we obtain the following ordinary differential equation for the radii:

$$\partial_t r_{\varepsilon,k}(t) = \frac{\varepsilon^{-N}}{c_s S_{N-1}(r_{\varepsilon,k}(t))}\int_{\Gamma_{\varepsilon,k}(t)} \varepsilon f(u_\varepsilon(t,x), r_{\varepsilon,k}(t))d\sigma_x. \qquad (12)$$

Combining the diffusion equation with the boundary condition (10) and the evolution of the radii given by (12) yields the following strong formulation:

$$\partial_t u_\varepsilon(t,x) - \operatorname{div}(D\nabla u_\varepsilon(t,x)) = f^{\mathrm{p}}(t,x) \qquad \text{in } \bigcup_{t\in S}\{t\}\times\Omega_\varepsilon(t), \tag{13}$$

$$-D\nabla u_\varepsilon(t,x)\cdot n(t,x) + \varepsilon\partial_t r_{\varepsilon,k}(t) u_\varepsilon(t,x) = \varepsilon f(u_\varepsilon(t,x), r_{\varepsilon,k}(t)) \qquad \text{on } \bigcup_{t\in S}\{t\}\times\Gamma_{\varepsilon,k}(t), \tag{14}$$

$$-D\nabla u_\varepsilon(t,x)\cdot n(t,x) = 0 \qquad \text{on } \partial\Omega, \tag{15}$$

$$\partial_t r_{\varepsilon,k}(t) = \frac{\varepsilon^{-N}}{c_s S_{N-1}(r_{\varepsilon,k}(t))} \int\limits_{\Gamma_{\varepsilon,k}(t)} \varepsilon f(u_\varepsilon(t,x), r_{\varepsilon,k}(t))\, d\sigma_x \qquad \text{for } k\in I_\varepsilon \tag{16}$$

for $\Omega_\varepsilon(t)$ given by (2), $n(t,x)$ the unit outer normal of $\Omega_\varepsilon(t)$ for every $t\in S$ and initial conditions $r_\varepsilon(0) = r_\varepsilon^{(0)} \in [r_{\min}, r_{\max}]^{|I_\varepsilon|}$, $u_\varepsilon(0,\cdot_x) = u_\varepsilon^{(0)} \in L^2(\Omega_\varepsilon(0))$.

We assume that f^{p} is Lipschitz continuous in every ε-scaled cell $\varepsilon(k+Y)$ for every $k\in I_\varepsilon$ and every $n\in\mathbb{N}$. Note that this does not necessarily imply $f^{\mathrm{p}} \in C(\Omega)$. We assume that there exists $r^{(0)}$ in $L^2(\Omega)$ such that $r^{(0)}_{\varepsilon,k_\varepsilon(\cdot_x)} \to r^{(0)}$ in $L^2(\Omega)$, where $k_\varepsilon(x)\in I_\varepsilon$ is the index of the cell in which x is located. Moreover, we assume that there exists $u_0^{(0)} \in L^2(\Omega)$ such that the extension of $u_\varepsilon^{(0)}$ by 0 to Ω two-scale converges with respect to the L^2-norm to $\chi_{Y^*_{r^{(0)}(\cdot_x)}}(\cdot_y) u_0^{(0)}(\cdot_x)$ for $Y_r^* := Y \setminus \overline{B_r(x_M)}$ and we assume that $\left\|u_\varepsilon^{(0)}\right\|_{L^\infty(\Omega_\varepsilon)} \le C$.

2.1. Weak formulation

We multiply (13) by φ and integrate over $\Omega_\varepsilon(t)$ and S. Then, we integrate the divergence term by parts and apply (14). Thus we obtain the boundary integral $\int\limits_S \int\limits_{\Gamma_\varepsilon(t)} \varepsilon\partial_t r_{\varepsilon,k_\varepsilon(x)}(t) u_\varepsilon(t,x) - \varepsilon f(u_\varepsilon(t,x), r_{\varepsilon,k_\varepsilon(x)}(t,x)) d\sigma_x dt$. The integration by parts of $\partial_t u_\varepsilon\varphi$ with respect to t cancels $\int\limits_S \int\limits_{\Gamma_\varepsilon(t)} \varepsilon\partial_t r_{\varepsilon,k_\varepsilon(x)}(t) u_\varepsilon(t,x) d\sigma_x dt$ due to the time-dependent domain $\Omega_\varepsilon(t)$ (cf. Reynold's transport theorem). Thus, we get (17). Furthermore, we multiply (16) by ϕ and integrate over

Swhich gives (18). Altogether, we obtain the following weak form of (2), (13)–(16): Find $(u_\varepsilon, r_\varepsilon)\in L^2(S;H^1(\Omega_\varepsilon(t)))\times W^{1,\infty}(S)^{|I_\varepsilon|}$ such that

$$-\int\limits_S\int\limits_{\Omega_\varepsilon(t)} u_\varepsilon(t,x)\partial_t\varphi(t,x)dxdt-\int\limits_{\Omega_\varepsilon(0)} u_\varepsilon^{(0)}(x)\varphi(0,x)dxdt$$

$$+\int\limits_S\int\limits_{\Omega_\varepsilon(t)} D\nabla u_\varepsilon(t,x)\cdot\nabla\varphi(t,x)dxdt=\int\limits_S\int\limits_{\Omega_\varepsilon(t)} f^{\mathrm{p}}(t,x)\varphi(t,x)dxdt$$

$$-\sum_{k\in I_\varepsilon}\int\limits_S\int\limits_{\Gamma_{\varepsilon,k}(t)} \varepsilon f(u_\varepsilon(t,x),r_{\varepsilon,k}(t))\varphi(t,x)d\sigma_x dt, \tag{17}$$

$$\int\limits_S \partial_t r_{\varepsilon,k}(t)\phi(t)dt=\int\limits_S \frac{\varepsilon^{-N}}{c_s S_{N-1}(r_{\varepsilon,k}(t))}\int\limits_{\Gamma_{\varepsilon,k}(t)} \varepsilon f(u_\varepsilon(t,x),r_{\varepsilon,k}(t))d\sigma_x\phi(t)dt, \tag{18}$$

$$r_\varepsilon(0)=r_\varepsilon^{(0)} \tag{19}$$

for all$\varphi\in C^1(\overline{\bigcup_{t\in S}\{t\}\times\Omega_\varepsilon(t)})$ with$\varphi(T;\cdot_x)=0$, all$k\in I_\varepsilon$, all$\phi\in L^1(S)^{|I_\varepsilon|}$ and all$t\in S$. Note that$r_\varepsilon\in W^{1,\infty}(S)^{|I_\varepsilon|}\subset C^{0,1}(S)^{|I_\varepsilon|}$, which allows us to evaluate$r_{\varepsilon,k}$ pointwise in time and ensures that$\Omega_\varepsilon(t)$ is well defined for every $t\in S$.

2.2. Transformation of the domain

We transform (17)–(19) from $\bigcup_{t\in S}\{t\}\times\Omega_\varepsilon(t)$, where$\Omega_\varepsilon(t)$ is given by (2), on the in time cylindrical and in space periodic domain$S\times\Omega_\varepsilon$ with$\Omega_\varepsilon:=\Omega\setminus\bigcup_{k\in I_\varepsilon}\overline{\varepsilon B_{r_0}(k+x_M)}$ forfixedr_0 with$r_{\min}\le r_0\le r_{\max}$. Thus, we can show the existence and uniqueness of a solution of (17)–(19) and pass to the limit $\varepsilon\to 0$. We define$\Gamma_{\varepsilon,k}:=\partial\varepsilon B_{r_0}(k+x_M)$ for$k\in I_\varepsilon$ and$\Gamma_\varepsilon:=\bigcup_{k\in I_\varepsilon}\Gamma_{\varepsilon,k}$.

Although the geometry of$\Omega_\varepsilon(t)$ is already completely defined by its boundary, we need a transformation of the whole space and not only of the boundary by means of the radii, in order to apply the two-scale-transformation method. Since$r_{\varepsilon,k}\le r_{\max}$, the solid obstacles remain inside their respective cells so that the transformation can be defined for eachε-scaled cell separately using a generic transformation defined on the reference cell.

2.2.1. Generic transformation of the reference cell

We define the pore space of the reference cell by$Y^*:=Y^*_{r_0}$ and the interface of the reference cell by$\Gamma:=\partial B_{r_0}(x_M)$. We construct a generic cell transformation $\psi \in C^2([r_{\min}, r_{\max}] \times \overline{Y})^N$, such that

$$\psi(r_\Gamma, Y^*) = Y^*_{r_\Gamma} \quad \text{for } r_\Gamma \in [r_{\min}, r_{\max}], \tag{20}$$
$$\psi(r_\Gamma, y) = y \text{ for } (r_\Gamma, y) \in [r_{\min}, r_{\max}] \times (\overline{Y^{\mathrm{p}}_{r_{\max}+\delta}} \cup B_{r_{\min}-\delta}(x_M)), \tag{21}$$
$$||\psi||_{C^2([r_{\min}, r_{\max}] \times \overline{Y})} \leq C, \tag{22}$$
$$y \mapsto \psi(r_\Gamma, y) \text{ is bijective from } \overline{Y} \text{ onto } \overline{Y}, \tag{23}$$
$$\det(D_y \psi(r_\Gamma, y)) \geq c_J > 0 \quad \text{for } (r_\Gamma, y) \in [r_{\min}, r_{\max}] \times \overline{Y} \tag{24}$$

for δ small enough.

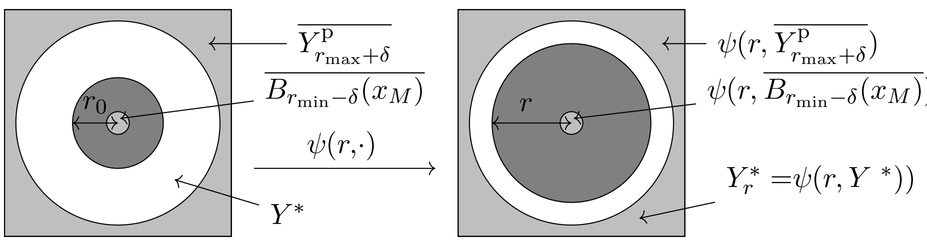

Figure 2: Generic cell transformation $\psi(r,\cdot)$

Note that due to (21), we can glue such cell transformations $\psi(r_\Gamma,\cdot)$ for different values of r_Γ next to each other. Such a generic cell transformation ψ can be easily constructed using the radial symmetry of the geometry in the reference cell. We define

$$\psi(r_\Gamma, y) := x_M + R(r_\Gamma, ||y - x_M||) \frac{y - x_M}{||y - x_M||} \tag{25}$$

for a smooth function $R \in C^\infty([r_{\min}, r_{\max}] \times [0, \infty))$, which scales the distance of y to x_M and fulfils

$$R(r_\Gamma, r_0) = r_\Gamma \quad \text{for } r_\Gamma \in [r_{\min}, r_{\max}], \tag{26}$$
$$R(r_\Gamma, r) = r \text{ for } (r_\Gamma, r) \in [r_{\min}, r_{\max}] \times (\mathbb{R} \setminus [r_{\min} - \delta, r_{\max} + \delta]), \tag{27}$$
$$||DR||_{C^2([r_{\min}, r_{\max}] \times [0, \infty))} \leq C, \tag{28}$$
$$\partial_r R(r_\Gamma, r) \geq c > 0 \text{ for } (r_\Gamma, r) \in [r_{\min}, r_{\max}] \times [0, \infty). \tag{29}$$

Such a mapping R can be obtained by linear interpolation and smoothing (cf. Figure 3). First we define

$$\check{R}(r_\Gamma, r) := \begin{cases} r & \text{for } r \leq r_{\min} - 2\tilde{\delta}, \\ c_1(r_\Gamma)(r - (r_{\min} - 2\tilde{\delta})) + r_{\min} - 2\tilde{\delta} & \text{for } r_{\min} - 2\tilde{\delta} \leq r \leq r_0 - \tilde{\delta}, \\ (r - r_0) + r_\Gamma & \text{for } r_0 - \tilde{\delta} \leq r \leq r_0 + \tilde{\delta}, \\ c_2(r_\Gamma)(r - (r_{\max} + 2\tilde{\delta})) + r_{\max} + 2\tilde{\delta} & \text{for } r_0 + \tilde{\delta} \leq r \leq r_{\max} + 2\tilde{\delta}, \\ r & \text{for } r \geq r_{\max} - 2\tilde{\delta} \end{cases} \tag{30}$$

for $r_\Gamma \in [r_{\min}, r_{\max}]$ with $c_1(r_\Gamma) := \frac{r_\Gamma - r_{\min} + \tilde{\delta}}{r_0 - r_{\min} + \tilde{\delta}}$ and $c_2(r_\Gamma) := \frac{r_{\max} - r_\Gamma + \tilde{\delta}}{r_{\max} - r_0 + \tilde{\delta}}$ and $\tilde{\delta} = \delta/3$.

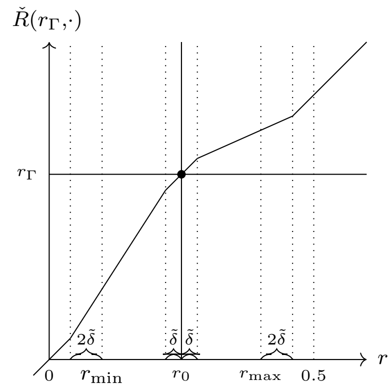

Figure 3: Construction of $\check{R}$

Then, we define

$$R(r_\Gamma, r) := \int_{\mathbb{R}} \check{R}(r_\Gamma, s)\eta\left(\frac{r-s}{\tilde{\delta}}\right) ds \tag{31}$$

for $\eta(x) := \left(\int_{\mathbb{R}} \exp\left(\frac{-1}{1-|y|^2}\right) dy\right)^{-1} \exp\left(\frac{-1}{1-|x|^2}\right)$. It can be shown easily that R fulfils (26)–(29).

We define the corresponding displacement field by $\check{\psi}(r_\Gamma, y) = \psi(r_\Gamma, y) - y$.

2.2.2. ε-scaling of the transformation

Scaling of ψ by ε and combining with the radii $r_{\varepsilon,k}$ for each cell gives a transformation for the ε-scaled porous medium:

$$\psi_\varepsilon(t, x) := [x]_{\varepsilon,Y} + \varepsilon\psi(r_{\varepsilon,k_\varepsilon(x)}(t), \{x\}_{\varepsilon,Y}) \tag{32}$$

where $[x]_{\varepsilon,Y} := \varepsilon \sum_{i=1}^{N} \left\lfloor \frac{x_i}{\varepsilon} \right\rfloor e_i$ is the position of the cell in which x is located and $\{x\}_{\varepsilon,Y} := \frac{1}{\varepsilon}(x - [x]_{\varepsilon,Y}(x))$ is the position inside the upscaled cell.

For the corresponding displacement field, we get

$$\begin{aligned}\check{\psi}_\varepsilon(t,x) &:= \psi_\varepsilon(t,x) - x = [x]_{\varepsilon,Y} + \varepsilon\psi(r_{\varepsilon,k_\varepsilon(x)}(t), \{x\}_{\varepsilon,Y}) - x \\ &= [x]_{\varepsilon,Y} + \varepsilon\,\check{\psi}(r_{\varepsilon,k_\varepsilon(x)}(t), \{x\}_{\varepsilon,Y}) + \varepsilon\{x\}_{\varepsilon,Y} - x = \varepsilon\,\check{\psi}(r_{\varepsilon,k_\varepsilon(x)}(t), \{x\}_{\varepsilon,Y})\end{aligned}$$

We denote the Jacobian matrix of ψ_ε and its determinant by

$$\Psi_\varepsilon(t,x) := D_x\psi_\varepsilon(t,x), \quad J_\varepsilon(t,x) = \det(\Psi_\varepsilon(t,x)). \tag{33}$$

Moreover, we obtain the following uniform estimates for ψ_ε:

Lemma 1 (Uniform boundedness of ψ_ε). *Let $r_\varepsilon \in W^{1,\infty}(S)^{|I_\varepsilon|}$ with $r_\varepsilon(t) \in [r_{\min}, r_{\max}]^{|I_\varepsilon|}$ for a.e. $t \in S$ and let ψ_ε be defined by (32). Then, $\psi_\varepsilon \in W^{1,\infty}(S; C^1(\overline{\Omega_\varepsilon})^N)$ and there exist constants $C, c_J, \alpha > 0$ independent of ε such that*

$$\begin{aligned}
&\varepsilon^{-1}\|\psi_\varepsilon - \mathrm{id}_{\Omega_\varepsilon}\|_{L^\infty(S\times\Omega_\varepsilon)} + \|\Psi_\varepsilon\|_{L^\infty(S\times\Omega_\varepsilon)} + \|J_\varepsilon\|_{L^\infty(S\times\Omega_\varepsilon)} \leq C, && (34)\\
&J_\varepsilon(t,x) \geq c_J, && (35)\\
&\varepsilon^{-1}\|\partial_t\psi_\varepsilon\|_{L^\infty(S;C(\overline{\Omega_\varepsilon}))} + \|\partial_t J_\varepsilon\|_{L^\infty(S;C(\overline{\Omega_\varepsilon}))} \leq \left\|\partial_t r_{\varepsilon,k_\varepsilon(\cdot x)}\right\|_{L^\infty(S\times\Omega_\varepsilon)}, && (36)\\
&\left\|\Psi_\varepsilon^{-1}\right\|_{L^\infty(S\times\Omega_\varepsilon)} \leq C, && (37)\\
&\xi^\top J_\varepsilon(t,x)\Psi_\varepsilon^{-1}(t,x)\Psi_\varepsilon^{-\top}(t,x)\xi \geq \alpha\|\xi\|^2 && (38)
\end{aligned}$$

for a.e. $(t,x) \in S \times \Omega_\varepsilon$ and every $\xi \in \mathbb{R}^N$.

Proof. The estimates (34)–(35) are a direct consequence of (20)–(24) and the cell-wise construction of ψ_ε. The estimate (37)–(38) follow from (34)–(35) by simple computations. The estimate (36) follows with (20)–(24) and the chain rule.

Furthermore, we obtain the following uniform Lipschitz estimates for ψ_ε with respect to the radii r_ε. Thereby, we abuse slightly the notation of r_ε by $r_\varepsilon(t,x) := r_{\varepsilon,k_\varepsilon(x)}(t)$. We will also use this notation in later proofs.

Lemma 2 (Lipschitz regularity of ψ_ε). *Let $p \in [1,\infty]$ and $r_{\varepsilon,i} \in W^{1,p}(S)^{|I_\varepsilon|}$ with $r_{\varepsilon,i}(t) \in [r_{\min}, r_{\max}]^{|I_\varepsilon|}$ for a.e. $t \in S$ and $i \in \{1,2\}$. Let $\psi_{\varepsilon,i}$ be defined by*

(32) with $r_\varepsilon = r_{\varepsilon,i}$ for $i\in\{1,2\}$. Then, there exists a constant C independent of ε such that

$$\varepsilon^{-1}\left|\left|\psi_{\varepsilon,1}-\psi_{\varepsilon,2}\right|\right|_{L^\infty(S\times\Omega_\varepsilon)} \leq C\left|\left|r_{\varepsilon,1}-r_{\varepsilon,2}\right|\right|_{L^\infty(S\times\Omega_\varepsilon)}, \tag{39}$$

$$\left|\left|\Psi_{\varepsilon,1}-\Psi_{\varepsilon,2}\right|\right|_{L^\infty(S\times\Omega_\varepsilon)} + \left|\left|J_{\varepsilon,1}-J_{\varepsilon,2}\right|\right|_{L^\infty(S\times\Omega_\varepsilon)} \leq C\left|\left|r_{\varepsilon,1}-r_{\varepsilon,2}\right|\right|_{L^\infty(S\times\Omega_\varepsilon)}, \tag{40}$$

$$\left|\left|\Psi^{-1}_{\varepsilon,1}-\Psi^{-1}_{\varepsilon,2}\right|\right|_{L^\infty(S\times\Omega_\varepsilon)} \leq C\left|\left|r_{\varepsilon,1}-r_{\varepsilon,2}\right|\right|_{L^\infty(S\times\Omega_\varepsilon)}, \tag{41}$$

$$\varepsilon^{-1}\left|\left|\partial_t(\psi_{\varepsilon,1}-\psi_{\varepsilon,2})\right|\right|_{L^p(S\times\Omega_\varepsilon)} + \left|\left|\partial_t(J_{\varepsilon,1}-J_{\varepsilon,2})\right|\right|_{L^p(S\times\Omega_\varepsilon)} \leq C\left|\left|\partial_t(r_{\varepsilon,2}-r_{\varepsilon,1})\right|\right|_{L^p(S\times\Omega_\varepsilon)}. \tag{42}$$

$$\left|\left|\nabla J_{\varepsilon,1}-\nabla J_{\varepsilon,2}\right|\right|_{L^\infty(S\times\Omega_\varepsilon)} \leq C\varepsilon^{-1}\left|\left|r_{\varepsilon,1}-r_{\varepsilon,2}\right|\right|_{L^\infty(S\times\Omega_\varepsilon)}. \tag{43}$$

PROOF. Lemma 2 can be proven by similar computations as in the proof of Lemma 1.

