Band width estimates with lower scalar curvature bounds

Dissertation

Zur Erlangung des akademischen Grades

Dr. rer. nat.

eingereicht an der

Mathematisch-Naturwissenschaftlich-Technischen Fakultät

der Universität Augsburg

von

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Augsburg, 2022

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Tag der mündlichen Prüfung: 12.09.2022

Abstract

A band is a connected compact manifold X together with a decomposition

$$\partial X = \partial_- X \sqcup \partial_+ X$$

where $\partial_\pm X$ are non-empty unions of boundary components. If X is equipped with a Riemannian metric, the pair (X, g) is called a Riemannian band and the width of (X, g) is defined to be the distance between $\partial_- X$ and $\partial_+ X$ with respect to g.

Following Gromov's seminal work on metric inequalities with scalar curvature, the study of Riemannian bands with lower curvature bounds has been an active field of research in recent years, which led to several breakthroughs on longstanding open problems in positive scalar curvature geometry and to a better understanding of the positive mass theorem in general relativity.

In the first part of this thesis we combine ideas of Gromov and Cecchini-Zeidler and use the variational calculus surrounding so called μ-bubbles to establish a scalar and mean curvature comparison principle for Riemannian bands with the property that no closed embedded hypersurface which separates the two ends of the band admits a metric of positive scalar curvature. The model spaces we use for this comparison are warped product over scalar flat manifolds with log-concave warping functions.

We employ ideas from surgery and bordism theory to deduce that, if Y is a closed orientable manifold which does not admit a metric of positive scalar curvature, $\dim(Y) \neq 4$ and $X^{n \leq 7} = Y \times [-1, 1]$, the width of X with respect to any Riemannian metric with scalar curvature $\geq n(n-1)$ is bounded from above by $\frac{2\pi}{n}$. This solves, up to dimension 7, a conjecture due to Gromov in the orientable case.

Furthermore, we adapt and extend our methods to show that, if Y is as before and $M^{n \leq 7} = Y \times \mathbb{R}$, then M does not admit a metric of positive scalar curvature. This solves, up to dimension 7 a conjecture due to Rosenberg and Stolz in the orientable case.

In the second part of this thesis we explore how these results transfer to the setting where the lower scalar curvature bound is replaced by a lower bound on the macroscopic scalar curvature of a Riemannian band. This curvature condition amounts to an upper bound on the volumes of all unit balls in the universal cover of the band.

We introduce a new class of orientable manifolds we call filling enlargeable and prove: If Y is filling enlargeable, $X^n = Y \times [-1, 1]$ and g is a Riemannian metric on X with the property that the volumes of all unit balls in the universal cover of (X, g) are bounded from above by a small dimensional constant ε_n, then $\mathrm{width}(X, g) \leq 1$.

Finally, we establish that whether or not a closed orientable manifold is filling enlargeable or not depends on the image of the fundamental class under the classifying map of the universal cover.

Acknowledgements

Throughout the last three years I have benefited from the expert advice, mentorship and encouragement of many colleagues.

I am grateful to my advisor Bernhard Hanke for his constant support. From day one onwards, he gave me the freedom to follow my mathematical interests and to choose and develop my projects accordingly. He shared his knowledge in many fruitful discussions and whenever it came time to write things down, he was a perceptive reader and helped me along with his detailed and insightful feedback, which had a lasting impact on the way I approach mathematical writing.

I thank Simone Cecchini and Rudolf Zeidler for their interest in my work and countless inspiring discussions which culminated in a joint project. Without their recent work, acting as a source of ideas, this thesis could not have been completed in its current form.

Special thanks also go to Johannes Ebert, whose input has greatly improved the topological results I was able to establish, Jan Metzger, who took the time to answer my questions regarding maximum principles, Georg Frenck, for carefully reading an earlier draft of this thesis, Roman Sauer, for his interest in my work and his confidence in my abilities and Thomas Schick, who invited me to speak in Göttingen early on and kindly agreed to act as second referee.

Furthermore, I thank everyone, who is, or was, a part of the differential geometry group during my time at Augsburg University for a friendly and stimulating work environment, be it virtual or in presence.

I am deeply grateful to my family and friends for their support and especially to Anna and Lilly for their neverending love and encouragement throughout good times and bad.

Funding Acknowledgement: Since April 2020, my work was generously funded by a doctoral scholarship of the 'Studienstiftung des deutschen Volkes'.

Notes on the text

This thesis grew in the writing and reflects my work on three different projects:

- ▷ Chapter 2 corresponds to my article [54], which is available as a preprint on the arXiv and currently submitted for publication.
- ▷ Chapter 3 corresponds to parts of joint work [9] with Simone Cecchini and Rudolf Zeidler, which is available as a preprint on the arXiv and currently submitted for publication.
- ▷ Chapter 4 corresponds to my article [55], which was first published in the journal *Algebraic & Geometric Topology* 22.1 (2022), pp. 405-432 by Mathematical Science Publishers.

Contents

Introduction

Given a smooth manifold X, a fundamental question in modern differential geometry is whether or not X admits a smooth Riemannian metric with positive scalar curvature. Furthermore, if such a metric g exists, it is intriguing to ask how positivity of $\mathrm{scal}(X, g)$ manifests itself in other geometric properties of (X, g).

Over the years, multiple ways to approach these problems have been developed. The first one is through index theory for spinor Dirac operators on spin manifolds and originates in the work of Lichnerowicz [48].

He observed that, by the so called Schrödinger-Lichnerowicz formula

$$\not{\mathfrak{D}}^2 = \nabla^*\nabla + \frac{\mathrm{scal}}{4},$$

the spinor Dirac operator on a closed Riemannian spin manifold is invertible if the scalar curvature is uniformly positive. In combination with the Atiyah-Singer index theorem one obtains topological obstructions to positive scalar curvature.

Building on this approach, and subsequent refinements due to Hitchin [40], Gromov and Lawson [29, 30] were able to prove that the torus T^n does not admit a metric with positive scalar curvature.

In dimension $n \leq 7$ this result had previously been established by Schoen and Yau [63, 64]. Their approach is built on the observation that if $\Sigma \subset (X^n, g)$ is a closed embedded stable minimal hypersurface, the second variation formula for the area functional yields

$$\int_\Sigma |\nabla_\Sigma \psi|^2 + \frac{1}{2}\,\mathrm{scal}(\Sigma, g)\psi^2 \geq \int_\Sigma \frac{1}{2}(\mathrm{scal}(X, g) + |A|^2)\psi^2$$

for all $\psi \in C^\infty(\Sigma)$, where A is the second fundamental form of Σ. If the scalar curvature of (X, g) is positive, the right hand side of the equation is positive for all ψ. By the Gauss-Bonnet theorem for $n = 3$, or a conformal change argument, inspired by Kazdan and Warner [44], for $n > 3$, it follows that Σ itself admits a metric with positive scalar curvature. If the homology of X is rich enough, this fact can be used in an inductive way to provide obstructions to positive scalar curvature and insight into the structure of metrics with non-negative scalar curvature on X.

Furthermore Gromov and Lawson [29] as well as Schoen and Yau [64] proved that the existence of a metric with positive scalar curvature on a smooth manifold X is preserved under surgery with codimension at least three. This result allows one to study the question which manifolds of dimension at least five admit metrics with positive scalar curvature via bordism theory. Consequently all simply connected closed manifolds in dimension ≥ 5 which admit a metric with positive scalar curvature were completely classified in [29] and the work of Stolz [66].

In dimension four there are additional obstructions and techniques coming from Seiberg-Witten theory. They play a role in this work insofar as they often provide counterexamples (see Remark 1.7) to conjectures regarding the structure of all manifolds which do not admit metrics with positive scalar curvature (see Remark 1.7). For an overview on this topic we refer the reader to [58, Section 1.1.3].

If the manifold $X^{n\geq 2}$ is not closed, it admits a Riemannian metric g with uniformly positive scalar curvature by Gromov's h-principle. In order to encounter interesting phenomena one has to adapt the fundamental questions accordingly. In this case the metric is required to be complete and, if $\partial X \neq 0$, to fulfill certain boundary conditions which are usually phrased in terms of mean curvature.[1]

This aspect of positive scalar curvature geometry is exemplified by a series of results, mostly due to Gromov and Lawson [28, 30]. For now we refrain from stating any result in their general form but focus instead on the special case of the torus.

Theorem 1.1 ([30, Corollary 6.13])**.** *Let $X = T^{n-1} \times \mathbb{R}$ and $n \geq 2$. Then X does not admit a complete metric with positive scalar curvature.*

Theorem 1.1 is indicative of the general principle that obstructions to positive scalar curvature on non-closed manifolds can be derived from obstructions on closed submanifolds which are embedded in a suitable manner. This phenomenon persists in codimension two. Although $\mathbb{R}^2$ admits a complete metric with positive scalar curvature and hence $T^{n-2} \times \mathbb{R}^2$ does as well (the scalar curvature of a product metric is equal to the sum of the scalar curvatures of its factors), the following holds true:

Theorem 1.2 ([30, 38, 75, Theorem 1.10])**.** *Let $X = T^{n-2} \times \mathbb{R}^2$ and $n \geq 3$. Then X does not admit a complete metric with uniformly positive scalar curvature.*

Regarding boundary conditions, Gromov and Lawson observed that if a compact manifold X admits a metric g with $\operatorname{scal}(X, g) > 0$ and $H(\partial X, g) > 0$, then the *double* of X admits a metric with positive scalar curvature [28, Theorem 5.7]. This result was later generalized by Almeida [1, Theorem 1.1] to allow for nonnegative mean curvature (see also the recent work of Bär and Hanke [6]). Put together with the fact that T^n does not admit positive scalar curvature this implies:

Theorem 1.3 ([1, 28])**.** *Let $X = T^{n-1} \times [-1, 1]$ and $n \geq 2$. If g is a Riemannian metric on X with $\operatorname{scal}(X, g) > 0$, then*

$$\inf_{x \in \partial X} H(\partial X, g)(x) < 0.$$

We adapt the viewpoint that Theorem 1.3 quantifies the fact that any metric with positive scalar curvature on the open manifold $T^{n-1} \times (-1, 1)$ is necessarily incomplete by Theorem 1.1. Recently, in an exciting development, a new way to study boundary conditions for positive scalar curvature and to quantify the failure of completeness on open manifolds has emerged. The following, so called, *band width estimate* was proven by Gromov [31, Section 2] in dimension $n \leq 7$ using the minimal hypersurface approach of Schoen and Yau [64] and in all dimensions by Zeidler [75, Theorem 1.4] and Cecchini [12, Theorem D] using Dirac operator methods:

[1] Throughout this thesis, $H(\partial X, g)$ is defined to be the trace of the second fundamental form of ∂X with respect to the inner unit normal vector field. With this convention the unit sphere $S^{n-1} \subset \mathbb{R}^n$ has mean curvature $(n-1)$.

Theorem 1.4 ([12, 31, 75])**.** *Let* $X = T^{n-1} \times [-1,1]$ *and* $n \geq 2$*. If* g *is a Riemannian metric on* X *with* $\operatorname{scal}(X,g) \geq n(n-1)$*, then*

$$\operatorname{width}(X,g) := \operatorname{dist}_g\left(T^{n-1} \times \{-1\}, T^{n-1} \times \{1\}\right) \leq \frac{2\pi}{n}. \tag{1.0.1}$$

In the context of classical Riemannian geometry, Theorem 1.4 can be compared with the Bonnet-Myers theorem, according to which a minimal geodesic in a complete Riemannian manifold with Ricci curvature bounded from below by $(n-1)\sigma > 0$ has length at most $\frac{\pi}{\sqrt{\sigma}}$. Cecchini [12, Theorem D] and Zeidler [76, Corollary 1.5] proved that equality can not be attained in (1.0.1). The estimate, however, is sharp:

Example 1.5. For $n \geq 2$ let $X = T^{n-1} \times [-1,1]$ and g_T be the standard flat metric on T^{n-1}. Let $\varphi(t) = \cos\left(\frac{nt}{2}\right)^{\frac{2}{n}}$ for $t \in (-\frac{\pi}{n}, \frac{\pi}{n})$. For any $\varepsilon > 0$ the warped product

$$\left(T^{n-1} \times \left[-\frac{\pi}{n} + \varepsilon, \frac{\pi}{n} - \varepsilon\right], \varphi(t)^2 g_T + dt^2\right)$$

has $\operatorname{scal} = n(n-1)$ and width $= \frac{2\pi}{n} - 2\varepsilon$.

Theorem 1.4 should be contextualized among Theorem 1.1, 1.2 and 1.3. It is important to point out that there is no direct formal implication between Theorem 1.1 and Theorem 1.4 as the latter only implies that $T^{n-1} \times \mathbb{R}$ does not admit a complete metric with *uniformly* positive scalar curvature. Nevertheless Zeidler [76, Corollary 1.5] realized that the Dirac operator approach towards the band width estimate could be adapted to prove Theorem 1.1 and Theorem 1.4 in unison.

For a complete Riemannian metric g on $X = T^{n-2} \times \mathbb{R}^2$ with positive scalar curvature Gromov [31, Section 2] established that the infimum of $\operatorname{scal}(X,g)$ in a concentric ball of radius R in X decays at least quadratically with the radius R. The key ingredient in his proof is Theorem 1.4. Hence it implies Theorem 1.2.

The relationship between Theorem 1.3 and Theorem 1.4 is more complex. On the one hand the band width estimate in dimension n implies that T^n does not admit a metric with positive scalar curvature; otherwise there is a covering space of the form $(T^{n-1} \times \mathbb{R}, \overline{g})$ with uniformly positive scalar curvature. Together with the aforementioned doubling result [1, Theorem 1.1, 6, 28, Theorem 5.7] for metrics with $\operatorname{scal} > 0$ and $H \geq 0$, one recovers Theorem 1.3.

On the other hand one can analyze Example 1.5 to see that for $\varepsilon \to 0$ the mean curvature of the corresponding metrics on $T^{n-1} \times \left[-\frac{\pi}{n} + \varepsilon, \frac{\pi}{n} - \varepsilon\right]$ diverges to $-\infty$ as their boundaries collapse to points. This behavior suggests that the width estimate can be strengthened under the assumption of an additional lower bound for the mean curvature. In this vein Cecchini and Zeidler [14] proved:

Theorem 1.6 ([14, Theorem 7.6])**.** *For* $n \geq 2$ *odd let* $X = T^{n-1} \times [-1,1]$ *and* g *be a Riemannian metric on* X*. If*

- ▷ $\operatorname{scal}(X,g) \geq n(n-1)$,
- ▷ $H(\partial_\pm X, g) \geq \mp(n-1)\tan\left(\frac{n\ell_\pm}{2}\right)$ *for some* $-\frac{\pi}{n} < \ell_- < \ell_+ < \frac{\pi}{n}$,[2]

then $\operatorname{width}(X,g) \leq \ell_+ - \ell_-$.

[2] In the following we denote $\partial_- X = T^{n-1} \times \{-1\}$ and $\partial_+ X = T^{n-1} \times \{1\}$

The restriction to odd dimensions originates from technical difficulties in Dirac operator approach to positive scalar curvature on manifolds with boundary as is explained in [14, Section 1.4]. Whenever it applies Theorem 1.6 interpolates between Theorem 1.3 and Theorem 1.4, which can be seen as its extremal cases.

Indeed, if g is a Riemannian metric on $X = T^{n-1} \times [-1, 1]$ with $\mathrm{scal}(X, g) \geq n(n-1)$, then $H(\partial_\pm X) \geq c$ for some constant $c \in \mathbb{R}$. As $(n-1)\tan\left(\frac{nt}{2}\right) \to \infty$ for $t \to \frac{\pi}{n}$ there are $-\frac{\pi}{n} < \ell_- < \ell_+ < \frac{\pi}{n}$ such that $H(\partial_\pm X, g) \geq c \geq \mp(n-1)\tan\left(\frac{n\ell_\pm}{2}\right)$. By Theorem 1.6, one concludes that $\mathrm{width}(X, g) \leq \ell_+ - \ell_- < \frac{2\pi}{n}$.

Regarding Theorem 1.3 we could assume for a contradiction that the mean curvature of (X, g) is non-negative. But then $\mathrm{width}(X, g) \leq \ell_+ - \ell_-$ for $\ell_- = 0$ and $\ell_+ > 0$ arbitrary by Theorem 1.6, contradicting the fact that $\mathrm{width}(X, g) > 0$.

From a conceptual standpoint, Theorem 1.6 can be viewed as a scalar and mean curvature comparison principle between arbitrary metrics on $T^{n-1} \times [-1, 1]$ and the warped product

$$(M, g_\varphi) := \left(T^{n-1} \times [\ell_-, \ell_+], \varphi(t)^2 g_T + dt^2\right)$$

from Example 1.5. Standard results for warped products (see 2.2) imply:

- $\mathrm{scal}(M, g_\varphi) = n(n-1)$,
- $H(\partial_\pm M, g_\varphi) = \mp(n-1)\tan\left(\frac{n\ell_\pm}{2}\right)$,
- $\mathrm{width}(M, g_\varphi) = \ell_+ - \ell_-$.

Thus Theorem 1.6 takes the following form: If $\mathrm{scal}(X, g) \geq \mathrm{scal}(M, g_\varphi)$ and $H(\partial_\pm X, g) \geq H(\partial_\pm M, g_\varphi)$, then $\mathrm{width}(X, g) \leq \mathrm{width}(M, g_\varphi)$.

1.1 A set of Conjectures

It is only natural to ask, whether the results we have introduced for the torus hold true for a larger class of, or perhaps all, manifolds which do not admit metrics with positive scalar curvature. As we mentioned in the beginning, conjectures of this kind tend to fail in dimension four, where the special obstructions coming from Seiberg-Witten theory can be used to provide counterexamples. The following is an instance of this phenomenon:

Remark 1.7. There exists a closed simply connected 4-manifold Y which does not admit a metric of positive scalar curvature while $Y \times S^1$ does (see [60, Counterexample 4.16]). But then $Y \times \mathbb{R}$, and consequently $Y \times \mathbb{R}^2$, admit complete metrics with uniformly positive scalar curvature.

In dimension $\neq 4$, the results for T^{n-1} are expected to hold in full generality:

Conjecture 1.8 ([59, Section 7])**.** Let Y^{n-1} be a closed manifold of dimension $\neq 4$ which does not admit a metric with positive scalar curvature. Then $Y \times \mathbb{R}$ does not admit a complete metric with positive scalar curvature.

Conjecture 1.9 ([59, Section 7])**.** Let Y^{n-2} be a closed manifold of dimension $\neq 4$ which does not admit a metric with positive scalar curvature. Then $Y \times \mathbb{R}^2$ does not admit a complete metric with uniformly positive scalar curvature.

Conjecture 1.10. Let Y^{n-1} be a closed manifold of dimension $\neq 4$ which does not admit a metric with positive scalar curvature and $X = Y \times [-1, 1]$. If g is a Riemannian metric on X with $\mathrm{scal}(X, g) \geq n(n-1)$, then

$$\inf_{x \in \partial X} H(\partial X, g)(x) < 0.$$

Conjecture 1.11 ([31, 11.12 Conjecture C])**.** Let Y^{n-1} be a closed manifold of dimension $\neq 4$ which does not admit a metric with positive scalar curvature and $X = Y \times [-1, 1]$. If g is a Riemannian metric on X with $\mathrm{scal}(X, g) \geq n(n-1)$, then

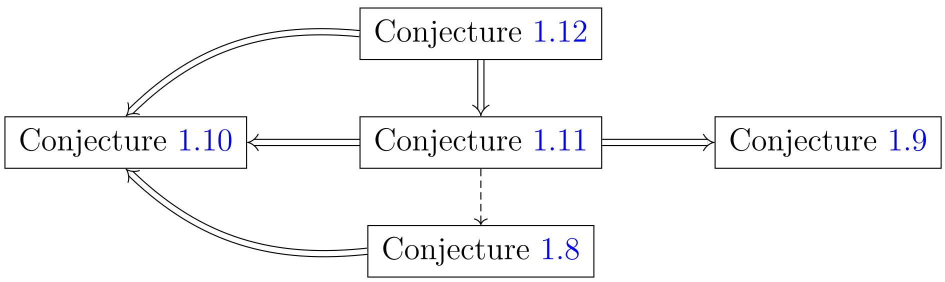
$$\mathrm{width}(X, g) := \mathrm{dist}_g(Y \times \{-1\}, Y \times \{1\}) \leq \frac{2\pi}{n}.$$

Conjecture 1.12. Let Y^{n-1} be a closed manifold of dimension $\neq 4$ which does not admit a metric with positive scalar curvature and $X = Y \times [-1, 1]$. If g is a Riemannian metric on X with

- ▷ $\mathrm{scal}(X, g) \geq n(n-1)$,
- ▷ $H(\partial_\pm X, g) \geq \mp(n-1)\tan\left(\frac{n\ell_\pm}{2}\right)$ for some $-\frac{\pi}{n} < \ell_- < \ell_+ < \frac{\pi}{n}$,

then $\mathrm{width}(X, g) \leq \ell_+ - \ell_-$.

The following chart illustrates the connections and implications between these conjectures, as they were discussed in the example $Y = T^{n-1}$:



The dashed arrow represents the fact that, while Conjecture 1.11 does not imply Conjecture 1.8, the techniques used to prove the former can likely be adapted to prove the latter as is exemplified by [76, Corollary 1.5].

The main goal of the first part of this thesis (Chapters 2 and 3) is to prove all of the conjectures above for orientable manifolds Y^{n-1} in dimension $n \leq 7$. We develop a general scalar and mean curvature comparison principle in Chapter 2, which implies Conjecture 1.12 for orientable manifolds Y^{n-1} in dimension $n \leq 7$. Once this is established, Conjectures 1.10, 1.11 and 1.9 will follow suit, as is indicated by the chart above.

Furthermore, we modify our techniques in Chapter 3 to prove, what we call, a *partitioned* scalar and mean curvature comparison principle, which implies Conjecture 1.8 for orientable manifolds Y^{n-1} in dimension $n \leq 7$.

Throughout this endeavor, we follow ideas of Gromov [32, Sections 3.6 & 5] and use a variation of the Schoen-Yau [64] approach to positive scalar curvature, where minimal hypersurfaces are replaced by hypersurfaces of *prescribed mean curvature*, which appear as the boundaries of so called *μ-bubbles* (see Section 2.3).

1.2 Overview of the Main Results

In this section we introduce the main results Theorem I and Theorem II of Chapters 2 and 3 respectively. We provide an overview over the context in which they were developed and their various applications towards the conjectures listed in Section 1.1. We do not present any proofs at this stage.

All results and conjectures we have mentioned so far are concerned with manifolds of the form $X = Y \times [-1, 1]$. To study the underlying phenomena in a general framework, Gromov introduced the concept of a *band* in [31, Section 2]. While he originally provides a broader definition, the following is enough for our purposes:

Definition 1.13. A *band* is a connected compact manifold X together with a decomposition

$$\partial X = \partial_- X \sqcup \partial_+ X,$$

where $\partial_\pm X$ are (non-empty) unions of boundary components. If X is equipped with a Riemannian metric g, we call (X, g) a *Riemannian band* and denote by $\mathrm{width}(X, g)$ the distance (with respect to g) between $\partial_- X$ and $\partial_+ X$.

Definition 1.14. A continuous map $f : X \to X'$ between two bands is called a *band map* if it maps $\partial_- X$ to $\partial_- X'$ and $\partial_+ X$ to $\partial_+ X'$.

Remark 1.15. The standard example of a band is $X = Y \times [-1, 1]$, where Y is a closed manifold. Such bands are called *trivial* throughout this thesis.

In [31, Section 2] Gromov observed that the band width estimate Theorem 1.4 holds true, not only for the trivial band $T^{n-1} \times [-1, 1]$, but for any oriented band X which admits a band map of non-zero degree to $T^{n-1} \times [-1, 1]$. Bands with this property are usually called *over-torical*.

Following his lead, Cecchini [12] and Zeidler [75] proved the band width estimate not only for $Y \times [-1, 1]$, where Y is spin and has non-vanishing Rosenberg index, but for a general class of possibly non-trivial spin bands (see [75, Theorem 3.1]).

In [14, Theorem 7.6] they established the comparison result Theorem 1.6 for spin bands with *infinite vertical $\hat{A}$-area*, which only occur in odd-dimensions.

What all of these bands have in common is the following:

Property A. No closed embedded hypersurface $\Sigma \subset X$ which separates $\partial_- X$ and $\partial_+ X$ admits a metric with positive scalar curvature.

Definition 1.16. Let X be a band and $\Sigma \subset X$ be a closed embedded hypersurface. We say that Σ separates $\partial_- X$ and $\partial_+ X$ if no connected component of $X \backslash \Sigma$ contains a path $\gamma : [0, 1] \to X$ with $\gamma(0) \in \partial_- X$ and $\gamma(1) \in \partial_+ X$. Furthermore Σ *properly* separates $\partial_- X$ and $\partial_+ X$ if every connected component of Σ can be connected to both $\partial_+ X$ and $\partial_- X$ inside $X \setminus \Sigma$.

The condition of being properly separating simply means that there are no superfluous components. We observe that any separating hypersurface contains a union of connected components which is properly separating (see Lemma 2.56).

Gromov [32, Section 3.6 & Section 5] pointed out that one can use a variation of the Schoen-Yau [64] approach to prove the band width estimate for all oriented bands with Property A in dimension ≤ 7.

Theorem 1.17 ([32, Section 3.6])**.** *Let $n \leq 7$ and X^n be an oriented band with Property A. If g is a Riemannian metric on X with* $\operatorname{scal}(X, g) \geq n(n-1)$*, then*

$$\operatorname{width}(X, g) \leq \frac{2\pi}{n}.$$

The idea of Gromov's proof can be summarized as follows: Since $\operatorname{scal}(X, g) > 0$ and X has Property A the band can not contain a stable minimal hypersurface which separates $\partial_- X$ and $\partial_+ X$ by the work of Schoen and Yau.

However, the only way to guarantee the existence of such an object is to assume that $H(\partial X, g) \geq 0$, which implies that $\partial_- X$ and $\partial_+ X$ act as barriers for the variation of area. Notice, that one could prove Theorem 1.3 for X in this way.

Gromov realized that, if the band is wide enough, one can compensate for the negativity of $H(\partial X, g)$ by augmenting the usual area functional with a contribution from a potential function $h : X \to \mathbb{R}$. If one assumes for a contradiction that $\operatorname{width}(X, g) > \frac{2\pi}{n}$ a clever choice of h guarantees the existence of a minimizer (call μ-bubble) for such a variational problem.

In dimension ≤ 7 classical regularity results ensure that the boundary $\Sigma \subset X$ of this minimizer is a closed smooth embedded hypersurface of *prescribed mean curvature*, ie $H(\Sigma) = h\big|_{\Sigma}$, which separates $\partial_- X$ and $\partial_+ X$.

The potential can be chosen in such a way that the contributions in terms of h in the second variation formula balance out with the positivity of $\operatorname{scal}(X, g)$. It follows that Σ itself admits a metric with positive scalar curvature, which is a contradiction.

Remark 1.18. If X is a 2-dimensional band, the condition that no closed embedded hypersurface which separates $\partial_- X$ and $\partial_+ X$ admits a metric with positive scalar curvature is vacuous. Hence every 2-dimensional band has Property A.

The proof of Theorem 1.17, as it is sketched above, is simplified in this case. As before one obtains a minimizer to a suitable variational problem if the band is assumed to be too wide. Its boundary Σ (a collection of circles) separates $\partial_- X$ and $\partial_+ X$ and by stability and the second variational formula one concludes $-\Delta_\Sigma > 0$. This is the desired contradiction.

Of course, in dimension 2, scalar curvature is equal to twice the sectional curvature and hence any statement we will derive about 2-dimensional bands with lower scalar curvature bounds can also be obtained via the classical means of triangle comparison and analysis of Jacobi fields.

Nevertheless it is conceptually pleasing to see that μ-bubbles can be used very effectively in the study of 2-dimensional manifolds with positive curvature.

In Chapter 2 we use the μ-bubble approach to establish a general scalar and mean curvature comparison principle, inspired by [14, Theorem 7.6], for oriented bands with Property A in dimension ≤ 7. As model spaces for this comparison, we use a class of warped products with constant scalar curvature (see Example 1.5).

Definition 1.19. A smooth function $\varphi : [a, b] \to \mathbb{R}_+$ is called log-*concave* if

$$\frac{d^2}{dt^2}\log(\varphi)(t) = \left(\frac{\varphi'(t)}{\varphi(t)}\right)' \leq 0$$

for all $t \in [a, b]$. If the inequality is strict we say that φ is *strictly* log-*concave*. In case of equality we say that φ is log-*affine*.

Definition 1.20. Let (N, g_N) be a closed Riemannian manifold with constant scalar curvature. A warped product

$$\left(M, g_\varphi\right) = \left(N \times [a, b], \varphi^2(t) g_N + dt^2\right)$$

with warping function $\varphi : [a, b] \to \mathbb{R}_+$ is called a *model space* if $\operatorname{scal}(M, g_\varphi)$ is constant and φ is strictly log-concave or log-affine.

Theorem I. *Let $n \leq 7$ and X^n be an oriented band with Property A. Let g be a Riemannian metric on X and (M^n, g_φ) be a model space over a scalar flat base with warping function $\varphi : [a, b] \to \mathbb{R}_+$. If*

- ▷ $\operatorname{scal}(X, g) \geq \operatorname{scal}(M, g_\varphi)$,
- ▷ $H(\partial_\pm X, g) \geq H(\partial_\pm M, g_\varphi)$,

we distinguish two cases:

1. *If φ is strictly* log-*concave, then* $\operatorname{width}(X, g) \leq \operatorname{width}(M, g_\varphi)$.
2. *If φ is* log-*affine, then (X, g) is isometric to a warped product*

$$\left(\hat{N} \times [c, d], \varphi^2 g_{\hat{N}} + dt^2\right),$$

where $(\hat{N}, g_{\hat{N}})$ is a closed Ricci flat Riemannian manifold.

Via the choice of model space Theorem I applies to a variety of geometric situations and provides results for bands with negative lower scalar curvature bounds as well. In particular, we can estimate the width of (X, g) if $\operatorname{scal}(X, g) \geq -n(n-1)$ and $H(\partial_+ X, g) > (n-1)$ (see Corollary 2.9). An in depth discussion of Theorem I and its various applications is included in Section 2.1.

For now, we point out that the warping function $\varphi(t) = \cos\left(\frac{nt}{2}\right)^{\frac{2}{n}}$ from Example 1.5 is strictly log-concave. Hence Theorem I implies the following generalization of Theorem 1.6 for bands with Property A in dimension $n \leq 7$.

Theorem 1.21. *Let $n \leq 7$ and X be an oriented band with Property A. Let g be a Riemannian metric on X. If*

- ▷ $\operatorname{scal}(X, g) \geq n(n-1)$
- ▷ $H(\partial_\pm X, g) \geq \mp(n-1)\tan\left(\frac{n\ell_\pm}{2}\right)$ *for some* $-\frac{\pi}{n} < \ell_- < \ell_+ < \frac{\pi}{n}$,

then $\operatorname{width}(X, g) \leq \ell_+ - \ell_-$.

In the discussion after Theorem 1.6 we explained that it implies Theorem 1.4 with strict inequality. Using the same idea, one can show that Theorem 1.21 yields the following improved version of Theorem 1.17:

Corollary 1.22. *Let $n \leq 7$ and X^n be an oriented band with Property A. If g is a Riemannian metric on X with $\mathrm{scal}(X, g) \geq n(n-1)$, then*

$$\mathrm{width}(X, g) < \frac{2\pi}{n}.$$

Theorem 1.21 is related to Conjecture 1.12. To bridge the gap between them we investigate under which conditions on Y, the trivial band $Y \times [-1, 1]$ has Property A in Sections 2.1.2 and 2.5.1. Our main result in this vein is:

Theorem 1.23. *Let $(n-1) \neq 4$ and Y^{n-1} be a closed orientable manifold which does not admit a metric with positive scalar curvature. Then $Y \times [-1, 1]$ has Property A.*

We combine Corollary 1.22 and Theorem 1.23 to establish:

Corollary 1.24. *Let $(n-1) \neq 4$ and $n \leq 7$. Let Y^{n-1} be a closed orientable manifold which does not admit a metric with positive scalar curvature and $X = Y \times [-1, 1]$. If g is a Riemannian metric on X with $\mathrm{scal}(X, g) \geq n(n-1)$, then*

$$\mathrm{width}(X, g) < \frac{2\pi}{n}.$$

Furthermore, we can use the doubling argument of Gromov and Lawson [28] and Almeida [1] to conclude the following:

Corollary 1.25. *Let $(n-1) \neq 4$ and $n \leq 7$. Let Y^{n-1} be a closed orientable manifold which does not admit a metric with positive scalar curvature and $X = Y \times [-1, 1]$. If g is a Riemannian metric on X with $\mathrm{scal}(X, g) \geq n(n-1)$, then*

$$\inf_{x \in \partial X} H(\partial X, g)(x) < 0.$$

Finaly, we notice that for $R > 0$ small enough and $X^n = Y^{n-2} \times \mathbb{R}^2$ the manifold $X \backslash U_R(Y \times \{0\})$ is diffeomorphic to $Y \times S^1 \times [0, \infty)$. We employ Corollary 1.22 to establish the following result with regards to Conjecture 1.9:

Corollary 1.26. *Let $(n-2) \neq 4$ and $n \leq 7$. Let Y^{n-2} be a closed orientable manifold which does not admit a metric with positive scalar curvature and $X = Y \times \mathbb{R}^2$. Then X does not admit a complete metric with uniformly positive scalar curvature.*

In Chapter 3 we combine the techniques we developed in Chapter 2 with ideas of Cecchini and Zeidler from [10, 75] to prove Theorem II, a *partitioned* scalar and mean curvature comparison principle for bands with Property A. Theorem II will then be used to establish Conjecture 1.8 for orientable manifolds in dimension $n \leq 7$. The results of Chapter 3 were obtained in joint work with Cecchini and Zeidler and correspond to parts of the preprint [9].

Gromov and Lawson [30, Section 6] first proved Conjecture 1.8 for enlargeable spin manifolds. Cecchini [11] and Zeidler [76] showed that these results can be generalized for spin manifolds with non-vanishing Rosenberg index. Recently, new results in this vein appeared in [15], in connection with the positive mass theorem.

In particular [15, Theorem 1.1] implies Conjecture 1.8 for closed aspherical manifolds of dimension ≤ 5, which do not admit positive scalar curvature by [16, 33].

We motivate the main idea behind Theorem II and our approach to Conjecture 1.8. Let $X^n = Y \times \mathbb{R}$ be as in Conjecture 1.8 with Y orientable and $n \leq 7$. Assume for a contradiction that g is a complete Riemannian metric on X with positive scalar curvature. The band width estimate Conjecture 1.11 does not imply Conjecture 1.8 since $\operatorname{scal}(X, g)$ may not be *uniformly* positive. However, if we consider a compact segment of (X, g), for example $Y \times [-1, 1] \subset X$, the scalar curvature is uniformly positive on this segment and non-negative on the segments $Y \times [-C, -1]$ and $Y \times [1, C]$ for any $C > 1$. On the other hand, our partitioned comparison result Theorem II will imply that the width of the segments $Y \times [-C, -1]$ and $Y \times [1, C]$ is bounded from above by a constant, which depends on the uniform lower bound on $\operatorname{scal}(X, g)$ in $Y \times [-1, 1]$ and the width of $Y \times [-1, 1]$. Since the metric g is assumed to be complete, we can choose $C > 1$ large enough to produce a contradiction.

As before, we work in the general setting where (X, g) has Property A. Furthermore, we assume that (X, g) is partitioned into multiple segments with possibly different lower scalar curvature bounds.

Definition 1.27. Let X be a band and Σ_i, for $i \in \{1, \dots, k\}$, be closed embedded hypersurfaces such that Σ_1 properly separates $\partial_- X$ and $\partial_+ X$ and Σ_i properly separates Σ_{i-1} and $\partial_+ X$ for $i \in \{2, \dots, k\}$. We call (X, Σ_i, k) a *partitioned band* and denote by V_j, for $j \in \{1, \dots, k+1\}$, the segments of X bounded by Σ_{j-1} and the Σ_j, where $\Sigma_0 = \partial_- X$ and $\Sigma_{k+1} = \partial_+ X$.

Theorem II. *Let $n \leq 7$ and (X^n, Σ_i, k) be an oriented partitioned band with Property A. Let g be a Riemannian metric on X and (M_j, g_{φ_j}) for $j \in \{1, \dots, k+1\}$ be strictly* log*-concave model spaces over a scalar flat base. If*

- $\operatorname{scal}(V_j, g) \geq \operatorname{scal}(M_j, g_{\varphi_j})$ *for all* $j \in \{1, \dots, k+1\}$,
- $H(\partial_- X, g) \geq H(\partial_- M_1, g_{\varphi_1})$ *and* $H(\partial_+ X, g) \geq H(\partial_+ M_{k+1}, g_{\varphi_{k+1}})$,
- $H(\partial_+ M_j, g_{\varphi_j}) = -H(\partial_- M_{j+1}, g_{\varphi_{j+1}})$ *for all* $j \in \{1, \dots, k\}$,

then $\operatorname{width}(V_j, g) \leq \operatorname{width}(M_j, g_{\varphi_j})$ *for at least one* $j \in \{1, \dots, k+1\}$.

Via a suitable choice of model spaces, Theorem II implies the following result concerning a Riemannian band with Property A and a positively curved segment.

Theorem 1.28. *Let $n \leq 7$ and $(X^n, \Sigma_i, 2)$ be an orientable partitioned band with Property A. Let g be a Riemannian metric on X and $\kappa > 0$ be a positive constant. If*

- $\operatorname{scal}(V_2, g) \geq \kappa n(n-1)$,
- $\operatorname{scal}(X, g) \geq 0$,

and we denote $d := \operatorname{width}(V_2, g) < \frac{2\pi}{\sqrt{\kappa} n}$, *then*

$$\min\{\operatorname{width}(V_1, g), \operatorname{width}(V_3, g)\} < \ell = \frac{2}{\sqrt{\kappa} n} \cot\left(\frac{\sqrt{\kappa} n d}{4}\right).$$

In combination with Theorem 1.23, a deformation result of Kazdan [45] and the Cheeger-Gromoll splitting theorem, Theorem 1.28 yields:

Theorem 1.29. *Let $(n-1) \neq 4$ and $n \leq 7$. Let Y^{n-1} be a closed orientable manifold which does not admit a metric with positive scalar curvature and $X = Y \times \mathbb{R}$. Then X does not admit a complete metric with positive scalar curvature. Moreover, if g is a metric on X with non-negative scalar curvature, then (X, g) is isometric to $(Y \times \mathbb{R}, g_Y + dt^2)$, where g_Y is a Ricci flat metric on Y.*

Thus we established Conjectures 1.8-1.12 for orientable manifolds in dimension $n \leq 7$. The dimensional restriction only enters the argument through the regularity theory for μ-bubbles. If this issue could be circumvented, our methods could be used to prove Conjectures 1.8-1.12 for orientable manifolds in all dimensions.

1.3 Macroscopic Scalar Curvature

In the second part of this thesis (Chapter 4) we explore a different aspect of positive scalar curvature and the band width estimate Theorem 1.4. Instead of asking whether the torus can be replaced by other manifolds which do not admit positive scalar curvature, we ask whether the curvature condition itself can be replaced by a lower bound on the *macroscopic scalar curvature* of $(T^{n-1} \times [-1, 1], g)$. This notion of curvature was introduced by Larry Guth in [34]. We summarize:

The value $\mathrm{scal}(X, g)(x)$ of the scalar curvature at a point x in a Riemannian manifold (X^n, g) appears as a coefficient in the Taylor expansion of the volume of a geodesic ball of radius R around x:

$$\mathrm{vol}(B_R(x)) = \omega_n R^n \left(1 - \frac{\mathrm{scal}(X, g)(x)}{6(n+2)} R^2 + \mathcal{O}(R^4)\right), \tag{1.3.1}$$

where ω_n is the volume of the unit ball in euclidean space $\mathbb{R}^n$. It follows that, if the scalar curvature of (X, g) at a point x is positive, there is a $\lambda(X, g, x) > 0$ such that all R-balls in (X, g) centered at X with $R < \lambda(X, g, x)$ have $\mathrm{vol}(B_R(x)) < \omega_n R^n$.

