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We study a family of action functionals whose critical points interpolate between frozen planet orbits for the helium atom with mean interaction between the electrons and the free fall. The rather surprising first result of this paper asserts that for the whole family, critical points are always nondegenerate. This implies that the frozen planet orbit with mean interaction is nondegenerate and gives a new proof of its uniqueness. As an application, we show that the integral count of frozen planet orbits with instantaneous interaction equals one. For this, we prove orientability of the determinant line bundle over the space of self-adjoint Fredholm operators with spectrum bounded from below, and use it to define an integer valued Euler characteristic for Fredholm sections whose linearization belongs to this class.

1. Introduction

Frozen planet orbits are periodic orbits in the helium atom in which both electrons move on a line on the same side of the nucleus. The inner electron undergoes consecutive collisions with the nucleus, while the outer electron (the actual "frozen planet") remains almost stationary at some distance. See [13; 14] for numerical evidence for such orbits and a discussion of their role in the semiclassical treatment of the helium atom.

When trying to prove the existence of frozen planet orbits, one faces the difficulty that they cannot be obtained as perturbations of the system without interaction between the electrons. In order to deal with this problem, the second author replaced in [9] the instantaneous interaction between the two electrons by a mean interaction and proved that in this case for every negative energy there exists a unique frozen planet orbit. Building on work of Barutello, Ortega and Verzini [2], we introduced in [4] two functionals $\mathcal{B}_{av}$ and $\mathcal{B}_{in}$ whose critical points correspond to the Levi–Civita regularizations of frozen planet orbits for the mean and instantaneous interaction, respectively. We proved that for each $r \in [0, 1]$ the L^2-gradient $\nabla\mathcal{B}_r$ of the interpolation $\mathcal{B}_r = r\mathcal{B}_{in} + (1 - r)\mathcal{B}_{av}$ is a C^1-Fredholm map of index 0.

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Now to each C^1-Fredholm map F of index 0 with compact zero set one can associate its mod 2 *Euler number* $\chi(F) \in \mathbb{Z}/2\mathbb{Z}$ counting its zeroes modulo 2 after perturbation; see [4, Appendix C]. Using the compactness result from [10] and homotopy invariance of the mod 2 Euler number, we deduced

$$\chi(\nabla \mathcal{B}_{in}) \equiv \chi(\nabla \mathcal{B}_{av}) \equiv 1 \pmod 2. \tag{1}$$

Here the functionals are considered on a suitable space of normalized simple symmetric loops. In particular, for each negative energy there exists a frozen planet orbit [4, Corollary C].

The proof that $\chi(\nabla \mathcal{B}_{av}) \equiv 1 \mod 2$ in [4, Theorem D.1] was based on a further deformation of $\nabla \mathcal{B}_{av}$ through Fredholm maps which are not gradients of functionals. Our first result in the present paper improves this to (see Corollary 4.6)

Theorem A. *For each negative energy there exists a unique normalized simple symmetric frozen planet orbit for the mean intersection functional $\mathcal{B}_{av}$. It is nondegenerate and of Morse index* 0.

To prove this, we introduce a family of functionals $\mathcal{F}_r$, $r \in [0, \infty)$, such that for $\rho = (\sqrt{2}-1)^2$ the critical points of $\mathcal{F}_\rho$ and their Hessians agree with those of $\mathcal{B}_{av}$, see Section 4B. On the other hand, $\mathcal{F}_0$ describes the *free fall* of an electron into the helium nucleus (again undergoing consecutive collisions which are regularized), which is easily seen to possess a unique simple periodic orbit that is nondegenerate of Morse index 0. This can be seen by looking at its Fourier expansion; see [11, Lemma 3.6]. Thus Theorem A (as well as the uniqueness of the frozen planet orbit for $\mathcal{B}_{av}$) will be a consequence of the following result (see Theorem 3.1):

Theorem B. *For every $r \in [0, \infty)$, each critical point of $\mathcal{F}_r$ is nondegenerate.*

In Section 3D we explain how requiring the additional properties "normalized", "simple", "symmetric" breaks the symmetries and leads to uniqueness. The main ingredient in the proof of Theorem B is an algebraic identity associated to critical points of $\mathcal{F}_r$ which can be solved in terms of elliptic integrals (see Proposition 3.4 and the Appendix.