2.3. Transformation of the weak form

Using the diffeomorphism ψ_ε, which is defined in (32), we define $\hat{f}^{\mathrm{p}}_\varepsilon(t,x) := f^{\mathrm{p}}(t,\psi_\varepsilon(t,x))$ and note that Lemma 1 implies the uniform estimate for $\hat{f}^{\mathrm{p}}_\varepsilon$ by

$$\begin{aligned}\left|\left|\hat{f}^{\mathrm{p}}_\varepsilon\right|\right|^2_{S\times\Omega_\varepsilon} &= \int_{S\times\Omega_\varepsilon} f^{\mathrm{p}}(t,\psi_\varepsilon(t,x))^2 dx dt = \int_S\int_{\Omega_\varepsilon(t)} J^{-1}_\varepsilon(t,\psi^{-1}_\varepsilon(t,x)) f^{\mathrm{p}}(t,x)^2 dx dt \\ &\leq c^{-1}_J \int_S\int_{\Omega_\varepsilon(t)} f^{\mathrm{p}}(t,x)^2 dx dt \leq C\left|\left|f^{\mathrm{p}}\right|\right|^2_{S\times\Omega}\end{aligned}$$

We define $A_\varepsilon := J_\varepsilon \Psi^{-1}_\varepsilon D \Psi^{-\top}_\varepsilon$ and $B_\varepsilon := J_\varepsilon\Psi^{-1}_\varepsilon \partial_t\psi_\varepsilon$. Then, we transform the weak form (2),(17)–(19) into the following equivalent weak form:

Find $(\hat{u}_\varepsilon, r_\varepsilon) \in L^2(S;H^1(\Omega_\varepsilon))\times W^{1,\infty}(S)^{|I_\varepsilon|}$ such that $\partial_t(J_\varepsilon u_\varepsilon)\in L^2(S;H^1(\Omega_\varepsilon)')$ and

$$\begin{aligned}&\int_S \langle\partial_t(J_\varepsilon(\tau)\hat{u}_\varepsilon(\tau)),\varphi(\tau)\rangle_{\Omega_\varepsilon} d\tau + (A_\varepsilon\nabla\hat{u}_\varepsilon,\nabla\varphi)_{S\times\Omega_\varepsilon} + (B_\varepsilon\hat{u}_\varepsilon,\nabla\varphi)_{S\times\Omega_\varepsilon} \\ &= (J_\varepsilon\hat{f}^{\mathrm{p}}_\varepsilon,\varphi)_{S\times\Omega_\varepsilon} - \sum_{k\in I_\varepsilon}\tfrac{r^{n-1}_{\varepsilon,k}}{r^{n-1}_0}(\varepsilon f(\hat{u}_\varepsilon, r_{\varepsilon,k}),\varphi)_{S\times\Gamma_{\varepsilon,k}}\end{aligned} \tag{44}$$

$$\int_S \partial_t r_{\varepsilon,k}(t)\phi(t)dt = \int_S \tfrac{\varepsilon^{-N}}{c_s S_{N-1}(r_0)}\int_{\Gamma_{\varepsilon,k}} \varepsilon f(\hat{u}_\varepsilon(t,x), r_{\varepsilon,k}(t))d\sigma_x\phi(t)dt, \tag{45}$$

$$r_\varepsilon(0) = r^{(0)}_\varepsilon, \hat{u}_\varepsilon(0) = \hat{u}^{(0)}_\varepsilon := u^{(0)}_\varepsilon \circ \psi^{-1}_0(0) \tag{46}$$

for $\varphi \in L^2(S;H^1(\Omega_\varepsilon))$, all $k \in I_\varepsilon$ and all $\phi \in L^1(S)^{|I_\varepsilon|}$, where ψ_ε depends on r_ε and is defined by (32) and $\Psi_\varepsilon, J_\varepsilon$ are defined by (33).

Lemma 3. *Let $\psi_\varepsilon, \Psi_\varepsilon, J_\varepsilon$ be given by (32) and (33), respectively. Then, $(u_\varepsilon, r_\varepsilon)$ is a solution of (2),(17)–(19) if and only if $\hat{u}_\varepsilon = u_\varepsilon(\cdot_t, \psi_\varepsilon(\cdot_t, \cdot_x))$ is a solution of (32)–(33),(44)–(46).*

Proof. The proof follows by a simple transformation and the density of $C^1(S \times \Omega_\varepsilon) \subset L^2(S;H^1(\Omega_\varepsilon)')$.

3. Existence and uniform a priori estimates

For the existence proof, we combine a fixed-point argumentation with the theory of monotone operators from [22].

Definition 1 (Monotone operator). Let V be a Banach space. A function $\mathcal{A}: V \to V'$ is *monotone* if $\langle \mathcal{A}(u) - \mathcal{A}(v), u - v \rangle_V \geq 0$ for every $u, v \in V$.

Definition 2 (Family of regular operators). Let W be a separable Hilbert space. A family of operators $\{B(t) | t \in \overline{S}\}$ with $B(t) \in L(W, W')$ for each $t \in \overline{S}$ and $B(\cdot)u(v) \in L^\infty(S)$ for each pair $u, v \in W$ is called *regular* if for each pair $u, v \in W$, the function $B(\cdot)u(v)$ is absolutely continuous on $\overline{S}$ and there is a $K \in L^1(S)$ such that

$$\left|\frac{d}{dt} B(t)u(v)\right| \leq K(t) ||u||_W ||u||_W \tag{47}$$

for every $u, v \in W$ and for a.e. $t \in \overline{S}$.

The monotone operator theory gives the following existence result for degenerate parabolic equations (cf. [22]).

Theorem 4. *Let V be a separable Hilbert space. Suppose that W is a Hilbert space containing V with dense and continuous injection $V \hookrightarrow W$. Let $\mathcal{V} := L^2(S;V)$ and $\mathcal{W} := L^2(S;W)$. We assume that for every $t \in \overline{S}$ there are given operators $\mathcal{A}(t) \in L(V, V')$ and $\mathcal{B}(t) \in L(W, W')$ such that $\mathcal{A}(\cdot)u(v) \in L^\infty(S)$ for each pair $u, v \in V$ and $\mathcal{B}(\cdot)u(v) \in L^\infty(S)$ for each pair $u, v \in W$.*

In addition, we assume that $\{\mathcal{B}(t) | t \in \overline{S}\}$ is a regular family of self-adjoint operators, $\mathcal{B}(0)$ is monotone and there are numbers $\lambda, c > 0$ such that

$$2\mathcal{A}(t)v(v) + \lambda B(t)v(v) + \mathcal{B}'(t)v(v) \geq c||v||_V \quad \text{for all } v \in V \text{ and all } t \in \overline{S}. \tag{48}$$

Then, for given $u^{(0)} \in W$ *and* $f \in L^2(0,T;\mathcal{V}')$ *there exists* $u \in \mathcal{V}$ *such that*

$$\tfrac{d}{dt}(\mathcal{B}(t)u(t)) + \mathcal{A}(t)u(t) = f(t) \text{ in } \mathcal{V}', \text{ with } (\mathcal{B}u)(0) = \mathcal{B}(0)u_0. \quad (49)$$

Combining Theorem 4 with a fixed point argument allows us to derive the existence and uniqueness of the solution of the system (32)–(33), (44)–(46) for ε small enough.

Theorem 5. *There exists a unique solution* $(\hat{u}_\varepsilon, r_\varepsilon) \in L^2(S;H^1(\Omega)) \times W^{1,\infty}(S)^{|I_\varepsilon|}$ *with* $\partial_t(J_\varepsilon \hat{u}_\varepsilon), \partial_t \hat{u}_\varepsilon \in L^2(S;H^1(\Omega_\varepsilon)')$ *of the system* (32)–(33), (44)–(46) *and thus* $\hat{u}_\varepsilon \in C^0(\overline{S};L^2(\Omega_\varepsilon))$. *Moreover, the following uniform estimates hold*

$$\|\hat{u}_\varepsilon\|_{C^0(\overline{S};L^2(\Omega_\varepsilon))} + \|\nabla \hat{u}_\varepsilon\|_{L^2(S\times\Omega_\varepsilon)} \le C, \quad (50)$$

$$\|\hat{u}_\varepsilon\|_{L^\infty(S\times\Omega_\varepsilon)} \le C, \quad (51)$$

$$r_{\varepsilon,k}(t) \in [r_{\min}, r_{\max}]^{|I_\varepsilon|} \text{ for every } t \in \overline{S} \text{ and every } k \in I_\varepsilon, \quad (52)$$

$$\|\partial_t r_{\varepsilon,k}\|_{L^\infty(S)} \le C_f c_s^{-1} \text{ for every } k \in I_\varepsilon. \quad (53)$$

Proof. In order to show the existence and uniqueness of the solution, we divide S in finitely many subintervals $S_i := (t_i, t_{i+1})$ with $0 = t_0 < t_1 < \dots < t_n = T$ for $i \in \{0, \dots, N_\varepsilon\}$ and N_ε large enough. Then, we show iteratively that there exists a unique solution $(\hat{u}_\varepsilon|_{S_i}, r_\varepsilon|_{S_i}) \in L^2(S_i;H^1(\Omega_\varepsilon)) \times W^{1,\infty}(S_i)^{|I_\varepsilon|}$ with $\partial_t(J_\varepsilon \hat{u}_\varepsilon|_{S_i}), \partial_t \hat{u}_\varepsilon|_{S_i} \in L^2(S_i;H^1(\Omega_\varepsilon)')$ such that

$$\int_{t_i}^{t_{i+1}} \langle \partial_t(J_\varepsilon(\tau)\hat{u}_\varepsilon|_{S_i}(\tau)), \varphi(\tau)\rangle_{\Omega_\varepsilon} d\tau + (A_\varepsilon \nabla \hat{u}_\varepsilon|_{S_i}, \nabla\varphi)_{(t_i,t_{i+1})\times\Omega_\varepsilon} + (B_\varepsilon \hat{u}_\varepsilon|_{S_i}, \nabla\varphi)_{(t_i,t_{i+1})\times\Omega_\varepsilon}$$

$$= (J_\varepsilon \hat{f}^{\mathrm{p}}_\varepsilon, \varphi)_{(t_i,t_{i+1})\times\Omega_\varepsilon} - \sum_{k\in I_\varepsilon} \left(\frac{r_{\varepsilon,k}^{n-1}}{r_0^{n-1}} \varepsilon f(\hat{u}_\varepsilon|_{S_i}, r_{\varepsilon,k}), \varphi \right)_{(t_i,t_{i+1})\times\Gamma_{\varepsilon,k}} \quad (54)$$

$$\int_{t_i}^{t_{i+1}} \partial_t r_{\varepsilon,k}(t)\phi(t)dt = \int_{t_i}^{t_{i+1}} \frac{\varepsilon^{-N}}{c_s S_{N-1}(r_0)} \int_{\Gamma_{\varepsilon,k}} \varepsilon f(\hat{u}_\varepsilon|_{S_i}(t,x), r_{\varepsilon,k}(t)) d\sigma_x \phi(t)dt, \quad (55)$$

holds for every $(\varphi, \phi) \in L^2(S_i;H^1(\Omega_\varepsilon)) \times L^2(S_i)^{|I_\varepsilon|}$ and the initial condition $(\hat{u}_\varepsilon|_{S_i}(t_i), r_\varepsilon|_{S_i}(t_i)) = (\hat{u}_\varepsilon^{(t_i)}, r_\varepsilon^{(t_i)})$ is fulfilled. For $i \ge 1$, the initial values are defined by means of the solution on the previous time interval, i.e. $(\hat{u}_\varepsilon^{(t_i)}, r_\varepsilon^{(t_i)}) := (\hat{u}_\varepsilon|_{S_{i-1}}(t_i), r_\varepsilon|_{S_{i-1}}(t_i))$. Then, we get the solution $(\hat{u}_\varepsilon, r_\varepsilon)$ for the whole interval S by concatenating the solutions.

First, we choose t_1 small enough such that we can apply Lemma 6. Then, we get a solution $(\hat{u}_\varepsilon|_{S_0}, r_\varepsilon|_{S_0}) \in L^2(S_0; H^1(\Omega)) \times W^{1,\infty}(S_0)^{|I_\varepsilon|}$ with $\partial_t \hat{u}_\varepsilon|_{S_0} \in L^2(S_0; H^1(\Omega_\varepsilon)')$. Now, we proceed inductively. We assume that we have a unique solution $(\hat{u}_\varepsilon|_{(0,t_i)}, r_\varepsilon|_{(0,t_i)})$ of (32)–(33), (44)–(46) for the time interval $(0, t_i)$ instead of S. Then, we claim that there exists also an unique solution on the time interval $(0, t_{i+1})$ where $t_{i+1} - t_i \geq \sigma_\varepsilon > 0$ for a constant σ_ε which depends neither on the iteration number i nor on the exact time t_i as long as $t_i \leq T$. Hence, we obtain after finitely many steps a solution for the whole interval S. In order to show this uniform bound σ_ε, we use Lemma 6 and note that we have only to show that

$$r_\varepsilon|_{(0,t_i)}(t_i) \in [r_{\min}, r_{\max}]^{|I_\varepsilon|}, \quad (56)$$

$$\left|\left| \hat{u}_\varepsilon|_{(0,t_i)}(t_i) \right|\right|_{\Omega_\varepsilon} \leq K, \quad (57)$$

for a constant K which is independent on the iteration number i and the time $t_i \leq T$. Then, we can construct the solution on (t_i, t_{i+1}) with Lemma 6 and can concatenate it with the solution on $(0, t_i)$. The estimate (56) follows directly from Lemma 6 since $r_\varepsilon|_{(0,t_i)}$ was constructed by Lemma 6. The estimate (57) can be derived like the estimates (80)–(87) but applied on $\hat{u}_\varepsilon|_{(0,t_i)}$. The crucial point is that the constant in (87) does not depend on t as long as $t \leq T$. It depends only on the initial value. Since we do not apply the estimates iteratively on the interval (t_l, t_{l+1}) for $l \in \{0, \ldots, i-1\}$ but only once on the whole interval $(0, t_i)$, we do not have to take care if the initial values multiply in a bad manner. However, we have to note that these estimates only bound $||\hat{u}_\varepsilon||_{L^\infty((0,t_i);L^2(\Omega_\varepsilon))}$ uniformly. In order to get the uniform bound not only for a.e. $t \in (0, t_i)$ but also for t_i, we use the following argument. Since $\partial_t \hat{u}_\varepsilon \in L^2((0, t_i); H^1(\Omega_\varepsilon)')$, the Lemma of Lions-Aubin gives $\hat{u}_\varepsilon \in C(\overline{(0, t_i)}; L^2(\Omega_\varepsilon))$ and since $||\hat{u}_\varepsilon||_{L^\infty((0,t_i);L^2(\Omega_\varepsilon))} = ||\hat{u}_\varepsilon||_{C(\overline{(0,t_i)};L^2(\Omega_\varepsilon))}$, we get the uniform bound for $\left|\left| \hat{u}_\varepsilon|_{(0,t_i)}(t_i) \right|\right|_{\Omega_\varepsilon}$.

Moreover, we note that the estimates (80)–(87) do not depend on ε. In fact, an ε-dependency would not be a problem for the proof of the existence and uniqueness of $\hat{u}_\varepsilon$ on the whole time interval. However, due to their ε-independency, they give us immediately the uniform bound (50) since the initial values $\hat{u}_\varepsilon^{(0)}$ are uniformly bounded. Furthermore, (100) implies directly (51).

Lemma 6. *Let $S_i = (t_i, t_{i+1})$ with $0 \le t_i < t_{i+1} \le T$. Let $\hat{u}_\varepsilon \in L^\infty((0,t_i);L^2(\Omega_\varepsilon)) \cap L^2((0,t_i);H^1(\Omega_\varepsilon))$ with $\partial_t \hat{u}_\varepsilon \in L^2((0,t_i);H^1(\Omega_\varepsilon)')$ be the solution of (44)–(46) on the time interval $(0,t_i)$ for $\varepsilon > 0$ and $\hat{u}_\varepsilon^{(t_i)} := \hat{u}_\varepsilon(t_i)$. Then, for every $K > 0$, there exists a constant $\sigma_{\varepsilon,K} > 0$, which depends only on ε and K, such that (32)–(33), (54)–(55), has a unique solution $(\hat{u}_\varepsilon, r_\varepsilon) \in L^2(S_i;H^1(\Omega_\varepsilon)) \times W^{1,\infty}(S)^{|I_\varepsilon|}$ with $\partial_t(J_\varepsilon \hat{u}_\varepsilon), \partial_t \hat{u}_\varepsilon \in L^2(S;H^1(\Omega_\varepsilon)')$, $\hat{u}_\varepsilon(t_i) = \hat{u}_\varepsilon^{(t_i)}$ and $r_\varepsilon(t_i) = r_\varepsilon^{(t_i)}$ for arbitrary $\hat{u}_\varepsilon^{(t_i)} \in L^2(\Omega_\varepsilon)$ and $r_\varepsilon^{(t_i)} \in [r_{\min}, r_{\max}]^{|I_\varepsilon|}$, if $\left|\left|\hat{u}_\varepsilon^{(t_i)}\right|\right|_{\Omega_\varepsilon} \le K$ and $|S_i| \le \max\{1, \sigma_{\varepsilon,K}\}$. Moreover, $r_\varepsilon(t) \in [r_{\min}, r_{\max}]^{|I_\varepsilon|}$, $|\partial_t r_\varepsilon(t)| \le C_f c_s^{-1}$ for a.e. $t \in S_i$ and $||\hat{u}_\varepsilon||_{L^\infty(S\times\Omega_\varepsilon)} \le C$.*

Proof. We show the existence and uniqueness by means of a fixed-point argument for $\hat{u}_\varepsilon \in L^2(S_i;H^1(\Omega_\varepsilon))$ with the fixed-point operator $L_\varepsilon : L^2(S_i;H^1(\Omega_\varepsilon)) \to L^2(S_i;H^1(\Omega_\varepsilon))$. First, L_ε inserts a given function ζ into the right-hand side of (54)–(55), which yields

$$\int_{t_i}^{t_{i+1}} \langle \partial_t(J_\varepsilon(t)\hat{u}_\varepsilon(t)), \varphi(t)\rangle_{\Omega_\varepsilon} dt + (A_\varepsilon \nabla \hat{u}_\varepsilon, \nabla\varphi)_{(t_i,t_{i+1})\times\Omega_\varepsilon} + (B_\varepsilon \hat{u}_\varepsilon, \nabla\varphi)_{(t_i,t_{i+1})\times\Omega_\varepsilon}$$

$$= (J_\varepsilon \hat{f}_\varepsilon^{\mathrm{p}}, \varphi)_{(t_i,t_{i+1})\times\Omega_\varepsilon} - \sum_{k\in I_\varepsilon} \left(\frac{r_{\varepsilon,k}^{n-1}}{r_0^{n-1}} \varepsilon f(\zeta, r_{\varepsilon,k}), \varphi \right)_{(t_i,t_{i+1})\times\Gamma_{\varepsilon,k}}, \quad (58)$$

$$\int_{t_i}^{t_{i+1}} \partial_t r_{\varepsilon,k}(t)\phi(t)dt = \int_{t_i}^{t_{i+1}} \frac{\varepsilon^{-N}}{c_s S_{N-1}(r_0)} \int_{\Gamma_{\varepsilon,k}} \varepsilon f(\zeta(t,x), r_{\varepsilon,k}(t)) d\sigma_x \phi(t) dt. \quad (59)$$

Then, it solves (59) for r_ε. This r_ε gives $\psi_\varepsilon, \Psi_\varepsilon, J_\varepsilon$ via (32)–(33) for (58). Then, $L_\varepsilon(\zeta) := \hat{u}_\varepsilon$ where $\hat{u}_\varepsilon$ is the solution of (58).

In order to show that L_ε is well defined and is a contraction, we rewrite $L_\varepsilon(\hat{u}_\varepsilon)$ by means of the following both operators. Let $V_{r,\varepsilon}(S_i) := \{r \in W^{1,2}(S_i)^{|I_\varepsilon|} \; |r(t) \in [r_{\min}, r_{\max}]^{|I_\varepsilon|}$ and $|\partial_t r(t)| \le C_f c_s^{-1}$ for a.e. $t \in S_i\}$ We define $L_{\varepsilon,1} : L^2(S_i;H^1(\Omega_\varepsilon)) \to V_{r,\varepsilon}(S_i)$ as the solution operator of (59), i.e. $L_{\varepsilon,1}(\zeta) := r_\varepsilon$, where $r_\varepsilon \in V_{r,\varepsilon}(S_i)$ is the solution of (58) for every $k \in I_\varepsilon$ and every $\phi \in L^2(S_i)$ with initial condition $r_\varepsilon(t_i) = r_\varepsilon^{(t_i)}$. Moreover, we define $L_{\varepsilon,2} : L^2(S_i;H^1(\Omega_\varepsilon)) \times V_{r,\varepsilon}(S_i) \to L^2(S_i;H^1(\Omega_\varepsilon))$ by $L_{\varepsilon,2}(\zeta_\varepsilon, r_\varepsilon) := \hat{u}_\varepsilon$, where $\hat{u}_\varepsilon$ is the solution of (58) for every $\varphi \in L^2(S;H^1(\Omega_\varepsilon))$ with initial condition $\hat{u}_\varepsilon(t_i) = \hat{u}_\varepsilon^{(t_i)}$. Hence, we get $L_\varepsilon(\zeta) = L_{\varepsilon,2}(\zeta, L_{\varepsilon,1}(\zeta))$.

Note, that $\hat{u}_\varepsilon$ is a fixed point of L_ε with $\partial_t(J_\varepsilon \hat{u}_\varepsilon), \partial_t \hat{u}_\varepsilon \in L^2(S;H^1(\Omega_\varepsilon)')$ and $r_\varepsilon = L_{\varepsilon,i}\hat{u}_\varepsilon$ with $\partial_t r_\varepsilon \in L^\infty(S_i)^{|I_\varepsilon|}$ if and only if $(\hat{u}_\varepsilon, r_\varepsilon)$ solves (54)–(55).

Hence, it is sufficient to show, that L_ε has a unique fixed point. First, we show, that $L_{\varepsilon,1}$ is well defined and Lipschitz continuous. Then, we do the same for $L_{\varepsilon,2}$. Thereby, we show that the Lipschitz constants of $L_{\varepsilon,1}$ and $L_{\varepsilon,2}$ tend to zero for $|S_i|\to 0$. Thus, we obtain that L_ε is a contraction for $|S_i|$ small enough and the contraction theorem gives the existence and uniqueness of a fixed point of L_ε.

• $L_{\varepsilon,1}$ *is well defined.* Since $(t,r)\mapsto \frac{\varepsilon^{-N}}{c_s S_{N-1}(r_0)}\int_{\Gamma_{\varepsilon,k}} \varepsilon f(\zeta(t,x),r)d\sigma_x$ is globally Lipschitz continuous with respect to r and measurable with respect to t if $\zeta\in L^2(S;H^1(\Omega_\varepsilon))$, Carathéodory's existence theorem yields the existence and uniqueness of a solution $r_\varepsilon\in W^{1,1}(S_i)^{|I_\varepsilon|}$ of (55). Moreover, the Assumption (3)–(4) ensure that $r_\varepsilon(t)\in[r_{\min},r_{\max}]^{|I_\varepsilon|}$ for a.e. $t\in S_i$ and (6) that $\left|\frac{\varepsilon^{-N}}{c_s S_{N-1}(r_0)}\int_{\Gamma_{\varepsilon,k}} \varepsilon f(\zeta(t,x),r)d\sigma_x\right|\leq C_f c_s^{-1}$. Thus $L_{\varepsilon,1}$ is well defined with $L_{\varepsilon,1}(\zeta)=r_\varepsilon\in V_{r,\varepsilon}(S_i)$.