Hence, for R small enough, the scalar curvature of (X, g) at X can be quantified by comparing the volumes of R-balls around x with their counterparts in S^n, $\mathbb{R}^n$ and $\mathbb{H}^n$, the standard simply connected manifolds with constant scalar curvature. If we carry out this volume comparison (see [34, Section 7]) for all radii $R > 0$ we get:

Definition 1.30. Let (X^n, g) be a Riemannian manifold and $x \in X$. The *macroscopic scalar curvature* at scale R at x, denoted by $\mathrm{scal}_R(x)$, is defined to be the number S such that the volume of the ball of radius R around any lift of x in the universal cover of X equals the volume of the ball of radius R in a simply connected space with constant curvature and with scalar curvature S.

The universal cover is used to ensure that the macroscopic scalar curvature of a flat torus is zero at any scale. If X is closed and one does not consider balls in the universal cover, but in (X, g) itself, then the macroscopic scalar curvature would be positive at a large enough scale. By (1.3.1), we have

$$\lim_{R \to 0} \mathrm{scal}_R(X, g)(x) = \mathrm{scal}(X, g)(x). \tag{1.3.2}$$

The study of macroscopic scalar curvature originates from [24]. In this work, Gromov's main conceit is the following: in a complete Riemannian manifold which is large, in a suitable sense, compared to $\mathbb{R}^n$, one can always find a ball of at least euclidean volume for any radius $R > 0$. To make this precise he introduced several notions of largeness for complete Riemannian manifolds and conjectured that each of them is sufficient for the existence of such large balls.

Phrased in terms of the macroscopic scalar curvature of a closed Riemannian manifold, this yields the following intriguing conjecture:

Conjecture 1.31 ([24]). Let (X^n, g) be a closed Riemannian manifold. If the universal cover $(\widetilde{X}, \widetilde{g})$ is large, then

$$\inf_{x \in X} \mathrm{scal}_R(X, g)(x) \leq 0$$

at all scales $R > 0$. In particular, there is a point $x_0 \in X$ with $\mathrm{scal}(X, g)(x_0) \leq 0$.

Remark 1.32. The conclusion that the scalar curvature of (X, g) has to vanish at some point is based on two observations:

- ▷ For a *closed* Riemannian manifold (X, g) with positive scalar curvature, we can find a *uniform* constant $\lambda(X, g)$ such that all R-balls in (X, g) with $R < \lambda(X, g)$ have $\mathrm{vol}(B_R(x)) < \omega_n R^n$ (compare the paragraph after (1.3.1)).
- ▷ For R small enough the R-balls in (X, g) agree with the R-balls in $(\widetilde{X}, \widetilde{g})$.

Furthermore, many notions of largeness introduced in [24] are preserved under bi-Lipschitz diffeomeorphisms. Since any two Riemannian metrics on a closed smooth manifold X are in bi-Lipschitz correspondence (the identity is a Lipschitz map), it follows that whether or not $(\widetilde{X}, \widetilde{g})$ is large, often does not depend on the particular choice of g. The following can be viewed as special case of Conjecture 1.31:

Conjecture 1.33 ([24]). Let X^n be a closed aspherical manifold. If g is any Riemannian metric on X, then

$$\inf_{x \in X} \mathrm{scal}_R(X, g)(x) \leq 0$$

at all scales $R > 0$. In particular, there is a point $x_0 \in X$ with $\mathrm{scal}(X, g)(x_0) \leq 0$.

The conclusion that a closed aspherical manifold X^n does not admit a metric with positive scalar curvature has been established in certain cases. By work of Rosenberg [57, Theorem 3.5] it holds true whenever $\pi_1(X)$ satisfies the strong Novikov conjecture. Furthermore, by recent work of Chodosh and Li [16] and Gromov [33], it holds true if $n \leq 5$.

Conjecture 1.33 on the other hand has not been established for any closed aspherical manifold and is widely considered to be out of reach with the present methods.

One possible way to approach Conjecture 1.33 is to study quantitative versions of it ie instead of searching for a ball of radius R in $(\widetilde{X}, \widetilde{g})$ with volume $\geq \omega_n R^n$, one replaces ω_n with a smaller constant ε_n. Remarkable breakthroughs in this vein have been made by Guth [35–37]. In [36, Corollary 3] he proved the following:

Theorem 1.34 ([36]). *For every $n \geq 1$ there is a constant $\varepsilon_n > 0$ such that the following holds. Let X^n be a closed aspherical manifold and g any Riemannian metric on X. Then for every $R > 0$ there is a point x in the universal cover $(\widetilde{X}, \widetilde{g})$ such that*

$$\mathrm{vol}(B_R(x)) \geq \varepsilon_n R^n.$$

Remark 1.35. We point out that, by rescaling the metric, it suffices to prove Theorem 1.34 for any fixed radius $R > 0$. For simplicity, we work with $R = 1$ from now on.

One can rephrase Theorem 1.34 to end up with:

Theorem 1.36 ([36]). *For every $n \geq 1$ there are constants $\varepsilon_n > 0$ and $S_n > 0$ such that the following holds. No closed aspherical manifold X^n admits a Riemannian metric g with the property that $\mathrm{scal}_1(X, g) > S_n$ or equivalently that all unit balls in the universal cover $(\widetilde{X}, \widetilde{g})$ have volume less than ε_n.*

This result does not suffice to prove the non-existence of metrics with positive scalar curvature on closed aspherical manifolds. If one rescales it to understand $\mathrm{scal}_R(X, g)$, one obtains $\inf_{x \in X} \mathrm{scal}_R(X, g)(x) \leq S_n R^{-2}$ which, in the limit $R \to 0$, yields nothing (compare (1.3.2)).

It does, however, imply the systolic inequality for a closed aspherical manifold X^n, according to which there is a dimensional constant C_n such that, for any metric g on X, the length of the shortest non-trivial loop in (X, g) is bounded from above by $C_n \mathrm{vol}(X, g)^{\frac{1}{n}}$. This result was first established by Gromov in [23].

Based on this observation, Guth [34, Section 7] proposed that the notion of macroscopic scalar curvature can be used as a metaphor to connect systolic geometry and the study of Riemannian manifolds with positive scalar curvature. Following this idea, one tries to identify results or conjectures concerning Riemannian manifolds with positive scalar curvature which, at least quantitatively, have their analogs in the macroscopic setting. In this sense Theorem 1.36 is the macroscopic analog of the conjecture that no closed aspherical manifold admits a metric with positive scalar curvature.

In Chapter 4 we investigate a macroscopic analog of the band width estimate Theorem 1.4 and Conjecture 1.11. The main conceit is that, if a closed manifold Y^{n-1} does not admit a Riemannian metric with $\mathrm{scal}_1 > S_{n-1}$ and g is a Riemannian metric on $X = Y \times [-1, 1]$ with $\mathrm{scal}_1(X, g) > S_n$, then the width of (X, g) should be bounded from above by a uniform constant, which is independent of Y or g.

We proceed in two steps. First, we observe that Theorem 1.36 holds not only for aspherical manifolds but also for *enlargeable* manifolds in the sense of [8, 13, 30]. We define a class of closed orientable manifolds we call *filling enlargeable* by combining the notion of the *filling radius* of a complete Riemannian manifold [23] with the definition of enlargeability. We prove that this class has the following properties:

- ▷ If a closed orientable manifold Y is enlargeable or aspherical, then it is filling enlargeable as well,
- ▷ If a closed orientable manifold is filling enlargeable, then $Y \times S^1$ is filling enlargeable as well.
- ▷ Theorem 1.36 holds true for filling enlargeable manifolds,

We then use a doubling trick, inspired by [1, Theorem 1.1, 28, Theorem 5.7] and their application to Theorem 1.3, to prove the following main result of Chapter 4:

Theorem III. *For all* $n \geq 1$ *there is a constant* $\varepsilon_n > 0$ *such that the following holds. Let* Y^{n-1} *be a closed filling enlargeable manifold and* $X = Y \times [-1, 1]$. *If* g *is a Riemannian metric on* X *with the property that all unit balls in the universal cover* $(\widetilde{X}, \widetilde{g})$ *have volume less than* $\frac{1}{2}\varepsilon_n$, *then* $\mathrm{width}(X, g) \leq 1$.

Furthermore, to put this result into perspective, we investigate how the class of filling enlargeable manifolds fits into the pantheon of large manifolds. We adapt ideas of Brunnbauer and Hanke [8] to prove that whether or not a closed oriented manifold is filling enlargeable, only depends on the image of its fundamental class under the classifying map of the universal cover. In particular we construct a vector subspace $H_n^{sm}(B\Gamma; \mathbb{Q}) \subset H_n(B\Gamma; \mathbb{Q})$ of 'small classes' in the rational group homology of a finitely generated group Γ such that:

Theorem 1.37. *Let* Y^n *be a closed oriented manifold. Then* Y *is filling enlargeable if and only if* $\phi_*[Y] \in H_n(B\pi_1(Y); \mathbb{Q})$ *is not contained in* $H_n^{sm}(B\pi_1(Y); \mathbb{Q})$.

As a consequence, we obtain the following metric characterization of rationally essential manifolds with residually finite fundamental groups:

Corollary 1.38. *A closed oriented manifold* Y^n *with residually finite fundamental group is rationally essential if and only if it is filling enlargeable.*

1.4 Organization of this thesis

In Chapter 2 we develop the scalar and mean curvature comparison principle and prove Theorem I in Section 2.4. We discuss its applications in Section 2.1.1 and list the topological ingredients for Theorem 1.23 in Section 2.1.2. In Section 2.2 we recall basic properties of warped products and motivate our choice of model spaces for the comparison principle. In Section 2.3 we discuss μ-bubbles, the variational theory surrounding them and some of the main tools we use to understand the geometry of bands with lower scalar curvature bounds. In Section 2.5.1 we prove Theorem 1.23. Furthermore, we investigate non-trivial bands with Property A and prove that the trivial band over a closed aspherical 4-manifold has Property A.

In Chapter 3 we use the techniques and results developed in Chapter 2 in combination with ideas from [10, 76] to prove the partitioned comparison principle Theorem II and Theorem 1.29.

The main result of Chapter 4 is Theorem III, which will be proved in Section 4.3 after a discussion of largeness properties of manifolds in Section 4.2, where the notion of filling enlargeability is introduced.

Scalar and Mean Curvature Comparison via μ-Bubbles

The main goal of this chapter, which corresponds to the preprint [54] available on the arXiv, is to prove Theorem I, our scalar and mean curvature comparison principle. Furthermore we show that Theorem I implies Conjecture 1.12 for orientable manifolds Y in dimension $n \leq 7$. As we indicated in Section 1.1, Conjectures 1.9, 1.10 and 1.11 will follow suit for these manifolds.

So far, Conjecture 1.12 has been established for spin bands X^n with *infinite vertical $\hat{A}$-area* by Cecchini and Zeidler [14, Theorem 7.6]. This class of, possibly non-trivial, odd-dimensional spin bands includes in particular $X = Y \times [-1, 1]$, where Y is a closed enlargeable spin manifold. Hence Conjecture 1.12 holds true for any even dimensional torus.

The methods in [14] are based around a variation of the index theoretic approach to positive scalar curvature, where the classical spinor Dirac operator is modified in terms of potential functions. Cecchini and Zeidler had previously used these techniques [12, 75] to prove Conjecture 1.11 for closed spin manifolds with non-vanishing Rosenberg index.

We will approach Conjecture 1.12 in two steps, as was outlined in Section 1.2. First, we establish Theorem I as a general principle regarding scalar and mean curvature comparison between Riemannian bands with Property A and warped products over closed scalar flat Riemannian manifolds. It provides a general framework, in the context of which the connections between previous results, regarding Riemannian bands with lower scalar curvature bounds, become apparent.

Via a choice of model space Theorem I implies Conjecture 1.12 for oriented bands X^n with Property A in dimension $n \leq 7$. The second step will be to investigate which topological conditions on a band X ensure that it has Property A. In particular, we prove Theorem 1.23 ie if Y^{n-1} is closed orientable, $n - 1 \neq 4$ and Y does not admit a metric with positive scalar curvature, then $X^n = Y^{n-1} \times [-1, 1]$ has Property A.

In dimension 4, where Seiberg-Witten theory provides counterexamples to Conjecture 1.12 (see Remark 1.7), we prove that $X = Y^4 \times [-1, 1]$ has Property A if Y is a closed aspherical manifold.

The formulation of Theorem I and our choice of model spaces is inspired by [14]. However, instead of relying on the Dirac operator, we follow ideas of Gromov, which already appear in [26, Section $5\frac{5}{6}$] and are developed further in [31, Section 9] and [32, Section 5], and use a version of the minimal surface approach involving so called μ-bubbles (see Section 2.3). Here the usual area functional is modified in terms of a potential function.

It is important to point out that the potential functions we use in this augmentation of the minimal surface approach also appear in the work of Cecchini and Zeidler, where they are used to modify the Dirac operator. This parallelism indicates further connections between these new methods.

We conclude these introductory remarks by reminding the reader of our mean curvature convention:

Remark 2.1. Let (X, g) be a Riemannian manifold with boundary $\partial X \neq \emptyset$. In this article $H(\partial X, g)$ denotes the trace of the second fundamental form of ∂X with respect to the inner unit normal vector field. With this convention the unit sphere $S^{n-1} \subset \mathbb{R}^n$ has mean curvature $(n-1)$.

2.1 Comparison Principle

The following section contains an in depth discussion of Theorem I, the main result of this chapter. We review its statement and survey its applications by providing various examples of model spaces and bands with Property A. Throughout this section we restate definitions and results from the Introduction for the convenience of the reader.

Definition 1.19. A smooth function $\varphi : [a, b] \to \mathbb{R}_+$ is called log-*concave* if

$$\frac{d^2}{dt^2} \log(\varphi)(t) = \left(\frac{\varphi'(t)}{\varphi(t)}\right)' \leq 0$$

for all $t \in [a, b]$. If the inequality is strict we say that φ is *strictly* log-*concave*. In case of equality we say that φ is log-*affine*.

Definition 1.20. Let (N, g_N) be a closed Riemannian manifold with constant scalar curvature. A warped product

$$\big(M, g_\varphi) = (N \times [a, b], \varphi^2(t) g_N + dt^2\big)$$

with warping function $\varphi : [a, b] \to \mathbb{R}_+$ is called a *model space* if $\operatorname{scal}(M, g_\varphi)$ is constant and φ is strictly log-concave or log-affine.

Theorem I. *Let $n \leq 7$ and X^n be an oriented band with Property A. Let g be a Riemannian metric on X and (M^n, g_φ) be a model space over a scalar flat base with warping function $\varphi : [a, b] \to \mathbb{R}_+$. If*

- ▷ $\operatorname{scal}(X, g) \geq \operatorname{scal}(M, g_\varphi)$,
- ▷ $H(\partial_\pm X, g) \geq H(\partial_\pm M, g_\varphi)$,

we distinguish two cases:

1. *If φ is strictly* log-*concave, then* $\operatorname{width}(X, g) \leq \operatorname{width}(M, g_\varphi)$.
2. *If φ is* log-*affine, then (X, g) is isometric to a warped product*

$$\left(\hat{N} \times [c, d], \varphi^2 g_{\hat{N}} + dt^2\right),$$

where $(\hat{N}, g_{\hat{N}})$ is a closed Ricci flat Riemannian manifold.

Remark 2.2. It is expected that part (1) of the Theorem I is rigid as well ie for φ strictly log-concave we have $\mathrm{width}(X, g) = \mathrm{width}(M, g_\varphi)$ if and only if (X, g) is isometric to a warped product

$$\left(\hat{N} \times [a, b], \varphi^2 g_{\hat{N}} + dt^2\right),$$

where $(\hat{N}, g_{\hat{N}})$ is a closed Ricci flat Riemannian manifold. For spin bands with $\hat{A}(\partial_- X) \neq 0$ this holds true by work of Cecchini and Zeidler [14, Theorem 8.3, Theorem 9.1]. In our case there are some obstacles yet to be overcome (see Remark 2.31). On the other hand the log-affine case ie part (2) of Theorem I is not treated in [14].

Remark 2.3. Even if rigidity in (1) can be established, both parts of Theorem I have to be treated separately, as the width of the band plays a role only in the strictly log-concave case. We point out that we have no control over the width of the band in part (2) ie the log-affine case and X can be isometric to a model space of arbitrary finite width.

Remark 2.4. It turns out that Theorem I holds true for any oriented band X in dimension $n = 2$, where the condition that no closed embedded hypersurface admits a positive scalar curvature metric is vacuous. This will become apparent in Section 2.4 (see Remark 2.54).

2.1.1 Model Spaces and Applications

To understand the different types of results we can deduce from Theorem I, we consider five exemplary model spaces. Throughout this subsection (N^{n-1}, g_N) is fixed to be a closed scalar flat Riemannian manifold.

For $-\frac{\pi}{n} < \ell_- < \ell_+ < \frac{\pi}{n}$ and the $\varphi(t) = \cos\left(\frac{nt}{2}\right)^{\frac{2}{n}}$ (strictly log-concave), the warped product

$$(N \times [\ell_-, \ell_+], \varphi^2(t) g_N + dt^2)$$

is a model space with scalar curvature $n(n-1)$. Plugging this into part (1) yields the following result which was already mentioned in the introduction:

Theorem 1.21. *Let $n \leq 7$ and X be an oriented band with Property A. Let g be a Riemannian metric on X. If*

- $\mathrm{scal}(X, g) \geq n(n-1)$
- $H(\partial_\pm X, g) \geq \mp(n-1)\tan\left(\frac{n\ell_\pm}{2}\right)$ *for some* $-\frac{\pi}{n} < \ell_- < \ell_+ < \frac{\pi}{n}$,

then $\mathrm{width}(X, g) \leq \ell_+ - \ell_-$.

In dimension $n \leq 7$, Theorem 1.21 generalizes Theorem 1.6, as $T^{n-1} \times [-1, 1]$ has Property A (see 2.1.2). In the discussion after Theorem 1.6 we saw that Theorem 1.6 implies Theorem 1.4 since $(n-1)\tan\left(\frac{nt}{2}\right) \to \infty$ as $t \to \frac{\pi}{n}$.

The same observation yields that Theorem 1.21 implies the following version of Gromov's band width inequality, which upgrades Theorem 1.17 to strict inequality.

Corollary 1.22. *Let $n \leq 7$ and X^n be an oriented band with Property A. If g is a Riemannian metric on X with* $\mathrm{scal}(X,g) \geq n(n-1)$*, then*

$$\mathrm{width}(X,g) < \frac{2\pi}{n}.$$

Furthermore Theorem 1.21 implies a generalization of Theorem 1.3 for $n \leq 7$:

Corollary 2.5. *Let $n \leq 7$ and X^n be an oriented band with Property A. If g is a Riemannian metric on X with* $\mathrm{scal}(X,g) \geq n(n-1)$*, then*

$$\inf_{x\in\partial_+ X} H(\partial_+ X, x) \quad + \inf_{x\in\partial_- X} H(\partial_- X, x) < 0.$$

For $0 < \ell_- < \ell_+ < \infty$ and $\varphi(t) = t^{\frac{2}{n}}$ (strictly log-concave) the warped product

$$(N \times [\ell_-, \ell_+], \varphi^2(t) g_N + dt^2)$$

is a scalar flat model space. Plugging this into part (1) yields:

Theorem 2.6. *Let $n \leq 7$ and X^n be an oriented band with Property A. Let g be a Riemannian metric on X. If*

- ▷ $\mathrm{scal}(X,g) \geq 0$
- ▷ $H(\partial_\pm X, g) \geq \pm\frac{2(n-1)}{n\ell_\pm}$ *for some* $0 < \ell_- < \ell_+ < \infty$*,*

then $\mathrm{width}(X,g) \leq \ell_+ - \ell_-$.

Since $\frac{2(n-1)}{nt} \to \infty$ as $t \to 0$ and $\frac{2(n-1)}{nt} \to 0$ as $t \to \infty$, Theorem 2.6 allows one to estimate the width of a Riemannian band (X,g) with Property A if its scalar curvature is nonnegative and $\partial_+ X$ (or $\partial_- X$) is strictly mean convex.

Corollary 2.7. *Let $n \leq 7$ and X^n be an oriented band with Property A. If g is a Riemannian metric on X with* $\mathrm{scal}(X,g) \geq 0$ *and $H(\partial_+ X) > 0$, then*

$$\mathrm{width}(X,g) < \frac{2(n-1)}{n\left(\inf_{x\in\partial_+ X} H(\partial_+ X, x)\right)}.$$

For $0 < \ell_- < \ell_+ < \infty$ and $\varphi(t) = \sinh(\frac{nt}{2})^{\frac{2}{n}}$ (strictly log-concave) the warped product

$$(N \times [\ell_-, \ell_+], \varphi^2(t) g_N + dt^2)$$

is a model space with scalar curvature $-n(n-1)$. Plugging this into part (1) yields:

Theorem 2.8. *Let $n \leq 7$ and X^n be an oriented band with Property A. Let g be a Riemannian metric on X. If*

- ▷ $\mathrm{scal}(X,g) \geq 0$
- ▷ $H(\partial_\pm X) \geq \pm(n-1)\coth\left(\frac{n\ell_\pm}{2}\right)$ *for some* $0 < \ell_- < \ell_+ < \infty$*,*

then $\mathrm{width}(X,g) \leq \ell_+ - \ell_-$.

Since $(n-1)\coth\left(\frac{nt}{2}\right) \to \infty$ as $t \to 0$ and $(n-1)\coth\left(\frac{nt}{2}\right) \to n-1$ as $t \to \infty$, Theorem 2.8 allows one to estimate the width of a Riemannian band (X, g) with Property A if its scalar curvature is bounded from below by $-n(n-1)$ and $\partial_+ X$ (or $\partial_- X$) has mean curvature $> n-1$. For $T^{n-1} \times [-1, 1]$ this had already been observed by Gromov [27, Section 4]:

Corollary 2.9. *Let $n \le 7$ and X^n be an oriented band with Property A. If g is a Riemannian metric on X with $\mathrm{scal}(X, g) \ge -n(n-1)$ and $H(\partial_+ X) > n-1$, then*

$$\mathrm{width}(X, g) < \frac{2}{n}\,\mathrm{arcoth}\left(\frac{1}{n-1}\inf_{x \in \partial_+ X} H(\partial_+ X, x)\right).$$

For $-\infty < \ell_- < \ell_+ < \infty$ and $\varphi(t) = 1$ (log-affine) the warped product

$$(N \times [\ell_-, \ell_+], g_N + dt^2)$$

is a scalar flat model space. Plugging this into part (2) yields the following rigidity result, which is probably well known to experts although we were not able to find a reference for it in the literature.

Theorem 2.10. *Let $n \le 7$ and X^n be an oriented band with Property A. Let g be a Riemannian metric on X. If*

- ▷ $\mathrm{scal}(X, g) \ge 0$,
- ▷ $H(\partial_\pm X, g) \ge 0$,

then (X, g) is isometric to a product $(\hat{N} \times [c, d], g_{\hat{N}} + dt^2)$, where $(\hat{N}, g_{\hat{N}})$ is a closed Ricci flat Riemannian manifold.

For $-\infty < \ell_- < \ell_+ < \infty$ and $\varphi(t) = \exp(t)$ (log-affine) the warped product

$$(N \times [\ell_-, \ell_+], \varphi^2(t) g_N + dt^2)$$

is a model space with scalar curvature $-n(n-1)$. Plugging this into part (2) yields:

Theorem 2.11. *Let $n \le 7$ and X^n be an oriented band with Property A. Let g be a Riemannian metric on X. If*

- ▷ $\mathrm{scal}(X, g) \ge -n(n-1)$,
- ▷ $H(\partial_\pm X, g) \ge \pm(n-1)$,

then (X, g) is isometric to a warped product $(\hat{N} \times [c, d], \exp(2t) g_{\hat{N}} + dt^2)$, where $(\hat{N}, g_{\hat{N}})$ is a closed Ricci flat Riemannian manifold.

Remark 2.12. Special cases of Theorem 2.11 appear in [26, Section $5\frac{5}{6}$, p. 57-58] and in [31, Section 9], where its relation to the hyperbolic positive mass theorem is explained. Furthermore there are cubical versions of Theorem 2.10 and Theorem 2.11, which involve the dihedral angle between adjacent faces. Li provided general results in this direction in [47] and [46, Theorem 1.3].

2.1.2 Topological Results

The results of Section 2.1.1 apply to oriented bands X with Property A. Gromov provides a list of examples for such bands in [32, Section 3.6], which we expand significantly. In particular we establish the following optimal result for trivial bands in dimension $n \geq 6$ (see Section 2.5.1).

Proposition 2.13. *Let $n \geq 6$ and Y^{n-1} be a closed connected oriented manifold which does not admit a metric with positive scalar curvature. Then $Y \times [-1, 1]$ has Property A.*

In the spin setting we recall an observation by Zeidler [75, 76].

Proposition 2.14 ([75, 76])**.** *Let $n \geq 2$ and Y^{n-1} be a closed connected oriented spin manifold with Rosenberg index $\alpha(Y) \neq 0 \in KO_{n-1}(C^*\pi_1 Y)$. Then $Y \times [-1, 1]$ has Property A.*

Since any closed orientable manifold of dimension ≤ 3 which does not admit a metric with positive scalar curvature is necessarily spin and has non-vanishing Rosenberg index, these two results suffice to establish the following general result which already appeared in the introduction:

Theorem 1.23. *Let $(n-1) \neq 4$ and Y^{n-1} be a closed orientable manifold which does not admit a metric with positive scalar curvature. Then $Y \times [-1, 1]$ has Property A.*

Furthermore we consider a class of bands which are not necessarily trivial.

Definition 2.15. A closed connected oriented manifold Y^{n-1} is called NPSC$^+$ if it can not be dominated by a manifold which admits a metric with positive scalar curvature. In other words: if Z^{n-1} is a closed oriented manifold and there exists a continuous map $f : Z \to Y$ with $\deg(f) \neq 0$, then Z does not admit a metric with positive scalar curvature.

Definition 2.16. A connected oriented band X^n is called over-NPSC$^+$ if there is a NPSC$^+$-manifold Y^{n-1} and a band map $f : X \to Y \times [-1, 1]$ with $\deg(f) \neq 0$.

Proposition 2.17. *A connected oriented over-*NPSC$^+$ *band has Property A.*

Remark 2.18. The two classical examples of NPSC$^+$-manifolds one should have in mind are *enlargeable* manifolds (compare [30, Theorem 5.8], [13, Theorem A] and [30, Proposition 5.7]) as well as *Schoen-Yau-Schick* manifolds (compare [64, Theorem 1], [61] and [16, Definition 23]).

Chodosh and Li [16, Theorem 2] and Gromov [33, Section 7] used μ-bubbles to prove that closed aspherical manifolds of dimension ≤ 5 do not admit metrics with positive scalar curvature. We implement Gromov's approach from [32] in the language of [16] and present a proof of the following in Section 2.5.2

Theorem 2.19. *All closed connected oriented aspherical 4-manifolds are* NPSC$^+$*.*

Remark 2.20. In the first arXiv version of [54] there was a mistake in the proof of Theorem 2.19, which was pointed out to us by Otis Chodosh and Chao Li (the missing piece was Proposition 2.68). In subsequent joint work with Yevgeny Liokumovich they classified sufficiently connected manifolds in dimension 4 and 5 which admits a positive scalar curvature metric. Their result implies Theorem 2.19 as well (see [17, Theorem 3]).

2.1.3 Synthesis

In Section 2.1.1 we explored the applications of Theorem I to several model spaces. In Section 2.1.2 we provided examples of bands with Property A. Here we combine both aspects and highlight some interesting applications of the general theory we have displayed.

A combination of Corollary 1.22 and Theorem 1.23 yields the following result towards Conjecture 1.11.

Corollary 1.24. *Let $(n-1) \neq 4$ and $n \leq 7$. Let Y^{n-1} be a closed orientable manifold which does not admit a metric with positive scalar curvature and $X = Y \times [-1, 1]$. If g is a Riemannian metric on X with $\operatorname{scal}(X, g) \geq n(n-1)$, then*

$$\operatorname{width}(X, g) < \frac{2\pi}{n}.$$

Remark 2.21. We point out that Corollary 1.24 implies the S^1-stability conjecture of Rosenberg [58, Conjecture 1.24] for closed connected orientable manifolds of dimension $(n-1) \in \{1, 2, 3, 5, 6\}$.

In dimension 4 a combination of Corollary 1.22, Proposition 2.17 and Theorem 2.19 yields a band width estimate for trivial bands over closed aspherical manifolds:

Corollary 2.22. *Let Y^4 be a closed connected aspherical manifold. If $X = Y \times [-1, 1]$ and g is a Riemannian metric on X with $\operatorname{scal}(X, g) \geq n(n-1)$, then*

$$\operatorname{width}(X, g) < \frac{2\pi}{n}.$$

Remark 2.23. In Corollary 2.22 Y may be nonorientable since we can pass to the orientable double cover, which is a closed connected aspherical manifold as well.

Finally we deduce our main result towards Conjecture 1.9:

Corollary 1.26. *Let $(n-2) \neq 4$ and $n \leq 7$. Let Y^{n-2} be a closed orientable manifold which does not admit a metric with positive scalar curvature and $X = Y \times \mathbb{R}^2$. Then X does not admit a complete metric with uniformly positive scalar curvature.*

To see this we point out that if $D_R \subset \mathbb{R}^2$ denotes the closed R-ball around the origin in the euclidean metric, then $Y \times (D_R \backslash \mathring{D}_1) \cong Y \times S^1 \times [-1, 1]$. Since Y is orientable, does not admit a metric with positive scalar curvature and $dim(Y) \neq 4$ we deduce from Corollary 1.24 that $Y \times S^1$ does not admit a metric with positive scalar curvature either.

Hence, if (X, g) is complete and has uniformly positive scalar curvature, the width of the band $(Y \times (D_R \backslash \mathring{D}_1), g)$ is bounded from above independent of R (if $dim(Y) \neq 3$, then we can apply Corollary 1.24 to $(Y \times (D_R \backslash \mathring{D}_1), g)$; if $dim(Y) = 3$, then both Y and $Y \times S^1$ are spin and have non-vanishing Rosenberg index and we can apply Corollary 1.22 and Proposition 2.14). For $R \to \infty$ this is a contradiction.

2.2 Warped products

In this section we recall some facts about warped products and develop the general framework for scalar and mean curvature comparison of Riemannian bands.

The following definitions and formulas are standard knowledge.

Definition 2.24. Let (N, g_N) be a closed Riemannian manifold and $\varphi : (a, b) \to \mathbb{R}_+$ be a smooth positive function. The *warped product* over (N, g_N) with warping function φ is

$$(M, g_\varphi) := \left(N \times (a, b), \varphi^2 g_N + dt^2\right).$$

The scalar curvature of (M, g_φ) is determined by the scalar curvature of (N, g_N) and the warping function φ. The following formula

$$\begin{aligned} \operatorname{scal}(M, g_\varphi)(p, t) = \frac{1}{\varphi^2(t)} \operatorname{scal}(N, g_N)(p) - 2(n-1)\frac{\varphi''(t)}{\varphi(t)} \\ - (n-1)(n-2)\left(\frac{\varphi'(t)}{\varphi(t)}\right)^2 \end{aligned} \tag{2.2.1}$$

is obtained by a straightforward calculation (see also [32, Section 2.4]).

If we denote $N_t := N \times \{t\}$ for $t \in (a, b)$ and consider N_t as the boundary of $N \times (a, t]$, then its second fundamental form with respect to the inner unit normal vector field is a diagonal matrix whose entries are all equal to

$$\frac{d}{dt} \log(\varphi)(t) = \frac{\varphi'(t)}{\varphi(t)}.$$

It follows that N_t is an umbilic hypersurface and its mean curvature is given by

$$H(N_t) = (n-1)\frac{\varphi'(t)}{\varphi(t)} =: h_\varphi(t). \tag{2.2.2}$$

Finally, we rearrange (2.2.1) in terms of h_φ to obtain:

$$\operatorname{scal}(M, g_\varphi)(p, t) + \frac{n}{n-1} h_\varphi(t)^2 + 2h'_\varphi(t) = \frac{1}{\varphi^2(t)} \operatorname{scal}(N, g_N)(p). \tag{2.2.3}$$

This formula, which combines information on scalar and mean curvature of a warped product, is the basis on which we build a comparison principle.

2.2.1 Comparison of two Warped Products

As a first step towards a scalar and mean curvature comparison principle for Riemannian bands, we compare two warped products (M_1, g_{φ_1}) and (M_2, g_{φ_2}) over the same base manifold (N, g_N). The results in this subsection are purely motivational and do not factor into the proof of Theorem I.

We start off with the simplest situation, where the warping functions φ_1 and φ_2 have the same domain $[a, b]$ and the base manifold (N, g_N) is scalar flat.

Here we observe the following prototypical result, which can be regarded as a first proof of concept for Theorem I and all the comparison results it implies:

Proposition 2.25. *Let (N, g_N) be a closed scalar flat Riemannian manifold. Let $\varphi_1 : [a, b] \to \mathbb{R}_+$ and $\varphi_2 : [a, b] \to \mathbb{R}_+$ be two smooth positive functions. If*

- ▷ $\mathrm{scal}(M, g_{\varphi_1}) \geq \mathrm{scal}(M, g_{\varphi_2})$,
- ▷ $H(\partial_\pm M, g_{\varphi_1}) \geq H(\partial_\pm M, g_{\varphi_2})$,

then $h_{\varphi_1} = h_{\varphi_2}$ ie equality holds in both conditions.

Even though the statement of Proposition 2.25 is geometric in nature, its proof is purely analytical and based on:

Lemma 2.26. *Let $\varphi_1 : [a, b] \to \mathbb{R}_+$ and $\varphi_2 : [a, b] \to \mathbb{R}_+$ be two smooth positive functions. Then $h_{\varphi_1} = h_{\varphi_2}$ if and only if*

- ▷ $\frac{n}{n-1} h^2_{\varphi_1} + 2h'_{\varphi_1} \leq \frac{n}{n-1} h^2_{\varphi_2} + 2h'_{\varphi_2}$,
- ▷ $h_{\varphi_1}(a) \leq h_{\varphi_2}(a)$ *and* $h_{\varphi_1}(b) \geq h_{\varphi_2}(b)$.

Proof. The idea is to reduce the statement to a comparison result for the Riccati equation which can be found in [5, Lemma 4.1]. Consider $\hat{\varphi}_i(t) = \varphi_i\left(2\sqrt{\frac{n-1}{n}}t\right)^{\frac{n}{2}}$ as functions $[\hat{a}, \hat{b}] \to \mathbb{R}_+$ where $\hat{a} := \frac{a\sqrt{n}}{2\sqrt{n-1}}$ and $\hat{b} := \frac{b\sqrt{n}}{2\sqrt{n-1}}$. We denote

$$\hat{h}_i(t) := \frac{\hat{\varphi}'_i(t)}{\hat{\varphi}_i(t)} = \sqrt{\frac{n}{n-1}}(n-1)\frac{\varphi'_i\left(2\sqrt{\frac{n-1}{n}}t\right)}{\varphi_i\left(2\sqrt{\frac{n-1}{n}}t\right)} = \sqrt{\frac{n}{n-1}}h_{\varphi_i}\left(2\sqrt{\frac{n-1}{n}}t\right).$$

Then

$$\hat{h}^2_i(t) + \hat{h}'_i(t) = \frac{n}{n-1}h^2_{\varphi_i}\left(2\sqrt{\frac{n-1}{n}}t\right) + 2h'_{\varphi_i}\left(2\sqrt{\frac{n-1}{n}}t\right).$$

Furthermore, if we denote $\kappa_i := -\hat{h}^2_i(t) - \hat{h}'_i(t)$, we see that $\kappa_2 \leq \kappa_1$ and

$$\hat{\varphi}''_i(t) + \kappa_i\hat{\varphi}_i(t) = 0.$$

At this point we are in the situation where we can apply [5, Lemma 4.1] to conclude.

For the convenience of the reader, we repeat the proof here. Hence

$$\begin{aligned} 0 &= \int_{\hat{a}}^t \hat{\varphi}_1(\hat{\varphi}''_2 + \kappa_2\hat{\varphi}_2) - (\hat{\varphi}''_1 + \kappa_1\hat{\varphi}_1)\hat{\varphi}_2 \\ &= (\hat{\varphi}_1\hat{\varphi}'_2 - \hat{\varphi}'_1\hat{\varphi}_2)\Big|_{\hat{a}}^t + \int_{\hat{a}}^t (\kappa_2 - \kappa_1)\hat{\varphi}_1\hat{\varphi}_2 \end{aligned}$$

and therefore

$$\hat{\varphi}_1(t)\hat{\varphi}'_2(t) - \hat{\varphi}'_1(t)\hat{\varphi}_2(t) = \hat{\varphi}_1(\hat{a})\hat{\varphi}'_2(\hat{a}) - \hat{\varphi}'_1(\hat{a})\hat{\varphi}_2(\hat{a}) + \int_{\hat{a}}^t (\kappa_1 - \kappa_2)\hat{\varphi}_1\hat{\varphi}_2. \tag{2.2.4}$$

Now $\hat{\varphi}_1(\hat{a})\hat{\varphi}'_2(\hat{a}) - \hat{\varphi}'_1(\hat{a})\hat{\varphi}_2(\hat{a}) \geq 0$ since $\hat{h}_1(\hat{a}) \leq \hat{h}_2(\hat{a})$ and the second term on the right hand side is nonnegative since $\kappa_2 \leq \kappa_1$ and $\hat{\varphi}_1\hat{\varphi}_2 > 0$. It follows that

$$\hat{\varphi}_1(t)\hat{\varphi}'_2(t) - \hat{\varphi}'_1(t)\hat{\varphi}_2(t) \geq 0 \Leftrightarrow \frac{\hat{\varphi}'_1(t)}{\hat{\varphi}_1(t)} \leq \frac{\hat{\varphi}'_2(t)}{\hat{\varphi}_2(t)} \Leftrightarrow \hat{h}_1(t) \leq \hat{h}_2(t) \tag{2.2.5}$$

for all $t \in [\hat{a}, \hat{b}]$. By (2.2.4) $\hat{h}_1(\hat{a}) = \hat{h}_2(\hat{a})$ if equality holds at t. We can replace $\hat{a}$ by any $t_0 \in [\hat{a}, t]$ in the argument above since $\hat{h}_1(t_0) \leq \hat{h}_2(t_0)$. Hence $\hat{h}_1 = \hat{h}_2$ on $[\hat{a}, t]$ if equality holds at t. Since $\hat{h}_1(\hat{b}) \geq \hat{h}_2(\hat{b})$ by assumption and $\hat{h}_1(\hat{b}) \leq \hat{h}_2(\hat{b})$ by (2.2.5), equality holds at $\hat{b}$. Hence $\hat{h}_1 = \hat{h}_2$ on $[\hat{a}, \hat{b}]$ and therefore $h_{\varphi_1} = h_{\varphi_2}$ on $[a, b]$. □

Proof of Proposition 2.25. Since (N, g_N) is scalar flat (2.2.3) implies

$$\begin{aligned}\frac{n}{n-1}h_{\varphi_1}^2(t) + 2h_{\varphi_1}'(t) &= -\operatorname{scal}(M, g_{\varphi_1})(p,t) \\ &\leq -\operatorname{scal}(M, g_{\varphi_2})(p,t) = \frac{n}{n-1}h_{\varphi_2}^2(t) + 2h_{\varphi_1}'(t).\end{aligned}$$

Furthermore we have

$$h_{\varphi_1}(a) = -H(\partial_- M, g_{\varphi_1}) \leq -H(\partial_- M, g_{\varphi_2}) = h_{\varphi_2}(a)$$

and

$$h_{\varphi_1}(b) = H(\partial_+ M, g_{\varphi_1}) \geq H(\partial_+ M, g_{\varphi_2}) = h_{\varphi_2}(a)$$

by (2.2.2). Thus $h_{\varphi_1} = h_{\varphi_2}$ by Lemma 2.26. □

Next, we allow the warping functions to have different domains. Let (N, g_N) be a closed scalar flat Riemannian manifold and $\varphi_1 : [a,b] \to \mathbb{R}_+$ and $\varphi_2 : [c,d] \to \mathbb{R}_+$ two positive functions.

To compare the scalar curvature of the warped products (M_1, g_{φ_1}) and (M_2, g_{φ_2}) pointwise, we need to choose a band map $\Phi : M_1 \to M_2$. In this setting the canonical choice is $\Phi = \mathrm{id}_N \times \phi$, where $\phi : [a,b] \to [c,d]$ is given by $t \mapsto \frac{(d-c)}{(b-a)}(t-a) + c$.

