The two-scale-transformation method

David Wiedemann

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David Wiedemann[1],*

University of Augsburg, 86135 Augsburg, Germany

Abstract

We prove the two-scale-transformation method which allows rigorous homogenisation of problems defined on locally periodic domains by transformation on periodic domains. The idea to consider periodic substitute problems was originally proposed for the homogenisation on evolving microstructure and is applied in several works. However, only the homogenisation of the periodic substitute problems was proven, whereas the method itself was just postulated (i.e. the equivalence to the homogenisation of the actual problem had to be assumed). In this work, we develop this idea further and formulate a rigorous two-scale convergence concept for microscopic transformation to prove this method. Moreover, we show a new two-scale transformation rule for gradients which allows to derive new limit problems that are now transformationally independent.

Keywords: Periodic homogenisation, two-scale convergence, two-scale-transformation method, locally periodic microstructure, evolving microstructure
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1. Introduction

Periodic homogenisation allows to derive effective macroscopic models for processes described on a fine heterogeneous microscopic structure as for example given in composite materials or porous media. Thus, effective physical,
5 biological or geological models, like the Darcy law, can be derived rigorously. The main assumption for periodic homogenisation is the microscopic periodic structure, which can be scaled by a parameter $\varepsilon > 0$ and arbitrarily refined in a limit process. However, this is too restrictive in many applications because it cannot capture local microscopic varieties or temporal changes of the
10 microstructure, which can have a considerable impact. For processes in which the periodic structure is only given by coefficients, as for example in composite

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materials, this assumption can be weakened. There, the convergence theory can handle coefficients $A_\varepsilon(t, x) = A\left(t, x, \frac{x}{\varepsilon}\right)$, which can capture spatial or temporal changes in the microstructure. However, if the microstructure is also given by the domain $\Omega_\varepsilon \subset \Omega \subset \mathbb{R}^N$ itself, as for example in porous media problems, the procedure can not be transferred directly. Instead, special compactness results are required. These, and often also the derivation of the solutions' uniform estimates, depend largely on the strict periodic structure (cf. [1], [2], [3]).

In order to overcome this strongly restricting microscopic periodicity, M. A. Peter proposed the following method in [4]. Instead of homogenising the actual problem, he transformed it into a substitute problem on a periodic domain. There, the domain's local periodicity becomes a local periodicity of the coefficients, which can be handled by the two-scale convergence. Then, the homogenised equation can be transformed back to an associated evolving domain. However, the homogenisation of the substitute problem is a priori not equivalent to the homogenisation of the actual problem, which would mean that (1) commutes. Therefore, the method itself was only proposed and has not been proven until now. Nevertheless, this method found wide application – in the sense that the back-transformations is done formally and only the homogenisation of the substitute problems is proven – since it allows to consider many interesting problems, particularly on domains evolving in time (see [5], [6], [7], [8], [9]).

In this work, we develop the idea of [4] further, introduce a two-scale concept for *locally periodic domains* and formulate a rigorous two-scale convergence concept for this transformation method. Thus, we can show that these transformations actually commute with the two-scale convergence. Hence, we call it *two-scale-transformation method*. This proves the method and also allows us to show that (1) commutes. Furthermore, we prove a new two-scale-transformation rule for gradients. Thus, we can improve the result for the limit problem in the case of a slow process (flux scaling by ε^0) significantly. Previously, the formal back-transformation could only tackle the homogenised problem and not the two-scale limit problem. As a consequence, the back-transformed limit result still depended on the chosen transformation. With the help of the new gradient transformation rule, we can directly transform the two-scale limit equations back and derive a new (equivalent) homogenised problem which is independent of the transformation.

Moreover, the results developed in this work allow us to translate two-scale compactness results from periodic domains to locally periodic domains. Consequently a direct homogenisation of the actual problem becomes possible if it is defined on locally periodic domains.

$$
\begin{array}{ccc}
\text{microproblem} & \xrightarrow{\text{homogenisation on locally periodic domains}} & \text{macroproblem} \\
\downarrow{\scriptstyle\text{transformation}} & & \uparrow{\scriptstyle\text{back-transformation}} \\
\text{transformed microproblem} & \xrightarrow{\text{homogenisation on periodic domains}} & \text{transformed macroproblem}
\end{array}
\tag{1}
$$

This paper is arranged as follows: In Section 2, we recap the two-scale convergence (cf. [1]) as well as the unfolding operator (cf. [10]) and state results about them, which we employ to prove the two-scale-transformation method. In Section 3, we introduce the two-scale-transformation method and formulate the assumptions on the locally periodic domain. Then, we show that the two-scale convergence and the two-scale transformation commute, which allows to perform (1) rigorously. Moreover, we derive the new two-scale-transformation rule for gradients, which improves the results of the back-transformation. In Section 4, we demonstrate the method by homogenising the following diffusion process on locally periodic domains $\Omega_\varepsilon \subset \Omega$, which are defined in Section 3. There, we consider the case of a fast flux ($l = 0$) as well as the case of a slow flux ($l = 2$). Let $A_\varepsilon \in L^\infty(\Omega_\varepsilon)^{N \times N}$ be bounded and uniformly coercive, i.e. there exist $C, \alpha > 0$ such that $||A_\varepsilon||_{L^\infty(\Omega_\varepsilon)} \leq C$ and $\xi^\top A_\varepsilon(x)\xi \geq \alpha ||\xi||^2$ for every $\varepsilon > 0$, a.e. $x \in \Omega_\varepsilon$ and every $\xi \in \mathbb{R}^N$ and let $f_\varepsilon \in L^2(\Omega_\varepsilon)$. Then, find $u_\varepsilon \in H^1(\Omega_\varepsilon)$ such that

$$\int_{\Omega_\varepsilon} \varepsilon^l A_\varepsilon(x) \nabla u_\varepsilon(x) \cdot \nabla \varphi(x) + u_\varepsilon(x)\varphi(x)dx = \int_{\Omega_\varepsilon} f_\varepsilon(x)\varphi(x)dx \qquad (2)$$

for all $\varphi \in H^1(\Omega_\varepsilon)$. First, we transform (2) on the periodic reference domain and show how to derive uniform estimates for the substitute problem. Afterwards, we pass to the limit $\varepsilon \to 0$ in the periodic substitute problem. By using the two-scale-transformation method and particularly the new transformation rule for the gradients, we transform the limit problems back. The results are the homogenised problem (26) (in the case of $l = 0$) and the two-scale problem (29) (in the case of $l = 2$).

In the following, we use $C > 0$ as generic constant, which is independent of ε. Let $(\varepsilon_n)_{n \in \mathbb{N}}$ be a fixed sequence of positive real numbers converging to 0 (when it is clear from the context, we omit the subscript n). Moreover, we write ε' for a subsequence $(\varepsilon_{n_k})_{k \in \mathbb{N}}$ of ε. We denote the Jacobian matrix for a (weakly) differentiable function u by Du and its transposed by $\nabla u = Du^\top$.

2. Two-scale convergence and the unfolding operator

For the homogenisation of (2) and the assumption on the domain Ω_ε, we use the two-scale convergence (cf. [1], [11]). Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let $Y := (0, 1)^N$ denote the reference cell. Nevertheless, all the arguments can be transferred to arbitrary reference parallelotopes $Y \subset \mathbb{R}^N$.

Definition 2.1. We say that a sequence u_ε in $L^1(\Omega)$ two-scale converges distributionally to $u_0 \in L^1(\Omega \times Y)$ if

$$\lim_{\varepsilon \to 0} \int_{\Omega} u_\varepsilon(x)\varphi\left(x, \frac{x}{\varepsilon}\right) dx = \iint_{\Omega Y} u_0(x, y)\varphi(x, y)dydx \qquad (3)$$

for every $\varphi \in D(\Omega; C_\#^\infty(Y))$. We write $u_\varepsilon \xrightarrow{D} u_0$.

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Definition 2.2. Let $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. We say that a sequence u_ε in $L^p(\Omega)$ two-scale converges weakly to $u_0 \in L^p(\Omega \times Y)$ if

$$\lim_{\varepsilon \to 0} \int\limits_\Omega u_\varepsilon(x) \varphi\left(x, \frac{x}{\varepsilon}\right) dx = \int\limits_\Omega \int\limits_Y u_0(x, y) \varphi(x, y) dy dx \tag{4}$$

for every $\varphi \in L^q(\Omega; C_\#(Y))$. We write $u_\varepsilon \xrightarrow{p} u_0$.

The main compactness result for the two-scale convergence is Theorem 2.3 (see [11, Theorem 7]).

Theorem 2.3. Let $p \in (1, \infty)$ and let u_ε be a bounded sequence in $L^p(\Omega)$. Then, there exists a subsequence ε' and $u_0 \in L^p(\Omega \times Y)$ such that $u_{\varepsilon'} \xrightarrow{p} u_0$.

This compactness result can be improved for sequences of weakly differentiable functions by the following two standard two-scale compactness results.

Theorem 2.4. Let $p \in (1, \infty)$ and let u_ε be a bounded sequence in $W^{1,p}(\Omega)$. Then, there exist a subsequence ε', $u_0 \in W^{1,p}(\Omega)$ and $u_1 \in L^p(\Omega; W^{1,p}(Y)/\mathbb{R})$ such that $u_{\varepsilon'} \xrightarrow{p} u_0$ and $\nabla u_{\varepsilon'} \xrightarrow{p} \nabla_x u_0 + \nabla_y u_1$.

Theorem 2.5. Let $p \in (1, \infty)$ and let u_ε be a sequence in $W^{1,p}(\Omega)$ such that $||u_\varepsilon||_{L^p(\Omega)} + \varepsilon\, ||\nabla u_\varepsilon||_{L^p(\Omega)} \leq C$. Then, there exist a subsequence ε' and $u_0 \in L^p(\Omega; W_\#^{1,p}(Y))$ such that $u_{\varepsilon'} \xrightarrow{p} u_0$ and $\nabla u_{\varepsilon'} \xrightarrow{p} \nabla_y u_0$.

Testing (4) with functions $\varphi \in L^q(\Omega)$ shows the following relation between the weak two-scale convergence and the weak convergence in $L^p(\Omega)$.

Proposition 2.6. Let $p \in (1, \infty)$. Let u_ε be a sequence in $L^p(\Omega)$ and $u_0 \in L^p(\Omega \times Y)$ such that $u_\varepsilon \xrightarrow{p} u_0$. Then, u_ε converges weakly in $L^p(\Omega)$ to $u(\cdot_x) = \int\limits_Y u_0(\cdot_x, y) dy$ and the sequence u_ε is bounded in $L^p(\Omega)$.

On the other hand, if the sequence is bounded, the set of test functions can be reduced to smooth and, in a certain way, dense test functions, similar to the weak L^p-convergence (cf. [11, Proposition 1])

Proposition 2.7. Let $p \in (1, \infty)$. Let u_ε be a bounded sequence in $L^p(\Omega)$ and $u_0 \in L^p(\Omega \times Y)$ such that $u_\varepsilon \xrightarrow{D} u_0$, then $u_\varepsilon \xrightarrow{p} u_0$.

One desirable property of the two-scale convergence is that the product and the limit process commute. The well known fact that in uniformly convex Banach spaces the weak convergence plus the convergence of the norms are equivalent to the strong convergence motivates the following definition of the strong two-scale convergence.

Definition 2.8. Let $p \in (1, \infty)$. We say that a sequence u_ε in $L^p(\Omega)$ two-scale converges strongly to $u_0 \in L^p(\Omega \times Y)$ if $u_\varepsilon \xrightarrow{p} u_0$ and $\lim_{\varepsilon \to 0} ||u_\varepsilon||_{L^p(\Omega)} = ||u_0||_{L^p(\Omega)}$. We write $u_\varepsilon \xrightarrow{p} u_0$.

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Proposition 2.9. *Let $1 < p, q, r < \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Let u_ε be a sequence in $L^p(\Omega)$ which two-scale converges strongly to $u_0 \in L^p(\Omega \times Y)$ and let v_ε be a sequence in $L^q(\Omega)$ which two-scale converges weakly (resp. strongly) to $v_0 \in L^q(\Omega \times Y)$. Then, $u_\varepsilon v_\varepsilon$ is a sequence of functions in $L^r(\Omega)$ which two-scale converges weakly (resp. strongly) to $u_0 v_0 \in L^r(\Omega \times Y)$.*

Using the unfolding operator $\mathcal{T}_\varepsilon$, which was introduced in [12], two-scale convergence can be translated into convergence in $L^p(\Omega \times Y)$. Thus, we can give a brief proof of Proposition 2.9 later.

In order to simplify the proofs of the two-scale convergence method, we introduce the following notations.

Notation 2.10. Let $p \in (1, \infty]$. If u_ε is a sequence in $L^q(\Omega)$ and $u_0 \in L^q(\Omega \times Y)$ for every $q \in (1, p)$ such that $u_\varepsilon \xrightarrow{q} u_0$ for every $q \in (1, p)$, we write $u_\varepsilon \xrightarrow{<p} u_0$. If additionally $u_\varepsilon \xrightarrow{q} u_0$ for every $q \in (1, p)$, we write $u_\varepsilon \xrightarrow{<p} u_0$.

Notation 2.11. Let $\Omega \subset \mathbb{R}^N$ and $x = \sum\limits_{i=1}^{N} x_i e_i \in \mathbb{R}^N$, where e_i denotes the euclidean unit vectors, then let

$$[x]_Y := \sum_{i=1}^{N} \lfloor x_i \rfloor e_i \,, \quad \{x\}_Y := x - [x]_Y \,, \quad [x]_{\varepsilon,Y} := \varepsilon \left[\frac{x}{\varepsilon} \right]_Y \,, \quad \{x\}_{\varepsilon,Y} := \left\{ \frac{x}{\varepsilon} \right\}_Y$$

$$I_\varepsilon := \{ k \in \varepsilon \mathbb{Z}^N, k + \varepsilon Y \subset \overline{\Omega} \} \,, \quad \tilde{\Omega}_\varepsilon := \text{int} \left(\bigcup_{k \in I_\varepsilon} k + \varepsilon Y \right) \,, \quad \Lambda_\varepsilon = \Omega \setminus \tilde{\Omega}_\varepsilon.$$

Definition 2.12. Let $1 \leq p \leq \infty$. We define the unfolding operator $\mathcal{T}_\varepsilon : L^p(\Omega) \to L^p(\Omega \times Y)$ by

$$\mathcal{T}_\varepsilon(\varphi)(x, y) := \begin{cases} \varphi([x]_{\varepsilon,Y} + \varepsilon y) & \text{for a.e. } (x, y) \in \tilde{\Omega}_\varepsilon \times Y, \\ \varphi(x) & \text{for a.e. } (x, y) \in \Lambda_\varepsilon \times Y. \end{cases} \tag{5}$$

Note that we have defined $\mathcal{T}_\varepsilon(\varphi)(x, y) = \varphi(x)$ on the cells that are not completely included in Ω and not $\mathcal{T}_\varepsilon(\varphi)(x, y) = 0$ as in [10]. By this slight modification, $\mathcal{T}_\varepsilon$ becomes isometric (cf. Theorem 2.13). Thus, we can not only translate between the two-scale convergence of u_ε and the weak convergence of $\mathcal{T}_\varepsilon(u_\varepsilon)$ in $L^p(\Omega \times Y)$, as shown in [10], but we can also translate between the strong two-scale convergence and the strong convergence in $L^p(\Omega \times Y)$.