• *Lipschitz estimate of* $L_{\varepsilon,1}$. Let $\zeta_1,\zeta_2\in L^2(S;H^1(\Omega_\varepsilon))$. We define $r_{\varepsilon,i}:= L_{\varepsilon,1}(\zeta_i)$ for $i\in\{1,2\}$ and test (59) for $\zeta=\zeta_i$ for $i\in\{1,2\}$ with $\chi_{(t_i,t)}(r_{\varepsilon,1,k}-r_{\varepsilon,2,k})$ for $t\in(t_i,t_{i+1})$. We subtract both equations. Then, we obtain with the Lipschitz condition (5) of f, the Young and the Cauchy–Schwarz inequalities

$$
\begin{aligned}
&\tfrac{1}{2}|r_{\varepsilon,1,k}(t)-r_{\varepsilon,2,k}(t)|^2 = (\partial_t(r_{\varepsilon,1,k}-r_{\varepsilon,2,k}),r_{\varepsilon,1,k}-r_{\varepsilon,2,k})_{(t_i,t)}\\
&=\tfrac{\varepsilon^{-N}}{c_s S_{N-1}(r_0)}\varepsilon(f(\zeta_1,r_{\varepsilon,1,k})-f(\zeta_2,r_{\varepsilon,2,k}),r_{\varepsilon,1,k}-r_{\varepsilon,2,k})_{(t_i,t)\times\Gamma_{\varepsilon,k}}\\
&\leq C\varepsilon^{-N+1}\int\limits_{(t_i,t)\times\Gamma_{\varepsilon,k}} C_{L_f}(|\zeta_1(\tau,x)-\zeta_2(\tau,x)|+|r_{\varepsilon,1,k}(\tau)-r_{\varepsilon,2,k}(\tau)|)(r_{\varepsilon,1,k}(\tau)-r_{\varepsilon,2,k}(\tau))d\sigma_x dt\\
&\leq C\varepsilon^{-N+1}||\zeta_1-\zeta_2||^2_{(t_i,t)\times\Gamma_{\varepsilon,k}}+C||r_{\varepsilon,1,k}-r_{\varepsilon,2,k}||^2_{(t_i,t)}
\end{aligned}
$$

After collecting all the constants and applying Gronwall's inequality, we get

$$|r_{\varepsilon,1,k}(t)-r_{\varepsilon,2,k}(t)|^2\leq C\varepsilon^{-N+1}||\zeta_1-\zeta_2||^2_{S_i\times\Gamma_\varepsilon} \tag{60}$$

for every $t\in S_i$, which implies with the ε-scaled trace inequality

$$||r_{\varepsilon,1,k}-r_{\varepsilon,2,k}||^2_{L^\infty((t_i,t))}\leq C\varepsilon^{-N}||\zeta_1-\zeta_2||^2_{S_i\times(\varepsilon k+\varepsilon Y^*)}+C\varepsilon^{-N+2}||\nabla(\zeta_1-\zeta_2)||^2_{S_i\times(\varepsilon k+\varepsilon Y^*)} \tag{61}$$

After multiplication by ε^N and summing over $k\in I_\varepsilon$, we get

$$
\begin{aligned}
||r_{\varepsilon,1}-r_{\varepsilon,2}||^2_{L^\infty((t_i,t);L^2(\Omega_\varepsilon))}&=C\varepsilon^N\sum_{k\in I_\varepsilon}||r_{\varepsilon,1,k}-r_{\varepsilon,2,k}||^2_{L^\infty((t_i,t))}\\
&\leq C||\zeta_1-\zeta_2||^2_{L^2(S_i;H^1(\Omega_\varepsilon))}. \qquad (62)
\end{aligned}
$$

Moreover, (61) gives an estimate of $r_{\varepsilon,1} - r_{\varepsilon,2}$ in the L^∞-norm with respect to space, but at the cost of an ε-dependency in the constant:

$$||r_{\varepsilon,1} - r_{\varepsilon,2}||_{L^\infty((t_i,t)\times\Omega_\varepsilon)} \leq C_\varepsilon ||\zeta_1 - \zeta_2||_{L^2(S_i;H^1(\Omega_\varepsilon))}. \quad (63)$$

Then, we test (59) for $\zeta = \zeta_i$ for $i \in \{1,2\}$ with $\partial_t r_{\varepsilon,1,k} - \partial_t r_{\varepsilon,2,k}$ and use again the Lipschitz condition (5):

$$\begin{aligned}
&||\partial_t r_{\varepsilon,1,k} - \partial_t r_{\varepsilon,2,k}||^2_{S_i} \\
&= \frac{\varepsilon^{-N}}{c_s S_{N-1}(r_0)} \varepsilon (f(\zeta_1, r_{\varepsilon,1,k}) - f(\zeta_2, r_{\varepsilon,2,k}), \partial_t r_{\varepsilon,1,k} - \partial_t r_{\varepsilon,2,k})_{S_i\times\Gamma_{\varepsilon,k}} \\
&\leq \varepsilon^{-N+1} C(||\zeta_1 - \zeta_2||_{S_i\times\Gamma_{\varepsilon,k}} + \varepsilon^{(N-1)/2} ||r_{\varepsilon,1,k} - r_{\varepsilon,2,k}||_{S_i}) \varepsilon^{(N-1)/2} ||\partial_t r_{\varepsilon,1,k} - \partial_t r_{\varepsilon,2,k}||_{S_i} \\
&\leq C(\varepsilon^{(-N+1)/2} ||\zeta_1 - \zeta_2||_{S_i\times\Gamma_{\varepsilon,k}} + C||r_{\varepsilon,1,k} - r_{\varepsilon,2,k}||_{S_i}) ||\partial_t r_{\varepsilon,1,k} - \partial_t r_{\varepsilon,2,k}||_{S_i}.
\end{aligned} \quad (64)$$

Inserting (60) in (64) and employing the continuity of the trace operator for $\Gamma_{\varepsilon,k}$ yields

$$\begin{aligned}
\varepsilon^{N/2} ||\partial_t r_{\varepsilon,1,k} - \partial_t r_{\varepsilon,2,k}||_{S_i} &\leq C\varepsilon^{1/2} ||\zeta_1 - \zeta_2||_{S_i\times\Gamma_{\varepsilon,k}} + \varepsilon^{N/2} C||r_{\varepsilon,1,k} - r_{\varepsilon,2,k}||_{S_i} \\
&\leq C(1 + \sqrt{|S_i|}) \varepsilon^{1/2} ||\zeta_1 - \zeta_2||_{S_i\times\Gamma_{\varepsilon,k}} \leq C\varepsilon^{1/2} ||\zeta_1 - \zeta_2||_{S_i\times\Gamma_{\varepsilon,k}}. \quad (65)
\end{aligned}$$

After summing over $k \in I_\varepsilon$ and applying the ε-scaled trace inequality, we get

$$\begin{aligned}
||\partial_t r_{\varepsilon,1} - \partial_t r_{\varepsilon,2}||^2_{S_i\times\Omega_\varepsilon} &= \varepsilon^N C \sum_{k\in I_\varepsilon} ||\partial_t r_{\varepsilon,1,k} - \partial_t r_{\varepsilon,2,k}||^2_{S_i} \\
&\leq \varepsilon C||\zeta_1 - \zeta_2||^2_{S_i\times\Gamma_{\varepsilon,k}} \leq C||\zeta_1 - \zeta_2||^2_{L^2(S_i;H^1(\Omega_\varepsilon))}. \quad (66)
\end{aligned}$$

Furthermore, we can conclude with the fundamental theorem of calculus and the Hölder inequality for every $t \in S_i$:

$$\begin{aligned}
|r_{\varepsilon,1,k}(t) - r_{\varepsilon,2,k}(t)| &= \int_{t_i}^{t} \partial_t (r_{\varepsilon,1,k} - r_{\varepsilon,2,k})(\tau) d\tau \leq ||1||_{S_i} ||\partial_t r_{\varepsilon,1,k} - \partial_t r_{\varepsilon,2,k}||_{S_i} \\
&\leq \sqrt{|S_i|} C_\varepsilon ||\zeta_1 - \zeta_2||_{L^2(S_i;H^1(\Omega_\varepsilon))}.
\end{aligned}$$

Thus,

$$||r_{\varepsilon,1} - r_{\varepsilon,2}||_{L^\infty(S_i\times\Omega_\varepsilon)} \leq \sqrt{|S_i|} C_\varepsilon ||\zeta_1 - \zeta_2||_{L^2(S_i;H^1(\Omega_\varepsilon))}. \quad (67)$$

Moreover, (65) gives with the trace inequality

$$||\partial_t r_{\varepsilon,1} - \partial_t r_{\varepsilon,2}||_{L^\infty(\Omega_\varepsilon;L^2(S_i))} \leq C_\varepsilon ||\zeta_1 - \zeta_2||_{L^2(S_i;H^1(\Omega_\varepsilon))}. \quad (68)$$

• $L_{\varepsilon,2}$ *is well defined.* First, we show the existence of a solution $\hat{u}_\varepsilon \in L^2(S_i;H^1(\Omega_\varepsilon))$ with $\partial_t(J_\varepsilon \hat{u}_\varepsilon) \in L^2(S_i;H^1(\Omega_\varepsilon)')$ of (58) using Theorem 4. With the regularity of J_ε, we can conclude $\partial_t \hat{u}_\varepsilon \in L^2(S_i;H^1(\Omega_\varepsilon)')$. Afterwards we test (58) with $\hat{u}_\varepsilon$ which shows the uniqueness of the solution of (58) and thus that $\hat{u}_\varepsilon = L_{\varepsilon,2}(\zeta, r_\varepsilon)$ is well defined for every $\zeta \in L^2(S_i;H^1(\Omega_\varepsilon))$.

Using the setting of Theorem 4, we set $V = H^1(\Omega_\varepsilon)$ and $W = L^2(\Omega_\varepsilon)$. Let $\psi_\varepsilon, \Psi_\varepsilon$ and J_ε be given by (32)–(33). For each $t \in [t_i, t_{i+1}]$ and $u, v \in V$, we define $\mathcal{A}_\varepsilon(t) : V \to V'$ by $(\mathcal{A}_\varepsilon(t)u)(v) := (A_\varepsilon(t)\nabla u, \nabla v)_{\Omega_\varepsilon} + (B_\varepsilon(t)u, \nabla v)_{\Omega_\varepsilon}$. For each $t \in [t_i, t_{i+1}]$ and $u, v \in W$, we define $\mathcal{B}_\varepsilon(t) : W \to W'$ by $(\mathcal{B}_\varepsilon(t)u)(v) := (J_\varepsilon(t)u, v)_{\Omega_\varepsilon}$. For $\zeta, v \in \mathcal{V}$, we define $f_\varepsilon(\zeta;\cdot) : \mathcal{V} \to \mathbb{R}$ by

$$f_\varepsilon(\zeta;v) := (J_\varepsilon \hat{f}_\varepsilon^{\mathrm{p}}, v)_{S_i \times \Omega_\varepsilon} - \sum_{k \in I_\varepsilon} \left(\frac{r_{\varepsilon,k}^{n-1}}{r_0^{n-1}} \varepsilon f(\zeta, r_{\varepsilon,k}), v \right)_{S_i \times \Gamma_{\varepsilon,k}}.$$

In order to apply Theorem 4, we verify its assumption in the following. The Lipschitz regularity of f and the continuous embedding $H^1(\Omega_\varepsilon) \hookrightarrow L^2(\Gamma_{\varepsilon,k})$ ensure that $f_\varepsilon(\zeta;\cdot) \in \mathcal{V}'$ for every $\zeta \in L^2(S;H^1(\Omega_\varepsilon))$. Moreover, it is clear that $\mathcal{A}_\varepsilon(t) \in L(V, V')$, $\mathcal{B}_\varepsilon(t) \in L(W, W')$ for every $t \in [t_i, t_{i+1}]$. Since $r_\varepsilon \in V_{r,\varepsilon}(S_i)$, we can conclude with Lemma 1 that $\mathcal{A}_\varepsilon(\cdot)u(v) \in L^\infty(S)$ for every pair $u, v \in V$ and $\mathcal{B}_\varepsilon(\cdot)u(v) \in L^\infty(S)$ for every pair $u, v \in W$. Furthermore, it is clear that $\{\mathcal{B}_\varepsilon(t) | t \in [t_i, t_{i+1}]\}$ is a family of self-adjoint operators. From Lemma 1, we get the time regularity of J_ε which can be transferred on $\mathcal{B}_\varepsilon$ so that $\{\mathcal{B}_\varepsilon(t) | t \in [t_i, t_{i+1}]\}$ is a family of regular operators. Using the uniform boundedness of J_ε from below given by Lemma 1, we get that $\mathcal{B}(0)$ is monotone. It remains to show the estimate (48). Using the coercivity of $J_\varepsilon \Psi_\varepsilon^{-1} \Psi_\varepsilon^{-\top}$ given by Lemma 1, we obtain for every $v \in H^1(\Omega_\varepsilon)$ and every $t \in \overline{S}$

$$(A_\varepsilon(t)\nabla v \nabla v)_{\Omega_\varepsilon} \geq \alpha \|\nabla v\|_{\Omega_\varepsilon}^2. \quad (69)$$

Using the estimates on Ψ_ε, J_ε and $\partial_t \psi_\varepsilon$ of Lemma 1 as well as the Hölder and Young inequalities, we get for every $\delta > 0$ a constant C_δ such that for every $v \in H^1(\Omega_\varepsilon)$ and every $t \in \overline{S}$

$$-(B_\varepsilon(t)v, \nabla v)_{\Omega_\varepsilon} \leq C\|v\|_{\Omega_\varepsilon} \|\nabla v\|_{\Omega_\varepsilon} \leq C_\delta \|v\|_{\Omega_\varepsilon}^2 + \delta \|\nabla v\|_{\Omega_\varepsilon}^2 \qquad (70)$$

Combing (69)–(70) with the definition of $\mathcal{A}_\varepsilon(t)$ yields for $\delta = \alpha/2$

$$\mathcal{A}_\varepsilon(t)v(v) = (A_\varepsilon(t)\nabla v, \nabla v)_{\Omega_\varepsilon} + (B_\varepsilon(t)v, \nabla v)_{\Omega_\varepsilon} \geq \alpha/2 \|\nabla v\|_{\Omega_\varepsilon}^2 - C_{\alpha/2} \|v\|_{\Omega_\varepsilon}^2 \quad (71)$$

The estimate on J_ε from below implies

$$\mathcal{B}_\varepsilon(t)v(v) \geq c_J ||v||^2_{\Omega_\varepsilon} \tag{72}$$

and the boundedness of $||\partial_t r_\varepsilon||_{L^\infty(S_i)} \leq C$ together with Lemma 1 gives

$$-\mathcal{B}'(t)v(v) = (\partial_t J_\varepsilon(t)v, v)_{\Omega_\varepsilon} \leq C||v||^2_{\Omega_\varepsilon} \tag{73}$$

Thus, we get

$$\lambda\mathcal{B}_\varepsilon(t)v(v) + \mathcal{B}'(t)v(v) \geq (\lambda c_J - C)||v||^2_{\Omega_\varepsilon}. \tag{74}$$

Combining (71)–(74) for $\lambda = (\alpha/2 + C - C_{\alpha/2})/c_J$ gives (48). Thus, we have shown that all prerequisites of Theorem 4 are fulfilled and we get a solution $\hat{u}_\varepsilon \in L^2(S_i; H^1(\Omega_\varepsilon))$ with $\partial_t (J_\varepsilon \hat{u}_\varepsilon) \in L^2(S_i; H^1(\Omega_\varepsilon)')$. Then, the regularity of J_ε implies that $\partial_t \hat{u}_\varepsilon = \langle \partial_t (J_\varepsilon \hat{u}_\varepsilon), J_\varepsilon^{-1} \cdot \rangle_{\Omega_\varepsilon} - (J_\varepsilon^{-1} \partial_t J_\varepsilon \hat{u}_\varepsilon, \cdot)_{\Omega_\varepsilon} \in L^2(S_i; H^1(\Omega_\varepsilon)')$.

In order to show that $L_{\varepsilon,2}$ is well defined, it remains to show the uniqueness of the solution of (58). Due to the linearity of the equation (58), it is sufficient to show that $\hat{u}_\varepsilon = 0$, if $\hat{u}_\varepsilon^{(t_i)} = 0$, $\hat{f}_\varepsilon^{\mathrm{p}} = 0$ and $f = 0$. Therefore, we test (58) with the solution $\chi_{(t_i,t)} \hat{u}_\varepsilon$ for $t \in S_i$, which yields

$$\int_{t_i}^{t} \langle \partial_t (J_\varepsilon(\tau) \hat{u}_\varepsilon(\tau)), \hat{u}_\varepsilon(\tau) \rangle_{\Omega_\varepsilon} d\tau + (A_\varepsilon \nabla \hat{u}_\varepsilon, \nabla \hat{u}_\varepsilon)_{(t_i,t)\times\Omega_\varepsilon} + (B_\varepsilon \hat{u}_\varepsilon, \nabla \hat{u}_\varepsilon)_{(t_i,t)\times\Omega_\varepsilon} = 0, \tag{75}$$

We note that the left-hand side of (75) can be rewritten to

$$\int_{t_i}^{t} \langle \partial_t (J_\varepsilon(\tau) \hat{u}_\varepsilon(\tau)), \hat{u}_\varepsilon(\tau) \rangle_{\Omega_\varepsilon} d\tau = \frac{1}{2} \left|\left| \sqrt{J_\varepsilon(t)} \hat{u}_\varepsilon(t) \right|\right|^2_{\Omega_\varepsilon} + \frac{1}{2} (\partial_t J_\varepsilon \hat{u}_\varepsilon, \hat{u}_\varepsilon)_{(t_i,t)\times\Omega_\varepsilon}, \tag{76}$$

thus (75) becomes

$$\begin{aligned} &\frac{1}{2} \left|\left| \sqrt{J_\varepsilon(t)} \hat{u}_\varepsilon(t) \right|\right|^2_{\Omega_\varepsilon} + (A_\varepsilon \nabla \hat{u}_\varepsilon, \nabla \hat{u}_\varepsilon)_{(t_i,t)\times\Omega_\varepsilon} \\ &= -(B_\varepsilon \hat{u}_\varepsilon, \nabla \hat{u}_\varepsilon)_{(t_i,t)\times\Omega_\varepsilon} - \frac{1}{2} (\partial_t J_\varepsilon \hat{u}_\varepsilon, \hat{u}_\varepsilon)_{(t_i,t)\times\Omega_\varepsilon}. \end{aligned} \tag{77}$$

Using the uniform boundedness from below of J_ε and the coercivity of A_ε given by Lemma 1, we can estimate the left-hand side of (77) by

$$\tfrac{1}{2}c_J\,\|\hat u_\varepsilon(t)\|^2_{\Omega_\varepsilon}+\alpha\|\nabla\hat u_\varepsilon\|^2_{(t_i,t)\times\Omega_\varepsilon}\le\tfrac{1}{2}\left\|\sqrt{J_\varepsilon(t)}\,\hat u_\varepsilon(t)\right\|^2_{\Omega_\varepsilon}+(A_\varepsilon\nabla\hat u_\varepsilon,\nabla\hat u_\varepsilon)_{(t_i,t)\times\Omega_\varepsilon}. \tag{78}$$

The right-hand side of (77) can be estimated with the Cauchy–Schwarz and Young inequalities for arbitrary $\delta>0$ and a constant C_δ by

$$\begin{aligned}&-(B_\varepsilon\hat u_\varepsilon,\nabla\hat u_\varepsilon)_{(t_i,t)\times\Omega_\varepsilon}-\tfrac{1}{2}(\partial_t J_\varepsilon\hat u_\varepsilon,\hat u_\varepsilon)_{(t_i,t)\times\Omega_\varepsilon}\\&\le\delta\|\nabla\hat u_\varepsilon\|^2_{(t_i,t)\times\Omega}+C_\delta\|\hat u_\varepsilon\|^2_{(t_i,t)\times\Omega}+C\|\hat u_\varepsilon\|^2_{(t_i,t)\times\Omega}\end{aligned} \tag{79}$$

After combining (77)–(79) and collecting all the constants, we get for $\delta=\alpha/2$

$$\frac{1}{2}c_J\,\|u_\varepsilon(t)\|^2_{\Omega_\varepsilon}+(\alpha-\alpha/2)\|\nabla\hat u_\varepsilon\|^2_{(t_i,t)\times\Omega_\varepsilon}\le(C_{\alpha/2}+C)\|\hat u_\varepsilon\|^2_{(t_i,t)\times\Omega_\varepsilon}$$

Then, Gronwall's inequality shows $\hat u_\varepsilon=0$ which gives the uniqueness of $\hat u_\varepsilon$ and thus $L_{\varepsilon,2}$ is well defined.