Theorem A allows us to upgrade (1) to an equality of *integer valued Euler numbers*. For this, we need to define a $\mathbb{Z}$-valued Euler number for a class of Fredholm sections including the $\nabla \mathcal{B}_r$ above. The important feature of these Fredholm sections is that their linearizations are *self-adjoint* and bounded from below with respect to the L^2-scalar product. The main ingredient is the following abstract result which may be of independent interest.

Let F be a real Hilbert space, and $E \subset F$ a dense linear subspace which is itself a Hilbert space (with a different inner product) such that the inclusion $E \hookrightarrow F$ is

compact. For a real number $\mathfrak{R}$ consider the spaces

$$\mathcal{F}_s^{>\mathfrak{R}}(E,F) \subset \mathcal{F}_s(E,F) \subset \mathcal{F}(E,F)$$

where $\mathcal{F}(E,F)$ is the space of Fredholm operators $E \to F$, $\mathcal{F}_s(E,F)$ the subspace of operators that are self-adjoint as unbounded operators on F with domain E, and $\mathcal{F}_s^{>\mathfrak{R}}(E,F)$ the subspace of operators whose spectrum is contained in $(\mathfrak{R},\infty)$. Recall (see e.g., [12]) that the determinants $\det(D) = \Lambda^{\max}(\ker D^*) \otimes \Lambda^{\max}(\ker D)$ of $D \in \mathcal{F}(X,Y)$ give rise to a real line bundle, the *determinant line bundle* $\det \to \mathcal{F}(X,Y)$. It is well-known that this line bundle is nonorientable, but we have (see Theorem 5.4):

Theorem C. *The restriction of the determinant line bundle to $\mathcal{F}_s^{>\mathfrak{R}}(E,F)$ carries a canonical orientation.*

Contrary to our initial expectation, the restriction of the determinant line bundle to $\mathcal{F}_s(E,F)$ is nonorientable; we construct an explicit loop over which the bundle is nontrivial in Proposition 5.7.

Consider now a Hilbert manifold X and a Hilbert space bundle $E \to X$ with a continuous bundle inclusion $TX \subset E$ such that $T_xX \subset E_x$ is dense and the inclusion $T_xX \hookrightarrow E_x$ is compact for each $x \in X$. Denote by $\mathrm{Func}_s^{>\mathfrak{R}}(X)$ the space of C^2-functions $f : X \to \mathbb{R}$ whose E-gradient $\nabla_E f : X \to E$ is of class C^1 such that $\nabla_E f(x) \in \mathcal{F}_s^{>\mathfrak{R}}(T_xX, E_x)$ for each critical point x of f. For this class of functions, Theorem C allows us to define a $\mathbb{Z}$-valued Euler number (see Theorem 5.12):

Corollary A. *To each $f \in \mathrm{Func}_s^{>\mathfrak{R}}(X)$ with compact zero set we can associate an* ***Euler number*** *$\chi(\nabla_E f) \in \mathbb{Z}$ which is uniquely characterized by suitable axioms of* (*Transversality*), (*Excision*), *and* (*Homotopy*).

Verifying that the $\mathcal{B}_r$ above belong to $\mathrm{Func}_s^{>\mathfrak{R}}(X)$ for a suitable bundle $E \to X$ and using the (Transversality) and (Homotopy) axioms, we deduce (see Corollary 6.2):

Corollary B. *The integral count of normalized simple symmetric frozen planet orbits equals*

$$\chi(\nabla \mathcal{B}_{in}) = \chi(\nabla \mathcal{B}_{av}) = 1 \in \mathbb{Z}.$$

Remark. Symmetric frozen planet orbits have two Morse indices: one as a symmetric frozen planet orbit, and one just as a frozen planet orbit forgetting about the symmetry. When we talk about Morse indices (which enter into the Euler number via the (Transversality) axiom in Theorem 5.12) we always mean the symmetric one. In fact, the two indices are different: the unique normalized simple symmetric periodic orbit for the regularized free fall (i.e., the functional $\mathcal{F}_r$ for $r = 0$) has index 0 as a symmetric orbit and index 1 just as a periodic orbit. As we show, the functional $\mathcal{F}_r$ is always nondegenerate in the symmetric as well as in the just periodic sense. Therefore, the symmetric Euler characteristic equals 1 while the

periodic Euler characteristic equals -1. However, to our knowledge, compactness for frozen planet orbits in the homotopy from mean to instantaneous interaction has only been established in the symmetric case [10]. Therefore, it is not clear whether Corollary B has an analogue if one forgets about the symmetry. We expect that it does and that the Euler characteristic for normalized simple (not necessarily symmetric) frozen planet orbits with instantaneous interaction is -1. This would fit with the findings of physicists [13; 14], who numerically detected a frozen planet orbit for instantaneous interaction which is stable and therefore has odd Conley–Zehnder index.

