Projective dynamics and an integrable Boltzmann billiard model

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The aim of this note is to explain the integrability of an integrable Boltzmann billiard model, previously established by Gallavotti and Jauslin [G. Gallavotti and I. Jauslin, A theorem on Ellipses, an integrable system and a theorem of Boltzmann, preprint (2020); arXiv:2008.01955], alternatively via the viewpoint of projective dynamics. We show that the energy of a corresponding spherical problem leads to an additional first integral of the system equivalent to Gallavotti–Jauslin's first integral. The approach also leads to a family of integrable billiard models in the plane and on the sphere defined through the planar and spherical Kepler–Coulomb problems.

1. Introduction

In [12], Gallavotti and Jauslin examined a billiard model derived from the Kepler problem in the plane, with a line not containing the attractive center as wall of reflection. The billiard model is defined through the Kepler dynamics with the usual law of reflection in the plane, namely, at a point of reflection, the tangential component of the velocity to the wall of reflection does not change while the normal component change its sign. In the case of the Kepler problem, since each Keplerian orbit with negative energy intersects the line of reflection generically at two points, the billiard dynamics on one side of the wall is the reverse of the billiard dynamics on the other side. Still it is preferred to take the dynamics on the other side of the line as the center to avoid the problem of collisions. Gallavotti and Jauslin showed that the billiard system is integrable in the sense that it has another first integral additional to the energy, which confirms a previous conjecture of Gallavotti [10], [11, Appendix D] on its integrability. This integrable Boltzmann model is shown to carry periodic and quasi-periodic dynamics by Felder [9] with algebraic geometric method related to the Poncelet theorem.

This model is a limiting case of a toy model considered by Boltzmann [6], defined such that an additional centrifugal force with strength inverse proportional to the cube of the distance to the center is added. Boltzmann assumed that this is an ergodic system which illustrates his "ergodic hypothesis". Actually this may only happen when the additional centrifugal force is sufficiently large, as by the analysis of Gallavotti–Jauslin and Felder, KAM and topological stability of orbits on energy hypersurfaces can be established via application of KAM theory.

In this note, we aim to explain the integrability of the integrable Boltzmann Model alternatively with the viewpoint of projective dynamics, as developed and illustrated in [14, 5, 2, 3]. We shall show that the integrable Boltzmann model admits an additional first integral derived from a corresponding spherical problem. We explain that the first integral of Gallavotti–Jauslin can be expressed from the energies of the planar problem and the corresponding spherical problem. Moreover, we explain that this viewpoint leads to certain integrable variants of this integrable Boltzmann model in the plane and on the sphere.

We note that the "projective method" has been previously used to obtain the integrability of the geodesic flow on an ellipsoid [18, 20] and the billiard system inside an ellipsoid [19]. Our result can be considered as an extension of this method to billiard systems defined through Kepler–Coulomb mechanical systems.

We organize this note as follows: We first recall projective dynamical properties of the Kepler–Coulomb problem in Sec. 2. In Secs. 3 and 4, we define certain billiard systems in the plane (which include the integrable Boltzmann model) and on the sphere with Kepler–Coulomb problem and prove their integrability, respectively, by showing that both energies of the planar and spherical problems lead to independent first integrals of these systems. In Sec. 5, we derive Gallavotti–Jauslin's first integral from the energies of the planar and spherical problems. Some further remarks are collected in Sec. 6.

2. Projective Dynamics of Kepler–Coulomb Problems

We start with a mechanical system (M, g, U) on a Riemannian manifold (M, g) with the force function U. Newton's equations of motion of this system are given by

$$\nabla_{\dot{q}}\dot{q} = \mathrm{grad}_g U,$$

in which ∇ denotes the Levi-Civita connection associated to g, and grad denotes the gradient. To a point $x \in M$ and a velocity $v \in T_x M$, the kinetic energy is given by the formula $T_x(v) := g_x(v, v)/2$ and the potential energy is $-U(x)$. The function $E = T - U$ is the (total) energy of the system and is a well-known conserved quantity. An orbit $\gamma(t)$ of the mechanical system (M, g, U) is a solution of Newton's equation in the configuration space M. Newton's equation can be lifted to a first order ordinary differential equation which defines a lifted vector field W on the tangent bundle TM. The orbit $\gamma(t)$ then lifts to an integral curve $\tilde{\gamma}(t) = (\gamma(t), \dot{\gamma}(t))$ of W.

Definition 2.1. Two mechanical systems $(M_1, g_1, U_1), (M_2, g_2, U_2)$ with lifted vector fields $W_i, i = 1, 2$ on $TM_i, i = 1, 2$ are called *in correspondence*, if there is a diffeomorphism $\phi : M_1 \to M_2$ with differential $\phi_* : TM_1 \to TM_2$ and a smooth rescaling function $\rho : M_1 \to \mathbb{R}_+$ with the associated rescaling diffeomorphism $P : TM_1 \to TM_1, (x, v) \mapsto (x, \rho v)$ such that $\phi_* \circ P : TM_1 \to TM_2$ sends integral curves of W_1 to integral curves of W_2 up to a time reparametrization.

Equivalently, $(\phi_* \circ P)_* : T(TM_1) \mapsto T(TM_2)$ sends $\rho^{-1} W_1$ to W_2.

Now if $\tilde{\gamma}_1(t) = (\gamma_1(t), \dot{\gamma}_1(t))$ is an integral curve of W_1, then we have by construction $d\gamma_1(t)/dt = \dot{\gamma}_1(t)$. Set $\tilde{\gamma}_2(t) = \phi_* \circ P \circ \gamma_1 = (\gamma_2(t), \dot{\gamma}_2(t))$. For a reparametrization $\tilde{\gamma}_2(\tau)$ of $\tilde{\gamma}_2(t)$ to be an integral curve of W_2, we need to have $d\gamma_2(\tau)/d\tau = \dot{\gamma}_2(\tau)$ which amounts to the equality $d\tau/dt = \rho(\gamma_1(t))^{-1}$ which implies the last assertion in the above definition.

