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Escape dynamics in an anisotropically driven Brownian magneto-system

I. Abdoli[1,2(a)], J.-U. Sommer[1,2], H. Löwen[3] and A. Sharma[1,2(b)]

[1] *Leibniz-Institut für Polymerforschung Dresden, Institut Theorie der Polymere - 01069 Dresden, Germany*
[2] *Technische Universität Dresden, Institut für Theoretische Physik - 01069 Dresden, Germany*
[3] *Heinrich-Heine-Universität Düsseldorf, Institut für Theoretische Physik II: Weiche Materie 40225 Düsseldorf, Germany*

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Abstract – Thermally activated escape of a Brownian particle over a potential barrier is well understood within Kramers theory. When subjected to an external magnetic field, the Lorentz force slows down the escape dynamics via a rescaling of the diffusion coefficient without affecting the exponential dependence on the barrier height. Here, we study the escape dynamics of a charged Brownian particle from a two-dimensional truncated harmonic potential under the influence of Lorentz force due to an external magnetic field. The particle is driven anisotropically by subjecting it to noises with different strengths along different spatial directions. We show that the escape time can largely be tuned by the anisotropic driving. While the escape process becomes anisotropic due to the two different noises, the spatial symmetry is restored in the limit of large magnetic fields. This is attributed to the Lorentz-force–induced coupling between the spatial degrees of freedom which makes the difference between two noises irrelevant at high magnetic fields. The theoretical predictions are verified by Brownian dynamics simulations. In principle, our predictions can be tested by experiments with a Brownian gyrator in the presence of a magnetic field.

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Introduction. – Escape of a particle from a metastable potential well is one of the most celebrated problems in statistical physics [1]. The escape problem was first conceived and studied by Kramers to model the rate of chemical reactions [2]. In the classical escape problem, which includes only thermal fluctuations, the escape rate decreases exponentially with the barrier height.

The classical trap model has been generalized to include fluctuations other than thermal fluctuations [3–7]. Such a scenario arises naturally in active gels [8,9], in which embedded tracer particles are subjected to both thermal fluctuations and motor-induced, athermal fluctuations [10], and for active particles [11–17]. While one models the thermal fluctuations in the usual fashion as white Gaussian noise, the other noise is often described as a colored noise to model the effect of active, athermal fluctuations. The presence of an additional noise can dramatically modify the escape dynamics of a particle [10,18–20].

Here we explore the escape dynamics of a single Brownian particle from a potential well subjected to two noises with different strengths along different spatial directions, *i.e.*, acting on the different Cartesian components of the momentum vector. Such a system constitutes a primitive Brownian engine composed of a gyrating Brownian particle [21]. This kind of anisotropic driving can be experimentally realized by applying a strongly fluctuating electric field to a charged Brownian particle in one direction mimicking the role of an additional temperature [22]. We show that breaking the spatial symmetry via two different noises results in anisotropic escape dynamics; more particles escape the potential well along the axis with larger noise strength which we refer to as the hot axis. This gives rise to a large tunability of the escape time by the difference between noise strengths. However, when the particle is further subjected to Lorentz force due to an

(a) E-mail: abdoli@ipfdd.de (corresponding author)
(b) E-mail: sharma@ipfdd.de (corresponding author)

external magnetic field, we show that the spatial symmetry is restored in the limit of large magnetic fields; the escape is driven by an effective noise with the average strength of the two noises. This is attributed to the Lorentz-force–induced coupling between the spatial degrees of freedom which makes the difference between noise strengths irrelevant at high magnetic fields.

In a spatially isotropic Brownian magneto-system, the Lorentz force slows down the escape dynamics via a trivial rescaling of the diffusion coefficient (see [23,24] and the data in the Supplementary Material `Supplementarymaterial.pdf` (SM)[1]) without affecting the exponential dependence on the barrier height. In contrast, in an anisotropically driven system, a magnetic field affects the dynamics in a qualitatively different way [25]. While curving the trajectory of a particle, there also occurs energy transfer in form of heat from the hot source to the cold source, which is mediated by the magnetic field [26]. As a consequence, at large magnetic fields, the two spatial degrees of freedom become identical regardless of the difference between noise strengths. In general, it is difficult to study the escape problem in a multidimensional potential well. Here we use an asymptotic method to compute the mean escape time for a large barrier height, obtaining an explicit analytical prediction.