Theorem 2.13. *Let $\varphi \in L^p(\Omega)$ for $1 \leq p \leq \infty$. Then*

$$\iint\limits_{\Omega \, Y} \mathcal{T}_\varepsilon(\varphi)(x, y) dy dx = \int\limits_{\Omega} \varphi(x) dx, \tag{6}$$

$$\|\mathcal{T}_\varepsilon(\varphi)\|_{L^p(\Omega \times Y)} = \|\varphi\|_{L^p(\Omega)}. \tag{7}$$

Proof. We split the integral on $\tilde{\Omega}_\varepsilon \times Y$ and $\Lambda_\varepsilon \times Y$ so that

$$\iint_{\Omega\, Y} \mathcal{T}_\varepsilon(\varphi)(x,y)dydx = \sum_{k\in I_\varepsilon} \int_{k+\varepsilon Y}\int_Y \varphi([x]_{\varepsilon,Y} + \varepsilon y)dydx + \int_{\Lambda_\varepsilon \times Y} \varphi(x)dydx.$$

Since $[x]_{\varepsilon,Y} = k$ on each cell $k + \varepsilon Y$, we obtain

$$\int_{k+\varepsilon Y}\int_Y \varphi([x]_{\varepsilon,Y} + \varepsilon y)dydx = \int_{k+\varepsilon Y}\int_Y \varphi(k + \varepsilon y)dydx = |\varepsilon Y| \int_Y \varphi(k+\varepsilon y)dy$$

$$= \int_{k+\varepsilon Y} \varphi(x)dx.$$

Combining these two equations yields

$$\iint_{\Omega\, Y} \mathcal{T}_\varepsilon(\varphi)(x,y)dydx = \sum_{k\in I_\varepsilon} \int_{k+\varepsilon Y} \varphi(x)dx + \int_{\Lambda_\varepsilon} \varphi(x)dx = \int_\Omega \varphi(x)dx.$$

Since $|\mathcal{T}_\varepsilon(\varphi)|^p = \mathcal{T}_\varepsilon(|\varphi|^p)$, (7) follows for $p < \infty$ by applying (6) to $|\varphi|^p$. For $p = \infty$, (7) follows directly from the definition of $\mathcal{T}_\varepsilon$. □

Proposition 2.14. *Let u_ε be a sequence in $L^p(\Omega)$ and $u_0 \in L^p(\Omega \times Y)$ with $1 < p < \infty$. Then, the following statements hold:*

1. *$u_\varepsilon \xrightarrow{D} u_0$ if and only if $\mathcal{T}_\varepsilon(u_\varepsilon)\varphi \to u_0\varphi$ in $L^1(\Omega \times Y)$ for every $\varphi \in D(\Omega; C^\infty_\#(Y))$,*
2. *$u_\varepsilon \xrightarrow{p} u_0$ if and only if $\mathcal{T}_\varepsilon(u_\varepsilon) \rightharpoonup u_0$ in $L^p(\Omega \times Y)$,*
3. *$u_\varepsilon \xrightarrow{p} u_0$ if and only if $\mathcal{T}_\varepsilon(u_\varepsilon) \to u_0$ in $L^p(\Omega \times Y)$.*

Proof. In order to prove the first equivalence, it is enough to show that

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x)\varphi\left(x, \frac{x}{\varepsilon}\right) dx = \lim_{\varepsilon \to 0} \iint_{\Omega\, Y} \mathcal{T}_\varepsilon(u_\varepsilon)(x,y)\mathcal{T}_\varepsilon\left(\varphi\left(\cdot_x, \frac{\cdot_x}{\varepsilon}\right)\right)(x,y)dydx$$

$$= \lim_{\varepsilon \to 0} \iint_{\Omega\, Y} \mathcal{T}_\varepsilon(u_\varepsilon)(x,y)\varphi(x,y)dydx \qquad (8)$$

for every smooth test function $\varphi \in D(\Omega; C^\infty_\#(Y))$. The first equality in (8) follows directly from the definition of $\mathcal{T}_\varepsilon$ and (6). For the second equality, it is enough to show that $\mathcal{T}_\varepsilon(\varphi(\cdot_x, \frac{\cdot_x}{\varepsilon}))$ converges strongly to φ in $L^q(\Omega \times Y)$ for $\frac{1}{p} + \frac{1}{q} = 1$. We note that for every $x \in \Omega$ there exists $\varepsilon_0 > 0$ small enough such that $x \in \tilde{\Omega}_\varepsilon$ for every $0 < \varepsilon < \varepsilon_0$. Thus, we obtain the pointwise convergence

$$\mathcal{T}_\varepsilon\left(\varphi\left(\cdot_x, \frac{\cdot_x}{\varepsilon}\right)\right)(x,y) = \varphi\left([x]_{\varepsilon,Y} + \varepsilon y, \frac{[x]_{\varepsilon,Y} + \varepsilon y}{\varepsilon}\right) = \varphi([x]_{\varepsilon,Y} + \varepsilon y, y) \xrightarrow[\varepsilon\to 0]{} \varphi(x,y)$$

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for every $(x, y) \in \Omega \times Y$. Since $|\mathcal{T}_\varepsilon(\varphi(\cdot_x, \frac{\cdot_x}{\varepsilon}))(x, y)|$ is also pointwise bounded for a.e. $(x, y) \in \Omega \times Y$ by $\|\varphi\|_{L^\infty(\Omega \times Y)}$, we can apply Lebesgue's convergence theorem and obtain the strong convergence of $\mathcal{T}_\varepsilon(\varphi(\cdot_x, \frac{\cdot_x}{\varepsilon}))$ to φ in $L^q(\Omega \times Y)$ for $\frac{1}{p} + \frac{1}{q} = 1$, which implies Proposition 2.14(1).

In order to prove Proposition 2.14(2), we note that both types of weak convergences are equivalent to the boundedness of the sequence plus the corresponding convergence of Proposition 2.14(1). Using the isometry of $\mathcal{T}_\varepsilon$, we can translate the boundedness of the sequences. Then, the equivalence of the weak convergences follows directly from Proposition 2.14(1).

For the equivalence of the strong convergences, we note that the strong convergence of $\mathcal{T}_\varepsilon(u_\varepsilon)$ is equivalent to the weak convergence of $\mathcal{T}_\varepsilon(u_\varepsilon)$ plus $\lim\limits_{\varepsilon \to 0} \|\mathcal{T}_\varepsilon(u_\varepsilon)\|_{L^p(\Omega \times Y)} = \|u_0\|_{L^p(\Omega \times Y)}$ since $L^p(\Omega \times Y)$ is a uniformly convex Banach space. Thus, Proposition 2.14(3) follows from Proposition 2.14(2) and the isometry of $\mathcal{T}_\varepsilon$. $\qquad\square$

Having these results about the unfolding operator, we can prove Proposition 2.9 as follows.

Proof of Proposition 2.9. We translate the strong two-scale convergence of u_ε and the weak (resp. strong) two-scale convergence of v_ε with the unfolding operator $\mathcal{T}_\varepsilon$ and Proposition 2.14 into the strong convergence of $\mathcal{T}_\varepsilon(u_\varepsilon)$ in $L^p(\Omega \times Y)$ and the weak (resp. strong) convergence of $\mathcal{T}_\varepsilon(v_\varepsilon)$ in $L^q(\Omega \times Y)$. Employing Hölder estimates, we obtain the weak (resp. strong) convergence of the product $\mathcal{T}_\varepsilon(u_\varepsilon v_\varepsilon) = \mathcal{T}_\varepsilon(u_\varepsilon)\mathcal{T}_\varepsilon(v_\varepsilon)$ to $u_0 v_0$ in $L^r(\Omega \times Y)$. Proposition 2.14 transfers this convergence back into the weak (resp. strong) two-scale convergences of $u_\varepsilon v_\varepsilon$ to $u_0 v_0 \in L^r(\Omega \times Y)$. $\qquad\square$

3. The two-scale-transformation method

In the following, let Y be divided into an open set $Y^* \subset Y$, which constitutes the material part of Y, and a hole $Y \setminus \overline{Y^*}$. Note, that the material part can also be the whole set, i.e. $Y^* = Y$. We assume that the Y-periodic extension of Y^*, denoted by $Y_\#^* := \text{int}\left(\bigcup\limits_{k \in \mathbb{Z}^N} k + \overline{Y^*}\right)$, has a Lipschitz boundary. Moreover, from now on we assume that the macroscopic domain $\Omega \subset \mathbb{R}^N$ is not only open and bounded, but also has a Lipschitz boundary. Let $\hat{\Omega}_\varepsilon := \Omega \cap \varepsilon Y_\#^*$ be the ε-scaled periodic reference domains. Then, we define the locally periodic domains Ω_ε by transforming the periodic reference domains.

Definition 3.1. We say that a sequence of open domains $\Omega_\varepsilon \subset \mathbb{R}^N$ is *locally periodic* with *two-scale limit domain* $Q = \bigcup\limits_{x \in \Omega} \{x\} \times Y_x^* \subset \Omega \times Y$, where $Y_x^* \subset Y$ is open for a.e. $x \in \Omega$, if there exists a sequence of locally periodic transformations ψ_ε (see Definition 3.2) with a limit transformation ψ_0 such that $\Omega_\varepsilon = \psi_\varepsilon(\hat{\Omega}_\varepsilon)$ and $Y_x^* = \psi_0(x, Y^*)$ for a.e. $x \in \Omega$.

In order to give the definition of locally periodic transformations, we have to consider the two-scale convergence for sequences u_ε defined on $\hat{\Omega}_\varepsilon$. Therefore, we extend them by 0 to Ω which we denote by $\widetilde{\cdot}$. Moreover, for functions defined on $\Omega \times Y^*$, we analogously denote their extension by 0 to $\Omega \times Y$ by $\widetilde{\cdot}$.

Definition 3.2. We say a sequence of C^1-diffeomorphisms $\psi_\varepsilon : \hat{\Omega}_\varepsilon \to \Omega_\varepsilon$, for $\Omega_\varepsilon := \psi_\varepsilon(\hat{\Omega}_\varepsilon) \subset \mathbb{R}^N$, is a sequence of *locally periodic transformations* if:

1. there exists $c_J > 0$ such that $J_\varepsilon \geq c_J$ with $J_\varepsilon := \det(\Psi_\varepsilon)$ and $\Psi_\varepsilon := D\psi_\varepsilon$,
2. there exists $C > 0$ such that $\varepsilon^{-1} \left|\left|\check{\psi}_\varepsilon\right|\right|_{C(\hat{\Omega}_\varepsilon)} + \left|\left|\nabla\check{\psi}_\varepsilon\right|\right|_{C(\hat{\Omega}_\varepsilon)} \leq C$, where $\check{\psi}_\varepsilon(x) := \psi_\varepsilon(x) - x$ are the corresponding displacement mappings,
3. there exists $\psi_0 \in L^\infty(\Omega; C^1(Y^*))^N$, which we call the *limit transformation*, such that
   (a) $\psi_0(x, \cdot_y) : Y \to Y$ are C^1-diffeomorphisms for a.e. $x \in \Omega$ with inverses $\psi_0^{-1}(x, \cdot_y)$, where $\psi_0^{-1} \in L^\infty(\Omega; C^1(Y_x^*))^N$,
   (b) the corresponding displacement mapping, defined for a.e. $x \in \Omega$ by $\check{\psi}_0(x, y) := \psi_0(x, y) - y$, can be extended Y-periodically such that $\check{\psi}_0 \in L^\infty(\Omega; C_\#^1(Y))^N$,
   (c) $\varepsilon^{-1}\widetilde{\check{\psi}_\varepsilon} \overset{<\infty}{\twoheadrightarrow} \widetilde{\check{\psi}_0}$ and $\widetilde{\nabla\check{\psi}_\varepsilon} \overset{<\infty}{\twoheadrightarrow} \widetilde{\nabla_y\check{\psi}_0}$.

For a.e. $x \in \Omega$, we denote the Jacobians of $\psi_0(x, \cdot_y)$ by $\Psi_0(x, \cdot_y) := D_y\psi_0(x, \cdot_y)$ and $J_0(x, \cdot_y) := \det(\Psi_0(x, \cdot_y))$. Moreover, we denote the displacement mappings of the back-transformations by $\check{\psi}_\varepsilon^{-1}(x) := \psi_\varepsilon^{-1}(x) - x$ and $\check{\psi}_0^{-1}(x, y) := \psi_0^{-1}(x, y) - y$.

We obtain the following uniform estimates and additional strong two-scale convergences as a direct consequence of the definition of the locally periodic transformations ψ_ε.