2. Levi–Civita transformation

In this section we recall some background on the Levi–Civita transformation. For details we refer the reader to [4].

We abbreviate by $S^1 = \mathbb{R}/\mathbb{Z}$ the circle. We denote the L^2-inner product of $z_1, z_2 \in L^2(S^1, \mathbb{R})$ by

$$\langle z_1, z_2 \rangle := \int_0^1 z_1(\tau) z_2(\tau)\, d\tau,$$

and the L^2-norm of $z \in L^2(S^1, \mathbb{R})$ by

$$\|z\| := \sqrt{\langle z, z \rangle}.$$

In the sequel we will work with Sobolev spaces $H^k = W^{k,2}$, but the only relevant norms and inner products will be the ones from L^2.

Consider two maps

$$q : S^1 \to \mathbb{R}_{\geq 0}, \quad z : S^1 \to \mathbb{R}$$

related by the *Levi–Civita transformation*

$$q(t) = z(\tau)^2 \tag{2}$$

for a time change $t \leftrightarrow \tau$ satisfying $0 \leftrightarrow 0$ and

$$\frac{dt}{q(t)} = \frac{d\tau}{\|z\|^2}. \tag{3}$$

This implies that the mean values of q and $1/q$ are given by

$$\bar{q} := \int_0^1 q(t)\, dt = \int_0^1 \frac{z(\tau)^4}{\|z\|^2}\, d\tau = \frac{\|z^2\|^2}{\|z\|^2} \tag{4}$$

and

$$\int_0^1 \frac{dt}{q(t)} = \frac{1}{\|z\|^2}. \tag{5}$$

We will denote derivatives with respect to t by a dot and derivatives with respect to τ by a prime. Then the first and second derivatives of q and z (where they are defined) are related by

$$\dot{q}(t) = 2z(\tau)z'(\tau)\frac{d\tau}{dt} = \frac{2\|z\|^2 z'(\tau)}{z(\tau)} \tag{6}$$

and

$$\ddot{q}(t) = 2\|z\|^2 \frac{z''(\tau)z(\tau) - z'(\tau)^2}{z(\tau)^2}\frac{d\tau}{dt} = \frac{2\|z\|^4}{z(\tau)^4}(z''(\tau)z(\tau) - z'(\tau)^2). \tag{7}$$

Substituting z^2 and z'^2 by (2) and (6) this becomes

$$\ddot{q}(t) = \frac{1}{q(t)}\left(2\|z\|^4 \frac{z''(\tau)}{z(\tau)} - \frac{\dot{q}(t)^2}{2}\right). \tag{8}$$

The L^2-norm of the derivative of q is given by

$$\|\dot{q}\|^2 = \int_0^1 \dot{q}(t)^2\, dt = \int_0^1 \frac{4\|z\|^4 z'(\tau)^2}{z(\tau)^2}\frac{z(\tau)^2}{\|z\|^2}\, d\tau = 4\|z\|^2\|z'\|^2. \tag{9}$$

We can now give the precise definition of the Levi–Civita transformation. Let $z \in C^0(S^1, \mathbb{R})$ be a continuous function with finite zero set

$$Z_z := z^{-1}(0).$$

We associate to z a C^1-map $t_z : S^1 \to S^1$ by

$$t_z(\tau) := \frac{1}{\|z\|^2}\int_0^\tau z(\sigma)^2\, d\sigma. \tag{10}$$

Note that $t_z(0) = 0$ and

$$t_z'(\tau) = \frac{z(\tau)^2}{\|z\|^2}. \tag{11}$$

Since z has only finitely many zeroes, this shows that t_z is strictly increasing and we conclude

Lemma 2.1. *If $z \in C^0(S^1, \mathbb{R})$ has only finitely many zeroes, then the map $t_z : S^1 \to S^1$ defined by (10) is a homeomorphism.* □