Consequently, when two mechanical systems $(M_1, g_1, U_1), (M_2, g_2, U_2)$ are in correspondence, then the diffeomorphism ϕ sends unparametrized orbits of (M_1, g_1, U_1) in M_1 to unparametrized orbits of (M_2, g_2, U_2) in M_2.

Now, we let $E_i, i = 1, 2$ be, respectively, the energies of the systems $(M_i, g_i, U_i), i = 1, 2$. The diffeomorphism $\phi_* \circ P : TM_1 \mapsto TM_1$ permits to pull back E_2 to a function defined on TM_1. Abusively we refer this function also as the energy of the system (M_2, g_2, U_2) (on TM_1).

Proposition 2.1. *The function $E_2 \circ \phi_* \circ P$ is a conserved quantity for the system (M_1, g_1, U_1).*

Proof. If $\tilde{\gamma}_1(t)$ is an integral curve of W_1, then $\phi_* \circ P \circ \tilde{\gamma}_1(t)$ can be reparametrized into an integral curve of W_2, along which E_2 is conserved. This means E_2 is conserved along $\phi_* \circ P \circ \tilde{\gamma}_1(t)$ and consequently $E_2 \circ \phi_* \circ P$ is conserved along $\tilde{\gamma}_1(t)$. □

We note that with this definition, a mechanical system (M, g, U) naturally have corresponding systems $(M, kg, k'U), k, k' \in \mathbb{R}_+$, as can be directly deduced from the corresponding Newton's equations. However these do not provide further information of the system, thus this type of correspondence is considered as trivial. A correspondence between two natural mechanical systems $(M_1, g_1, U_1), (M_2, g_2, U_2)$ is called nontrivial if their energies lead to independent first integrals, i.e. if the energies E_1 and $E_2 \circ \phi_* \circ P$ are functionally independent on TM_1. When a mechanical system has a nontrivial corresponding system, then the system itself can be realized as a quasi-bi-Hamiltonian system, i.e. it admits two different Hamiltonian formalisms up to time reparametrization and in this case both Hamiltonians are first integrals of the system [7]. See, also [4].

Remark 2.1. We use the terminology "correspondence" to follow Painlevé [16] in a mechanical context. It is closely related to the notion of "projective equivalence" (as have been used in [18]) in a more geometric context. Remark that the system (M, g, U) is just the geodesic flow system on (M, g) when U is constant.

Example 2.1. To illustrate, we consider a simple example of geodesic motions in a horizontal plane

$$V = \{(x, y, -1)\} \subset \mathbb{R}^3.$$

We set $S_{\mathrm{SH}} := \{(x, y, z) \in S : z < 0\}$ the south hemisphere of the unit sphere $S \subset \mathbb{R}^3$. The central projection from the point $(0, 0, 0)$ induces a projective equivalence between V and S_{SH}, both equipped with their induced metrics from $\mathbb{R}^3$: it sends unparametrized geodesics in V to unparametrized geodesics on S_{SH}, and thus geodesics in V to geodesics on S_{SH} when the time is appropriately reparametrized. Therefore, in this example the central projection induces a correspondence between the geodesic flow systems on V and on S_{SH}. The energy of the geodesic flow in V is $(\dot{x}^2 + \dot{y}^2)/2$, while the energy of the spherical geodesic flow system can be written in the gnomonic chart V of S_{SH} as (Eq. 11):

$$\frac{1}{2}((1 + y^2)\dot{x}^2 - 2xy\dot{x}\dot{y} + (1 + x^2)\dot{y}^2) = \frac{\dot{x}^2 + \dot{y}^2}{2} + \frac{(\dot{x}y - \dot{y}x)^2}{2}.$$

We see that the two functions are functionally independent. In this example, the spherical energy implies the existence of the angular momentum first integral $C = \dot{x}y - \dot{y}x$ for the planar geodesic system in V.

A planar central force problem is a mechanical system $(\mathbb{R}^2 \backslash Z, g_{\mathrm{flat}}, U)$ such that the force function U is invariant under the $SO(2)$-action by rotations in $\mathbb{R}^2$ fixing the center $Z \in \mathbb{R}^2$. The mass-factor of such a system refers to that of the center, which can nevertheless take both signs to allow both attractive and repulsive forces. The mass of the moving particle is taken as the unit of mass. A central force problem on a sphere $S \subset \mathbb{R}^3$ is defined analogously, by imposing that the force function is invariant under rotations along an axis passing through the center of the sphere S.

We now consider two types of diffeomorphisms given by central projections in $\mathbb{R}^3$ and see how they transform central force problems. We follow the presentations in [1, 2]. Note that the center of projection is always the point $O = (0, 0, 0) \in \mathbb{R}^3$.

The first type of projection projects affine planes to affine planes in $\mathbb{R}^3$ (see Fig. 1). Let $V_1, V_2 \subset \mathbb{R}^3$ be two non-parallel, non-perpendicular hyperplanes in $\mathbb{R}^3$ given, respectively, by the equations $\langle h_1, q\rangle = 1$ and $\langle h_2, q\rangle = 1$, in which $h_1, h_2 \in {\mathbb{R}^3}^* \backslash O \cong \mathbb{R}^3 \backslash O$. For a central force system on V_1, its center Z_1 is projected to a point $Z_2 \in V_2$. A point $q_2 \in V_2$ is projected from a point $q_1 \in V_1$ such that $q_2 := \lambda(q_1)^{-1} q_1$. With the equations we determine this function as $\lambda(q_1) = \langle h_2, q_1\rangle$. We consider the projection in the region where $\lambda(q_1) = \langle h_2, q_1\rangle = \langle h_1, q_2\rangle^{-1} > 0$, which projects the half plane $V_1^+ := \{q_1 \in V_1, \langle h_2, q_1\rangle > 0\}$ to the half plane $V_2^+ := \{q_2 \in V_2, \langle h_1, q_2\rangle > 0\}$. Without loss of generality we assume that the centers Z_1, Z_2 lie in these half planes, respectively. Note that the case $\lambda(q_1) < 0$ can be considered analogously.