Lemma 3.3. *Let ψ_ε be locally periodic transformations with limit transformation ψ_0. Then, there exist constants $c_J, C > 0$ such that*

$$\left|\left|\Psi_\varepsilon\right|\right|_{C(\hat{\Omega}_\varepsilon)} + \left|\left|\Psi_\varepsilon^{-1}\right|\right|_{C(\hat{\Omega}_\varepsilon)} + \left|\left|J_\varepsilon\right|\right|_{C(\hat{\Omega}_\varepsilon)} \leq C,$$

$$\left|\left|\Psi_0\right|\right|_{L^\infty(\Omega; C(Y^*))} + \left|\left|\Psi_0^{-1}\right|\right|_{L^\infty(\Omega; C(Y^*))} + \left|\left|J_0\right|\right|_{L^\infty(\Omega; C(Y^*))} \leq C,$$

$$J_0(x, y) \geq c_J \text{ for a.e. } x \in \Omega \text{ and every } y \in Y^*.$$

Furthermore,

$$\widetilde{\Psi_\varepsilon} \overset{<\infty}{\twoheadrightarrow} \widetilde{\Psi_0}, \qquad \widetilde{\Psi_\varepsilon^{-1}} \overset{<\infty}{\twoheadrightarrow} \widetilde{\Psi_0^{-1}}, \qquad \widetilde{J_\varepsilon} \overset{<\infty}{\twoheadrightarrow} \widetilde{J_0}, \qquad \widetilde{J_\varepsilon^{-1}} \overset{<\infty}{\twoheadrightarrow} \widetilde{J_0^{-1}}.$$

Proof. The uniform estimate of $\nabla\check{\psi}_\varepsilon$ directly gives one for $\Psi_\varepsilon = D_x\check{\psi}_\varepsilon + \mathbb{1}$. Since J_ε and the entries of Ψ_ε^{-1} are polynomials in J_ε^{-1} and the entries of Ψ_ε, the uniform estimates for these follow with the additional uniform bound of $J_\varepsilon \geq c_J$ from below.

We rewrite $\Psi_\varepsilon = D\psi_\varepsilon = D\check{\psi}_\varepsilon + Dx = D\check{\psi}_\varepsilon + \mathbb{1}$ and $\Psi_0 = D_y\psi_0 = D_y\check{\psi}_0 + \mathbb{1}$. Then, $\widetilde{D\check{\psi}_\varepsilon} \overset{<\infty}{\twoheadrightarrow} \widetilde{D_y\check{\psi}_0}$ together with $\chi_{\hat{\Omega}_\varepsilon} \overset{<\infty}{\twoheadrightarrow} \chi_{Y^*}$ imply $\widetilde{\Psi_\varepsilon} \overset{<\infty}{\twoheadrightarrow} \widetilde{\Psi_0}$. Note

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that $\chi_{\hat{\Omega}_\varepsilon} \overset{<\infty}{\twoheadrightarrow} \chi_{Y^*}$ holds because we can write $\chi_{\hat{\Omega}_\varepsilon}(x) = \chi_{Y_\#^*}(\frac{x}{\varepsilon})$ for $\chi_{Y_\#^*} \in L_\#^q(Y; C(\overline{\Omega}))$ for every $q \in (1, \infty)$ (cf. [11, Theorem 3]).

Since $\widetilde{J}_\varepsilon$ and $\widetilde{J}_0$ are polynomials with respect to the entries of $\widetilde{\Psi}_\varepsilon$ and $\widetilde{\Psi}_0$, respectively, Proposition 2.9 implies $\widetilde{J}_\varepsilon \overset{<\infty}{\twoheadrightarrow} \widetilde{J}_0$.

The uniform boundedness of $J_\varepsilon \geq c_J$ from below gives $\mathcal{T}_\varepsilon(\widetilde{J}_\varepsilon)(x, y) \geq c_J$ for a.e. $(x, y) \in \Omega \times Y^*$. Then, the strong convergence of $\mathcal{T}_\varepsilon(\widetilde{J}_\varepsilon)$ to $\widetilde{J}_0$ in $L^p(\Omega \times Y)$ transfers the uniform boundedness from below to $J_0(x, y) \geq c_J$ for a.e. $(x, y) \in \Omega \times Y^*$. By Definition 3.2, $J_0 \in L^\infty(\Omega; C(Y^*))$, which implies $J_0(x, y) \geq c_J$ for a.e. $x \in \Omega$ and every $y \in Y^*$.

We rewrite $\mathcal{T}_\varepsilon(\widetilde{J_\varepsilon^{-1}}) = (\mathcal{T}_\varepsilon(\widetilde{J}_\varepsilon)|_{\Omega \times Y^*})^{-1}$ and obtain

$$\left\| \mathcal{T}_\varepsilon(\widetilde{J_\varepsilon^{-1}}) - \widetilde{J_0^{-1}} \right\|_{L^p(\Omega Y)} = \left\| (J_0 - \mathcal{T}_\varepsilon(\widetilde{J}_\varepsilon))/(J_0 \mathcal{T}_\varepsilon(\widetilde{J}_\varepsilon)) \right\|_{L^p(\Omega \times Y^*)}$$

$$\leq \frac{1}{c_J^2} \left\| J_0 - \mathcal{T}_\varepsilon(\widetilde{J}_\varepsilon) \right\|_{L^p(\Omega \times Y^*)} = \frac{1}{c_J^2} \left\| \widetilde{J_0} - \mathcal{T}_\varepsilon(\widetilde{J}_\varepsilon) \right\|_{L^p(\Omega \times Y)} \to 0,$$

which implies $\widetilde{J_\varepsilon^{-1}} \overset{<\infty}{\twoheadrightarrow} \widetilde{J_0^{-1}}$.

Since the entries of $\widetilde{\Psi_\varepsilon^{-1}}$ are polynomials in $\widetilde{J_\varepsilon^{-1}}$ and the entries of $\widetilde{\Psi}_\varepsilon$, the strong two-scale convergence can be directly transferred to the strong two-scale convergence of $\widetilde{\Psi_\varepsilon^{-1}}$ to $\widetilde{\Psi_0^{-1}}$.

Moreover, we obtain that $\left\| \Psi_\varepsilon^{-1} \right\|_{L^\infty(\Omega; C(Y^*))} \leq C$ and $\| J_0 \|_{L^\infty(\Omega; C(Y^*))} \leq C$ from the fact that $\| \Psi_0 \|_{L^\infty(\Omega; C(Y^*))} \leq C$ and $J_0(x, y) \geq c_J$ for a.e. $x \in \Omega$ and every $y \in Y^*$ by using the same argumentation as for J_ε and Ψ_ε^{-1}. $\qquad\square$

For example, a family of diffeomorphisms ψ_ε which are locally periodic transformations in the sense of Definition 3.2 can be obtained as follows.

Example 3.4. Let $\Theta : \Omega \to [\Theta_{\min}, \Theta_{\max}] \subset \mathbb{R}$ be a continuous function, which describes, for example, the local porosity. Let $\psi : [\Theta_{\min}, \Theta_{\max}] \times \overline{Y} \to \overline{Y}$ be a smooth mapping such that, for every $\Theta \in [\Theta_{\min}, \Theta_{\max}]$, $\psi(\Theta, Y^*)$ gives a cell with porosity Θ and $\psi(\Theta, \cdot) : \overline{Y} \to \overline{Y}$ is a C^1-diffeomorphism. Moreover, we assume that there exist $C, c_J > 0$ such that $\| D_y \psi \|_{C([\Theta_{\min,\max}] \times \overline{Y})} \leq C$, $\det(D_y \psi) \geq c_J$ and that the corresponding displacement mapping $\check{\psi}(\Theta, y) = \psi(\Theta, y) - y$ has compact support in the interior of Y. Then, $\psi_\varepsilon(x) := x + \varepsilon \check{\psi}(\Theta([x]_{\varepsilon, Y}), \{x\}_{\varepsilon, Y})$ are locally periodic transformations with limit transformation $\psi_0(x, y) := \check{\psi}(\Theta(x), y)$ in the sense of Definition 3.2.

Before we continue with the transformation of the two-scale convergence, we recap the main two-scale compactness results for periodic domains. Since the extension by 0 does not preserve the $W^{1,p}$-regularity, these compactness results cannot be derived directly from the previous $W^{1,p}$-compactness results (cf. Theorem 2.4 and Theorem 2.5). Instead, their derivation extensively utilises the domains' periodic structures.

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Proposition 3.5. *Let u_ε be a bounded sequence in $L^p(\hat\Omega_\varepsilon)$ for $p \in (1, \infty)$. Then, there exists a subsequence ε' and $u_0 \in L^p(\Omega \times Y^*)$ such that $\widetilde{u_{\varepsilon'}} \xrightarrow{p} \widetilde{u_0}$.*

Moreover, if u_ε is a sequence in $W^{1,p}(\hat\Omega_\varepsilon)$ such that

$$\|u_\varepsilon\|_{L^p(\hat\Omega_\varepsilon)} + \varepsilon^{l/2} \|\nabla u_\varepsilon\|_{L^p(\hat\Omega_\varepsilon)} \le C$$

for $l \in \{0, 2\}$. Then, the following statements hold:

1. *If $l = 0$ and $Y_\#^*$ is connected, then there exist a subsequence ε', $u_0 \in L^p(\Omega)$ and $u_1 \in L^p(\Omega; W_\#^{1,p}(Y^*)/\mathbb{R})$ such that $\widetilde{u_{\varepsilon'}} \xrightarrow{p} \chi_{Y^*} u_0$ and $\widehat{\nabla u_{\varepsilon'}} \xrightarrow{p} \chi_{Y^*} \nabla_x u_0 + \widetilde{\nabla_y u_1}$.*

2. *If $l = 2$, then there exist a subsequence ε' and $u_0 \in L^p(\Omega; W_\#^{1,p}(Y^*))$ such that $\widetilde{u_{\varepsilon'}} \xrightarrow{p} \widetilde{u_0}$ and $\varepsilon'\widehat{\nabla u_{\varepsilon'}} \xrightarrow{p} \widetilde{\nabla_y u_0}$.*

Proof. Let u_ε be bounded, then $\widetilde{u_\varepsilon}$ is bounded as well and Theorem 2.3 gives a subsequence which two-scale converges weakly to a limit function $u_0 \in L^p(\Omega \times Y)$. Employing two-scale test functions φ which are 0 in $\Omega \times Y^*$ yields $u_0 = 0$ in $Y \setminus Y^*$. Thus, we can rewrite the limit as $\widetilde{u_0}$ for $u_0|_{\Omega \times Y^*} \in L^p(\Omega \times Y^*)$.

A proof of Proposition 3.5(1) is given in [2, Theorem 4.6] for the case $p = 2$. It can be generalised to arbitrary $p \in (1, \infty)$ in the same way as the standard H^1-two-scale compactness result.

Proposition 3.5(2) can be proven analogously to Theorem 2.5 by using the $L^p(\Omega_\varepsilon)$ compactness result of Proposition 3.5 instead of Theorem 2.3. $\qquad\square$

Before we can analyse the two-scale convergence under the two-scale transformation, we have to consider what two-scale convergence of sequences on Ω_ε means. We note that the definition of the locally periodic transformations ψ_ε does not ensure that $\Omega_\varepsilon = \psi_\varepsilon(\hat\Omega_\varepsilon)$ is contained in Ω. However, it ensures that $\left\|\check\psi_\varepsilon\right\|_{C(\hat\Omega_\varepsilon)} \le \varepsilon C$, which implies that $|\Omega_\varepsilon \setminus \Omega| \le \varepsilon C$ as well as $\Omega_\varepsilon \subset \{x \in \mathbb{R}^N \mid \mathrm{dist}(x, \Omega) \le \varepsilon C\}$. Therefore, we expect a limit defined on the macroscopic domain Ω, which could suggest to formulate the two-scale convergence for functions defined on Ω_ε by restricting them on $\Omega \cap \Omega_\varepsilon$ and then extending them by 0 to Ω. However, it turns out that this ansatz would not yield a natural translation between the two-scale convergence in the untransformed and the transformed setting. Instead, we consider $\Omega^{(\delta)} := \{x \in \mathbb{R}^N \mid \mathrm{dist}(x, \Omega) < \delta\}$ for fixed $0 < \delta << 1$ as the macroscopic domain and note that $\Omega_\varepsilon \subset \Omega^{(\delta)}$ for ε small enough. We extend functions defined on Ω_ε by 0 to $\Omega^{(\delta)}$, which we denote by $\widetilde{\cdot}$. Then, we can use the normal two-scale convergence, but for the macroscopic domain $\Omega^{(\delta)}$ instead of Ω. However, we will show that the corresponding two-scale limits have support on $Q \subset \Omega \times Y$.

Therefore, we introduce the following Banach spaces spaces. Let $p \in [1, \infty]$ and $q \in [1, \infty)$, then we define

$$L^p(\Omega; L^q(Y_x^*)) := \{f(\cdot_x, \psi_0^{-1}(\cdot_x, \cdot_y)) \mid f \in L^p(\Omega; L^q(Y^*))\},$$
$$L^p(\Omega; W^{1,q}(Y_x^*)) := \{f(\cdot_x, \psi_0^{-1}(\cdot_x, \cdot_y)) \mid f \in L^p(\Omega; W^{1,q}(Y^*))\},$$
$$L^p(\Omega; W_\#^{1,q}(Y_x^*)) := \{f(\cdot_x, \psi_0^{-1}(\cdot_x, \cdot_y)) \mid f \in L^p(\Omega; W_\#^{1,q}(Y^*))\},$$

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with the corresponding norms

$$\|u\|_{L^p(\Omega;L^q(Y_x^*))} := \begin{cases} \left(\int_\Omega \|u(x)\|_{L^q(Y_x^*)}^p \, dx\right)^{\frac{1}{p}} & \text{if } p < \infty, \\ \operatorname*{ess\,sup}_{x\in\Omega} \|u(x)\|_{L^q(Y)} & \text{if } p = \infty, \end{cases}$$

$$\|u\|_{L^p(\Omega;W^{1,q}(Y_x^*))} := \|u\|_{L^p(\Omega;L^q(Y_x^*))} + \|D_y u\|_{L^p(\Omega;L^q(Y_x^*))},$$

$$\|u\|_{L^p(\Omega;W_{\#}^{1,q}(Y_x^*))} := \|u\|_{L^p(\Omega;L^q(Y_x^*))} + \|D_y u\|_{L^p(\Omega;L^q(Y_x^*))}.$$

Note that, the integrability of $\|u(x)\|_{L^q(Y_x^*)}^p$ over Ω is ensured by the transformation $\|u(x)\|_{L^q(Y_x^*)}^p = \int_{Y^*} J_0(x,y)u(x,\psi_0(x,y))dy$. Moreover, these function spaces and the corresponding norms does nether depend on the chosen reference domain Y^* nor on the diffeomorphism ψ_0. Furthermore, the assumptions on ψ_0 ensure that Q is measurable with respect to the $2N$-dimensional Lebesgue measure and $L^p(\Omega;L^q(Y_x^*))$ can be identified with the subset of $L^p(\Omega;L^q(Y))$ which contains functions that are zero on $(x, Y \setminus Y_x^*)$ for a.e. $x \in \Omega$.

