An investigation on the use of GeoGebra in university level calculus

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Geogebra has shown great potential in school mathematics and some areas of university level math. However, it seems unclear how integration into a more abstract calculus should be done. The purpose of this paper is to describe the approach taken and reports some promising results.

Keywords: Calculus, Geogebra, logic, formal mathematics

INTRODUCTION

The shift to formal mathematics is a major obstacle for first year university students and it is currently of broad interest in Germany (e.g. Hoppenbrock et al. 2016) and internationally (e.g. Gonzales-Martin et al. 2017). In Germany students enter university after having gained a high school diploma. The curriculum of these schools includes some basic calculus (derivatives and integral) but on a very informal level where proofs play almost no role at all. Thus, when starting at the university they experience a substantial gap that results in high failure rates in examinations after the first term (typical failure rates 70-80%).

The learning of calculus has been investigated by many researchers. A recent overview is given by Bressoud et al. (2016). Insight has been gained into many problems that students face when learning university level calculus, e.g. problems with logic (e.g. Selden& Selden (1995), Shipman (2016)) and proofs (e.g. Stavrou (2014)). A wider overview is also given in (Winslow 2018).

The use of technology is discussed in a variety of papers as well. Tall (2003) has argued that technology allows for an embodied approach to teaching calculus by making notions dynamic and visible. Similarly, Moreno-Armella (2014) argued that the traditional teaching approach is not able to bridge the tension between intuition and formalism. He suggests some dynamic activities that illustrate limiting processes and involve differentials as small changes.

A lot of research has investigated the use of dynamic math software such as Geogebra (Hohenwarter 2019) for the learning of mathematics in general and also of calculus. However, the majority of research concentrates on the high school level. Beyond high school college calculus is investigated to some extent but there are only a few investigations about using Geogebra at the university level of analysis. In Tall et al. (2008) an overview is given that aims mainly at the high school level but presents also ideas beyond that. One paper that focusses on university level analysis is Attorps et al. (2016). They find positive effect in teaching Taylor approximations using a variation-theory based approach. d'Azevedo Breda and dos Santos dos Santos (2015) investigated complex numbers. Nobre et al. (2016) have positively evaluated the use of Geogebra in a calculus course for computer science students. However, the topics touched are more of the college style calculus. Much the same can be said about Machromah et al. (2018).

The contribution of this paper is new as it addresses rigorous university analysis.

THE STUDY DESIGN

The course

The course "Analysis I" was taught by the author in the summer term 2019 (duration 14 weeks). 180 students were enrolled into the course with 141 taking the examination at the end. Students' age and sex was not recorded for reasons of privacy but age was approximately 20 and sex distribution almost equal.

The main learning objective of this course is to introduce students to the rigour of mathematics. This course is taken mainly by students aiming at a bachelor in mathematics, but also students from physics and trainee teachers for high schools. The content includes logic, axiomatic theory of natural, rational, real and complex numbers, sequences, series and convergence, limits, continuity, differentiability, sequences of functions, Taylor series, and integral. The approach is rigorous, i.e. all statements are proven and exercise for students included a lot of proof tasks. The course consists of 4 h lecture per week, 2 h exercises in a huge group, and homework exercises which are graded and discussed in small groups (2 h / week).

The setting implies that many concepts have to be re-learned by the students, e.g. in high school the sine and cosine functions are defined geometrically while in this course they are defined by the exponential series. Traditionally, computers are practically absent from such courses. However, for the redesigned course reported here, computers were used to some extent (also Mathematica). This paper concentrates on the use of Geogebra. About half of the students reported that they knew Geogebra from high school. The use of Geogebra was twofold:

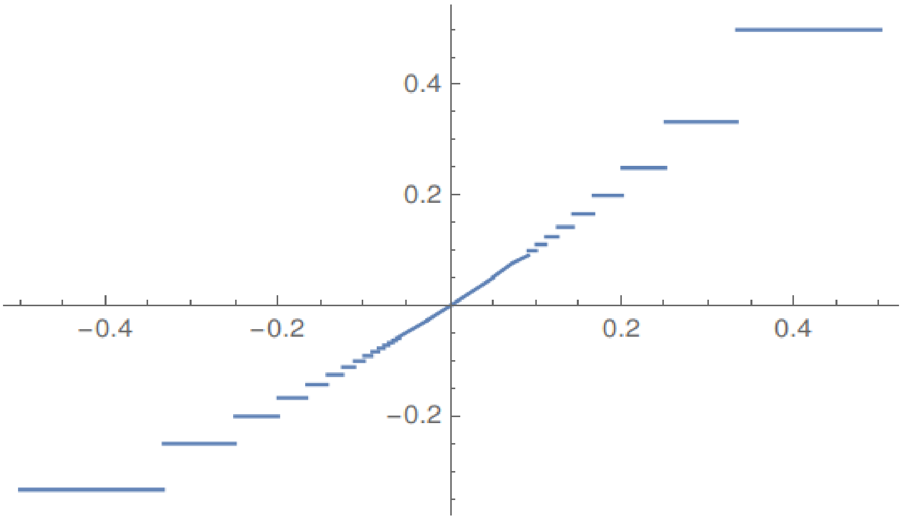
- Demonstrations in the lectures. Many concepts, e.g. addition and multiplication of complex numbers, epsilon-strip-concept of convergence, convergence of function sequences (in general and particular for Taylor series), epsilon-delta-definition of continuity, local linearity of differentiable functions etc were visualized.
- Non-mandatory home work. Every week a set of homework assignments were given and some of them were mandatory and graded, however, for legal reasons, the computer assignments were voluntary.

Task design

The following example illustrates the use of Geogebra in homework assignments:

Exercise: Investigate where the function $f\colon \left(\frac{-1}{2}, \frac{1}{2}\right) \to \mathbb{R}, f(0) := 0; f(x \neq 0) := \frac{1}{\left\lceil \frac{1}{x} \right\rceil}$ has a derivative.

In doing this it is very useful to plot the function as this gives the idea that it may be differentiable in the origin with $f'(0) = 1$, which is a bit contra intuitive. (I learned this nice example from Peter Quast, Augsburg).



The didactical principle behind this task design is a kind of variation theory (Maton & Booth 1997). In mathematics education this theory has been mainly applied in elementary school mathematics. A very typical example is the use in a teaching experiment on logarithms (O'Neil & Doerr 2015). In my own conceptualization the theory says that learning materials should be arranged to allow the individual genesis of a concept by contrasting examples and counter-examples, experience relations to hold of a variety of examples, identify single aspects, exclude counter examples and fuse several aspects to the general concept. Applied to the concept of differentiability this leads to the following learning trajectory: Students learned the concept definition $f'(x) := \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ already in high school. This definition emphasizes the aspect of rate of change which is applied to determine the slope of tangents. Thus, in my course the derivative was introduced in the following varied manner: f is differentiable in x_0 if there is a function $q: U \to \mathbb{R}, x_o \in U$ defined on some open neighbourhood of x_0 and continuous in x_0 such that $f(x) - f(x_0) = q(x) \cdot (x - x_0)$, i.e. $\Delta y = q(x) \cdot \Delta x$. This definition emphasizes local linearity and students were demonstrated in the lecture that graphs of differentiable functions appear straight when zoomed in at a sufficient scaling factor. Variation theory then suggested to explore a bunch of functions to sharpen the concept. The example given above in the example is the most challenging in this series.

Assessment

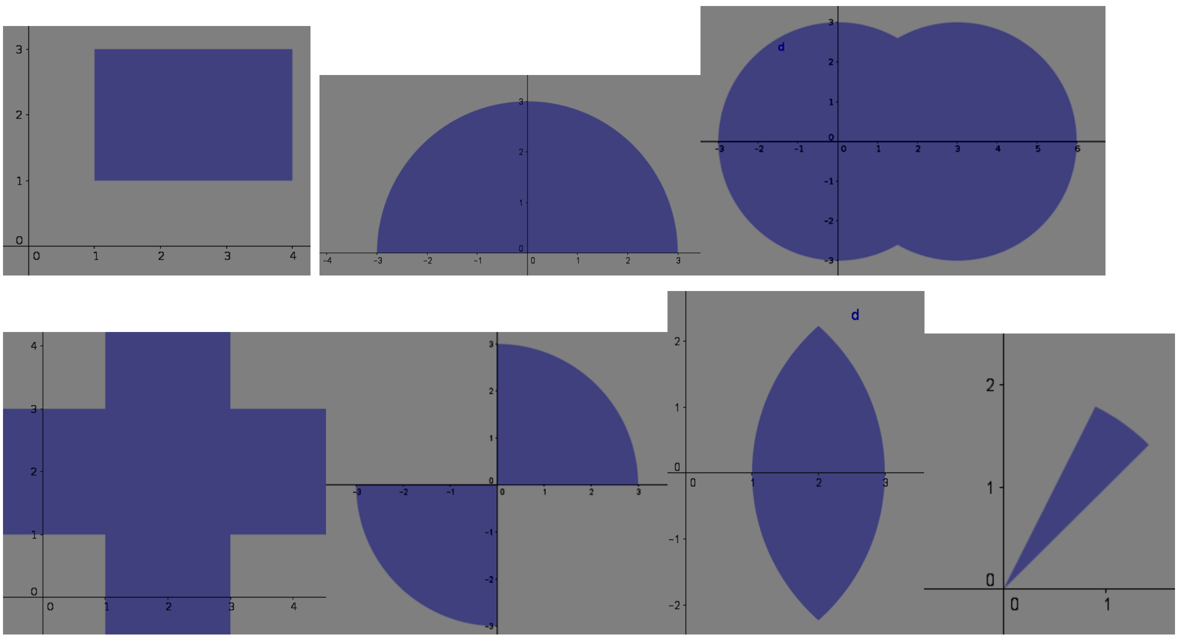
The general research question would be if this kind of using Geogebra helps students to master the course. In this generality, of course, the question cannot be answered empirically, and more precise questions will be posed later on.

