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Compact Quotients of Homogeneous Negatively Curved Riemannian Manifolds*

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1. Introduction

As it is well known (Borel [2]), a symmetric space of negative curvature admits compact quotients, also called compact Clifford-Klein forms. It is the purpose of this note to prove:

Theorem. *Let M be a connected homogeneous riemannian manifold of strictly negative curvature. If M admits a quotient of finite volume, then M is symmetric.*

This may be reformulated as follows:

Theorem′. *A locally homogeneous riemannian manifold of strictly negative curvature and finite volume is locally symmetric.*

The proof of the Theorem is a consequence of two results due to Eberlein-O'Neill [4] and Chen [3].

2. Preliminaries

Let H be a simply connected, complete n-dimensional riemannian manifold of curvature $K \leqq 0$ (Hadamard manifold). Let $\bar{H}$ denote its naturally defined compactification (Klingenberg [6], Eberlein-O'Neill [4]), which is homeomorphic to the closed unit ball of $\mathbb{R}^n$. The group of isometries $I(H)$ of H acts as a group of homeomorphisms on $\bar{H}$, extending its action on H. If G is a subgroup of $I(H)$ denote its limit set by $L(G)$. This is defined by $L(G) = \overline{G(p)} \cap H(\infty)$ where p is an arbitrary point in H, $H(\infty) = \bar{H} \setminus H$ are the points at infinity and $\overline{G(p)}$ is the closure in $\bar{H}$ of the orbit $G(p)$. The limit set is well defined, i.e. independent of p.

We will use the following two results:

Theorem (Eberlein-O'Neill). *Let M be a connected, simply connected complete riemannian manifold of curvature $K \leqq c < 0$ and Γ a group of isometries acting freely and properly discontinuously on M. Assume Γ has a fixed point x in $M(\infty)$. Then either:*

(i) *x is the only fixed point of Γ and $L(\Gamma) = \{x\}$ or*

(ii) *Γ has exactly two fixed points x and y and $L(\Gamma) = \{x, y\}$.*

This is a combination of Proposition 8.9 P and 8.9 A in [4] observing that Γ can have at most two fixed points. Actually both propositions are stated there under a slightly weaker curvature assumption.

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Theorem (Chen [3]). *Let M be a connected, simply connected complete riemannian manifold with curvature $K \leqq c < 0$. Suppose G is a connected subgroup of the group of isometries with $L(G) = M(\infty)$. If G does not have a common fixed point in $M(\infty)$, then G is semisimple.*

Chen's proof is based strongly on his Theorem 3.1 [3], whose proof seems to contain a gap at the end. But this may be filled using Lemma 4.7 of Eberlein-O'Neill [4].

3. Proof of the Theorem

Let M be a connected homogeneous riemannian manifold of strictly negative curvature. Let $\bar{M} = M/\Gamma$ be a quotient of finite volume, where Γ is a group of isometries acting freely and properly discontinuously on M. We may assume $\Gamma \subset I_0(M)$, since $I(M)$ has only finitely many components (M is homogeneous, thus $I(M) = K \cdot I_0(M)$, where K is *a* (compact) isotropy group). If $I_0(M)$ is semisimple, then M is known to be symmetric. If $I_0(M)$ is not semisimple, then it has a fixed point in the boundary by Chen's Theorem and the same holds for Γ. Thus $L(\Gamma)$ contains at most two elements. On the other hand it is well known that $L(\Gamma) = M(\infty)$, if M/Γ has finite volume; a contradiction.

4. Remark

With results of the author [5] and Chen's Theorem it is possible to determine the group of isometries of a homogeneous manifold of strictly negative curvature M. If M is not symmetric, then $I_0(M) = K x_\sigma G$, i.e. $I(M)$ is the semidirect product of a compact subgroup K and a normal solvable subgroup G, which is simply transitive on M. Furthermore G and hence $I_0(M)$ are not unimodular. This implies again – by a result of Siegel [6] – that M has no quotients of finite volume. Recently Azencott and Wilson [1] announced that they have generalized this result to a class of homogeneous manifolds of non-positive curvature. Therefore we don't want to give the proof of the above statement.

References

1. Azencott, R., Wilson, R.: Varietés homogènes à courbure négative. C.r. Acad. Sci., Paris **278**, 561–562 (1974)
2. Borel, A.: Compact Clifford-Klein forms of symmetric spaces. Topology **2**, 111–122 (1963)
3. Chen, S.S.: Complete homogeneous riemannian manifolds of negative curvature. To appear in Commentarii math. Helvet.
4. Eberlein, P., O'Neill, B.: Visibility manifolds. Pacific J. Math. **46**, 45–109 (1973)
5. Heintze, E.: On homogeneous manifolds of negative curvature. To appear in Math. Ann.
6. Klingenberg, W.: Geodätischer Fluß auf Mannigfaltigkeiten vom hyperbolischen Typ. Inventiones Math. **14**, 63–82 (1971)
7. Siegel, C.: Discontinuous groups. Ann. of Math. II. Ser. **44**, 674–689 (1943)

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