In order to use these function spaces for the two-scale convergence, we extend the corresponding functions by 0 to $\Omega^{(\delta)} \times Y$, which we denote by $\widetilde{\cdot}$. If $\Omega_\varepsilon \subset \Omega$, we do not have to enlarge Ω and all the following results hold for $\Omega^{(\delta)} = \Omega$.

Note that the assumptions on ψ_0 and the uniform estimates given in Lemma 3.3 ensure that the transformations by ψ_0 and ψ_0^{-1} induce an continuous isomorphism between $L^p(\Omega;L^q(Y_x^*))$ and $L^p(\Omega;L^q(Y^*))$ in the sense that

$$\|f(\cdot_x, \psi_0(\cdot_x, \cdot_y))\|_{L^p(\Omega;L^q(Y^*))} \le C \|f\|_{L^p(\Omega;L^q(Y_x^*))},$$

$$\left\|\hat{f}(\cdot_x, \psi_0^{-1}(\cdot_x, \cdot_y))\right\|_{L^p(\Omega;L^q(Y_x^*))} \le C \left\|\hat{f}\right\|_{L^p(\Omega;L^q(Y^*))}$$

for every $f \in L^p(\Omega;L^q(Y_x^*))$ and $\hat{f} \in L^p(\Omega;L^q(Y_x^*))$. Moreover, ψ_0 and ψ_0^{-1} give also an continuous isomorphism between $L^p(\Omega;W^{1,q}(Y^*))$ and $L^p(\Omega;W^{1,q}(Y_x^*))$ as well as between $L^p(\Omega;W_{\#}^{1,q}(Y^*))$ and $L^p(\Omega;W_{\#}^{1,q}(Y_x^*))$.

In order to shorten out notation, we define

$$\varphi_{\varepsilon,\psi_\varepsilon}(\cdot_x) := \varphi\left(\psi_\varepsilon(\cdot_x), \frac{\psi_\varepsilon(\cdot_x)}{\varepsilon}\right), \quad \varphi_{\psi_0}(\cdot_x, \cdot_y) := \varphi(\cdot_x, \psi_0(\cdot_x, \cdot_y)),$$

$$\varphi_{\psi_0^{-1}}(\cdot_x, \cdot_y) := \varphi(\cdot_x, \psi_0^{-1}(\cdot_x, \cdot_y))$$

for functions which depend on x and y. For functions, which have already an index themselves, we write $u_{0,\psi_0}(\cdot_x, \cdot_y) = u_0(\cdot_x, \psi_0(\cdot_x, \cdot_y))$ and $\hat{u}_{0,\psi_0^{-1}}(\cdot_x, \cdot_y) = \hat{u}_0(\cdot_x, \psi_0^{-1}(\cdot_x, \cdot_y))$.

First, we consider the two-scale convergence of continuous functions under the locally periodic transformation ψ_ε.

Lemma 3.6. *Let $\varphi \in C(\Omega^{(\delta)}; C_{\#}(Y))$. Then, $\widetilde{\varphi_{\varepsilon,\psi_\varepsilon}} \xrightarrow{\,<\infty\,} \widetilde{\varphi_{\psi_0}}$.*

Proof. Because of Proposition 2.14, it is enough to show that $\mathcal{T}_\varepsilon(\widetilde{\varphi_{\varepsilon,\psi_\varepsilon}})$ converges strongly to φ_{ψ_0} in $L^p(\Omega \times Y)$ for every $p \in (1,\infty)$. It can be reduced to the strong convergence in $L^p(\Omega \times Y^*)$ since $\operatorname{supp}(\mathcal{T}_\varepsilon(\widetilde{\varphi_{\varepsilon,\psi_\varepsilon}})), \operatorname{supp}(\widetilde{\varphi_{\psi_0}}) \subset \overline{\Omega \times Y^*}$.

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Using $\psi_\varepsilon(x) = x + \check\psi_\varepsilon(x)$ and the Y-periodicity of φ, we can rewrite, for a.e. $(x,y) \in \Omega \times Y^*$ and $\varepsilon > 0$ small enough such that $x \in \tilde\Omega_\varepsilon$

$$
\mathcal{T}_\varepsilon(\widetilde{\varphi_{\varepsilon,\psi_\varepsilon}}) = \mathcal{T}_\varepsilon\left(\varphi\left(\psi_\varepsilon(\cdot_x), \widetilde{\frac{\psi_\varepsilon(\cdot_x)}{\varepsilon}}\right)\right)(x,y) = \varphi\left(\psi_\varepsilon([x]_{\varepsilon,Y} + \varepsilon y), \frac{\psi_\varepsilon([x]_{\varepsilon,Y} + \varepsilon y)}{\varepsilon}\right)
$$

$$
= \varphi\left([x]_{\varepsilon,Y} + \varepsilon y + \check\psi_\varepsilon([x]_{\varepsilon,Y} + \varepsilon y), \frac{[x]_{\varepsilon,Y} + \varepsilon y + \check\psi_\varepsilon([x]_{\varepsilon,Y} + \varepsilon y)}{\varepsilon}\right)
$$

$$
= \varphi\left([x]_{\varepsilon,Y} + \varepsilon y + \mathcal{T}_\varepsilon(\widetilde{\check\psi_\varepsilon})(x,y), y + \frac{\mathcal{T}_\varepsilon(\widetilde{\check\psi_\varepsilon})(x,y)}{\varepsilon}\right).
$$

The strong two-scale convergence of $\frac{1}{\varepsilon}\widetilde{\check\psi_\varepsilon}$ to $\widetilde{\check\psi_0}$ implies the strong convergence of $\frac{1}{\varepsilon}\mathcal{T}_\varepsilon(\widetilde{\check\psi_\varepsilon})$ to $\widetilde{\check\psi_0}$ in $L^p(\Omega \times Y)$. Then, there exists a subsequence ε' such that $\frac{1}{\varepsilon'}\mathcal{T}_{\varepsilon'}(\widetilde{\check\psi_{\varepsilon'}})(x,y) \to \widetilde{\check\psi_0}(x,y)$ for a.e. $(x,y) \in \Omega \times Y$. Moreover, $[x]_{\varepsilon,Y}$ converges to x and εy to 0. Since $\varphi \in C(\Omega^{(\delta)}; C_\#(Y))$, we can carry over these pointwise convergences to the pointwise convergence

$$
\varphi\left([x]_{\varepsilon',Y} + \varepsilon' y + \mathcal{T}_{\varepsilon'}(\widetilde{\check\psi_{\varepsilon'}})(x,y), y + \frac{\mathcal{T}_{\varepsilon'}(\widetilde{\check\psi_{\varepsilon'}})(x,y)}{\varepsilon'}\right)
$$

$$
\to \varphi(x, y + \check\psi_0(x,y)) = \varphi(x, \psi_0(x,y))
$$

for every $(x,y) \in \Omega \times Y^*$. Furthermore, $\left|\varphi\left(\mathcal{T}_{\varepsilon'}(\widetilde{\check\psi_{\varepsilon'}})(x,y), y + \frac{\mathcal{T}_{\varepsilon'}(\widetilde{\check\psi_{\varepsilon'}})(x,y)}{\varepsilon'}\right)\right| \leq \|\varphi\|_{C^\infty(\Omega^{(\delta)} \times Y)} \leq C$ for a.e. $(x,y) \in \Omega \times Y^*$. Thus, we can apply Lebesgue's dominated convergence theorem and get the strong convergence of $\mathcal{T}_{\varepsilon'}(\widetilde{\varphi_{\varepsilon',\psi_{\varepsilon'}}})$ to $\widetilde{\varphi_{\psi_0}}$ in $L^p(\Omega \times Y^*)$. Indeed, this argumentation holds for every arbitrary subsequence, too, which implies the strong convergence for the whole sequence. $\qquad\square$

The next lemma shows that the transformations by ψ_ε and ψ_ε^{-1} are uniformly continuous. Together with Lemma 3.6, this allows to translate between the weak two-scale convergence of sequences defined on Ω_ε and $\hat\Omega_\varepsilon$, respectively.

Lemma 3.7. *Let $p \in (1,\infty)$. Let u_ε and $\hat u_\varepsilon = u_\varepsilon \circ \psi_\varepsilon$ be sequences of measurable functions on Ω_ε and $\hat\Omega_\varepsilon$, respectively. Then, the following statements hold:*

1. *The sequence u_ε is bounded in $L^p(\Omega_\varepsilon)$ if and only if the sequence $\hat u_\varepsilon$ is bounded in $L^p(\hat\Omega_\varepsilon)$.*

2. *Let $l \in \{0,2\}$. Then the sequence $\varepsilon^{l/2}\nabla u_\varepsilon$ is bounded in $L^p(\Omega_\varepsilon)$ if and only if the sequence $\varepsilon^{l/2}\nabla\hat u_\varepsilon$ is bounded in $L^p(\hat\Omega_\varepsilon)$.*

Proof. Transforming the following integrals by ψ_ε and using the uniform esti-

mates on J_ε and J_ε^{-1} imply Lemma 3.7(1)

$$||u_\varepsilon||_{L^p(\Omega_\varepsilon)}^p = \int\limits_{\Omega_\varepsilon} |u_\varepsilon(x)|^p dx = \int\limits_{\hat\Omega_\varepsilon} J_\varepsilon(x)|\hat u_\varepsilon(x)|^p dx \leq C \int\limits_{\hat\Omega_\varepsilon} |\hat u_\varepsilon(x)|^p dx = C \, ||\hat u_\varepsilon||_{L^p(\hat\Omega_\varepsilon)}^p,$$

$$||\hat u_\varepsilon||_{L^p(\hat\Omega_\varepsilon)}^p = \int\limits_{\Omega_\varepsilon} J_\varepsilon^{-1}(\psi_\varepsilon^{-1}(x))|u_\varepsilon(x)|^p dx \leq c_J \int\limits_{\Omega_\varepsilon} |u_\varepsilon(x)|^p dx = c_J \, ||u_\varepsilon||_{L^p(\Omega_\varepsilon)}^p.$$

From the chain rule, we get $(\nabla u_\varepsilon)(\psi_\varepsilon(x)) = \Psi_\varepsilon^{-\top}(x)\nabla\hat u_\varepsilon(x)$. Using the uniform estimates for the Jacobians and their inverses (cf. Definition 3.2 and Lemma 3.3), we can estimate as follows

$$||\nabla u_\varepsilon||_{L^p(\Omega_\varepsilon)}^p = \int\limits_{\Omega_\varepsilon} |\nabla u_\varepsilon(x)|^p dx = \int\limits_{\hat\Omega_\varepsilon} J_\varepsilon(x)|\Psi_\varepsilon^{-\top}(x)\nabla\hat u_\varepsilon(x)|^p dx$$

$$\leq C \int\limits_{\hat\Omega_\varepsilon} \left|\left|\Psi_\varepsilon^{-\top}(x)\right|\right|^p |\nabla\hat u_\varepsilon(x)|^p dx \leq C \int\limits_{\hat\Omega_\varepsilon} \left|\left|\Psi_\varepsilon^{-\top}(x)\right|\right|_{L^\infty(\hat\Omega_\varepsilon)}^p |\nabla\hat u_\varepsilon(x)|^p dx$$

$$\leq C \int\limits_{\hat\Omega_\varepsilon} |\nabla\hat u_\varepsilon(x)|^p dx = C \, ||\nabla\hat u_\varepsilon||_{L^p(\hat\Omega_\varepsilon)}^p,$$

$$||\nabla\hat u_\varepsilon||_{L^p(\hat\Omega_\varepsilon)}^p = \int\limits_{\hat\Omega_\varepsilon} |\nabla\hat u_\varepsilon(x)|^p dx = \int\limits_{\Omega_\varepsilon} J_\varepsilon^{-1}(\psi_\varepsilon^{-1}(x))|\Psi_\varepsilon^\top(\psi_\varepsilon^{-1}(x))\nabla u_\varepsilon(x)|^p dx$$

$$\leq c_J \int\limits_{\Omega_\varepsilon} \left|\left|\Psi_\varepsilon^\top(\psi_\varepsilon^{-1}(x))\right|\right|^p |\nabla u_\varepsilon(x)|^p dx \leq c_J \int\limits_{\Omega_\varepsilon} \left|\left|\Psi_\varepsilon^\top\right|\right|_{L^\infty(\hat\Omega_\varepsilon)}^p |\nabla u_\varepsilon(x)|^p dx$$

$$\leq C \int\limits_{\Omega_\varepsilon} |\nabla u_\varepsilon(x)|^p dx = C \, ||\nabla u_\varepsilon||_{L^p(\Omega_\varepsilon)}^p,$$

which yields Lemma 3.7(2). $\qquad\square$

Now, we give a rigorous translation between the weak two-scale convergence of sequences defined on Ω_ε and the corresponding sequences defined on $\hat\Omega_\varepsilon$. First, we prove the following back-transformation, which was proposed in [4].

Theorem 3.8. *Let* $p \in (1,\infty)$. *Let* u_ε *be a sequence in* $L^p(\hat\Omega_\varepsilon)$ *and* $\hat u_\varepsilon = u_\varepsilon \circ \psi_\varepsilon$. *Then,* $\widetilde{u_\varepsilon} \xrightarrow{p} \widetilde{u_0}$ *for* $u_0 \in L^p(\Omega; L^p(Y_x^*))$ *if and only if* $\widetilde{\hat u_\varepsilon} \xrightarrow{p} \widetilde{\hat u_0}$ *for* $\hat u_0 \in L^p(\Omega \times Y^*)$. *Moreover,* $\hat u_0 = u_{0,\psi_0}$ *holds and equivalently* $u_0 = \hat u_{0,\psi_0^{-1}}$.