Remark 2.2. We note that nothing essential changes when we only impose the condition $\langle h_2, q_1\rangle \neq 0$ on V_1, but this causes disconnectedness of the regions, as well as disconnectedness of those orbits in V_1 which intersect the set $\{q_1 \in V_1 : \langle h_2, q_1\rangle = 0\}$ transversally.

Proposition 2.2. *A Kepler–Coulomb problem with mass-factor m_1 and center Z_1 defined in the half-plane $V_1^+ = \{q_1 \in V_1, \langle h_2, q_1\rangle > 0\}$ is centrally projected to a Kepler–Coulomb problem with mass-factor $m_2 = m_1\langle h_1, Z_2\rangle^{-1}$ and center Z_2 in the half-plane $V_2^+ = \{q_2 \in V_2, \langle h_1, q_2\rangle > 0\}$ (with a proper choice of metric on V_2^+).*

Proof. We compute

$$\dot{q}_2 = \lambda(q_1)^{-2}(\dot{q}_1\lambda(q_1) - \langle \mathrm{grad}\lambda(q_1), \dot{q}_1\rangle q_1) = \lambda(q_1)^{-2}(\dot{q}_1\lambda(q_1) - \langle h_2, \dot{q}_1\rangle q_1).$$

We now change time in V_2 according to the law $\frac{d}{d\tau} = \lambda(q_1)^2\frac{d}{dt}$. We denote the time derivative with respect to τ by $'$. We may then write the above expression as

$$q_2' = \dot{q}_1\lambda(q_1) - \langle h_2, \dot{q}_1\rangle q_1$$

and

$$q_2'' = \lambda(q_1)^2(\lambda(q_1)\ddot{q}_1 - \langle h_2, \ddot{q}_1\rangle q_1). \tag{1}$$

We also have $q_1 = \langle h_1, q_2\rangle^{-1} q_2$. The right-hand side can thus be written in a form depending only on q_2 and therefore defines a force field on V_2.

We now assume that $U_1 = m_1\|q_1 - Z_1\|_1^\alpha$, in which the distance is defined through a norm $\|\cdot\|_1$ on V_1 induced from the Euclidean norm of $\mathbb{R}^3$, obtained by restricting the Euclidean distance to V_1 and (auxiliary) specifying an origin.

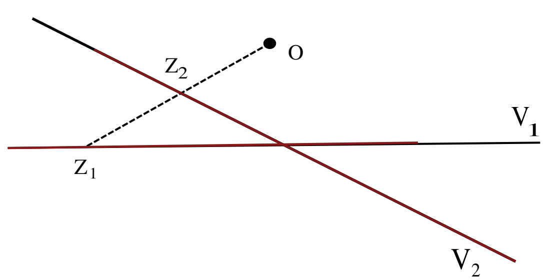

Fig. 1. A plane–plane projection.

We thus have

$$\ddot{q}_1 = \text{grad } U_1 = \alpha m_1 \|q_1 - Z_1\|_1^{\alpha-2}(q_1 - Z_1).$$

Thus Eq. (1) takes the form

$$\begin{aligned} q_2'' &= \lambda(q_1)^2(\lambda(q_1)\ddot{q}_1 - \langle h_2, \ddot{q}_1\rangle q_1) \\ &= \alpha m \langle h_1, q_2\rangle^{-3}\langle h_1, Z_2\rangle^{-1}\|q_2\langle h_1, q_2\rangle^{-1} - Z_2\langle h_1, Z_2\rangle^{-1}\|_1^{\alpha-2}(q_2 - Z_2). \quad (2) \end{aligned}$$

To proceed we would like to sort out the factor $\langle h_1, q_2\rangle^{-1}$ from the expression of distance that appeared in the above expression. On the other hand, we observe that the vector $Z_2\langle h_1, q_2\rangle\langle h_1, Z_2\rangle^{-1} \in \mathbb{R}^3$ does not necessarily lie in V_2.

To define a system in V_2, it is thus desired to define a new norm $\|\cdot\|_2$ (and thus a new metric) on V_2 such that

$$\|q_2 - Z_2\|_2 = \langle h_1, q_2\rangle\|q_2\langle h_1, q_2\rangle^{-1} - Z_2\langle h_1, Z_2\rangle^{-1}\|_1. \quad (3)$$

In order to achieve this, we extend the Euclidean norm $\|\cdot\|_1$ to $\mathbb{R}^3$ in a non-standard way: Each vector $v \in \mathbb{R}^3$ is decomposed as $v = v_1 + c \cdot Z_1$ in which v_1 is parallel to V_1. By parallel transport we may consider v_1 as a vector in V_1 and set $\|v\|_* = \|v_1\|_1$. Consequently we set $\|\cdot\|_2$ as the restriction of $\|\cdot\|_*$ to V_2. Clearly $\|\cdot\|_2$, which is non-degenerate whenever V_1 and V_2 are not perpendicular, has the desired property (3).

We thus deduce from Eq. (2), by setting $m_2 = m_1\langle h_1, Z_2\rangle^{-1}$, that

$$\begin{aligned} q_2'' &= \alpha m_1\langle h_1, Z_2\rangle^{-1}\langle h_1, q_2\rangle^{-3-(\alpha-2)}\|q_2 - Z_2\|^{\alpha-2}(q_2 - Z_2) \\ &= \langle h_1, q_2\rangle^{-1-\alpha}\text{grad } m_2\|q_2 - Z_2\|_2^{\alpha}, \quad (4) \end{aligned}$$

which is again the force field of a central force problem with potential when $\alpha = -1$, i.e. when the system is the Kepler–Coulomb problem. □

We now discuss central projection from a hemisphere to a plane. We take S to be the unit sphere in $\mathbb{R}^3(x, y, z)$. We consider the plane $V := \{z = -1\}$ (so $h = (0, 0, -1)$) tangent to S at its south pole $(0, 0, -1)$ which we take as the center Z. The central projection from the origin $O \in \mathbb{R}^3$ defines a diffeomorphism from the south-hemisphere $S_{\text{SH}} := S \cap \{z < 0\}$ to V. It is well known that this projection induces a projective equivalence between these two Riemannian manifolds, *i.e.* it sends unparametrized geodesics to unparametrized geodesics.