To prove a comparison result like Proposition 2.25 we want to apply Lemma 2.26 to the functions h_{φ_1} and $\widetilde{h}_{\varphi_2} = h_{\varphi_2} \circ \phi = h_{\widetilde{\varphi}_2}$ where

$$\widetilde{\varphi}_2 : [a,b] \to \mathbb{R}_+ \quad t \mapsto \varphi_2(\phi(t))^{\frac{b-a}{d-c}}.$$

Hence we need to ensure that $\operatorname{scal}(M_1, g_{\varphi_1})(p,t) \geq \operatorname{scal}(M_2, g_{\varphi_2})(p, \phi(t))$ implies

$$\frac{n}{n-1}h_{\varphi_1}^2(t) + 2h_{\varphi_1}'(t) \leq \frac{n}{n-1}\widetilde{h}_{\varphi_2}^2(t) + 2\widetilde{h}_{\varphi_2}'(t).$$

for all $t \in [a,b]$. By (2.2.3) this works if

$$h_{\varphi_2}'(\phi(t)) \leq \widetilde{h}_{\varphi_2}'(t) = h_{\varphi_2}'(\phi(t))\phi'(t),$$

which in turn holds true if $h_{\varphi_2}'(\phi(t)) = 0$ or $h_{\varphi_2}'(\phi(t)) < 0$ and $\phi'(t) \leq 1$ ie $b-a \geq d-c$.

For this reason we consider strictly log-concave or log-affine warping functions in our comparison results.

Proposition 2.27. *Let (N, g_N) be a closed scalar flat Riemannian manifold. Let $\varphi_1 : [a,b] \to \mathbb{R}_+$ and $\varphi_2 : [c,d] \to \mathbb{R}_+$ be two positive functions. Consider the warped products (M_1, g_{φ_1}) and (M_2, g_{φ_2}) and the map $\phi : [a,b] \to [c,d]$ given by $t \mapsto \frac{(d-c)}{(b-a)}(t-a) + c$. If φ_2 is* log-affine*,*

- ▷ $\operatorname{scal}(M_1, g_{\varphi_1})(p,t) \geq \operatorname{scal}(M_2, g_{\varphi_2})(p, \phi(t))$,
- ▷ $H(\partial M_1, g_{\varphi_1}) \geq H(\partial M_2, g_{\varphi_2})$,

then $h_{\varphi_1} = h_{\varphi_2} \circ \phi$ ie equality holds in both conditions.

Proof of Proposition 2.27. Denote $\widetilde{h}_{\varphi_2} = h_{\varphi_2} \circ \phi : [a, b] \to \mathbb{R}$. By (2.2.3)

$$\operatorname{scal}(M_1, g_{\varphi_1})(p, t) = -\frac{n}{n-1} h^2_{\varphi_1}(t) - 2h'_{\varphi_1}(t)$$

as well as

$$\operatorname{scal}(M_2, g_{\varphi_2})(p, \phi(t)) = -\frac{n}{n-1} h^2_{\varphi_2}(\phi(t)) - 2h'_{\varphi_2}(\phi(t)) = -\frac{n}{n-1} \widetilde{h}^2_{\varphi_2}(t),$$

since φ_2 is assumed to be log-affine ie $h'_{\varphi_2} = 0$. Hence

$$\frac{n}{n-1} h^2_{\varphi_1}(t) + 2h'_{\varphi_1}(t) \leq \frac{n}{n-1} \widetilde{h}^2_{\varphi_2}(t) = \frac{n}{n-1} \widetilde{h}^2_{\varphi_2}(t) + 2\widetilde{h}'_{\varphi_2}(t).$$

Furthermore $h_{\varphi_1}(a) \leq \widetilde{h}_{\varphi_2}(a)$ and $h_{\varphi_1}(b) \geq \widetilde{h}_{\varphi_2}(b)$ (this follows from (2.2.2) and the assumption on mean curvature). Now Lemma 2.26 implies $h_{\varphi_1} = \widetilde{h}_{\varphi_2}$. □

Proposition 2.28. *Let (N, g_N) be a scalar flat Riemannian manifold. Let $\varphi_1 : [a, b] \to \mathbb{R}_+$ and $\varphi_2 : [c, d] \to \mathbb{R}_+$ be two positive functions. Consider the warped products (M_1, g_{φ_1}) and (M_2, g_{φ_2}) and the map $\phi : [a, b] \to [c, d]$ given by $t \mapsto \frac{(d-c)}{(b-a)}(t - a) + c$. If φ_2 is strictly* log-*concave,*

- ▷ $\operatorname{scal}(M_1, g_{\varphi_1})(p, t) \geq \operatorname{scal}(M_2, g_{\varphi_2})(p, \phi(t))$,
- ▷ $H(\partial M_1, g_{\varphi_1}) \geq H(\partial M_2, g_{\varphi_2})$,
- ▷ $\operatorname{width}(M_1, g_{\varphi_1}) \geq \operatorname{width}(M_2, g_{\varphi_2})$,

then $b - a = d - c$ and $h_{\varphi_1} = h_{\varphi_2} \circ \phi$ ie equality holds in all three conditions.

Proof of Proposition 2.28. Denote $\widetilde{h}_{\varphi_2} = h_{\varphi_2} \circ \phi : [a, b] \to \mathbb{R}$. As before:

$$\begin{aligned} \frac{n}{n-1} h^2_{\varphi_1}(t) + 2h'_{\varphi_1}(t) &\leq \frac{n}{n-1} h^2_{\varphi_2}(\phi(t)) + 2h'_{\varphi_2}(\phi(t)) \\ &\leq \frac{n}{n-1} \widetilde{h}^2_{\varphi_2}(t) + 2\widetilde{h}'_{\varphi_2}(t), \end{aligned} \tag{2.2.6}$$

where we used that φ_2 is log-concave and ϕ is 1-Lipschitz for the last inequality.

Furthermore $h_{\varphi_1}(a) \leq \widetilde{h}_{\varphi_2}(a)$ and $h_{\varphi_1}(b) \geq \widetilde{h}_{\varphi_2}(b)$. If $b - a > d - c$ ie ϕ is strictly 1-Lipschitz, the last inequality in (2.2.6) would be strict since $h'_{\varphi_2} < 0$. This is impossible because Lemma 2.26 implies $h_{\varphi_1} = \widetilde{h}_{\varphi_2}$. □

In the following we want to generalize Proposition 2.27 and Proposition 2.28, the prototypes for the two parts of Theorem I, to allow for the comparison of Riemannian bands with warped products over closed Riemannian manifolds with constant scalar curvature.

2.2.2 Structural Maps

To compare the scalar and mean curvature of two Riemannian bands (X, g) and (V, τ) pointwise, one has to choose a band map $\Phi : X \to V$. If $\operatorname{scal}(V, \tau)$ is constant and $H(\partial V, \tau)$ is constant on $\partial_- V$ resp. $\partial_+ V$, the outcome does not depend on the choice of Φ.

If (V,τ) is a warped product (M,g_φ) over a closed Riemannian manifold (N,g_N) with warping function $\varphi:[a,b]\to\mathbb{R}_+$, the second condition is always satisfied as the mean curvature of $\partial M=\partial_-M\sqcup\partial_+M$ with respect to g_φ is constant equal to $\pm h_\varphi(a)$ on $\partial_\pm(M)$ (see (2.2.2)).

Furthermore, if $\mathrm{scal}(N,g_N)$ is constant, then $\mathrm{scal}(M,g_\varphi)(p,t)$ only depends on the t-coordinate (see (2.2.3)) and therefore the scalar curvature comparison between (X,g) and (M,g_φ) only depends on $\phi:=\mathrm{pr}_{[a,b]}\circ\Phi:X\to[a,b]$.

This is the situation we focus on for the rest of this chapter ie (X^n,g) will be a Riemannian band, (N^{n-1},g_N) will be a closed Riemannian manifold with constant scalar curvature and (M^n,g_φ) will be a warped product over (N,g_N) with warping function $\varphi:[a,b]\to\mathbb{R}_+$.

To compare (X,g) and (M,g_φ) we fix a point $p_0\in N$, choose a band map $\phi:X\to[a,b]$ and define $\Phi:X\to M$ by $x\mapsto(p_0,\phi(x))$.

While every choice of ϕ enables us to compare the scalar and mean curvature of (X,g) and (M,g_φ) pointwise, we will need ϕ to preserve some geometric structure to prove comparison results like Proposition 2.27 or Proposition 2.28. We denote $h=h_\varphi\circ\phi:X\to\mathbb{R}$ and consider a 'pullback' version of equation (2.2.3) on (X,g).

Definition 2.29. A band map $\phi:X\to[a,b]$, which is used to compare (X,g) and (M,g_φ), is called *structural* if it is smooth and for any closed embedded hypersurface Σ with outward unit normal field ν which separates ∂_-X and ∂_+X the inequality

$$\mathrm{scal}(X,g)(x)+\frac{n}{n-1}h^2(x)+2g(\nabla h(x),\nu(x))\geq\frac{1}{\varphi^2(\phi(x))}Sc(N,g_N)\tag{2.2.7}$$

holds at all points $x\in\Sigma$.

In Section 2.3 we will use μ-bubbles to prove the following Proposition:

Proposition 2.30. *Let $n\leq 7$ and (X,g) be an oriented Riemannian band. Let (N,g_N) be a closed oriented Riemannian manifold with constant scalar curvature and (M,g_φ) the warped product over (N,g_N) with warping function $\varphi:[a,b]\to\mathbb{R}_+$. If there is a structural band map $\phi:X\to[a,b]$ and*

$$H(\partial_\pm X,g)>H(\partial_\pm M,g_\varphi),$$

there is a hypersurface $\Sigma\subset X$, which separates ∂_-X and ∂_+X such that:

$$-\Delta_\Sigma+\frac{1}{2}\mathrm{scal}(\Sigma,g)\geq\frac{1}{2\varphi^2(\phi)}Sc(N,g_N).$$

Remark 2.31. From a conceptual perspective one should be able to relax the assumption $H(\partial_\pm X,g)>H(\partial_\pm M,g_\varphi)$ in Proposition 2.30 to $H(\partial_\pm X,g)\geq H(\partial_\pm M,g_\varphi)$ if φ is log-concave. However, we are only able to do so whenever φ is log-affine (see Section 2.3.1). If φ is strictly log-concave we can work around certain aspects of the problem but fall short of the desired result. The reason for this indiscrepancy is the lack of a strong maximum principle for μ-bubbles.

In light of Proposition 2.30 we try to identify situations where there are structural maps to compare (X,g) and (M,g_φ). As in Proposition 2.27 and Proposition 2.28 we assume φ to be strictly log-concave or log-affine.

Lemma 2.32. *Let* (X, g) *be a Riemannian band,* (N, g_N) *be a closed Riemannian manifold with constant scalar curvature and* (M, g_φ) *be a warped product over* (N, g_N) *with warping function* $\varphi : [a, b] \to \mathbb{R}_+$*. If* φ *is* log-*affine, then any smooth band map* $\phi : X \to [a, b]$ *such that* $\mathrm{scal}(X, g)(x) \geq \mathrm{scal}(M, g_\varphi)(p_0, \phi(x))$ *is structural.*

Proof. Let Σ be a hypersurface which separates $\partial_- X$ and $\partial_+ X$ in X. If $\phi : X \to [a, b]$ is smooth, and $\mathrm{scal}(X, g)(x) \geq \mathrm{scal}(M, g_\varphi)(p_0, \phi(x))$, then

$$\mathrm{scal}(X, g)(x) + \frac{n}{n-1} h^2(x) + 2g(\nabla h(x), \nu(x)) = \mathrm{scal}(X, g)(x) + \frac{n}{n-1} h^2(x)$$

since $\nabla h = 0$ (φ is log-affine) and

$$\begin{aligned}\mathrm{scal}(X, g)(x) + \frac{n}{n-1} h^2(x) &\geq \mathrm{scal}(M, g_\varphi)(p_0, \phi(x)) + \frac{n}{n-1} h_\varphi^2(\phi(x)) \\ &= \frac{1}{\varphi^2(\phi(x))} Sc(N, g_N),\end{aligned}$$

which implies that ϕ is structural. □

Lemma 2.33. *Let* (X, g) *be a Riemannian band,* (N, g_N) *be a closed Riemannian manifold with constant scalar curvature and* (M, g_φ) *be a warped product over* (N, g_N) *with warping function* $\varphi : [a, b] \to \mathbb{R}_+$*. If*

- ▷ $\phi : X \to [a, b]$ *is a smooth 1-Lipschitz band map,*
- ▷ φ *is* log-*concave and*
- ▷ $\mathrm{scal}(X, g)(x) \geq \mathrm{scal}(M, g_\varphi)(p_0, \phi(x))$,

then ϕ *is structural.*

Proof. Let Σ be a hypersurface which separates $\partial_- X$ and $\partial_+ X$ in X. Since φ is log-concave ie $h'_\varphi \leq 0$ and ϕ is 1-Lipschitz we have

$$\begin{aligned}\frac{n}{n-1} h^2(x) + 2g(\nabla h(x), \nu(x)) &\geq \frac{n}{n-1} h_\varphi^2(\phi(x)) + 2h'_\varphi(\phi(x))|\nabla\phi| \\ &\geq \frac{n}{n-1} h_\varphi^2(\phi(x)) + 2h'_\varphi(\phi(x)).\end{aligned}$$

Together with $\mathrm{scal}(X, g)(x) \geq \mathrm{scal}(M, g_\varphi)(p_0, \phi(x))$ and (2.2.3) it follows that ϕ is structural. □

The following can also be found in [79, Lemma 4.1] and [14, Lemma 7.2].

Lemma 2.34. *Let* (X, g) *be a Riemannian band. If* $\mathrm{width}(X, g) > a - b$*, there is a smooth band map* $\phi : (X, g) \to [a, b]$ *with* $\mathrm{Lip}(\phi) < 1$.

A combination of Lemma 2.33 and Lemma 2.34 yields:

Lemma 2.35. *Let* (X, g) *be a Riemannian band,* (N, g_N) *be a closed Riemannian manifold with constant scalar curvature and* (M, g_φ) *be a warped product over* (N, g_N) *with warping function* $\varphi : [a, b] \to \mathbb{R}_+$*. If* φ *is* log-*concave,* $\mathrm{scal}(M, g_\varphi)$ *is constant, and the following holds true*

▷ $\mathrm{scal}(X, g) \geq \mathrm{scal}(M, g_\varphi)$,

▷ $\mathrm{width}(X, g) > \mathrm{width}(M, g_\varphi)$,

there exists a structural band map $\phi : X \to [a, b]$. □

Remark 2.36. If $\mathrm{Lip}(\phi) < 1$ in Lemma 2.33 and φ is strictly log-concave, we get strict inequality in (2.2.7). This observation is important as it allows us to obtain strict inequality for the operator in Proposition 2.30 later (see Remark 2.42). In particular this applies to the band map ϕ we get from Lemma 2.35.

2.2.3 Model Spaces over Spheres

The notion of a model space (compare Definition 1.20) for scalar and mean curvature comparison of Riemannian bands is motivated by our observations so far. In addition to those we introduced in Section 2.1.1 one considers annuli in simply connected space forms.

Let $(S^n, g_1)\backslash\{p_1, p_2\}$ be the round unit n-sphere with two antipodal points removed. This has constant scalar curvature equal to $n(n-1)$ and can be written as a warped product

$$\left(S^{n-1} \times (-\frac{\pi}{2}, \frac{\pi}{2}), \cos^2(t) g_1 + dt^2\right),$$

where (S^{n-1}, g_1) is the unit sphere in one dimension less. Since $\cos(t)$ is strictly log-concave we see that for $-\frac{\pi}{2} < \ell_- < \ell_+ < \frac{\pi}{2}$ the warped product

$$\left(S^{n-1} \times [\ell_-, \ell_+], \cos^2(t) g_1 + dt^2\right)$$

is a model space.

Let $(\mathbb{R}^n, g_{\mathrm{std}})\backslash\{0\}$ be euclidean space with the origin removed. This is scalar flat and can be written as a warped product

$$\left(S^{n-1} \times (0, \infty), t^2 g_1 + dt^2\right)$$

Since t is stricly log-concave we see that for $0 < \ell_- < \ell_+ < \infty$ the warped product

$$\left(S^{n-1} \times [\ell_-, \ell_+], t^2 g_1 + dt^2\right)$$

is a model space.

Let $(\mathbb{H}^n, g_{-1})\backslash\{p\}$ be hyperbolic space with a point removed. This has constant scalar curvature equal to $-n(n-1)$ and can be written as a warped product

$$\left(S^{n-1} \times (0, \infty), \sinh^2(t) g_1 + dt^2\right).$$

Since $\sinh(t)$ is strictly log-concave we see that for $0 < \ell_- < \ell_+ < \infty$ the warped product

$$\left(S^{n-1} \times [\ell_-, \ell_+], \sinh^2(t) g_1 + dt^2\right)$$

is a model space.

Remark 2.37. Let (X^n, g) be a Riemannian spin band and (M, g_φ) one of the above model spaces. Let $\Phi : X \to M$ be a smooth 1-Lipschitz band map with degree non zero. In [14, Corollary 10.4] Cecchini and Zeidler prove the following: If $\mathrm{scal}(X, g) \geq \mathrm{scal}(M, g)$ and $H(\partial_\pm X, g) \geq H(\partial_\pm M, g)$, then Φ is an isometry.

As is explained in [32, Section 5.5] one can recreate similar results using μ-bubbles and a stabilized version of Llarull's theorem [50] in dimension $3 \leq n \leq 7$ (one does not need to assume that n is odd).

However, as rigidity for strictly log-concave warping functions remains problematic in our setting (see Remark 2.2) the best result we could obtain at this moment is: If $\mathrm{scal}(X, g) \geq \mathrm{scal}(M, g)$ and $H(\partial_\pm X, g) \geq H(\partial_\pm M, g)$, there is no smooth band map $\Phi : X \to M$ with degree non-zero and $\mathrm{Lip}(\Phi) < 1$.

2.3 μ-Bubbles

We briefly introduce the most important definitions and results (cf. [16, Section 3], [32, Section 5.1] and [79, Section 2]) concerning μ-bubbles. As a good reference for the theory of Caccioppoli sets, which will be used freely throughout the rest of this section, we recommend [22, Chapter 1].

Let (X, g) be an oriented Riemannian band and h be a smooth function on X. Denote by $\mathcal{C}(X)$ the set of all Caccioppoli sets in X which contain an open neighborhood of $\partial_- X$ and are disjoint from $\partial_+ X$.

For $\hat{\Omega} \in \mathcal{C}(X)$ consider the functional

$$\mathcal{A}_h(\hat{\Omega}) = \mathcal{H}^{n-1}(\partial^* \hat{\Omega} \cap \mathring{X}) - \int_{\hat{\Omega}} h d\mathcal{H}^n,$$

where $\partial^* \hat{\Omega}$ is the reduced boundary [22, Chapter 3, 4] of $\hat{\Omega}$.

We denote

$$\mathcal{I} := \inf\{\mathcal{A}_h(\hat{\Omega}) \big| \hat{\Omega} \in \mathcal{C}(X)\}$$

and call a Caccioppoli set $\Omega \in \mathcal{C}(X)$ a *μ-bubble* if $\mathcal{A}_h(\Omega) = \mathcal{I}$ ie Ω minimizes the $\mathcal{A}_h$-functional among all Caccioppoli sets in X, which contain a neighborhood of $\partial_- X$ and are disjoint from $\partial_+ X$.

Remark 2.38. To preempt any confusion we remind reader of our mean curvature convention in Remark 2.1, according to which $H(\partial_- X)$ is the trace of the second fundamental form with respect to the inner unit normal field.

However, if $\hat{\Omega}$ is a smooth Caccioppoli set which contains an open neighborhood of $\partial_- X$ and $\hat{\Sigma}$ is a connected component of $\partial \hat{\Omega} \cap \mathring{X}$, then the mean curvature $H(\hat{\Sigma})$ is the trace of the second fundamental form with respect to the unit normal field pointing into $\hat{\Omega}$. Hence, if $\hat{\Sigma}$ approaches $\partial_- X$ then $H(\hat{\Sigma})$ approaches $-H(\partial_- X)$ and if $\hat{\Sigma}$ approaches $\partial_+ X$, then $H(\hat{\Sigma})$ approaches $H(\partial_+ X)$.

Lemma 2.39 (see [32, Section 5.1])**.** *If $n \leq 7$ and $H(\partial_\pm X) > \pm h$ on $\partial_\pm X$, there is a smooth μ-bubble Ω ie a smooth Caccioppoli set $\Omega \in \mathcal{C}(X)$, with $\mathcal{A}_h(\Omega) = \mathcal{I}$.*

Proof. We adapt the proofs of [79, Proposition 2.1] and [16, Proposition 12]. For $t > 0$ denote by $\Omega_\pm^t$ the t-neighborhoods of $\partial_\pm X$. Since $\partial_\pm X$ is smooth $\Omega_\pm^t$ has a foliation by smooth equidistant hypersurfaces $\Sigma_\pm^{s \leq t}$ for t small enough.

Denote by $\nu^s_\pm$ the unit normal vector field to $\Sigma^s_\pm$ pointing in the direction of $\partial_+ X$ and by $H(\Sigma^s_\pm)$ the trace of the second fundamental form of $\Sigma^s_\pm$ with respect to $-\nu^s_\pm$. By possibly making t even smaller we can guarantee

$$\operatorname{div}(\nu^s_-) = H(\Sigma^s_-) < h(x) \text{ on } \Omega^t_- \text{ and } \operatorname{div}(\nu^s_+) = H(\Sigma^s_+) > h(x) \text{ on } \Omega^t_+.$$

Let $\hat\Omega$ be any Caccioppoli set with $\partial_- X \subset \hat\Omega$ and $\partial_+ X \cap \hat\Omega = \emptyset$.

We want to see the following: if we add Ω^t_- to $\hat\Omega$ or substract Ω^t_+ from $\hat\Omega$ we do not increase the value of $\mathcal{A}_h$.

$$\mathcal{A}_h((\hat\Omega \cup \Omega^t_-)\backslash \Omega^t_+) - \mathcal{A}_h(\hat\Omega) = \mathcal{H}^{n-1}(\partial\Omega^t_- \backslash \hat\Omega) - \mathcal{H}^{n-1}(\partial^*\hat\Omega \cap \Omega^t_-) + \mathcal{H}^{n-1}(\partial\Omega^t_+ \cap \hat\Omega) - \mathcal{H}(\partial^*\hat\Omega \cap \Omega^t_+) - \int_{\Omega^t_- \backslash \hat\Omega} h d\mathcal{H}^n + \int_{\Omega^t_+ \cap \hat\Omega} h d\mathcal{H}^n.$$

The divergence theorem and our assumption on h implies

$$\begin{aligned}\int_{\Omega^t_- \backslash \hat\Omega} h d\mathcal{H}^n &> \int_{\Omega^t_- \backslash \hat\Omega} \operatorname{div} \nu^s_- d\mathcal{H}^n \\ &= \int_{\partial^*(\Omega^t_- \backslash \hat\Omega)} \langle \nu^s_-, \nu\rangle d\mathcal{H}^{n-1} \\ &\geq \mathcal{H}^{n-1}(\partial\Omega^t_- \backslash \hat\Omega) - \mathcal{H}^{n-1}(\partial^*\hat\Omega \cap \Omega^t_-)\end{aligned}$$

and

$$\begin{aligned}\int_{\Omega^t_+ \cap \hat\Omega} h d\mathcal{H}^n &< \int_{\Omega^t_+ \cap \hat\Omega} \operatorname{div} \nu^s_+ d\mathcal{H}^n \\ &= \int_{\partial^*(\Omega^t_+ \cap \hat\Omega)} \langle \nu^s_+, \nu\rangle d\mathcal{H}^{n-1} \\ &\leq -\mathcal{H}^{n-1}(\partial\Omega^t_+ \cap \hat\Omega) + \mathcal{H}^{n-1}(\partial^*\hat\Omega \cap \Omega^t_+).\end{aligned}$$

We conclude that

$$\mathcal{A}_h((\hat\Omega \cup \Omega^t_-)\backslash \Omega^t_+) - \mathcal{A}_h(\hat\Omega) < 0, \tag{2.3.1}$$

which implies that it is enough to search for a minimizer among all Caccioppoli sets in X with $\Omega^t_- \subset \hat\Omega$ and $\Omega^t_+ \cap \hat\Omega = \emptyset$.

If C is a constant such that $|h| < C$ on X, then for any such Caccioppoli set we have $\mathcal{A}_h(\hat\Omega) > -C\mathcal{H}^n(X) > -\infty$. We choose a minimizing sequence $\hat\Omega_k$. By compactness for Caccioppoli sets (compare [22, Theorems 1.19 & 1.20]) $\hat\Omega_k$ subconverges to a minimizing Caccioppoli set Ω which contains an open neighborhood of $\partial_- X$ and is disjoint from $\partial_+ X$. Smoothness of Ω follows from the regularity theorem [78, Theorem 2.2]. □

If $\hat\Omega \in \mathcal{C}(X)$ is smooth and $\hat\Sigma$ is a connected component of $\partial\hat\Omega \cap \mathring{X}$, we denote by ν the outwards pointing unit normal vector field, by A the second fundamental form with respect to $-\nu$ and by H the trace of A. We present the first and second variation formula for the $\mathcal{A}_h$-functional (cf. [79, Proof of Theorem 1.4] or [78, Equation 1.2]).

Lemma 2.40 (First variation formula)**.** *For any smooth function ψ on $\hat\Sigma$ let V_ψ be a vector field on X, which vanishes outside a small neighborhood of $\hat\Sigma$ and agrees with $\psi\nu$ on $\hat\Sigma$. If we denote by Φ_t the flow generated by V_ψ, then*

$$\frac{d}{dt}\Big|_{t=0} \mathcal{A}_h(\Phi_t(\hat\Omega)) = \int_{\hat\Sigma} (H - h)\psi d\mathcal{H}^{n-1}. \tag{2.3.2}$$

Lemma 2.41 (Second variation formula)**.** *For any smooth function ψ on $\hat{\Sigma}$ let V_ψ be a vector field on X, which vanishes outside a small neighborhood of $\hat{\Sigma}$ and agrees with $\psi\nu$ on $\hat{\Sigma}$. If we denote by Φ_t the flow generated by V_ψ, then*

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathcal{A}_h(\Phi_t(\hat{\Omega})) = \int_{\hat{\Sigma}} |\nabla_{\hat{\Sigma}}\psi|^2 + (H^2 - Ric(\nu,\nu) - |A|^2 - Hh - g(\nabla_X h, \nu))\psi^2,$$

which is equal to

$$\int_\Sigma |\nabla_\Sigma \psi|^2 - \frac{1}{2}(\mathrm{scal}(X,g) - \mathrm{scal}(\hat{\Sigma}, g) - H^2 + |A|^2)\psi^2 - (Hh + g(\nabla_X(h), \nu))\psi^2 \quad (2.3.3)$$

Proof. We differentiate the first variation formula employing the following Leibniz rule: If f is a smooth function on X, then

$$\frac{d}{dt}\Big|_{t=0} \int_{\hat{\Sigma}_t} f\psi d\mathcal{H}^{n-1} = \int_{\hat{\Sigma}} (Hf + g(\nabla_X f, \nu))\psi^2 d\mathcal{H}^{n-1}.$$

Furthermore we use the formula

$$\int_{\hat{\Sigma}} g(\nabla_X H_{\hat{\Sigma}_t}, \nu)\psi^2 d\mathcal{H}^{n-1} = \int_{\hat{\Sigma}} |\nabla_{\hat{\Sigma}}\psi|^2 - (Ric(\nu,\nu) + |A|^2)\psi^2 d\mathcal{H}^{n-1},$$

and the standard trick to rewrite

$$Ric(\nu,\nu) = \frac{1}{2}(\mathrm{scal}(X,g) - \mathrm{scal}(\Sigma,g) + H^2 - |A|^2)$$

from [64, p. 165]. □

If Ω is the μ-bubble we get from Lemma 2.39, then the mean curvature H of Σ is equal to h by Lemma 2.40 and by stability and Lemma 2.41, we see that

$$\begin{aligned} 0 &\le \int_\Sigma |\nabla_\Sigma\psi|^2 - \frac{1}{2}(\mathrm{scal}(X,g) - \mathrm{scal}(\Sigma,g) - H^2 + |A|^2)\psi^2 - (Hh + g(\nabla_X h,\nu))\psi^2 \\ &= \int_\Sigma |\nabla_\Sigma\psi|^2 - \frac{1}{2}(\mathrm{scal}(X,g) - \mathrm{scal}(\Sigma,g) + H^2 + |A|^2)\psi^2 - g(\nabla_X h,\nu)\psi^2 \\ &\le \int_\Sigma |\nabla_\Sigma\psi|^2 - \frac{1}{2}(\mathrm{scal}(X,g) - \mathrm{scal}(\Sigma,g) + \frac{n}{n-1}h^2 + 2g(\nabla_X h,\nu))\psi^2, \end{aligned}$$

where we used $|A|^2 \ge \frac{H^2}{n-1}$ for the last inequality. By rearranging terms, we conclude

$$\int_\Sigma |\nabla_\Sigma\psi|^2 + \frac{1}{2}\mathrm{scal}(\Sigma,g)\psi^2 d\mathcal{H}^{n-1} \ge \int_\Sigma \frac{1}{2}(\mathrm{scal}(X,g) + \frac{n}{n-1}h^2 + 2g(\nabla_X h,\nu))\psi^2 d\mathcal{H}^{n-1}. \quad (2.3.4)$$

We are now ready to prove Proposition 2.30:

Proof of Proposition 2.30. By assumption there is a structural band map $\phi : X \to [a,b]$. Thus $h = h_\varphi \circ \phi$ is a smooth function on X and by assumption $H(\partial_\pm X) > \pm h$. Since $n \le 7$ Lemma 2.39 yields a smooth minimizer Ω for the $\mathcal{A}_h$-functional, which contains a neighborhood of $\partial_- X$ and is disjoint from $\partial_+ X$. Hence $\Sigma = \partial\Omega$ separates $\partial_- X$ and $\partial_+ X$ in X. Furthermore by (2.3.4)

$$\int_\Sigma |\nabla_\Sigma\psi|^2 + \frac{1}{2}\mathrm{scal}(\Sigma,g)\psi^2 d\mathcal{H}^{n-1} \ge \int_\Sigma \frac{1}{2}(\mathrm{scal}(X,g) + \frac{n}{n-1}h^2 + 2g(\nabla_X h,\nu))\psi^2 d\mathcal{H}^{n-1}$$

for any $\psi \in C^\infty(\Sigma)$. Since ϕ is structural this implies

$$\int_\Sigma |\nabla_\Sigma \psi|^2 + \frac{1}{2}\operatorname{scal}(\Sigma, g)\psi^2 d\mathcal{H}^{n-1} \geq \int_\Sigma \frac{1}{2\varphi^2(\phi)} Sc(N, g_N)\psi^2$$

and hence

$$-\Delta_\Sigma + \frac{1}{2}\operatorname{scal}(\Sigma, g) \geq \frac{1}{2\varphi^2(\phi)} Sc(N, g_N).$$

□

Remark 2.42. Let (X, g) be an oriented Riemannian band, (M, g_φ) a warped product over (N, g_N) and $\phi : X \to [a, b]$ a smooth band map. If φ is strictly log-concave, $\operatorname{scal}(X, g)(x) \geq \operatorname{scal}(M, g_\varphi(p_0, \phi(x))$ and ϕ has $\operatorname{Lip}(\phi) < 1$, then ϕ is structural by Lemma 2.33. As was observed in Remark 2.36 we even get strict inequality in (2.2.7) in this case as $g(\nabla_X h, \nu) > h'_\varphi$. Hence the argument above yields

$$-\Delta_\Sigma + \frac{1}{2}\operatorname{scal}(\Sigma, g) > \frac{1}{2\varphi^2(\phi)} Sc(N, g_N).$$

2.3.1 Constant Mean Curvature Surfaces

If Ω is a smooth minimizer for the $\mathcal{A}_h$ functional, then $\Sigma = \partial\Omega \cap \mathring{X}$ is often called a prescribed mean curvature (or short PMC) surface in the literature (see for example [78]). This terminology is based on the observation that $H(\Sigma) = h\big|_\Sigma$ by the first variation formula.

In the following we assume h to be a constant function. In this case Σ is called a constant mean curvature (or short CMC) surface and our main goal is to understand what happens if we relax the strict boundary condition $H(\partial_\pm X) > \pm h$ to $H(\partial_\pm X) \geq \pm h$ in Lemma 2.39. In the proof of Lemma 2.39 the assumption $H(\partial_\pm X) > \pm h$ was used to show that there is a minimizing sequence $\hat{\Omega}_k$ in $\mathcal{C}(X)$ which converges to a Caccioppoli set $\Omega \in \mathcal{C}(X)$.

This fails if we relax to $H(\partial_\pm X) \geq \pm h$, since it might happen that the limit Ω of any minimizing sequence $\hat{\Omega}_k$ in $\mathcal{C}(X)$ contains points of $\partial_+ X$ or does not contain a neighborhood of $\partial_- X$ any more. However, for h constant, we can use a strong maximum principle to address this issue.

To make this precise we slightly change our set up. Let (X, g) be an oriented Riemannian band and h be constant function on X. Without loss of generality we can assume h to be nonnegative (otherwise we just change the roles of $\partial_- X$ and $\partial_+ X$). For some $\delta > 0$ we glue on collars $\partial_- X \times (-\delta, 0]$ and $\partial_+ X \times [0, \delta)$ on both sides of X and extend the metric g smoothly to produce a Riemannian manifold (X_δ, g_δ). This can be done in such a way that $\operatorname{vol}(X_\delta, g_\delta) < \operatorname{vol}(X, g) + \delta$.

Let $\mathcal{C}(X_\delta)$ be the set of all Caccioppoli sets in X_δ, which contain $\partial_- X \times (-\delta, 0]$ and are disjoint from $\partial_+ X \times (0, \delta)$. We replace $\mathcal{H}^{n-1}(\partial^* \hat{\Omega} \cap \mathring{X})$ by $\mathcal{H}^{n-1}(\partial^* \hat{\Omega})$ in the $\mathcal{A}_h$-functional and define $\mathcal{I}_\delta := \inf\{\mathcal{A}_h(\hat{\Omega}) \big| \hat{\Omega} \in \mathcal{C}(X_\delta)\}$.

Proposition 2.43. *Let $h \geq 0$ be constant and $n \leq 7$. If $H(\partial_\pm X) \geq \pm h$ and $\Omega \in \mathcal{C}(X_\delta)$ is a minimizer ie $\mathcal{A}_h(\Omega) = \mathcal{I}_\delta$, then any connected component of $\partial\Omega$ is either contained in $\mathring{X}$ or agrees with a connected component of $\partial_- X$ resp. $\partial_+ X$.*

Proof. The main issue we face is that, a priori, $\partial\Omega$ might not be smooth at the points where it touches $\partial_\pm X$. This is due to the fact that we minimize the $\mathcal{A}_h$ functional with respect to the obstacles $\partial_- X$ and $\partial_+ X$.

The best a priori regularity result we can apply is [41, Theorem 1.3] (based on [68]) according to which $\partial\Omega$ is a $C^{1,\frac{1}{2}}$-submanifold of X and hence has a $C^{0,\frac{1}{2}}$ outer unit normal vector field ν. If $h = 0$, then $\partial\Omega$ is even $C^{1,1}$. For a related discussion see also [73, Paragraph after Theorem 1].

This, however, is not quite good enough (we would need C^2) to use the following observation which would imply Proposition 2.43 if $\partial\Omega$ were smooth and had constant mean curvature equal to h.

Claim. Let Ω_1 and Ω_2 be two smooth Caccioppoli sets in $\mathcal{C}(X_\delta)$ such that $\Omega_1 \subset \Omega_2$ and their boundaries Σ_1 resp. Σ_2 touch at a point p (ie p is contained in both Σ_1 and Σ_2 and the interior normal vector fields to Σ_1 resp. Σ_2 at p agree). Assume furthermore that $H(\Sigma_2)$ is constant. Then, if $H(\Sigma_1) \leq H(\Sigma_2)$, the connected components of Σ_1 and Σ_2 which contain p coincide.

Proof of Claim. We adapt the proof of [77, Lemma 2.7]. Since the tangent planes to Σ_1 and Σ_2 agree at p, there is a small ball U around p where both hypersurfaces may be written as smooth graphs u_1, u_2 in the ν direction over the common tangent plane $T_p\Sigma_1 = T_p\Sigma_2$.

There is a positive definite second order elliptic operator L with smooth coefficients such that the difference $u = u_2 - u_1$ satisfies $Lu \geq 0$. Since Σ_1 lies above Σ_2 (with respect to ν) we have $u \leq 0$ and $u = 0$ at p. Hence we can apply the Hopf maximum principle [21, Theorem 3.5] (it would suffice if the u_i were only C^2) to see that u is constant equal to zero in U and thus Σ_1 coincides with Σ_2 in an open neighborhood of p. It follows that the set where Σ_1 and Σ_2 agree is non-empty, open and closed in Σ_i. Hence the connected components of Σ_1 and Σ_2 which contain p coincide. □

To overcome this problem, we turn to [65] and [73], where strong maximum principles, which do not require $\partial\Omega$ to be smooth, are established. For further generalizations see also [74, Theorem 7.3] and [47, Section 3.1] for a free boundary result in polyhedral domains.

We realize that, if $h = 0$, Proposition 2.43 follows directly from [73, Theorem 4] or the main result of [65] (for the unfamiliar reader we point out, as it is done in [73], that one may substitute 'varifold' by 'C^1-submanifold' in these results).

Moreover, if $h > 0$, one can apply [73, Theorem 7] to show that a connected component of $\partial\Omega$ can only touch a connected component of $\partial_+ X$ if they coincide.

To see this we refer to [78, Equation (1.2)] for the general first variation formula of the $\mathcal{A}_h$-functional (compare with Lemma 2.40), according to which for any vector field in X with $g(X, \nu_\pm) \geq 0$ at the boundary $\partial_\pm X$, we have

$$\int_{\partial\Omega} \operatorname{div}_{\partial\Omega} V \, d\mathcal{H}^{n-1} - \int_{\partial\Omega} hg(V,\nu) d\mathcal{H}^{n-1} = 0,$$

since Ω minimizes the $\mathcal{A}_h$-functional in X. Hence

$$\int_{\partial\Omega} \operatorname{div}_{\partial\Omega} V \, d\mathcal{H}^{n-1} = \int_{\partial\Omega} hg(V,\nu) d\mathcal{H}^{n-1} \geq -\int_{\partial\Omega} h|V| d\mathcal{H}^{n-1},$$

and the assumptions of [73, Theorem 7] are satisfied.

Thus it remains for us to prove that a connected component of $\partial\Omega$ can not touch a connected component of $\partial_- X$, unless they coincide.

The proofs of [73, Theorems 4 & 7] as well as those of [74, Theorem 7.3] and [47, Theorem 3.1] follow the same two step procedure, which was first developed in [65].

The first step is to show that the minimizing object can not touch the boundary at a point where the mean curvature barrier condition is strict. This is sometimes called a weak maximum principle. It is usually proved by contradiction; if the minimizer were to touch the boundary, one could explicitly construct a test vector field such that the first variation of the minimizer in the direction of this vector field would be negative, which is impossible (cf. [73, Theorems 1 & 5], [74, Theorem 7.1] and [47, Proposition 3.3]).

The second step aims to reduce the strong maximum principle to the weak maximum principle. Again, this is done by contradiction; if the minimizer were to touch the boundary at a point p but does not coincide with the boundary in any open neighborhood of p, one could use the implicit function theorem for differentiable maps between Banach spaces to find, in a small enough neighborhood of p, a smooth hypersurface with the strict mean curvature barrier condition, which touches the minimizer as well. This, of course, is impossible by the weak maximum principle.

The main source for this part of the argument is [65, Step 1, p. 687]. In fact, it seems to be accepted in the literature that once the weak maximum principle is established (cf. [73, Theorems 1 & 5], [74, Theorem 7.1]) the corresponding strong maximum principles (cf. [73, Theorems 4 & 7], [74, Theorem 7.3]) can be obtained by just repeating the argument in [65, Step 1, p. 687] involving the implicit function theorem with slight modifications. For the convenience of the reader, we will give more details.