Proof. First, we assume that $\widetilde{\hat u_\varepsilon}$ two-scale converges to $\widetilde{\hat u_0}$ in $L^p(\Omega)$ for $1 < p < \infty$. Proposition 2.6 implies that $\widetilde{\hat u_\varepsilon}$ is bounded and by Lemma 3.7(1), $\widetilde{u_\varepsilon}$ is bounded as well. Moreover, $\widetilde{\hat u_{0,\psi_0^{-1}}} \in L^p(\Omega^{(\delta)} \times Y)$. Therefore, it is enough to show the distributional two-scale convergence, i.e.

$$\lim_{\varepsilon\to 0} \int\limits_{\Omega^{(\delta)}} \widetilde{u_\varepsilon}(x)\varphi\left(x,\frac{x}{\varepsilon}\right) dx = \int\limits_{\Omega^{(\delta)}} \int\limits_{Y} \widetilde{\hat u_{0,\psi_0^{-1}}}(x,y)\varphi(x,y)dydx \qquad (9)$$

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for every smooth function $\varphi \in D(\Omega^{(\delta)}; C_\#^\infty(Y))$. We transform the integrand of the left-hand side by ψ_ε

$$\int\limits_{\Omega^{(\delta)}} \widetilde{u_\varepsilon}(x)\varphi\left(x, \frac{x}{\varepsilon}\right)dx = \int\limits_{\Omega_\varepsilon} u_\varepsilon(x)\varphi\left(x, \frac{x}{\varepsilon}\right)dx = \int\limits_{\hat{\Omega}_\varepsilon} J_\varepsilon(x)\hat{u}_\varepsilon(x)\varphi_{\varepsilon,\psi_\varepsilon}(x)dx$$

$$= \int\limits_{\Omega} \widetilde{J_\varepsilon}(x)\widetilde{\hat{u}_\varepsilon}(x)\widetilde{\varphi_{\varepsilon,\psi_\varepsilon}}(x)dx.$$

We note that $\varphi \in C(\Omega^{(\delta)}; C_\#(Y))$ and Lemma 3.6 imply $\widetilde{\varphi_{\varepsilon,\psi_\varepsilon}} \overset{\infty}{\rightharpoonup} \widetilde{\varphi_{\psi_0}}$. Using the strong two-scale convergence $\widetilde{J_\varepsilon} \overset{<\infty}{\rightrightarrows} \widetilde{J_0}$, we can pass to the limit $\varepsilon \to 0$ and get

$$\lim_{\varepsilon \to 0}\int\limits_{\Omega} \widetilde{J_\varepsilon}(x)\widetilde{\hat{u}_\varepsilon}(x)\widetilde{\varphi_{\varepsilon,\psi_\varepsilon}}(x)dx = \iint\limits_{\Omega\,Y} \widetilde{J_0}(x,y)\widetilde{\hat{u}_0}(x,y)\widetilde{\varphi_{\psi_0}}(x,y)dydx.$$

Then, we transform the Y-integral back with $\psi_0(x, \cdot_y)$

$$\iint\limits_{\Omega\,Y} \widetilde{J_0}(x,y)\widetilde{\hat{u}_0}(x,y)\widetilde{\varphi_{\psi_0}}(x,y)dydx = \iint\limits_{\Omega\,Y^*} J_0(x,y)\hat{u}_0(x,y)\varphi(x,\psi_0(x,y))dydx$$

$$= \iint\limits_{\Omega\,Y_x^*} \hat{u}_0(x,\psi_0^{-1}(x,y))\varphi(x,y)dydx = \int\limits_{\Omega^{(\delta)}}\int\limits_{Y} \widetilde{\hat{u}_{0,\psi_0^{-1}}}(x,y)\varphi(x,y)dydx.$$

Combining these equations shows (9).

Now, we assume that $\widetilde{u_\varepsilon} \overset{p}{\rightharpoonup} \widetilde{u_0}$. By using Proposition 2.6 and Lemma 3.7(1), we obtain the boundedness of $\hat{u}_\varepsilon$. Then, Proposition 3.5 gives the existence of a subsequence ε' and a function $\hat{u}_0 \in L^p(\Omega \times Y^*)$ such that $\widetilde{\hat{u}_{\varepsilon'}} \overset{p}{\rightharpoonup} \widetilde{\hat{u}_0}$. The previous argumentation applied to this subsequence yields $\hat{u}_0 = u_{0,\psi_0}$. Since this argumentation also holds for every subsequence, the whole sequence $\widetilde{\hat{u}_\varepsilon}$ two-scale converges weakly to $\widetilde{\hat{u}_0}$ for $\hat{u}_0 = u_{0,\psi_0}$. $\qquad\square$

In the next step, we consider the weak two-scale convergence for weakly differentiable functions. We start with the case of large gradients, i.e. $\varepsilon\nabla u_\varepsilon \leq C$, and show that the same transformation rule as for the functions themselves hold.

Theorem 3.9. *Let $p \in (1,\infty)$. Let u_ε be a sequence in $W^{1,p}(\Omega_\varepsilon)$ and $\hat{u}_\varepsilon = u_\varepsilon \circ \psi_\varepsilon$ a sequence in $W^{1,p}(\hat{\Omega}_\varepsilon)$ such that u_ε is bounded in $L^p(\Omega_\varepsilon)$. Then, $\varepsilon\widetilde{\nabla u_\varepsilon} \overset{p}{\rightharpoonup} \widetilde{\nabla_y u_0}$ for $u_0 \in L^p(\Omega; W_\#^{1,p}(Y_x^*))$ if and only if $\varepsilon\widetilde{\nabla\hat{u}_\varepsilon} \overset{p}{\rightharpoonup} \widetilde{\nabla_y\hat{u}_0}$ for $\hat{u}_0 \in L^p(\Omega; W_\#^{1,p}(Y^*))$. Moreover, $\hat{u}_0 = u_{0,\psi_0}$ holds and equivalently $u_0 = \hat{u}_{0,\psi_0^{-1}}$.*

Proof. First, we assume that $\varepsilon\widetilde{\nabla\hat{u}_\varepsilon} \overset{p}{\rightharpoonup} \widetilde{\nabla_y\hat{u}_0}$. Proposition 2.6 implies that $\varepsilon\widetilde{\nabla\hat{u}_\varepsilon}$ is bounded and by Lemma 3.7(2), $\varepsilon\widetilde{\nabla u_\varepsilon}$ is bounded as well. Moreover,

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$\widetilde{\nabla_y \hat{u}_{0,\psi_0^{-1}}} \in L^p(\Omega^{(\delta)} \times Y)$. Therefore, it is enough to show the distributional two-scale convergence, i.e.

$$\lim_{\varepsilon \to 0} \int\limits_{\Omega^{(\delta)}} \varepsilon \widetilde{\nabla u_\varepsilon}(x) \cdot \varphi\left(x, \frac{x}{\varepsilon}\right) dx = \int\limits_{\Omega^{(\delta)}} \int\limits_Y \widetilde{\nabla_y \hat{u}_{0,\psi_0^{-1}}}(x,y) \cdot \varphi(x,y) dy dx \qquad (10)$$

for test functions $\varphi \in D(\Omega^{(\delta)}; C_\#^\infty(Y))^N$, where $\nabla_y \hat{u}_{0,\psi_0^{-1}}$ denotes the gradient of $y \mapsto \hat{u}_0(x, \psi_0^{-1}(x,y))$. We transform the integral on the left-hand side by ψ_ε and use the chain rule, which gives $\nabla u_\varepsilon(\psi_\varepsilon^{-1}(x)) = \Psi_\varepsilon^{-\top}(x) \nabla \hat{u}_\varepsilon(x)$. Thus, we get

$$\int\limits_{\Omega^{(\delta)}} \varepsilon \widetilde{\nabla u_\varepsilon}(x) \cdot \varphi\left(x, \frac{x}{\varepsilon}\right) dx = \int\limits_{\Omega} \widetilde{J}_\varepsilon(x) \widetilde{\Psi_\varepsilon^{-\top}}(x) \varepsilon \widetilde{\nabla \hat{u}_\varepsilon}(x) \cdot \widetilde{\varphi_{\varepsilon,\psi_\varepsilon}}(x) dx.$$

In order to pass to the limit $\varepsilon \to 0$, we proceed as in the proof of Theorem 3.8 and additionally use the strong two-scale convergence of $\widetilde{\Psi_\varepsilon^{-\top}}$. In the limit, we transform the Y-integral back with ψ_0^{-1} and obtain

$$\lim_{\varepsilon \to 0} \int\limits_{\Omega^{(\delta)}} \varepsilon \widetilde{\nabla u_\varepsilon}(x) \cdot \varphi\left(x, \frac{x}{\varepsilon}\right) dx = \lim_{\varepsilon \to 0} \int\limits_{\Omega} \widetilde{J}_\varepsilon(x) \widetilde{\Psi_\varepsilon^{-\top}}(x) \varepsilon \widetilde{\nabla \hat{u}_\varepsilon}(x) \cdot \widetilde{\varphi_{\varepsilon,\psi_\varepsilon}}(x) dx$$

$$= \int\limits_{\Omega} \int\limits_Y \widetilde{J}_0(x,y) \widetilde{\Psi_0^{-\top}}(x,y) \widetilde{\nabla_y \hat{u}_0}(x,y) \cdot \widetilde{\varphi_{\psi_0}}(x,y) dy dx$$

$$= \int\limits_{\Omega^{(\delta)}} \int\limits_Y \widetilde{\nabla_y \hat{u}_{0,\psi_0^{-1}}}(x,y) \cdot \varphi(x,y) dy dx.$$

Now, we assume that $\varepsilon \widetilde{\nabla u_\varepsilon} \overset{p}{\rightharpoondown} \widetilde{\nabla_y u_0}$. Using Proposition 2.6 and Lemma 3.7(2), we obtain the boundedness of $\varepsilon \nabla \hat{u}_\varepsilon$. Moreover, Proposition 3.7(1) transfers the boundedness of u_ε in $L^p(\Omega_\varepsilon)$ to the boundedness of $\hat{u}_\varepsilon$ in $L^p(\hat{\Omega}_\varepsilon)$. Then, Proposition 3.5 implies the existence of $\hat{u}_0 \in L^2(\Omega; H_\#^1(Y^*))$ and a subsequence ε' such that $\varepsilon' \widetilde{\nabla u_{\varepsilon'}} \overset{p}{\rightharpoondown} \widetilde{\nabla_y \hat{u}_0}$. The previous argumentation applied to this subsequence yields $\hat{u}_0 = u_{0,\psi_0}$. Since this argumentation holds for every subsequence, it holds for the whole sequence. $\qquad\square$

The last part of the two-scale transformation is the case of small gradients, i.e. $\|\nabla u_\varepsilon\|_{L^p(\Omega_\varepsilon)} \leq C$. Following the approach of the case of large gradients yields $\Psi_0^{-\top}(x,y)\nabla_x \hat{u}_0(x) + \Psi_0^{-\top}(x,y)\nabla_y \hat{u}_1(x,y)$, which has to be transformed back. However, the Jacobian $\Psi_0^{-\top}$ only vanishes by the back-transformation of the y-gradient and remains in front of the x-gradient. This remaining Jacobian is basically the reason why the back-transformation did not yield a transformationally independent limit problem in the hitherto existing works. In order to overcome this problem, we separate the purely macroscopic part of $\Psi_0^{-\top}(x)\nabla_x \hat{u}_0(x)$ and put the remaining part into the transformation rule for the y-gradient. Thus, we can prove the following new transformation rule.

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Theorem 3.10. *Let $p \in (1, \infty)$ and assume that $Y_{\#}^*$ is connected. Let u_ε be a sequence in $W^{1,p}(\Omega_\varepsilon)$ and $\hat{u}_\varepsilon = u_\varepsilon \circ \psi_\varepsilon$ a sequence in $W^{1,p}(\hat{\Omega}_\varepsilon)$ such that u_ε is bounded in $L^p(\Omega_\varepsilon)$. Then, $\widetilde{\nabla u_\varepsilon} \xrightarrow{p} \chi_Q \widetilde{\nabla_x u_0} + \widetilde{\nabla_y u_1}$ for $u_0 \in W^{1,p}(\Omega)$ and $u_1 \in L^p(\Omega; W_{\#}^{1,p}(Y_x^*)/\mathbb{R})$ if and only if $\widetilde{\nabla \hat{u}_\varepsilon} \xrightarrow{p} \chi_{Y^*} \nabla_x \hat{u}_0 + \widetilde{\nabla_y \hat{u}_1}$ for $\hat{u}_0 \in W^{1,p}(\Omega)$ and $\hat{u}_1 \in L^p(\Omega; W_{\#}^{1,p}(Y^*)/\mathbb{R})$. Moreover, $\hat{u}_0 = u_0$ holds and also $u_1 = \hat{u}_{1,\psi_0^{-1}} + \check{\psi}_0^{-1} \cdot \nabla_x \hat{u}_0$, which is equivalent to $\hat{u}_1 = u_{1,\psi_0} + \check{\psi}_0 \cdot \nabla_x u_0$.*

Proof. First, we assume that $\widetilde{\nabla \hat{u}_\varepsilon} \xrightarrow{p} \chi_{Y^*} \hat{u}_0 + \widetilde{\nabla_y \hat{u}_1}$. Proposition 2.6 implies that $\widetilde{\nabla \hat{u}_\varepsilon}$ is bounded and by Lemma 3.7(2), $\widetilde{\nabla u_\varepsilon}$ is bounded as well. Moreover, $\widetilde{\nabla_y u_1} \in L^p(\Omega^{(\delta)} \times Y)$ for $u_1 = \hat{u}_{1,\psi_0^{-1}} + \check{\psi}_0^{-1} \cdot \nabla_x \hat{u}_0$. Therefore, it is enough to show the distributional two-scale convergence, i.e.