In [17], Serret defined a central force system on S with a force function of the form $m \cot\theta$, in which θ denotes the central angle of a moving particle on the sphere to a fixed point on the sphere, which serves as a center of the problem. Its antipodal point is another center of the problem. The system is referred to as the spherical Kepler–Coulomb problem in this note.

In the spherical Kepler–Colomb problem, one of the centers is attractive and the other one is repulsive. The sign of the mass-factor m now determines which center is attractive and which is repulsive. The orbits of this system are (possibly degenerate) spherical (conics and more precisely) ellipses. Elliptic, parabolic and hyberbolic orbits of the Kepler–Coulomb problem in V correspond, respectively, to intersections with $S_{\rm SH}$ of spherical ellipses lying entirely in $S_{\rm SH}$, tangent to the equator and intersecting transversely the equator, respectively. In this sense, the spherical Kepler–Coulomb problem completes and extends the corresponding planar Kepler–Coulomb problem. Also, the system is seen to possess the Bertrand property: just as the planar Kepler problem, all bounded non-singular orbits are closed. We refer to [1] for more detailed analysis on the spherical Kepler–Coulomb problem.

When one of these centers lie in $S_{\rm SH}$, by restriction we get a mechanical problem on the hemisphere $S_{\rm SH}$.

Proposition 2.3. *A Kepler–Coulomb problem in V with mass-factor m and center at the south pole $(0,0,-1)$ is centrally projected to a spherical Kepler–Coulomb problem in $S_{\rm SH}$ with mass-factor m and with center at $(0,0,-1)$.*

Proof. We let $q_1 \in V$ and $q_2 \in S_{\rm SH}$ be related by the central projection as follows:

$$q_2 = (1 + \|q_1 - Z\|^2)^{-1/2} q_1 := \lambda(q_1)^{-1} q_1. \tag{5}$$

Again, we have

$$\dot{q}_2 = \lambda(q_1)^{-2}(\dot{q}_1 \lambda(q_1) - \langle \text{grad } \lambda(q_1), \dot{q}_1 \rangle q_1)$$

and we may again change time according to $\frac{d}{d\tau} = \lambda(q_1)^2 \frac{d}{dt}$ which then leads to the equation

$$q_2'' = \lambda(q_1)^2(\lambda(q_1)\ddot{q}_1 - \langle \text{grad } \lambda(q_1), \ddot{q}_1 \rangle q_1). \tag{6}$$

The last term of the right-hand side of this equation is proportional to q_2, thus is vertical to $S_{\rm SH}$. It can be seen as a force of constraint which keeps the force to be tangent to $S_{\rm SH}$. By replacing $\ddot{q}_1$ with a force field, we see that with this computation, any force field in V is transformed into a force field of $S_{\rm SH}$ and vice versa.

Now starting with the Kepler–Coulomb problem in V, i.e.

$$\ddot{q}_1 = -m\|q_1 - Z\|^{-3}(q_1 - Z),$$

we have

$$\lambda(q_1)^3 \ddot{q}_1 = -m \sin^{-3}\theta \cdot (q_1 - Z),$$

in which θ denotes the angle $\angle ZOq_1$, whose component tangent to $S_{\rm SH}$ at q_2 gives the force, which has norm $|m| \sin^{-2}\theta$, points toward Z when $m > 0$, and points in the reverse direction when $m < 0$. This is a central force system on $S_{\rm SH}$ with force function $m \cot\theta$ and is exactly the spherical Kepler–Coulomb problem on $S_{\rm SH}$. ◻

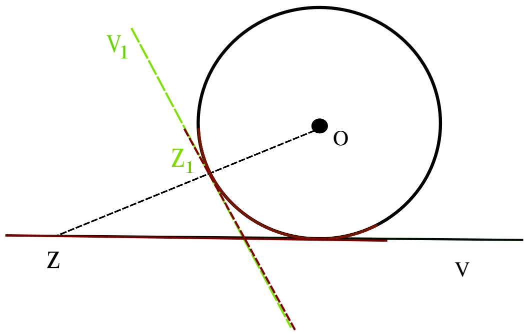

Fig. 2. A plane-sphere projection.

We now construct a projection in which the south pole is not necessarily the center of the spherical Kepler–Coulomb problem. We let $Z = (0, a, -1) \in V$ be the center in V, which is the projection of the center $Z_1 := (0, \frac{a}{\sqrt{1+a^2}}, -\frac{1}{\sqrt{1+a^2}})$ in S_{SH}.

Proposition 2.4. *The planar Kepler–Coulomb problem in V with mass-factor m and center $(0, a, -1)$ is centrally projected from the spherical Kepler–Coulomb problem on S_{SH} with mass-factor $m' = m\sqrt{1+a^2}$ and center $(0, \frac{a}{\sqrt{1+a^2}}, -\frac{1}{\sqrt{1+a^2}})$.*

Proof. With the central projection procedure, the spherical Kepler–Coulomb force field is projected to an analytic force field F in V. To deduce F without additional computation, we proceed with steps. We set $h_1 = (0, \frac{a}{\sqrt{1+a^2}}, -\frac{1}{\sqrt{1+a^2}})$. The affine plane $V_1 = \{q : \langle h_1, q \rangle = 1\}$ is now tangent to S_{SH} at Z_1. We first project from S_{SH} to V_1, then from V_1 to V (see Fig. 2). Note that due to the composition of two different projections, *a priori* only an open subset of S_{SH} is projected to an open subset of V. Nevertheless, the previous analysis shows that the force field F coincides with that of the planar Kepler–Coulomb problem with mass-factor m on an open subset of V. Consequently by analyticity, F is derived from the Kepler–Coulomb potential with mass-factor m in V. The conclusion thus follows by combining Propositions 2.2 and 2.3. □

As a consequence to these and Proposition 2.1, we get the following corollary.