• *Uniform bound of $L_{\varepsilon,2}(\zeta,r_\varepsilon)$.* In order to derive a uniform bound for $\hat u_\varepsilon$, we test (58) with $\chi_{(t_i,t)}\hat u_\varepsilon$ for a.e. $t\in S_i$, which gives

$$\begin{aligned}&\int_{t_i}^{t}\langle\partial_t(J_\varepsilon(\tau)\hat u_\varepsilon(\tau)),\hat u_\varepsilon(\tau)\rangle_{\Omega_\varepsilon}d\tau+(A_\varepsilon\nabla\hat u_\varepsilon,\nabla\hat u_\varepsilon)_{(t_i,t)\times\Omega_\varepsilon}+(B_\varepsilon\hat u_\varepsilon,\nabla\hat u_\varepsilon)_{(t_i,t)\times\Omega_\varepsilon}\\&\qquad=(J_\varepsilon\hat f^{\mathrm{p}}_\varepsilon,\hat u_\varepsilon)_{(t_i,t)\times\Omega_\varepsilon}-\sum_{k\in I_\varepsilon}\left(\frac{r^{n-1}_{\varepsilon,k}}{r^{n-1}_0}\varepsilon f_\varepsilon(\zeta,r_{\varepsilon,k}),\hat u_\varepsilon\right)_{(t_i,t)\times\Gamma_{\varepsilon,k}}.\end{aligned} \tag{80}$$

We rewrite the first term of (80), similar to (76), by

$$\begin{aligned}&\int_{t_i}^{t}\langle\partial_t(J_\varepsilon(\tau)\hat u_\varepsilon(\tau)),\hat u_\varepsilon(\tau)\rangle_{\Omega_\varepsilon}d\tau\\&=\tfrac{1}{2}\left\|\sqrt{J_\varepsilon(t)}\,\hat u_\varepsilon(t)\right\|^2_{\Omega_\varepsilon}-\tfrac{1}{2}\left\|\sqrt{J_\varepsilon(t_i)}\,\hat u_\varepsilon(t_i)\right\|^2_{\Omega_\varepsilon}+\tfrac{1}{2}(\partial_t J_\varepsilon\hat u_\varepsilon,\hat u_\varepsilon)_{(t_i,t)\times\Omega_\varepsilon}.\end{aligned}$$

Thus, (80) can be rewritten into

$$
\begin{aligned}
&\tfrac{1}{2}\left|\left|\sqrt{J_\varepsilon(t)}\,\hat{u}_\varepsilon(t)\right|\right|^2_{\Omega_\varepsilon} + (A_\varepsilon \nabla \hat{u}_\varepsilon, \nabla \hat{u}_\varepsilon)_{(t_i,t)\times\Omega_\varepsilon} = (J_\varepsilon \hat{f}^{\mathrm{p}}_\varepsilon, \hat{u}_\varepsilon)_{(t_i,t)\times\Omega_\varepsilon} \\
&- \sum_{k\in I_\varepsilon} \left(\frac{r^{n-1}_{\varepsilon,k}}{r_0^{n-1}} \varepsilon f(\zeta, r_{\varepsilon,k}), \hat{u}_\varepsilon \right)_{(t_i,t)\times\Gamma_{\varepsilon,k}} - (B_\varepsilon \hat{u}_\varepsilon, \nabla \hat{u}_\varepsilon)_{(t_i,t)\times\Omega_\varepsilon} \\
&- \tfrac{1}{2}(\partial_t J_\varepsilon \hat{u}_\varepsilon, \hat{u}_\varepsilon)_{(t_i,t)\times\Omega_\varepsilon} + \tfrac{1}{2}\left|\left|\sqrt{J_\varepsilon(t_i)}\,\hat{u}^{(t_i)}_\varepsilon\right|\right|^2_{\Omega_\varepsilon}. \quad (81)
\end{aligned}
$$

Thefirst two terms of the right-hand side of (81) can be estimated with the Cauchy–Schwarz and Young inequalities and theε-scaled trace operator (117) by

$$
\begin{aligned}
&(J_\varepsilon \hat{f}^{\mathrm{p}}_\varepsilon, \hat{u}_\varepsilon)_{(t_i,t)\times\Omega_\varepsilon} - \sum_{k\in I_\varepsilon} \left(\frac{r^{n-1}_{\varepsilon,k}}{r_0^{n-1}} \varepsilon f(\zeta, r_{\varepsilon,k}), \hat{u}_\varepsilon \right)_{(t_i,t)\times\Gamma_{\varepsilon,k}} \\
&\le C \left|\left|\hat{f}^{\mathrm{p}}_\varepsilon\right|\right|^2_{(t_i,t)\times\Omega_\varepsilon} + ||\hat{u}_\varepsilon||^2_{(t_i,t)\times\Omega_\varepsilon} + C\varepsilon ||f_{\max}||^2_{(t_i,t)\times\Gamma_\varepsilon} + \varepsilon ||\hat{u}_\varepsilon||^2_{(t_i,t)\times\Gamma_\varepsilon} \\
&\le C + ||\hat{u}_\varepsilon||^2_{(t_i,t)\times\Omega_\varepsilon} + C f^2_{\max} \varepsilon |S_i||\Gamma_\varepsilon| + C_\delta ||\hat{u}_\varepsilon||^2_{(t_i,t)\times\Omega_\varepsilon} + \delta\varepsilon ||\nabla \hat{u}_\varepsilon||^2_{(t_i,t)\times\Omega_\varepsilon} \\
&\le C + C_\delta ||\hat{u}_\varepsilon||^2_{(t_i,t)\times\Omega_\varepsilon} + \delta ||\nabla \hat{u}_\varepsilon||^2_{(t_i,t)\times\Omega_\varepsilon} \qquad (82)
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
&-(B_\varepsilon \hat{u}_\varepsilon, \nabla \hat{u}_\varepsilon)_{(t_i,t)\times\Omega_\varepsilon} \le C_\delta ||\hat{u}_\varepsilon||^2_{(t_i,t)\times\Omega_\varepsilon} + \delta ||\nabla \hat{u}_\varepsilon||^2_{(t_i,t)\times\Omega_\varepsilon}, \quad (83) \\
&- \tfrac{1}{2}(\partial_t J_\varepsilon \hat{u}_\varepsilon, \hat{u}_\varepsilon)_{(t_i,t)\times\Omega_\varepsilon} \le C ||\hat{u}_\varepsilon||^2_{(t_i,t)\times\Omega_\varepsilon}, \quad (84) \\
&\left|\left|\sqrt{J_\varepsilon(t_i)}\,\hat{u}^{(t_i)}_\varepsilon\right|\right|^2_{\Omega_\varepsilon} \le C \left|\left|\hat{u}^{(t_i)}_\varepsilon\right|\right|^2_{\Omega_\varepsilon} \le C_K. \qquad (85)
\end{aligned}
$$

Combining the estimates (78), (82)–(85) with (81) yields forδsmall enough and after collecting all the constants

$$
||\hat{u}_\varepsilon(t)||^2_{\Omega_\varepsilon} + ||\nabla \hat{u}_\varepsilon||^2_{(t_i,t)\times\Omega_\varepsilon} \le C_K + C ||\hat{u}_\varepsilon||^2_{(t_i,t)\times\Omega_\varepsilon}. \quad (86)
$$

Then, Gronwall's inequality implies

$$
||\hat{u}_\varepsilon(t)||^2_{\Omega_\varepsilon} + ||\nabla \hat{u}_\varepsilon||^2_{S_i\times\Omega_\varepsilon} \le C_K \qquad (87)
$$

for a.e.$t \in S_i$.

By employing (87), we get from (58)

$$||\partial_t(J_\varepsilon \hat{u}_\varepsilon)||_{L^2(S;H^1(\Omega_\varepsilon)')} \leq C_K. \quad (88)$$

Moreover, we get with $\partial_t \hat{u}_\varepsilon = \langle \partial_t(J_\varepsilon \hat{u}_\varepsilon), J^{-1}\cdot\rangle_{\Omega_\varepsilon} - (\partial_t J_\varepsilon \hat{u}_\varepsilon, J_\varepsilon^{-1}\cdot)_{\Omega_\varepsilon}$

$$\begin{aligned}||\partial_t \hat{u}_\varepsilon||_{L^2(S;H^1(\Omega_\varepsilon)')} &\leq ||\partial_t(J_\varepsilon \hat{u}_\varepsilon)||_{L^2(S;H^1(\Omega_\varepsilon)')} \left|\left|J_\varepsilon^{-1}\right|\right|_{W^{1,\infty}(S\times\Omega_\varepsilon)} \\ &+ ||\partial_t J_\varepsilon||_{L^\infty(S\times\Omega_\varepsilon)} ||\hat{u}_\varepsilon||_{S\times\Omega_\varepsilon} \left|\left|J_\varepsilon^{-1}\right|\right|_{L^\infty(S\times\Omega_\varepsilon)} \leq C_K C\varepsilon^{-1} + C_K \leq C_{K,\varepsilon}. \quad (89)\end{aligned}$$

• *L^∞-estimate of $L_{\varepsilon,2}(\zeta, r_\varepsilon)$.* Let $\hat{u}_\varepsilon|_{S_i} := L_{\varepsilon,2}(\zeta, r_\varepsilon)$ for $r_\varepsilon \in V_{r,\varepsilon}((0, t_{i+1}))$. Then, $\hat{u}_\varepsilon$ can be extended to a solution of (44) on the time interval $(0, t_{i+1})$. We define

$$\hat{u}_\varepsilon^{(k)}(t,x) := \begin{cases} \hat{u}_\varepsilon(t,x) - k & \text{if } \hat{u}_\varepsilon(t,x) \geq 0, \\ 0 & \text{if } \hat{u}_\varepsilon(t,x) < 0 \end{cases} \quad (90)$$

for $k \in \mathbb{R}$ with $k \geq \max_{\varepsilon\in(0,1)} \left|\left|u_\varepsilon^{(0)}\right|\right|_{L^\infty(\Omega_\varepsilon)} + 1$. Testing (44) by $\chi_{(0,t)} \hat{u}_\varepsilon^{(k)}$ yields

$$\begin{aligned}\int_0^t \langle \partial_t(J_\varepsilon \hat{u}_\varepsilon)(\tau), \hat{u}_\varepsilon^{(k)}(\tau)\rangle_{\Omega_\varepsilon} d\tau + (A_\varepsilon \nabla \hat{u}_\varepsilon^{(k)}, \nabla \hat{u}_\varepsilon^{(k)})_{(0,t)\times\Omega_\varepsilon} + (B_\varepsilon \hat{u}_\varepsilon, \nabla \hat{u}_\varepsilon^{(k)})_{(0,t)\times\Omega_\varepsilon} \\ = (J_\varepsilon \hat{f}_\varepsilon^{\mathrm{p}}, \hat{u}_\varepsilon^{(k)})_{(0,t)\times\Omega_\varepsilon} - \sum_{k\in I_\varepsilon} \left(\frac{r_{\varepsilon,k}^{n-1}}{r_0^{n-1}} \varepsilon f(\hat{u}_\varepsilon, r_{\varepsilon,k}), \hat{u}_\varepsilon^{(k)}\right)_{(0,t)\times\Gamma_{\varepsilon,k}}.\end{aligned}$$

We rewrite the first term by

$$\begin{aligned}\int_0^t \langle \partial_t(J_\varepsilon \hat{u}_\varepsilon)(\tau), \hat{u}_\varepsilon^{(k)}(\tau)\rangle_{H^1_\Gamma(\Omega_\varepsilon)} d\tau &= \frac{1}{2} \left|\left|\sqrt{J_\varepsilon}(t) \hat{u}_\varepsilon^{(k)}(t)\right|\right|^2_{\Omega_\varepsilon} \\ &- \frac{1}{2} \left|\left|\sqrt{J}(0) \hat{u}_\varepsilon^{(k)}(0)\right|\right|^2_{\Omega_\varepsilon} + \frac{1}{2}(\partial_t J_\varepsilon \hat{u}_\varepsilon, \hat{u}_\varepsilon^{(k)})_{(0,t)\times\Omega_\varepsilon} + \frac{1}{2}(\partial_t J_\varepsilon k, \hat{u}_\varepsilon^{(k)})_{(0,t)\times\Omega_\varepsilon}\end{aligned}$$

Since $k \geq ||u_\varepsilon(0)||_{L^\infty(\Omega_\varepsilon)}$, we get

$$\begin{aligned}\frac{1}{2} \left|\left|\sqrt{J_\varepsilon}(t) \hat{u}_\varepsilon^{(k)}(t)\right|\right|^2_{\Omega_\varepsilon} + (A_\varepsilon \nabla \hat{u}_\varepsilon^{(k)}, \nabla \hat{u}_\varepsilon^{(k)})_{(0,t)\times\Omega_\varepsilon} &= -(B_\varepsilon \hat{u}_\varepsilon, \nabla \hat{u}_\varepsilon^{(k)})_{(0,t)\times\Omega_\varepsilon} \\ &- \frac{1}{2}(\partial_t J_\varepsilon \hat{u}_\varepsilon, \hat{u}_\varepsilon^{(k)}(\tau))_{(0,t)\times\Omega_\varepsilon} - \frac{1}{2}(\partial_t J_\varepsilon k, \hat{u}_\varepsilon^{(k)})_{(0,t)\times\Omega_\varepsilon} \\ &+ (J_\varepsilon \hat{f}_\varepsilon^{\mathrm{p}}, \hat{u}_\varepsilon^{(k)})_{(0,t)\times\Omega_\varepsilon} - \sum_{k\in I_\varepsilon} \left(\frac{r_{\varepsilon,k}^{n-1}}{r_0^{n-1}} \varepsilon f(\hat{u}_\varepsilon, r_{\varepsilon,k}), \hat{u}_\varepsilon^{(k)}\right)_{(0,t)\times\Gamma_{\varepsilon,k}}.\end{aligned} \quad (91)$$

Using Lemma 1, we can estimate the left-hand side of (91) by

$$\frac{1}{2}\left|\left|\sqrt{J_\varepsilon}(t)\hat{u}_\varepsilon^{(k)}(t)\right|\right|^2_{\Omega_\varepsilon} + (A_\varepsilon \nabla \hat{u}_\varepsilon^{(k)}, \nabla \hat{u}_\varepsilon^{(k)})_{(0,t)\times\Omega_\varepsilon}$$
$$\geq \frac{1}{2}\left|\left|\hat{u}_\varepsilon^{(k)}(t)\right|\right|^2_{\Omega_\varepsilon} + \alpha \left|\left|\nabla \hat{u}_\varepsilon^{(k)}\right|\right|^2_{(0,t)\times\Omega_\varepsilon} \tag{92}$$

For the right-hand side of (91), we get with Lemma 1

$$-(B_\varepsilon \hat{u}_\varepsilon, \nabla \hat{u}_\varepsilon^{(k)})_{(0,t)\times\Omega_\varepsilon} = -(B_\varepsilon \hat{u}_\varepsilon^{(k)}, \nabla \hat{u}_\varepsilon^{(k)})_{(0,t)\times\Omega_\varepsilon} - (B_\varepsilon k, \nabla \hat{u}_\varepsilon^{(k)})_{(0,t)\times\Omega_\varepsilon}$$
$$\leq \varepsilon C \left|\left|\hat{u}_\varepsilon^{(k)}\right|\right|_{(0,t)\times\Omega_\varepsilon} \left|\left|\nabla \hat{u}_\varepsilon^{(k)}\right|\right|_{(0,t)\times\Omega_\varepsilon} + \varepsilon ||k||_{\{\hat{u}_\varepsilon \geq k\}} \left|\left|\nabla \hat{u}_\varepsilon^{(k)}\right|\right|_{(0,t)\times\Omega_\varepsilon}$$
$$\leq \varepsilon C \left|\left|\hat{u}_\varepsilon^{(k)}\right|\right|^2_{(0,t)\times\Omega_\varepsilon} + \varepsilon C \left|\left|\nabla \hat{u}_\varepsilon^{(k)}\right|\right|^2_{(0,t)\times\Omega_\varepsilon} + \varepsilon ||k||^2_{\{\hat{u}_\varepsilon \geq k\}}, \tag{93}$$

where $\{\hat{u}_\varepsilon \geq k\} := \{(t,x) \in (0,t_{i+1}) \times \Omega_\varepsilon \,|\, \hat{u}_\varepsilon(t,x) \geq k\}$. Similarly, we get

$$-\frac{1}{2}(\partial_t J_\varepsilon \hat{u}_\varepsilon^{(k)}, \hat{u}_\varepsilon^{(k)})_{(0,t)\times\Omega_\varepsilon} \leq C \left|\left|\hat{u}_\varepsilon^{(k)}\right|\right|^2_{(0,t)\times\Omega_\varepsilon}, \tag{94}$$

$$(J_\varepsilon \hat{f}_\varepsilon^{\mathrm{p}}, \hat{u}_\varepsilon^{(k)})_{(0,t)\times\Omega_\varepsilon} \leq ||C||_{\{\hat{u}_\varepsilon \geq k\}} \left|\left|\hat{u}_\varepsilon^{(k)}\right|\right|_{(0,t)\times\Omega_\varepsilon} \leq C||1||^2_{\{\hat{u}_\varepsilon \geq k\}} + \left|\left|\hat{u}_\varepsilon^{(k)}\right|\right|^2_{(0,t)\times\Omega_\varepsilon}, \tag{95}$$

$$-\sum_{k\in I_\varepsilon} \left(\frac{r_{\varepsilon,k}^{n-1}}{r_0^{n-1}} \varepsilon f(\hat{u}_\varepsilon, r_{\varepsilon,k}), \hat{u}_\varepsilon^{(k)}\right)_{(0,t)\times\Gamma_{\varepsilon,k}} \leq C\varepsilon \left|\left|\hat{u}_\varepsilon^{(k)}\right|\right|_{L^1((0,t)\times\Gamma_\varepsilon)}$$
$$\leq C \left|\left|\hat{u}_\varepsilon^{(k)}\right|\right|_{L^1((0,t)\times\Omega_\varepsilon)} + \varepsilon C \left|\left|\nabla \hat{u}_\varepsilon^{(k)}\right|\right|_{L^1((0,t)\times\Omega_\varepsilon)}$$
$$\leq C||1||_{\{\hat{u}_\varepsilon \geq k\}} \left|\left|\hat{u}_\varepsilon^{(k)}\right|\right|_{(0,t)\times\Omega_\varepsilon} + \varepsilon C||1||_{\{\hat{u}_\varepsilon \geq k\}} \left|\left|\nabla \hat{u}_\varepsilon^{(k)}\right|\right|_{(0,t)\times\Omega_\varepsilon}$$
$$\leq C_\delta ||1||^2_{\{\hat{u}_\varepsilon \geq k\}} + \left|\left|\hat{u}_\varepsilon^{(k)}\right|\right|^2_{(0,t)\times\Omega_\varepsilon} + \delta \left|\left|\nabla \hat{u}_\varepsilon^{(k)}\right|\right|^2_{(0,t)\times\Omega_\varepsilon}. \tag{96}$$

We choose delta small enough, combine (91)–(96) and collect all the constants:

$$\left|\left|\hat{u}_\varepsilon^{(k)}(t)\right|\right|^2_{\Omega_\varepsilon} + \left|\left|\nabla \hat{u}_\varepsilon^{(k)}\right|\right|^2_{(0,t)\times\Omega_\varepsilon} \leq C \left|\left|\hat{u}_\varepsilon^{(k)}\right|\right|^2_{(0,t)\times\Omega_\varepsilon} + Ck^2|\{u_\varepsilon \geq k\}| \tag{97}$$

Then, Gronwall's inequality implies

$$\left|\left|\hat{u}_\varepsilon^{(k)}\right|\right|^2_{L^\infty((0,t_{i+1});L^2(\Omega_\varepsilon))} + \left|\left|\nabla \hat{u}_\varepsilon^{(k)}\right|\right|^2_{(0,t_{i+1})\times\Omega_\varepsilon} \leq Ck^2|\{u_\varepsilon \geq k\}| \tag{98}$$

Likewise, it can be shown that

$$\left\|(-\hat{u}_\varepsilon)^{(k)}\right\|^2_{L^\infty((0,t_{i+1});L^2(\Omega_\varepsilon))} + \left\|\nabla(-\hat{u}_\varepsilon)^{(k)}\right\|^2_{(0,t_{i+1})\times\Omega_\varepsilon} \leq Ck^2|\{u_\varepsilon \geq k\}| \quad (99)$$

Thus, we can conclude with [23, Theorem 6.1]

$$\|\hat{u}_\varepsilon\|_{L^\infty((0,t_{i+1})\times\Omega_\varepsilon)} \leq C. \quad (100)$$

We note that the constant C in (100) is explicitly given in [23, Theorem 6.1] and depends also on the embedding constant of

$$L^\infty((0,T);L^2(\Omega_\varepsilon)) \cap L^2((0,T);H^1(\Omega_\varepsilon)) \hookrightarrow L^r((0,T)\times\Omega_\varepsilon)$$

for suitable r. However, using the extension from Corollary 14 this embedding constant can be chosen independent of ε (cf. [24] for a more detailed discussion).