To prove that a connected component of $\partial\Omega$ can not touch a connected component of $\partial_- X$, unless they coincide, we follow this two step procedure.

For the weak maximum principle we assume that there is a point $p \in \partial_- X$ with $H(\partial_- X, g)(p) > \eta > -h$. By [73, Theorem 2] there is a compactly supported vector field V on X_δ such that $V(p)$ is a nonzero normal to $\partial_- X$ and

$$\int_{\partial\Omega} \operatorname{div}_{\partial\Omega} V d\mathcal{H}^{n-1} + \int_{\partial\Omega} \eta |V| d\mathcal{H}^{n-1} \leq 0.$$

If we choose the support of V small enough and assume that $\partial\Omega$ touches $\partial_- X$ at p ie $p \in \partial\Omega$ and the normal vectors coincide, then

$$\begin{aligned} 0 &= \int_{\partial\Omega} \operatorname{div}_{\partial\Omega} V d\mathcal{H}^{n-1} - \int_{\partial\Omega} hg(V,\nu) d\mathcal{H}^{n-1} \\ &< \int_{\partial\Omega} \operatorname{div}_{\partial\Omega} V d\mathcal{H}^{n-1} + \int_{\partial^*\Omega} \eta |V| d\mathcal{H}^{n-1} \leq 0, \end{aligned}$$

which is a contradiction. Hence $p \notin \partial\Omega$ and the weak maximum principle is established.

For the strong maximum principle we now follow the ideas from [65, Step 1, p. 687, Additional Remarks pp. 690-691]. Furthermore, we draw some inspiration from [47, Lemma 3.4].

We will assume that $\partial\Omega$ is connected. Otherwise we could treat each connected component seperately. Let $p \in \partial_- X$ be arbitrary and assume that $\partial\Omega$ touches $\partial_- X$ at p but does not coincide with $\partial_- X$ in any neighborhood of p.

For $R > 0$ let $B^n_R(0)$ be the R-ball around the origin in $T_p X_\delta$. Our convention is that $T_p\partial_- X \subset T_p X$ corresponds to the plane $x_n = 0$ and the upper half space are the directions which point into X. For R small enough the exponential map at p restricted to $B^n_R(0)$ is a diffeomorphism onto its image in X_δ. We pull back the metric via this diffeomorphism and denote the resulting Riemannian manifold by $(B^n_R(0), g)$. For R small enough and some even smaller $0 < r << R$ the intersection of some neighborhood of p in $\partial_- X$ with the image of $B^n_R(0)$ in X_δ, corresponds via exponential map to the graph of a function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ restricted to $B^{n-1}_r(0) = B^n_r(0) \cap T_p\partial_- X$ with $f(0) = 0$ and such that the derivative of f vanishes at the origin.

For some fixed $R > 0$ and varying $0 < r << R$, we consider the rescaled manifold $(B^n_R(0), r^{-2}g)$ and within it the 10-ball

$$(D^n_{10}, g_r) := \Big\{x \in (B^n_R(0), r^{-2}g) \text{ s.t. } \operatorname{dist}(x, 0) \leq 10\Big\}$$

with the restricted metric. Of course this is just extra notation and (D^n_{10}, g_r) is nothing but $(B^n_{10r}(0), r^{-2}g)$. Within (D^n_{10}, g_r) we consider

$$(D^{n-1}_1, g_r) := \Big\{x \in (B^n_R(0), r^{-2}g) \text{ s.t. } x_n = 0; \operatorname{dist}(x, 0) \leq 1\Big\}$$

and for r small enough we know that a neighborhood of p in $\partial_- X$ corresponds to a graph of a function $f_r(x) := f(rx)$ over D^{n-1}_1 within D^n_{10}.

We proceed as in [71, Appendix] or [47, Lemma 3.4] and define a map:

$$q : \mathbb{R} \times \mathbb{R} \times C^{2,\alpha}(D^{n-1}_1) \times C^{2,\alpha}_0(D^{n-1}_1) \to C^{0,\alpha}(D^{n-1}_1)$$

by $q(r, t, w, u) = H\left(\operatorname{graph}(f_r + u + w + t), g_r\right) + r(h - s)$, where $s \in (-\varepsilon, \varepsilon)$ for some small $\varepsilon > 0$, which we will choose later. It is verified in [72, Theorem in Section 1.3, Appendix] that q is indeed a C^1-map.

Furthermore, since (D^n_{10}, g_r) converges to a euclidean 10-ball and f_r to the zero function as $r \to 0$, we have the following linearized operator:

$$Dq_{(0,t,0,0)}(0, 0, 0, u) = -\Delta u,$$

which has trivial kernel in $C^{2,\alpha}(D^{n-1}_1)$ and restricts to an isomorphism of Banach spaces $C^{2,\alpha}(D_1) \to C^{0,\alpha}(D_1)$ (the Poisson-problem for the unit ball in $\mathbb{R}^{n-1}$ is uniquely solvable).

By the implicit function theorem we find, for any choice of r and w small enough (ie $r \in (-\varepsilon^s_1, \varepsilon^s_1)$ and $\|w\|_{2,\alpha} < \varepsilon^s_2$ for ε^s_1, ε^s_2 small enough, depending on s) and each $t \in [-\frac{1}{2}, \frac{1}{2}]$ a function $u_{s,r,t,w} \in C^{2,\alpha}_0(D^{n-1}_1)$ such that

$$H(\operatorname{graph}(f_r + u_{s,r,t,w} + w + t), g_r)) + r(h - s) = 0.$$

For fixed s, r and w and varying t, the graphs of $f_r + u_{s,r,t,w} + w + t$ foliate a neighborhood of p. We still have the freedom to choose $\varepsilon > 0$ such that $s \in (-\varepsilon, \varepsilon)$. As it is pointed out in [71, Note, p. 254], we can rid ourselves of the dependency of ε^s_i on s by choosing ε small enough ie we find uniform bounds ε_1 and ε_2 for the sizes of r resp. w which work for all $s \in (-\varepsilon, \varepsilon)$.

With these foliations at hand we can now tackle the strong maximum principle. We assumed for a contradiction that $\partial\Omega$ touches $\partial_- X$ at p but does not coincide with p in any open neighborhood of p. We choose $\varepsilon > 0$ small enough and $s \in (-\varepsilon, \varepsilon)$. For some $r < \varepsilon_1$, $\partial_- X$ corresponds to the graph $f : \mathbb{R}^{n-1} \to \mathbb{R}$ restricted to $B_r^{n-1}(0)$ and the set $\{x \in \partial B_r^{n-1}(0) : (x, u(x)) \notin \partial\Omega\}$ is non empty.

We rescale the metric and look at the situation over (D_1^{n-1}, g_r) (here $\partial_- X$ is the graph of f_r over D_1^{n-1}). We find a non-zero function $w \geq 0 \in C^{2,\alpha}(D_1^{n-1})$ with $\|w\|_{2,\alpha} < \varepsilon_2$ such that on ∂D_1^{n-1} it is supported in $\{x \in \partial D_1^{n-1} : (x, u(x)) \notin \partial\Omega\}$ and such that the graph of $f_r + w$ lies below $\partial\Omega$ (ie if $(x, y) \in \partial\Omega$ with $x \in \partial D_1^{n-1}$, then $f_r(x) + w(x) \leq y$).

For $t \in [-\frac{1}{2}, \frac{1}{2}]$ we find functions $v_{s,r,t,w} = f_r + u_{s,r,t,w} + w + t$ with $u_{s,r,t,w} \in C_0^{2,\alpha}(D_1^{n-1})$ whose graphs foliate a neighborhood of the origin with

$$H(\mathrm{graph}(v_{s,r,t,w}), g_r) = H(\mathrm{graph}(f_r + u_{s,r,t,w} + w + t), g_r)) = -r(h - s).$$

From now on we keep r and w fixed and just vary s and t. Hence we denote $v_{s,t} = v_{s,r,t,w}$.

We rescale back and work over $(B_r^{n-1}(0), g)$. The $v_{s,t}$ correspond to functions on $B_r^{n-1}(0)$. We will denote them by $v_{s,t}$ as well. For a fixed $s \in (-\varepsilon, \varepsilon)$ and $t \in [-\frac{r}{2}, \frac{r}{2}]$ the $v_{s,t}$ foliate a neighborhood of the origin and $H(\mathrm{graph}(v_{s,t}), g) = -h + s$.

We claim that $v_{0,0}(0) > f(0) = 0$. Indeed, on $\partial B_r^{n-1}(0)$, we have $v_{0,0} = f + w \geq f$ and the inequality is strict at some points. Furthermore $H(\mathrm{graph}(v_{s,t}), g) = -h \leq H(\mathrm{graph}(f), g)$. As in the proof of the first claim $L(f - v_{0,0}) \geq 0$ for some positive definite second order elliptic operator L with smooth coefficients. On the other hand $f - v_{0,0}$ is ≤ 0 on $\partial B_r^{n-1}(0)$ and the inequality is strict at some points. By the Hopf maximum principle [21, Theorem 3.5] we see that $f - v_{0,0}$ can not attain a maximum at the origin. Hence $f(0) - v_{0,0}(0) < 0$ ie $0 = f(0) < v_{0,0}(0)$. Therefore we can find some $s_0 > 0$ small enough such that $v_{s,0}(0) > 0$.

Now let t_0 be the smallest value of t such that the graph of v_{s_0,t_0} intersects $\partial\Omega$. Since $\partial\Omega$ contains the origin and $v_{s_0,0}(0) > 0$ it follows that t_0 is negative. It follows that the graph of v_{s_0,t_0} touches $\partial\Omega$, but since $v_{s_0,t_0} = f + w + t_0$ on $\partial B_r^{n-1}(0)$ we see that $\partial\Omega$ touches the graph of v_{s_0,t_0} in the interior of $B_r^{n-1}(0)$. On the other hand $H(\mathrm{graph}(v_{s_0,t_0}), g) = -h + s_0$. This contradicts the weak maximum principle and therefore concludes the proof of Proposition 2.43. □

Lemma 2.44. *If h is constant, $n \leq 7$ and $H(\partial_\pm X) \geq \pm h$ on $\partial_\pm X$, there is a smooth Caccioppoli set $\Omega \in \mathcal{C}(X_\delta)$, with $\mathcal{A}_h(\Omega) = \mathcal{I}_\delta$.*

Proof. Since $\mathrm{vol}(X_\delta, g_\delta) < \mathrm{vol}(X, g) + \delta$ the $\mathcal{A}_h$-functional is bounded from below on $\mathcal{C}(X_\delta)$. By compactness for Caccioppoli sets there is a minimizer $\Omega \in \mathcal{C}(X_\delta)$. By Proposition 2.43 any connected component of $\partial\Omega$ is either contained in $\mathring{X}$ and hence smooth by the regularity theorem [78, Theorem 2.2] or agrees with a connected component of $\partial_- X$ resp. $\partial_+ X$. □

Let Ω be the minimizer from Lemma 2.44 and $\Sigma = \partial\Omega$. It is important to note that Ω is only stationary and stable for variations which preserve X which is the case if and only if the variation vector field has nonnegative scalar product with the interior normal vector fields to $\partial_\pm X$.

Let $\Sigma_0 \subset \Sigma$ be a connected component. If $\Sigma_0 \subset \mathring{X}$ all variation vector fields V_ψ are admissible and we conclude $H(\Sigma_0) = h\big|_{\Sigma_0}$ by the first variation formula. By the second variation formula and stability (2.3.4) holds for all $\psi \in C^\infty(\Sigma_0)$.

If Σ_0 agrees with a component of $\partial_- X$ (the case $\Sigma_0 \subset \partial_+ X$ follows in analogous fashion), we only consider variation vector fields V_ψ with $\psi \geq 0$. By the first variation formula

$$\int_{\hat{\Sigma}} (H - h)\psi d\mathcal{H}^{n-1} \geq 0$$

for all nonnegative $\psi \in C^\infty(\Sigma_0)$.

Since $(H - h)$ is nonpositive on $\partial_- X$ by assumption (remember Remark 2.38 ie $H(\Sigma_0) = -H(\partial_- X)$) this implies $H(\Sigma_0) = h\big|_{\Sigma_0}$. By stability and the second variation formula (2.3.4) holds for all $\psi \in C^\infty(\Sigma_0)$ with $\psi \geq 0$. Since the first eigenfunction of the operator

$$-\Delta_{\Sigma_0} + \frac{1}{2}\operatorname{scal}(\Sigma_0, g) - \frac{1}{2}(\operatorname{scal}(X, g) + \frac{n}{n-1}h^2 + 2g(\nabla_X h, \nu))$$

does not change sign (follows from elliptic regularity and the Hopf maximum principle), this implies that the operator is nonnegative.

Remark 2.45. With Lemma 2.44 and the argument above we can prove Proposition 2.30 for constant h_φ with the weakened boundary condition $H(\partial_\pm X) \geq \pm h$.

2.3.2 Warped μ-Bubbles

The following version of μ-bubbles was introduced in [16, Section 3]. The results of this subsection will be used exclusively in Section 2.5.2.

Let (X, g) be an oriented Riemannian band. Let $u > 0$ be a smooth function on X and h be a smooth function on $\mathring{X}$ respectively X. We fix a Caccioppoli set Ω_0 with smooth boundary, which contains an open neighborhood of $\partial_- X$ and is disjoint from $\partial_+ X$. Hence all components of $\partial\Omega_0$, which are not part of $\partial_- X$ are contained in $\mathring{X}$. We consider

$$\mathcal{A}_h^u(\hat{\Omega}) = \int_{\partial^*\hat{\Omega}} u d\mathcal{H}^{n-1} - \int_X (\chi_{\hat{\Omega}} - \chi_{\Omega_0}) h u d\mathcal{H}^n$$

for all Caccioppoli sets $\hat{\Omega}$ with $\hat{\Omega}\Delta\Omega_0$ contained in the interior of X (this implies in particular, that $\hat{\Omega}$ contains an open neighborhood of $\partial_- X$ and is disjoint from $\partial_+ X$).

A Caccioppoli set, which is minimizing $\mathcal{A}_h^u$ in this class, is called a *warped μ-bubble*. The following existence and regularity result is [79, Proposition 2.1] and [16, Proposition 12].

Lemma 2.46. *If $n \leq 7$ and $h(x) \to \pm\infty$ as $x \to \partial_\mp X$, there exists a smooth minimizer Ω for $\mathcal{A}_h^u$, such that $\Omega\Delta\Omega_0$ is contained in the interior of X.*

The first and second variation formulas for the $\mathcal{A}_h^u$-functional are given in [16, Lemmas 13 & 14]. One can obtain the second variation formula from the first variation formula in the same way we obtained Lemma 2.41 from 2.40. In doing so we reorder the terms in a slightly different way than it is stated in [16, Lemma 14].

Lemma 2.47 (Warped first variation formula). *For any smooth function ψ on $\hat{\Sigma}$ let V_ψ be a vector field on X, which vanishes outside a small neighborhood of $\hat{\Sigma}$ and agrees with $\psi\nu$ on $\hat{\Sigma}$. If we denote by Φ_t the flow generated by V_ψ, then*

$$\frac{d}{dt}\Big|_{t=0} \mathcal{A}_h^u(\Phi_t(\hat{\Omega})) = \int_{\hat{\Sigma}} (Hu + g(\nabla_X u, \nu) - hu)\psi d\mathcal{H}^{n-1}. \tag{2.3.5}$$

Lemma 2.48 (Warped second variation formula). *For any smooth function ψ on $\hat{\Sigma}$ let V_ψ be a vector field on X, which vanishes outside a small neighborhood of $\hat{\Sigma}$ and agrees with $\psi\nu$ on $\hat{\Sigma}$. If we denote by Φ_t the flow generated by V_ψ, then*

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathcal{A}_h^u(\Phi_t(\hat{\Omega})) = \int_{\hat{\Sigma}} |\nabla_{\hat{\Sigma}}\psi|^2 u + (H^2 - Ric(\nu,\nu) - |A|^2)\psi^2 u + \\ + (2Hg(\nabla_X u, \nu) + \frac{d^2 u}{d\nu^2} - Hhu - g(\nabla_X(hu), \nu)\psi^2,$$

which is equal to

$$\int_{\hat{\Sigma}} |\nabla_{\hat{\Sigma}}\psi|^2 u - \frac{1}{2}(\mathrm{scal}(X,g) - \mathrm{scal}(\hat{\Sigma}, g) - H^2 + |A|^2)\psi^2 u \\ + (2Hg(\nabla_X u, \nu) + \frac{d^2 u}{d\nu^2} - Hhu - g(\nabla_X(hu), \nu))\psi^2. \tag{2.3.6}$$

2.4 Proof of the Comparison Principle

In this section we prove parts (1) and (2) of Theorem I using the techniques from Section 2.2 and 2.3. Regarding part (1) of Theorem I, we establish:

Theorem 2.49. *Let $n \leq 7$ and (X^n, g) be an oriented Riemannian band with the property that no closed embedded hypersurface Σ which separates $\partial_- X$ and $\partial_+ X$ has $-\Delta_\Sigma + \frac{1}{2}\mathrm{scal}(\Sigma, g) > 0$. Let (M, g_φ) be a model space over a scalar flat base with warping function $\varphi : [a,b] \to \mathbb{R}_+$. If φ is strictly* log-concave*,*

- ▷ $\mathrm{scal}(X, g) \geq \mathrm{scal}(M, g_\varphi)$,
- ▷ $H(\partial_\pm X, g) \geq H(\partial_\pm M, g_\varphi)$,

then $\mathrm{width}(X, g) \leq \mathrm{width}(M, g_\varphi)$.

Proof. If we assume for a contradiction that $\mathrm{width}(X,g) > \mathrm{width}(M, g_\varphi) = b - a$, there is a small $\varepsilon > 0$, such that $\mathrm{width}(X, g) > b - a + 2\varepsilon$. Let $(M_\varepsilon, g_\varphi^\varepsilon)$ be the model space

$$\left(N \times [a-\varepsilon, b+\varepsilon], \varphi^2 g_N + dt^2\right).$$

We compare (X, g) and $(M_\varepsilon, g_\varphi^\varepsilon)$. According to Lemma 2.35 there is a structural map $\phi : (X, g) \to [a-\varepsilon, b+\varepsilon]$ with $\mathrm{Lip}(\phi) < 1$. Since φ is strictly log-concave and $H(\partial_\pm X, g) \geq H(\partial_\pm M, g_\varphi)$ we have $H(\partial_\pm X, g) > H(M_\varepsilon, g_\varphi^\varepsilon)$. Proposition 2.30, together with Remark 2.36 and Remark 2.42, implies the existence of a hypersurface Σ, which separates $\partial_- X$ and $\partial_+ X$ and has $-\Delta_\Sigma + \frac{1}{2}\mathrm{scal}(\Sigma, g) > 0$. This is a contradiction. □

Theorem 2.49 implies part (1) of Theorem I with the help of the following classical result of Kazdan-Warner [44] and Schoen-Yau [64]:

Lemma 2.50. *Let $(\Sigma^{n\geq 2}, g)$ be a closed connected oriented manifold. If $-\Delta_\Sigma + \frac{1}{2}\operatorname{scal}(\Sigma, g)$ is positive, then Σ admits a metric with positive scalar curvature.*

Proof. The proof is standard so we only recall the main ideas. Since the operator is positive

$$\int_\Sigma -\psi\Delta_\Sigma\psi + \frac{1}{2}\operatorname{scal}(\Sigma, g)\psi^2 > 0$$

for all $\psi \in C^2(\Sigma)$. If $n = 2$ we choose $\psi \equiv 1$ and use Gauss-Bonnet to see that

$$0 < \int_\Sigma \frac{1}{2}\operatorname{scal}(\Sigma, g) = 2\pi\chi(\Sigma).$$

It follows that Σ is a 2-sphere and hence admits a metric with positive scalar curvature.

If $n \geq 3$, we consider the conformal Laplacian $L_g = -\Delta_\Sigma + \frac{n-2}{4(n-1)}\operatorname{scal}(\Sigma, g)$. It is easy to see that this operator is positive as well. Hence the first eigenvalue $\lambda_1(L_g)$ is positive. It follows from elliptic regularity and the strong maximum principle that the first eigenfunction $u \in C^\infty(\Sigma)$ can be chosen positive.

We then use this function for a conformal change of metric ie $\hat{g} = u^{\frac{4}{n-2}}g$. We conclude

$$\operatorname{scal}(\Sigma, \hat{g}) = u^{-\frac{n+2}{n-2}}\frac{4(n-1)}{n-2}L_g u > 0$$

using the standard formula for scalar curvature under a conformal change of metric. □

Regarding part (2) of Theorem I we establish:

Theorem 2.51. *Let $n \leq 7$ and (X^n, g) be an oriented Riemannian band with Property A. Let (M, g_φ) be a model space over a scalar flat base with warping function $\varphi : [a, b] \to \mathbb{R}_+$. If φ is* log-*affine,*

- ▷ $\operatorname{scal}(X, g) \geq \operatorname{scal}(M, g_\varphi)$,
- ▷ $H(\partial_\pm X, g) \geq H(\partial_\pm M, g_\varphi)$,

then (X, g) is isometric to a warped product

$$\left(\hat{N} \times [c, d], \varphi^2 g_{\hat{N}} + dt^2\right),$$

where $(\hat{N}, g_{\hat{N}})$ is a closed scalar flat Riemannian manifold.

Proof. The following proof is an adaptation of the rigidity arguments presented in [2, Section 2] and [79, Section 3]. Let $\phi : X \to [a, b]$ be a band map. According to Lemma 2.32 ϕ is structural. Following the proof Proposition 2.30, together with Remark 2.45, we see that there is a hypersurface Σ, which separates $\partial_- X$ and $\partial_+ X$ and has

$$-\Delta_\Sigma + \frac{1}{2}\operatorname{scal}(\Sigma, g) \geq \frac{1}{2}\left(\operatorname{scal}(X, g) + h^2 + |A|^2\right) \geq 0,$$

where $h = h_\varphi \circ \phi$. We start off by proving an infinitesimal splitting result.

Claim 1. [cf. [2, Proposition 2.2]] For any connected component $\Sigma_0 \subset \Sigma$ which does not admit a metric with positive scalar curvature, the following holds true:

- ▷ Σ_0 is umbilic; all principal curvatures of Σ_0 are equal to $\frac{h}{n-1}$,
- ▷ $\operatorname{scal}(\Sigma_0, g) = 0$ and $\operatorname{scal}(X, g) = \operatorname{scal}(M, g_\varphi)$ along Σ_0.

Proof of Claim. Let $\Sigma_0 \subset \Sigma$ be a connected component which does not admit a metric with positive scalar curvature. Considering Lemma 2.50, we conclude that the first eigenvalue of $-\Delta_{\Sigma_0} + \frac{1}{2}\operatorname{scal}(\Sigma_0, g)$ is equal to zero.

If w_1 is the corresponding positive first eigenfunction, then

$$\int_{\Sigma_0} \frac{1}{2}\left(\operatorname{scal}(X, g) + h^2 + |A|^2\right) w_1^2 = 0.$$

Consequently $\operatorname{scal}(X, g) + h^2 + |A|^2 = 0$ which is equivalent to $-h^2 - |A|^2 = \operatorname{scal}(X, g)$.

On the other hand $\operatorname{scal}(X, g) \geq \operatorname{scal}(M, g_\varphi) = -\frac{n}{n-1}h_\varphi^2 = -\frac{n}{n-1}h^2$ by (2.2.3). Since $|A|^2 \geq \frac{H^2}{n-1} = \frac{h^2}{n-1}$ we conclude that $\operatorname{scal}(X, g) = \operatorname{scal}(M, g_\varphi)$ along Σ_0 and $|A|^2 = \frac{H^2}{n-1} = \frac{h^2}{n-1}$.

At every point $p \in \Sigma_0$ the last equality forces A to be a diagonal matrix with all entries equal to $\frac{h}{n-1}$ with respect to any orthonormal basis at p ie Σ_0 is umbilic with all principal curvatures equal to $\frac{h}{n-1}$.

Regarding $\operatorname{scal}(\Sigma_0, g)$ we distinguish three cases: If $n = 2$, the term $\operatorname{scal}(\Sigma_0, g)$ does not appear. If $n = 3$, we choose $\psi \equiv 1$ in

$$\int_{\Sigma_0} |\nabla_{\Sigma_0}\psi|^2 + \frac{1}{2}\operatorname{scal}(\Sigma_0, g)\psi^2 \geq 0.$$

By Gauss-Bonnet Σ_0 is a torus and $\operatorname{scal}(\Sigma_0, g) = 0$.

If $n > 3$ we proceed as in [64, p. 166] and consider the first positive eigenfunction $w_2 \in C^\infty(\Sigma_0)$ corresponding to the first eigenvalue λ_0 of the conformal Laplacian

$$L_g = -\Delta_{\Sigma_0} + \frac{(n-3)}{4(n-2)}\operatorname{scal}(\Sigma_0, g).$$

If λ_0 were positive, one could use w_2 for a conformal change of metric which would result in a metric with positive scalar curvature on Σ_0 (compare the proof of Lemma 2.50). Since this is impossible we conclude that $\lambda_0 \leq 0$. Hence

$$\frac{2(n-2)}{n-3}\int_{\Sigma_0} |\nabla_{\Sigma_0} w_2|^2 = -\int_{\Sigma_0} \frac{1}{2}\operatorname{scal}(\Sigma_0, g) w_2^2 + \frac{2\lambda_0(n-2)}{n-3}\int_{\Sigma_0} w_2^2 \leq \int_{\Sigma_0} |\nabla_{\Sigma_0} w_2|^2.$$

Since $\frac{2(n-2)}{n-3} > 1$ we see that $\lambda_0 = 0$ and w_2 is a constant function. Consequently $\operatorname{scal}(\Sigma_0, g)$ is constant as well. Since the first eigenvalue of $-\Delta_{\Sigma_0} + \frac{1}{2}\operatorname{scal}(\Sigma_0, g)$ is equal to zero we conclude $\operatorname{scal}(\Sigma_0, g) = 0$.

Furthermore, we observe that the Jacobi operator associated to Σ_0 is

$$-\Delta_{\Sigma_0} - (\operatorname{Ric}(\nu, \nu) + |A|^2) = -\Delta_{\Sigma_0} - \frac{1}{2}(\operatorname{scal}(X, g) - \operatorname{scal}(\Sigma_0, g) + H^2 + |A|^2) = -\Delta_{\Sigma_0}.$$

□

Before we continue we point out the following topological fact:

Claim 2. There is a component $\Sigma_0 \subset \Sigma$ such that $\Sigma_0 \times [-1,1]$ has Property A.

Proof of Claim. If this were not the case, we could replace each component of Σ inside its tubular neighborhood by a closed embedded hypersurface which admits a metric with positive scalar curvature. The union of all these components would then be a closed embedded hypersurface which separates $\partial_- X$ and $\partial_+ X$ and admits a metric with positive scalar curvature. This is impossible, since X is assumed to have Property A. □

Next, we use the infinitesimal splitting to establish that the desired warped product splitting of (X, g) can be found locally around suitable components of Σ.

Claim 3. [cf. [2, Theorem 2.3]] There is a connected component $\Sigma_0 \subset \Sigma$ and a tubular neighborhood U of Σ_0 which is isometric to the warped product

$$\left(\Sigma_0 \times (-\varepsilon, \varepsilon), \exp(2s\frac{h}{n-1})g_0 + ds^2\right)$$

for some small $\varepsilon > 0$, where g_0 denotes the metric on Σ_0 induced by g.

Proof of Claim. Since X has Property A, there is a connected component $\Sigma_0 \subset \Sigma$ which does not admit a metric with positive scalar curvature and such that $\Sigma_0 \times [-1,1]$ has Property A (see Claim 2). The conclusions of Claim 1 hold for Σ_0.

We follow the proof of [2, Theorem 2.3] respectively [79, Lemma 3.4] and use the implicit function theorem to show that there is a foliation $\{\Sigma_s\}_{-\delta<s<\delta}$ around Σ_0 such that

- ▷ each Σ_s is a graph over Σ_0 with graph function u_s along the outward unit normal field ν with

$$\frac{d}{ds}\Big|_{s=0} u_s = 1 \text{ and } \int_{\Sigma_0} u_s d\mathcal{H}^{n-1} = s; \tag{2.4.1}$$

- ▷ $H_s = H(\Sigma_s) - h$ is a constant function on Σ_s.

For $s \in [0, \delta)$ let Ω_s be the union of Ω and the region bounded by Σ_0 and Σ_s. By choosing δ small enough, we can guarantee that the region bounded by Σ_0 and Σ_s does not intersect $\Sigma \backslash \Sigma_0$.

Since Ω minimizes the $\mathcal{A}_h$ functional, there is a $0 < \delta' \leq \delta$ such that $H_s - h \geq 0$ for all $s \in [0, \delta')$. There are two possibilities: either $H_s - h > 0$ for some $s \in [0, \delta')$ or $H_s - h = 0$ for all $s \in [0, \delta')$.

In the first case we choose a constant $0 \leq h < \hat{h} < H_s$ and consider the $\mathcal{A}_{\hat{h}}$-functional on the Riemannian band $\hat{X}$ bounded by Σ_0 and Σ_s, which is diffeomorphic to $\Sigma_0 \times [-1,1]$ and hence has Property A. By Lemma 2.39 there is a smooth hypersurface $\hat{\Sigma}$ which separates Σ_0 and Σ_s with $H(\hat{\Sigma}) = \hat{h}$ and by stability and the second variation formula we see

$$-\Delta_{\hat{\Sigma}} + \frac{1}{2}\operatorname{scal}(\hat{\Sigma}, g) \geq \frac{1}{2}\left(\operatorname{scal}(X, g) + \frac{n}{n-1}\hat{h}^2\right) > 0.$$

By Lemma 2.50, $\hat{\Sigma}$ admits a metric with positive scalar curvature. Since $\hat{X}$ has Property A, this is a contradiction.

It follows that $H_s - h = 0$ for all $s \in [0, \delta')$. We show that Ω_s is a minimizer for the $\mathcal{A}_h$-functional as well:

$$\mathcal{A}_h(\Omega_s) - \mathcal{A}_h(\Omega) = \mathcal{H}^{n-1}(\Sigma_s) - \mathcal{H}^{n-1}(\Sigma) - \int_{\hat{X}} h d\mathcal{H}^n = \int_0^s \int_{\Sigma_t} f_t(H_t - h) d\mathcal{H}^{n-1} dt = 0,$$

where $f_t = g(\frac{d}{dt}u_t, \nu_t)$ is the lapse function of Σ_t moving along the foliation.

As Ω_s is a minimizer we can apply Claim 1 to $\partial\Omega_s$. Since Σ_s is diffeomorphic to Σ_0 and does not admit a metric with positive scalar curvature, we conclude that $\mathrm{scal}(X, g) = \mathrm{scal}(M, g_\varphi)$ along Σ_s and

$$|A_s|^2 = \frac{H_s^2}{n-1} = \frac{h^2}{n-1}$$

ie Σ_s is umbilic and all principal curvatures of Σ_s are equal to $\frac{h}{n-1}$. Furthermore $\mathrm{scal}(\Sigma_s, g) = 0$ and the Jacobi operator of Σ_s is $-\Delta_{\Sigma_s}$.

By an analogous argument for negative values of s, we see that there is some $0 < \delta'' < \delta$ such that the above holds true for all Σ_s with $s \in (-\delta'', 0]$. We choose some small $0 < \varepsilon \leq \min\{\delta', \delta''\}$.

With the foliation we can write the metric as $g = g_s + f_s^2 ds^2$, where $g_s = g\big|_{\Sigma_s}$. As the lapse function f_s satisfies the Jacobi equation [41, Equation (1.2)], which reduces to $\Delta_{\Sigma_s} f_s = 0$, we see that f_s is constant. By rescaling the s-coordinate, if necessary, we can assume $f_s = 1$ and hence Σ_s is s-equidistant to Σ_0. Since Σ_s is umbilic for all $s \in (-\varepsilon, \varepsilon)$ we conclude that the map

$$S : \left(\Sigma_0 \times (-\varepsilon, \varepsilon), \exp(2s\frac{h}{n-1})g_0 + ds^2\right) \to (X, g)$$

which is defined by

$$(p, s) \mapsto \exp_p(s\nu_0)$$

is an isometry onto its image, which we denote by U. □

Let Σ_0 be the component we get from Claim 3. We want to show that there is a maximal interval $[c, d]$ containing $(-\varepsilon, \varepsilon)$ such that

$$S : \left(\Sigma_0 \times [c, d], \exp(2s\frac{h}{n-1})g_0 + ds^2\right) \to (X, g)$$

is an isometry. Note that if the map S is defined on $\Sigma_0 \times (c', d')$ it is also defined on $\Sigma_0 \times [c', d']$ since the normal geodesic can always be extended to times c' resp. d'.

Claim 4. Assume that for some real number $0 < d'$ the map

$$S : \left(\Sigma_0 \times [0, d'), \exp(2s\frac{h}{n-1})g_0 + ds^2\right) \to (X, g)$$

which is defined by

$$(p, s) \mapsto \exp_p(s\nu_0)$$

is an isometry onto its image. Assume further that $S\left(\Sigma_0 \times [0, d')\right)$ does not intersect $\Sigma \backslash \Sigma_0$. Then the following holds true:

▷ $S(\Sigma_0 \times \{d'\})$ does not intersect $\Sigma\backslash\Sigma_0$,

▷ if $S(\Sigma_0 \times \{d'\})$ intersects $\partial_+ X$, it coincides with a component of $\partial_+ X$,

▷ $S : \left(\Sigma_0 \times [0,d'], \exp(2s\frac{h}{n-1})g_0 + ds^2\right) \to (X,g)$ is an isometry onto its image,

Proof of Claim. Consider an increasing sequence $s_k \to d'$ in $[0,d')$ and the corresponding sequence $\Sigma_{s_k} = S(\Sigma_0 \times \{s_k\})$ of embeddings of Σ_0. We denote by Ω_{s_k} the union of Ω and the region bounded by Σ_0 and Σ_{s_k}. As we have seen before Ω_{s_k} minimizes the $\mathcal{A}_h$-functional for any $k \in \mathbb{N}$.

Furthermore $|A_{s_k}|^2 = \frac{h^2}{n-1} = const$ for all k and $\mathcal{H}^{n-1}(\Sigma_{s_k}) \leq \mathcal{I} + h\text{vol}(X,g) < \infty$. By [7, Theorem 1.1] (and the comments thereafter) and the compactness theorem [77, Theorem 2.11] for stable CMC-hypersurface, the limit $S(\Sigma_0 \times \{d'\})$ of these embeddings is an immersion. In fact $S(\Sigma_0 \times \{d'\})$ is an *almost embedded* [77, Definition 2.3] stable CMC-surface with mean curvature equal to h.

To show that $S(\Sigma_0 \times \{d'\})$, which we will denote by $\Sigma_{d'}$, does not intersect $\Sigma\backslash\Sigma_0$, we distinguish two cases. If $\Sigma_{d'}$ is embedded and coincides with a component Σ' of $\Sigma\backslash\Sigma_0$ (with the opposite orientation), then $h = 0$ (the normal vector fields to Σ' and $\Sigma_{d'}$ are inverse to each other and both have constant mean curvature equal to h). If we consider the minimizing sequence Ω_{s_k} for the $\mathcal{A}_h$-functional, we see that it converges to an open set Ω' with boundary $\Sigma\backslash(\Sigma_0 \cup \Sigma')$ (Σ' and $\Sigma_{d'}$ cancel each other out). Hence $\mathcal{A}_h(\Omega') < \mathcal{A}_h(\Omega)$ which contradicts the minimality of Ω.

If $\Sigma_{d'}$ intersects a component of $\Sigma\backslash\Sigma_0$ but they do not coincide, then the minimizing sequence Ω_{s_k} converges to an open set which is minimizing the $\mathcal{A}_h$-functional but has non-smooth boundary. This contradicts the regularity result [78, Theorem 2.2]. Hence $\Sigma_{d'}$ does not intersect $\Sigma\backslash\Sigma_0$.

We denote by $\Omega_{d'}$, the region in X which is bounded by $\Sigma_{d'}$ together with $\Sigma\backslash\Sigma_0$ and $\partial_- X$. Of course $\Omega_{d'}$ is a minimizer for the $\mathcal{A}_h$-functional, as the limit of the minimizing sequence Ω_{s_k}. If $\Sigma_{d'}$ touches $\partial_+ X$ it coincides with a connected component of $\partial_+ X$ by the strong maximum principle Proposition 2.43. If $\Sigma_{d'}$ does not touch $\partial_+ X$ it is embedded as a boundary component of the minimizer $\Omega_{d'}$ by the regularity result [78, Theorem 2.2]. In both cases $\Sigma_{d'}$ is embedded and the smooth limit of the embeddings Σ_{s_k}.

It is important to point out that, since the image $S(\Sigma_0 \times (0,d'])$ does not intersect $\Sigma\backslash\Sigma_0$, it is contained in $X\backslash\Omega$. Hence $\Sigma_{d'}$ is disjoint from Σ_0 (the normal geodesics to Σ_0 do not close up).

We conclude that $S : \left(\Sigma_0 \times [0,d'], \exp(2s\frac{h}{n-1})g_0 + ds^2\right) \to (X,g)$ is an isometry (and not just a local isometry) onto its image. □

Let $s_{\max} \geq 0$ be maximal with the property that

$$S : \left(\Sigma_0 \times [0,s_{\max}), \exp(2s\frac{h}{n-1})g_0 + ds^2\right) \to (X,g)$$

is an isometry onto its image and that this image does not intersect $\Sigma\backslash\Sigma_0$. We know that $s_{\max} > 0$. We show that $s_{\max}$ is finite and that $S(\Sigma_0 \times \{s_{\max}\})$ touches $\partial_+ X$.

If $s_{\max} = \infty$, then a connected component of $X\backslash\Sigma$ is isometric to

$$\left(\Sigma_0 \times [0,\infty), \exp(2s\frac{h}{n-1})g_0 + ds^2\right)$$

which is impossible since X is compact.

Hence $s_{\max}$ is finite. By Claim 4 the map

$$S : \left(\Sigma_0 \times [0, s_{\max}], \exp(2s\frac{h}{n-1})g_0 + ds^2\right) \to (X, g)$$

is an isometry onto its image. We denote $\Sigma_{s_{\max}} := S(\Sigma_0 \times \{s_{\max}\})$. As before, let $\Omega_{s_{\max}}$ be the union of Ω and $S(\Sigma \times [0, s_{\max}])$. Since $\Omega_{s_{\max}}$ minimizes the $\mathcal{A}_h$-functional, the metric splits infinitesimally around $\Sigma_{s_{\max}}$ by Claim 1.

Assume for a contradiction that $\Sigma_{s_{\max}}$ does not touch $\partial_+ X$. Then $\Sigma_{s_{\max}}$ is contained in $\mathring{X}$ (remember that $\Sigma_{s_{\max}}$ can not touch $\partial_- X$ since the normal geodesics never cross $\Sigma \backslash \Sigma_0$) and therefore a tubular neighborhood of $\Sigma_{s_{\max}}$ is contained in $\mathring{X}$.

Since $\Sigma_0 \times [-1, 1]$ has Property *A* we can repeat the proof of Claim 3 to show that the metric splits locally around $\Sigma_{s_{\max}}$. Thus, there is some $q > 0$ such that $S : \left(\Sigma_0 \times [0, s_{\max} + q), \exp(2s\frac{h}{n-1})g_0 + ds^2\right) \to (X, g)$ is an isometry onto its image and such that the image does not intersect $\Sigma \backslash \Sigma_0$. This contradicts the assumed maximality of $s_{\max}$.

We conclude that $S(\Sigma \times \{d\})$ touches $\partial_+ X$. In this case we have already seen in Claim 4 that $S(\Sigma \times \{d\})$ coincides with a component of $\partial_+ X$ by the strong maximum principle Proposition 2.43. We define $d := s_{\max}$.

Using a version of Claim 4 for negative values of s, and by an analogous argument involving a minimal value $s_{\min}$, we see that for $c := s_{\min}$ the map

$$S : \left(\Sigma_0 \times [c, d], \exp(2s\frac{h}{n-1})g_0 + ds^2\right) \to (X, g)$$

is an isometry onto its image and that $S(\Sigma_0 \times \{c\})$ resp. $S(\Sigma_0 \times \{d\})$ are components of $\partial_- X$ resp. $\partial_+ X$. Hence the image is open and closed in X. Since X is connected, we conclude that (X, g) is isometric to

$$\left(\Sigma_0 \times [c, d], \exp(2s\frac{h}{n-1})g_0 + ds^2\right),$$

where g_0 is a scalar flat metric on Σ_0.