Corollary 2.1. *The energies $E_{\mathrm{pl}}, E_{\mathrm{sph}}$ of the planar and the spherical problems induce two first integrals of the planar Kepler–Coulomb problem.*

We now derive explicit formulas for the functions $E_{\mathrm{pl}}, E_{\mathrm{sph}}$.

Let S be the unit sphere in $\mathbb{R}^3$ and V_1 be the affine plane tangent to S at the point $Z_1 = (0, \frac{a}{\sqrt{1+a^2}}, -\frac{1}{\sqrt{1+a^2}})$ which is the center of the spherical Kepler–Coulomb problem with mass m'. We equip both S and V_1 with the induced metrics from the standard Euclidean metric of $\mathbb{R}^3$. We take $V = \{z = -1\}$, which is tangent to S at $(0, 0, -1)$ on which there defines a Kepler–Coulomb problem with center $Z := (0, a, -1)$ and with mass m. The metric $\|\cdot\|_2$ in V will not be the one induced from the standard Euclidean metric of $\mathbb{R}^3$, but the one defined by Proposition 2.2, which we now compute.

According to the construction we first extend an Euclidean norm $\|\cdot\|_1$ of V_1, with Z_1 as the origin, to a non-standard form $\|\cdot\|_*$ on $\mathbb{R}^3$, by setting for $v \in \mathbb{R}^3$, that $\|v\|_* = \|v - \langle v, Z_1\rangle Z_1\|_1$. Then for $v = (x, y, -1) \in V$, we have $\|v\|_2 = \|v\|_*$. We compute

$$v - \langle v, Z_1\rangle Z_1 = \left(x, y - \frac{a(ay+1)}{1+a^2}, -1 + \frac{ay+1}{1+a^2}\right) = \left(x, \frac{y-a}{1+a^2}, \frac{a(y-a)}{1+a^2}\right),$$

thus

$$\|v\|_2 = \|v\|_* = \sqrt{x^2 + \frac{(y-a)^2}{1+a^2}}.$$

Note that since Z, Z_1 and O are collinear, we have $Z - \langle Z, Z_1\rangle Z_1 = 0$ and thus for $v \in V$ there holds $\|v\|_2 = \|v - Z_1\|_2$: It is thus suggestive to consider Z_1 as playing the role of the origin in V and to view v as represents the vector $v - Z_1$ based at Z_1.

We therefore have that the energy of the planar Kepler–Coulomb problem in $(V, \|\cdot\|_2)$ is

$$E_{pl} = \frac{1}{2}\left(\dot{x}^2 + \frac{\dot{y}^2}{1+a^2}\right) - \frac{m}{\sqrt{x^2 + \frac{(y-a)^2}{1+a^2}}}. \tag{7}$$

We now express the energy of the spherical Kepler–Coulomb problem in S_{SH} with center Z_1 and with mass m' in V, by viewing V as a gnomonic chart for S_{SH} as given by the central projection.

In the gnomonic chart V, the metric of S_{SH} takes the form

$$\frac{1}{(1+x^2+y^2)^2}((1+y^2)dx^2 - 2xydxdy + (1+x^2)dy^2). \tag{8}$$

The kinetic energy of the system thus takes the form

$$\frac{1}{2(1+x^2+y^2)^2}((1+y^2){x'}^2 - 2xyx'y' + (1+x^2)d{y'}^2), \tag{9}$$

in which $'$ denotes the derivative with respect to the time parameter on S_{SH}. The factor of time change is obtained via (5) as follows:

$$(x', y') = (1+x^2+y^2)(\dot{x}, \dot{y}). \tag{10}$$

Thus the kinetic energy is expressed equivalently as

$$\frac{1}{2}((1+y^2)\dot{x}^2 - 2xy\dot{x}\dot{y} + (1+x^2)\dot{y}^2) = \frac{1}{2}((\dot{x}^2+\dot{y}^2) + (x\dot{y} - y\dot{x})^2). \tag{11}$$

We see that this is the sum of the kinetic energy in the plane and half of the square of the angular momentum in the plane with respect to the point $(x, y) = (0, 0)$. It is worthwhile to remark that this angular momentum agrees with the angular momentum on the sphere with respect to the vertical axis [3], as can be directly verified.

The potential on the sphere is $-m' \cot\theta$, where θ is the angle between the position of the moving particle $\frac{1}{\sqrt{x^2+y^2+1}}(x, y, -1)$ on S_{SH} and the center Z_1. This is expressed in terms of (x, y) as

$$-m' \frac{ay+1}{\sqrt{(y-a)^2 + (1+a^2)x^2}},$$

thus the spherical energy has expression

$$E_{sph} := \frac{1}{2}((1+y^2)\dot{x}^2 - 2xy\dot{x}\dot{y} + (1+x^2)\dot{y}^2) - m' \frac{ay+1}{\sqrt{(y-a)^2 + (1+a^2)x^2}}. \qquad (12)$$

In particular, we derive from these formulas that E_{pl} and E_{sph} are functionally independent.

3. Integrable Billiard Models with Potentials in the Plane

We now define certain billiard models in the plane with Kepler–Coulomb problem with mass-factor m and center Z. For this, it is enough to specify the wall of reflection and on which side of the wall the dynamics defining the billiard system takes place. At this generality, not all orbits starting from a point on the wall comes back to hit the wall again, so that the billiard mapping is possibly not always defined. Nevertheless we also accept such cases for the purpose of uniformity.

A billiard system as such is called *integrable* when there exist two functionally independent conserved quantities. Note that the energy is always a conserved quantity. Therefore the system is integrable when an additional conserved quantity functionally independent from the energy can be found.