Anisotropically driven Brownian magneto-system. – We consider a Brownian magneto-system, which is made of a single diffusing particle of the charge q and mass m in the presence of a constant magnetic field B in the z-direction. The x and y positions of the particle, which we indicate in the vectorial form $\boldsymbol{r} = (x, y)^\top$, are coupled to noises with different strengths proportional to T_x and T_y, respectively (see fig. 1). The transpose is indicated by $\top$. Since the Lorentz force does not affect the motion of the particle in the direction of the applied magnetic field, we effectively investigate a two-dimensional system in the xy plane.

The dynamics of the particle with velocity $\boldsymbol{v} = (v_x, v_y)^\top$ which is trapped in a potential in the form of $V(\boldsymbol{r}) = \frac{1}{2}\boldsymbol{r}^\top \cdot \hat{U} \cdot \boldsymbol{r}$ can be described by the following underdamped Langevin equation:

$$m\dot{\boldsymbol{q}}(t) = -\hat{F}\boldsymbol{q}(t) + \boldsymbol{\xi}(t), \tag{1}$$

where $\boldsymbol{q}(t) = (x(t), y(t), v_x(t), v_y(t))^\top$ and $\boldsymbol{\xi}(t) = (0, 0, \xi_x(t), \xi_y(t))^\top$ is Gaussian white noise with zero mean and time correlation $\langle \boldsymbol{\xi}(t)\boldsymbol{\xi}^\top(t')\rangle = 2\gamma\hat{T}\delta(t-t')$, where γ is the friction coefficient. Throughout the paper we set the Boltzmann constant k_B to unity. For an isotropic potential $\hat{U} = k\hat{\mathbb{I}}$ with k being the stiffness of the potential. Length and time are measured in units of $\sqrt{\bar{T}/k}$ and γ/k, respectively. Here $\bar{T} = (T_x + T_y)/2$ where $\gamma\bar{T}$ is the average strength of the noises. Here $\hat{T} = \mathrm{diag}(0, 0, T_x, T_y)$ is a diagonal matrix and the matrix $\hat{F}$ is defined as

$$\hat{F} = \begin{pmatrix} \hat{0} & -m\hat{\mathbb{I}} \\ \hat{U} & \hat{G} \end{pmatrix}, \quad \text{where} \quad \hat{G} = \gamma \begin{pmatrix} 1 & -\kappa \\ \kappa & 1 \end{pmatrix}. \tag{2}$$

Here $\hat{\mathbb{I}}$ is the identity matrix and $\kappa = qB/\gamma$ is the diffusive Hall parameter which quantifies the strength of the Lorentz force relative to the frictional force. Note that the hat over the symbols indicates the matrices and the vectors are shown by bold symbols.

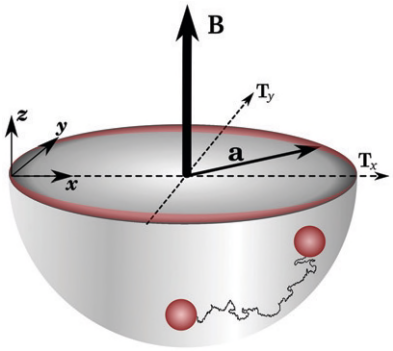

Fig. 1: A single charged Brownian particle, subjected to an external magnetic field B and trapped in an isotropic harmonic potential $V(x, y) = k(x^2 + y^2)/2$ with the parameter k. The particle is simultaneously subjected to two noises with different strengths along its x and y degrees of freedom. The particle escapes more along the y-axis due to the broken spatial symmetry by the two noises, which is shown in reddish color. We show that escape distribution can be tuned and symmetrized by varying the magnitude of the applied magnetic field.

We are interested in the overdamped dynamics of the particle. Usually this is done by setting the inertia term in eq. (1) to zero. However, in the presence of a magnetic field, this yields an incorrect description of the overdamped dynamics [27,28]. A careful small-mass limit of the Langevin equation reveals that the overdamped dynamics, though diffusive in nature, are described an unusual odd-diffusion tensor that has both even (diagonal) and odd (off-diagonal) elements. We note that the variance of the particle position is determined only by the even part of the diffusion tensor [24,25]. The odd part of the tensor, which has its origin in the broken time reversal symmetry due to the magnetic field, gives rise to additional Lorentz (rotational) fluxes which are missed if one takes the route of setting the inertial term to zero. The Lorentz fluxes can play a crucial role in the dynamical properties of Brownian systems [24,25].