• *Lipschitz estimate of $L_{\varepsilon,2}$.* Let $r_{\varepsilon,i} \in V_{r,\varepsilon}(S_i)$ with $r_{\varepsilon,1}(t_i) = r_{\varepsilon,2}(t_i)$ and $\zeta_i \in L^2(S_i;H^1(\Omega_\varepsilon))$ for $i \in \{1,2\}$. We define $\hat{u}_{\varepsilon,i} = L_{\varepsilon,2}(\zeta_i, r_{\varepsilon,i})$ for $i \in \{1,2\}$ as well as $\psi_{\varepsilon,i}$ and $\Psi_{\varepsilon,i}$, $J_{\varepsilon,i}$ by (32)–(33) for $r_\varepsilon = r_{\varepsilon,i}$ and $A_{\varepsilon,i} := J_{\varepsilon,i}\Psi^{-1}_{\varepsilon,i}\Psi^{-\top}_{\varepsilon,i}$, $B_{\varepsilon,i} := J_{\varepsilon,i}\Psi^{-1}_{\varepsilon,i}\partial_t\psi_{\varepsilon,i}$. We test (58) for $i \in \{1,2\}$ with $\chi_{(t_i,t)}(\hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2})$ and subtract the corresponding equations:

$$\begin{aligned}
&\int_{t_i}^{t} \langle \partial_t(J_{\varepsilon,1}(\tau)\hat{u}_{\varepsilon,1}(\tau) - J_{\varepsilon,2}(\tau)\hat{u}_{\varepsilon,2}(\tau)), \hat{u}_{\varepsilon,1}(\tau) - \hat{u}_{\varepsilon,2}(\tau)\rangle_{\Omega_\varepsilon} d\tau \\
&+ (A_{\varepsilon,1}\nabla\hat{u}_{\varepsilon,1} - A_{\varepsilon,2}\nabla\hat{u}_{\varepsilon,2}, \nabla(\hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2}))_{(t_i,t)\times\Omega_\varepsilon} \\
&+ (B_{\varepsilon,1}\hat{u}_{\varepsilon,1} - B_{\varepsilon,2}\hat{u}_{\varepsilon,2}, \nabla(\hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2}))_{(t_i,t)\times\Omega_\varepsilon} \\
&= (J_{\varepsilon,1}f^{\mathrm{p}}(\cdot_t, \psi_{\varepsilon,1}(\cdot_t, \cdot_x)) - J_{\varepsilon,2}f^{\mathrm{p}}(\cdot_t, \psi_{\varepsilon,2}(\cdot_t, \cdot_x)), \hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2})_{(t_i,t)\times\Omega_\varepsilon} \\
&- \varepsilon \sum_{k\in I_\varepsilon} \left(\frac{r^{n-1}_{\varepsilon,1,k}}{r_0^{n-1}} f(\zeta_1, r_{\varepsilon,1,k}) - \frac{r^{n-1}_{\varepsilon,2,k}}{r_0^{n-1}} f(\zeta_2, r_{\varepsilon,2,k}), \hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2} \right)_{(t_i,t)\times\Gamma_{\varepsilon,k}}. \quad (101)
\end{aligned}$$

Employing $r_{\varepsilon,1}(t_i) = r_{\varepsilon,2}(t_i)$ and $\hat{u}_{\varepsilon,1}(t_i) = \hat{u}_{\varepsilon,2}(t_i)$, we can rewrite the first term of (101) into

$$
\begin{aligned}
&\int_{t_i}^{t} \langle \partial_t (J_{\varepsilon,1}(\tau)\hat{u}_{\varepsilon,1}(\tau) - J_{\varepsilon,2}(\tau)\hat{u}_{\varepsilon,2}(\tau)), \hat{u}_{\varepsilon,1}(\tau) - \hat{u}_{\varepsilon,2}(\tau)\rangle_{\Omega_\varepsilon} d\tau \\
&= \frac{1}{2}\left|\left|\sqrt{J_{\varepsilon,1}}(t)(\hat{u}_{\varepsilon,1}(t) - \hat{u}_{\varepsilon,2}(t))\right|\right|^2_{\Omega_\varepsilon} + \frac{1}{2}(\partial_t J_{\varepsilon,1}(\hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2}), \hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2})_{(t_i,t)\times\Omega_\varepsilon} \\
&+ (\partial_t (J_{\varepsilon,1} - J_{\varepsilon,2})\hat{u}_{\varepsilon,2}, \hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2})_{(t_i,t)\times\Omega_\varepsilon} \\
&+ \int_{t_i}^{t} \langle \partial_t \hat{u}_{\varepsilon,2}(\tau), (J_{\varepsilon,1}(\tau) - J_{\varepsilon,2}(\tau))(\hat{u}_{\varepsilon,1}(\tau) - \hat{u}_{\varepsilon,2}(\tau))\rangle_{\Omega_\varepsilon}
\end{aligned}
$$

Thus, we can rewrite (101) by:

$$
\begin{aligned}
&I_1 + I_2 := \frac{1}{2}\left|\left|\sqrt{J_{\varepsilon,1}}(t)(\hat{u}_{\varepsilon,1}(t) - \hat{u}_{\varepsilon,2}(t))\right|\right|^2_{\Omega_\varepsilon} + (A_{\varepsilon,1}\nabla(\hat{u}_{\varepsilon,2} - \hat{u}_{\varepsilon,2}), \nabla(\hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2}))_{(t_i,t)\times\Omega_\varepsilon} \\
&= -\frac{1}{2}(\partial_t J_{\varepsilon,1}(\hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2}), \hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2})_{(t_i,t)\times\Omega_\varepsilon} - (\partial_t (J_{\varepsilon,1} - J_{\varepsilon,2})\hat{u}_{\varepsilon,2}, \hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2})_{(t_i,t)\times\Omega_\varepsilon} \\
&- \int_{t_i}^{t} \langle \partial_t \hat{u}_{\varepsilon,2}(\tau), (J_{\varepsilon,1}(\tau) - J_{\varepsilon,2}(\tau))(\hat{u}_{\varepsilon,1}(\tau) - \hat{u}_{\varepsilon,2}(\tau))\rangle_{\Omega_\varepsilon} d\tau \\
&- ((A_{\varepsilon,1} - A_{\varepsilon,2})\nabla\hat{u}_{\varepsilon,2}, \nabla(\hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2}))_{(t_i,t)\times\Omega_\varepsilon} - (B_{\varepsilon,1}\hat{u}_{\varepsilon,1} - B_{\varepsilon,2}\hat{u}_{\varepsilon,2}, \nabla(\hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2}))_{(t_i,t)\times\Omega_\varepsilon} \\
&+ (J_{\varepsilon,1} f^{\mathrm{p}}(\cdot_t, \psi_{\varepsilon,1}(\cdot_t, \cdot_x)) - J_{\varepsilon,2} f^{\mathrm{p}}(\cdot_t, \psi_{\varepsilon,2}(\cdot_t, \cdot_x)), \hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2})_{(t_i,t)\times\Omega_\varepsilon} \\
&- \varepsilon \sum_{k\in I_\varepsilon} \left(\frac{r^{n-1}_{\varepsilon,1,k}}{r_0^{n-1}} f(\zeta_1, r_{\varepsilon,1,k}) - \frac{r^{n-1}_{\varepsilon,2,k}}{r_0^{n-1}} f(\zeta_2, r_{\varepsilon,2,k}), \hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2}\right)_{(t_i,t)\times\Gamma_{\varepsilon,k}} \\
&=: I_3 + I_4 + I_6 + I_7 + I_8 + I_9
\end{aligned}
\tag{102}
$$

In the next step, we estimate I_1, I_2 from below and $I_3, \ldots, I_8$ from above:

I_1, I_2: Lemma 1 implies:

$$
\left|\left|\sqrt{J_{\varepsilon,1}}(t)(\hat{u}_{\varepsilon,1}(t) - \hat{u}_{\varepsilon,2}(t))\right|\right|^2_{\Omega_\varepsilon} \geq c_J ||\hat{u}_{\varepsilon,1}(t) - \hat{u}_{\varepsilon,2}(t)||^2_{\Omega_\varepsilon} \tag{103}
$$

$$
(A_{\varepsilon,1}\nabla(\hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2}), \nabla(\hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2}))_{(t_i,t)\times\Omega_\varepsilon} \geq \alpha ||\nabla(\hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2})||^2_{(t_i,t)\times\Omega_\varepsilon}. \tag{104}
$$

I_3: Using Lemma 1, we get:

$$
-\frac{1}{2}(\partial_t J_{\varepsilon,1}(\hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2}), \hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2})_{(t_i,t)\times\Omega_\varepsilon} \leq C ||\hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2}||^2_{(t_i,t)\times\Omega_\varepsilon}. \tag{105}
$$

I_4**:**Application of Lemma 2, (87), the Cauchy–Schwarz and the Young inequalities yields for every $\mu > 0$ a constant C_μ such that:

$$
\begin{aligned}
&-(\partial_t(J_{\varepsilon,1} - J_{\varepsilon,2})\hat{u}_{\varepsilon,2}, \hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2})_{(t_i,t)\times\Omega_\varepsilon} \\
&\leq C\|\partial_t(r_{\varepsilon,1} - r_{\varepsilon,2})\|_{L^\infty(\Omega_\varepsilon;L^2(t_i,t))}\|\hat{u}_{\varepsilon,2}\|_{L^\infty((t_i,t);L^2(\Omega_\varepsilon))}\|\hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2}\|_{(t_i,t)\times\Omega_\varepsilon} \\
&\leq C\|\partial_t(r_{\varepsilon,1} - r_{\varepsilon,2})\|_{L^\infty(\Omega_\varepsilon;L^2(t_i,t))} C_K \|\hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2}\|_{(t_i,t)\times\Omega_\varepsilon} \\
&\leq \mu\|\partial_t(r_{\varepsilon,1} - r_{\varepsilon,2})\|^2_{L^\infty(\Omega_\varepsilon;L^2(t_i,t))} + C_K C_\mu \|\hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2}\|^2_{(t_i,t)\times\Omega_\varepsilon}
\end{aligned} \tag{106}
$$

I_5**:**Application of (89), Lemma 2 the Cauchy–Schwarz and the Young inequalities yields for every $\delta > 0$:

$$
\begin{aligned}
&-\int_{t_i}^{t} \langle \partial_t \hat{u}_{\varepsilon,2}(\tau), (J_{\varepsilon,1}(\tau) - J_{\varepsilon,2}(\tau))(\hat{u}_{\varepsilon,1}(\tau) - \hat{u}_{\varepsilon,2}(\tau))\rangle_{\Omega_\varepsilon} d\tau \\
&\leq \|\partial_t \hat{u}_{\varepsilon,2}\|_{L^2((t_i,t);H^1(\Omega_\varepsilon)')}(\|J_{\varepsilon,1} - J_{\varepsilon,2}\|_{L^\infty((t_i,t)\times\Omega_\varepsilon)} + \|\nabla(J_{\varepsilon,1} - J_{\varepsilon,2})\|_{L^\infty((t_i,t)\times\Omega_\varepsilon)}) \\
&(\|\hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2}\|_{(t_i,t)\times\Omega_\varepsilon} + \|\nabla(\hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2})\|_{(t_i,t)\times\Omega_\varepsilon}) \\
&\leq C_{K,\varepsilon}\|r_{\varepsilon,1} - r_{\varepsilon,2}\|_{L^\infty((t_i,t)\times\Omega_\varepsilon)}(\|\hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2}\|_{(t_i,t)\times\Omega_\varepsilon} + \|\nabla(\hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2})\|_{(t_i,t)\times\Omega_\varepsilon}) \\
&\leq C_{K,\varepsilon,\delta}\|r_{\varepsilon,1} - r_{\varepsilon,2}\|^2_{L^\infty((t_i,t)\times\Omega_\varepsilon)} + \|\hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2}\|^2_{(t_i,t)\times\Omega_\varepsilon} + \delta\|\nabla(\hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2})\|^2_{(t_i,t)\times\Omega_\varepsilon}.
\end{aligned} \tag{107}
$$

I_6**:**We estimate similar as (107) and use (87) and Lemma 2:

$$
\begin{aligned}
&-((A_{\varepsilon,1} - A_{\varepsilon,2})\nabla\hat{u}_{\varepsilon,2}, \nabla(\hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2}))_{(t_i,t)\times\Omega_\varepsilon} \\
&\leq \|r_{\varepsilon,1} - r_{\varepsilon,2}\|_{L^\infty((t_i,t)\times\Omega_\varepsilon)}\|\nabla\hat{u}_{\varepsilon,2}\|_{(t_i,t)\times\Omega_\varepsilon}\|\nabla(\hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2})\|_{(t_i,t)\times\Omega_\varepsilon} \\
&\leq C_{K,\delta}\|r_{\varepsilon,1} - r_{\varepsilon,2}\|^2_{L^\infty((t_i,t)\times\Omega_\varepsilon)} + \delta\|\nabla(\hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2})\|^2_{(t_i,t)\times\Omega_\varepsilon}
\end{aligned} \tag{108}
$$

I_7**:**The Cauchy–Schwarz inequality gives

$$
\begin{aligned}
&-(B_{\varepsilon,1}\hat{u}_{\varepsilon,1} - B_{\varepsilon,2}\hat{u}_{\varepsilon,2}, \nabla(\hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2}))_{(t_i,t)\times\Omega_\varepsilon} \\
&\leq \|B_{\varepsilon,1}\hat{u}_{\varepsilon,1} - B_{\varepsilon,2}\hat{u}_{\varepsilon,2}\|_{(t_i,t)\times\Omega_\varepsilon}\|\nabla(\hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2})\|_{(t_i,t)\times\Omega_\varepsilon}
\end{aligned} \tag{109}
$$

Using (100), Lemma 1 and Lemma 2, we get

$$
\begin{aligned}
&\|B_{\varepsilon,1}\hat{u}_{\varepsilon,1} - B_{\varepsilon,2}\hat{u}_{\varepsilon,2}\|_{(t_i,t)\times\Omega_\varepsilon} = \|(B_{\varepsilon,1} - B_{\varepsilon,2})\hat{u}_{\varepsilon,1}\|_{(t_i,t)\times\Omega_\varepsilon} + \|B_{\varepsilon,2}(\hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2})\|_{(t_i,t)\times\Omega_\varepsilon} \\
&\leq \|B_{\varepsilon,1} - B_{\varepsilon,2}\|_{(t_i,t)\times\Omega_\varepsilon} + \varepsilon C\|\hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2}\|_{(t_i,t)\times\Omega_\varepsilon} \\
&\leq \varepsilon C\|r_{\varepsilon,1} - r_{\varepsilon,2}\|_{L^\infty((t_i,t)\times\Omega_\varepsilon)} + \varepsilon C\|\partial_t(r_{\varepsilon,1} - r_{\varepsilon,2})\|_{(t_i,t)\times\Omega_\varepsilon} \\
&+ \varepsilon C\|\hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2}\|_{(t_i,t)\times\Omega_\varepsilon}
\end{aligned} \tag{110}
$$

Inserting (110) in (109) and applying the Young inequality yields

$$
\begin{aligned}
&-(B_{\varepsilon,1}\hat{u}_{\varepsilon,1} - B_{\varepsilon,2}\hat{u}_{\varepsilon,2}, \nabla(\hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2}))_{(t_i,t)\times\Omega_\varepsilon} \\
&\leq \varepsilon ||r_{\varepsilon,1} - r_{\varepsilon,2}||^2_{L^\infty((t_i,t)\times\Omega_\varepsilon)} + \varepsilon C ||\partial_t (r_{\varepsilon,1} - r_{\varepsilon,2})||^2_{(t_i,t)\times\Omega_\varepsilon} + \varepsilon C ||\hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2}||^2_{(t_i,t)\times\Omega_\varepsilon} \\
&+ \varepsilon ||\nabla(\hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2})||^2_{(t_i,t)\times\Omega_\varepsilon} . \qquad (111)
\end{aligned}
$$

I_8 : By the same procedure as in the estimate of I_6 and employing that f^{p} is Lipschitz continuous in each ε-scaled cell, we can estimate:

$$
\begin{aligned}
&(J_{\varepsilon,1} f^{\mathrm{p}}(\cdot_t, \psi_{\varepsilon,1}(\cdot_t, \cdot_x)) - J_{\varepsilon,2} f^{\mathrm{p}}(\cdot_t, \psi_{\varepsilon,2}(\cdot_t, \cdot_x)), \hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2})_{(t_i,t)\times\Omega_\varepsilon} \\
&= ((J_{\varepsilon,1} - J_{\varepsilon,2}) f^{\mathrm{p}}(\cdot_t, \psi_{\varepsilon,1}(\cdot_t, \cdot_x)) + J_{\varepsilon,2}(f^{\mathrm{p}}(\cdot_t, \psi_{\varepsilon,1}(\cdot_t, \cdot_x)) - f^{\mathrm{p}}(\cdot_t, \psi_{\varepsilon,2}(\cdot_t, \cdot_x))), \hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2})_{(t_i,t)\times\Omega_\varepsilon} \\
&\leq C ||r_{\varepsilon,1} - r_{\varepsilon,2}||^2_{L^\infty((t_i,t)\times\Omega_\varepsilon)} + C ||\psi_{\varepsilon,1} - \psi_{\varepsilon,2}||^2_{L^\infty((t_i,t)\times\Omega_\varepsilon)} + C ||\hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2}||^2_{(t_i,t)\times\Omega_\varepsilon} \\
&\leq C ||r_{\varepsilon,1} - r_{\varepsilon,2}||^2_{L^\infty((t_i,t)\times\Omega_\varepsilon)} + C ||\hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2}||^2_{(t_i,t)\times\Omega_\varepsilon} \qquad (112)
\end{aligned}
$$

I_9: Using the Cauchy–Schwarz inequality gives

$$
\begin{aligned}
&\varepsilon \sum_{k\in I_\varepsilon} \left(\frac{r^{n-1}_{\varepsilon,1,k}}{r^{n-1}_0} f(\zeta_1, r_{\varepsilon,1,k}) - \frac{r^{n-1}_{\varepsilon,2,k}}{r^{n-1}_0} f(\zeta_2, r_{\varepsilon,2,k}), \hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2} \right)_{(t_i,t)\times\Gamma_{\varepsilon,k}} \\
&\leq \varepsilon \left|\left| \frac{r^{n-1}_{\varepsilon,1}}{r^{n-1}_0} f(\zeta_1, r_{\varepsilon,1}) - \frac{r^{n-1}_{\varepsilon,2}}{r^{n-1}_0} f(\zeta_2, r_{\varepsilon,2}) \right|\right|_{(t_i,t)\times\Gamma_\varepsilon} ||\hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2}||_{(t_i,t)\times\Gamma_\varepsilon} . (113)
\end{aligned}
$$

We estimate the first factor of the right-hand side of (113) using the Lipschitz continuity of f and the boundedness of $r_{\varepsilon,1}$ and $r_{\varepsilon,2}$

$$
\begin{aligned}
&\left|\left| \frac{r^{n-1}_{\varepsilon,1}}{r^{n-1}_0} f(\zeta_1, r_{\varepsilon,1}) - \frac{r^{n-1}_{\varepsilon,2}}{r^{n-1}_0} f(\zeta_2, r_{\varepsilon,2}) \right|\right|_{(t_i,t)\times\Gamma_\varepsilon} \leq f_{\max} \left|\left| \frac{r^{n-1}_{\varepsilon,1} - r^{n-1}_{\varepsilon,2}}{r^{n-1}_0} \right|\right|_{(t_i,t)\times\Gamma_\varepsilon} \\
&+ C ||f(\zeta_1, r_{\varepsilon,1}) - f(\zeta_2, r_{\varepsilon,1})||_{(t_i,t)\times\Gamma_\varepsilon} + C ||f(\zeta_2, r_{\varepsilon,1}) - f(\zeta_2, r_{\varepsilon,2})||_{(t_i,t)\times\Gamma_\varepsilon} \\
&\leq \varepsilon^{-1/2} C ||r_{\varepsilon,1} - r_{\varepsilon,2}||_{L^\infty((t_i,t)\times\Omega_\varepsilon)} + \varepsilon^{-1/2} C ||\zeta_1 - \zeta_2||_{L^2(S_i;H^1(\Omega_\varepsilon))} . (114)
\end{aligned}
$$

Combining (113)–(114), applying the Young and the ε-scaled trace inequality (117) as well as the estimate (87) yields:

$$
\begin{aligned}
&\varepsilon \sum_{k\in I_\varepsilon} \left(\frac{r^{n-1}_{\varepsilon,1}}{r^{n-1}_0} f(\zeta_1, r_{\varepsilon,1,k}) - \frac{r^{n-1}_{\varepsilon,2,k}}{r^{n-1}_0} f(\zeta_2, r_{\varepsilon,2,k}), \hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2} \right)_{(t_i,t)\times\Gamma_{\varepsilon,k}} \\
&\leq C(||r_{\varepsilon,1} - r_{\varepsilon,2}||_{L^\infty((t_i,t)\times\Omega_\varepsilon)} + ||\zeta_1 - \zeta_2||_{L^2(S_i;H^1(\Omega_\varepsilon))}) \\
&(C_\delta ||\hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2}||_{(t_i,t)\times\Omega_\varepsilon} + \varepsilon^{1/2}\delta ||\nabla(\hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2})||_{(t_i,t)\times\Omega_\varepsilon}) \\
&\leq C ||r_{\varepsilon,1} - r_{\varepsilon,2}||^2_{L^\infty((t_i,t)\times\Omega_\varepsilon)} + \mu ||\zeta_1 - \zeta_2||^2_{L^2(S_i;H^1(\Omega_\varepsilon))} \\
&+ C_\mu C_\delta ||\hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2}||^2_{(t_i,t)\times\Omega_\varepsilon} + C_\mu \varepsilon^{1/2}\delta ||\nabla(\hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2})||^2_{(t_i,t)\times\Omega_\varepsilon} \qquad (115)
\end{aligned}
$$

Now, we combine (102) with (103)–(115). Then, we choose ε and δ small enough such that the gradient terms on the right-hand side, which arise in the estimates of (107), (108) and (111) can be compensated by the gradient on the left-hand side. After collecting the constants, we get for $\varepsilon \leq 1$

$$
\begin{aligned}
&\|\hat{u}_{\varepsilon,1}(t)-\hat{u}_{\varepsilon,2}(t)\|^2_{\Omega_\varepsilon} + (1-\varepsilon C-\delta C_\mu)\|\nabla(\hat{u}_{\varepsilon,1}-\hat{u}_{\varepsilon,2})\|^2_{(t_i,t)\times\Omega_\varepsilon}\\
&\leq (C_{K,\mu}+C_\delta C_\mu)\|\hat{u}_{\varepsilon,1}-\hat{u}_{\varepsilon,2}\|^2_{(t_i,t)\times\Omega_\varepsilon} + \mu C\|\partial_t(r_{\varepsilon,1}-r_{\varepsilon,2})\|^2_{L^\infty(\Omega_\varepsilon;L^2((t_i,t)))}\\
&+\varepsilon C\|\partial_t(r_{\varepsilon,1}-r_{\varepsilon,2})\|^2_{(t_i,t)\times\Omega_\varepsilon} + C_{K,\varepsilon,\delta}\|r_{\varepsilon,1}-r_{\varepsilon,2}\|^2_{L^\infty((t_i,t)\times\Omega_\varepsilon)} + \mu C\|\zeta_1-\zeta_2\|^2_{L^2(S_i;H^1(\Omega_\varepsilon))}.
\end{aligned}
$$

Then, Gronwall's inequality gives

$$
\begin{aligned}
&\|\hat{u}_{\varepsilon,1}(t)-\hat{u}_{\varepsilon,2}(t)\|^2_{\Omega_\varepsilon} + (1-\varepsilon C-\delta C_\mu)\|\nabla(\hat{u}_{\varepsilon,1}-\hat{u}_{\varepsilon,2})\|^2_{S_i\times\Omega_\varepsilon}\\
&\leq \exp(|S_i|(C_{K,\mu}+C_\delta C_\mu))\Big(\mu C\|\partial_t(r_{\varepsilon,1}-r_{\varepsilon,2})\|^2_{L^\infty(\Omega_\varepsilon;L^2(S_i))} + \varepsilon C\|\partial_t(r_{\varepsilon,1}-r_{\varepsilon,2})\|^2_{S_i\times\Omega_\varepsilon}\\
&+C_{K,\varepsilon,\delta}\|r_{\varepsilon,1}-r_{\varepsilon,2}\|^2_{L^\infty(S_i\times\Omega_\varepsilon)} + \mu C\|\zeta_1-\zeta_2\|^2_{L^2(S_i;H^1(\Omega_\varepsilon))}\Big) \qquad (116)
\end{aligned}
$$

• *Lipschitz estimate of L_ε.* We combine (116) with (66),(67),(68) and get for $r_{\varepsilon,i} := L_{\varepsilon,1}(\zeta_i)$ for $i\in\{1,2\}$:

$$
\begin{aligned}
&(1-\varepsilon C^{(1)}-\delta C^{(2)}_\mu)\|L_\varepsilon(\zeta_1)-L_\varepsilon(\zeta_2)\|^2_{L^2(S_i;H^1(\Omega_\varepsilon))}\\
&= (1-\varepsilon C^{(1)}-\delta C^{(2)}_\mu)\|L_{\varepsilon,2}(\zeta_1,r_{\varepsilon,1})-L_{\varepsilon,2}(\zeta_2,r_{\varepsilon,2})\|^2_{L^2(S_i;H^1(\Omega_\varepsilon))}\\
&\leq \exp(|S_i|(C_{K,\mu}+C_\delta C_\mu))\Big(\mu C\|\partial_t(r_{\varepsilon,1}-r_{\varepsilon,2})\|^2_{L^\infty(\Omega_\varepsilon;L^2(S_i))} + \varepsilon C\|\partial_t(r_{\varepsilon,1}-r_{\varepsilon,2})\|^2_{S_i\times\Omega_\varepsilon}\\
&+C_{K,\varepsilon,\delta}\|r_{\varepsilon,1}-r_{\varepsilon,2}\|^2_{L^\infty(S_i\times\Omega_\varepsilon)} + \mu C\|\zeta_1-\zeta_2\|^2_{L^2(S_i;H^1(\Omega_\varepsilon))}\Big)\\
&\leq \exp(|S_i|(C^{(3)}_{K,\mu}+C^{(4)}_\delta C^{(5)}_\mu))(\varepsilon C^{(6)}+C^{(7)}_{K,\varepsilon,\delta}|S_i|+\mu C^{(8)}_\varepsilon)\|\zeta_1-\zeta_2\|^2_{L^2(S;H^1(\Omega_\varepsilon))},
\end{aligned}
$$

where we have added the superscript at the constants in order to clarify the following choices of the parameter μ and δ. First, we assume $\varepsilon \leq \max\{(4C^{(1)})^{-1},(24C^{(6)})^{-1}\}$. Then, we choose $\mu \leq (24C^{(8)}_\varepsilon)^{-1}$, afterwards we choose $\delta \leq (4C^{(2)}_\mu)^{-1}$. Finally, we choose $\sigma_{\varepsilon,K} \leq \max\{\ln(2)(C^{(3)}_{K,\mu}+C^{(4)}_\delta C^{(5)}_\mu)^{-1},(24C^{(7)}_{K,\varepsilon,\delta})^{-1}\}$ and L_ε becomes a contraction for $|S_i| \leq \sigma_{\varepsilon,K}$. Hence there exists a unique solution of (32)–(33), (54)–(55).