We consider the following billiard models with potentials (see Fig. 3): We take a Kepler–Coulomb problem in the plane, in which the mass-factor can take either positive or negative signs. The wall of reflection is taken either as a circle $\mathcal{C}$ centered at the center of the Kepler–Coulomb problem (in what follows, we call such $\mathcal{C}$ centered circle), or a line $\mathcal{L}$, which can take any positions and the dynamics on either sides of the wall can be taken to define the billiard system.

In order to deal with possible collisions with the center Z in the line case, we have to invoke a regularization. Standard Levi-Civita regularization [15] does the work well, and gives a continuation of an orbit running into a collision with the center by an orbit ejecting from the collision by elastic bouncing. When the wall of reflection contains the center Z, any orbit which passes through a point in the line of reflection different from Z will never meet Z, therefore it is also harmless to delete Z from the line of reflection and consider only the rest, non-collisional orbits.

The integrable Boltzmann model corresponds to the case that the mass-factor is negative, with a line as the wall of reflection, and (preferably) with the dynamics on the side of the line not containing the center Z.

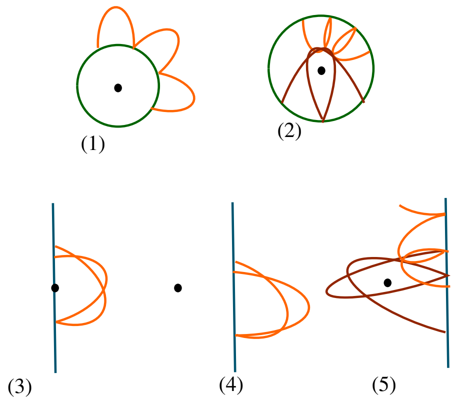

Fig. 3. Some planar Billiard systems.

Theorem 3.1. *Any planar billiard system defined through the Kepler–Coulomb problem with a line or a centered circle as wall of reflection is integrable. The first integrals are its energy together with the first integral induced by the energy of a spherical corresponding system.*

Proof. The integrability of these systems is explained conveniently using projective dynamics. We let $V = \{z = -1\} \in \mathbb{R}^3$ be the plane in which the system is defined.

We first consider the case that the wall is a centered circle $\mathcal{C}$. We choose it to have its center at the south pole $(0, 0, -1)$ of S which is thus also a center of the Kepler–Coulomb problem. Thus $a = 0$ in (7), (12) so we have

$$E_{\mathrm{pl}} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{m}{\sqrt{x^2 + y^2}}$$

and

$$E_{\mathrm{sph}} := \frac{1}{2}((1 + y^2)\dot{x}^2 - 2xy\dot{x}\dot{y} + (1 + x^2)\dot{y}^2) - \frac{m'}{\sqrt{x^2 + y^2}}.$$

In the plane, the natural law of reflection reflects the normal component of the velocity to the wall of reflection while keeping the tangent component of the velocity invariant. Consequently the norm of the velocity does not change and thus E_{pl} is invariant. To see that E_{sph} is also invariant at a centered circle wall of reflection, observe that by rotational symmetry we may assume that the reflection is at a point $(b, 0)$ on the positive x-axis. At this point, we have

$$E_{\mathrm{sph}} := \frac{1}{2}(\dot{x}^2 + (1 + b^2)\dot{y}^2) - \frac{m'}{\sqrt{x^2 + y^2}}$$

and the law of reflection is $(\dot{x}, \dot{y}) \mapsto (-\dot{x}, \dot{y})$ under which E_{sph} is invariant. By rotational symmetry E_{sph} is invariant at reflections at a centered circle.

Note that in this case, it is also direct to see that the square of the angular momentum $C^2 = (\dot{x}y - \dot{y}x)^2$ is also preserved under reflections. Writing

$$E_{sph} := \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}(\dot{x}y - \dot{y}x)^2 - \frac{m'}{\sqrt{x^2 + y^2}}, \tag{13}$$

shows that in this case

$$E_{\mathrm{sph}} = E_{pl} + \frac{C^2}{2}. \tag{14}$$

We now treat the case that the wall of reflection is a line $\mathcal{L}$. We choose the line $\mathcal{L}$ to cross the south pole $(0, 0, -1)$ of S such that $(0, 0, -1)$ is the nearest point to the center of the system Z on $\mathcal{L}$, i.e. the vector from Z to $(0, 0, -1)$ is perpendicular to $\mathcal{L}$. The case $Z = (0, 0, -1)$ is accepted as well.

By translation and rotation we may set the line of reflection to $\mathcal{L} = \{(x, 0, -1)\}$, and we may put the center at $(0, a, -1)$. The planar energy E_{pl} is invariant under reflections at the wall. To have integrability, we expect that the spherical energy E_{sph} is so as well. For this to hold, it is enough that the spherical kinetic energy is invariant under reflections at $\mathcal{L}$. But the spherical kinetic energy $\frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}(\dot{x}y - \dot{y}x)^2$ restricts to the expression $\frac{1}{2}(\dot{x}^2 + (1 + x^2)\dot{y}^2)$ on $\mathcal{L}$, in which the cross-terms containing $\dot{x}\dot{y}$ vanishes. We are now left with a quadratic function on $(\dot{x}, \dot{y})$ without cross-terms which we could expect to be invariant under reflections at $\mathcal{L}$.

Precisely, we have from (7), (12), that, after restricting to $\mathcal{L} = \{(x, 0, -1)\}$ the energies have expressions

$$E_{\mathrm{pl}} = \frac{1}{2}\left(\dot{x}^2 + \frac{\dot{y}^2}{1 + a^2}\right) - \frac{m}{\sqrt{x^2 + \frac{a^2}{1+a^2}}}.$$

$$E_{\mathrm{sph}} := \frac{1}{2}(\dot{x}^2 + (1 + x^2)\dot{y}^2) - \frac{m'}{\sqrt{a^2 + (1 + a^2)x^2}}.$$

The law of reflection in the plane is derived from the norm $\|\cdot\|_2$. We see that at a point on $\mathcal{L}$ it still takes the form $(\dot{x}, \dot{y}) \mapsto (\dot{x}, -\dot{y})$. Both functions E_{pl} and E_{sph} are invariant under this reflection and therefore both functions are conserved quantities.