Since $(n-1)(\log \varphi(s))' = h$, we see that

$$\varphi(s) = \exp\left(s\frac{h}{n-1} + C\right) = \exp\left(s\frac{h}{n-1}\right)\exp(C)$$

for some constant $C \in \mathbb{R}$. We define $(\hat{N}, g_{\hat{N}})$ to be $(\Sigma_0, \exp(-2C)g_0)$. □

Theroem 2.51 implies part (2) of Theorem I. The last ingredient we need is the following observation that appears in [28, Theorem 2.3] and is attributed to J. P. Bourguignon:

Proposition 2.52. *Let Σ be closed connected Riemannian manifold. If Σ does not admit a metric with positive scalar curvature and g is a Riemannian metric on Σ with* $\mathrm{scal}(\Sigma, g) \geq 0$*, then (Σ, g) is Ricci flat.*

Remark 2.53. The rigidity analysis in the Proof of Theorem 2.51 could be adapted to prove rigidity of part (1) of Theorem I in case $\mathrm{width}(X, g) = \mathrm{width}(M, g_\varphi)$ if there was a way to guarantee the existence of a μ-bubble in this situation. This is connected to Remark 2.2 and Remark 2.31.

Remark 2.54. As we already alluded to in Remarks 1.18 and 2.4, Theorem 2.49 as well as Theorem 2.51 apply to any oriented band X in dimension $n = 2$, as for any closed hypersurface Σ (a collection of circles), which separates $\partial_\pm X$, the operator $-\Delta_\Sigma + \frac{1}{2}\,\mathrm{scal}(\Sigma, g) = -\Delta_\Sigma$ has first eigenvalue equal to zero.

2.5 Proof of the Topological Results

2.5.1 Separating Hypersurfaces

Lemma 2.55. *Let Y^{n-1} be a closed connected oriented manifold and $X = Y \times [-1, 1]$. If Σ is a closed embedded hypersurface in X, which separates $\partial_- X$ and $\partial_+ X$, there is one connected component Σ_0 of Σ that separates $\partial_- X$ and $\partial_+ X$.*

Proof. Without loss of generality Σ can be assumed to be oriented, since nonorientable components are non-separating. Furthermore we can assume that $\Sigma \subset \mathring{X}$ (otherwise we isotope Σ by flowing along the interior unit normal vector field to ∂X for a short time).

The relative homology group $H_1(X, \partial X)$ is generated by paths $\gamma : [0, 1] \to X$ with $\gamma(0) \in \partial_- X$ and $\gamma(1) \in \partial_+ X$. Since the hypersurface Σ separates $\partial_- X$ and $\partial_+ X$ it has nonzero algebraic intersection with every such path γ. It follows by Lefschetz duality that $[\Sigma] \neq 0 \in H_{n-1}(X) \cong H_{n-1}(Y) = \mathbb{Z}$.

Of course $[\Sigma]$ is nothing but $[\Sigma_0] + \ldots + [\Sigma_m]$, where Σ_i are the connected components of Σ. Since $[\Sigma] \neq 0$ it follows that $[\Sigma_i] \neq 0$ for some $i \in \{0, \ldots, m\}$ (w.l.o.g. we can assume $[\Sigma_0] \neq 0$). Going back, by Lefschetz duality, Σ_0 has nonzero algebraic intersection with any path γ which connects $\partial_- X$ and $\partial_+ X$ and therefore it separates $\partial_- X$ and $\partial_+ X$. □

Lemma 2.56. *Let $\Sigma \subset X$ be a separating hypersurface in a band X. Then there exists a union of components of Σ which is a properly separating hypersurface in X.*

Proof. Suppose that Σ is a separating hypersurface that contains a component not connected to both $\partial_- X$ and $\partial_+ X$ inside $X \setminus \Sigma$. Then the hypersurface Σ' obtained from Σ by deleting this component is still a separating hypersurface. This shows that there is a minimal collection of components of Σ such that its union is still separating yields the desired properly separating hypersurface. □

Lemma 2.57. *Let X^n be a connected oriented band and $X' = Y^{n-1} \times [-1, 1]$, where Y is a closed connected oriented manifold. Let $f : X \to X'$ be a band map with $\deg(f) = d \neq 0$ and Σ^{n-1} be a closed embedded hypersurface in X, which separates $\partial_- X$ and $\partial_+ X$. Then there is one connected component Σ_0 of Σ such that the map $(\mathrm{pr}_Y \circ f) : \Sigma_0 \to Y$ has nonzero degree.*

Proof. Without loss of generality Σ can be assumed to be oriented, since nonorientable components are non-separating. Furthermore we can assume that $\Sigma \subset \mathring{X}$ (otherwise we isotope Σ by flowing along the interior unit normal vector field to ∂X for a short time).

By Lemma 2.56 there is a union of components of Σ which is a properly separating hypersurface. We denote this union of components by Σ'. By construction every path $\gamma : [0, 1] \to X$ with $\gamma(0) \in \partial_- X$ and $\gamma(1) \in \partial_+ X$ has algebraic intersection number equal to one with Σ'.

Since f is a band map $f \circ \gamma$ connects $\partial_- X'$ and $\partial_+ X'$. It follows that $[\Sigma']$ is Lefschetz dual to $f^*\alpha$, where α is the generator of $H^1(X', \partial X') \cong \mathbb{Z}$.

Consider the diagram:

$$
\begin{array}{ccccc}
H^1(X, \partial X; \mathbb{Z}) & \xrightarrow[\cong]{\cap [X, \partial X]} & H_{n-1}(X; \mathbb{Z}) & & \\
\big\uparrow {\scriptstyle f^*} & & \big\downarrow {\scriptstyle f_*} & & \\
H^1(X', \partial X'; \mathbb{Z}) & \xrightarrow{\cap d[X', \partial X']} & H_{n-1}(X'; \mathbb{Z}) & \xrightarrow{\mathrm{pr}_{Y*}} & H_{n-1}(Y; \mathbb{Z}).
\end{array}
$$

We conclude that $(\mathrm{pr} \circ f)_*[\Sigma'] = d[Y]$. Hence there is one connected component Σ_0 of Σ' with $(\mathrm{pr} \circ f)_*[\Sigma'] \neq 0$. By construction Σ_0 is also a component of Σ. □

Proposition 2.58. *Let Y be a closed connected oriented manifold of dimension $n - 1 \geq 5$ and $X = Y \times [-1, 1]$. Let Σ_0 be a closed connected oriented hypersurface separating $\partial_- X$ and $\partial_+ X$ in X. If Σ_0 admits a metric with positive scalar curvature, then so does Y.*

Proof. The proof uses standard results and ideas from high dimensional topology. We can assume that $\Sigma_0 \subset \mathring{X}$ (otherwise we isotope Σ_0 by flowing along the interior normal vector field to ∂X for a short time).

We want to see that Y can be obtained from Σ_0 by a finite sequence of surgeries in codimension ≥ 3 and hence, by the well known argument of Gromov and Lawson [29, Theorem A], a positive scalar curvature metric on Σ_0 can be transported to Y. See [18] for full details of the proof of [29, Theorem A].

We denote by W the connected component of $X \backslash \Sigma_0$ which contains $\partial_- X$. Then W is a cobordism $W : Y \rightsquigarrow \Sigma_0$. We restrict the projection $X \to Y$ to W and obtain a retract map $r : W \to Y$.

Claim. The cobordism $W : Y \rightsquigarrow \Sigma_0$ and the retract map $r : W \to Y$ can be improved via surgery in the interior of W to a cobordism $W_2 : Y \rightsquigarrow \Sigma_0$ with a retract map $r_2 : W_2 \to Y$, which is 3-connected. The inclusion $\iota : Y \hookrightarrow W_2$ will be 2-connected since $\iota \circ r_2 = id_M$.

Proof of Claim. If $\nu(Y)$ denotes the stable normal bundle of Y, there is a stable trivialization of $r^*\nu(Y) \oplus TW$. Since r is a retract map the induced map $\pi_1(r) : \pi_1(W) \to \pi_1(Y)$ is already surjective and its kernel is finitely generated as a normal subgroup of $\pi_1(W)$, since $\pi_1(W)$ is finitely generated and $\pi_1(Y)$ is finitely presented (see [62, Lemma 3.2]). Let α be a generator of $\ker(\pi_1(r))$. Since $1 < n/2$ we can represent α by an embedding $S^1 \hookrightarrow \mathring{W}$. Since W is oriented the normal bundle of this embedding is trivial and hence we can kill α by surgery in the interior of W. We obtain a cobordism $W_\alpha : Y \rightsquigarrow \Sigma_0$ and a retract map $r_\alpha : W_\alpha \to Y$. After repeating this step finitely many times we end up with $W_1 : Y \rightsquigarrow \Sigma_0$ and a retract map $r_1 : W_1 \to Y$, which is 2-connected.

Next we need to kill the kernel of $\pi_2(r_1) : \pi_2(W_1) \to \pi_2(Y)$. In order to do so one has to argue that this is possible with finitely many surgeries along elements of $\ker(\pi_2(r_1))$. We proceed similarly as in the proof of [62, Proposition 3.1], which in turn is based on [67, Lemma 5.6] and [69, Lemma 1.1]. Since Y and W_1 are compact manifolds, if one starts with a handle decomposition of W_1 relative to Y one can use handle cancellation [70] and the fact that $Y \hookrightarrow W_1$ induces an isomorphism on π_0

and π_1 to get rid of 0-handles and 1-handles. All the (finitely many) 2-handles in this new handlebody are attached to Y via contractible maps (otherwise they would kill elements in π_1). Hence the 2-skeleton $(W_1, Y)^{(2)}$ arising from this new handlebody is homotopy equivalent to $Y \vee (\bigvee_{j\in J} S^2)$. The 2-spheres in this wedge product finitely generate $\ker(\pi_2(r_1))$ as a $\mathbb{Z}[\pi_1(Y)]$-module over the common fundamental group $\pi_1(Y) = \pi_1(W_1)$.

Since $2 < n/2$ we can represent each of those generators by an embedding $f : S^2 \hookrightarrow \mathring{W}_1$. Since $r \circ f(S^2)$ is contractible, there is a map $g : D^3 \to Y$ such that the following diagram commutes:

$$\begin{array}{ccc} S^2 & \xrightarrow{i} & D^3 \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ W_1 & \xrightarrow{r_1} & Y. \end{array}$$

The stable trivialization of $r_1^*\nu(Y) \oplus TW_1$ induces a stable trivialization of $f^*r_1^*\nu(Y) \oplus f^*TW_1 = i^*g^*\nu(Y) \oplus f^*TW_1$. But $i^*g^*\nu(Y)$ is trivial since D^3 is contractible and hence $f^*TW_1 \cong \nu(S^2, W_1) \oplus TS^2$ is stably trivial. Since TS^2 is stably trivial it follows that $\nu(S^2, W_1)$ is stably trivial and since $2 < (n-1)/2$ we conclude that $\nu(S^2, W_1)$ is trivial.

Hence we can kill $\ker(\pi_2(r_1))$ in finitely many surgery steps. We end up with a cobordism $W_2 : Y \rightsquigarrow \Sigma$ and a retract map $r_2 : W_2 \to Y$ which is 3-connected. Consequently the inclusion $\iota : Y \hookrightarrow W_2$ is 2-connected. □

If we start with a handle decomposition of W_2 with respect to Y we can use handle cancellation [70] to get rid of all the 0-, 1- or 2-handles since the inclusion $\iota : Y \hookrightarrow W_2$ is 2-connected. Turning this upside down this handle decomposition can be interpreted as a handle decomposition of W_2 with respect to Σ_0. In this interpretation the dimension of each handle becomes its codimension.

Consequently W_2 can be obtained from $\Sigma_0 \times [-1, 1]$ by attaching handles of codimension ≥ 3 and Y can be obtained from Σ_0 by a finite sequence of surgeries in codimension ≥ 3. Thus, by [29, Theorem A], Y admits a metric of positive scalar curvature if Σ_0 does. □

We have all the ingredients to prove the main results of Section 2.1.2. Proposition 2.13 follows directly from Lemma 2.55 and Proposition 2.58. Further Proposition 2.17 follows directly from Lemma 2.55 and Definition 2.15. For the convenience of the reader we also include a proof of Proposition 2.14, which heavily draws on the work of Zeidler [75, 76].

Proof of Proposition 2.14. By Lemma 2.55 there is one connected component Σ_0 of Σ, which separates $\partial_- X$ and $\partial_+ X$. We can assume that $\Sigma_0 \subset \mathring{X}$ (otherwise we isotope Σ_0 by flowing along the interior normal vector field to ∂X for a short time).

We consider the real Miščenko bundle $\mathcal{L}_Y \to Y$, which is the flat bundle of finitely generated projective Hilbert-$C^*\pi_1 Y$-modules associated to the representation of $\pi_1 Y$ on $C^*\pi_1 Y$ by left multiplication. Recall (see for example [76, Section 2]) that the Rosenberg index $\alpha(Y) \in KO_{n-1}(C^*\pi_1 Y)$ is then the (K-theoretic) index of the Dirac operator on the spinor bundle of Y twisted with $\mathcal{L}_Y$. We pull back $\mathcal{L}_Y$ to X via the projection $X \to Y$ and restrict this pullback bundle to the connected component W of X which is bounded by $\partial_- X$ and Σ_0. We denote the resulting bundle by $\mathcal{E} \to W$.

Since Y is spin, so are W and Σ_0. If we restrict $\mathcal{E}$ to Σ_0, the index of the Dirac operator on the spinor bundle of Σ_0 twisted with the restriction of $\mathcal{E}$ is an element in $KO_{n-1}(C^*\pi_1 Y)$ which we denote by $\alpha_{\mathcal{E}}(\Sigma_0)$. By bordism invariance of the index $\alpha_{\mathcal{E}}(\Sigma_0) = \alpha(Y) \neq 0$ and hence Σ_0 does not admit a metric with positive scalar curvature as $\mathcal{E}$ is a flat bundle and by the usual argument involving the Lichnerowicz-Weitzenböck formula. □

2.5.2 Aspherical 4-Manifolds

In this section we present a detailed proof of Theorem 2.19. To do so we implement the ideas of [33, Section 7, Main Theorem] using the techniques developed in [16].

Remark 2.59. To unburden the notation in this section we will denote the scalar curvature of a Riemannian manifold (M, g) by R_M. If $B \subset M$ is an embedded submanifold we will denote the scalar curvature of the induced metric $g\big|_B$ by R_B.

Definition 2.60. Let M^n be a band. Let $\alpha \neq 0 \in H_{n-2}(M;\mathbb{Z})$ be a non torsion homology class. We say that α is a *band class* if there are $\alpha^+ \in H_{n-2}(\partial_+ M;\mathbb{Z})$ and $\alpha^- \in H_{n-2}(\partial_- M;\mathbb{Z})$ with $\alpha = \iota_*(\alpha^\pm)$, where $\iota : \partial M \to M$ denotes the inclusion of the boundary.

The main analytical tool we need to develop is the following proposition, which is reminiscent of what Gromov, in [33, Section 3], calls *Richard's Lemma* in reference to [56]. The proof, however, follows in the line of [16, Sections 6.1 & 6.2].

Proposition 2.61. *Let (M^4, g) be an oriented Riemannian band and $\alpha \in H_2(M;\mathbb{Z})$ be a band class. If $R_M > \sigma > 0$ and $\operatorname{width}(M, g) > \frac{2\pi}{\sqrt{\sigma}}$, there is a closed oriented embedded submanifold Σ which represents α and each connected component Σ_0 of Σ is homeomorphic to a 2-sphere with*

$$\operatorname{diam}(\Sigma_0, g|_{\Sigma_0}) \leq \pi\sqrt{\frac{2}{\inf R_M - \sigma}}.$$

Proof. Denote $\ell = \frac{\pi}{\sqrt{\sigma}}$. Let $\beta \in H_3(M, \partial M;\mathbb{Z})$ be a relative class with $\partial\beta = \alpha^+ - \alpha^-$. Let B^3 be a smooth embedded stable minimal hypersurface in the class β.

One can obtain B^3 by directly minimizing area in β, as it is done in [30, Proof of Theorem 12.1, pp. 398-399] and later [31, Induction Step, p.652]. However as it is noted there, while the minimizer is smooth in $\mathring{M}$, it might not be smooth at the points where it intersects the boundary ∂M. This can be overcome by shaving off an arbitrarily small collar neighborhood of ∂M. The strict inequality $\operatorname{width}(M, g) > \frac{2\pi}{\sqrt{\sigma}}$ can be preserved in this process.

Alternatively one can double M along its boundary (we denote the result by $\hat{M}$). By a simple Mayer-Vietoris argument, there is an absolute homology class $\hat{\beta} \in H_3(\hat{M}, \mathbb{Z})$ which maps to $\alpha^+ - \alpha^-$ under the Mayer-Vietoris boundary map.

Hence if we subdivide a representative of $\hat{\beta}$ in such a way that it can be written as the sum of two chains contained in the two copies of M with common boundary along ∂M, then each of these chains represents a suitable class $\beta \in H_3(M, \partial M;\mathbb{Z})$.

If we were to smooth the metric on $\hat{M}$ in an arbitrarily small neighborhood of the common boundary we could minimize area in $\hat{\beta}$ and restrict the resulting closed smooth embedded minimal surface to a copy of (M, g), with the small neighborhood we used for smoothing cut off, to obtain B^3. Again, the strict inequality $\mathrm{width}(M, g) > \frac{2\pi}{\sqrt{\sigma}}$ can be preserved in this process.

By stability of B and the classical second variation formula for the area functional we see that

$$\int_B |\nabla_B \psi|^2 - \frac{1}{2}(R_M - R_B + |A|^2)\psi^2 d\mathcal{H}^3 \geq 0,$$

for all $\psi \in C_0^1(B)$. The first eigenfunction of the associated operator

$$-\Delta_B - \frac{1}{2}(R_M - R_B + |A|^2)$$

will be smooth and can be chosen in such a way that it is positive on $\mathring{B}$ ie there is a function $u \in C_0^\infty(B)$ with $u > 0$ on $\mathring{B}$ and

$$\Delta_B u \leq -\frac{1}{2}(R_M - R_B + |A|^2)u. \tag{2.5.1}$$

Furthermore $(B, g|_B)$ is a Riemannian band with $\mathrm{width}(B, g|_B) > 2\ell$. We can shrink B a little bit from both sides such that width $> 2\ell$ remains true but $\partial B \subset \mathring{M}$ (this guarantees $u > 0$ on B). Let $\phi : B \to [-\ell, \ell]$ be the map produced by Lemma 2.34 and set $h(x) = -\frac{\pi}{\ell}\tan(\frac{\pi}{2\ell}\phi(x))$.

We define $\Omega_0 = \phi^{-1}[-\ell, 0]$ and consider the functional

$$\mathcal{A}(\hat{\Omega}) = \int_{\partial^*\hat{\Omega}} u d\mathcal{H}^2 - \int_B (\chi_{\hat{\Omega}} - \chi_{\Omega_0}) hu d\mathcal{H}^3$$

for all Caccioppoli sets $\hat{\Omega}$ with $\hat{\Omega}\Delta\Omega_0$ contained in the interior of B.

By Lemma 2.46 we find a μ-bubble $\Omega \subset B$ with smooth boundary $\Sigma = (\partial\Omega \backslash \partial_- B)$, which represents the class $\alpha \in H_2(M; \mathbb{Z})$. Stability of Ω and Lemma 2.48 imply that for each connected component Σ_0 of Σ we have:

$$\int_{\Sigma_0} |\nabla_{\Sigma_0}\psi|^2 u - \frac{1}{2}\left(R_B - R_{\Sigma_0} - H^2 + |A|^2\right)\psi^2 u \\ + \left(2H\langle\nabla_B u, \nu\rangle + \frac{d^2 u}{d\nu^2} - Hhu - \langle\nabla_B(hu), \nu\rangle\right)\psi^2 \geq 0,$$

for all $\psi \in C^\infty(\Sigma_0)$.

Claim. $\Delta_B u = \Delta_{\Sigma_0} u + H\langle\nabla_B u, \nu\rangle + \frac{d^2 u}{d\nu^2}$.

Proof of Claim. We check this in local coordinates. Let $e_1, \ldots, e_{n-1}$ be a local orthonormal frame of $T\Sigma_0$. We extend ν to a unit vector field in a small tubular neighborhood of Σ_0 via the normal exponential map. The extension will be the velocity vector field of normal geodesics to Σ_0. In the following $\nabla_B u = \nabla_B^\perp u + \nabla_B^\nu u$

denotes the decomposition of $\nabla_B u$ into its normal and tangential part.

$$\begin{aligned}
\Delta_B u = \operatorname{div}(\nabla_B u) &= \sum_{i=1}^{n-1}\langle \nabla_{e_i}(\nabla_B u), e_i\rangle + \langle \nabla_\nu(\nabla_B u), \nu\rangle \\
&= \sum_{i=1}^{n-1}\langle \nabla_{e_i}(\nabla_B^{\perp} u), e_i\rangle + \sum_{i=1}^{n-1}\langle \nabla_{e_i}(\nabla_B^{\nu} u), e_i\rangle + \langle \nabla_\nu(\nabla_B u), \nu\rangle \\
&= \Delta_{\Sigma_0} u + \sum_{i=1}^{n-1}\langle \nabla_{e_i}(\langle \nabla_B u, \nu\rangle \nu), e_i\rangle + \langle \nabla_\nu(\nabla_B u), \nu\rangle \\
&= \Delta_{\Sigma_0} u + \langle \nabla_B u, \nu\rangle \sum_{i=1}^{n-1}\langle \nabla_{e_i}\nu, e_i\rangle + \langle \nabla_\nu(\nabla_B u), \nu\rangle \\
&= \Delta_{\Sigma_0} u + H\langle \nabla_B u, \nu\rangle + \langle \nabla_\nu(\nabla_B u), \nu\rangle \\
&= \Delta_{\Sigma_0} u + H\langle \nabla_B u, \nu\rangle + \frac{d^2u}{d\nu^2} - \langle \nabla_B u, \nabla_\nu \nu\rangle \\
&= \Delta_{\Sigma_0} u + H\langle \nabla_B u, \nu\rangle + \frac{d^2u}{d\nu^2},
\end{aligned}$$

where we used that ν is the velocity vector field of normal geodesics to Σ_0. □

Consequently we combine with (2.5.1) to get:

$$\frac{d^2u}{d\nu^2} + H\langle \nabla_B u, \nu\rangle = \Delta_B u - \Delta_{\Sigma_0} u \le -\frac{1}{2}(R_M - R_B + |A|^2)u - \Delta_{\Sigma_0} u.$$

By the first variation formula Lemma 2.47, we have $hu = Hu + \langle \nabla_B u, \nu\rangle$ and therefore

$$Hhu = H^2 u + H\langle \nabla_B u, \nu\rangle,$$

as well as

$$\begin{aligned}
\frac{1}{2}H^2\psi^2 u &= \frac{1}{2}\left(h - u^{-1}\langle \nabla_B u, \nu\rangle\right)^2 \psi^2 u \\
&= \frac{1}{2}u^{-1}\langle \nabla_B u, \nu\rangle^2\psi^2 - h\langle \nabla_B u, \nu\rangle\psi^2 + \frac{1}{2}h^2\psi^2 u.
\end{aligned}$$

Plugging all of this, step by step, into the stability inequality yields:

$$\begin{aligned}
0 &\le \int_{\Sigma_0} |\nabla_{\Sigma_0}\psi|^2 u - \frac{1}{2}(R_M - R_{\Sigma_0} - H^2 + 2|A|^2)\psi^2 u \\
&\quad - (\Delta_{\Sigma_0} u - H\langle \nabla_B u, \nu\rangle + Hhu + h\langle \nabla_B(u), \nu\rangle + u\langle \nabla_B h, \nu\rangle)\psi^2 \\
&\le \int_{\Sigma_0} |\nabla_{\Sigma_0}\psi|^2 u - \frac{1}{2}(R_M - R_{\Sigma_0} + H^2 + 2|A|^2)\psi^2 u \\
&\quad - (\Delta_{\Sigma_0} u + h\langle \nabla_B(u), \nu\rangle + u\langle \nabla_B h, \nu\rangle)\psi^2 \\
&\le \int_{\Sigma_0} |\nabla_{\Sigma_0}\psi|^2 u - \frac{1}{2}(R_M - R_{\Sigma_0} + 2|A|^2)\psi^2 u - (\Delta_{\Sigma_0} u)\psi^2 \\
&\quad - \frac{1}{2}(h^2 + 2\langle \nabla_B h, \nu\rangle)\psi^2 u \\
&\le \int_{\Sigma_0} |\nabla_{\Sigma_0}\psi|^2 u - \frac{1}{2}(R_M - \sigma - R_{\Sigma_0})\psi^2 u - (\Delta_{\Sigma_0} u)\psi^2 \\
&\quad - \frac{1}{2}(\sigma + h^2 + 2\langle \nabla_B h, \nu\rangle)\psi^2 u
\end{aligned}$$

for all $\psi \in C^\infty(\Sigma_0)$.

Using $\ell = \frac{\pi}{\sqrt{\sigma}}$, we can estimate

$$\sigma + h^2 + 2\langle \nabla_B h, \nu \rangle \geq \sigma + h^2 - 2|\nabla_B h| > 0.$$

We conclude

$$0 \leq \int_{\Sigma_0} |\nabla_{\Sigma_0}\psi|^2 u - \frac{1}{2}(R_M - \sigma - R_{\Sigma_0})\psi^2 u - (\Delta_{\Sigma_0} u)\psi^2. \tag{2.5.2}$$

If we choose $\psi = u^{-\frac{1}{2}}$, we can use

$$\operatorname{div}(u^{-1}\nabla_{\Sigma_0} u) = -u^{-2}|\nabla_{\Sigma_0} u|^2 + u^{-1}\Delta_{\Sigma_0} u$$

to integrate by parts and see

$$\begin{aligned} 0 &\leq \int_{\Sigma_0} |\nabla_{\Sigma_0} u^{-\frac{1}{2}}|^2 u - \frac{1}{2}(R_M - \sigma - R_{\Sigma_0}) - (\Delta_{\Sigma_0} u)u^{-1} \\ &= \int_{\Sigma_0} -\frac{3}{4}u^{-2}|\nabla_{\Sigma_0} u|^2 - \frac{1}{2}(R_M - \sigma - R_{\Sigma_0}) \\ &\leq \int_{\Sigma_0} -\frac{1}{2}(R_M - \sigma - R_{\Sigma_0}), \end{aligned}$$

or equivalently

$$\frac{1}{2}\int_{\Sigma_0} R_M - \sigma \leq \frac{1}{2}\int_{\Sigma_0} R_{\Sigma_0} = 2\pi\chi(\Sigma_0),$$

which implies that Σ_0 is a sphere with $\operatorname{area}(\Sigma_0) \leq \frac{8\pi}{\inf R_M - \sigma}$.

To finish the proof we return to (2.5.2). Let $w \in C^\infty(\Sigma_0)$ be the first positive eigenfunction to the associated operator. Thus

$$\operatorname{div}_{\Sigma_0}(u\nabla_{\Sigma_0} w) \leq -\frac{1}{2}(R_M - \sigma - R_{\Sigma_0})wu - (\Delta_{\Sigma_0} u)w.$$

If we set $\lambda = uw$, then

$$\begin{aligned} \Delta_{\Sigma_0}\lambda &= \operatorname{div}_{\Sigma_0}(u\nabla_{\Sigma_0} w) + \operatorname{div}_{\Sigma_0}(w\nabla_{\Sigma_0} u) \\ &\leq -\frac{1}{2}(R_M - \sigma - R_{\Sigma_0})\lambda - (\Delta_{\Sigma_0} u)w + \operatorname{div}_{\Sigma_0}(w\nabla_{\Sigma_0} u) \\ &= -\frac{1}{2}(R_M - \sigma - R_{\Sigma_0})\lambda - (\Delta_{\Sigma_0} u)w + w\operatorname{div}_{\Sigma_0}(\nabla_{\Sigma_0} u) + \langle \nabla_{\Sigma_0} u, \nabla_{\Sigma_0} w\rangle \\ &\leq -\frac{1}{2}(R_M - \sigma - R_{\Sigma_0})\lambda + \langle \nabla_{\Sigma_0} u, \nabla_{\Sigma_0} w\rangle \\ &\leq -\frac{1}{2}(R_M - \sigma - R_{\Sigma_0})\lambda + \langle \nabla_{\Sigma_0} u, \nabla_{\Sigma_0} w\rangle + \frac{1}{2}u^{-1}w|\nabla_{\Sigma_0} u|^2 + \frac{1}{2}w^{-1}u|\nabla_{\Sigma_0} w|^2 \\ &\leq -\frac{1}{2}(R_M - \sigma - R_{\Sigma_0})\lambda + \frac{1}{2}\lambda^{-1}|\nabla_{\Sigma_0}\lambda|^2. \end{aligned}$$

Now $\operatorname{diam}(\Sigma_0, g|_{\Sigma_0}) \leq \pi\sqrt{\frac{2}{\inf R_M - \sigma}}$ follows directly from the next lemma. □

Lemma 2.62 ([16, Lemma 16])**.** *Let (N^2, g) be a closed 2-dimensional Riemannian manifold. If there is a smooth function $\lambda > 0$ on Σ_0 with*

$$\Delta_N\lambda \leq -\frac{1}{2}(C - R_N)\lambda + \frac{1}{2}\lambda^{-1}|\nabla_N\lambda|^2$$

for some $C > 0$, then $\operatorname{diam}(N, g) \leq \sqrt{\frac{2}{C}}\pi$.

Proof. Let $\ell = \sqrt{\frac{2}{C}}\pi$. Assume for a contradiction, that there are $p, q \in N$ with $\mathrm{dist}_g(p,q) > \ell$. For $\varepsilon > 0$ small enough $M = N\backslash(B_\varepsilon(p) \cup B_\varepsilon(q))$ is a band with $\mathrm{width}(M, g) > \ell$. Let $\phi : M \to [-\frac{\ell}{2}, \frac{\ell}{2}]$ be the map we get from Lemma 2.34 and define $h(x) = -\frac{2\pi}{\ell}\tan(\frac{\pi}{\ell}\phi(x))$.

Let $\Omega_0 = \phi^{-1}[-\frac{\ell}{2}, 0]$ (w.l.o.g 0 is a regular value of ϕ) and consider the functional

$$\mathcal{A}(\hat{\Omega}) = \int_{\partial^*\hat{\Omega}} \lambda - \int_M (\chi_{\hat{\Omega}} - \chi_{\Omega_0}) h\lambda d\mathcal{H}^2$$

for all Caccioppoli sets $\hat{\Omega}$ with $\hat{\Omega}\Delta\Omega_0$ contained in the interior of M. We repeat most of the steps from the proof of the previous Lemma, with u replaced by λ.

By Lemma 2.46 there is a minimizer Ω with smooth boundary $(\partial\Omega\backslash\partial_- M)$ and by Lemma 2.47 every connected component Σ satisfies

$$H = -\lambda^{-1}\langle\nabla_M\lambda, \nu\rangle + h.$$

Note that in this case H is the geodesic curvature.

By stability and Lemma 2.48 we see

$$0 \le \int_\Sigma |\nabla_\Sigma\psi|^2\lambda - \frac{1}{2}(R_M - H^2 + |A|^2)\psi^2\lambda + (2H\langle\nabla_M\lambda, \nu\rangle + \frac{d^2\lambda}{d\nu^2} - Hh\lambda - \langle\nabla_M(h\lambda), \nu\rangle)\psi^2$$

for all $\psi \in C^\infty(\Sigma)$. We use as before:

$$\frac{d^2\lambda}{d\nu^2} + H\langle\nabla_M u, \nu\rangle = \Delta_M\lambda - \Delta_\Sigma\lambda \le -\Delta_\Sigma\lambda - \frac{1}{2}(C - R_M)\lambda + \frac{1}{2}\lambda^{-1}|\nabla_M\lambda|^2,$$

as a consequence of Lemma 2.47

$$Hh\lambda = H^2\lambda + H\langle\nabla_M\lambda, \nu\rangle,$$

as well as

$$\frac{1}{2}H^2\psi^2\lambda = \frac{1}{2}\lambda^{-1}\langle\nabla_M\lambda, \nu\rangle^2\psi^2 - h\langle\nabla_M\lambda, \nu\rangle\psi^2 + \frac{1}{2}h^2\psi^2\lambda.$$

If we plug all of the above in the stability inequality, it follows that

$$0 \le \int_\Sigma |\nabla_\Sigma\psi|^2\lambda + \frac{1}{2}\lambda^{-1}|\nabla_M\lambda|^2\psi^2 - \frac{1}{2}\lambda^{-1}\langle\nabla_M\lambda, \nu\rangle^2\psi^2 - (\Delta_\Sigma\lambda)\psi^2 - \frac{1}{2}(C + h^2 + 2\langle\nabla_M h, \nu\rangle)\psi^2\lambda$$

for all $\psi \in C^\infty(\Sigma)$ and since $C + h^2 + 2\langle\nabla_M h, \nu\rangle > 0$ (remember that $\mathrm{Lip}(\phi) < 1$) by construction of h, we see

$$0 < \int_\Sigma |\nabla_\Sigma\psi|^2\lambda + \frac{1}{2}\lambda^{-1}|\nabla_M\lambda|^2\psi^2 - \frac{1}{2}\lambda^{-1}\langle\nabla_M\lambda, \nu\rangle^2\psi^2 - (\Delta_\Sigma\lambda)\psi^2$$

If we choose $\psi = \lambda^{-\frac{1}{2}}$, we can use

$$\mathrm{div}(\lambda^{-1}\nabla_{\Sigma_0}\lambda) = -\lambda^{-2}|\nabla_{\Sigma_0}\lambda|^2 + \lambda^{-1}\Delta_{\Sigma_0}\lambda$$

to integrate by parts as before and see

$$\begin{aligned}
0 &< \int_\Sigma -\frac{3}{4}\lambda^{-2}|\nabla_\Sigma\lambda|^2 + \frac{1}{2}\lambda^{-2}|\nabla_M\lambda|^2 - \frac{1}{2}\lambda^{-2}\langle\nabla_M\lambda, \nu\rangle^2 \\
&= \int_\Sigma -\frac{3}{4}\lambda^{-2}|\nabla_\Sigma\lambda|^2 + \frac{1}{2}\lambda^{-2}\left(|\nabla_\Sigma\lambda|^2 + \langle\nabla_M\lambda, \nu\rangle^2\right) - \frac{1}{2}\lambda^{-2}\langle\nabla_M\lambda, \nu\rangle^2 \\
&= \int_\Sigma -\frac{1}{4}\lambda^{-2}|\nabla_\Sigma\lambda|^2,
\end{aligned}$$

which is a contradiction. $\square$

Definition 2.63. Let (X^n, g) be a complete oriented Riemannian manifold and c a locally finite singular k-cycle. Then the the *filling radius* of c in (X, g) is defined to be

$$\mathrm{FillRad}_{\mathbb{Z}}(c, X) = \inf\{r > 0 | [c] = 0 \in H_k^{lf}(U_r(c); \mathbb{Z})\},$$

where $U_r(c)$ denotes the open r-neighborhood of c in (X, g). If one replaces $\mathbb{Z}$ by $\mathbb{Q}$ the same definition yields the rational filling radius.

Remark 2.64. In [23, Section 1] Gromov generalizes the above and defines the filling radius $\mathrm{FillRad}_{\mathbb{Z}}(X, g)$ of a complete oriented Riemannian manifold (X, g), with respect to the Kuratowski embedding of (X, d_g). Furthermore he proves two results we will need in the following:

- $\mathrm{FillRad}_{\mathbb{Z}}(\mathbb{R}, g_{std}) = \infty$ (see [23, Section 4.4.C])
- if X is isometrically embedded in a Metric space (S, d) and one defines $\mathrm{FillRad}_{\mathbb{Z}}(X, S)$ appropriately as $\inf\{r > 0 | [X] = 0 \in H_k^{lf}(U_r(X); \mathbb{Z})\}$, then $\mathrm{FillRad}_{\mathbb{Z}}(X, S) \geq \mathrm{FillRad}_{\mathbb{Z}}(X, g)$ (see [23, Section 1]).

Both points remain true with rational coefficients.

Lemma 2.65 ([42, Theorem IX.4.7])**.** *Let X^n be an oriented manifold. Let $C \subset X$ be a closed subset such that $\partial C = C \backslash \mathring{C}$ is a smooth submanifold. Then $H_1^{lf}(C; \mathbb{Z}) \cong H^{n-1}(X, X \backslash C; \mathbb{Z})$.*

Lemma 2.66 (Codim 2 Linking Lemma [32, Lemma 4.G])**.** *Let Y^n be a closed aspherical manifold and g a Riemannian metric on Y. For every $\sigma > 0$ there is a compact band M in the universal cover $(\widetilde{Y}, \widetilde{g})$ such that:* $\mathrm{width}(M, \widetilde{g}) > \frac{2\pi}{\sqrt{\sigma}}$*, there is a band class $\alpha \in H_{n-2}(M; \mathbb{Z})$ and for every cycle $c \subset M$ representing a nonzero multiple of α we have* $\mathrm{FillRad}_{\mathbb{Z}}(c, \widetilde{Y}) > \frac{2\pi}{\sqrt{\sigma}}$.

Proof. Let $\sigma > 0$ be arbitrary. By [16, Lemma 6] there is a geodesic line

$$\gamma : \mathbb{R} \to (\widetilde{Y}, \widetilde{g}).$$

Since γ is an isometric embedding of the real line,

$$\mathrm{FillRad}_{\mathbb{Z}}(\gamma, \widetilde{Y}) \geq \mathrm{FillRad}_{\mathbb{Z}}(\mathbb{R}, g_{Std}) = \infty$$

(see Remark 2.64), hence for all $r > 0$ the line γ represents a non-zero class in $H_1^{lf}(U_r(\gamma), \mathbb{Z})$, where $U_r(\gamma)$ denotes the open r-neighborhood of γ in $(\widetilde{Y}, \widetilde{g})$. Since

$$\mathrm{FillRad}_{\mathbb{Q}}(\gamma, \widetilde{Y}) \geq \mathrm{FillRad}_{\mathbb{Q}}(\mathbb{R}, g_{Std}) = \infty$$

we see by the same argument, that $[\gamma]$ is non torsion in $H_1^{lf}(U_r(\gamma), \mathbb{Z})$.

For some $\varepsilon > 0$ let ρ be a smooth approximation of $\mathrm{dist}(\gamma, \cdot)$ which is ε-close. There is a sequence $(r_k)_{k \in \mathbb{N}}$ of regular values of ρ with $r_k \to \infty$ and the property that for all $k \in \mathbb{N}$ we have $r_k > 2\varepsilon$ and $r_{k+1} - r_k > 2\varepsilon$.

Denote $\overline{U}_k = \rho^{-1}[0, r_k]$. By construction $[\gamma] \neq 0 \in H_1^{lf}(\overline{U}_k, \mathbb{Z})$ for all k and the class is non torsion. Thus by Lemma 2.65, the fact that $\widetilde{Y}$ is contractible we see:

$$0 \neq [\gamma] \in H_1^{lf}(\mathrm{U}_k, \mathbb{Z}) \cong H^{n-1}(\widetilde{Y}, \widetilde{Y} \backslash \mathrm{U}_k; \mathbb{Z}) \cong H^{n-2}(\widetilde{Y} \backslash \mathrm{U}_k; \mathbb{Z}).$$

Furthermore, since $[\gamma]$ is non-torsion its image in $H^{n-2}(\widetilde{Y}\setminus \mathrm{U}_k;\mathbb{Z})$ corresponds by the UCT to an element $\alpha_k \neq 0 \in H_{n-2}(\widetilde{Y}\setminus \mathrm{U}_k;\mathbb{Z})$. If we represent α_k by a closed smooth submanifold $N_k \subset \widetilde{Y}\setminus \mathrm{U}_k$, then N_k is linked with γ and $\mathrm{dist}(\gamma, N_k) > r_k - \varepsilon > r_{k-1} + \varepsilon$. That N_k is linked with γ means that $[N_k] \neq 0 \in H_{n-2}(\widetilde{Y}\setminus\gamma;\mathbb{Z})$ ie every fill-in of N_k in $\widetilde{Y}$, which exists because $\widetilde{Y}$ is contractible, intersects γ.