Rescaling the trace inequality of the reference cell onto Ω_ε yields for every $\delta > 0$ a constant C_δ such that for every ε and $u \in H^1(\Omega_\varepsilon)$

$$
\|u\|^2_{\partial\Omega_\varepsilon} \leq \varepsilon\delta\|\nabla u\|^2_{\Omega_\varepsilon} + \varepsilon^{-1}C_\delta\|u\|^2_{L^2(\Omega_\varepsilon)}. \qquad (117)
$$

4. Derivation of the limit problem for the periodic substitute problem

We use the notion of two-scale convergence which was introduced in [8] and [9].

Definition 3 (Two-scale convergence). Let $p, q, p_s, q_s \in (1,\infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $\frac{1}{p_s} + \frac{1}{q_s} = 1$. We say that a sequence u_ε in $L^{p_s}(S; L^p(\Omega))$ two-scale converges weakly to $u_0 \in L^{p_s}(S; L^p(\Omega \times Y))$ if

$$\lim_{\varepsilon \to 0} \int_S \int_\Omega u_\varepsilon(t,x) \varphi\left(t, x, \tfrac{x}{\varepsilon}\right) dx dt = \int_S \int_\Omega \int_Y u_0(t,x,y) \varphi(t,x,y) dy dx dt \tag{118}$$

for every $\varphi \in L^{q_s}(S; L^q(\Omega; C_\#(Y)))$. In this case, we write $u_\varepsilon \overset{p_s,p}{\rightharpoonup\rightharpoonup} u_0$.

Moreover, we say that u_ε two-scale converges strongly to u_0 if additionally $\lim_{\varepsilon \to 0} ||u_\varepsilon||_{L^{p_s}(S; L^p(\Omega))} = ||u||_{L^{p_s}(S; L^p(\Omega \times S))}$. In this case, we write $u_\varepsilon \overset{p_s,p}{\to\to} u_0$.

The notion of two-scale convergence provides the following compactness results. Proposition 7 and Proposition 8 are time dependent versions of compactness results that can be found in [8].

Proposition 7. *Let $p_s, p \in (1,\infty)$ and let u_ε be a bounded sequence in $L^{p_s}(S; L^p(\Omega))$. Then, there exists a subsequence ε and $u_0 \in L^{p_s}(S; L^p(\Omega \times Y))$ such that $u_\varepsilon \overset{p_s,p}{\rightharpoonup\rightharpoonup} u_0$.*

For the sake of simplicity, let in the following proposition the domain Ω_ε be given as in the previous sections and $Y^* = Y \setminus B_{r_0}(x_M)$ (for more general domains cf. [8]). We use $\tilde{\cdot}$ in order to denote the extension of functions which are defined on Ω_ε or $\Omega_\varepsilon(t)$ by 0 to Ω. We use it also for the extension by 0 to Y for functions which are defined on Y^* or on Y_r^* with $r \in [r_{\min}, r_{\max}]$.

Proposition 8. *Let $p_s, p \in (1,\infty)$ and let u_ε be a bounded sequence in $L^{p_s}(S; W^{1,p}(\Omega_\varepsilon))$. Then, there exists a subsequence ε and $(u_0, u_1) \in L^{p_s}(S; W^{1,p}(\Omega)) \times L^{p_s}(S; L^p(\Omega; W^{1,p}_\#(Y^*)/\mathbb{R}))$ such that $\tilde{u}_\varepsilon \overset{p_s,p}{\rightharpoonup\rightharpoonup} \chi_{Y^*} u_0$ and $\widetilde{\nabla u_\varepsilon} \overset{p_s,p}{\rightharpoonup\rightharpoonup} \chi_{Y^*} \nabla_x u_0 + \widetilde{\nabla_y u_1}$.*

In order to have (1) commutative, we use the concept of locally periodic transformations, which was introduced for the stationary case in [14] and is extended to the time-dependent case here:

Definition 4. We say a sequence $\psi_\varepsilon : S \times \overline{\Omega} \to \overline{\Omega}$, is a sequence of locally periodic transformations if

1. $\psi_\varepsilon \in L^\infty(S; C^1(\overline{\Omega}))^N$,
2. there exists a constant c_J such that $J_\varepsilon(t) \geq c_J$ for a.e. $t \in S$ with $J_\varepsilon(t,x) := \det(\Psi_\varepsilon(t,x))$ and $\Psi_\varepsilon := D_x\psi_\varepsilon(t,x)$,
3. there exists a constant $C > 0$ such that $\varepsilon^{i-1} \left|\left|\check{\psi}_\varepsilon\right|\right|_{L^\infty(S;C^i(\overline{\Omega}))} \leq C$ for $i \in \{0,1\}$, where $\check{\psi}_\varepsilon(t,x) := \psi_\varepsilon(t,x) - x$ is the corresponding displacement mapping,
4. there exists $\psi_0 \in L^\infty(S \times \Omega; C^1(\overline{Y}))^N$, which we call limit transformation, such that
   (a) $\psi_0(t,x,\cdot_y) : Y \to Y$ are C^1-diffeomorphisms for a.e. $(t,x) \in S \times \Omega$ with inverses $\psi_0^{-1}(t,x,\cdot_y)$ for $\psi_\varepsilon^{-1} \in L^\infty(S \times \Omega; C^1(\overline{Y}))^N$,
   (b) the corresponding displacement mapping, defined for a.e. $(t,x) \in S \times \Omega$ by $\check{\psi}_0(t,x,y) := \psi_0(t,x,y) - y$, can be extended Y-periodically such that $\check{\psi}_0 \in L^\infty(S \times \Omega; C^1_\#(\overline{Y}))^N$,
   (c) $\varepsilon^{-1}\check{\psi}_\varepsilon \xrightarrow{p,p} \check{\psi}_0$ and $\nabla\check{\psi}_\varepsilon \xrightarrow{p,p} \nabla_y\check{\psi}_0$ for every $p \in (1,\infty)$.

For a.e. $(t,x) \in S \times \Omega$, we denote the Jacobian matrix and determinant of $y \mapsto \psi_0(t,x,y)$ by $\Psi_0(t,x,y) := D_y\psi_0(t,x,y)$ and $J_0(t,x,y) := \det(\Psi_0(t,x,y))$. Moreover, we denote the displacement mappings of the back-transformations by $\check{\psi}_\varepsilon^{-1}(t,x) := \psi_\varepsilon^{-1}(t,x) - x$ and $\check{\psi}_0^{-1}(t,x,y) := \psi_0^{-1}(t,x,y) - y$.

Notation 1. For a function u, we introduce the following notations:

$$\begin{aligned} u_{\psi_\varepsilon}(t,x) &:= u(t,\psi_\varepsilon(t,x)), & u_{\psi_\varepsilon^{-1}}(t,x) &:= u(t,\psi_\varepsilon^{-1}(t,x)), \\ u_{\psi_0}(t,x,y) &:= u(t,x,\psi_0(t,x,y)), & u_{\psi_0^{-1}}(t,x,y) &:= u(t,x,\psi_0^{-1}(t,x,y)). \end{aligned}$$

Remark 1. Let ψ_ε be a sequence of locally periodic transformations in the sense of Definition 4. If additionally $\partial_t\psi_\varepsilon \in L^{p_s}(S; C(\overline{\Omega}))^N$ for $p_s > 1$, we can conclude $\psi_\varepsilon \in C(\overline{S}; C(\overline{\Omega}))^N$, which allows us to evaluate $\Omega_\varepsilon(t) = \psi_\varepsilon(t, \Omega_\varepsilon)$ for every $t \in \overline{S}$. Moreover, if $\partial_t\psi_0 \in L^\infty(\Omega; L^{p_s}(S; C(\overline{Y})))^N$, we get $\psi_0 \in L^\infty(\Omega; C(\overline{S}; C(\overline{Y})))^N$ which allows defining the local reference cell $Y_x^*(t) := \psi_0(t,x,Y^*)$ for a.e. $x \in \Omega$ and every $t \in \overline{S}$.

In our case, where ψ_ε is given by (32), we can show that ψ_ε is a locally periodic transformation in the sense of Definition 4 if r_ε converges strongly. In order to prove this, we use the unfolding operator

$$L^{p_s}(S; L^p(\Omega)) \to L^{p_s}(S; L^p(\Omega \times Y)), \mathcal{T}_\varepsilon u(t,x,y) := u(t, [x]_{\varepsilon,Y} + \varepsilon y).$$

It allows us to rewrite, two-scale convergence as convergence in $L^{p_s}(S;L^p(\Omega\times Y))$, i.e. $u_\varepsilon \xrightharpoonup{p_s,p} u_0$ if and only if $\mathcal{T}_\varepsilon u_\varepsilon \rightharpoonup u_0$ in $L^{p_s}(S;L^p(\Omega\times Y))$. In our case $\mathcal{T}_\varepsilon$ is isometric, because Ω consist only on whole ε-scaled cells. Thus, $u_\varepsilon \xrightarrow{p_s,p} u_0$ if and only if $\mathcal{T}_\varepsilon u_\varepsilon \to u_0$ in $L^{p_s}(S;L^p(\Omega\times Y))$. Moreover, the unfolding operator can be defined for the periodic boundary in the same way, i.e.

$$L^{p_s}(S;L^p(\Gamma_\varepsilon))\to L^{p_s}(S;L^p(\Omega\times\Gamma)), \mathcal{T}_\varepsilon u(t,x,y) := u(t,[x]_{\varepsilon,Y}+\varepsilon y).$$

In the limit process, we use the following properties of $\mathcal{T}_\varepsilon$ which can be found in [25]: For $u\in L^{p_s}(S;W^{1,p}(\Omega))$ it holds $\varepsilon^{-1}\nabla_y\mathcal{T}_\varepsilon u = \mathcal{T}_\varepsilon\nabla_x u$ and for $u\in L^{p_s}(S;L^p(\Gamma_\varepsilon))$ it holds $\varepsilon\int_S\int_{\Gamma_\varepsilon} u(t,x)d\sigma_x dt = \int_S\int_\Omega\int_\Gamma \mathcal{T}_\varepsilon u(t,x,y)d\sigma_y dx dt$.

Lemma 9. *Let ψ be defined by (25) and ψ_ε by (32), where R fulfils the assumptions (26)–(29). Let $r_{\min}\le r_{\varepsilon,k}(t)\le r_{\max}$ for every $k\in I_\varepsilon$ and a.e. $t\in S$ and assume that $r_{\varepsilon,k_\varepsilon(\cdot x)}$ converges strongly to r in $L^1(S\times\Omega)$. Then, ψ_ε is a sequence of locally periodic transformations in the sense of Definition 4 with limit transformation*

$$\psi_0(t,x,y) = \psi(r(t,x),y). \quad (119)$$

Proof. The properties 1–3 of Definition 4 follow directly from the construction of ψ_ε and the uniform boundedness of $r_{\varepsilon,k}(t)$ from above and below. The strong convergence of $r_{\varepsilon,k_\varepsilon(\cdot x)}$ to r transfers this boundedness to its limit: $r_{\min}\le r(t,x)\le r_{\max}$ for a.e. $(t,x)\in S\times\Omega$. Thus, the properties 4a–4b of Definition 4 follow from the construction of ψ_0.

It remains to show Property 4c, which is equivalent to the strong convergences $\varepsilon^{-1}\mathcal{T}_\varepsilon\check{\psi}_\varepsilon\to\check{\psi}_0$ and $\mathcal{T}_\varepsilon\nabla_x\check{\psi}_\varepsilon = \varepsilon^{-1}\nabla_y\mathcal{T}_\varepsilon\check{\psi}_\varepsilon\to\nabla_y\check{\psi}_0$ in $L^p(\Omega\times Y)$. Due to the strong convergence of $r_{\varepsilon,k_\varepsilon(\cdot x)}$, we can pass to a subsequence ε such that $r_{\varepsilon,k_\varepsilon(x)}(t)\to r(t,x)$ for a.e. $(t,x)\in S\times\Omega$. Employing the continuity of $r\mapsto\check{\psi}(r,y)$ and $r\mapsto\nabla_y\check{\psi}(r,y)$, we obtain:

$$\varepsilon^{-1}\mathcal{T}_\varepsilon\check{\psi}_\varepsilon(t,x,y) = \check{\psi}(r_{\varepsilon,k_\varepsilon([x]_{\varepsilon,Y}+\varepsilon y)}(t),\{[x]_{\varepsilon,Y}+\varepsilon y\}_{\varepsilon,Y}) = \check{\psi}(r_{\varepsilon,k_\varepsilon(x)}(t),y)\to\check{\psi}(r(t,x),y),$$
$$\mathcal{T}_\varepsilon\nabla_x\check{\psi}_\varepsilon(t,x,y) = \varepsilon^{-1}\nabla_y\mathcal{T}_\varepsilon\check{\psi}_\varepsilon(t,x,y) = \nabla_y\check{\psi}(r_{\varepsilon,k_\varepsilon(x)}(t),y)\to\nabla_y\check{\psi}(r(t,x),y)$$

for a.e. $(t,x)\in S\times\Omega$. Since these functions are uniformly bounded in $L^\infty(S\times\Omega\times Y)$ we obtain the convergence of these functions in $L^p(S\times\Omega\times Y)$. Because the argumentation holds for every arbitrary subsequence, we obtain the desired convergences.

We obtain the strong convergence of the Jacobian matrix and determinant of the transformations:

Lemma 10. *Let ψ_ε be a locally periodic transformation in the sense of Definition 4 with limit transformation ψ_0 and Jacobian matrices and determinants $\Psi_\varepsilon, J_\varepsilon, \Psi_0, J_0$. Then, $\Psi_\varepsilon \xrightarrow{p,p} \Psi_0$, $J_\varepsilon \xrightarrow{p,p} J_0$, $\Psi_\varepsilon^{-1} \xrightarrow{p,p} \Psi_0^{-1}$ for every $p \in (1,\infty)$.*

Proof. Lemma 10 is the time-dependent version of [14, Lemma 3.3] and can be proven analogously to the stationary case there.

Using the notation of locally periodic transformations, we can show that the two-scale limit and the transformation commutes in the following sense:

Proposition 11 (Two-scale transformation). *Let ψ_ε be a locally periodic transformation in the sense of Definition 4 with limit transformation ψ_0. Let $p_s, p \in (1,\infty)$ and $u_\varepsilon, \hat{u}_\varepsilon = u_\varepsilon(\cdot_t, \psi_\varepsilon(\cdot_t, \cdot_x)) \in L^{p_s}(S; L^p(\Omega))$. Then, the following statements hold:*

1. *$u_\varepsilon \overset{p_s,p}{\rightharpoonup} u_0$ for $u_0 \in L^{p_s}(S; L^p(\Omega \times Y))$ if and only if $\hat{u}_\varepsilon \overset{p_s,p}{\rightharpoonup} \hat{u}_0$ for $\hat{u}_0 = u_{0,\psi_0}$ and equivalently $u_0 = \hat{u}_{0,\psi_0^{-1}}$,*
2. *$u_\varepsilon \xrightarrow{p_s,p} u_0$ for $u_0 \in L^{p_s}(S; L^p(\Omega \times Y))$ if and only if $\hat{u}_\varepsilon \xrightarrow{p_s,p} \hat{u}_0$ for $\hat{u}_0 = u_{0,\psi_0}$ and equivalently $u_0 = \hat{u}_{0,\psi_0^{-1}}$.*

Proof. Statement 1 is the time-dependent version of [14, Theorem 3.8] and statement 2 is the time-dependent version of [14, Theorem 3.14]. They can be proven analogously to the stationary case there.

We can apply Proposition 11 for functions defined on the porous subset by extending them by 0 to Ω. However, this can not be transferred directly to the case of weakly differentiable functions because the extension by 0 is not regularity preserving. Therefore, we use the following transformation rule for functions defined on the porous domain.

For the sake of simplicity, in the following Proposition let the domains Ω_ε and $\Omega_\varepsilon(t)$ be given as in the previous sections and $Y_x^*(t) := \psi_0(t, x, Y^*)$.

Proposition 12 (Two-scale transformation of gradients). *Let ψ_ε be a locally periodic transformation in the sense of Definition 4 with limit transformation ψ_0. Let $p_s, p \in (1,\infty)$ and $u_\varepsilon \in L^{p_s}(S; W^{1,p}(\Omega_\varepsilon(t)))$, $\hat{u}_\varepsilon = u_\varepsilon(\cdot_t, \psi_\varepsilon(\cdot_t, \cdot_x)) \in$*

$L^{p_s}(S;W^{1,p}(\Omega_\varepsilon))$, where $\Omega_\varepsilon(t) := \psi_\varepsilon(t,\Omega_\varepsilon)$ for a.e. $t\in S$. Then, $\widetilde{\nabla u_\varepsilon} \xrightarrow{p_s,p} \chi_{Y^*_{(\cdot_x)}(\cdot_t)}(\cdot_y)\nabla_x u_0 + \widetilde{\nabla_y u_1}$ for $(u_0, u_1)\in L^{p_s}(S;W^{1,p}(\Omega))\times L^{p_s}(S;L^p(\Omega;W^{1,p}_{\#}(Y^*_x(t))/\mathbb{R}))$ if and only if $\widetilde{\nabla \hat{u}_\varepsilon} \xrightarrow{p_s,p} \chi_{Y^*}\nabla_x \hat{u}_0 + \widetilde{\nabla_y \hat{u}_1}$ for $\hat{u}_0 = u_0$ and $\hat{u}_1 = u_{1,\psi_0} + \chi_{Y^*}\check{\psi}_0 \cdot \nabla_x u_0$, which is equivalent to $u_1 = \hat{u}_{1,\psi_0^{-1}} + \chi_{Y^*_{(\cdot_x)}(\cdot_t)}\check{\psi}_0^{-1}\cdot\nabla_x \hat{u}_0$.

Proof. Proposition 12 is the time-dependent version of [14, Theorem 3.10] and can be proven analogously to the stationary case there.

In order to homogenise the non-linear boundary terms of (17)–(18), we need a strong convergence of $\hat{u}_\varepsilon$. This can be achieved by extending the functions with the following result (cf. [26], [27]):

Proposition 13. *There exists a family of extension operator $E_\varepsilon \in L(H^1(\Omega_\varepsilon);H^1(\Omega))$ such that*

$$||E_\varepsilon u_\varepsilon||_\Omega \leq C||u_\varepsilon||_{\Omega_\varepsilon}, ||\nabla E_\varepsilon u_\varepsilon||_\Omega \leq C||\nabla u_\varepsilon||_{\Omega_\varepsilon}$$

for every $u_\varepsilon \in H^1(\Omega_\varepsilon)$.

Applying Proposition 13 for a.e. $t\in S$ gives the following time-dependent version of this extension operator.

Corollary 14. *Let $p_s \in [1,\infty]$. There exists a family of linear extension operators E_ε from $L^{p_s}(S;H^1(\Omega_\varepsilon))$ to $L^{p_s}(S;H^1(\Omega))$ such that*

$$||E_\varepsilon u||_{L^{p_s}(S;L^2(\Omega))} \leq C||u||_{L^{p_s}(S;L^2(\Omega_\varepsilon))}, \tag{120}$$
$$||\nabla E_\varepsilon u||_{L^{p_s}(S;L^2(\Omega))} \leq C||\nabla u||_{L^{p_s}(S;L^2(\Omega_\varepsilon))}, \tag{121}$$
$$||E_\varepsilon u(t)||_\Omega \leq C||u(t)||_{\Omega_\varepsilon} \tag{122}$$

for every $u\in L^{p_s}(S;H^1(\Omega_\varepsilon))$ and a.e. $t\in S$.