Consequently both the planar and spherical energies are first integrals of the system in both cases. □

Remark 3.1. For the purpose of simplicity, the above proof does not make reference to a spherical billiard system. Geometrically, the line $\mathcal{L}$ is centrally projected from a half great circle $\{x^2 + z^2 = 1, z < 0\}$ in S_{SH}. The spherical energy is invariant under the natural spherical reflection law at this half great circle. In general it is not true that the natural law of reflection on the sphere projects to the natural law of reflection in the plane, however when this is true, then the spherical energy is also invariant under the planar reflection law. This holds in our situation, and will be explained in the proof of Theorem 4.1 . We thus derive E_{sph} as an additional first integral for the planar billiard system.

4. Integrable Billiard Models on the Sphere

The construction in the last section above also suggests a family of integrable billiard systems on the sphere.

We define billiard systems on the sphere with the spherical Kepler–Coulomb problem with either a circle whose centers agree with the centers of the Kepler–Coulomb problem (henceforth centered circles), or with any choice of a great circle as the wall of reflection, and with the natural law of reflection on the sphere, with dynamics on either side of the circle. When the antipodal pair of spherical Kepler–Coulomb singularities are contained in the circle of reflection, then we may either regularize collisions with the attractive center with elastic bouncing, or remove them from the circle. Note that the repulsive center does not lie in any finite energy hypersurface of the spherical Kepler problem.

Theorem 4.1. *Any spherical billiard system defined through a spherical Kepler–Coulomb problem, with a centered circle or a great circle as wall of reflection is integrable. The first integrals are its energy together with the first integral induced by the energy of a planar corresponding system.*

Proof. We first examined the case of centered circles. This is the spherical analogue of the planar Kepler–Coulomb billiard system with a centered circle as wall of reflection. We put the circle horizontal. In the case of a small circle in S_{SH} we project it centrally to $V := \{z = -1\}$. In this case, we see just as in the planar case that the laws of reflections in the plane and on the sphere correspond to each other and thus both E_{sph} and E_{pl} are first integrals as in the proof of the planar case, Theorem 3.1. In the case of a horizontal great circle, we may simply approximate the billiard system thus obtained by a family of billiard systems on the sphere with horizontal small circles. Note that by our construction the function E_{pl} might not be well defined up to the horizontal great circle. We note that this pair of first integrals are equivalent to E_{sph} together with the square of the angular momentum with respect to the axis passing through the centers. This last quantity is clearly also a conserved quantity and can be extended to define on the whole sphere which also implies the extendability of E_{pl} (an alternative argument is also provided in what follows). Thus by a limiting argument both E_{sph} and E_{pl} are the first integrals also in this case.

When the great circle is not centered on the sphere, by rotation we may set the circle wall of reflection to lie in the plane $\{(x, 0, z)\}$ and the centers to lie in the plane $\{(0, y, z)\}$. The intersection of the great circle wall of reflection with S_{SH} is then centrally projected to a line in the planar system in V and the relative position of the projected center in V with respect to the south pole is perpendicular to the line in V (see Fig. 4). We consider this case as the spherical analogue of the planar Kepler–Coulomb billiard system with a straight line as wall of reflection. In this case, the laws of reflection in the plane and on the sphere correspond to each other as has been observed in the proof of the planar case, Theorem 3.1: To see this, we consider $V := \{z = -1\}$ as an gnomonic chart for S_{SH}, in which the lower half of the great circle $\{x^2 + z^2 = 1, z < 0\}$ is represented by the line $\mathcal{L} := \{y = 0, z = -1\}$ in V. The spherical metric (8) restricted to $\mathcal{L}$ takes the form

$$\frac{1}{(1+x^2)^2}(dx^2 + (1+x^2)dy^2),$$

which gives the spherical law of reflection $(x', y') \mapsto (x', -y')$ in the chart V, in which $'$ refers to time-derivative for the time on the sphere. Since the time change factor (10) depends only on the positions, this is equivalent to the planar law of reflection $(\dot{x}, \dot{y}) \mapsto (\dot{x}, -\dot{y})$. Since the planar energy is invariant under the planar law of reflection, it is also invariant under the spherical law of reflection.

Therefore the planar and spherical energies are functionally independent conserved quantities of the corresponding billiard system on S_{SH}.

Finally, it can be verified directly from (7) that the planar energy E_{pl} extends analytically to a first integral of the billiard system on the sphere: Note that the expression $\dot{x}^2 + \dot{y}^2$ can be expressed from the spherical kinetic energy and the square of the angular momentum C^2 from (13), which is therefore extendable analytically to the tangent bundle of the whole sphere since the angular momentum agrees with the angular momentum on the sphere with respect to the vertical axis [3]. Consequently, by restriction, the expressions $\dot{x}^2$ and $\dot{y}^2$, and thus the expression

$$\dot{x}^2 + \frac{\dot{y}^2}{1+a^2},$$

extend to the tangent bundle of the whole sphere as well. As for the force function, by setting $(x = -X/Z, y = -Y/Z)$ for $(X, Y, Z) \in S_{\mathrm{SH}}$ we see that the planar force function is expressed in these variables as

$$\frac{mZ(1+a^2)}{\sqrt{(1+a^2)X^2 + (Y - aZ)^2}},$$

which extends to an analytic function on S singular at two points, which are just the attractive and repulsive centers of the spherical Kepler–Coulomb problem.

□

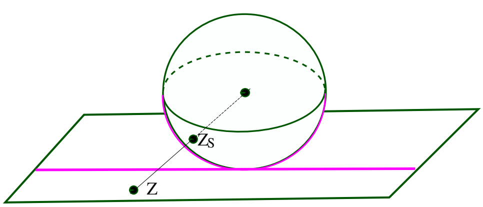

Fig. 4. Central projection correspondence between a planar and a spherical Billiard systems.