Let V_k be a smoothed version (as before) of the closed $(r_{k-1}/2)$-neighborhood of N_k. Then $0 \neq \alpha_k \in H_{n-1}(V_k;\mathbb{Z})$, since N_k is linked with γ and $V_k \subset (\widetilde{Y}\setminus\gamma)$. Furthermore any cycle $c \subset V_k$, which represents a nonzero multiple of α_k is linked with γ (since α_k is non torsion) and hence $\mathrm{FillRad}_{\mathbb{Z}}(c, \widetilde{Y}) \geq \mathrm{dist}(\gamma, c) \geq (r_{k-1}/2)$. We want to see that there is a class $\alpha_k^+ \in H_{n-1}(\partial V_k;\mathbb{Z})$ with $\iota(\alpha_k^+) = \alpha_k$, where $\iota : \partial V_k \to V_k$ denotes the inclusion of the boundary.

Consider the ε-neighborhood $U_\varepsilon(\gamma)$ of γ. If we choose ε small enough, there is a closed smooth submanifold $N_0 \subset U_\varepsilon(\gamma)$ with $[N_0] = [N_k] \in H_{n-2}(\widetilde{Y}\setminus\gamma;\mathbb{Z})$ (this N_0 will be the image under the exponential map of small sphere around the origin in the fiber over $\gamma(0)$ in the normal bundle of γ). Hence we can find a smooth oriented submanifold B^{n-1} with $\partial B = N_0 \cup N_k$ and by possibly deforming B a little bit we can ensure that B intersects ∂V_k transversely in a closed $(n-2)$-dimensional submanifold N_k^+. Then $[N_k^+]$ is homologous to $[N_k]$ in $H_{n-2}(V_k;\mathbb{Z})$ and we can set $\alpha_k^+ = [N_k^+] \in H_{n-2}(\partial V_k;\mathbb{Z})$.

Finally, for $\delta > 0$ small enough, $U_\delta(N_k)$ is isometric via the exponential map to the normal δ-disc bundle $D_\delta(N_k)$. It follows that $\partial U_\delta(N_k) = N_k \times S^1$ and we set $\alpha_k^- = [N_k \times \{0\}] \in H_{n-2}(N_k \times S^1;\mathbb{Z})$. We conclude that $M_k = V_k\setminus U_\delta(N_k)$ is a compact band with boundary $\partial_+ M_k = \partial V_k$ and $\partial_- M_k = \partial U_\delta(N_k) = N_k \times S^1$, which has $\mathrm{width}(M_k, \widetilde{g}) \geq \frac{r_{k-1}}{2} - \varepsilon - \delta$. Furthermore the triple $(\alpha_k, \alpha_k^+, \alpha_k^-)$ represents a band class in M_k. For k big enough $\mathrm{width}(M_k, \widetilde{g}) \geq \frac{r_{k-1}}{2} - \varepsilon - \delta > \frac{2\pi}{\sqrt{\sigma}}$. □

Lemma 2.67. *Let M' and M be two smooth connected oriented n-dimensional bands and $f : M' \to M$ be a map of degree $d \neq 0$ with $f(\partial_\pm M') = \partial_\pm M$. If $\alpha \in H_{n-2}(M;\mathbb{Z})$ is a band class, there is a band class $\alpha' \in H_{n-2}(M';\mathbb{Z})$ with $f_*(\alpha') = d\alpha$.*

Proof. Consider the following diagram:

$$\begin{array}{ccc} H_{n-2}(M';\mathbb{Z}) & \xleftarrow[\cong]{\cap[M',\partial M']} & H^2(M',\partial M';\mathbb{Z}) \\ \big\downarrow f_* & & f^*\big\uparrow \\ H_{n-2}(M;\mathbb{Z}) & \xleftarrow{\cap d[M,\partial M]} & H^2(M,\partial M;\mathbb{Z}). \end{array}$$

The diagram commutes since $f_*[M',\partial M'] = d[M,\partial M]$. By Lefschetz duality there is a unique cohomology class $\eta \in H^2(M,\partial M;\mathbb{Z})$ with $\eta \cap d[M,\partial M] = d\alpha \neq 0$ since α is non torsion. Then $\alpha' := f^*\eta \cap [M',\partial M']$ is such that $f_*\alpha' = d\alpha$.

The (co)homology of ∂M and $\partial M'$ splits as the direct sum of the (co)homology of the components $\partial_\pm M$ and $\partial_\pm M'$ i. e. $H_*(\partial M) = H_*(\partial_- M) \oplus H_*(\partial_+ M)$ and $H^*(\partial M) = H^*(\partial_- M) \oplus H^*(\partial_+ M)$. The induced maps f_* and f^* split into components as well. By comparing the components of

$$f_*([\partial_+ M']) - f_*([\partial_- M']) = \partial f_*[M',\partial M'] = \partial d[M,\partial M] = d[\partial_+ M] - d[\partial_- M],$$

we conclude that $f_*([\partial_+ M']) = d[\partial_+ M]$ and $f_*([\partial_- M']) = d[\partial_- M]$, so the restricted maps between the boundary components have degree d as well.

For both boundary components we separately write down a diagram as above

$$\begin{array}{ccc} H_{n-2}(\partial_\pm M';\mathbb{Z}) & \xleftarrow[\cong]{\cap[\partial_\pm M']} & H^1(\partial_\pm M';\mathbb{Z}) \\ \downarrow{\scriptstyle f_*} & & \uparrow{\scriptstyle f^*} \\ H_{n-2}(\partial_\pm M;\mathbb{Z}) & \xleftarrow{\cap d[\partial_\pm M]} & H^1(\partial_\pm M;\mathbb{Z}) \end{array}$$

and we find unique cohomology classes $\eta^\pm \in H^1(\partial_\pm M;\mathbb{Z})$ with $\eta^\pm \cap d[\partial_\pm M] = \pm d\alpha^\pm$.

We then consider

$$\begin{array}{ccc} H^1(\partial_- M;\mathbb{Z}) \oplus H^1(\partial_+ M;\mathbb{Z}) & \xrightarrow{\cap[\partial M]} & H_{n-2}(\partial_- M;\mathbb{Z}) \oplus H_{n-2}(\partial_+ M;\mathbb{Z}) \\ \downarrow & & \downarrow{\scriptstyle \iota_*} \\ H^2(M,\partial M;\mathbb{Z}) & \xrightarrow{\cap[M]} & H_{n-2}(M;\mathbb{Z}) \end{array}$$

By comparing components we see that η^- and η^+ map to η under the connecting homomorphism $H^1(\partial M;\mathbb{Z}) \to H^2(M,\partial M;\mathbb{Z})$.

Thus we define classes $\alpha'^\pm := f^*\eta^\pm \cap [\partial_\pm M']$. Finally

$$\begin{array}{ccccc} H^1(\partial M;\mathbb{Z}) & \xrightarrow{f^*} & H^1(\partial M';\mathbb{Z}) & \xrightarrow{\cap[\partial M']} & H_{n-2}(\partial M';\mathbb{Z}) \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \iota_*} \\ H^2(M,\partial M;\mathbb{Z}) & \xrightarrow{f^*} & H^2(M',\partial M';\mathbb{Z}) & \xrightarrow{\cap[M]} & H_{n-2}(M';\mathbb{Z}) \end{array}$$

implies that $\iota_*(\alpha'^\pm) = \alpha'$. □

The following Proposition is well known. A proof can be found in [17, Section 4].

Proposition 2.68. *Let Y^n and Z^n be closed connected oriented manifolds and $f : Z \to Y$ a smooth map with $\deg(f) \neq 0$. The pullback $pr : \hat{Z} \to Z$ of the universal covering $\widetilde{Y} \to Y$ has the following properties:*

- *$\hat{Z}$ is non compact,*
- *the map $f \circ pr : \hat{Z} \to Y$ can be lifted to a map $\hat{f} : \hat{Z} \to \widetilde{Y}$,*
- *$\hat{f}$ is proper and $\deg(\hat{f}) = \deg(f)$.*

Finally we have assembled all the tools we need to prove Theorem 2.19.

Proof of Theorem 2.19. Let Y^4 be a closed oriented aspherical manifold. Let Z^4 be a closed oriented manifold and $f : Z \to Y$ a continuous map with $\deg(f) \neq 0$.

We remind the reader that for any metric on Y the universal cover, equipped with the pullback metric, is uniformly contractible (see for example [23, Section 4.5.D]) i. e. for every radius $R > 0$ there is a radius $C(R) > 0$ such that for every point $y \in \widetilde{Y}$ the ball $B_R(y)$ is contractible within $B_{C(R)}(y)$.

Assume for a contradiction that Z admits a Riemannian metric g_1 with $Sc(Z,g) = R_Z > \sigma > 0$. By possibly replacing it with a homotopic map, we can assume that $f : Z \to Y$ is smooth. Let g_2 be a Riemannian metric on Y. Then $f : (Z;g_1) \to (Y,g_2)$ is a Lipschitz map and by possibly rescaling g_2, we can assume that it is distance decreasing.

By Proposition 2.68 there is a covering space $\hat{Z}$ of Z and a lift $\hat{f}:\hat{Z}\to\widetilde{Y}$ such that $\hat{f}$ is proper and $\deg(\hat{f})=\deg(f)$. By Lemma 2.66 we can find a compact band M in $(\widetilde{Y},\widetilde{g_2})$ with $\mathrm{width}(M,\widetilde{g_2})>\frac{2\pi}{\sqrt{\sigma}}$ and a band class $\alpha\in H_{n-2}(M;\mathbb{Z})$, such that for every cycle $c\subset M$ representing a nonzero multiple of α we have $\mathrm{FillRad}_{\mathbb{Z}}(c,\widetilde{Y})>\frac{2\pi}{\sqrt{\sigma}}$. By transversality we can deform the map $\hat{f}$ by an arbitrarily small amount to make it transverse to M_k as well as $\partial M_k=\partial_+M\cup\partial_-M$, while remaining distance decreasing.

Then $M'=\hat{f}^{-1}(M)$ is a compact band in $\hat{Z}$ with smooth boundary

$$\partial M'=\hat{f}^{-1}(\partial M)=\hat{f}^{-1}(\partial_+M)\cup\hat{f}^{-1}(\partial_-M)=:\partial_+M'\cup\partial_-M'$$

and since $\hat{f}$ is distance decreasing $\mathrm{width}(M',\hat{g}_1)>\frac{2\pi}{\sqrt{\sigma}}$. Furthermore $\hat{f}$ restricts to a map $M'\to M$ of degree non-zero. By Lemma 2.67, there is a band class $\alpha'\in H_{n-2}(M';\mathbb{Z})$ with $\hat{f}_*\alpha'=\deg(\hat{f})\alpha\neq 0$.

By Proposition 2.61 there is a smooth oriented submanifold Σ which represents α' and each connected component Σ_0 of Σ is a 2-sphere with

$$\mathrm{diam}(\Sigma_0,\hat{g}|_{\Sigma_0})\leq\sqrt{\frac{2}{\inf R_Z-\sigma}}\pi.$$

Then $\hat{f}(\Sigma)$ is a cycle c_0 in M, which represents $\deg(\hat{f})\alpha$ and $\mathrm{FillRad}_{\mathbb{Z}}(c_0,\widetilde{Y})\leq C\left(\sqrt{\frac{2}{\inf R_Z-\sigma}}\pi\right)$, since $\hat{f}$ is distance decreasing and $(\widetilde{Y},\widetilde{g})$ is uniformly contractible. For $\sigma>0$ small enough this yields

$$\mathrm{FillRad}_{\mathbb{Z}}(c_0,\widetilde{Y})>\frac{2\pi}{\sqrt{\sigma}}>C\left(\sqrt{\frac{2}{\inf R_Z-\sigma}}\pi\right)\geq\mathrm{FillRad}_{\mathbb{Z}}(c_0,\widetilde{Y}),\qquad(2.5.3)$$

which is a contradiction. □

Partitioned Scalar and Mean Curvature Comparison

The main goal of this chapter, which corresponds to parts of joint work [9] with Simone Cecchini and Rudolf Zeidler is to prove the partitioned scalar and mean curvature comparison principle Theorem II and to see that it implies Conjecture 1.8 of Rosenberg and Stolz for orientable manifolds in dimension ≤ 7. We recall:

Conjecture 1.8 ([59, Section 7])**.** Let Y^{n-1} be a closed manifold of dimension $\neq 4$ which does not admit a metric with positive scalar curvature. Then $Y \times \mathbb{R}$ does not admit a complete metric with positive scalar curvature.

The conjecture builds on the work [29, Section 6] of Gromov and Lawson, who established this behavior for enlargeable spin manifolds Y in all dimensions. Cecchini generalized this result in [11], where he proved Conjecture 1.8 for all spin manifolds Y with non-vanishing Rosenberg index.

Following up on [11, 12, 75], Zeidler [76, Theorem 1.4] realized that there is a general geometric statement underlying the band width conjecture and the non-existence of complete metrics with positive scalar curvature on any connected spin manifold X without boundary which contains a closed incompressible hypersurface $Y \subset X$ with trivial normal bundle and non-vanishing Rosenberg index.

Definition 3.1. Let X be a connected manifold and $\Sigma \subset X$ be an embedded hypersurface. The Σ is called *incompressible* if the map $\iota_* : \pi_1(\Sigma) \to \pi_1(X)$, induced by the inclusion $\iota : Y \hookrightarrow X$, is injective.

Recently Chen, Liu, Shi and Zhu [15, Theorem 1.1] used μ-bubbles to establish the non-existence of complete metrics with uniformly positive scalar curvature on any connected orientable manifold $X^{\leq 7}$ without boundary which contains a closed incompressible hypersurface $Y \subset X$ which is aspherical and NPSC$^+$. We remind the reader that any closed aspherical manifold of dimension ≤ 5 is NPSC$^+$ by our Theorem 2.19 and [17].

In the following we will combine the methods we developed in Chapter 2 with ideas from [76] to prove a scalar and mean curvature comparison principle for bands with Property A which are partitioned into multiple segments.

Definition 1.27. Let X be a band and Σ_i, for $i \in \{1, \ldots, k\}$, be closed embedded hypersurfaces such that Σ_1 properly separates $\partial_- X$ and $\partial_+ X$ and Σ_i properly separates Σ_{i-1} and $\partial_+ X$ for $i \in \{2, \ldots, k\}$. We call (X, Σ_i, k) a *partitioned band* and denote by V_j, for $j \in \{1, \ldots, k+1\}$, the segments of X bounded by Σ_{j-1} and the Σ_j, where $\Sigma_0 = \partial_- X$ and $\Sigma_{k+1} = \partial_+ X$.

The main conceit is, that positivity of the scalar curvature in a single segment can have global effects on the geometry of a partitioned Riemannian band.

3.1 Partitioned Comparison Principle

The following section contains an in depth discussion of Theorem II, the main result of this chapter. We review its statement, its applications and provide the necessary context. Throughout this section we restate definitions and results from the introduction for the convenience of the reader.

Definition 1.19. A smooth function $\varphi : [a, b] \to \mathbb{R}_+$ is called log-*concave* if

$$\frac{d^2}{dt^2} \log(\varphi)(t) = \left(\frac{\varphi'(t)}{\varphi(t)}\right)' \leq 0$$

for all $t \in [a, b]$. If the inequality is strict we say that φ is *strictly* log-*concave*. In case of equality we say that φ is log-*affine*.

Definition 1.20. Let (N, g_N) be a closed Riemannian manifold with constant scalar curvature. A warped product

$$\left(M, g_\varphi\right) = \left(N \times [a, b], \varphi^2(t) g_N + dt^2\right)$$

with warping function $\varphi : [a, b] \to \mathbb{R}_+$ is called a *model space* if $\operatorname{scal}(M, g_\varphi)$ is constant and φ is strictly log-concave or log-affine.

Theorem II. *Let $n \leq 7$ and (X^n, Σ_i, k) be an oriented partitioned band with Property A. Let g be a Riemannian metric on X and (M_j, g_{φ_j}) for $j \in \{1, \dots, k+1\}$ be strictly* log-*concave model spaces over a scalar flat base. If*

- ▷ $\operatorname{scal}(V_j, g) \geq \operatorname{scal}(M_j, g_{\varphi_j})$ *for all* $j \in \{1, \dots, k+1\}$,
- ▷ $H(\partial_- X, g) \geq H(\partial_- M_1, g_{\varphi_1})$ *and* $H(\partial_+ X, g) \geq H(\partial_+ M_{k+1}, g_{\varphi_{k+1}})$,
- ▷ $H(\partial_+ M_j, g_{\varphi_j}) = -H(\partial_- M_{j+1}, g_{\varphi_{j+1}})$ *for all* $j \in \{1, \dots, k\}$,

then $\operatorname{width}(V_j, g) \leq \operatorname{width}(M_j, g_{\varphi_j})$ *for at least one* $j \in \{1, \dots, k+1\}$.

3.1.1 Applications

The following result is more or less a direct application of Theorem II:

Theorem 1.28. *Let $n \leq 7$ and $(X^n, \Sigma_i, 2)$ be an orientable partitioned band with Property A. Let g be a Riemannian metric on X and $\kappa > 0$ be a positive constant. If*

- ▷ $\operatorname{scal}(V_2, g) \geq \kappa n(n-1)$,
- ▷ $\operatorname{scal}(X, g) \geq 0$,

and we denote $d := \operatorname{width}(V_2, g) < \frac{2\pi}{\sqrt{\kappa n}}$, *then*

$$\min\{\operatorname{width}(V_1, g), \operatorname{width}(V_3, g)\} < \ell = \frac{2}{\sqrt{\kappa n}} \cot\left(\frac{\sqrt{\kappa n} d}{4}\right).$$

If, instead of $\mathrm{scal}(X,g) \geq 0$, one assumes that the scalar curvature of the partitioned band is bounded from below by a negative constant, Theorem II provides the following estimate, which is very much in the same spirit as Theorem 1.28 and should be compared with Zeidler's result [76, Theorem 1.4] in the spin setting.

Theorem 3.2. *Let $n \leq 7$ and $(X, \Sigma_i, 2)$ be an orientable partitioned n-dimensional band with Property A. Let g be a metric on X and $\kappa > 0$ be a positive constant. If*

- ▷ $\mathrm{scal}(V_2, g) \geq \kappa n(n-1)$,
- ▷ $\mathrm{scal}(X,g) \geq -\sigma > -\kappa n(n-1)\tan\left(\frac{\sqrt{\kappa}nd}{4}\right)^2$, *where* $d := \mathrm{width}(V_2, g) < \frac{2\pi}{\sqrt{\kappa}n}$,

then $\min\{\mathrm{width}(V_1,g), \mathrm{width}(V_3,g)\} < \ell$, *where ℓ is such that*

$$\sqrt{\kappa}(n-1)\tan\left(\frac{\sqrt{\kappa}nd}{4}\right) = \sqrt{\frac{\sigma(n-1)}{n}}\coth\left(\frac{\sqrt{\sigma}n\ell}{2\sqrt{n-1}}\right).$$

We want to use Theorem 1.28 or Theorem 3.2 to attack Conjecture 1.8. If $X = Y \times \mathbb{R}$ and g is a complete metric on X, we consider the compact segment $Y \times [-C, C]$ for any $C > 0$, which is partitioned in to the bands $Y \times [-C, -1]$, $Y \times [-1, 1]$ and $Y \times [1, C]$. If the scalar curvature of (X, g) is assumed to be positive and $Y \times [-C, C]$ has Property A, the minimum of the widths of $(Y \times [-C, -1], g)$ and $(Y \times [1, C], g)$ is bounded from above in terms of the width of $(Y \times [-1, 1], g)$ and the infimum of $\mathrm{scal}(Y \times [-1, 1], g)$. For $C > 0$ large enough, this produces a contradiction.

We try to formulate the most general result that one can prove in this manner. To do so, we make use of the Freudenthal end compactification [20] of a connected manifold M, denoted by $\mathcal{F}M = M \cup \mathcal{E}M$, where $\mathcal{E}M$ is the space of ends. We introduce the following class of non-compact manifolds without boundary:

Definition 3.3. An *open band* is a connected non-compact manifold M without boundary, together with a decomposition

$$\mathcal{E}M = \mathcal{E}_-M \sqcup \mathcal{E}_+M,$$

where $\mathcal{E}_\pm M$ are non-empty closed[1] subsets $\mathcal{E}_\pm M \subset \mathcal{E}M$ ie M has at least two ends.

The standard example of an open band is $M = \mathring{X}$ where X is a band. In the spirit of Conjecture 1.8 and Theorem II we are interested in open bands with:

Property B. No closed embedded hypersurface $\Sigma \subset M$ which separates $\mathcal{E}_-M$ and $\mathcal{E}_+M$ admits a metric with positive scalar curvature.

We point out that, if X is a band with Property A, then $M = \mathring{X}$ is an open band with Property B. Conversely if M is an open band with Property B and $\Sigma_\pm$ are two closed embedded hypersurfaces such that Σ_- properly separates $\mathcal{E}_-M$ and $\mathcal{E}_+M$ and Σ_+ properly separates Σ_- and $\mathcal{E}_+M$, then the band $X \subset M$ which is bounded by $\Sigma_\pm$ has Property A.

In this way Theorem 1.28 or Theorem 3.2 yield obstructions to the existence of a complete metric with positive scalar curvature on an open band with Property B. Since any open band M has at least two ends by definition, we can go even further. We include a proof of the following standard result in Section 3.3.

[1] While the space of end is totally disconnected, it is in general not necessarily discrete. In this case it is important to assume $\mathcal{E}_\pm M$ to be closed (and thus clopen) subsets of $\mathcal{E}M$.

Proposition 3.4. *Let M be an open band. If g is a complete Riemannian metric on M, there is a geodesic line which connects $\mathcal{E}_-M$ and $\mathcal{E}_+M$.*

With a deformation result due to Kazdan [45] and the classical Cheeger-Gromoll splitting theorem (see e.g. [53, Theorem 7.3.5]) we arrive at the following conclusion:

Theorem 3.5. *Let $n \leq 7$ and M^n be an open band with Property B. If g is a complete metric on M with nonnegative scalar curvature, then (M, g) is isometric to*

$$(Y \times \mathbb{R}, g_Y + dt^2),$$

where (Y, g_Y) is a closed Ricci flat manifold.

3.1.2 Topological Results

Building on Section 2.1.2 we provide a list of open bands with Property B.

Proposition 3.6. *Let M^n be a connected orientable manifold without boundary. If*

- ▷ *$M = Y \times \mathbb{R}$, where Y is a closed orientable manifold which does not admit a metric with positive scalar curvature and $n \geq 6$, then M is an open band with Property B.*
- ▷ *there is a proper continuous map $f : M \to Y \times \mathbb{R}$ with nonzero degree, where Y is closed oriented and* NPSC^+, *then M is an open band with Property B.*
- ▷ *M is spin and $Y \subset M$ is a closed embedded incompressible hypersurface with trivial normal bundle and $\alpha(Y) \neq 0 \in KO_{n-1}(C^*\pi_1(Y))$, there is a covering space $\hat{M}$ which is an open band with Property B.*
- ▷ *$n \leq 6$ and $Y \subset X$ is a closed embedded incompressible hypersurface with trivial normal bundle which dominates an aspherical manifold, there is a covering space $\hat{M}$ which is an open band with Property B.*

Together with Theorem 3.5 we conclude:

Theorem 1.29. *Let $(n-1) \neq 4$ and $n \leq 7$. Let Y^{n-1} be a closed orientable manifold which does not admit a metric with positive scalar curvature and $X = Y \times \mathbb{R}$. Then X does not admit a complete metric with positive scalar curvature. Moreover, if g is a metric on X with non-negative scalar curvature, then (X, g) is isometric to $(Y \times \mathbb{R}, g_Y + dt^2)$, where g_Y is a Ricci flat metric on Y.*

3.2 Combining Potentials

As we did for part (1) of Theorem I, we will prove Theorem II by contradiction. Under the assumption that $\mathrm{width}(V_j, g) > \mathrm{width}(M_j, g_{\varphi_j})$ for all $j \in \{1, \dots, k+1\}$, we will produce a closed embedded hypersurface $\Sigma \subset X$ which separates $\partial_- X$ and $\partial_+ X$ and admits a metric with positive scalar curvature.

As before, Σ will appear as the boundary of a μ-bubble. The key ingredient for the corresponding functional is the potential function $h : X \to \mathbb{R}$. We can use the construction from Chapter 2 for each band (V_j, g) and model space (M_j, g_{φ_j})

separately to produce $h_j : V_j \to \mathbb{R}$ as the concatenation of a strictly 1-Lipschitz band map $(V_j, g) \to (M_j, g_{\varphi_j})$ and the function $h_{\varphi_j} : M_j \to \mathbb{R}$. Subsequently we use a gluing construction to paste all of the h_j together to obtain a smooth function $h : X \to \mathbb{R}$ which is suitable for our purposes.

The idea to combine potential functions in this way, was already used in [10, 76]. The gluing construction is based on the following result:

Lemma 3.7. *Let $h : [a, b] \to \mathbb{R}$ be a smooth strictly monotonously decreasing function such that*

$$-\frac{n}{n-1}h^2 - 2h' = \sigma,$$

for some constant $\sigma \in \mathbb{R}$. For every $\varepsilon > 0$ there is a function $\hat{h} : [a - \varepsilon, b + \varepsilon] \to \mathbb{R}$ such that:

- ▷ $\hat{h}(t) = h(t)$ *for* $t \in [a + \varepsilon, b - \varepsilon]$,
- ▷ $\hat{h}(t) = h(a)$ *in a neighborhood of* $a - \varepsilon$ *and* $\hat{h}(t) = h(b)$ *in a neighborhood of* $b + \varepsilon$,
- ▷ $\hat{h}' \leq 0$,
- ▷ $-\frac{n}{n-1}\hat{h}^2 - 2\hat{h}' \leq \sigma$ *and* $-\frac{n}{n-1}\hat{h}^2(t) - 2\hat{h}'(t) < \sigma$ *if* $\hat{h}'(t) = 0$.

Proof. Let $\rho : \mathbb{R} \to [a, b]$ be a smooth function with:

- ▷ $\rho(t) = a$ for $t \in (-\infty, a - \frac{\varepsilon}{2}]$, $\rho(t) = t$ for $t \in [a + \frac{\varepsilon}{2}, b - \frac{\varepsilon}{2}]$ and $\rho(t) = b$ for $t \in [b + \frac{\varepsilon}{2}, \infty)$.
- ▷ $0 < \rho'(t) < 1$ for $t \in (a - \frac{\varepsilon}{2}, a + \frac{\varepsilon}{2})$ and $t \in (b - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2})$.

Then the function $\hat{h} : [a - \varepsilon, b + \varepsilon] \to \mathbb{R}$ defined by $\hat{h} = h \circ \rho$ has all of the desired properties. The first two are immediate from the definition. The third one holds since $\hat{h}'(t) = h'(\rho(t))\rho'(t)$ and $h' < 0$ while $\rho' \geq 0$. To check the last property we point out that

$$-\frac{n}{n-1}\hat{h}^2(t) - 2\hat{h}'(t) = -\frac{n}{n-1}h(\rho(t)) - 2h'(\rho(t))\rho'(t) = \sigma + 2h'(\rho(t))(1 - \rho'(t)).$$

Since $h'(\rho(t)) < 0$ and $0 \leq \rho' \leq 1$ the above is always $\leq \sigma$ and it is $< \sigma$ if $\rho'(t) < 1$ which holds true in particular where $\hat{h}'(t) = 0$ ie $\rho'(t) = 0$. □

For the convenience of the reader we recall Lemma 2.34. Furthermore we summarize those results concerning μ-bubbles (see Section 2.3) which are needed to prove the partitioned comparison principle in Proposition 3.8.

Lemma 2.34. *Let (X, g) be a Riemannian band. If* $\operatorname{width}(X, g) > a - b$, *there is a smooth band map $\phi : (X, g) \to [a, b]$ with* $\operatorname{Lip}(\phi) < 1$.

Proposition 3.8. *Let $n \leq 7$ and (X^n, g) be an oriented Riemannian band. Let $h : X \to \mathbb{R}$ be a smooth function with the property that*

$$\operatorname{scal}(X, g) + \frac{n}{n-1}h^2 - 2|\nabla h| > 0.$$

If the mean curvature satisfies

$$H(\partial_\pm X, g) > \pm h\Big|_{\partial_\pm X},$$

there is a closed embedded hypersurface Σ *which separates* $\partial_- X$ *and* $\partial_+ X$ *such that*

$$-\Delta_\Sigma + \frac{1}{2}\operatorname{scal}(\Sigma, g) > 0.$$

Proof. Denote by $\mathcal{C}(X)$ the set of all Caccioppoli sets in X which contain an open neighborhood of $\partial_- X$ and are disjoint from $\partial_+ X$. For $\hat{\Omega} \in \mathcal{C}(X)$ consider the functional

$$\mathcal{A}_h(\hat{\Omega}) = \mathcal{H}^{n-1}(\partial^* \hat{\Omega} \cap \mathring{X}) - \int_{\hat{\Omega}} h d\mathcal{H}^n,$$

where $\partial^* \hat{\Omega}$ is the reduced boundary [22, Chapters 3, 4] of $\hat{\Omega}$.

By Lemma 2.39 there is a smooth so called *μ-bubble* $\Omega \in \mathcal{C}(X)$ ie a smooth Caccioppoli set with

$$\mathcal{A}_h(\Omega) = \mathcal{I} := \inf\{\mathcal{A}_h(\hat{\Omega}) \big| \hat{\Omega} \in \mathcal{C}(X)\}.$$

Denote $\partial\Omega \cap \mathring{X} = \Sigma$ and let ν be the outward pointing unit normal vector field to Σ. By the first variation formula for $\mathcal{A}_h$ (see Lemma 2.40) the mean curvature of Σ (computed with respect to $-\nu$) is equal to $h\big|_\Sigma$. By stability and the second variation formula (see Lemma 2.41) we conclude that

$$\begin{aligned}\int_\Sigma |\nabla_\Sigma \psi|^2 + \frac{1}{2}\operatorname{scal}(\Sigma, g)\psi^2 &\geq \int_\Sigma \frac{1}{2}(\operatorname{scal}(X, g) + \frac{n}{n-1}h^2 + 2g(\nabla_X h, \nu))\psi^2 \\ &> 0\end{aligned} \tag{3.2.1}$$

for all $\psi \neq 0 \in C^\infty(\Sigma)$. ☐

3.3 Proof of the Partitioned Comparison Principle

We have gathered all the ingredients we need to prove Theorem II.

Proof of Theorem II. Assume for a contradiction that $\operatorname{width}(V_j, g) > \operatorname{width}(M_j, g_{\varphi_j})$ for all $j \in \{1, \ldots, k+1\}$. By Lemma 2.34 there are smooth maps

$$\beta_j : V_j \to [a_j - \varepsilon, b_j + \varepsilon]$$

such that $\beta_j(\partial_- V_j) = a_j - \varepsilon$, $\beta_j(\partial_+ V_j) = b_j + \varepsilon$ and $\operatorname{Lip}(\beta_j) < 1$. Consider the functions

$$h_{\varphi_j}(t) = (n-1)\frac{\varphi_j'(t)}{\varphi_j(t)} : [a_j, b_j] \to \mathbb{R};$$

Since the φ_j are strictly log-concave the h_j are strictly monotonously decreasing and the scalar curvature of (M_j, g_{φ_j}) is given by:

$$\sigma_j = \operatorname{scal}(M_j, g_{\varphi_j}) = -\frac{n}{n-1}h_{\varphi_j}^2 - 2h_{\varphi_j}'. \tag{3.3.1}$$

For $j \in \{2, \ldots, k\}$ we apply Lemma 3.7 to h_{φ_j} and obtain smooth functions

$$\hat{h}_{\varphi_j} : [a_j - \varepsilon, b_j + \varepsilon] \to \mathbb{R}$$

with the aforementioned properties.

For $j = 1$ we extend the domain of h_{φ_j} to $[a_1 - \varepsilon, b_1]$ and apply the interpolation procedure of Lemma 3.7 only on the right hand side of the interval to produce $\hat{h}_{\varphi_1} : [a_1 - \varepsilon, b_1 + \varepsilon] \to \mathbb{R}$. For $j = k+1$ we extend the domain of $h_{\varphi_{k+1}}$ to $[a_{k+1}, b_{k+1} + \varepsilon]$ and apply the interpolation procedure of Lemma 3.7 only on the left hand side of the interval to produce $\hat{h}_{\varphi_{k+1}} : [a_{k+1} - \varepsilon, b_{k+1} + \varepsilon] \to \mathbb{R}$.

Finally, we define $h : X \to \mathbb{R}$ by $h(x) = \hat{h}_{\varphi_j} \circ \beta_j(x)$ if $x \in V_j$. The function h is continuous since

$$\hat{h}_{\varphi_j}(b_j + \varepsilon) = H(\partial_+ M_j, g_{\varphi_j}) = -H(\partial_- M_{j+1}, g_{\varphi_{j+1}}) = \hat{h}_{\varphi_{j+1}}(a_{j+1} - \varepsilon)$$

for all $j \in \{1, \ldots, k\}$. It is smooth since the $\hat{h}_{\varphi_j} \circ \beta_j$ are constant in a neighborhood of the separating hypersurfaces Σ_i, which partition the band.

Furthermore

$$\begin{aligned} &\operatorname{scal}(X, g) + \frac{n}{n-1} h^2 - 2|\nabla h| \\ \geq &\operatorname{scal}(M_j, g_{\varphi_j}) + \frac{n}{n-1}(\hat{h}_{\varphi_j} \circ \beta_j)^2 + 2|\nabla \beta_j|((\hat{h}'_{\varphi_j}) \circ \beta_j) \\ > &- \frac{n}{n-1} h^2_{\varphi_j} - 2h'_{\varphi_j} - \sigma_j = 0 \end{aligned}$$

by (3.3.1), the chain rule, the third and fourth property of the $\hat{h}_j$ from Lemma 3.7 and since $\operatorname{Lip}(\beta_j) < 1$ ie $|\nabla \beta_j| < 1$.

For the mean curvature of the boundary, the following holds true:

$$H(\partial_- X, g) \geq H(\partial_- M_1, g_{\varphi_1}) = -\hat{h}_{\varphi_1}(a_1) > -\hat{h}_{\varphi_1}(a_1 - \varepsilon) = -h\Big|_{\partial_- X}$$

and

$$H(\partial_+ X, g) \geq H(\partial_+ M_{k+1}, g_{\varphi_{k+1}}) = \hat{h}_{\varphi_{k+1}}(b_{k+1}) > \hat{h}_{\varphi_{k+1}}(b_{k+1} + \varepsilon) = h\Big|_{\partial_+ X}.$$

By Proposition 3.8 there is a closed embedded hypersurface Σ^{n-1} such that

$$-\Delta_\Sigma + \frac{1}{2}\operatorname{scal}(\Sigma, g) > 0$$

and Σ separates $\partial_- X$ and $\partial_+ X$.

If $n = 2$, this yields an immediate contradiction (choose $\psi = 1$). If $n \geq 3$, then Σ admits a metric of positive scalar curvature by Lemma 2.50 which contradicts the fact that X has Property A. □

Theorem 1.28 and Theorem 3.2 are more or less direct applications of Theorem II for a suitable choice of partition and model spaces (M_j, g_{φ_j}). The main examples of strictly log-concave model spaces we use, were introduced in Section 2.1.2.

Proof of Theorem 1.28. Consider the function

$$\varphi_2 : \left(-\frac{\pi}{\sqrt{\kappa n}}, \frac{\pi}{\sqrt{\kappa n}}\right) \to \mathbb{R}_+ \quad t \mapsto \cos\left(\frac{\sqrt{\kappa n} t}{2}\right)^{\frac{2}{n}},$$

which is strictly log-concave and has

$$h_{\varphi_2}(t) = -\sqrt{\kappa}(n-1)\tan\left(\frac{\sqrt{\kappa n} t}{2}\right).$$

Consider the function

$$\varphi_1 : \mathbb{R}_+ \to \mathbb{R}_+ \quad t \mapsto t^{\frac{2}{n}},$$

which is strictly log-concave and has

$$h_{\varphi_1}(t) := \frac{2(n-1)}{nt}.$$

Since $h_{\varphi_1}(t) \to \infty$ as $t \to 0$ there is a value $t_- > 0$ such that $H(\partial_- X, g) \geq -h_{\varphi_1}(t_-)$. By continuity there are $\delta_1, \delta_2 > 0$ small enough such that

$$h_{\varphi_2}\left(\frac{-d+\delta_1}{2}\right) = h_{\varphi_1}(\ell + \delta_2),$$

while $\delta_2 < t_-$ and hence $\ell + \delta_2 - t_- < \ell$. Let (N, g_N) be a closed scalar flat Riemannian manifold. We fix the model space:

$$(M_1, g_{\varphi_1}) = (N \times [t_-, \ell + \delta_2], \varphi_1^2(t) g_N + dt^2)$$

with scalar curvature equal to zero and width $< \ell$.

Let $\varphi_3 : \mathbb{R}_- \to \mathbb{R}_+$ be defined by $\varphi_3(t) = \varphi_1(-t)$. This function is strictly log-concave and $h_{\varphi_3}(t) = -h_{\varphi_1}(-t)$. Since $h_{\varphi_3}(t) \to -\infty$ as $t \to 0$ there is a value $t_+ < 0$ such that $H(\partial_X, g) \geq h_{\varphi_3}(t_+)$. Similarly as before we find $\delta_3, \delta_4 > 0$ such that

$$h_{\varphi_2}\left(\frac{d-\delta_3}{2}\right) = h_{\varphi_3}(-\ell - \delta_4),$$

while $\delta_4 < -t_+$ and hence $\ell + \delta_4 + t_+ < \ell$. We fix the model space:

$$(M_3, g_{\varphi_3}) = (N \times [t_-, \ell + \delta_4], \varphi_3^2(t) g_N + dt^2)$$

with scalar curvature equal to zero and width $< \ell$.

Finally we fix the model space

$$(M_2, g_{\varphi_2}) = \left(N \times \left[\frac{-d+\delta_1}{2}, \frac{d-\delta_3}{2}\right], \varphi_2^2(t) g_N + dt^2\right)$$

with scalar curvature equal to $\kappa n(n-1)$ and width $< d$.

It follows from Theorem II that $\text{width}(V_j, g) \leq \text{width}(M_j, g_{\varphi_j})$ for at least one $i \in \{1, 2, 3\}$. Since $d = \text{width}(V_2, g) > \text{width}(M_2, g_{\varphi_2})$ we conclude that

$$\min\{\text{width}(V_1, g), \text{width}(V_3, g)\} < \ell,$$

which is what we wanted to prove. □

Proof of Theorem 3.2. Consider the function

$$\varphi_2 : (-\frac{\pi}{\sqrt{\kappa n}}, \frac{\pi}{\sqrt{\kappa n}}) \to \mathbb{R}_+ \quad t \mapsto \cos\left(\frac{\sqrt{\kappa n} t}{2}\right)^{\frac{2}{n}},$$

which is strictly log-concave and has

$$h_{\varphi_2}(t) = -\sqrt{\kappa}(n-1)\tan\left(\frac{\sqrt{\kappa n} t}{2}\right).$$

Consider the function

$$\varphi_1 : \mathbb{R}_+ \to \mathbb{R}_+ \quad t \mapsto \sinh\left(\frac{\sqrt{\sigma n} t}{2\sqrt{n-1}}\right)^{\frac{2}{n}},$$

which is strictly log-concave and has

$$h_{\varphi_1}(t) := \sqrt{\frac{\sigma(n-1)}{n}} \coth\left(\frac{\sqrt{\sigma n} t}{2\sqrt{n-1}}\right).$$

Since $h_{\varphi_1}(t) \to \infty$ as $t \to 0$, there is a value $t_- > 0$ such that the mean curvature $H(\partial_- X, g)$ is greater or equal to $-h_{\varphi_1}(t_-)$. By continuity there are $\delta_1, \delta_2 > 0$ small enough such that $-\sigma > -\kappa n(n-1)\tan\left(\frac{\sqrt{\kappa n}(d-2\delta_1)}{4}\right)^2$ and

$$h_{\varphi_2}\left(\frac{-d+\delta_1}{2}\right) = h_{\varphi_1}(\ell + \delta_2),$$

while $\delta_2 < t_-$ and hence $\ell + \delta_2 - t_- < \ell$. Let (N, g_N) be a closed scalar flat Riemannian manifold. We fix the model space:

$$(M_1, g_{\varphi_1}) = (N \times [t_-, \ell + \delta_2], \varphi_1^2(t) g_N + dt^2)$$

with scalar curvature equal to $-\sigma$ and width $< \ell$.