Escape from an isotropic trap. – We consider a trapped particle in an isotropic potential $V(\boldsymbol{r})$ and postulate that the particle escapes the trap when it reaches the boundary, truncated at $r = a$, where $r = |\boldsymbol{r}|$ is the distance from the origin, as shown in fig. 1. An exact calculation of the mean escape time is possible in a single temperature magneto-system, as derived in details in the SM, taking advantage of the spatial isotropy. However,

[1]In the SM we show that the large-barrier approximation for a system with temperature $\bar{T}$ gives $\langle t_{esc}\rangle_{eq} \approx (1+\kappa^2)\gamma e^{\delta_E}/2k\delta_E$ where the temperature difference is zero and the escape occurs due to the equilibrium fluctuations.

in an anisotropically driven magneto-system, the broken spatial symmetry prevents such an approach. We therefore use an asymptotic method to investigate the escape dynamics in our Brownian magneto-system. We assume that the barrier height $\Delta E = \frac{1}{2}ka^2$, is sufficiently large that the particle leaks out slowly across the trap and settles into a quasistationary state: the escape is a Poisson process with the inverse rate of mean escape time $\langle t_{esc}\rangle$. The quasistationary probability density is given by $P(\boldsymbol{r},t) \sim \rho_{ss}(\boldsymbol{r})e^{-t/\langle t_{esc}\rangle}$ where $\rho_{ss}(\boldsymbol{r})$ is the steady-state probability density obtained in the limit of $a \to \infty$ which we have obtained in previous works [25,26] and in the SM.

The total outgoing flux at $r = a$ is given by $J(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{2\pi}\int_0^a P(r,\theta,t)r\mathrm{d}r\mathrm{d}\theta$ where $P(r,\theta,t)$ is the quasistationary probability distribution in the polar coordinates. The outgoing flux can be alternatively defined as the probability $a\int_0^{2\pi}\rho_{ss}(a,\theta)e^{-t/\langle t_{esc}\rangle}\mathrm{d}\theta$ to be on the boundary at time t, times the velocity of the fluctuation path leading to the boundary [10,29]. Using the two equivalent definitions of the outgoing flux the mean escape time is equal to the inverse of the stationary-state probability density on the boundary besides a prefactor. The prefactor can be determined from the exact analytical result for a spatially isotropic magneto-system, as shown in the SM. By doing so, we obtain the mean escape time (see the SM), which can be written as

$$\langle t_{esc}\rangle^{-1} \approx \frac{2}{1+\kappa^2}\sqrt{\frac{\delta_E^2}{1-\delta_\kappa^2}}e^{-\frac{\delta_E}{1-\delta_\kappa^2}}I_0\left(-\frac{\delta_\kappa\delta_E}{1-\delta_\kappa^2}\right), \quad (3)$$

where $I_0(x)$ is the modified Bessel function of the first kind of the order zero, $\delta_E = \Delta E/\bar{T}$ is the scaled barrier height, and

$$\delta_\kappa = \frac{\Delta T}{2\bar{T}\sqrt{1+\kappa^2}} \quad (4)$$

is a dimensionless parameter quantifying the difference between the noise strengths, $\gamma\Delta T$ relative to the average noise strength, $\gamma\bar{T}$ scaled by $2\sqrt{1+\kappa^2}$ where $\Delta T = T_y - T_x$.

In a spatially isotropic system the mean escape time in eq. (3) reduces to the familiar expression from Kramers theory $\langle t_{esc}\rangle^{-1} \approx (1+\kappa^2)^{-1}2\delta_E\exp(-\delta_E)$, as shown in the SM. In the general case of two different noises, the escape time also depends on the difference between the strengths of noises via the parameter δ_κ. For a fixed κ, increasing the noise strength difference accentuates the spatial anisotropy of the escape dynamics. However, in the limit of large magnetic fields $\delta_\kappa \to 0$, independent of the difference between the two noises, in which case one again obtains the Kramers escape rate for a spatially isotropic system with temperature $\bar{T}$. The κ-governed crossover from anisotropic to isotropic escape can be qualitatively understood as follows. The Lorentz force curves the trajectory of a moving particle. However, since the two spatial degrees of freedom are subjected to noises with different strengths, the curving of the trajectory also results in energy transfer in the form of heat from the hot source to the cold one. Consequently, in the limit of large magnetic fields, the two spatial degrees of freedom become identical regardless of the difference between noise strengths.