In general, empirical intervention studies at university level are not easy to carry out. Ideally, one would randomly split courses into groups with different treatment and measure results. However, splitting a course requires teaching resources that are rarely available and spitting also raises the ethical issue if some students are offered better conditions than others. In this situation the problem that computer exercises could not be made mandatory turned out to offer a new possibility for research: Students themselves decided if they did the computer exercises or not. Hence, this provided two groups without ethical problems. However, one should not assume these two groups to be equivalent. It seems likely that students doing the exercises might be more interested, more motivated and thus stronger overall. The methodological trick to solve this problem was to give two different kinds of tasks: One that could potentially profit from the computer exercises because the mathematical content was related and another that was not expected to benefit from doing the computer exercises. The categorization of the tasks into these groups was done by my own expertise; the tasks used for assessment are detailed below. They were chosen to reflect some of the many teaching goals of this course, especially they should assess the understanding of the logical argumentation about sets, sequences and functions.

FIRST STUDY

The mathematical topics dealt with in the beginning were logic and sets. During the first week the following (non-mandatory) computer exercise was given:

> **Task (voluntary)**: Logic with Geogebra
>
> a) Geogebra can plot the set of solutions of certain (not too complex) inequalities in the two variables x, y. Try this out using the following inequalities:
>
> 1) $x + 1 > y - x/2$ 2) $x \cdot y < 4$ 3) $x^2 + y^2 < 9$ 4) $2x^2 + y^2 > 5$
>
> b) One may also plot logical combinations of inequalities. Try out: $x > 1 \land x < 4,\ x > 1 \lor x < -1,\ x > 0 \land \neg y > 0,\ x > 0 \rightarrow y > 0$
>
> c) Find ways to describe these sets:



The rationale behind this task should be obvious: Students should have the opportunity to work in visually appealing setting with logical operators that gives direct feedback. The importance of feedback is widely acknowledged (e.g. Hattie & Timperley 2007), so this should be effective.

During the second week the students had to do homework exercise that had to be done on paper and were graded. Two of these mandatory exercises are given below:

Exercise 1

a) Prove: $M = N \Leftrightarrow M \subset N \wedge N \subset M$.

b) Prove both *de Morgan* laws for sets.

c) Illustrate the symmetric set difference $M \Delta N \coloneqq (M \cup N) \setminus (M \cap N)$ and prove: $M \setminus N = M\Delta(M \cap N)$.

Exercise 2

Find pairs of equal sets and prove equality resp. inequality:

$M_1 = \{(x,y)|\neg(x > 2 \wedge x < 3)\}, M_2 = \{(x,y)|x \cdot y > 0\}$

$M_3 = \{(x,y)|x > 2 \wedge y > 0 \vee y < 0\}$

$M_4 = \{(x,y)|x > 0 \wedge y > 0 \vee x < 0 \wedge y < 0\}$

$M_5 = \{(x,y)|x \leq 2 \vee x \geq 3\}, M_6 = \{(x,y)|\neg((x \leq 2 \vee y \leq 0) \wedge y \geq 0)\}$

My expert classification was that Exercise 2 might benefit from doing the Geogebra task, while little effect of Geogebra use on exercise 1 was to be expected. Thus, the hypothesis was that students who decided to do the Geogebra tasks would perform substantially better on exercise 3 but not better or only slightly better on the other tasks.

To assess which students took the voluntary Geogebra task students were asked explicitly to indicate if they did do the Geogebra task and then they were asked to rank the intensity on a Likert scale from 0 (not done) to 5 (intensely).