It follows that $t_z\colon S^1 \to S^1$ has a continuous inverse

$$\tau_z := t_z^{-1}\colon S^1 \to S^1.$$

Since t_z is of class C^1, the function τ_z is also of class C^1 on the complement of the finite set $t_z(Z_z)$ with derivative

$$\dot{\tau}_z(t) = \frac{\|z\|^2}{z(\tau_z(t))^2}, \quad t \in S^1 \setminus t_z(Z_z). \tag{12}$$

We define a continuous map $q : S^1 \to \mathbb{R}_{\geq 0}$ by

$$q(t) := z(\tau_z(t))^2. \tag{13}$$

Then the two maps z, q are related by the Levi–Civita transformation (2) with $\tau = \tau_z$. Their zero sets

$$Z_z = z^{-1}(0) \quad \text{and} \quad Z_q := q^{-1}(0) = t_z(Z_z)$$

are in bijective correspondence via t_z (or equivalently τ_z). Moreover, by (5) we have

$$\int_0^1 \frac{ds}{q(s)} = \frac{1}{\|z\|^2} < \infty.$$

Conversely, suppose we are given a map $q \in C^0(S^1, \mathbb{R}_{\geq 0})$ with finite zero set Z_q satisfying $\int_0^1 \frac{ds}{q(s)} < \infty$. We associate to q the time reparametrization $\tau_q : S^1 \to S^1$,

$$\tau_q(t) := \left(\int_0^1 \frac{ds}{q(s)} \right)^{-1} \int_0^t \frac{1}{q(s)}\, ds. \tag{14}$$

Then $\tau_q(1) = 1$, τ_q is of class C^1 outside the zero set $Z_q = q^{-1}(0)$ with derivative

$$\tau_q'(t) = \left(\int_0^1 \frac{ds}{q(s)} \right)^{-1} \frac{1}{q(t)}, \quad t \in S^1 \setminus Z_q. \tag{15}$$

By [2, Lemma 2.1], the map $\tau_q : S^1 \to S^1$ is a homeomorphism whose inverse $t_q := \tau_q^{-1}$ is of class C^1 and satisfies $t_q(1) = \tau_q^{-1}(1) = 1$ and

$$t_q'(\tau) = \left(\int_0^1 \frac{ds}{q(s)} \right) q(t_q(\tau)), \quad \tau \in S^1. \tag{16}$$

Suppose that $z : S^1 \to \mathbb{R}$ is a continuous function satisfying

$$z(\tau)^2 = q(t_q(\tau)). \tag{17}$$

Then z has finite zero set $Z_z = \tau_q(Z_q)$, so we can associate to z the homeomorphism $t_z : S^1 \to S^1$ defined by (10) and its inverse τ_z. We claim that

$$\tau_q = \tau_z \quad \text{and} \quad t_q = t_z. \tag{18}$$

It is enough to check the second equality. For this we compute

$$\int_0^\tau z(\sigma)^2\, d\sigma = \int_0^\tau q(t_q(\sigma))\, d\sigma \overset{(*)}{=} \left(\int_0^1 \frac{ds}{q(s)} \right)^{-1} \int_0^{t_q(\tau)} ds = \left(\int_0^1 \frac{ds}{q(s)} \right)^{-1} t_q(\tau),$$

where $(*)$ follows from the coordinate change $\sigma = \tau_q(s)$ and (15). Evaluating at $\tau = 1$ gives us

$$\frac{1}{\|z\|^2} = \int_0^1 \frac{ds}{q(s)}. \tag{19}$$

Therefore,

$$t_z(\tau) = \frac{1}{\|z\|^2} \int_0^\tau z(\sigma)^2 d\sigma = t_q(\tau)$$

and (18) is established. Hence q is the Levi–Civita transform of z defined by (13).

Equation (17) does not uniquely determine z for given q because the sign of z can be arbitrarily chosen on each connected component of $S^1 \setminus Z_z$. If Z_z consists of an *even* number of points, then we can determine z up to a global sign by the requirement that z changes its sign at each zero. If Z_z consists of an *odd* number of points, then the requirement that z changes its sign at each zero leads to $z(\tau+1) = -z(\tau)$, so z has period 2 rather than 1. Therefore, the preceding discussion shows