$$\lim_{\varepsilon \to 0} \int\limits_{\Omega^{(\delta)}} \widetilde{\nabla u_\varepsilon}(x) \cdot \varphi\left(x, \frac{x}{\varepsilon}\right) dx$$

$$= \int\limits_{\Omega^{(\delta)}} \int\limits_Y \left(\chi_Q(x,y)\widetilde{\nabla_x \hat{u}_0}(x) + \widetilde{\nabla_y u_1}(x,y)\right) \cdot \varphi(x,y)dydx \qquad (11)$$

for $u_1 = \hat{u}_{1,\psi_0^{-1}} + \check{\psi}_0^{-1} \cdot \nabla_x \hat{u}_0$ and for test functions $\varphi \in D(\Omega^{(\delta)}; C_{\#}^\infty(Y))^N$. We transform the integral on the left-hand side and pass to the limit $\varepsilon \to 0$ like in the proof of Theorem 3.9. After transforming the Y-integral back, we use $\Psi_{0,\psi_0^{-1}}^{-\top}(x,y) = \Psi_0^{-\top}(x, \psi_0^{-1}(x,y)) = \nabla_y \psi_0^{-1}(x,y) = \mathbb{1} + \nabla_y \check{\psi}_0^{-1}(x,y)$

$$\lim_{\varepsilon \to 0} \int\limits_{\Omega^{(\delta)}} \widetilde{\nabla u_\varepsilon}(x) \cdot \varphi\left(x, \frac{x}{\varepsilon}\right) dx = \lim_{\varepsilon \to 0} \int\limits_\Omega \widetilde{J_\varepsilon}(x)\widetilde{\Psi_\varepsilon^{-\top}}(x)\widetilde{\nabla \hat{u}_\varepsilon}(x) \cdot \widetilde{\varphi_{\varepsilon,\psi_\varepsilon}}(x)dx$$

$$= \int\limits_\Omega \int\limits_Y \widetilde{J_0}(x,y)\widetilde{\Psi_0^{-\top}}(x,y)(\chi_{Y^*}(x)\nabla_x \hat{u}_0(x) + \widetilde{\nabla_y \hat{u}_1}(x,y)) \cdot \widetilde{\varphi_{\psi_0}}(x,y)dydx$$

$$= \int\limits_{\Omega^{(\delta)}} \int\limits_Y \left(\widetilde{\Psi_{0,\psi_0^{-1}}^{-\top}}(x,y)\chi_Q(x,y)\widetilde{\nabla_x \hat{u}_0}(x) + \widetilde{\nabla_y \hat{u}_{1,\psi_0^{-1}}}(x,y)\right) \cdot \varphi(x,y)dydx$$

$$= \int\limits_{\Omega^{(\delta)}} \int\limits_Y \left(\chi_Q(x,y)\widetilde{\nabla_x \hat{u}_0}(x) + \widetilde{\nabla_y \check{\psi}_0^{-1}}(x,y)\widetilde{\nabla_x \hat{u}_0}(x) + \widetilde{\nabla_y \hat{u}_{1,\psi_0^{-1}}}(x,y)\right) \cdot \varphi(x,y)dydx,$$

where $\nabla_y \hat{u}_{1,\psi_0^{-1}}$ denotes the gradient of $y \mapsto \hat{u}_1(x, \psi_0^{-1}(x,y))$. Thus, $\widetilde{\nabla u_\varepsilon} \xrightarrow{p} \chi_Q \widetilde{\nabla_x u_0} + \widetilde{\nabla_y u_1}$ with $u_0 = \hat{u}_0$ and $u_1 = \hat{u}_{1,\psi_0^{-1}} + \check{\psi}_0^{-1} \cdot \nabla_x \hat{u}_0$.

Now, we assume that $\widetilde{\nabla u_\varepsilon} \xrightarrow{p} \chi_Q \nabla_x u_0 + \widetilde{\nabla_y u_1}$. Then, we obtain $\widetilde{\hat{u}_\varepsilon} \xrightarrow{p} \chi_{Y^*} \nabla_x \hat{u}_0 + \widetilde{\nabla \hat{u}_1}$ with $\hat{u}_0 = u_0$ and $\hat{u}_1 = u_{1,\psi_0} - \check{\psi}_{0,\psi_0}^{-1} \cdot \nabla_x u_0$ by the same argumentation as in the proof of Theorem 3.9. Rewriting $\check{\psi}_0^{-1}(x, \psi_0(x,y)) = \psi_0^{-1}(x, \psi_0(x,y)) - \psi_0(x,y) = y - \psi_0(x,y) = -\check{\psi}_0(x,y)$ gives $\hat{u}_1 = u_{1,\psi_0} + \check{\psi}_0 \cdot \nabla_x u_0$. $\square$

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With the transformation results Theorem 3.8, Theorem 3.9 and Theorem 3.10, we can translate the two-scale compactness results for periodic domains Theorem 2.3, Theorem 2.4 and Theorem 2.5 directly into the following compactness results for locally periodic domains.

Theorem 3.11. *Let $p \in (1, \infty)$. Let u_ε be a bounded sequence in $L^p(\Omega_\varepsilon)$. Then, there exist $u_0 \in L^p(\Omega; L^p(Y_x^*))$ and a subsequence ε' such that $\widetilde{u_{\varepsilon'}} \xrightarrow{p} \widetilde{u_0}$.*

Theorem 3.12. *Let $p \in (1, \infty)$. Let u_ε be a sequence in $W^{1,p}(\Omega_\varepsilon)$ such that $||u_\varepsilon||_{L^p(\Omega_\varepsilon)} \leq C$ and $\varepsilon ||\nabla u_\varepsilon||_{L^p(\Omega_\varepsilon)} \leq C$. Then, there exist $u_0 \in L^p(\Omega; W_{\#}^{1,p}(Y_x^*))$ and a subsequence ε' such that $\widetilde{u_{\varepsilon'}} \xrightarrow{p} \widetilde{u_0}$ and $\varepsilon' \widetilde{\nabla u_{\varepsilon'}} \xrightarrow{p} \widetilde{\nabla_y u_0}$.*

Theorem 3.13. *Let $p \in (1, \infty)$ and assume that $Y_{\#}^*$ is connected. Let u_ε be a sequence in $W^{1,p}(\Omega_\varepsilon)$ such that $||u_\varepsilon||_{L^p(\Omega_\varepsilon)} + ||\nabla u_\varepsilon||_{L^p(\Omega_\varepsilon)} \leq C$. Then, there exist $u_0 \in W^{1,p}(\Omega)$, $u_1 \in L^p(\Omega; W_{\#}^{1,p}(Y_x^*)/\mathbb{R})$ and a subsequence ε' such that $\widetilde{u_{\varepsilon'}} \xrightarrow{p} \chi_Q \widetilde{u_0}$ and $\widetilde{\nabla u_{\varepsilon'}} \xrightarrow{p} \chi_Q \widetilde{\nabla_x u_0} + \widetilde{\nabla_y u_1}$.*

Now, we consider the transformational behaviour of the strong two-scale convergence. This allows us to translate the strong two-scale convergence of the coefficients into the strong two-scale convergence of the transformed coefficients.

Theorem 3.14. *Let $p \in (1, \infty)$. Let u_ε be a sequence in $L^p(\hat{\Omega}_\varepsilon)$ and $\hat{u}_\varepsilon = u_\varepsilon \circ \psi_\varepsilon$. Then, $\widetilde{u_\varepsilon} \xrightarrow{p} \widetilde{u_0}$ for $u_0 \in L^p(\Omega; L^p(Y_x^*))$ if and only if $\widehat{\tilde{u}_\varepsilon} \xrightarrow{p} \widehat{\tilde{u}_0}$ for $\hat{u}_0 \in L^p(\Omega \times Y^*)$. Moreover, $\hat{u}_0 = u_{0,\psi_0}$ holds and equivalently $u_0 = \hat{u}_{0,\psi_0^{-1}}$.*

Proof. Assume that $\widetilde{u_\varepsilon} \xrightarrow{p} u_0$. Because of Theorem 3.8 it is sufficient to show that $\lim_{\varepsilon \to 0} \left|\left|\widehat{\tilde{u}_\varepsilon}\right|\right|_{L^p(\Omega)} = \left|\left|\widehat{\tilde{u}_0}\right|\right|_{L^p(\Omega \times Y)}$. We transform via ψ_ε and obtain

$$\left|\left|\widehat{\tilde{u}_\varepsilon}\right|\right|_{L^p(\Omega)}^p = \int_\Omega |\widehat{\tilde{u}_\varepsilon}(x)|^p dx = \int_{\Omega^{(\delta)}} \widetilde{J_\varepsilon^{-1} \circ \psi_\varepsilon^{-1}}(x) |\widetilde{u_\varepsilon}(x)|^p dx$$

$$= \int_{\Omega^{(\delta)}} \int_Y \mathcal{T}_\varepsilon(\widetilde{J_\varepsilon^{-1} \circ \psi_\varepsilon^{-1}}(x) |\widetilde{u_\varepsilon}(x)|^p) dx = \int_{\Omega^{(\delta)}} \int_Y \mathcal{T}_\varepsilon(\widetilde{J_\varepsilon^{-1} \circ \psi_\varepsilon^{-1}}(x)) |\mathcal{T}_\varepsilon(\widetilde{u_\varepsilon}(x))|^p) dx$$

$$\tag{12}$$

In order to pass to the limit $\varepsilon \to 0$, we note that $\widetilde{J_\varepsilon^{-1}} \xrightarrow{<\infty} \widetilde{J_0^{-1}}$. Then, Theorem 3.8 implies that $\widetilde{J_\varepsilon^{-1} \circ \psi_\varepsilon^{-1}} \xrightarrow{<\infty} \widetilde{J_{0,\psi_0^{-1}}^{-1}}$ and thus $\mathcal{T}_\varepsilon(\widetilde{J_\varepsilon^{-1} \circ \psi_\varepsilon^{-1}})$ converges weakly in $L^p(\Omega^{(\delta)} \times Y)$, for every $p \in (1, \infty)$, to $\widetilde{J_{0,\psi_0^{-1}}^{-1}}$. Because $\left|\left|\mathcal{T}_\varepsilon(\widetilde{J_\varepsilon^{-1} \circ \psi_\varepsilon^{-1}})\right|\right|_{L^\infty(\Omega^{(\delta)})}$ is uniformly bounded by c_J^{-1}, $\mathcal{T}_\varepsilon(\widetilde{J_\varepsilon^{-1} \circ \psi_\varepsilon^{-1}})$ converges weakly-* in $L^\infty(\Omega \times Y)$. Moreover, $\widetilde{u_\varepsilon} \xrightarrow{p} u_0$ implies that $\mathcal{T}_\varepsilon(\widetilde{u_\varepsilon})$ converges strongly in $L^p(\Omega \times Y)$ to u_0, which gives the strong convergence of $|\mathcal{T}_\varepsilon(\widetilde{u_\varepsilon})|^p$ to

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$|\widetilde{u_0}|^p$. Thus, we can pass to the limit in (12) and obtain

$$\lim_{\varepsilon \to 0} \left\| \widetilde{\hat{u}_\varepsilon} \right\|_{L^p(\Omega)}^p = \lim_{\varepsilon \to 0} \int_{\Omega^{(\delta)} \times Y} \mathcal{T}_\varepsilon(\widetilde{J_\varepsilon^{-1} \circ \psi_\varepsilon^{-1}}(x)) |\mathcal{T}_\varepsilon(\widetilde{u_\varepsilon}(x))|^p dx$$

$$= \int_{\Omega^{(\delta)} \times Y} \widetilde{J_{0,\psi_0^{-1}}^{-1}}(x,y) |\widetilde{u_0}(x,y)|^p dx = \int_{\Omega \times Y} |\widetilde{u_{0,\psi_0}}(x,y)|^p dx = \int_{\Omega \times Y} |\widehat{u}_0(x,y)|^p dx$$

The proof of the other implication follows in the same way. □

365   Note that Theorem 3.8 transfers $\chi_{\hat{\Omega}_\varepsilon} \overset{\stackrel{<\infty}{\twoheadrightarrow}}{} \chi_{Y^*}$ into $\chi_{\Omega_\varepsilon} \overset{\stackrel{<\infty}{\twoheadrightarrow}}{} \chi_Q$. This implies that Q is uniquely defined up to a set $N \subset \Omega \times Y$ of measure zero. Moreover, since Definition 3.1 postulates that Y_x^* is open for a.e. $x \in \Omega$, Y_x^* is uniquely defined for a.e. $x \in \Omega$. Therefore, the two-scale limit domain should not be seen as subset of $\Omega \times Y$ but rather as the sets Y_x^* for a.e. $x \in \Omega$.

370   Furthermore, Theorem 3.14 implies also the strong two-scale convergence of the displacement fields of the back-transformations. After rewriting $\check{\psi}_\varepsilon^{-1} \circ \psi_\varepsilon = -\check{\psi}_\varepsilon$, we get $\frac{1}{\varepsilon} \widetilde{\check{\psi}_\varepsilon^{-1}} \overset{\stackrel{<\infty}{\twoheadrightarrow}}{} -\widetilde{\chi_{Y^p, \psi_0^{-1}} \check{\psi}_{0,\psi_0^{-1}}} = \chi_Q \widetilde{\check{\psi}_0^{-1}}$.

4. Homogenisation on locally periodic domains

In order to pass to the limit $\varepsilon \to 0$ in (2), we have to assume that there exists
375     $A_0 \in L^\infty(Q)^{N \times N}$, which is coercive, such that $\widetilde{A_\varepsilon} \overset{\stackrel{<\infty}{\twoheadrightarrow}}{} \widetilde{A_0}$ and that there exists $f_0 \in L^2(\Omega; L^2(Y_x^*))$ such that $\widetilde{f_\varepsilon} \overset{2}{\rightharpoonup} \widetilde{f_0}$. Note that it is not necessary to assume that these two-scale limits are 0 outside of Q. It is sufficient to assume only the existence of the two-scale limits A_0 and f_0. Then, the two-scale compactness results for locally periodic domains ensures that $A_0 = \chi_Q A_0$ and $f_0 = \chi_Q f_0$. In
380   the following, we assume that $Y_\#^*$ is connected if $l = 0$.

4.1. The periodic substitute problem

We transform the coefficients A_ε and the source functions f_ε into $\hat{A}_\varepsilon := A_\varepsilon \circ \psi_\varepsilon$ and $\hat{f}_\varepsilon := f_\varepsilon \circ \psi_\varepsilon$, respectively. Lemma 3.8 implies that $\widetilde{\hat{f}_\varepsilon} \overset{2}{\rightharpoonup} \widetilde{\hat{f}_0}$ with $\hat{f}_0 := f_{0,\psi_0}$ and Lemma 3.14 implies that $\widetilde{\hat{A}_\varepsilon} \overset{\stackrel{<\infty}{\twoheadrightarrow}}{} \widetilde{\hat{A}_0}$ with $\hat{A}_0 := A_{0,\psi_0}$. Moreover,
385   note that these transformations carry the uniform boundedness and coercivity from A_ε over to $\hat{A}_\varepsilon$ as well as from A_0 over to $\hat{A}_0$. Then, the transformation of (2) with ψ_ε gives the following weak form (cf. Proposition 4.2):
Find $\hat{u}_\varepsilon \in H^1(\hat{\Omega}_\varepsilon)$ such that

$$\int_{\hat{\Omega}_\varepsilon} \varepsilon^l J_\varepsilon(x) \Psi_\varepsilon^{-1}(x) \hat{A}_\varepsilon(x) \Psi_\varepsilon^{-\top}(x) \nabla \hat{u}_\varepsilon(x) \cdot \nabla \hat{\varphi}(x) + J_\varepsilon(x) \hat{u}_\varepsilon(x) \hat{\varphi}(x) dx$$

$$= \int_{\hat{\Omega}_\varepsilon} J_\varepsilon(x) \hat{f}_\varepsilon(x) \hat{\varphi}(x) dx \qquad (13)$$

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for every $\hat{\varphi} \in H^1(\hat{\Omega}_\varepsilon)$.