Unfortunately, several of the master students that ranked the students' papers forgot to write down these engagement variables and due to privacy issues, it was not possible to get this information. Hence, the usable data set consists of a rather small sample of n=23 students, 11 of them indicated that they had worked on the Geogebra task (group G), 12 indicated that they didn't (group N). Statistics (all done in R, www.r-project.org) is thus limited but here are the results:

E1, E2 denote the score students achieved at exercises 1 and 2 respectively. These variables can be considered to be normally distributed as the Shapiro test gives p-values for $E1$ of 0.33 for the whole group and of 0.48, respectively 0.45 for the N and G groups. However, $E2$ cannot be considered to be distributed normally. Thus, the Wilcox test is applied to discover group differences between the N and G groups.

Exercise	Wilcox-Test	Cohen *d*: G-N
*E*1	0.27	-0.371
E2	0.014 *	0.842

Conclusion: The students who worked on the Geogebra task scored significantly better on the third tasked, as expected. The fact that they performed worse (although not significantly) on exercise 1 came as a surprise and there is no good explanation yet. It is likely that this is just due to the small number of students, but it may also be that good and theoretically-minded students did not do the computer exercises.

Another way to explore the findings statistically is to use a linear regression model that includes the information (provided by the students) on the intensity of their technology use T. Although this is not normally distributed, a linear model was devised: $E2 \sim T + E1$ and it turned out, that T is significant, the whole explained variance is $R^2=0.44$. Given the fact that many other issues influence performance on such tasks this should be regarded as being rather high.

SECOND STUDY

The second study was conducted almost at the end of the course, in week 12. The methodology was the same as in the first study. While the first study focussed on a very small intervention the second study was more designed to account for the whole learning effect during the term.

A total of n=97 students' exercise responses could be used in the statistics. First, there were two Likert-scale items to judge agreement with a statement from 0 to 10:

a) "I used Geogebra regularly for this course." Mean: 3.4, Std. dev.: 3.0

b) "Geogebra is a useful tool for learning in this course." Mean: 6.7, Std.: 2.6

Those students who marked 5 or more on the first question were considered to be the Geogebra user group (G, 37 students), the others the non-users (N: 60 students)

The marked mandatory exercises that were used in this study were the following:

Exercise 1 Prove for which $k \in \{1,2,3\}$ the functions $f_k: \mathbb{R} \to \mathbb{R}, f_k(x) \coloneqq \begin{cases} \sin(x) + x^k \cdot \sin\left(\frac{1}{x}\right), x \neq 0 \\ 0, x = 0 \end{cases}$ are differentiable and if the derivatives are continuous.

Exercise 2: Prove: If $f: \mathbb{R}^+ \to \mathbb{R}$ is differentiable and $\exists c > 0 \exists d > 0: \forall x \geq d: f'(x) > c$, then $\lim_{x\to\infty} f(x) = \infty$. Give an example that shows that the conclusion is not valid if one only demands: $\forall x \geq d: f'(x) > 0$.

The choice of these tasks was mainly driven to match the topics of the lecture in that week but some theoretical considerations came into play: Given the application of Geogebra to explore the concept of derivative (explained above) I assumed that

Geogebra-affine students can use the tool to foster their intuition about what is going on here. Thus, it was expected that this task benefits from using Geogebra. However, it seems not obvious that the transfer from the graphical setting to a written proof that as required here can be made. The second task does not invite for plotting as no concrete function is given. Moreover, it deals with quantifiers that are not touched on in any Geogebra activity. As above, students' solutions were marked and graded by points by master students. For all these variables, the hypothesis of normal distribution was checked using the Shapiro test and had to be rejected.

Our general hypothesis is that students who used Geogebra regularly performed better than others. More specifically: Use of Geogebra should boost results of Exercise 1 because students who used Geogebra regularly could be expected to investigate this functions' graphs and used zooming in to investigate the limit empirically. For exercise 2 I didn't expect a benefit of using Geogebra besides the baseline effect caused by the fact that Geogebra use was likely to correlate with motivation and engagement. Wilcox tests were performed to test this hypothesis.

Exercise	Wilcox-Test	Cohen d: G-N
E1	0.00 **	0.52
E2	0.63	0.05

These results nicely confirmed the hypothesis.

DISCUSSION AND CONCLSION

Geogebra is a tool that can be used both in high school and at the university level and thus offering the advantage that students experience some continuity in the tool as the experience the rather radical change of mathematical culture from school to university. The study adds evidence to the proposition that Geogebra can be used to boost students' performance on certain tasks of rigorous analysis. Besides the usefulness to visualize graphs in this study the plotting of solutions to logical combinations of inequalities proved to be a useful teaching tools that should be studied in more detail.

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