Lemma 2.2. *The Levi–Civita transformation $z \mapsto q$ given by (13) defines for each* ***even*** *integer m a surjective 2-to-1 map*

$$\mathcal{L}\colon \{z \in C^0(S^1, \mathbb{R}) \mid z \text{ has precisely } m \text{ zeroes and switches sign at each zero}\} \to \left\{ q \in C^0(S^1, \mathbb{R}_{\geq 0}) \;\middle|\; z \text{ has precisely } m \text{ zeroes and } \int_0^1 \frac{ds}{q(s)} < \infty \right\},$$

and for each ***odd*** *integer m a surjective 2-to-1 map*

$$\mathcal{L}\colon \left\{z \in C^0(\mathbb{R}/2\mathbb{Z}, \mathbb{R}) \mid z \text{ has precisely } 2m \text{ zeroes and switches sign at each zero}, \; z(\tau+1) = -z(\tau) \text{ for all } \tau\right\} \to \{q \in C^0(S^1, \mathbb{R}_{\geq 0}) \mid z \text{ has precisely } m \text{ zeroes and } \int_0^1 \frac{ds}{q(s)} < \infty\}.$$

3. The functionals $\mathcal{F}_r$ and their critical points

We denote by $S^1 = \mathbb{R}/\mathbb{Z}$ the circle and abbreviate by

$$H^1_*(S^1, \mathbb{R}) = H^1(S^1, \mathbb{R}) \setminus \{0\}, \quad H^2_*(S^1, \mathbb{R}) = H^2(S^1, \mathbb{R}) \setminus \{0\}$$

the open subsets of the Hilbert space $H^1(S^1, \mathbb{R})$ respectively $H^2(S^1, \mathbb{R})$ where the origin is removed. For $r \in [0, \infty)$ we consider the functional

$$\mathcal{F}_r\colon H^1_*(S^1, \mathbb{R}) \to \mathbb{R}, \quad z \mapsto 2\|z\|^2\|z'\|^2 + \frac{2}{\|z\|^2} + r\frac{\|z\|^2}{\|z^2\|^2},$$

where as before $\|z\|$ is the L^2-norm of the loop z. If $z \in H^1_*(S^1, \mathbb{R})$ and $\xi \in H^1(S^1, \mathbb{R})$, then the differential of $\mathcal{F}_r$ at z in direction of ξ is given by

$$D\mathcal{F}_r(z)\xi = 4\|z\|^2\langle z', \xi'\rangle + 4\|z'\|^2\langle z, \xi\rangle - \frac{4\langle z, \xi\rangle}{\|z\|^4} + r\left(\frac{2\langle z, \xi\rangle}{\|z^2\|^2} - \frac{4\|z\|^2\langle z^3, \xi\rangle}{\|z^2\|^4}\right),$$

where $\langle \cdot, \cdot \rangle$ denotes the L^2-inner product. Therefore, integration by parts shows that for $z \in H^2_*(S^1, \mathbb{R})$ the functional $\mathcal{F}_r$ possess an L^2-gradient given by the formula

$$\begin{aligned} \nabla \mathcal{F}_r(z) &= -4\|z\|^2(z'' + bz + 2az^3), \\ b &= \frac{1}{\|z\|^6} - \frac{\|z'\|^2}{\|z\|^2} - \frac{r}{2\|z\|^2 \cdot \|z^2\|^2}, \\ a &= \frac{r}{2\|z^2\|^4}. \end{aligned} \tag{20}$$

We see that $\nabla \mathcal{F}_r : H^2_*(S^1, \mathbb{R}) \to L^2(S^1, \mathbb{R})$ is differentiable; we call its derivative the *Hessian* of $\mathcal{F}_r$. Critical points of $\mathcal{F}_r$ are solutions of the problem

$$z'' = -bz - 2az^3. \tag{21}$$

A standard bootstrapping argument implies that critical points of $\mathcal{F}_r$ are in fact smooth. If z is a solution of (21), then for $n \in \mathbb{N}$ the loop z_n defined as

$$z_n(\tau) = n^{-1/3} z(n\tau), \quad \tau \in S^1$$

is another solution of (21) with $a_n = n^{8/3}a$ and $b_n = n^2 b$. We say that a critical point z is *multiply covered* if there exists a critical point w and $n > 1$ such that $z = w_n$. Otherwise we call the critical point *simple*. Moreover, the functional $\mathcal{F}_r$ is invariant under the S^1-action given by time shift,

$$\sigma_* z(\tau) = z(\tau + \sigma), \quad \tau \in S^1,$$

where $\sigma \in S^1$ and $z \in H^1_*(S^1, \mathbb{R})$. In particular, its critical points are invariant under time shift as well. Therefore, if z is a critical point of $\mathcal{F}_r$, then z' lies in the kernel of the Hessian of $\mathcal{F}_r$ at z.