In order to show the strong convergence of $E_\varepsilon \hat{u}_\varepsilon$, we show the uniform convergence of $\delta_h \hat{u}_\varepsilon$ to 0 for $h\to 0$, where we define $\delta_h\varphi(t) = \varphi(t+h)-\varphi(t)$ for $h>0$ and time dependent functions φ. Then, we apply the compactness result of [21]. This approach has been presented in the context of homogenisation in [28]. However, in our setting the uniform convergence of $\delta_h \hat{u}_\varepsilon$ can not be concluded from a uniform bound of the time derivative as in [28] since $\partial_t \hat{u}_\varepsilon = \langle\partial_t(J_\varepsilon \hat{u}_\varepsilon), J^{-1}\cdot\rangle_{\Omega_\varepsilon} - (\partial_t J_\varepsilon \hat{u}_\varepsilon, J_\varepsilon^{-1}\cdot)_{\Omega_\varepsilon}$ is not uniformly bounded in our setting. The critical point is the ε^{-1}-scaling of ∇J_ε. Therefore, we derive the uniform convergence of $\delta_h \hat{u}_\varepsilon$ to 0 from the weak form in Lemma 16. In order to get rid of the ∇J_ε term, we integrate the solution $\hat{u}_\varepsilon$ over a by h-scaled time interval before we use it as test function in the weak form. Thus, we can shift the time derivative on this more regular test function and it becomes sufficient to estimate J_ε instead of ∇J_ε.

Lemma 15. *Let $\hat{u}_\varepsilon$ be a sequence in $L^2(S;H^1(\Omega_\varepsilon))$ such that*

$$||\hat{u}_\varepsilon||_{L^2(S;H^1(\Omega_\varepsilon))} \leq C, \tag{123}$$

$$||\delta_h \hat{u}_\varepsilon||_{(0,T-h)\times\Omega_\varepsilon} \xrightarrow{h\to 0} 0 \text{ uniformly with respect to } \varepsilon. \tag{124}$$

Then, there exists $\hat{u}_0 \in L^2(S\times\Omega)$ and a subsequence ε such that $E_\varepsilon \hat{u}_\varepsilon$ converges strongly to $\hat{u}_0$ in $L^2(S\times\Omega)$.

Proof. Let $E_\varepsilon \hat{u}_\varepsilon$ be the extension of $\hat{u}_\varepsilon$. Then,

$$||\delta_h E_\varepsilon \hat{u}_\varepsilon||_{(0,T-h)\times\Omega} \leq C||\delta_h \hat{u}_\varepsilon||_{(0,T-h)\times\Omega_\varepsilon} \to 0 \tag{125}$$

converges uniformly (with respect to ε) to zero for $h\to 0$. Moreover, we can estimate for every $0\leq t_1 < t_2 \leq S$ with the Hölder inequality

$$\left|\left|\int_{t_1}^{t_2} E_\varepsilon \hat{u}_\varepsilon(t)dt\right|\right|^2_{H^1(\Omega)} = \int_\Omega \Big(\int_{t_1}^{t_2} E_\varepsilon \hat{u}_\varepsilon(t,x)dt\Big)^2 dx + \int_\Omega \Big(\int_{t_1}^{t_2} \nabla E_\varepsilon \hat{u}_\varepsilon(t,x)dt\Big)^2 dx$$

$$\leq \int_\Omega ||1||^2_S \int_S (E_\varepsilon \hat{u}_\varepsilon)^2(t,x)dtdx + \int_\Omega ||1||^2_S \int_S (\nabla E_\varepsilon \hat{u}_\varepsilon)^2(t,x)dtdx$$

$$= |S|\, ||E_\varepsilon \hat{u}_\varepsilon||^2_{L^2(S;H^1(\Omega))} \leq C||\hat{u}_\varepsilon||^2_{L^2(S;H^1(\Omega_\varepsilon))} \leq C.$$

Since $\int_{t_1}^{t_2} E_\varepsilon \hat{u}_\varepsilon(t)dt$ is uniformly bounded in $H^1(\Omega)$, it is compact in $L^2(\Omega)$. Thus, we can conclude with [21, Theorem 1] that $E_\varepsilon \hat{u}_\varepsilon$ is compact in $L^2(S;L^2(\Omega)) = L^2(S\times\Omega)$.

Lemma 16. *Let $\hat{u}_\varepsilon$ be the solution of* (32)–(33), (44)–(46), *then*

$$||\delta_h \hat{u}_\varepsilon||_{(0,T-h)\times\Omega_\varepsilon} \to 0 \tag{126}$$

for $h\to 0$ uniformly with respect to ε, i.e. there exists a continuous monotonically decreasing function ω: $[0,\infty)\to\mathbb{R}$ with $\omega(0) = 0$ such that

$$||\delta_h \hat{u}_\varepsilon||_{(0,T-h)\times\Omega_\varepsilon} \leq \omega(h)$$

for every $\delta>0$.

Proof. First we note that

$$\delta_h(J_\varepsilon \hat{u}_\varepsilon) = J_\varepsilon \delta_h \hat{u}_\varepsilon + \delta_h J_\varepsilon \hat{u}_\varepsilon(\cdot + h).$$

Thus,

$$\begin{aligned}
c_J \|\delta_h \hat{u}_\varepsilon\|^2_{(0,T-h)\times\Omega_\varepsilon} &\leq (J_\varepsilon \delta_h \hat{u}_\varepsilon, \delta_h \hat{u}_\varepsilon)_{(0,T-h)\times\Omega_\varepsilon} \\
&\leq |(\delta_h(J_\varepsilon \hat{u}_\varepsilon), \delta_h \hat{u}_\varepsilon)_{(0,T-h)\times\Omega_\varepsilon}| + |(\delta_h J_\varepsilon \hat{u}_\varepsilon(\cdot+h), \delta_h \hat{u}_\varepsilon)_{(0,T-h)\times\Omega_\varepsilon}|. \quad (127)
\end{aligned}$$

Since $\|\partial_t r_\varepsilon\|_{L^\infty(S\times\Omega_\varepsilon)} \leq C$, Lemma 1 implies $\|\partial_t J_\varepsilon\|_{L^\infty(S;C(\overline{\Omega_\varepsilon}))} \leq C$ and thus, we can estimate the last term of (127) by

$$\begin{aligned}
|(\delta_h J_\varepsilon \hat{u}_\varepsilon(\cdot+h), \delta_h \hat{u}_\varepsilon)_{(0,T-h)\times\Omega_\varepsilon}| &\leq |Ch(\hat{u}_\varepsilon(\cdot+h), \delta_h \hat{u}_\varepsilon)_{(0,T-h)\times\Omega_\varepsilon}| \\
&\leq Ch \|\hat{u}_\varepsilon(\cdot_t + h)\|_{(0,T-h)\times\Omega_\varepsilon} \|\hat{u}_\varepsilon\|_{(0,T-h)\times\Omega_\varepsilon} \leq Ch.
\end{aligned}$$

Hence, it is sufficient to show that $(\delta_h(J_\varepsilon \hat{u}_\varepsilon), \delta_h \hat{u}_\varepsilon)_{(0,T-h)\times\Omega_\varepsilon}$ converges uniformly to zero for $h \to 0$.

We note that we can rewrite the first term in (44) for $\varphi \in L^2(S;H^1(\Omega_\varepsilon))$ with $\partial_t \varphi \in L^2(S;H^1(\Omega_\varepsilon))$ by

$$\begin{aligned}
&\int_S \langle \partial_t(J_\varepsilon(t)\hat{u}_\varepsilon(t)), \varphi(t)\rangle_{\Omega_\varepsilon} dt \\
&= -(\partial_t \varphi, J_\varepsilon \hat{u}_\varepsilon)_{S\times\Omega_\varepsilon} + (J_\varepsilon(T)\hat{u}_\varepsilon(T), \varphi(T))_{\Omega_\varepsilon} - (J_\varepsilon(0)\hat{u}_\varepsilon(0), \varphi(0))_{\Omega_\varepsilon}.
\end{aligned}$$

Now, we assume that $\varphi \in H^1((-h,T);H^1(\Omega_\varepsilon))$ with $\varphi(-h) = \varphi(T) = 0$. Then, we test (44) with $\delta_{-h}\varphi$ and use

$$(\partial_t \varphi(\cdot_t - h), J_\varepsilon \hat{u}_\varepsilon)_{S\times\Omega_\varepsilon} = (\partial_t \varphi, J_\varepsilon(\cdot_t + h)\hat{u}_\varepsilon(\cdot_t + h))_{(-h,T-h)\times\Omega_\varepsilon}, \quad (128)$$

which yields

$$\begin{aligned}
&(\partial_t \varphi, \delta_h(J_\varepsilon \hat{u}_\varepsilon))_{(0,T-h)\times\Omega_\varepsilon} \\
&= -(\partial_t \varphi, J_\varepsilon(\cdot_t + h)\hat{u}_\varepsilon(\cdot_t + h))_{(-h,0)\times\Omega_\varepsilon} + (\partial_t \varphi, J_\varepsilon \hat{u}_\varepsilon)_{(T-h,T)\times\Omega_\varepsilon} \\
&+ (J_\varepsilon(0)\hat{u}_\varepsilon(0), \varphi(0))_{\Omega_\varepsilon} - (J_\varepsilon(T)\hat{u}_\varepsilon(T), \varphi(T-h))_{\Omega_\varepsilon} \\
&+ (A_\varepsilon(t)\nabla \hat{u}_\varepsilon(t), \nabla\delta_{-h}\varphi)_{S\times\Omega_\varepsilon} + (B_\varepsilon(t)\hat{u}_\varepsilon(t), \nabla\delta_{-h}\varphi)_{S\times\Omega_\varepsilon} \\
&- (J_\varepsilon(t)\hat{f}^{\mathrm{p}}_\varepsilon(t), \delta_{-h}\varphi)_{S\times\Omega_\varepsilon} + \sum_{k\in I_\varepsilon} \frac{r^{n-1}_{\varepsilon,k}}{r^{n-1}_0} (\varepsilon f(\hat{u}_\varepsilon(t), r_{\varepsilon,k}(t)), \delta_{-h}\varphi)_{S\times\Gamma_{\varepsilon,k}} \\
&=: M_1 + \cdots + M_8. \qquad (129)
\end{aligned}$$

Now we choose

$$\varphi(t) = h^{-1}\int_t^{t+h} \hat{u}_\varepsilon(\tau)\,d\tau \tag{130}$$

where we implicitly extend $\hat{u}_\varepsilon(\tau)$ by 0 for $\tau > T$ and for $\tau < 0$. Thus, we get for a.e. $t \in S$

$$\partial_t\varphi(t) = \begin{cases} h^{-1}\hat{u}_\varepsilon(t+h) & t < 0, \\ h^{-1}(\hat{u}_\varepsilon(t+h) - \hat{u}_\varepsilon(t)) & 0 < t < T-h, \\ -h^{-1}\hat{u}_\varepsilon(t) & t > T-h. \end{cases} \tag{131}$$

Then, the left-hand side of (129) can be rewritten by

$$(\partial_t\varphi, \delta_h(J_\varepsilon\hat{u}_\varepsilon))_{(0,T-h)\times\Omega_\varepsilon} = h^{-1}(\delta_h\hat{u}_\varepsilon, \delta_h(J_\varepsilon\hat{u}_\varepsilon))_{(0,T-h)\times\Omega_\varepsilon}. \tag{132}$$

Hence, it is sufficient to show that $M_1, M_2, \ldots, M_8$ are uniformly bounded for φ given by (130).

• $M_1, \ldots, M_4$. Since $\|\hat{u}_\varepsilon\|_{C^0(\overline{S};L^2(\Omega_\varepsilon))} \leq C$, we can estimate

$$M_1 = -(h^{-1}\hat{u}_\varepsilon(\cdot_t + h), J_\varepsilon(\cdot_t + h)\hat{u}_\varepsilon(\cdot_t + h))_{(-h,0)\times\Omega_\varepsilon} \leq C, \tag{133}$$

$$M_2 = (h^{-1}\hat{u}_\varepsilon, J_\varepsilon\hat{u}_\varepsilon)_{(T-h,T)\times\Omega_\varepsilon} \leq C, \tag{134}$$

$$M_3 = \left(J_\varepsilon(0)\hat{u}_\varepsilon(0), h^{-1}\int_0^h \hat{u}_\varepsilon(\tau)\,d\tau\right)_{\Omega_\varepsilon} \leq C, \tag{135}$$

$$M_4 = -\left(J_\varepsilon(T)\hat{u}_\varepsilon(T), \int_{T-h}^T \hat{u}_\varepsilon(\tau)\,d\tau\right)_{\Omega_\varepsilon} \leq C. \tag{136}$$

• M_5, M_6 *and* M_7. We show the estimate for M_5. The estimate for M_6 follows analogously and the estimate for M_7 is similar. We rewrite

$$M_5 = h^{-1}\int_S \left(A_\varepsilon(t)\nabla\hat{u}_\varepsilon(t), \int_{t-h}^t \nabla\hat{u}_\varepsilon(\tau)\,d\tau\right)_{\Omega_\varepsilon} dt$$
$$- h^{-1}\int_S \left(A_\varepsilon(t)\nabla\hat{u}_\varepsilon(t), \int_t^{t+h} \nabla\hat{u}_\varepsilon(\tau)\,d\tau\right)_{\Omega_\varepsilon} dt =: M_{5a} + M_{5b}.$$

Then, we get with the Hölder inequality

$$
\begin{aligned}
M_{5a} &\leq h^{-1} \int_0^h \int_S C \|\nabla \hat{u}_\varepsilon(t)\|_{\Omega_\varepsilon} \|\nabla \hat{u}_\varepsilon(t-h+\tau)\|_{\Omega_\varepsilon} \, dt\, d\tau \\
&\leq C h^{-1} \int_0^h \|\nabla \hat{u}_\varepsilon\|_{S\times\Omega_\varepsilon} \|\nabla \hat{u}_\varepsilon(\cdot_t - h+\tau)\|_{S\times\Omega_\varepsilon} \, d\tau \leq C \|\nabla \hat{u}_\varepsilon\|^2_{S\times\Omega_\varepsilon} \leq C
\end{aligned}
$$

and by the same argumentation we can estimate M_{5b}.

• M_8. We split M_8 into two sums as we already did for M_5. We show the estimate for the first summand. The estimate for the second summand can be done in the same way. With the ε-scaled trace inequality, the uniform bound of r_ε and f we get

$$
\begin{aligned}
&h^{-1} \sum_{k\in I_\varepsilon} \left(\varepsilon \frac{r^{n-1}_{\varepsilon,k}}{r_0^{n-1}} f(\hat{u}_\varepsilon(t), r_{\varepsilon,k}(t)), \int_{t-h}^{t} \hat{u}_\varepsilon(\tau)\, d\tau \right)_{S\times\Gamma_{\varepsilon,k}} \\
&\leq h^{-1} \int_0^h C \sum_{k\in I_\varepsilon} \varepsilon \int_{S\times\Gamma_{\varepsilon,k}} |\hat{u}_\varepsilon(t-h+\tau, x)| \, d\tau\, dx\, dt \\
&\leq \varepsilon C \|\hat{u}_\varepsilon\|_{L^1(S\times\Gamma_\varepsilon)} \leq C \|\hat{u}_\varepsilon\|_{L^1(S\times\Omega_\varepsilon)} + C\varepsilon \|\nabla \hat{u}_\varepsilon\|_{L^1(S\times\Omega_\varepsilon)} \leq C.
\end{aligned}
$$

Combining the estimates of $M_1, M_2, \ldots, M_8$ shows, that $h^{-1}(\delta_h \hat{u}_\varepsilon, \delta_h (J_\varepsilon \hat{u}_\varepsilon))_{(0,T-h)\times\Omega_\varepsilon}$ is uniformly bounded and hence that $(\delta_h \hat{u}_\varepsilon, \delta_h (J_\varepsilon \hat{u}_\varepsilon))_{(0,T-h)\times\Omega_\varepsilon}$ converges uniformly to 0.

Theorem 17. *Let $(\hat{u}_\varepsilon, r_\varepsilon)$ be the unique solution of (32)–(33), (44)–(46). Then, there exists for every subsequence a further subsequence ε such that*

$$
\begin{aligned}
&\widetilde{\hat{u}_\varepsilon} \xrightarrow{2,2} \chi_{Y^*} \hat{u}_0 && \text{with respect to the } L^2\text{-norm}, \quad (137)\\
&\widetilde{\nabla \hat{u}_\varepsilon} \xrightarrow{2,2} \chi_{Y^*} \nabla \hat{u}_0 + \widetilde{\nabla_y \hat{u}_1} && \text{with respect to the } L^2\text{-norm}, \quad (138)\\
&r_{\varepsilon,k_\varepsilon(\cdot_x)} \to r \text{ in } L^\infty(S; L^p(\Omega)) && \text{for every } p \in [1,\infty), \quad (139)\\
&\partial_t r_{\varepsilon,k_\varepsilon(\cdot_x)} \to \partial_t r \text{ in } L^p(S\times\Omega) && \text{for every } p \in [1,\infty), \quad (140)
\end{aligned}
$$

where $(\hat{u}_0, \hat{u}_1, r) \in L^2(S; H^1(\Omega)) \times L^2(S; L^2(\Omega; H^1_\#(Y^*)/\mathbb{R})) \times W^{1,\infty}(S; L^2(\Omega))$ is a solution of (141)–(142).

Find $(\hat{u}_0, \hat{u}_1, r) \in L^2(S;H^1(\Omega))\times L^2(S;L^2(\Omega;H^1_\#(Y^*)/\mathbb{R}))\times W^{1,\infty}(S;L^2(\Omega))$ such that

$$\int_\Omega (1-V_N(r(0,x)))\hat{u}_0^{(0)}(x)\varphi(0,x)dx - \int_S\int_\Omega (1-V_N(r(t,x)))\hat{u}_0(t,x)\partial_t\varphi(t,x)dxdt$$

$$+\int_S\int_\Omega\int_{Y^*} A_0(t,x,y)(\nabla_x\hat{u}_0(t,x)+\nabla_y\hat{u}_1(t,x,y))\cdot(\nabla_x\varphi(t,x)+\nabla_y\varphi_1(t,x,y))dydxdt$$

$$=\int_S\int_\Omega (1-V_N(r(t,x)))f^{\mathrm{p}}(t,x)\varphi(t,x)-\partial_t V_N(r(t,x))c_s\varphi(t,x)dxdt \quad (141)$$

$$\int_S\int_\Omega \partial_t r(t,x)\phi(t,x)dxdt = \int_S\int_\Omega \frac{1}{c_s}f(\hat{u}_0(t,x),r(t,x))\phi(t,x)dxdt \quad (142)$$

for every $(\varphi,\varphi_1,\phi)\in H^1(S\times\Omega)\times L^2(S;L^2(\Omega;H^1_\#(Y^*)/\mathbb{R}))\times L^2(S\times\Omega)$ with initial values $r(0)=r^{(0)}$ and $\hat{u}_0^{(0)}=u^{(0)}$.