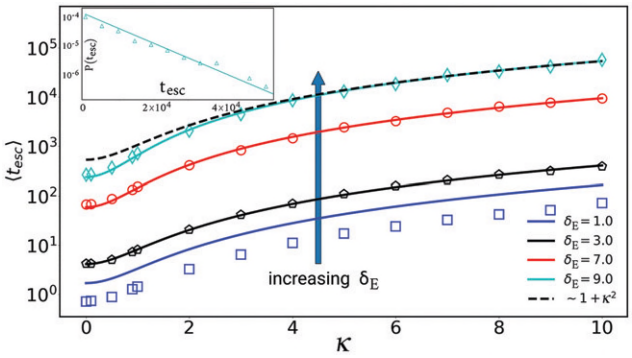

Fig. 2: The mean escape time as a function κ for different barrier heights δ_E with $\Delta T = 2.0$ and $\bar{T} = 3.0$. Solid lines show the theoretical prediction from eq. (3) and the symbols depict the simulation results. The black, dashed line is a plot of the prefactor $1+\kappa^2$. In the limit of large magnetic fields, the overall trend of the mean escape time is determined by this prefactor. The inset shows the distribution of the escape time for $\kappa = 4.0$ and $\delta_E = 9.0$. The solid line is the Poisson distribution $P(t) = \langle t_{esc}\rangle^{-1}\exp(-t/\langle t_{esc}\rangle)$ and the symbols depict the simulation results. The theoretical predictions overestimate the escape time for small barrier heights.

To validate our theoretical predictions we perform Brownian dynamics simulations using the Langevin equations of motion. It has been shown that the overdamped Langevin equation for Brownian motion in the presence of a magnetic field can give rise to unphysical values for velocity-dependent variables such as fluxes [28]. Therefore, we use the underdamped Langevin equation, given in eq. (1), with a sufficiently small mass. In the simulation, we use $\mathrm{d}t = 5\times10^{-7}\gamma/k$. The choice of $\mathrm{d}t$ is based on the following consideration. In a single integration step, the change in velocity due to the noise is of the order $\sqrt{\mathrm{d}t\gamma T_i/m^2}$ where $i = x, y$. This should be much smaller than the typical velocity that scales as $\sqrt{T_i/m}$. This yields $\mathrm{d}t \ll m/\gamma$. For the choice of mass $m = 10^{-3}\gamma^2/k$, this yields $\mathrm{d}t \ll 10^{-3}\gamma/k$. We consider a total number of particles of 1000.

Figure 2 shows the mean escape time as a function of the diffusive Hall parameter κ for different values of the scaled barrier height δ_E. The data are obtained from simulations of a system with $\bar{T} = 3.0$ and $\Delta T = 2.0$. Our theoretical predictions are in good agreement with the simulation results. In the SM we show that the large barrier approximation of the exact expression for escape time in a single temperature system is highly accurate compared to the known Kramers results when the expression is scaled by an empirical factor 1.2. We use the same factor in our asymptotic approach and observe that our theoretical predictions are in good agreement with the simulation results.