Using the uniform estimates of the transformations, we show the existence and uniqueness of solutions of (13) as well as their uniform boundedness.

Proposition 4.1. *For every $\varepsilon > 0$, there exists a unique solution $\hat{u}_\varepsilon \in H^1(\hat{\Omega}_\varepsilon)$ of the weak form* (13) *such that*

$$||\hat{u}_\varepsilon||_{L^2(\hat{\Omega}_\varepsilon)} + \varepsilon^{l/2} ||\nabla \hat{u}_\varepsilon||_{L^2(\hat{\Omega}_\varepsilon)} \leq C. \tag{14}$$

Proof. Using the uniform bounds of the Jacobians of ψ_ε, we can estimate

$$||\nabla u||_{L^2(\hat{\Omega}_\varepsilon)}^2 \leq \frac{1}{c_J} \left|\left| \sqrt{J_\varepsilon} \Psi_\varepsilon^\top \Psi_\varepsilon^{-\top} \nabla u \right|\right|_{L^2(\hat{\Omega}_\varepsilon)}^2 \leq \frac{1}{c_J} ||\Psi_\varepsilon^\top||_{C(\hat{\Omega}_\varepsilon)}^2 \left|\left| \sqrt{J_\varepsilon} \Psi_\varepsilon^{-\top} \nabla u \right|\right|_{L^2(\hat{\Omega}_\varepsilon)}^2$$

$$\leq C \left|\left| \sqrt{J_\varepsilon} \Psi_\varepsilon^{-\top} \nabla u \right|\right|_{L^2(\hat{\Omega}_\varepsilon)}^2 \leq \frac{C}{\alpha} \int_{\hat{\Omega}_\varepsilon} J_\varepsilon(x) \Psi_\varepsilon^{-1}(x) \hat{A}_\varepsilon(x) \Psi_\varepsilon^{-\top}(x) \nabla u(x) \cdot \nabla u(x) dx$$

for every $u \in H^1(\hat{\Omega}_\varepsilon)$. This implies the ε-independent coercivity of the left-hand side of (13) in $H^1(\hat{\Omega}_\varepsilon)$

$$\int_{\hat{\Omega}_\varepsilon} \varepsilon^l J_\varepsilon(x) \Psi_\varepsilon^{-1}(x) \hat{A}_\varepsilon(x) \Psi_\varepsilon^{-\top}(x) \nabla u(x) \cdot \nabla u(x) + J_\varepsilon(x) u(x) u(x) dx$$

$$\geq C \left(\varepsilon^l ||\nabla u||_{L^2(\hat{\Omega}_\varepsilon)}^2 + ||u||_{L^2(\hat{\Omega}_\varepsilon)}^2 \right). \tag{15}$$

Furthermore, the left-hand side of (13) can be estimated for every $u, v \in H^1(\hat{\Omega}_\varepsilon)$ with the Cauchy inequality and the uniform estimates of the transformations

$$\int_{\hat{\Omega}_\varepsilon} \varepsilon^l J_\varepsilon(x) \Psi_\varepsilon^{-1}(x) \hat{A}_\varepsilon(x) \Psi_\varepsilon^{-\top}(x) \nabla v(x) \cdot \nabla u(x) + J_\varepsilon(x) v(x) u(x) dx$$

$$\leq \varepsilon^l C \left|\left| \sqrt{J_\varepsilon} \Psi_\varepsilon^{-\top} \nabla v \right|\right|_{L^2(\hat{\Omega}_\varepsilon)} \left|\left| \sqrt{J_\varepsilon} \Psi_\varepsilon^{-\top} \nabla u \right|\right|_{L^2(\hat{\Omega}_\varepsilon)} + C ||v||_{L^2(\hat{\Omega}_\varepsilon)} ||u||_{L^2(\hat{\Omega}_\varepsilon)}$$

$$\leq \varepsilon^l C ||\nabla v||_{L^2(\hat{\Omega}_\varepsilon)} ||\nabla u||_{L^2(\hat{\Omega}_\varepsilon)} + C ||v||_{L^2(\hat{\Omega}_\varepsilon)} ||u||_{L^2(\hat{\Omega}_\varepsilon)}.$$

This implies the continuity of the left-hand side.

The right-hand side of (13) can be estimated with the uniform estimates from Lemma 3.3

$$\int_{\hat{\Omega}_\varepsilon} J_\varepsilon(x) \hat{f}_\varepsilon(x) \hat{\varphi}(x) dx \leq \left|\left| J_\varepsilon \hat{f}_\varepsilon \right|\right|_{L^2(\hat{\Omega}_\varepsilon)} ||\hat{\varphi}||_{L^2(\hat{\Omega}_\varepsilon)}$$

$$\leq ||J_\varepsilon||_{C(\hat{\Omega}_\varepsilon)} \left|\left| \hat{f}_\varepsilon \right|\right|_{L^2(\hat{\Omega}_\varepsilon)} ||\hat{\varphi}||_{L^2(\hat{\Omega}_\varepsilon)} \leq C ||\hat{\varphi}||_{L^2(\hat{\Omega}_\varepsilon)}. \tag{16}$$

Note that $\left|\left| \hat{f}_\varepsilon \right|\right|_{L^2(\hat{\Omega}_\varepsilon)}$ is bounded since $\widetilde{\hat{f}}_\varepsilon$ two-scale converges in $L^2(\Omega)$.

These estimates allow us to apply the Theorem of Lax–Milgram, which gives the existence and uniqueness of a solution $\hat{u}_\varepsilon \in H^1(\hat{\Omega}_\varepsilon)$. Combining (13) with (15) and (16) for $\hat{\varphi} = \hat{u}_\varepsilon$ and employing the Young inequality yield the uniform estimate (14). $\qquad \square$

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Proposition 4.2. *Let $u_\varepsilon \in H^1(\Omega_\varepsilon)$ be the solution of (2) and let $\hat{u}_\varepsilon \in H^1(\hat{\Omega}_\varepsilon)$ be the solution of (13). Then, $\hat{u}_\varepsilon = u_\varepsilon \circ \psi_\varepsilon$.*

Proof. The Theorem of Lax–Milgram ensures the existence of a unique solution of (2).

Testing (2) with $\hat{\varphi} \circ \psi_\varepsilon^{-1}$, gives

$$\int\limits_{\Omega_\varepsilon} \varepsilon^l A_\varepsilon(x)\nabla u_\varepsilon(x) \cdot \nabla \hat{\varphi} \circ \psi_\varepsilon^{-1}(x) + u_\varepsilon(x)\hat{\varphi} \circ \psi_\varepsilon^{-1}(x)dx = \int\limits_{\Omega_\varepsilon} f_\varepsilon(x)\hat{\varphi} \circ \psi_\varepsilon^{-1}(x)dx.$$

$$(17)$$

Transforming the integrals with ψ_ε and using the product rule yield

$$\int\limits_{\hat{\Omega}_\varepsilon} \varepsilon^l J_\varepsilon(x)\Psi_\varepsilon^{-1}(x)\hat{A}_\varepsilon(x)\Psi_\varepsilon^{-\top}(x)\nabla u_\varepsilon \circ \psi_\varepsilon(x) \cdot \nabla \hat{\varphi}(x) + u_\varepsilon \circ \psi_\varepsilon(x)\hat{\varphi}(x)dx$$

$$= \int\limits_{\hat{\Omega}_\varepsilon} J_\varepsilon(x)f_\varepsilon(\psi_\varepsilon(x))\hat{\varphi}(x)dx.$$

It follows by the uniqueness of the solution of (13) that $\hat{u}_\varepsilon = u_\varepsilon \circ \psi_\varepsilon$. $\qquad\square$

4.2. Homogenisation of the periodic substitute problem

In the following, we pass to the homogenisation limit $\varepsilon \to 0$ in (13) by using the compactness result for periodic domains (cf. Proposition 3.5).

Proposition 4.3. *Let $l = 0$ and let $Y_\#^*$ be connected. Let $\hat{u}_\varepsilon$ be the solutions of (13) given by Proposition 4.1. Then, $\widetilde{\hat{u}_\varepsilon} \xrightarrow{2} \chi_{Y^*}\hat{u}_0$ and $\widetilde{\nabla \hat{u}_\varepsilon} \xrightarrow{2} \chi_{Y^*}\nabla_x\hat{u}_0 + \widetilde{\nabla_y \hat{u}_1}$, where $(\hat{u}_0, \hat{u}_1) \in H^1(\Omega) \times L^2(\Omega, H_\#^1(Y^*)/\mathbb{R})$ is the unique solution of*

$$\iint\limits_{\Omega\, Y^*} J_0(x,y)\Psi_0^{-1}(x,y)\hat{A}_0(x,y)\Psi_0^{-\top}(x,y)(\nabla_x\hat{u}_0(x) + \nabla_y\hat{u}_1(x,y))$$

$$\cdot (\nabla_x\hat{\varphi}_0(x) + \nabla_y\hat{\varphi}_1(x,y))dydx$$

$$+ \iint\limits_{\Omega\, Y^*} J_0(x,y)\hat{u}_0(x)\hat{\varphi}_0(x)dydx = \iint\limits_{\Omega\, Y^*} J_0(x,y)\hat{f}_0(x,y)\hat{\varphi}_0(x)dydx \qquad (18)$$

for every $(\hat{\varphi}_0, \hat{\varphi}_1) \in H^1(\Omega) \times L^2(\Omega, H_\#^1(Y^*)/\mathbb{R})$.

Proof. Testing (13) with $\hat{\varphi}_0 + \varepsilon\hat{\varphi}_1\left(\cdot_x, \frac{\cdot_x}{\varepsilon}\right)$ for $\hat{\varphi}_0 \in C^\infty(\overline{\Omega})$ and $\hat{\varphi}_1 \in C^\infty(\overline{\Omega}; C_\#^\infty(Y))$ gives

$$\int\limits_{\Omega} \widetilde{J_\varepsilon}(x)\widetilde{\Psi_\varepsilon^{-1}}(x)\hat{A}_\varepsilon(x)\widetilde{\Psi_\varepsilon^{-\top}}(x)\widetilde{\nabla\hat{u}_\varepsilon}(x) \cdot \left(\nabla_x\hat{\varphi}_0(x) + \varepsilon\nabla_x\hat{\varphi}_1\left(x, \frac{x}{\varepsilon}\right) + \nabla_y\hat{\varphi}_1\left(x, \frac{x}{\varepsilon}\right)\right)$$

$$+ \widetilde{J_\varepsilon}(x)\widetilde{\hat{u}_\varepsilon}(x)\left(\hat{\varphi}_0(x) + \varepsilon\hat{\varphi}\left(x, \frac{x}{\varepsilon}\right)\right)dx = \int\limits_{\Omega} \widetilde{J_\varepsilon}(x)\widetilde{\hat{f}_\varepsilon}(x)\left(\hat{\varphi}_0(x) + \varepsilon\hat{\varphi}\left(x, \frac{x}{\varepsilon}\right)\right)dx.$$

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The uniform estimate of u_ε, given by Proposition 4.1, and the compactness result for periodic domains (cf. Proposition 3.5) yield the existence of $(\hat{u}_0, \hat{u}_1) \in H^1(\Omega) \times L^2(\Omega; H^1_\#(Y^*)/\mathbb{R})$ and a subsequence ε' and such that $\widetilde{\hat{u}_{\varepsilon'}} \xrightarrow{2} \chi_{Y^*}\hat{u}_0$ and $\widetilde{\nabla \hat{u}_{\varepsilon'}} \xrightarrow{2} \chi_{Y^*}\nabla_x\hat{u}_0 + \widetilde{\nabla_y\hat{u}_1}$. Then, we pass to the limit $\varepsilon' \to 0$ and obtain (18) for smooth test functions. By a density argument, (18) follows for test functions in $H^1(\Omega) \times L^2(\Omega; H^1_\#(Y^*)/\mathbb{R})$.

The existence and uniqueness of the solution $(\hat{u}_0, \hat{u}_1) \in H^1(\Omega) \times L^2(\Omega, H^1_\#(Y^*)/\mathbb{R})$ follow from the Theorem of Lax–Milgram. The necessary uniform coercivity and continuity estimates of the left-hand side can be proven in a standard way, while the uniform coercivity of $J_0\Psi_0^{-1}\hat{A}_0\Psi_0^{-\top}$ can be proven like in Proposition 4.1.

Since this argumentation holds for every subsequence, the uniqueness of the solution of (18) implies that the convergences hold for the whole sequence. $\square$

We rewrite the two-scale limit problem (18) into the following homogenised problem, which is defined on the cylindrical two-scale domain $\Omega \times Y^*$ and contains the transformation coefficients J_0 and Ψ_0.

Proposition 4.4. *Let $\hat{u}_0 \in H^1(\Omega)$ be the solution of* (18). *Then, it solves*

$$\int\limits_\Omega \hat{B}_0(x)\nabla\hat{u}_0(x) \cdot \nabla\hat{\varphi}(x) + \Theta(x)\hat{u}_0(x)\hat{\varphi}(x)dx = \int\limits_\Omega\int\limits_{Y^*} J_0(x,y)\hat{f}_0(x,y)dy \ \hat{\varphi}(x)dx$$

$$(19)$$

for every $\hat{\varphi} \in H^1(\Omega)$, where $\Theta(x) = \int\limits_{Y^*} J_0(x,y)dy$ and $\hat{B}_0 \in L^\infty(\Omega)^{N \times N}$ is given by

$$(\hat{B}_0)_{ij}(x) := \int\limits_{Y^*} J_0(x,y)\Psi_0^{-1}(x,y)\hat{A}_0(x,y)\Psi_0^{-\top}(x,y)(e_j + \nabla_y\hat{w}_j(x,y)) \cdot e_i dy$$

$$(20)$$

and $\hat{w}_j$ is defined as the unique solution in $L^2(\Omega; H^1_\#(Y^*)/\mathbb{R})$ such that

$$\int\limits_\Omega\int\limits_{Y^*} J_0(x,y)\Psi_0^{-1}(x,y)\hat{A}_0(x,y)\Psi_0^{-\top}(x,y)(\nabla_y\hat{w}_j(x,y) + e_j) \cdot \nabla_y\hat{\varphi}_1(x,y)dydx = 0$$

$$(21)$$

for every $\hat{\varphi}_1 \in L^2(\Omega; H^1_\#(Y^*)/\mathbb{R})$.