Proof. Having the uniform estimates (50), we can apply Proposition 8, which gives $\hat{u}_0\in L^2(S;H^1(\Omega))$, $\hat{u}_1\in L^2(S\times\Omega;H^1_\#(Y^*)/\mathbb{R})$ such that for a subsequence:

$$\widetilde{\hat{u}_\varepsilon} \xrightarrow[Y^*]{2,2} \hat{u}_0\chi, \quad \widetilde{\nabla u_\varepsilon} \xrightarrow[Y^*]{2,2} \chi\nabla\hat{u}_0+\widetilde{\nabla_y\hat{u}_1}. \quad (143)$$

With (50) and (126), we can apply Lemma 15 and get (after passing to a further subsequence and identifying the limits)

$$E_\varepsilon\hat{u}_\varepsilon\to\hat{u}_0\in L^2(S\times\Omega). \quad (144)$$

Thus the first convergence of (143) is strong. Moreover, this implies

$$\mathcal{T}_\varepsilon E_\varepsilon\hat{u}_\varepsilon\to\hat{u}_0 \text{ in } L^2(S\times\Omega\times Y) \quad (145)$$

and we get with $\mathcal{T}_\varepsilon\nabla E_\varepsilon\hat{u}_\varepsilon=\varepsilon^{-1}\nabla_y\mathcal{T}_\varepsilon\hat{u}_\varepsilon$, the isometry of $\mathcal{T}_\varepsilon$ and the uniform boundedness of $||\nabla\hat{u}_\varepsilon||_{S\times\Omega}$:

$$||\nabla_y\mathcal{T}_\varepsilon E_\varepsilon\hat{u}_\varepsilon||_{S\times\Omega\times Y}=\varepsilon||\mathcal{T}_\varepsilon\nabla E_\varepsilon\hat{u}_\varepsilon||_{S\times\Omega\times Y}=\varepsilon||\nabla E_\varepsilon\hat{u}_\varepsilon||_{S\times\Omega}\leq C\varepsilon||\nabla\hat{u}_\varepsilon||_{S\times\Omega}\to 0.$$

Thus, we can conclude with the trace operator on Γ

$$\begin{aligned}&||\mathcal{T}_\varepsilon\hat{u}_\varepsilon-\hat{u}_0||_{S\times\Omega\times\Gamma}=||\mathcal{T}_\varepsilon E_\varepsilon\hat{u}_\varepsilon-\hat{u}_0||_{S\times\Omega\times\Gamma}\\&\leq C||\mathcal{T}_\varepsilon E_\varepsilon\hat{u}_\varepsilon-\hat{u}_0||_{S\times\Omega\times Y^*}+C||\nabla_y\mathcal{T}_\varepsilon E_\varepsilon\hat{u}_\varepsilon-\nabla_y\hat{u}_0||_{S\times\Omega\times Y^*}\to 0. \quad (146)\end{aligned}$$

In order to pass to the limit $\varepsilon\to 0$ in the non-linear bulk and boundary terms, we show the strong convergence $r_{\varepsilon,k_\varepsilon(\cdot_x)}\to r$ at first. We define $r\in W^{1,\infty}(S;L^2(\Omega))$ as the unique solution of (142) with initial value $r(0)=r^{(0)}$ and $\hat{u}_0$ given by (144). Then, we test (45) by $\chi_{(0,t)}(r_{\varepsilon,k_\varepsilon(x)}-r(x))$ for a.e. $t\in S$, a.e. $x\in\varepsilon k+\varepsilon Y$ and every $k\in I_\varepsilon$, integrate over $\varepsilon k+\varepsilon Y$ and sum over $k\in I_\varepsilon$:

$$
\begin{aligned}
&(\partial_t r_{\varepsilon,k_\varepsilon(\cdot_x)}, r_{\varepsilon,k_\varepsilon(\cdot_x)}-r)_{(0,t)\times\Omega}\\
&=\int_0^t\int_\Omega \frac{\varepsilon^{-N}}{c_s S_{N-1}(r_0)}\int_{\Gamma_{\varepsilon,k_\varepsilon(x)}}\varepsilon f(\hat{u}_\varepsilon(\tau,y), r_{\varepsilon,k_\varepsilon(x)})d\sigma_y(r_{\varepsilon,k_\varepsilon(x)}(\tau)-r(\tau,x))dxd\tau\\
&=\left(\frac{1}{c_s S_{N-1}(r_0)}f(\mathcal{T}_\varepsilon\hat{u}_\varepsilon, r_{\varepsilon,k_\varepsilon(\cdot_x)}), r_{\varepsilon,k_\varepsilon(\cdot_x)}-r\right)_{(0,t)\times\Omega\times\Gamma}
\end{aligned}
\tag{147}
$$

We test (142) with $\chi_{(0,t)}(r_{\varepsilon,k_\varepsilon(\cdot_x)}-r)$ and subtract it from (147):

$$
\begin{aligned}
&(\partial_t(r_{\varepsilon,k_\varepsilon(\cdot_x)}-r), r_{\varepsilon,k_\varepsilon(\cdot_x)}-r)_{(0,t)\times\Omega}\\
&=\frac{1}{c_s S_{N-1}(r_0)}(f(\mathcal{T}_\varepsilon\hat{u}_\varepsilon, r_{\varepsilon,k_\varepsilon(\cdot_x)})-f(\hat{u}_0,r), r_{\varepsilon,k_\varepsilon(\cdot_x)}-r)_{(0,t)\times\Omega\times\Gamma}
\end{aligned}
$$

Then, we rewrite the left-hand side and estimate the right-hand side using the Cauchy–Schwarz inequality, the Lipschitz condition (5) and the Young inequality:

$$
-\frac{1}{2}\left\|r^{(0)}_{\varepsilon,k_\varepsilon(\cdot_x)}-r^{(0)}\right\|^2_\Omega+\frac{1}{2}\left\|r_{\varepsilon,k_\varepsilon(\cdot_x)}(t)-r(t)\right\|^2_\Omega\le C\|\mathcal{T}_\varepsilon\hat{u}_\varepsilon-u_0\|^2_{(0,t)\times\Omega\times\Gamma}+C\left\|r_{\varepsilon,k_\varepsilon(\cdot_x)}-r\right\|^2_{(0,t)\times\Omega}
$$

We estimate further with Gronwall's inequality and pass to the limit using (146) and the strong convergence of the initial values:

$$
\left\|r_{\varepsilon,k_\varepsilon(\cdot_x)}-r_0\right\|^2_{L^\infty(S;L^2(\Omega))}\le C\|\mathcal{T}_\varepsilon\hat{u}_\varepsilon-u_0\|^2_{S\times\Omega\times\Gamma}+C\left\|r^{(0)}_{\varepsilon,k_\varepsilon(\cdot_x)}-r^{(0)}\right\|^2_\Omega\to 0.
$$

Since $r_{\varepsilon,k_\varepsilon(\cdot_x)}$ and r_0 are uniformly bounded in $L^\infty(S\times\Omega)$, we get $\left\|r_{\varepsilon,k_\varepsilon(\cdot_x)}-r_0\right\|_{L^\infty(S;L^p(\Omega))}$ for every $p\in[1,\infty)$. Thus, Lemma 9 shows that ψ_ε are locally periodic transformations in the sense of Definition 4 and we can conclude with Lemma 10 the strong two-scale convergence of $J_\varepsilon,\Psi_\varepsilon,\Psi_\varepsilon^{-1}$, which we need in order to pass to the limit $\varepsilon\to 0$ in (44). Moreover, Definition 4, Proposition 11 and 10 can be also formulated for the two-scale convergence without time parameter (cf. [14]). Thus, we can conclude the strong two-scale convergence also for the initial data, i.e. $J_\varepsilon(0)$ two-scale converges strongly to $J_0(0)$ and $\widetilde{\hat{u}^{(0)}_\varepsilon}$ two scale converges to $\chi_{Y^*_{(\cdot_x)}(\cdot_t)}(\cdot_y)\hat{u}^{(0)}_0(\cdot_x)$ with $\hat{u}^{(0)}_0=u^{(0)}_0(\cdot_x,\psi_0(0,\cdot_x,\cdot_y))$.

The strong convergence $\partial_t r_{\varepsilon,k_\varepsilon(\cdot_x)} \to \partial_t r$ follows similarly. By testing (45) and (142) with $\partial_t (r_{\varepsilon,k_\varepsilon(\cdot_x)} - r)$ and then subtracting the equations, we can conclude the strong convergence in $L^2(S\times\Omega)$. Subsequently, the boundedness in $L^\infty(S\times\Omega)$ implies the strong convergence in $L^p(S\times\Omega)$ for every $p\in[1,\infty)$. However, we do not need this strong convergence in order to pass to the limit, although the term $B_\varepsilon = J_\varepsilon \Psi_\varepsilon^{-1}\partial_t\psi_\varepsilon$ contains the time derivative of ψ_ε. The reason is that $||\partial_t\psi_\varepsilon||_{L^\infty(S\times\Omega)} \leq \varepsilon C \left|\left|\partial_t r_{\varepsilon,k_\varepsilon(\cdot_x)}\right|\right|_{L^\infty(S\times\Omega)}$ and thus the boundedness of $\left|\left|\partial_t r_{\varepsilon,k_\varepsilon(\cdot_x)}\right|\right|_{L^\infty(S\times\Omega)}$ is already sufficient for the limit process.

In order to pass to the limit in (44), we test it by $\varphi(\cdot_t,\cdot_x) + \varepsilon\varphi_1(\cdot_t,\cdot_x,\frac{\cdot_x}{\varepsilon})$ for $(\varphi,\varphi_1)\in C^\infty(S;C^\infty(\Omega))\times D(S;C^\infty(\Omega;C^\infty_\#(Y)))$ with $\varphi(T) = 0$ and integrate the time derivative term by parts:

$$
\begin{aligned}
&\int\limits_{\Omega_\varepsilon} J_\varepsilon(t,x,y)\hat{u}_\varepsilon^{(0)}(x)\left(\varphi(0,x) + \varepsilon\varphi_1\left(0,x,\tfrac{x}{\varepsilon}\right)\right)dx \\
&- \int\limits_S\int\limits_{\Omega_\varepsilon} J_\varepsilon(t,x)\hat{u}_\varepsilon(t,x)\left(\partial_t\varphi(t,x) + \partial_t\varphi_1\left(t,x,\tfrac{x}{\varepsilon}\right)\right)dxdt \\
&+ \int\limits_S\int\limits_{\Omega_\varepsilon} A_\varepsilon(t,x,y)\nabla\hat{u}_\varepsilon(t,x)\cdot\left(\nabla_x\varphi(t,x) + \varepsilon\nabla_x\varphi_1\left(t,x,\tfrac{x}{\varepsilon}\right) + \nabla_y\varphi_1\left(t,x,\tfrac{x}{\varepsilon}\right)\right)dxdt \\
&+ \int\limits_S\int\limits_{\Omega_\varepsilon} B_\varepsilon(t,x,y)\hat{u}_\varepsilon(t,x)\cdot\left(\nabla_x\varphi(t,x) + \varepsilon\nabla_x\varphi_1\left(t,x,\tfrac{x}{\varepsilon}\right) + \nabla_y\varphi_1\left(t,x,\tfrac{x}{\varepsilon}\right)\right)dxdt \\
&= \int\limits_S\int\limits_{\Omega_\varepsilon} J_\varepsilon(t,x)\hat{f}_\varepsilon^{\mathrm{p}}(t,x)\left(\varphi(t,x) + \varepsilon\varphi_1\left(t,x,\tfrac{x}{\varepsilon}\right)\right)dxdt \\
&- \sum_{k\in I_\varepsilon}\int\limits_S\int\limits_{\Gamma_{\varepsilon,k}} \varepsilon\frac{r_{\varepsilon,k}^{n-1}(t)}{r_0^{n-1}} f(\hat{u}_\varepsilon(t,x), r_{\varepsilon,k}(t))(\varphi(t,x) + \varepsilon\varphi_1(t,x))d\sigma_x dt.
\end{aligned}
$$

We rewrite the boundary integral with the unfolding operator $\mathcal{T}_\varepsilon$, so that we can pass to the limit $\varepsilon\to 0$ using the strong convergences of $\mathcal{T}_\varepsilon u_\varepsilon$ and $r_{\varepsilon,k_\varepsilon(\cdot_x)}$

and the continuity of f:

$$\sum_{k\in I_\varepsilon}\int_S\int_{\Gamma_{\varepsilon,k}} \varepsilon\frac{r_{\varepsilon,k}^{n-1}(t)}{r_0^{n-1}} f(\hat{u}_\varepsilon(t,x), r_{\varepsilon,k}(t))(\varphi(t,x)+\varepsilon\varphi_1(t,x,y))d\sigma_y dxdt$$
$$=\int_S\int_\Omega\int_\Gamma \frac{r_{\varepsilon,k_\varepsilon(x)}^{n-1}(t)}{r_0^{n-1}} f(\mathcal{T}_\varepsilon\hat{u}_\varepsilon(t,x,y), r_{\varepsilon,k_\varepsilon(x)}(t))$$
$$\left(\mathcal{T}_\varepsilon\varphi(t,x)+\varepsilon\mathcal{T}_\varepsilon\left(\varphi_1\left(\cdot_t,\cdot_x,\tfrac{\cdot_x}{\varepsilon}\right)\right)(t,x,y)\right)d\sigma_y dxdt$$
$$\to\int_S\int_\Omega\int_\Gamma \frac{r^{n-1}(t,x)}{r_0^{n-1}} f(\hat{u}_0(t,x), r(t,x))\varphi(t,x)d\sigma_y dxdt \qquad (148)$$

Using (142) and $S_{n-1}(r)=\partial_r V_N(r)$, we can rewrite the right-hand side of (148):

$$\int_S\int_\Omega\int_\Gamma \frac{r^{n-1}(t,x)}{r_0^{n-1}} f(\hat{u}_0(t,x), r(t,x))\varphi(t,x)d\sigma_y dxdt=\int_S\int_\Omega \partial_t V_N(r(t,x))c_s\varphi(t,x)dxdt$$

Moreover, the uniform boundedness of $\partial_t r_\varepsilon$ given by (52) implies $\partial_t\psi_\varepsilon\to 0$ in $L^\infty(S\times\Omega)$. Thus, $B_\varepsilon\hat{u}_\varepsilon$ vanishes in the limit $\varepsilon\to 0$ of (148) and we obtain

$$\int_\Omega\int_{Y^*} J_0(0,x,y)\hat{u}_0^{(0)}(x)\varphi(0,x)dydx-\int_S\int_\Omega\int_{Y^*} J_0(t,x,y)\hat{u}_0(t,x)\partial_t\varphi(t,x)dydxdt$$
$$+\int_S\int_\Omega\int_{Y^*} A_0(t,x,y)(\nabla_x\hat{u}_0(t,x)+\nabla_y\hat{u}_1(t,x,y))\cdot(\nabla_x\varphi(t,x)+\nabla_y\varphi_1(t,x,y))dydxdt$$
$$=\int_S\int_\Omega\int_{Y^*}(J_0(t,x,y)f^{\mathrm{p}}(t,x)dy-\partial_t V_N(r(t,x))c_s)\varphi(t,x)dxdt$$

which can be rewritten into (141). By a density argument it holds for every $(\varphi,\varphi_1)\in H^1(S\times\Omega))\times L^2(S;L^2(\Omega;H^1_\#(Y^*)/\mathbb{R}))$.

5. Backtransformation

Now we transform the two-scale limit problem back from its substitute domain to its actual two-scale domain and obtain the following transformation-independent weak two-scale formulation.

Theorem 18 (Two-scale limit problem). *Let $(u_\varepsilon, r_\varepsilon)$ be the solution of (2),(17)–(19). Then, there exists for every subsequence a further subsequence ε such that*

$$\tilde{u}_\varepsilon \xrightarrow{2,2} \chi_{Y^*_{r(\cdot_t,\cdot_x)}}(\cdot_y)u_0 \qquad \text{with respect to the } L^2-\text{norm}, \tag{149}$$

$$\widetilde{\nabla u_\varepsilon} \xrightarrow{2,2} \chi_{Y^*_{r(\cdot_t,\cdot_x)}}(\cdot_y)\nabla_x u_0 + \widetilde{\nabla_y u_1} \qquad \text{with respect to the } L^2-\text{norm} \tag{150}$$

and the convergences (139)–(140) hold, where $(u_0, u_1, r) \in L^2(S;H^1(\Omega)) \times L^2(S;L^2(\Omega;H^1_\#(Y^*_{r(t,x)})/\mathbb{R})) \times L^2(S;L^2(\Omega))$ is a solution of the following weak form:

Find $(u_0, u_1, r) \in L^2(S;H^1(\Omega)) \times L^2(S;L^2(\Omega;H^1_\#(Y^*_{r(t,x)})/\mathbb{R})) \times W^{1,\infty}(S;L^2(\Omega))$ with $\partial_t((1-V_n(r))u_0) \in L^2(S;H^1(\Omega)')$ such that

$$\int_S \langle \partial_t((1-V_N(r(t)))u_0(t)), \varphi(t)\rangle_\Omega dt$$

$$+ \int_S \int_\Omega \int_{Y^*_{r(t,x)}} (\nabla_x u_0(t,x) + \nabla_y u_1(t,x,y)) \cdot (\nabla_x \varphi(t,x) + \nabla_y \varphi_1(t,x,y)) dy dx dt$$

$$= \int_S \int_\Omega (1-V_N(r(t,x))) f^{\mathrm{p}}(t,x)\varphi(t,x) - \partial_t V_N(r(t,x)) c_s \varphi(t,x) dx dt \tag{151}$$

$$\int_S \int_\Omega \partial_t r(t,x)\phi(t,x) dx dt = \int_S \int_\Omega \frac{1}{c_s} f(u_0(t,x), r(t,x))\phi(t,x) dx dt \tag{152}$$

hold for every $(\varphi, \varphi_1, \phi) \in L^2(S;H^1(\Omega)) \times L^2(S\times\Omega;H^1(Y^*_{r(t,x)})/\mathbb{R}) \times L^2(S\times\Omega)$ with initial values $r(0) = r^{(0)}$ and $((1-V_N(r))u_0)(0) = (1-V_N(r^{(0)}))u_0^{(0)}$.

Proof. We test (141) with $(\varphi, \varphi_{1,\psi_0} + \check{\psi}_0 \cdot \nabla_x \varphi)$ for $(\varphi, \varphi_1) \in C^\infty(S;C^\infty(\Omega)) \times C^\infty(S;C^\infty(\Omega;H^1_\#(Y))$ with $\varphi(T) = 0$. Then, we transform the Y^* integral in (141) with $\psi_0^{-1}(t,x)$ by

$$\int_S \int_\Omega \int_{Y^*} A_0(t,x,y)(\nabla_x \hat{u}_0(t,x) + \nabla_y \hat{u}_1(t,x,y))$$

$$\cdot(\nabla_x \varphi(t,x) + \nabla_y(\varphi_{1,\psi_0} + \check{\psi}_0(t,x,y)\cdot\nabla_x\varphi(t,x)))dy dx dt$$

$$= \int_S \int_\Omega \int_{Y^*_{r(t,x)}} (\Psi^{-1}_{0,\psi_0^{-1}}(t,x,y)\nabla_x \hat{u}_0(t,x) + \nabla_y \hat{u}_{1,\psi_0^{-1}}(t,x,y))$$

$$\cdot(\Psi^{-1}_{0,\psi_0^{-1}}(t,x,y)\nabla_x\varphi(t,x) + \nabla_y(\varphi_1(t,x,y) + \check{\psi}_{0,\psi_0^{-1}}(t,x,y)\cdot\nabla_x\varphi(t,x)))dy dx dt$$

Using $\Psi^{-1}_{0,\psi_0^{-1}}(t,x,y) = \mathbb{1} + \nabla_y \check{\psi}_0^{-1}(t,x,y)$, we can rewrite

$$
\begin{aligned}
&\Psi^{-1}_{0,\psi_0^{-1}}(t,x,y)\nabla_x u_0(t,x) + \nabla_y \hat{u}_{1,\psi_0^{-1}}(t,x,y)\\
&= \nabla_x \hat{u}_0(t,x) + \nabla_y(\hat{u}_{1,\psi_0^{-1}}(t,x,y) + \check{\psi}_0^{-1}(t,x,y)\cdot\nabla_x u_0(t,x))\\
&= \nabla_x u_0(t,x) + \nabla_y u_1(t,x,y))
\end{aligned}
$$

for a.e. $(t,x)\in S\times\Omega$ and a.e. $y\in Y^*_{r(t,x)}$ with $u_0 = \hat{u}_0$ and $u_1 = \hat{u}_{1,\psi_0^{-1}} + \chi_{Y^*_{r(t,x)}}\check{\psi}_0^{-1}\cdot\nabla_x\hat{u}_0$. Using the fact that

$$
\begin{aligned}
\check{\psi}_{0,\psi_0^{-1}}(t,x,y) = \check{\psi}_0(t,x,\psi_0^{-1}(t,x,y)) &= \psi_0(t,x,\psi_0^{-1}(t,x,y)) - \psi_0^{-1}(t,x,y)\\
&= y - \psi_0^{-1}(t,x,y) = -\check{\psi}_0^{-1}(t,x,y)
\end{aligned}
$$

we get

$$
\begin{aligned}
&\Psi^{-1}_{0,\psi_0^{-1}}(t,x,y)\nabla_x\varphi(t,x) + \nabla_y(\varphi_1(t,x,y) + \check{\psi}_{0,\psi_0^{-1}}(t,x,y)\cdot\nabla_x\varphi(t,x))\\
&= \nabla_x\varphi(t,x) + \nabla_y(\check{\psi}_0^{-1}(t,x,y)\cdot\nabla_x\varphi(t,x) + \varphi_1(t,x,y) + \check{\psi}_{0,\psi_0^{-1}}(t,x,y)\cdot\nabla_x\varphi(t,x))\\
&= \nabla_x\varphi(t,x) + \nabla_y\varphi_1(t,x,y).
\end{aligned}
$$

Thus, we get

$$
\begin{aligned}
&\int_S\int_\Omega\int_{Y^*} A_0(t,x,y)(\nabla_x\hat{u}_0(t,x) + \nabla_y\hat{u}_1(t,x,y))\\
&\cdot(\nabla_x\varphi(t,x) + \nabla_y(\varphi_{1,\psi_0} + \check{\psi}_0(t,x,y)\cdot\nabla_x\varphi(t,x)))dydxdt\\
&= \int_S\int_\Omega\int_{Y^*_{r(t,x)}} (\nabla_x u_0(t,x) + \nabla_y u_1(t,x,y))\cdot(\nabla_x\varphi(t,x) + \nabla_y\varphi_1(t,x,y))dydxdt,
\end{aligned}
$$

which allows us to rewrite (141) into (151) after integrating the second term of (141) by parts with respect to time. By a density argument (151) holds for all $(\varphi,\varphi_1)\in L^2(S;H^1(\Omega))\times L^2(S\times\Omega;H^1(Y^*_{r(t,x)})/\mathbb{R})$. Then, the two-scale-convergences (149)–(150) follows from Proposition 11 and Proposition 12.

Theorem 19 (Homogenised limit problem). *A tuple* (u_0, r) *is part of a solution of the two-scale limit problem* (151)–(152) *given by Theorem 18 if* *and only if it solves*

$$\int_S \langle \partial_t(1-V_N(r(t))u_0(t)),\varphi(t)\rangle_\Omega dt + (A_{\mathrm{hom}}(r)\nabla_x u_0,\nabla_x\varphi)_{S\times\Omega}$$
$$= ((1-V_N(r))f^{\mathrm{p}} - \partial_t V_N(r(t,x))c_s,\varphi)_{S\times\Omega} \tag{153}$$

and (152) *for every* $(\varphi,\phi)\in L^2(S;H^1(\Omega))\times L^2(S\times\Omega)$ *with initial value* $(1-V_N(r(0)))u_0(0) = (1-V_N(r^{(0)}))u_0^{(0)}$, *where* A *is given by*

$$(A_{\mathrm{hom}})_{ij}(r) := \int_{Y_r^*} \delta_{ij} + \partial_{y_i} w_j(r;y)dy \tag{154}$$

and $w_j(r)$ *is the unique solution in* $H^1_\#(Y_r^*)/\mathbb{R}$ *such that*

$$\int_{Y_r^*} (\nabla_y w_j(r;y) + e_j)\cdot\nabla_y\varphi(y)dy = 0. \tag{155}$$

for every $\varphi\in H^1_\#(Y_r^*)$.

Proof. Choosing $\varphi = 0$ in (151) implies $u_1(t,x,y) = \sum_{i=1}^{N} \partial_{x_j} u_0(t,x) w_j(r(t,x),y)$. Inserting this in (151) yields (153) for A_{hom} given by (154).

Note that we formulate the initial condition in Theorem 18 and Theorem 19 only for $(1-V_N(r))u_0$ and not for u_0. The reason is that $1-V_N(r)$ is a priori not regular enough in space in order to transfer the time regularity of $(1-V_N(r))u_0$ on u_0. However, this is not a drawback since $(1-V_N(r))u_0$ is the actual physically measurable quantity.

In our model the total mass is given by the sum of the mass in the pore space and the mass in the solid space. Thus, the conservation of mass reads $\partial_t((1-V_N(r))u_0)+\partial_t V_N(r)c_s =$ density of external sources. Testing our limit model (153) with $\varphi\in C^\infty(S)$ yields exactly this

$$\partial_t((1-V_N(r))u_0) = (1-V_N(r))f^p - \partial_t V_N(r)c_s. \tag{156}$$

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