Let $\varphi_3 : \mathbb{R}_- \to \mathbb{R}_+$ be defined by $\varphi_3(t) = \varphi_1(-t)$. This function is strictly log-concave and $h_{\varphi_3}(t) = -h_{\varphi_1}(-t)$. Since $h_{\varphi_3}(t) \to -\infty$ as $t \to 0$ there is a value $t_+ < 0$ such that $H(\partial_X, g) \geq h_{\varphi_3}(t_+)$. Similarly as before we find $\delta_3, \delta_4 > 0$ such that $-\sigma > -\kappa n(n-1)\tan\left(\frac{\sqrt{\kappa n}(d-2\delta_3)}{4}\right)^2$ and

$$h_{\varphi_2}\left(\frac{d-\delta_3}{2}\right) = h_{\varphi_3}(-\ell - \delta_4),$$

while $\delta_4 < -t_+$ and hence $\ell + \delta_4 + t_+ < \ell$. We fix the model space

$$(M_3, g_{\varphi_3}) = (N \times [-\ell - \delta_4, t_+], \varphi_3^2(t) g_N + dt^2)$$

with scalar curvature equal to $-\sigma$ and width $< \ell$.

Finally we fix the model space

$$(M_2, g_{\varphi_2}) = (N \times \left[\frac{-d+\delta_1}{2}, \frac{d-\delta_3}{2}\right], \varphi_2^2(t) g_N + dt^2)$$

with scalar curvature equal to $\kappa n(n-1)$ and width $< d$.

It follows from Theorem II that $\mathrm{width}(V_j, g) \leq \mathrm{width}(M_j, g_{\varphi_j})$ for at least one $i \in \{1,2,3\}$. Since $d = \mathrm{width}(V_2, g) > \mathrm{width}(M_2, g_{\varphi_2})$ we conclude that

$$\min\{\mathrm{width}(V_1, g), \mathrm{width}(V_3, g)\} < \ell,$$

which is what we wanted to prove. □

With Theorem II and its applications established, we now focus on open bands for the rest of this chapter. First of all we prove that in an open band we can always find a closed embedded hypersurface which properly separates the two ends $\mathcal{E}_-M$ and $\mathcal{E}_+M$ and we can find them arbitrarily far out.

Lemma 3.9. *Let M be an open band and $K \subset M$ be an arbitrary compact subset. Then there exists a properly separating hypersurface $\Sigma \subset M$ which also separates K from $\mathcal{E}_+M$ (or $\mathcal{E}_-M$, respectively).*

Proof. Note that the end compactification $\mathcal{F}M$ is a compact Hausdorff space which in our case of a connected manifold is also second countable (see [19, Proposition 2.5]). Thus, since $\mathcal{E}_\pm M$ are two disjoint closed subsets of $\mathcal{F}M$, Urysohn's lemma implies the existence of a continuous function $f\colon \mathcal{F}M \to [-1,1]$ such that $\mathcal{E}_\pm M = f^{-1}(\pm 1)$. Since $K \subseteq M$ is compact, there exists $0 < r < 1$ such that $K \subseteq f^{-1}([-r,r])$. Choose $s \in (r,1)$. Then $f^{-1}(s) \subseteq M$ is a compact subset which separates $\mathcal{E}_-M$ from $\mathcal{E}_+M$. Now choose a connected compact n-dimensional submanifold $V \subset M$ with boundary, where $n = \dim(M)$, such that $f^{-1}(s) \subseteq \mathring{V} \subseteq f^{-1}([s-\varepsilon, s+\varepsilon])$ for some $\varepsilon > 0$ with $r < s - \varepsilon$. Then ∂V is a separating hypersurface and it contains a properly separating hypersurface $\Sigma \subseteq \partial V$ by Lemma 2.56. Since by construction $f(x) \le r < s - \varepsilon \le f(y) \le s + \varepsilon < 1$ for each $x \in K$ and $y \in \Sigma$, it follows that Σ must separate K from $\mathcal{E}_+M$. A completely analogous argument also provides a properly separating hypersurface that separates K from $\mathcal{E}_-M$. □

Next, we prove Proposition 3.4 ie the assertion that for any complete Riemannian metric on an open band M there is a geodesic line which connects $\mathcal{E}_-M$ and $\mathcal{E}_+M$.

Proof of Proposition 3.4. Consider an exhaustion K^i of M by compact sets. By Lemma 3.9 we can assume without loss of generality that each K_i has smooth boundary ∂K^i and that $M \backslash K^i = U_- \sqcup U_+$, where $U_\pm$ are open sets with $\mathcal{E}_\pm M \subset U_\pm$. We decompose

$$\partial K^i = \partial_- K^i \sqcup \partial_+ K^i$$

accordingly as $\partial_\pm K^i := \partial U_\pm$. Let γ_i be a length minimizing geodesic which connects $\partial_- K^i$ and $\partial_+ K^i$.

On each compact set K^i the sequence $\gamma_i^{j \ge i}$ of geodesics connecting $\partial_- K^i$ and $\partial_+ K^i$ subconverges to a length minimizing geodesic by Arzelà-Ascoli. Now a diagonal sequence (over i) converges to a geodesic line in (M, g). □

Finally, we have all the necessary ingredients to prove Theorem 3.5.

Proof of Theorem 3.5. If (M, g) is not Ricci flat, then M admits a complete metric $\hat{g}$ with positive scalar curvature by [45, Theorem B]. Let $\Sigma \subset M$ be a closed embedded proper separating hypersurface which separates $\mathcal{E}_-M$ and $\mathcal{E}_+M$ in M. It exists by Lemma 3.9. There is a $\kappa > 0$ such that $\mathrm{scal}(M, \hat{g}) \ge \kappa n(n-1)$ in a neighborhood of width $d < \frac{2\pi}{\sqrt{\kappa n}}$, which is bounded by two closed embedded hypersurfaces Σ_1 and Σ_2. For every $C > 0$ we can find closed embedded hypersurfaces $\Sigma_\pm^C$ such that $\Sigma_\pm^C$ properly separates Σ and $\mathcal{E}_\pm M$ and

$$\mathrm{dist}_{\hat{g}}(\Sigma_-^C, \Sigma_1) = C = \mathrm{dist}_{\hat{g}}(\Sigma_2, \Sigma_+^C).$$

Let X^C be the compact band bounded by $\Sigma_\pm^C$. For C large enough Theorem 1.28 or 3.2 applied to X^C yields a contradiction. Hence (X, g) is Ricci flat.

By Proposition 3.4 and the Cheeger-Gromoll splitting theorem (M, g) is isometric to $(Y \times \mathbb{R}, g_Y + dt^2)$, where (Y, g_Y) is a Ricci flat manifold. Furthermore M has at least two ends by definition of an open band and hence Y is compact. □

3.4 Proof of the Topological Results

We conclude this chapter with a proof of Proposition 3.6. Hereby, we mostly adapt ideas from Section 2.5.1 to the non-compact setting:

Proof of Proposition 3.6. Since $Y \times \mathbb{R}$ is diffeomorphic to $Y \times (-1, 1)$ the first point follows directly from Proposition 2.13.

To prove the second point we denote $X' = Y \times \mathbb{R}$. Since f is proper it maps ends to ends and we define $\mathcal{E}_\pm M = f^{-1}(\mathcal{E}_\pm X')$. Since f has non-zero degree $\mathcal{E}_\pm M$ are both non empty closed collections of ends. From here we proceed as in Lemma 2.57 (replace $H^1(X', \partial X'; \mathbb{Z})$ by $H^1_c(X'; \mathbb{Z})$ and use the fact that f is proper) to see that any closed embedded separating Σ which separates $\mathcal{E}_- M$ and $\mathcal{E}_+ M$ has a connected component Σ_0 such that $(f \circ \pi)_*[\Sigma_0] = \deg(f)[Y]$ where $\pi : Y \times \mathbb{R} \to Y$ is the projection. Then Σ_0 does not admit a metric with positive scalar curvature, since Y is NPSC$^+$.

For the third point let $\hat{M}$ be the covering space with $\pi_1(\hat{M}) = \iota_*(\pi_1(Y))$ where $\iota : Y \hookrightarrow M$ is the inclusion. Since Y is incompressible $\iota_*(\pi_1(Y))$ is isomorphic to $\pi_1(Y)$. We denote by $\hat{Y}$ a copy of Y in $\hat{M}$. We point out that $\hat{Y}$ separates $\hat{M}$ into two components U_- and U_+. If this were not the case we could find a loop $\gamma \subset \hat{M}$ which has algebraic intersection number 1 with Σ. Since $\pi_1(\hat{Y}) \to \pi_1(\hat{M})$ is surjective there is a loop γ' in $\hat{Y}$, which is homotopic to γ. Since $\hat{Y}$ has trivial normal bundle γ' can be pushed out of $\hat{Y}$. Since the intersection number is invariant under homotopy this is a contradiction. The two components $U_\pm$ are unbounded since otherwise $\alpha(Y)$ would be zero by bordism invariance of the index. We define the ends of U_- to be $\mathcal{E}_-\hat{M}$ and the ends of U_+ to be $\mathcal{E}_+\hat{M}$. If Σ is a closed embedded hypersurface which separates $\mathcal{E}_-\hat{M}$ and $\mathcal{E}_+\hat{M}$ there is a union Σ' of components of Σ which is a properly separating hypersurface by Lemma 3.9. Let $K \subset \hat{M}$ be a band which contains $\Sigma' \cup \hat{Y}$ and such that $\partial_- K$ separates $\mathcal{E}_-\hat{M}$ and $\Sigma' \cup \hat{Y}$ (to find K one can use Lemma 3.9). Let $W \subset \hat{M}$ be the cobordism between $\partial_- K$ and $\hat{Y}$ and $W' \subset \hat{M}$ be the cobordism between $\partial_- K$ and Σ'.

Let $E = \mathcal{L}_{\hat{M}} \to \hat{M}$ be the Miščenko bundle, which is the flat bundle of finitely generated projective Hilbert-$\mathrm{C}^*\pi_1\hat{M}$-modules associated to the representation of $\pi_1\hat{M}$ on $\mathrm{C}^*\pi_1\hat{M}$ by left multiplication. Since $\pi_1(\hat{Y}) = \pi_1(Y) = \pi_1(\hat{M})$ the Rosenberg index $\alpha(\hat{Y}) \in \mathrm{KO}_{n-1}(\mathrm{C}^*\pi_1(\hat{Y}))$ is equal to the (K-theoretic) index $\alpha_E(\hat{Y})$ of the Dirac operator on the spinor bundle of $\hat{Y}$ twisted with the restriction of E to $\hat{Y}$. In the same way we can compute the indices $\alpha_E(\partial_- K) \in \mathrm{KO}_{n-1}(\mathrm{C}^*\pi_1(\hat{Y}))$ and $\alpha_E(\Sigma') \in \mathrm{KO}_{n-1}(\mathrm{C}^*\pi_1(\hat{Y}))$. By bordism invariance of the index

$$0 \neq \alpha(\hat{Y}) = \alpha_E(\hat{Y}) = \alpha_E(\partial_- K) = \alpha_E(\Sigma').$$

Consequently there is a connected component of Σ' which does not admit a metric with positive scalar curvature.

The last point is explained in [15, Section 2]. Let $\hat{M}$ be the covering space with $\pi_1(\hat{M}) = \iota_*(\pi_1(Y))$, where $\iota : Y \hookrightarrow M$ is the inclusion, and let $\hat{Y}$ and $\mathcal{E}_\pm\hat{M}$ be defined as before. By assumption there is an aspherical manifold Z and a map $f : \hat{Y} \to Z$ of non-zero degree. Consider the induced homorphism $f_* : \pi_1(\hat{Y}) = \pi_1(\hat{M}) \to \pi_1(Z)$. Since Z is aspherical we can define an extension $\hat{f} : \hat{M} \to Z$ of f with $\hat{f}_* = f_*$. Then any closed embedded hypersurface Σ which separates $\mathcal{E}_-\hat{M}$ and $\mathcal{E}_+\hat{M}$ has a connected component Σ_0 such that $\hat{f}_*([\Sigma_0]) \neq 0 \in H_{n-1}(Z;\mathbb{Z})$. Since Z is NPSC$^+$ we conclude that Σ_0, and therefore Σ, does not admit a metric with positive scalar curvature. □

Proof of Theorem 1.29. We use the fact that any closed connected oriented manifold Y in dimension ≤ 3 which does not admit a metric with positive scalar curvature is enlargeable and hence NPSC$^+$. By Proposition 3.6, M is an open band with Property B. Therefore we can use Theorem 3.5 to conclude the proof. □

Macroscopic Band Width Inequalities

In this chapter, which corresponds to the article [55] which was first published in the journal *Algebraic & Geometric Topology* 22-1 (2022) by Mathematical Science Publishers, we investigate an analog of the band width inequalities we have seen so far in the setting of *macroscopic scalar curvature*.

For a detailed introduction of this concept, the ideas connected to it and a detailed overview of the results we establish in the following, we refer to Section 1.3 in the introduction.

Remark 4.1. We point out that the notation in this chapter varies slightly from the rest of this thesis. In particular, it differs from the one we used in Section 1.3. In the following, bands will usually be denoted by V and closed manifolds by M or N, while X resp. Y are reserved for Riemannian polyhedra.

4.1 Main results

We introduce a class of orientable manifolds we call *filling enlargeable* (see Definiton 4.22) in Section 4.2. This notion combines Gromov and Lawson's [28] classical definition of *enlargeability* with the *filling radius* of a complete oriented Riemannian manifold, a metric invariant that was introduced by Gromov in [23, Section 1].

Some important features of filling enlargeable manifolds we establish in this paper are:

- ▷ If a closed orientable manifold is enlargeable or aspherical, then it is filling enlargeable (see Propositions 4.24 and 4.28).
- ▷ Closed filling enlargeable manifolds are essential. In fact we prove an even stronger statement in Theorem 4.4 (see also Remark 4.39 and Remark 4.54).
- ▷ For all $n \geq 1$ there is a constant $\varepsilon_n > 0$ such that the following holds. Let M^n be a closed filling enlargeable manifold and g a Riemannian metric on M. For any radius $R > 0$ there is a point p in the universal cover $(\widetilde{M}, \widetilde{g})$ with $\mathrm{vol}(B_R(p)) \geq \varepsilon_n R^n$ (see Proposition 4.34).

In light of the third point and in the context of macroscopic scalar curvature we prove, in Section 4.3, the following macroscopic analog of the band width inequalities with scalar curvature for trivial bands over filling enlargeable manifolds:

Theorem 4.2 (cf. Theorem III)**.** *For all $n \geq 1$ there is a constant $\varepsilon_n > 0$ such that the following holds. Let M^{n-1} be a closed filling enlargeable manifold and $V := M \times [0,1]$. If g is a Riemannian metric on V with the property that all unit balls in the universal cover $(\widetilde{V}, \widetilde{g})$ have volume less than $\frac{1}{2}\varepsilon_n$, then* $\operatorname{width}(V, g) \leq 1$.

It is a natural question to ask whether Theorem 4.2 holds true for all essential manifolds. However, as it turns out, there are some immediate counterexamples (see Example 4.42). In order to obtain similar results one has to add further assumptions regarding the systole (the length of the shortest noncontractible loop) of (V, g):

Theorem 4.3. *For all $n \geq 1$ there is a constant $\varepsilon_n > 0$ such that the following holds. Let M^{n-1} be a closed essential manifold and $V := M \times [0,1]$. Let g be a Riemannian metric on V and $0 < R < \frac{1}{2}\operatorname{sys}(V, g)$. If every ball of radius R in $(\widetilde{V}, \widetilde{g})$ has volume less than $\frac{1}{2}\varepsilon_n R^n$, then* $\operatorname{width}(V, g) \leq R$.

In Section 4.4 we follow ideas of Brunnbauer and Hanke [8] and study some functorial properties of filling enlargeable manifolds. In particular we construct a vector subspace $H_n^{sm}(B\Gamma; \mathbb{Q}) \subset H_n(B\Gamma; \mathbb{Q})$ of 'small classes' in the rational group homology of a finitely generated group Γ and prove:

Theorem 4.4 (cf. Theorem 1.37)**.** *Let M^n be a closed oriented manifold. Then M is filling enlargeable if and only if $\phi_*[M] \in H_n(B\pi_1(M); \mathbb{Q})$ is not contained in $H_n^{sm}(B\pi_1(M); \mathbb{Q})$.*

As a consequence, we obtain the following metric characterization of rationally essential manifolds with residually finite fundamental groups:

Corollary 4.5 (cf. Corollary 1.38)**.** *A closed oriented manifold M^n with residually finite fundamental group is rationally essential if and only if it is filling enlargeable.*

4.2 Largeness properties of manifolds

Even though our results are stated for smooth manifolds, almost all of our actual work takes place in the setting of simplicial complexes with piecewise smooth metrics (see Babenko [4, Section 2]).

Definition 4.6. By a Riemannian metric on a k-simplex Δ^k we understand the pullback of an arbitrary Riemannian metric on $\mathbb{R}^k$ via an affine linear embedding $\Delta^k \hookrightarrow \mathbb{R}^k$. A *Riemannian metric* on a simplicial complex X is given by a Riemannian metric g_τ on every simplex, such that $g_{\tau'} \equiv g_\tau|_{\tau'}$ for $\tau' \subset \tau$. We call (X, g) a *Riemannian polyhedron.*

A Riemannian metric g enables us to measure the lengths of piecewise smooth curves in a simplicial complex X^n. As for Riemannian manifolds one obtains a path metric d_g on X^n if X^n is connected. Moreover there is an obvious notion of n-dimensional Riemannian volume, coinciding with the n-dimensional Hausdorff measure. In the following we introduce some classical metric invariants used to describe the size of (X, g).

We remind the reader that a metric space is called *proper* if every closed and bounded subset is compact. Furthermore a continuous map between metric spaces is called *proper* if the preimage of every bounded set is bounded. It is a classical result, that a path metric space is proper if and only if it is locally compact and complete.

Remark 4.7. In most of the literature a continuous map between topological spaces is called proper, if the preimage of every compact set is compact. Our definition coincides with the classical one for continuous maps between proper metric spaces.

While Theorem 4.2 and Theorem 4.3 are concerned with compact metric spaces, the relevance of proper metric spaces is that we will be talking about possibly infinite-sheeted covering spaces of these compact spaces.

Definition 4.8. The k-dimensional Alexandrov width $\mathrm{UR}_k(X,g)$ of a proper Riemannian polyhedron (X,g) is the infimum over all values $R>0$, such that: there is a continuous map $f : X \to Y^k$ to a locally finite k-dimensional simplicial complex Y, with the property that the preimage $f^{-1}(\tau)$ of every simplex $\tau \subset Y$ is contained in a metric ball of radius R in X.

While originally appearing in the context of topological dimension theory, this notion was popularized in Riemannian Geometry by works of Gromov (see [23, Appendix 1] or [25]).

Over the course of this paper we will often use an equivalent definition of the k-dimensional Alexandrov width in terms of open covers, namely: $\mathrm{UR}_k(X,g)$ is the infimum over all R such that X admits a locally finite cover by open sets of radii $\leq R$ (ie contained in a metric ball of radius R) and multiplicity $\leq k+1$.

The following Lemma, which (only in the compact case) appears in [25, Section (H)], modulates between this definition and Definition 4.8. For the convenience of the reader, and since we need to be a little bit more careful, when considering non-compact polyhedra, we include a proof of this result.

Lemma 4.9. *A proper Riemannian polyhedron (X,g) has k-dimensional Alexandrov width $\mathrm{UR}_k(X,g) < R$ if and only if there is a locally finite open cover of X with multiplicity $\leq k+1$ and radius $< R$*

Proof. Suppose that X has a locally finite open cover O_i with multiplicity $\leq k+1$ and radius $< R$. If we denote by N the nerve of this cover, the nerve map ϕ associated to a partition of unity, subordinate to O_i, is a continuous map $X \to N$ and for each simplex $\tau \subset Y$ the preimage $\phi^{-1}(\tau)$ is contained in one of the sets O_i.

For the other direction suppose that (X,g) has $\mathrm{UR}_k(X,g) < R$ ie there is a locally finite simplicial complex Y^k and a continuous map $f : X \to Y$ such that for each simplex $\tau \subset Y$ the preimage $f^{-1}(\tau)$ is contained in a ball of radius $\mathrm{UR}_k(X,g)+\delta < R$ for some small $\delta > 0$.

We equip Y with the canonical path metric ie we equip every simplex with the standard euclidean metric and consider the induced path metric on Y. First we want to show that the map f is in fact proper. A bounded subset $K \subset Y$ intersects only finitely many simplices of Y. In particular it is contained in a finite subcomplex $K' \subset Y$. Since (X,g) is proper and the preimage of every simplex is closed and bounded we conclude that $f^{-1}(K')$ is bounded. But then $f^{-1}(K)$ is a closed subset of a bounded set and thus it is bounded as well.

Since Y is k-dimensional, we prove the following: there is an open cover O_i of Y with multiplicity $\leq k+1$ such that $f^{-1}(O_i)$ is contained in a ball of radius $< R$ for all i. We start by claiming that for every simplex $\tau \subset Y$ there is a constant $\varepsilon(\tau) > 0$ such that $f^{-1}(U_{\varepsilon(\tau)}(\tau))$ is contained in a ball of radius $< R$ in (X,g). By assumption $f^{-1}(\tau)$ is contained in a ball of radius $\mathrm{UR}_k(X,g)+\delta < R$ around a point $p \in X$.

Now we assume for a contradiction, that for every $\ell \in \mathbb{N}$ there is a point $y_\ell \in U_{1/\ell}(\tau)$ with $f^{-1}(y_\ell) \not\subset B_{R-1/\ell}(p)$ and choose a preimage $x_\ell \notin B_{R-1/\ell}(p)$. Up to a subsequence $(y_\ell)_{\ell\in\mathbb{N}}$ converges to a point $y \in \tau$. Since $(x_\ell)_{\ell\in\mathbb{N}}$ is contained in the compact set $f^{-1}(\overline{U_1(\tau)})$ there is a subsequence converging to a point $x \in X\backslash B_R(p)$, which contradicts the continuity of f.

To construct O_i we consider the skeleta of Y one at a time, starting with the vertices. If $v \in Y$ is a vertex then $f^{-1}(B_{\varepsilon(v)}(v))$ is contained in a ball of radius $< R$ in (X, g) and we can assume that for two vertices v_1 and v_2 the balls $B_{\varepsilon(v_1)}(v_1)$ and $B_{\varepsilon(v_2)}(v_2)$ are disjoint. Now let $\gamma \subset Y$ be an edge connecting vertices v_1 and v_2. Denote $\gamma' = \gamma\backslash(B_{\varepsilon(v_1)}(v_1) \cup B_{\varepsilon(v_2)}(v_2))$. Then $f^{-1}(U_{\varepsilon(\gamma)}(\gamma'))$ is contained in a ball of radius $< R$ in (X, g) and by possibly making $\varepsilon(\gamma)$ smaller we can arrange that $U_{\varepsilon(\gamma)}(\gamma')$ does not intersect $U_{\varepsilon(\eta)}(\eta')$ for any other edge $\eta \subset Y$.

We continue this process for all higher skeleta of Y and make sure that for each skeleton all newly introduced open sets are disjoint, which provides the upper bound on the multiplicativity of this cover. Finally $f^{-1}(O_i)$ is the required cover of (X, g). □

For us the most important result concerning the Alexandrov width of a proper Riemannian polyhedron (X^n, g) is the following theorem (which is a version of [52, Theorem 3.1]), providing an upper bound on $\mathrm{UR}_{n-1}(X, g)$ under the assumption that for a fixed radius $R > 0$ all R-balls in (X, g) have very small volume.

Theorem 4.10. *For all $n \geq 1$ there is a constant $\varepsilon_n > 0$ such that the following holds: If (X^n, g) is a proper Riemannian polyhedron and $R > 0$ is a radius such that for every $x \in X$ the volume of the ball $B_R(x)$ is bounded from above by $\varepsilon_n R^n$, then* $\mathrm{UR}_{n-1}(X, g) \leq R$. □

Remark 4.11. In the case of complete smooth Riemannian manifolds this was first proved by Guth in [37]. His theorem was generalized by Liokumovich, Lishak, Nabutovsky and Rotman [49] to metric spaces and Hausdorff content instead of volume. Building on ideas from [35] the proof of this result was significantly simplified by Papasoglu [52, Theorem 3.1]. Finally Nabutovsky [51, Theorem 2.6] was able to improve the constant ε_n from an exponential bound to a linear one in the case of compact Riemmanian polyhedra.

Remark 4.12. Any smooth Riemannian manifold (M, g) becomes a Riemannian polyhedron by choosing a smooth triangulation of M. For the rest of this paper, a Riemannian metric on M, if not explixitly required to be smooth, is a polyhedral metric with respect to some smooth triangulation of M. Furthermore all manifolds are assumed to be connected.

Next, we revisit the *filling radius* of a complete oriented Riemannian manifold (M^n, g) (we mostly follow [8, Section 2]). The orientation corresponds to a fundamental class $[M] \in H_n^{lf}(M; \mathbb{Z})$ in locally finite homology (ie we consider infinite chains with the property that each bounded subset intersects only finitely many singular simplices).

Denote by $L^\infty(M)$ the vector space of all functions $M \to \mathbb{R}$ with the uniform 'norm' $\|-\|_\infty$. We consider the affine subspace $L^\infty(M)_b$ of $L^\infty M$, that is parallel to the Banach space of all bounded functions on M and contains the distance function $d_g(x, -)$ for some $x \in M$. Notice that $\|-\|_\infty$ defines an actual metric on $L^\infty(M)_b$.

The *Kuratowski embedding*

$$\iota_g : (M, d_g) \hookrightarrow L^\infty(M)_b \quad x \mapsto d_g(x, -)$$

is isometric by the triangle inequality.

Definition 4.13. The *filling radius* of (M, g) is defined as

$$\mathrm{FillRad}(M, g) := \inf \{r > 0 | \iota_{g*}[M] = 0 \in H_n^{lf}(U_r(\iota_g M); \mathbb{Q})\},$$

where $U_r(\iota_g M)$ denotes the open r-neighborhood of $\iota_g M$ in $L^\infty(M)_b$. If the set on the right hand side is empty we say that (M, g) has infinite filling radius.

If M is closed, then $L^\infty(M)_b$ is the vector space of all bounded functions and the above definition coincides with the classical one from [23, Section 1]. We remind the reader that for an arbitrary metric space S, the space $L^\infty(S)$ of all functions has the following universal property.

Lemma 4.14. *If $Y \subset X$ is a subspace of a metric space and if $f : Y \to L^\infty(S)$ is an L-Lipschitz map, then there exists an extension $F : X \to L^\infty(S)$ which is also L-Lipschitz.* □

This is due to Gromov [23, Page 8]. There it is only stated for closed Riemannian manifolds, but the proof works in the general setting. The extension F is given by

$$F_x(v) := \inf_{y \in Y} (f_y(v) + L \cdot d(x, y)).$$

For our purposes we also need the following property of F:

Lemma 4.15. *If f is proper and $d(\cdot, Y)$ is uniformly bounded in X, then F is proper as well.*

Proof. Let $K \subset L^\infty(S)$ be a bounded subset. Let $C > 0$ be a uniform upper bound of $d(\cdot, Y)$ in X. Since F is L-Lipschitz we have $F_x \in U_{LC}(\mathrm{im}(f))$ for all $x \in X$. It follows that if K does not intersect $U_{LC}(\mathrm{im}(f))$, then $F^{-1}(K) = \emptyset$.

Hence assume that $K \subset U_{LC}(\mathrm{im}(f))$. Consider the bounded set

$$A := U_{LC}(K) \cap \mathrm{im}(f).$$

Since f is proper, $f^{-1}(A) \subset Y$ is bounded as well. We claim that $F^{-1}(K) \subset U_C(f^{-1}(A))$.

Let $x \notin U_C(f^{-1}(A))$ be arbitrary. Since $d(\cdot, Y)$ is uniformly bounded in X, there is a $y \in Y$ with $d(x, y) < C$. Consequently $y \notin f^{-1}(A)$. But then

$$d(F_x, F_y) = d(F_x, f_y) \leq L \cdot d(x, y) < LC$$

and thus $F_x \notin K$ because otherwise $f_y \in A$. □

Let (N^n, h) be another complete oriented Riemannian manifold with fundamental class $[N] \in H_n^{lf}(N; \mathbb{Z})$. The mapping degree of a continuous map $f : M \to N$ is well defined if f is proper or N is closed and f is almost proper (ie constant outside a compact set).

A fundamental property of the filling radius, which follows from Lemma 4.14 and Lemma 4.15 and will be used throughout this paper, is that it behaves well under Lipschitz maps of non-zero degree.

Lemma 4.16. *If $f : (M, g) \to (N, h)$ be a 1-Lipschitz proper (or almost proper) map with* $\deg(f) \neq 0$*, then* $\mathrm{FillRad}(M, g) \geq \mathrm{FillRad}(N, h)$*.*

Proof. Assume that for some $R > 0$ the fundamental class $\iota_{g*}[M]$ bounds a locally finite chain in $U_R(\iota_g M) \subset L^\infty(M)_b$. By Lemma 4.14 the 1-Lipschitz map $\iota_{g'} \circ f : M \to L^\infty(N)_b$ extends to a 1-Lipschitz map $F : L^\infty(M)_b \to L^\infty(N)_b$, which maps $U_R(\iota_g M)$ to $U_R(\iota_h N)$. Since $\iota_{g'} \circ f$ is proper, F, when restricted to $U_R(\iota_g M)$, is proper as well (see Lemma 4.15) ie preimages of bounded sets are bounded. By construction, F maps $\iota_{g*}[M] \in H_n^{lf}(U_R(\iota_g M); \mathbb{Q})$ to $\deg(f)\iota_{h*}[N] \in H_n^{lf}(U_R(\iota_h N); \mathbb{Q})$. Hence $\iota_{h*}[N]$ vanishes in $U_R(\iota_h N) \subset L^\infty(N)_b$ □

Remark 4.17. Here it is important that we chose rational coefficients for our definition of the filling radius. For integral coefficients we would need to restrict to the case $\deg(f) = 1$ in the Lemma above.

The next lemma regarding the relationship between the filling radius and the Alexandrov width is taken from [23, Appendix 1, Example in (B) and (D)].

Lemma 4.18. *For a complete Riemannian manifold (M^n, g) we have the following relation:* $\mathrm{FillRad}(M, g) \leq \mathrm{UR}_{n-1}(M, g)$*.* □

Finally Theorem 4.3 is related to systolic geometry, the study of the following metric invariant:

Definition 4.19. The *systole* $\mathrm{sys}(X, g)$ of a Riemannian polyhedron (X, g) is defined to be the infimum of all lengths of noncontractible closed piecewise smooth curves in X.

Definition 4.20. A connected finite n-dimensional simplicial complex X is called *essential* if the classifying map $f : X \to K(\pi_1(X), 1)$ induces a homomorphism

$$f_* : H_n(X; G) \to H_n(K(\pi_1(X), 1); G)$$

with non-trivial image for coefficients $G = \mathbb{Z}$ or $\mathbb{Z}_2$. A closed manifold M is called *essential* if any smooth triangulation of M produces an essential simplicial complex.

It is a central result in systolic geometry [23, Appendix 2, (B1')] that for any compact essential Riemannian polyhedron (X^n, g) the following, so called, *isosystolic inequality* holds true:

$$\mathrm{sys}(X, g) \leq C(n)\mathrm{vol}(X, g)^{\frac{1}{n}}, \tag{4.2.1}$$

where $C(n)$ is a constant that only depends on the dimension n. For the special case of closed essential manifolds see [23, Theorem 0.1.A].

4.2.1 Filling enlargeable manifolds

With all invariants in play we now want to explain the notion of a *filling enlargeable* manifold featured in Theorem 4.2. It is based on the concept of an *enlargeable* manifold, introduced by Gromov and Lawson [28].

Definition 4.21. A closed orientable manifold M^n is called *enlargeable*, if for every Riemannian metric g on M and every $r > 0$ there is a Riemannian covering $\overline{M}_r$ of (M, g) and a distance decreasing almost proper (ie constant outside of a compact set) map $f_r : \overline{M}_r \to S^n(r)$ to the round sphere of radius r with $\deg(f_r) \neq 0$.

We consider the following notion, which combines the ideas of Definition 4.21 and the filling radius of a complete oriented Riemannian manifold:

Definition 4.22. A closed orientable manifold M^n is called *filling enlargeable*, if for every Riemannian metric g on M and every $r > 0$ there is a Riemannian covering $\overline{M}_r$ of M with $\mathrm{FillRad}(\overline{M}_r, \overline{g}) \geq r$, where $\overline{g}$ denotes the lifted metric.

Remark 4.23. To check whether or not a closed orientable manifold is (filling) enlargeable it suffices to consider one fixed metric g, since all Riemannian metrics on a closed manifold are in bi-Lipschitz correspondence. Furthermore we want to stress the fact that the covering spaces $\overline{M}_r$ might very well be infinite-sheeted ie non-compact (see Remark 4.7).

We begin our discussion of this new class of large manifolds by proving that it contains all closed enlargeable and all orientable closed aspherical manifolds.

Proposition 4.24. *If (M, g) is a closed enlargeable manifold, then (M, g) is filling enlargeable.*

Proof. Katz [43] proved that the filling radius of the unit sphere $S^n(1)$ is $\frac{1}{2}\arccos(-\frac{1}{n+1})$. Let $r > 0$ be arbitrary and denote $r' = \frac{r}{\mathrm{FillRad}(S^n(1))}$. Since M is enlargeable there is a Riemannian covering $\overline{M}_{r'}$ of M and a distance decreasing almost proper map $f_r : \overline{M}_{r'} \to S^n(r')$ of nonzero degree. By Lemma 4.16 it follows that $\mathrm{FillRad}(\overline{M}_{r'}, \overline{g}) \geq \mathrm{FillRad}(S^n(r')) = r$. □

To prove the same for orientable closed aspherical manifolds we need some more preparation.

Definition 4.25. A proper Riemannian polyhedron (X, g) is said to be *uniformly contractible* if there exists a function $C : [0, \infty) \to \mathbb{R}_{\geq 0}$ such that every ball of radius R in X is contractible within the ball with the same center and radius $C(R)$.

Lemma 4.26. *Let (X, g) be a contractible proper Riemannian polyhedron and assume there exists a subgroup G of the isometry group that acts cocompactly on X. Then (X, g) is uniformly contractible.*

Proof. Let $\pi : X \to X/G$ be the quotient projection and for any $x \in X$ we consider the open ball $B_1(x)$. The projection is an open map: If $U \subset X$ is open, then

$$\pi^{-1}(\pi(U)) = \{x \in X : \pi(x) \in \pi(U)\} = \bigcup_{g \in G} gU$$

is the union of open sets. Hence the image $\pi(B_1(x))$ is an open neighborhood of $\pi(x) \in X/G$. Since X/G is compact there are finitely many points $x_1, \ldots, x_k$ in X such that $X/G \subset \bigcup_{i=1}^k \pi(B_1(x_i))$. We define $K := \bigcup_{i=1}^k B_1(x_i)$. Then for any $x \in X$ there is a $\alpha \in G$ such that $\alpha x \in K$. Furthermore $\overline{K}$ is compact.

Let $R > 0$ be arbitrary. Since X is contractible there is a homotopy $H : X \times [0, 1] \to X$ connecting id_X to the constant map $X \mapsto p$ where $p \in X$ is a basepoint. Without loss of generality we assume $p \in K$. Now $H(\overline{B}_{R+\mathrm{diam}(K)}(p) \times [0, 1])$ is a compact subset of X containing p. As compact sets are bounded $H(\overline{B}_{R+\mathrm{diam}(K)}(p) \times [0, 1])$ is contained in a ball of radius $C(R) - \mathrm{diam}(K)$ around p for some $R \leq C(R) < \infty$.

Now let $x \in X$ be arbitrary and $\alpha \in G$ such that $\alpha x \in K$. Then $\alpha(B_R(x)) \subset B_{R+\mathrm{diam}(K)}(p)$ and thus $B_R(x)$ is contractible within $B_{C(R)-\mathrm{diam}(K)}(\alpha^{-1}p) \subset B_{C(R)}(x)$. □

Lemma 4.27 ([24])**.** *Let (X^n, g) be a complete oriented Riemannian manifold. If X is uniformly contractible then* $\operatorname{FillRad}(X, g) = \infty$*.* □

Proposition 4.28. *If (M, g) is an orientable closed aspherical manifold, then (M, g) is filling enlargeable.*

Proof. The universal cover of a closed orientable aspherical manifold is uniformly contractible by Lemma 4.26 and, if we fix an orientation, has infinite filling radius by Lemma 4.27. Hence closed oriented aspherical manifolds are filling enlargeable. □

We remind the reader that the product of two enlargeable manifolds is enlargeable [28, Introduction]. As, later on, our proof of Theorem 4.2 will rely on a doubling procedure, where we glue two bands $M \times [0, 1]$ and obtain a copy of $M \times S^1$, we investigate whether or not the same holds true for filling enlargeable manifolds.

Lemma 4.29. *Let (M^n, g) be a complete oriented Riemannian manifold and (S^1, g_r) the standard round circle of radius r. If* $\operatorname{FillRad}(S^1, g_r) \geq \operatorname{FillRad}(M, g)$*, then*

$$\operatorname{FillRad}(M \times S^1, g \oplus g_r) \geq \operatorname{FillRad}(M, g).$$

Proof. We assume for a contradiction that $\operatorname{FillRad}(M \times S^1, g \oplus g_r) < \operatorname{FillRad}(M, g)$. Let $\varepsilon > 0$ be such that $\operatorname{FillRad}(M \times S^1, g \oplus g_r) < \varepsilon < \operatorname{FillRad}(M, g)$.

Using Lemma 4.14 we extend the projections

$$p_1 : M \times S^1 \to M \text{ and } p_2 : M \times S^1 \to S^1$$

to some nonexpanding maps

$$P_1 : L^\infty(M \times S^1)_b \to L^\infty(M)_b \text{ and } P_2 : L^\infty(M \times S^1)_b \to L^\infty(S^1).$$

By our choice of ε the class $\iota_{g_r *}[S^1]$ does not vanish in the ε-neighborhood of $\iota_{g_r}(S^1)$ in $L^\infty(S^1)$. Since we are working with field coefficients the universal coefficient theorem tells us that there is a cohomology class $[\alpha] \in H^1(U_\varepsilon(\iota_{g_r}(S^1)); \mathbb{Q})$ dual to $\iota_{g_r *}[S^1]$ that extends the fundamental cohomology class of S^1.

We pull back $[\alpha]$ via P_2 to get a cohomology class

$$P_2^*[\alpha] \in H^1(U_\varepsilon(\iota_{g \oplus g_r}(M \times S^1)); \mathbb{Q}).$$

Denote by σ the locally finite fundamental cycle $\iota_{g \oplus g_r *}[M \times S^1]$. By choice of ε we know that σ bounds a chain in its ε-neighbourhood. Consequently the cap product

$$\sigma \cap P_2^* \alpha \in C_n^{lf}(U_\varepsilon(\iota_{g \oplus g_r}(M \times S^1)); \mathbb{Q})$$

is a boundary as well. By construction $\sigma \cap P_2^* \alpha$ represents the same locally finite homology class as $\iota_{g \oplus g_r}[M \times \{*\}]$. We conclude that $0 = \iota_{g \oplus g_r *}[M \times \{*\}] \in H_n^{lf}(U_\varepsilon(\iota_{g \oplus g_r}(M \times S^1)); \mathbb{Q})$.

Finally P_1 maps $U_\varepsilon(\iota_{g \oplus g_r}(M \times S^1))$ to $U_\varepsilon(\iota_g(M))$ and is proper when restricted accordingly. Thus

$$0 = P_{1*}(\iota_{g \oplus g_r *}[M \times \{*\}]) = \iota_{g*}[M] \in H_n^{lf}(U_\varepsilon(\iota_g(M)); \mathbb{Q}).$$

This contradicts our assumption that $\varepsilon < \operatorname{FillRad}(M, g)$. □

Proposition 4.30. *Let M be a closed filling enlargeable manifold. Then $M \times S^1$ is filling enlargeable as well.*

Proof. By Remark 4.23 it is enough to consider the case that $M \times S^1$ is equipped with a product metric $g \oplus g_1$, where g is a Riemannian metric on M and g_1 is the standard metric on S^1 with radius 1. Let $r > 0$ be arbitrary.