It is natural to consider these spherical systems on S as extensions of the planar billiard systems proposed in Sec. 3, in a similar way as the spherical Kepler–Coulomb problem extends the corresponding planar Kepler–Coulomb problem. In particular, the unbounded orbits in the integrable Boltzmann system is completed in its spherical corresponding system.

5. Relation with Gallavotti–Jauslin's First Integral

In this section, we show that the pair $\{E_{pl}, E_{sph}\}$ is equivalent to $\{E_{pl}, D\}$ where D is Gallavotti–Jauslin's first integral for the integrable Boltzmann billiard model, constructed with an analysis of the geometry of ellipses [12]. The result extends to parabolic and hyperbolic orbits of the Kepler problem [9]. Our following result provides an alternative proof uniform in the context of Kepler–Coulomb problems.

We put coordinates in the plane $\mathbb{R}^2$ by (ξ, η). The plane is equipped with the standard Euclidean norm. We consider the Kepler–Coulomb problem with center at $(0,0)$ and with mass-factor m. We set $L = \xi\dot{\eta} - \eta\dot{\xi}$ the angular momentum with respect to $(\xi,\eta) = (0,0)$ and $A_\eta = -L\dot{\xi} - \frac{m\eta}{\sqrt{\xi^2+\eta^2}}$ the η-component of the Laplace–Runge–Lenz vector. Let h be the distance of a horizontal line of reflection $\{\eta = k\}, k \geq 0$ to $(0,0)$. Finally, let $D = L^2 - 2hA_\eta$: This is the first integral identified by Gallavotti and Jauslin.

Proposition 5.1. *The pair of independent first integrals* $\{E_{\mathrm{pl}}, D\}$ *is equivalent to the pair of independent first integrals* $\{E_{\mathrm{pl}}, E_{\mathrm{sph}}\}$.

Proof. In the proof in the line wall of reflection case of Theorem 3.1, we have considered the Kepler–Coulomb problem in $V = \{(x, y, -1)\}$, with mass m' and center at $(0, a, -1)$. The line of reflection is put at $\mathcal{L} := \{(x, 0, -1)\}$. Our assumption that $k \geq 0$ states that the line of reflection lies "not below" the center, which clearly does not change under the affine transformation we are going to use, translates into the requirement that $a \leq 0$ and thus $|a| = -a$.

The norm on V is a non-standard one with which we have obtained in (7), (12) the expressions for the planar and spherical energies as

$$E_{pl} = \frac{1}{2}\left(\dot{x}^2 + \frac{\dot{y}^2}{1+a^2}\right) - \frac{m}{\sqrt{x^2 + \frac{(y-a)^2}{1+a^2}}}$$

and

$$E_{\mathrm{sph}} := \frac{1}{2}((1+y^2)\dot{x}^2 - 2xy\dot{x}\dot{y} + (1+x^2)\dot{y}^2) - m'\frac{ay+1}{\sqrt{(y-a)^2 + (1+a^2)x^2}}.$$

The distance from the center $(0, a, -1)$ to the line $\mathcal{L} := \{(x, 0, -1)\}$ in V with respect to the non-standard norm is given by $h = \frac{|a|}{\sqrt{1+a^2}} = -\frac{a}{\sqrt{1+a^2}}$.

We now normalize the norm in V to a standard Euclidean one via the affine transformation

$$(\xi, \eta) \mapsto (x = \xi, y = \sqrt{1+a^2}\eta + a),$$

which implies $(\dot{x}, \dot{y}) = (\dot{\xi}, \sqrt{1+a^2}\dot{\eta})$. The planar and spherical energies now, respectively, take the forms (with relation $m' = \sqrt{1+a^2}\, m$)

$$E_{\mathrm{pl}} = \frac{1}{2}(\dot{\xi}^2 + \dot{\eta}^2) - \frac{m}{\sqrt{\xi^2+\eta^2}}$$

and

$$\begin{aligned} E_{\mathrm{sph}} &= (1+a^2)\left(\frac{1}{2}(\dot{\xi}^2 + \dot{\eta}^2) - \frac{m}{\sqrt{\xi^2+\eta^2}}\right) \\ &\quad + \frac{(1+a^2)}{2}\left((\xi\dot{\eta} - \eta\dot{\xi})^2 - \frac{2a}{\sqrt{1+a^2}}\left((\xi\dot{\eta} - \eta\dot{\xi})\dot{\xi} + \frac{m\eta}{\sqrt{\xi^2+\eta^2}}\right)\right). \end{aligned}$$

We thus have

$$E_{\mathrm{sph}} = (1+a^2)\left(E_{\mathrm{pl}} + \frac{D}{2}\right). \tag{15}$$

□

Remark 5.1. An anonymous referee suggests that a somehow more geometrical way to normalize the norm in V is to project V orthogonally onto V_1. Indeed a direct verification shows that after this projection, the non-standard norm in V projects now to the standard norm in V_1 as inherited from $\mathbb{R}^3$.

6. Some Further Remarks

We conclude with a few remarks.

First, we remark that with similar construction we may get several families of integrable billiard systems with potentials on the pseudosphere as well.

Next, we remark that conformal transformations induce correspondences of mechanical systems [13, 8], which can be used to obtain other integrable billiard systems. As an example, with the conformal mapping $\mathbb{C} \to \mathbb{C}, z \mapsto z^2$, one gets from the integrable Boltzmann model an integrable billiard system in the plane with the harmonic potential of a pair of isotropic harmonic oscillators, with a hyperbola as wall of reflection. Moreover, the method of Darboux [8] can be used to obtain integrable billiard systems on a cone or on certain other surfaces of revolutions. We wish to address these more precisely in a separate paper.

Finally, it can be interesting to better understand the integrable dynamics of all these systems. In particular, it can be interesting to see to which extend the algebraic geometric method in [9] can be applied to the systems on the sphere.

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