Proof. Choosing $\hat{\varphi}_0 = 0$ in (18) yields

$$\int\limits_\Omega\int\limits_{Y^*} J_0(x,y)\Psi_0^{-1}(x,y)\hat{A}_0(x,y)\Psi_0^{-\top}(x,y)(\nabla_x\hat{u}_0(x) + \nabla_y\hat{u}_1(x,y)) \cdot \nabla_y\hat{\varphi}_1(x,y)dydx = 0,$$

$$(22)$$

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which implies $\hat{u}_1 = \sum_{j=1}^{N} \partial_{x_j}\hat{u}_0\hat{w}_j$, where $\hat{w}_j$ is the unique solution of the cell problem (21). Inserting $\hat{u}_1 = \sum_{j=1}^{N} \partial_{x_j}\hat{u}_0\hat{w}_j$ in (18) and choosing $\hat{\varphi}_1 = 0$ yield (19) for $\hat{B}_0$ given by (20). $\qquad\square$

Proposition 4.5. *Let $l = 2$ and let $\hat{u}_\varepsilon$ be the solution of (13) given by Proposition 4.1. Then, $\widetilde{\hat{u}_\varepsilon} \xrightarrow{2} \widetilde{\hat{u}_0}$ and $\varepsilon\widetilde{\nabla\hat{u}_\varepsilon} \xrightarrow{2} \widetilde{\nabla_y\hat{u}_0}$, where $\hat{u}_0$ is the unique solution in $L^2(\Omega, H^1_{\#}(Y^*))$ such that*

$$\int\limits_{\Omega}\int\limits_{Y^*} \hat{D}_0(x,y)\nabla_y\hat{u}_0(x,y) \cdot \nabla_y\hat{\varphi}(x,y) + J_0(x,y)\hat{u}_0(x,y)\hat{\varphi}(x,y)dydx$$

$$= \int\limits_{\Omega}\int\limits_{Y^*} J_0(x,y)\hat{f}_0(x,y)\hat{\varphi}(x,y)dydx \qquad (23)$$

for every $\hat{\varphi} \in L^2(\Omega; H^1_{\#}(Y^*))$, where $\hat{D}_0 = J_0\Psi_0^{-1}\hat{A}_0\Psi_0^{-\top}$.

Proof. The uniform estimate of $\hat{u}_\varepsilon$, given by Proposition 4.1, and the compactness result for periodic domains (cf. Proposition 3.5) imply the existence of a subsequence ε' and $\hat{u}_0 \in L^2(\Omega; H^1_{\#}(Y^*))$ such that $\widetilde{\hat{u}_{\varepsilon'}} \xrightarrow{2} \widetilde{\hat{u}_0}$ and $\varepsilon'\widetilde{\nabla\hat{u}_{\varepsilon'}} \xrightarrow{2} \widetilde{\nabla_y\hat{u}_0}$. We test (13) with $\hat{\varphi}\left(\cdot_x, \frac{\cdot_x}{\varepsilon}\right)$ for $\hat{\varphi} \in C^\infty(\overline{\Omega}; C^\infty_{\#}(Y))$. After passing to the limit $\varepsilon' \to 0$, we obtain (23) for smooth test functions. Since $C^\infty(\overline{\Omega}; C^\infty_{\#}(Y))$ is dense in $L^2(\Omega; H^1_{\#}(Y^*))$, (23) holds for any $\hat{\varphi} \in L^2(\Omega; H^1_{\#}(Y^*))$.

The existence and uniqueness of the solution $\hat{u}_0 \in L^2(\Omega, H^1_{\#}(Y^*))$ follow from the Theorem of Lax–Milgram. The necessary uniform coercivity and continuity estimates can be proven in a standard way, while the uniform coercivity of $J_0\Psi_0^{-1}\hat{A}_0\Psi_0^{-\top}$ can be proven like in Proposition 4.1.

Since this argumentation holds for every subsequence, the uniqueness of the solution of (23) implies that the convergences hold for the whole sequence. $\qquad\square$

4.3. Back-transformation

Using Theorem 3.8, Theorem 3.10 and Theorem 3.9, we can transform the two-scale limit problems back. Thus, we can derive the two-scale limit problems of (2) for $l = 0$ and $l = 2$. Moreover, these limit problems do not depend on the chosen transformations ψ_ε and ψ_0.

Theorem 4.6. *Let $l = 0$ and let $Y^*_{\#}$ be connected. Let u_ε be the solution of (2). Then, $\widetilde{u_\varepsilon} \xrightarrow{2} \chi_Q\widetilde{u_0}$ and $\widetilde{\nabla u_\varepsilon} \xrightarrow{2} \chi_Q\widetilde{\nabla_x u_0} + \widetilde{\nabla_y u_1}$, where (u_0, u_1) is the unique*

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solution in $H^1(\Omega) \times L^2(\Omega; H^1_\#(Y^*_x)/\mathbb{R})$ of

$$\int_\Omega \int_{Y^*_x} A_0(x,y)(\nabla_x u_0(x) + \nabla_y u_1(x,y)) \cdot (\nabla_x \varphi_0(x) + \nabla_y \varphi_1(x,y)) + u_0(x)\varphi_0(x)dydx$$

$$= \int_\Omega \int_{Y^*_x} f_0(x,y)dy \; \varphi_0(x)dx \tag{24}$$

for every $(\varphi_0, \varphi_1) \in H^1(\Omega) \times L^2(\Omega; H^1_\#(Y^*_x)/\mathbb{R})$.

Proof. Proposition 4.2 shows that $\hat{u}_\varepsilon = u_\varepsilon \circ \psi_\varepsilon$, where $\hat{u}_\varepsilon$ is the unique solution of (19). By Theorem 3.8, we obtain $\widetilde{u_\varepsilon} \xrightarrow{2} \chi_Q \widetilde{u_0}$ and, by Theorem 3.10, $\widetilde{\nabla u_\varepsilon} \xrightarrow{2} \chi_Q \widetilde{\nabla_x u_0} + \widetilde{\nabla_y u_1}$ for $u_0 = \hat{u}_0$ and $u_1 = \hat{u}_{1,\psi_0^{-1}} + \check{\psi}_0^{-1} \cdot \nabla_x u_0$, where $\hat{u}_0$ and $\hat{u}_1$ determine the two-scale limits of $\hat{u}_\varepsilon$ and $\nabla \hat{u}_\varepsilon$.

Then, we test (18) by $(\varphi_0, \varphi_{1,\psi_0} + \check{\psi}_0 \cdot \nabla_x \varphi_0)$ for $(\varphi_0, \varphi_1) \in H^1(\Omega) \times L^2(\Omega; H^1_\#(Y^*_x))$ and transform the Y^*-integral by ψ_0^{-1} so that

$$\int_\Omega \int_{Y^*_x} A_0(x,y)\left(\Psi_0^{-\top}(x, \psi_0^{-1}(x,y))\nabla_x \hat{u}_0(x) + \nabla_y \hat{u}_1(x, \psi_0^{-1}(x,y)) \right)$$

$$\cdot \left(\Psi_0^{-\top}(x, \psi_0^{-1}(x,y))\nabla_x \varphi_0(x) + \nabla_y \left(\varphi_1(x,y) + \check{\psi}_0(x, \psi_0^{-1}(x,y)) \cdot \nabla_x \varphi_0(x) \right) \right)dydx$$

$$+ \int_\Omega \int_{Y^*_x} u_0(x)\varphi_0(x)dydx = \int_\Omega \int_{Y^*_x} f_0(x,y)\varphi_0(x)dydx, \tag{25}$$

where $\nabla_y \check{\psi}_0(x, \psi_0^{-1}(x,y))$ denotes the gradient of $y \mapsto \check{\psi}_0(x, \psi_0^{-1}(x,y))$. Using that $\Psi_0^{-\top}(x, \psi_0^{-1}(x,y)) = \mathbb{1} + \nabla_y \check{\psi}_0^{-1}(x,y)$, we can rewrite

$$\Psi_0^{-\top}(x, \psi_0^{-1}(x,y))\nabla_x \hat{u}_0(x) + \nabla_y \hat{u}_1(x, \psi_0^{-1}(x,y))$$
$$= \nabla_x \hat{u}_0(x) + \nabla_y(\hat{u}_1(x, \psi_0^{-1}(x,y)) + \check{\psi}_0^{-1} \cdot \nabla_x u_0(x)) = \nabla_x u_0(x) + \nabla_y u_1(x,y).$$

Employing that $\check{\psi}_0(x, \psi_0^{-1}(x,y)) = y - \psi_0^{-1}(x,y) = -\check{\psi}_0^{-1}(x,y)$, we get

$$\Psi_0^{-\top}(x, \psi_0^{-1}(x,y))\nabla_x \varphi_0(x) + \nabla_y(\varphi_1(x,y) + \check{\psi}_0(x, \psi_0^{-1}(x,y)) \cdot \nabla_x \varphi_0(x))$$
$$= \nabla_x \varphi_0(x) + \nabla_y(\varphi_1(x,y) + \check{\psi}_0(x, \psi_0^{-1}(x,y)) \cdot \nabla_x \varphi_0(x) + \check{\psi}_0^{-1}(x,y) \cdot \nabla_x \varphi_0(x))$$
$$= \nabla_x \varphi_0(x) + \nabla_y \varphi_1(x,y).$$

Thus, (25) can be simplified to (24). □

From (24), we can derive the following homogenised limit problem, which is defined on Ω with cell problems defined on Y^*_x. However, in contrast to previous works, it does not contain Jacobians of the chosen deformation ψ_0.

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Theorem 4.7. *Let $u_0 \in H^1(\Omega)$ be the solution of* (18)*. Then, it solves*

$$\int_\Omega B_0(x)\nabla u_0(x) \cdot \nabla\varphi(x) + \Theta(x)u_0(x)\varphi(x)dx = \int_\Omega \int_{Y_x^*} f_0(x,y)dy \; \varphi(x)dx \quad (26)$$

for every $\varphi \in H^1(\Omega)$, where $\Theta(x) := \int_{Y_x^*} 1 dy = |Y_x^*|$ and $B_0 \in L^\infty(\Omega)^{N \times N}$ is given by

$$(B_0)_{ij}(x) := \int_{Y_x^*} \delta_{ij} + \partial_{y_i} w_j(x,y)dy \quad (27)$$

and w_j is defined as the unique solution $w_j \in L^2(\Omega; H^1_\#(Y_x^*)/\mathbb{R})$ such that

$$\int_\Omega \int_{Y_x^*} (\nabla_y w_j(x,y) + e_j) \cdot \nabla_y \varphi(x,y)dydx = 0 \quad (28)$$

for every $\varphi \in L^2(\Omega; H^1_\#(Y_x^*)/\mathbb{R})$.

Proof. The proof of Theorem 4.7 runs as the proof of Proposition 4.4. $\qquad\square$

Note that Θ in (26) is the same as in (19) and gives the local porosity of the domain. With Lemma 3.3, we see that Θ is bounded from below by c_J and Θ is obviously bounded from above by 1.

The back-transformation of the two-scale limit problem (23) (the case $l = 2$) in its actual two-scale domain is straightforward and yields the following limit problem.

Theorem 4.8. *Let $l = 2$ and let u_ε be the solutions of* (2)*. Then, $\widetilde{u_\varepsilon} \overset{2}{\rightharpoonup} \chi_Q \widetilde{u_0}$ and $\varepsilon\widetilde{\nabla u_\varepsilon} \overset{2}{\rightharpoonup} \widetilde{\nabla_y u_0}$, where u_0 is the unique solution of the following weak form. Find $u_0 \in L^2(\Omega; H^1_\#(Y_x^*))$ such that*

$$\int_\Omega \int_{Y_x^p} A_0(x,y)\nabla_y u(x,y) \cdot \nabla_y \varphi(x,y) + u(x,y)\varphi(x,y)dydx = \int_\Omega \int_{Y_x^p} f(x,y)\varphi(x,y)dydx$$

$$(29)$$

for every $\varphi \in L^2(\Omega; H^1_\#(Y_x^*))$.

Theorem 4.8 follows by testing (23) with $\varphi(\cdot_x, \psi_0(\cdot_x, \cdot_y))$ and back-transformation with ψ_0^{-1}.

5. Direct homogenisation on the locally periodic domains and further comments

The compactness results for locally periodic domains which we have developed in this work (cf. Theorem 3.8, Theorem 3.10 and Theorem 3.9) allow us to

pass directly to the limit $\varepsilon \to 0$ in (2). The argumentation is the same as in the periodic case. However, the problem which we have considered is an easy linear problem and uniform a-priori estimates can also be derived on the locally periodic domain. If the homogenisation of a more difficult problem is considered, for instance the Stokes problem or non-linear problems, the derivation of further two-scale compactness results as well as the derivation of uniform estimates can be easier in the strict periodic setting. Indeed, the results of this work allow to transform those back from the periodic into the locally periodic domain. Therefore, the passage to the limit $\varepsilon \to 0$ can be done in the locally periodic setting as well. Moreover, it can be reasonable to transform the limit problem from the actual non-cylindrical two-scale domain to the cylindrical coordinates $\Omega \times Y^*$ in order to derive uniform estimates on the homogenised tensor and the cell problems.

The original motivation for this two-scale-transformation method originates from problems on evolving microstructures. There, problems are considered on a time interval S and a time dependent domain $\Omega_\varepsilon(t)$ (cf. [4], [5], [6], [7], [8], [9]). The two-scale-transformation method is basically the same for these problems since time is only a parameter in the concept of the two-scale convergence. Thus, our results can be carried over to these problems, where the domain $\Omega_\varepsilon(t) := \psi_\varepsilon(t, \hat{\Omega}_\varepsilon)$ is defined with a family of locally periodic transformations $\psi_\varepsilon : S \times \hat{\Omega}_\varepsilon \to \Omega$ which are dependent on time.

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