It follows from Proposition 3.3 below that for each critical point z of $\mathcal{F}_r$, $r \geq 0$, there exists $\tau_0 \in S^1$ such that $z(\tau_0) = 0$. Indeed, otherwise there exists $t_0 \in S^1$, such that the Levi–Civita transform q of z attains a minimum $q(t_0) > 0$ at t_0. Therefore, $\ddot{q}(t_0) \geq 0$. On the other hand, according to Proposition 3.3, the function q satisfies (31). For $t = t_0$ this implies that $\ddot{q}(t_0) < 0$. We then necessarily have $z'(\tau_0) \neq 0$, since otherwise by (21) the loop z would be the constant loop at the origin which does not lie in $H^1_*(S^1, \mathbb{R})$. Thus a critical point of $\mathcal{F}_r$ is never a fixed point of the S^1-action on the free loop space. In particular, its nullity, i.e., the dimension of the kernel of its Hessian, is at least one. We say that a critical point is *nondegenerate* if its nullity is precisely one. Our first result asserts that nondegeneracy always holds true for nonnegative r. It corresponds to Theorem B from the introduction and will be proved in Section 3C.

Theorem 3.1. *For every $r \in [0, \infty)$, each critical point of $\mathcal{F}_r$ is nondegenerate.*

3A. ***Levi–Civita transform of critical points.*** In this section we apply the Levi–Civita transformation to critical points of $\mathcal{F}_r$. Let $z \in H^1_*(S^1, \mathbb{R})$ be a solution of (21) and $q(t) := z(\tau_z(t))^2$ its Levi–Civita transform. We compute at points $t \in S^1 \setminus t_z(Z_z)$:

$$\begin{aligned}\ddot{q} &\overset{(A)}{=} \frac{1}{q}\left(-2\|z^4\|(b + 2az^2) - \frac{\dot{q}^2}{2}\right)\\ &\overset{(B)}{=} \frac{1}{q}\left(-\frac{2}{\|z\|^2} + 2\|z\|^2\|z'\|^2 + r\frac{\|z\|^2}{\|z^2\|^2} - 2r\frac{\|z\|^4}{\|z^2\|^4}q - \frac{\dot{q}^2}{2}\right)\\ &\overset{(C)}{=} \frac{1}{q}\left(-\int_0^1 \frac{2}{q(s)}ds + \frac{\|\dot{q}\|^2}{2} + \frac{r}{\bar{q}} - 2r\frac{q}{\bar{q}^2} - \frac{\dot{q}^2}{2}\right).\end{aligned}$$

Equality (A) follows from substituting z'' by (21) in (8); equality (B) uses the expressions for a and b in (21); equality (C) uses $q = z^2$ as well as (3), (4) and (9). Thus q satisfies the ODE

$$\ddot{q} = \left(c - \frac{\dot{q}^2}{2}\right)\frac{1}{q} - \frac{2r}{\bar{q}^2} \tag{22}$$

with the constant

$$c = \frac{\|\dot{q}\|^2}{2} - \int_0^1 \frac{2}{q(s)}\,ds + \frac{r}{\bar{q}}. \tag{23}$$

At the global maximum $t_{\max}$ of q, (22) becomes

$$\frac{c}{q(t_{\max})} + \frac{2r}{\bar{q}^2} = \ddot{q}_1(t_{\max}) \le 0,$$

hence

$$c \le -\frac{2rq(t_{\max})}{\bar{q}^2}. \tag{24}$$

Let now $t_- < t_+$ be adjacent zeroes of q and consider the smooth map

$$\beta := \frac{\ddot{q} + r/\bar{q}^2}{q} \colon (t_-, t_+) \to \mathbb{R}.$$

From (22) we obtain

$$\beta q^2 = c - \frac{\dot{q}^2}{2} - \frac{rq}{\bar{q}^2}.$$

With inequality (24) this implies

$$\beta q^2 \le -\frac{2rq(t_{\max})}{\bar{q}^2} - \frac{\dot{q}^2}{2} - \frac{rq}{\bar{q}^2} < 0,$$

hence $\beta < 0$ on (t_-, t_+). Differentiating both sides of the equation for βq^2 we get

$$\dot\beta q^2 + 2\beta q\dot q = -\ddot q\dot q - \frac{r\dot q}{\bar q^2} = -\beta q\dot q,$$

and therefore

$$\dot\beta q = -3\beta\dot q. \tag{25}$$

We need the following

Lemma 3.2. *Equation (25) for functions $q > 0$ and $\beta < 0$ on (t_-, t_+) implies that*

$$\beta = -\frac{\mu}{q^3} \tag{26}$$

on (t_-, t_+) for some constant $\mu > 0$.