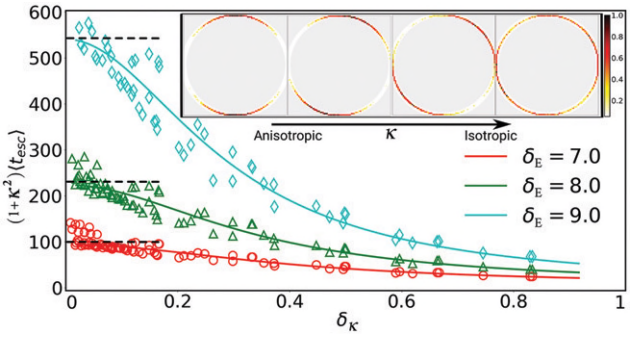

Fig. 3: The mean escape time as function of the scaled difference between the two noise strengths δ_κ for different barrier heights. Solid lines show the results from the theoretical prediction in eq. (3) and the symbols depict the simulation results. While a small δ_κ corresponds to either large magnetic fields or noises with the same strength, $\delta_\kappa \to 1$ corresponds to either small magnetic fields or large difference between the strengths of the two noises. Simulation data is obtained for $\Delta T = 1.0, 2.0, 3.0, 4.0, 5.0$ with $\bar{T} = 3.0$. The diffusive Hall parameter varies from 0.0 to 10.0. The dashed lines show the saturation to the system with temperature $\bar{T}$ corresponding to either two noises with the same strength or large-magnetic-field limit. The inset shows the angular distribution for $\Delta T = 2.0$ and $\bar{T} = 3.0$ and different values of the diffusive Hall parameter, $\kappa = 0.0, 0.5, 2.0, 10.0$ from left to right, respectively. In the absence of a magnetic field the particle mostly escapes along the y-axis due to the broken spatial symmetry via the two noises with different strengths.

While in the limit of large magnetic fields the overall trends of the curves are determined by the prefactor $1+\kappa^2$, as shown in fig. 2 by dashed lines, it deviates from this prefactor for small magnetic fields. In other words, while the anisotropic driving of the system affects the escape dynamics for small magnetic fields, the spatial symmetry is restored in the limit of large magnetic fields.

Figure 3 shows the scaled mean escape time as a function of the parameter δ_κ. The Brownian dynamics simulation has been done for systems with ΔT varying from 1.0 to 5.0 and the same noise strength average such that $\bar{T} = 3.0$. The diffusive Hall parameter is varying from $\kappa = 0.0$ to 10.0. In the limit of large magnetic fields or small temperatures, the noise strength difference parameter $\delta_\kappa \to 0$. Small magnetic fields or large noise strength difference correspond to $\delta_\kappa \to 1$. The scaled mean escape time decreases with increasing parameter δ_κ. The dashed lines show the saturation to a system with temperature $\bar{T}$ corresponding to two noises with the same strength or alternatively to large–magnetic-field limit. In the inset, we show the angular distribution from simulations for different values of the diffusive Hall parameter. As can be seen, in the absence of a magnetic field the particle mostly escapes along the y-axis due to the broken spatial symmetry via the two different noises. The escape becomes symmetric in the limit of a large magnetic field, namely the difference between the two noise strengths becomes irrelevant.

Conclusions. – We studied the escape dynamics in an anisotropically driven Brownian magneto-system. We derived an approximate expression for the mean escape time taking into account two noises with different strength along different spatial directions. We showed that Lorentz force induces a coupling between the spatial degrees of freedom which, in the limit of large magnetic fields, restores the spatial symmetry; the two spatial degrees of freedom become identical regardless of the difference between the two noise strengths.

While experimental realisation of the proposed magneto-system might be particularly difficult in colloidal systems, it is more realistic in a complex plasma where extremely large magnetic fields can be generated [30]. In such a set up, not only can one study the escape dynamics, one can also measure the momenta governed energy flow between the hot source and the cold bath. Such heat currents have been predicted theoretically, however, the studies employed toy models in which a particle is coupled to multiple thermostats [31,32]. A complex plasma, with anisotropic driving, thus presents a potentially realizable system to study momenta governed heat currents. The anisotropic noise strength can also be realized by putting the charged object in contact with active particles that provide the noise. If the latter are propelling in an anisotropic way, as for instance on an anisotropic patterned substrate [33] or in a nematic liquid crystalline cell [34], their effect on the charged particle would realize an anisotropic noise strength. Other possible realizations in which a confined particle experiences random kicks in an anisotropic environment under the additional action of a Lorentz, Coriolis or Magnus force are also conceivable.

In future work we aim to extend our model to the escape through a narrow hole [35–37]. An interesting generalization would be to replace the isotropic potential by an anisotropic potential where the stationary state of the system is characterized by not only a non-Boltzmann probability distribution, but also additional Lorentz fluxes [26]. Also, it could be interesting to study the escape in an anisotropically driven Brownian magneto-system from a bistable state, which might be realized by trapping the particle using two optical tweezers [38]. Another study of interest would be a system in the presence of a fluctuating magnetic field where a new turnover is observed as a generic signature of the system [39,40]. Finally, our analysis might be applicable to other systems such as chiral colloidal microswimmers in parabolic potentials [41], active Janus particles in a complex plasma [42], and even rotating skyrmions [43,44].

* * *

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Data availability statement: The data that support the findings of this study are available upon reasonable request from the authors.

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