On the one hand, since M is filling enlargeable, there is a Riemannian covering $(\overline{M}_r, \overline{g})$ with $\mathrm{FillRad}(\overline{M}_r, \overline{g}) \geq r$. On the other hand there is a radius ℓ such that (S^1, g_ℓ) is a Riemannian covering of (S^1, g_1) and $\mathrm{FillRad}(S^1, g_\ell) \geq \mathrm{FillRad}(\overline{M}_r, \overline{g})$. Using Lemma 4.29 we conclude that $\mathrm{FillRad}(\overline{M}_r \times S^1, \overline{g} \oplus g_\ell) \geq r$.

Since $(\overline{M}_r \times S^1, \overline{g} \oplus g_\ell)$ is a Riemannian covering of $(M \times S^1, g \oplus g_1)$ this proves the proposition. □

Remark 4.31. Lemma 4.29 remains true if we replace (S^1, g_r) with any other closed oriented Riemannian manifold (N, h) such that $\mathrm{FillRad}(N, h) \geq \mathrm{FillRad}(M, g)$.

This is, however, not enough to answer the general question whether the product of two filling enlargeable manifolds is filling enlargeable, as in the definition of filling enlargeability the Riemannian coverings are not required to be compact.

4.2.2 Width enlargeable manifolds

In the proof of our main Theorem 4.2, it will be convenient to consider an even more general class of manifolds, which we obtain by replacing the filling radius with the Alexandrov width in the definiton of filling enlargeability:

Definition 4.32. A closed manifold M^n is called *width enlargeable*, if for every Riemannian metric g on M and every $r > 0$ there is a covering manifold $\overline{M}_r$ of M with $\mathrm{UR}_{n-1}(\overline{M}_r, \overline{g}) \geq r$, where $\overline{g}$ denotes the lifted metric.

Remark 4.33. In contrast to Definition 4.22 we don't have to restrict ourselves to orientable manifolds in this case, since the definition of the Alexandrov width does not involve a fundamental class.

Any filling enlargeable manifold is also width enlargeable by Lemma 4.18. Furthermore all closed aspherical manifolds, including the non-orientable ones, are width enlargeable (the universal cover has $UR_{n-1} = \infty$ by Lemma 4.27 and Lemma 4.18).

It is not clear to us whether a product result like Proposition 4.30 also holds for width enlargeable manifolds. If so, our band width inequality Theorem 4.2 would be true for this even more general class of manifolds.

The main property of width enlargeable manifolds we are interested in is:

Proposition 4.34. *For all $n \geq 1$ there is a constant $\varepsilon_n > 0$ such that the following holds. Let M^n be a width enlargeable manifold and g any Riemannian metric on M. Then for every $R > 0$ there is a point p in the universal cover $(\widetilde{M}, \widetilde{g})$ such that $\mathrm{vol}(B_R(p)) \geq \varepsilon_n R^n$.*

Proof. Let ε_n be the constant from Theorem 4.10. Assume for a contradiction, that there is a radius $R > 0$ such that the volume of all balls of radius R in $(\widetilde{M}, \widetilde{g})$ is bounded from above by $\varepsilon_n R^n$. Since M is width-enlargeable there is a covering $\overline{M}_R$ with $\mathrm{UR}_{n-1}(\overline{M}_r, \overline{g}) > R$.

On the other hand the volume of a ball of radius R centered at a point p in $(\overline{M}_r, \overline{g})$ is bounded from above by the volume of the R-ball around any lift $\widetilde{p}$ of p in the universal cover $(\widetilde{M}, \widetilde{g})$, which is smaller than $\varepsilon_n R^n$ by assumption. Since $(\overline{M}_r, \overline{g})$ is a complete locally compact path metric space it is proper by the Hopf-Rinow Theorem and hence $UR_{n-1}(\overline{M}_r, \overline{g}) \leq R$ by Theorem 4.10, which is a contradiction. □

We spend the rest of this section proving that for closed width enlargeable manifolds the isosystolic inequality 4.2.1 holds true. Using [3, Corollary 8.3] we conclude that orientable closed width enlargeable manifolds are essential.

This result is not necessary to prove Theorem 4.2, but it is interesting to see how our newly introduced classes fit into the hierarchy of large manifolds.

For closed orientable manifolds we get:

Aspherical, Enlargeable $\subseteq$ Filling enlargeable $\subseteq$ Width enlargeable $\subseteq$ Essential

Lemma 4.35. *Let (M, g) be a Riemannian manifold and $(\overline{M}, \overline{g})$ be a Riemannian cover. Let $B_R(p)$ be a ball with the property that the inclusion homomorphism $\pi_1(B_R(p)) \to \pi_1(M)$ is trivial. Then $B_R(p)$ lifts to a collection of disjoint open sets in $\overline{M}$, each of which is the ball of radius R around some lift p' of p.*

Proof. Choose a preimage p' of p under the covering projection Since $\pi_1(B_R(p)) \to \pi_1(M)$ is trivial, there is a unique lift $f' : B_R(p) \to (\overline{M}, \overline{g})$ with $f'(p) = p'$. Let $x \in B_R(p)$ be arbitrary. There is a path $\gamma : [0, 1] \to M$ with $\gamma(0) = p$, $\gamma(1) = x$ and $\text{length}(\gamma) < R$. Now γ lifts to a path $f' \circ \gamma$ connecting p' and $f'(x)$ and $\text{length}(f' \circ \gamma) = \text{length}(\gamma) < R$. Thus $f'(B_R(p))$ is contained in $B_R(p')$.

Let $x' \in B_R(p')$ be arbitrary. There is a path $\sigma : [0, 1] \to \overline{M}$ with $\sigma(0) = p'$ and $\sigma(1) = x'$ and $\text{length}(\sigma) < R$. But then $\pi \circ \sigma$ is a path in M connecting p and $\pi(x')$ with the property $\text{length}(\pi \circ \sigma) = \text{length}(\sigma) < R$. Thus $\pi(B_R(p')) \subset B_R(p)$.

Furthermore π is injective on $B_R(p')$. If it were not then $B_R(p)$ would contain a loop that is not contractible in M. Together with the fact that $\pi \circ f'$ is the identity on $B_R(p)$ this implies that both inclusions $f'(B_R(p)) \subset B_R(p')$ and $\pi(B_R(p')) \subset B_R(p)$ are in fact equalities.

If p'' is a different preimage of p and $f'' : B_R(p) \to (\overline{M}, \overline{g})$ the respective lift of $B_R(p)$ with $f''(p) = p''$ then $f(B_R(p)) \cap f''(B_R(p)) = \emptyset$. If not then p' and p'' could be joined by a path within $\pi^{-1}(B_R(p))$ which would project to a noncontractible loop in $B_R(p)$. □

Proposition 4.36. *Let (M, g) be a Riemannian manifold and assume that $\mathrm{UR}_k(M, g) < \frac{1}{2}\,\mathrm{sys}(M, g)$. If $(\overline{M}, \overline{g})$ is any Riemannian cover of (M, g), then*

$$\mathrm{UR}_k(\overline{M}, \overline{g}) \leq \mathrm{UR}_k(M, g)$$

Proof. Denote $\mathrm{UR}_k(M, g) = r$ and let $(U_i)_{i \in \mathcal{I}}$ be an open cover of radius $\leq r + \delta$ and multiplicity $\leq k + 1$, where $\delta < \frac{1}{3}(\mathrm{sys}(M, g) - 2r)$. For a fixed $i \in \mathcal{I}$ there is a point $p_i \in M$ such that $U_i \subset B_{r+\delta}(p_i)$. We claim that the inclusion homomorphism $\pi_1(B_{r+\delta}(p_i)) \to \pi_1(M)$ is trivial.

Let γ be a loop in $B_{r+\delta}(p_i)$. There is a subdivision $0 = t_0 < \ldots < t_\ell = 1$ of the unit interval such that $d_g(\gamma(t_i), \gamma(t_{i+1})) < \delta$. We can connect each $\gamma(t_i)$ to p by a minimizing geodesic σ_i of length $< R + \delta$ and thus γ is homotopic to the concatenation of ℓ 'thin' triangles $\sigma_i(1 - t) \cdot \gamma|_{[t_i, t_{i+1}]} \cdot \sigma_i$. Each of these triangles is a loop based at p of length less than $\mathrm{sys}(X, g)$ and thus contractible, which implies that γ is contractible as well.

Lemma 4.35 the ball $B_{r+\delta}(p_i)$ lifts to a collection of disjoint $(r+\delta)$-balls around the preimages of p_i. Thus U_i also lifts to a collection of disjoint open sets in $\overline{M}$, each of which is contained in a ball of radius $r+\delta$. If we do this for each $i \in \mathcal{I}$ we produce an open cover of $(\overline{M}, \overline{g})$ with radius $\leq r+\delta$. The multiplicity of this cover is still $\leq k+1$.

To see this assume for a contradiction, that a point $x \in \overline{M}$ is contained in $k+2$ open sets $U_1, \dots, U_{k+2}$ of the cover. Then $\pi(x)$ is contained in the sets $\pi(U_1), \dots, \pi(U_{k+2})$. But each of these sets $\pi(U_j)$ corresponds to an open set in the cover of M. Since only $k+1$ of these sets can contain x we can assume without loss of generality that $\pi(U_1) = \pi(U_2) = U$. Let $p_1 \in U_1$ and $p_2 \in U_2$ be such that $\pi(p_1) = \pi(p_2)$. Since $U_1 \cap U_2 \neq \emptyset$, it follows that $U_1 = U_2$. □

Corollary 4.37. *If a closed manifold M^n is width enlargeable, then $\frac{1}{2}\,\mathrm{sys}(M, g) \leq \mathrm{UR}_{n-1}(M, g)$ for any metric g on M.* □

Proposition 4.38. *Let M^n be a closed width enlargeable manifold and g be a Riemannian metric on M. Then:*

$$\mathrm{sys}(M, g) \leq C(n)\mathrm{vol}(M, g)^{1/n},$$

where $C(n)$ is a constant that only depends on the dimension n.

Proof. If M is width enlargeable and g is a metric on M then $\mathrm{sys}(M, g) \leq 2\,\mathrm{UR}_{n-1}(M, g)$ by Corollary 4.37. If we take the radius R to be $\varepsilon_n^{-\frac{1}{n}}\mathrm{vol}(M, g)^{\frac{1}{n}}$ in Theorem 4.10, then the assumption automatically holds true and we conclude that $UR_{n-1}(M, g) \leq \varepsilon^{-\frac{1}{n}}\mathrm{vol}(M, g)^{1/n}$. Thus the isosystolic inequality 4.2.1 holds with $C(n) = 2\varepsilon_n^{-\frac{1}{n}}$. □

Remark 4.39. It follows directly from [3, Corollary 8.3] that an orientable closed width enlargeable manifold is essential.

4.3 Proof of the main results

Let M^{n-1} be a manifold and g a Riemannian metric on $V := M \times [0, 1]$. The key idea in our proof of Theorem 4.2 is to construct a Riemannian polyhedron (D, g_d) from a Riemannian band (V, g) by taking its *metric double*, which is constructed like this:

We fix a smooth triangulation of V. Let (V_1, g_1) be the Riemannian polyhedron obtained from (V, g) via this triangulation. Let $V_2 := M \times [-1, 0]$ and g_2 be the pullback metric under the diffeomorphism $s : V_2 \to V_1$ $(x, y) \mapsto (x, -y)$. Since s is a diffeomorphism we can also pull back the smooth triangulation from V_1 to V_2 via s, giving (V_2, g_2) the structure of a Riemannian polyhedron.

In order to get (D, g_d) we take the disjoint union of (V_1, g_1) and (V_2, g_2), and glue them together along their (isometric) boundaries ie $M \times \{-1\} \sim_s M \times \{1\}$ and $M \times \{0\} \sim_{id} M \times \{0\}$. The result is a simplicial complex D. Every simplex of D is a proper subset of the subcomplexes V_1 or V_2 (here we identify V_1 and V_2 with their images under the quotient map from their disjoint union to D). Thus we can define a Riemannian metric g_d on D by setting $g_d|_\tau = g_i|_\tau$ for any simplex τ in D depending on whether τ is a subset of V_1 or V_2.

To get acquainted with the notion of the double we establish the following:

Proposition 4.40. *Let γ be a closed noncontractible piecewise smooth curve in (D, g_d). Then either* $\text{length}(\gamma) \geq 2\,\text{width}(V, g)$ *or there is a closed noncontractible piecewise smooth curve $\tilde{\gamma}$ in (V, g) with* $\text{length}(\tilde{\gamma}) = \text{length}(\gamma)$.

Proof. As above, consider V_1 and V_2 as subsets of D. Their intersection is the disjoint union of two copies of M, call them M_0 and $M_{\pm 1}$. Without loss of generality we assume $\gamma(0) = \gamma(1) \in V_1$.

There is a partition $0 \leq t_1 < \cdots < t_{2k} \leq 1$ of $[0, 1]$ (k might of course be 0), such that the following holds: $\gamma(t_i) \in M_0 \cup M_{\pm 1}$ and $\gamma([t_{2i-1}, t_{2i}]) \subset V_2$ while $\gamma([t_{2i}, t_{2i+1}]) \subset V_1$. Furthermore $\gamma([0, t_1]) \subset V_1$ and $\gamma([t_{2k}, 1]) \subset V_1$.

Of course if $t_i \in M_0$ and $t_{i+1} \in M_{\pm 1}$ (or the other way round) for some index $i = 1, \ldots, 2k - 1$ then $\gamma([t_i, t_{i+1}])$ as well as $\gamma([t_{i+1}, 1]) \cdot \gamma([0, t_i])$ are curves that connect M_0 and $M_{\pm 1}$. It follows that $\text{length}(\gamma) \geq 2 \cdot \text{width}(V, g)$.

Thus we assume that for all i we have $\gamma(t_i) \in M_0$. Denote by $r : D \to V_1$ the map which is the identity on V_1 and on $V_2 = M \times [-1, 0]$ has the form $(x, y) \mapsto (x, -y)$ (where we consider D as $M \times [-1, 1]/_\sim$). This is a continuous retraction, preserving the length of curves. Let $i = 1, \ldots, k$ and take the curve $c := \gamma([t_{2i-1}, t_{2i}]) \subset V_2$. We claim that c is homotopic to $r(c) \in V_1$ with fixed end points.

Let $c = (c_1, c_2)$. By assumption $c_2(t_{2i-1}) = 0 = c_2(t_{2i})$. Furthermore $c_2(t) \in [-1, 0]$ as $c \subset V_2$. Now

$$H_1 : [t_{2i-1}, t_{2i}] \times [0, 1] \to D \quad (t, s) \mapsto (c_1(t), (1 - s)c_2(t))$$

is a homotopy between c and the curve $(c_1(t), 0) \subset M_0$. In the same way

$$H_2 : [t_{2i-1}, t_{2i}] \times [0, 1] \to D \quad (t, s) \mapsto (c_1(t), (s - 1)c_2(t))$$

is a homotopy between $(c_1, -c_2) = r(c)$ and $(c_1(t), 0) \subset M_0$. As the endpoints are fixed in both H_1 and H_2, we can concatenate them to get a homotopy between c and $r(c)$.

Finally, $\tilde{\gamma} := r(\gamma) = \gamma([0, t_1]) \cdot r(\gamma([t_1, t_2])) \cdot \gamma([t_2, t_3]) \cdots r(\gamma([t_{2k-1}, t_{2k}])) \cdot \gamma([t_{2k}, 1])$ is a closed curve in V_1 homotopic to γ and therefore noncontractible. As r preserves the length of curves, we get $\text{length}(\tilde{\gamma}) = \text{length}(\gamma)$ and since we can view $(V_1, g_d|_{V_1}) \subset (D, g_d)$ as a copy of (V, g), this proves the lemma. □

Now we have all the necessary tools to prove our main Theorem 4.2:

Proof of Theorem 4.2. Let ε_n be the constant from Theorem 4.10. Denote by $(\widetilde{V}, \widetilde{g})$ the universal cover of (V, g) and let (D, g_d) be the metric double of $(\widetilde{V}, \widetilde{g})$. We claim that the volume of every unit ball in (D, g_d) is bounded from above by ε_n:

To see this let $p_1 \in \widetilde{V_1}$ be arbitrary and p_2 be the mirror image of p_1 in $\widetilde{V_2}$ (here $\widetilde{V_1}$ and $\widetilde{V_2}$ are the two copies of $\widetilde{V}$ used in the doubling procedure). If we denote by $q : \widetilde{V_1} \coprod \widetilde{V_2} \to D$ the quotient projection, it turns out that $B_R(q(p_1)) \subset q(B_R(p_1) \cup B_R(p_2))$. In fact let $x \in B_R(q(p_1))$ ie there is a path σ of length less than R connecting $q(p)$ and x in D. Now if $x \in q(\widetilde{V_1})$, then, using the same techniques as in the proof of Proposition 4.40, σ can be modified to a path of the same length that connects $q(p_1)$ and x in $q(\widetilde{V_1})$. Hence $x \in q(B_R(p_1))$. If, on the other hand, $x \in q(\widetilde{V_2})$, the σ can be modified to a path of the same length that connects $q(p_2)$ and x in $q(\widetilde{V_2})$.

Assume that $\frac{1}{2}\,\text{sys}(D, g_d) = \text{width}(V, g) > 1$. Let p be an arbitrary point in the universal cover $(\widetilde{D}, \widetilde{g}_d)$. If $\pi : \widetilde{D} \to D$ denotes the covering projection, then π is injective on $B_1(p)$ as two points p_1 and p_2 in $\widetilde{D}$ with $p_1 \neq p_2$ and $\pi(p_1) = \pi(p_2)$ have $\operatorname{dist}(p_1, p_2) \geq \operatorname{sys}(D, g_d) > 2$ and any two points in $B_1(p)$ are of distance less than 2 from each other. It follows that $B_1(p)$ is isometric to $B_1(\pi(p))$ and thus $\operatorname{vol}(B_1(p)) < \varepsilon_n$.

Now $(\widetilde{D}, \widetilde{g}_d)$ is also the universal cover of the metric double of (V, g), which, as a manifold, is just $M \times S^1$ and hence it is width enlargeable by Proposition 4.30 and Lemma 4.18. Furthermore $(\widetilde{D}, \widetilde{g_d})$ is a complete, locally compact path metric space and thus proper by the Hopf-Rinow-Theorem. Since the volume of all unit balls in $(\widetilde{D}, \widetilde{g}_d)$ is bounded from above by ε_n this contradicts Proposition 4.34. □

Remark 4.41. The same proof also works for bands over closed width-enlargeable manifolds except for the fact that we do not know whether a product result like Proposition 4.30 also holds for width enlargeable manifolds ie whether $M \times S^1$ is width enlargeable if M is width enlargeable (see Remark 4.33).

However, since the product of an aspherical manifold with S^1 is aspherical as well, Theorem 4.2 holds true for all closed aspherical manifolds, not only the orientable ones.

As we stated in the introduction Theorem 4.2 does not hold true for all essential manifolds. This is due to the fact that some essential manifolds, for example $\mathbb{R}P^{n-1}$, actually do admit metrics with uniformly positive (macroscopic) scalar curvature at all scales. This leads to the following:

Example 4.42. Let g be the standard metric on $\mathbb{R}P^{n-1}$ induced by the round metric on the unit sphere S^{n-1}. Consider the direct product with an interval of arbitrary length $[0, \ell]$. We can produce a metric with any given lower bound on $Sc_1(p)$ for all $p \in \mathbb{R}P^{n-1} \times [0, \ell]$ by just rescaling the metric on $\mathbb{R}P^{n-1}$ to be very small, since the volume of a unit ball in the universal cover of the product is bounded from above by the volume of a unit ball in S^{n-1}, which becomes very small when rescaling the metric with a small constant.

We spend the rest of this section proving Theorem 4.3. It will be crucial to relate the systoles of (V, g) and (D, g_d).

Lemma 4.43. *Let M be a closed manifold and $V := M \times [0, 1]$. If g is a Riemannian metric on V, then*

$$\operatorname{sys}(D, g_d) \geq \min\{\operatorname{sys}(V, g), 2\operatorname{width}(V, g)\} \tag{4.3.1}$$

where (D, g_d) denotes the metric double of (V, g).

Proof. As the systole is defined to be the infimum over the lengths of all closed noncontractible piecewise smooth curves, this follows immediately from Proposition 4.40. □

Lemma 4.43 implies that to estimate the systole or the width of a Riemannian band (V, g) from above in terms of some $R > 0$, it is enough to estimate the systole of the metric double (D, g_d) from above in terms of R.

In order to achieve this we use Theorem 4.10 and the next two lemmata, the second of which appears in [51] and is attributed to Roman Karasev.

Lemma 4.44. *Let M^{n-1} be a closed essential manifold and $V = M \times [0,1]$ a band over M. Then the double D is essential as well.*

Proof. Since M^{n-1} is closed and essential the classifying map $f : M \to K(\pi_1(M), 1)$ induces a homomorphism $f_* : H_{n-1}(M; G) \to H_{n-1}(K(\pi_1(M), 1); G)$ with non-trivial image for coefficients $G = \mathbb{Z}$ or $\mathbb{Z}_2$. Since D is homeomorphic to $M \times S^1$ we have $\pi_1(D) = \pi_1(M) \times \mathbb{Z}$.

Hence a $K(\pi_1(D), 1)$-space is given by $K(\pi_1(M), 1) \times S^1$. The classifying map $g : D \to K(\pi_1(M), 1) \times S^1$ is the direct product of f and id_{S^1}. The general Künneth formula tells us that the cross product maps

$$H_{n-1}(M; G) \otimes H_1(S^1; G) \to H_n(M \times S^1; G)$$

and

$$H_{n-1}(K(\pi_1(M), 1); G) \otimes H_1(S^1; G) \to H_n(K(\pi_1(M), 1) \times S^1; G)$$

are injective and commute with the maps induced by f, id_{S^1} and g. Let $\alpha \in H_n(M; G)$ be a class with $f_*\alpha \neq 0$ and denote by e the generator of $H_1(S^1; G)$. Then $g_*(\alpha \times e) = f_*\alpha \times e \neq 0$, proving the lemma. □

Lemma 4.45. *If (X^n, g) is an essential Riemannian polyhedron, then* $\operatorname{sys}(X, g) \leq 2\,\mathrm{UR}_{n-1}(X, g)$.

Proof. Assume that $\operatorname{sys}(X, g) > 2\,\mathrm{UR}_{n-1}(X, g) := 2R$. Choose a covering of X with multiplicity $\leq n$ by connected open sets U_α of radii $\leq R + \delta$ with $\delta < \frac{1}{3}(\operatorname{sys}(X, g) - 2R)$. Let γ be a loop contained in one of the U_α and p be the center of a ball of radius $R + \delta$ that contains U_α.

There is a subdivision $0 = t_0 < \ldots < t_k = 1$ of the unit interval such that $d_g(\gamma(t_i), \gamma(t_{i+1})) < \delta$. We can connect each $\gamma(t_i)$ to p by a minimizing geodesic σ_i of length $< R + \delta$ and thus γ is homotopic to the concatenation of k 'thin' triangles $\sigma_i(1 - t) \cdot \gamma|_{[t_i, t_{i+1}]} \cdot \sigma_i$. Each of these triangles is a loop based at p of length less than $\operatorname{sys}(X, g)$ and thus contractible, which implies that γ is contractible as well.

It follows that the inclusion homomorphisms $\pi_1(U_\alpha) \to \pi_1(X)$ are trivial and hence each U_α lifts to a collection of disjoint open sets $(\widetilde{U}_\alpha)_g$ (with $g \in \pi_1(X)$), homeomorphic to U_α, in the universal cover $(\widetilde{X}, \widetilde{g})$ (for more details see Lemma 4.35).

If we consider the nerves N of the covering $\{U_\alpha\}$ of X and N' of the covering $\{(\widetilde{U}_\alpha)_g\}$ of $\widetilde{X}$, we see that N' is a covering of N and the following diagram commutes:

$$\begin{array}{ccc} \widetilde{X} & \longrightarrow & N' \\ \downarrow & & \downarrow \\ X & \longrightarrow & N. \end{array}$$

Here the vertical arrows are the covering projections while the horizontal arrows are the nerve maps. Let $p \in X$ be a point and γ a noncontractible loop in X based at p. Then γ lifts to a path $\widetilde{\gamma}$ connecting two different points $\widetilde{p}_1$ and $\widetilde{p}_2$ in the fiber over p. By construction $\widetilde{p}_1$ and $\widetilde{p}_2$ are not contained in a common set $(\widetilde{U}_\alpha)$.

Hence $\widetilde{\gamma}$ is mapped to a path connecting different points in the nerve N'. When projected to N this yields a noncontractible loop, which agrees with the image of γ under the nerve map $X \to N$. It follows that the induced map $\pi_1(X) \to \pi_1(N)$ is injective. Since this map is always surjective it is an isomorphism. Therefore the classifying map $X \to K(\pi_1(X), 1)$ factors through the $(n-1)$-dimensional nerve N, so X is not essential. □

With these ingredients we can prove Theorem 4.3:

Proof of Theorem 4.3. Consider the double (D, g_d) of (V, g) as before. Every R-ball in V has volume smaller than $\frac{1}{2}\varepsilon_n R^n$, and hence every R-ball in D has volume smaller than $\varepsilon_n R^n$ (see the proof of Theorem 4.2).

Now Theorem 4.10 implies that $\mathrm{UR}_{n-1}(D, g_d) \leq R$. As M is essential, D is essential as well by Lemma 4.44, and hence Lemma 4.45 implies

$$\mathrm{sys}(D, g_d) \leq 2\,\mathrm{UR}_{n-1}(D, g_d) \leq 2R.$$

Finally it follows from Lemma 4.43, that $\min\{\mathrm{sys}(V, g), 2\,\mathrm{width}(V, g)\} \leq 2R$, which proves the theorem as we assumed $\mathrm{sys}(V, g) > 2R$. □

Remark 4.46. There are two ways to look at this result: on the one hand, if we assume that $2R < \mathrm{sys}(V, g)$ (like we did in Theorem 4.3) the above proof produces a band width estimate.

On the other hand, if we replace the assumption that $2R < \mathrm{sys}(V, g)$ by $\mathrm{width}(V, g) > R$, the above proof produces an estimate for the systole. One could consider this to be an extension of the classical isosystolic inequality 4.2.1 to bands over essential manifolds which are wide enough.

4.4 Homological invariance of filling enlargeability

In this section we closely follow the arguments of Brunnbauer and Hanke [8, Section 3] to prove Theorem 4.4. In order to do this, we need to extend our notion of filling enlargeability from closed oriented manifolds to rational homology classes of simplicial complexes.

In the following if $p : \overline{X} \to X$ is a (not necessarily connected) cover of a simplicial complex X and $c \in H_n(X; \mathbb{Q})$ is a (simplicial) homology class, the *transfer* $p^!(c) \in H_n^{lf}(\overline{X}; \mathbb{Q})$ is represented by the formal sum of all possible lifts of simplices in a chain representative of c, where every lift of a simplex $\sigma : \Delta^n \to X$ is added with the same coefficient as σ. For more information on the transfer homomorphism (in the case of finite coverings) see for example Hatcher [39, 3.G].

Definition 4.47. A connected subcomplex S of a simplicial complex X is called π_1-*surjective* if the inclusion induces a surjection on fundamental groups and we say that S *carries* a homology class $c \in H_*(X; \mathbb{Q})$ if c lies in in the image of the map in homology induced by the inclusion.

Definition 4.48. (Compare [8, Definition 3.1]) Let X be a simplicial complex with finitely generated fundamental group and let $c \in H_n(X; \mathbb{Q})$ be a (simplicial) homology class. Choose a finite connected π_1-surjective subcomplex $S \subset X$ carrying c. (This subcomplex exists because $\pi_1(X)$ is finitely generated).

The class $c \in H_n(X; \mathbb{Q})$ is called *filling enlargeable*, if the following holds: For any $r > 0$, there is a connected cover $p : \overline{X} \to X$ such that the class $p^!(c) \in H_n^{lf}(\overline{S}; \mathbb{Q})$ does not vanish in the r-neighborhood of the Kuratowski embedding $\iota(\overline{S})$. Here $\overline{S} = p^{-1}(S)$ (which is connected since S is π_1-surjective) is equipped with the canonical path metric.

Remark 4.49. A closed oriented manifold M^n is filling enlargeable (as in Definition 4.22) if and only if its fundamental class $[M]$ is filling enlargeable (choose $S = M$ in Definition 4.48).

As in [8] we need to show that this definition does not depend on the choice of S. Let $S' \subset S$ be a smaller π_1-surjective subcomplex carrying c and $r > 0$ be arbitrary. Let $\overline{S}$ be a covering of S such that $p^!(c)$ does not vanish in the r-neighborhood of $\iota(\overline{S})$. The lifted inclusion $\overline{S'} \hookrightarrow \overline{S}$ is 1-Lipschitz and extends to a nonexpanding map $L^\infty(\overline{S'}) \to L^\infty(\overline{S})$. By naturality of $p^!$ the class $p^!(c) \in H_n^{lf}(\overline{S'}; \mathbb{Q})$ can not vanish in the r-neighborhood of $\iota(\overline{S'})$.

Now for two different π_1-surjective subcomplexes carrying c there is always a third one containing both. By the above it now remains to show that if $T \supset S$ is a larger π_1-surjective subcomplex carrying c then we can pass from S to T in Definition 4.48. This will be shown by induction on the skeleta $T^{(k)}$ of T. At the start of the induction we treat the cases $k = 0, 1$ simultaneously.

For the inductive step we will need the following Lemma, which works as a substitute for [8, Lemma 3.2].

Lemma 4.50. *Let X be a connected simplicial complex and $c \in H_n(X; \mathbb{Q})$ be a (simplicial) homology class. Let $Y \subset X$ be a subcomplex carrying c such that $X \backslash Y$ is the disjoint union of possibly infinitely many copies of the interior of a k-dimensional simplex with $k \geq 2$. There is a constant δ_k such that the following holds true: If the class c does not vanish in the r-neighborhood of $\iota(Y)$, then c does not vanish in the $(\delta_k r - 1)$-neighborhood of $\iota(X)$.*

Proof. Let Δ^k be the standard simplex endowed with the canonical path metric d_{Δ^k}. Consider its boundary $\partial\Delta^k$ and the canonical map $(\partial\Delta^k, d_{\Delta^k}) \to \partial\Delta^k$ with Lipschitz constant $\frac{1}{\delta_k}$ for some $0 < \delta_k \leq 1$.

If we rescale the canonical path metric on Y by δ_k then the map $(Y, d_X) \to (Y, \delta_k d_Y)$ is non expanding. To see this let v and v' be two points in Y. By definition there is a path γ in X connecting v and v' with $\text{length}(\gamma) = d_X(v, v')$. By replacing all segments of γ lying the interior of a copy of Δ^k with shortest paths connecting the endpoints in $\partial\Delta^k$ we construct a path $\gamma' \subset Y$ of length $\leq \frac{1}{\delta_k}\,\text{length}(\gamma)$. Hence $d_Y(v, v') \leq \frac{1}{\delta_k} d_X(v, v')$, which proves the claim.

If c does not vanish in the r-neighborhood of $\iota(Y)$ then it does not vanish in the $\delta_k r$-neighborhood of $\iota(Y, \delta_k d_Y)$. Using Lemma 4.16, we conclude that c does not vanish in the $\delta_k r$-neighborhood of $\iota(Y, d_X)$. As the $(\delta_k r - 1)$-neighborhood of $\iota(X)$ is contained in the $\delta_k r$-neighborhood of $\iota(Y, d_X)$, this proves the Lemma. □

Now we can start the induction process from [8]: First assume that $T \backslash S$ contains only one vertex v. Let $\overline{V} \subset \overline{T}$ be the set of lifts of v. For each $\overline{v} \in \overline{V}$ let $F(\overline{v}) \subset \overline{S}$ be the set of vertices having a common edge with $\overline{v}$. Note that $F(\overline{v})$ is nonempty and finite since $\overline{T}$ is connected and locally finite.

Let $F(\widetilde{v}) \subset \widetilde{S}$ be the subset defined in an analogous fashion as $F(\overline{v})$ but with $\overline{S}$ replaced by the universal cover $\widetilde{S} \to S$ (and v by a point $\widetilde{v}$ over v) and set

$$d := \text{diam}(F(\widetilde{v}))$$

measured with respect to canonical the path metric in $\widetilde{S}$. Then d is independent of the choice of $\widetilde{v}$ and r and furthermore

$$\text{diam} F(\overline{v}) \leq d.$$

Now the Lipschitz constant of the canonical map $(\overline{S}, d_{\overline{T}}) \to \overline{S}$ is smaller or equal $\frac{d}{2}$.

As in the proof of Lemma 4.50 we rescale the canonical path metric on $\overline{S}$ by $\frac{2}{d}$. If $p^!(c)$ does not vanish in the r-neighborhood of $\iota(\overline{S})$, then the same holds true for the $\frac{2r}{d}$-neighborhood of $\iota(\overline{S}, \frac{2}{d}d_{\overline{S}})$. Hence $p^!(c)$ does not vanish in the $\frac{2r}{d}$-neighborhood of $\iota(\overline{S}, d_{\overline{T}})$ and since the $(\frac{2r}{d} - 1)$-neighborhood of $\iota(\overline{T})$ is contained in the $\frac{2r}{d}$-neighborhood of $\iota(\overline{S}, d_{\overline{T}})$ we found a lower bound for the filling radius of $p^!(c)$ in $\overline{T}$ which only depends on S and T.

If $T\backslash S$ contains more than one vertex, we apply this procedure inductively, where in each induction step we pick a vertex which has a common edge with some vertex in the subcomplex of $\overline{T}$ that has already been treated. As T and S are both finite this produces (if r is sufficiently large) a constant δ_1' such that $p^!(c)$ does not vanish in the $\delta_1' r$-neighborhood of $\overline{S \cup T^1}$. For the inductive step we assume that this holds true for the $\delta_k' r$-neighborhood of $\overline{S \cup T^k}$. By Lemma 4.50 we conclude that $p^!(c)$ does not vanish in the $(\delta_{k+1}\delta_k' r)$-neighborhood of $\overline{S \cup T^{k+1}}$.

In the end we get a constant δ only depending on S and T (and not on r), such that c does not vanish in the δr-neighborhood of $\iota(\overline{T})$. Hence the notion of filling enlargeability is well defined.

Next we study functorial properties of filling enlargeable homology classes (compare [8, Proposition 3.4]).

Proposition 4.51. *Let X and Y be connected simplicial complexes with finitely generated fundamental groups and let $\phi : X \to Y$ be a continuous map. Then following implications hold:*

- *If ϕ induces a surjection of fundamental groups and $\phi_*(c)$ is filling enlargeable, then c is filling enlargeable.*
- *If ϕ induces an isomorphism of fundamental groups and c is filling enlargeable, then also $\phi_*(c)$ is filling enlargeable.*

Proof. First assume that $\phi_*(c)$ is filling enlargeable and ϕ is surjective on π_1. Let $S \subset X$ be a finite connected π_1-surjective subcomplex carrying c.

Then $\phi(S)$ is contained in a finite π_1-surjective subcomplex $T \subset Y$ carrying $\phi_*(c)$. As S and T are both compact the map $\phi : S \to T$ is Lipschitz with Lipschitz constant $\frac{1}{\lambda}$. Hence, if we rescale the canonical path metric on T by λ, this map is nonexpanding.

Let $r > 0$ and choose a connected cover $p_Y : \overline{Y} \to Y$ such that $p^!(\phi_*(c))$ does not vanish in the $\frac{1}{\lambda}r$-neighborhood of $\iota(\overline{T})$. Then the same holds true for the r-neighborhood of $\iota(\overline{T}, \lambda d_{\overline{T}})$. Let $p_X : \overline{X} \to X$ be the pullback of the covering to X. This will be connected since ϕ is surjective on π_1 and we get a map of covering spaces

$$\begin{array}{ccc} \overline{S} & \xrightarrow{\overline{\phi}} & \overline{T} \\ \downarrow{\scriptstyle p_X} & & \downarrow{\scriptstyle p_Y} \\ S & \xrightarrow{\phi} & T, \end{array}$$

which restricts to a bijection on each fibre. In particular it is nonexpanding, if we rescale the canonical path metric on $\overline{T}$ by λ, and by naturality it maps $p^!(c)$ to $p^!(\phi_*(c))$. Hence by Lemma 4.16, we see that $p^!(c)$ does not vanish in the r-neighborhood of $\iota(\overline{S})$ and c is filling enlargeable.

If ϕ induces an isomorphism of fundamental groups, then, by the first part, we can replace Y by a homotopy equivalent complex and hence we may assume that ϕ is an inclusion. Let $S \subset X$ be a finite π_1-surjective subcomplex carrying c. Then S is also a subcomplex of Y and it carries $\phi_*(c)$. Because ϕ induces an isomorphism on fundamental groups each connected cover of X can be written as the restriction of a connected cover of Y. This shows that $\phi_*(c)$ is filling enlargeable. □

As a corollary we get homological invariance of filling enlargeability. Notice that, by the above, the filling enlargeable classes form a well defined subset in the group homology $H_*(\Gamma; \mathbb{Q}) = H_*(B\Gamma; \mathbb{Q})$ of a finitely generated group Γ, since a homotopy equivalence between two different simplicial models of $B\Gamma$ identifies the subsets of filling enlargeable classes.

Corollary 4.52. *Let M be a closed oriented manifold of dimension n. Then M is filling enlargeable if and only if $\phi_*[M] \in H_n(B\pi_1(M); \mathbb{Q})$ is filling enlargeable.* □

Our Theorem 4.4 follows directly from Corollary 4.52 and the next proposition (compare [8, Theorem 3.6]).

Proposition 4.53. *Let X be a connected simplicial complex with finitely generated fundamental group. Then the non filling enlargeable classes in $H_n(X; \mathbb{Q})$ form a rational vector subspace.*

Proof. The class $0 \in H_n(X; \mathbb{Q})$ is not filling enlargeable. This follows directly from Definition 4.48 (every finite π_1-surjective subcomplex S carries 0 but of course the 0-class vanishes in any neighbourhood of $\iota(\overline{S})$ for all $\overline{S} \to S$). Furthermore if a class is not filling enlargeable then clearly no rational multiple of it can be filling enlargeable.

Finally we need to show that the subset of non filling enlargeable classes is closed under addition. Let $c, d \in H_n(X; \mathbb{Q})$ be non filling enlargeable and assume that $c+d$ is filling enlargeable. Then by definition there is a finite π_1-surjective subcomplex S carrying $c+d$ such that for every $r > 0$ there is a connected cover $\overline{X} \to X$ such that $p^!(c+d) = p^!(c) + p^!(d)$ does not vanish in the r-neighborhood of $\iota(\overline{S})$.

But this implies that either $p^!(c)$ or $p^!(d)$ does not vanish in the r neighborhood of $\overline{S}$. If we consider all natural numbers $k \geq 1$ for values of r then for infinitely many k either $p^!(c)$ or $p^!(d)$ does not vanish in the k-neighborhood of $\overline{S}$. Since S carries both c and d we conclude that either c or d is filling enlargeable. □

Remark 4.54. While we have already seen in Remark 4.39 that an orientable closed width-enlargeable manifold M^n is essential ie $\phi_*[M] \neq 0 \in H_n(B\pi_1(M); \mathbb{Z})$, it follows from Corollary 4.52 and Proposition 4.53 that if M^n is also filling enlargeable, then it is even *rationally essential* ie $\phi_*[M] \neq 0 \in H_n(B\pi_1(M); \mathbb{Q})$. This conclusion is strictly stronger since there are essential manifolds which are not rationally essential, for example $\mathbb{R}P^3$.

Conversely if M^n is a closed rationally essential manifold and $\pi_1(M)$ is residually finite, then for any metric g on M and any $r > 0$ there is a compact covering $(\overline{M}_r, \overline{g})$ with $\mathrm{sys}(\overline{M}_r, \overline{g}) \geq r$. Since M is rationally essential $\overline{M}_r$ is rationally essential as well and by [23, Lemma 1.2.B] we conclude that $6\,\mathrm{FillRad}(\overline{M}_r, \overline{g}) \geq \mathrm{sys}(\overline{M}_r, \overline{g}) \geq r$. Hence M is filling enlargeable. Together with Remark 4.54 this proves Corollary 4.5.

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