Proof. Dividing both sides of (25) by $q\beta$ yields

$$\frac{d}{dt}\log(-\beta) = -3\frac{d}{dt}\log(q),$$

which by integration implies the lemma. □

By Lemma 3.2 we get (26) on (t_-, t_+) for some constant $\mu > 0$. By definition of β, this yields the following equation for q:

$$\ddot q(t) = -\frac{\mu}{q(t)^2} - \frac{r}{\bar q^2} \tag{27}$$

for $t \in (t_-, t_+)$. It remains to compute μ. Plugging this into (22) we infer

$$\mu = -\left(c - \frac{\dot q(t)^2}{2}\right)q(t) + \frac{rq(t)^2}{\bar q^2} \tag{28}$$

for $t \in (t_-, t_+)$. In particular, using (6) and $q(t_\pm) = 0$ we obtain

$$\mu = \lim_{t\to t_\pm}\frac{\dot q(t)^2 q(t)}{2} = 2\|z\|^4 z'(\tau_z(t_\pm))^2. \tag{29}$$

We deduce from this that (27) holds on $S^1 \setminus t_z(Z_z)$ with a fixed μ independent of the connected component in $S^1 \setminus t_z(Z_z)$. Dividing (28) by $q(t)$ and inserting c from (23), we get

$$\frac{\mu}{q(t)} = -\frac{\|\dot q\|^2}{2} + \int_0^1 \frac{2}{q(s)}ds - \frac{r}{\bar q} + \frac{\dot q(t)^2}{2} + \frac{rq(t)}{\bar q^2}. \tag{30}$$

Integrating this equation yields

$$\mu\int_0^1 \frac{1}{q(t)}\,dt = 2\int_0^1 \frac{1}{q(s)}\,ds,$$

and therefore

$$\mu = 2.$$

This gives us the following statement.

Proposition 3.3. *Assume that $z \in H^1_*(S^1, \mathbb{R})$ is a critical point of the frozen functional $\mathcal{F}_r$. Then the Levi–Civita transform $q(t) = z(\tau_z(t))^2$ of z satisfies the differential equation*

$$\ddot{q}(t) = -\frac{2}{q(t)^2} - \frac{r}{\bar{q}^2}. \tag{31}$$

Equation (31) explains the physical meaning of critical points of $\mathcal{F}_r$: The orbit $q(t)$ describes an electron on the line attracted by a doubly positively charged nucleus at the origin and subject to an additional force depending on its mean $\bar{q}$. For $r = 0$ the mean interaction force vanishes and $q(t)$ describes the *free fall* of the electron into the nucleus, regularized by elastic reflection as it hits the nucleus. For $r > 0$ the mean interaction force pushes the electron towards the origin, which can be interpreted as the effect of a second electron further away and on the same side of the nucleus. Indeed, we will show in Section 4 that for a suitable value of r the system describes frozen planet orbits in helium with mean interaction between the electrons. For $r < 0$ the mean interaction force pushes the electron away from the origin, which can be interpreted as the effect of a second electron on the other side of the nucleus. However, we will not consider the case $r < 0$ in this paper.

3B. ***Analysis of critical points.*** Let $r \geq 0$ and $z \in H^1_*(S^1, \mathbb{R})$ be a critical point of $\mathcal{F}_r$, i.e., a solution of (21). We denote by $\|z\|_0$ the maximum norm of z and introduce the quantities

$$v := \frac{\|z\|^2}{\|z\|_0^2}, \quad w := \frac{\|z\|^2 \|z\|_0^2}{\|z^2\|^2}. \tag{32}$$

For $n \in \mathbb{N}_0$ we consider the elliptic integral

$$I_n \colon (-\infty, 1) \to \mathbb{R}. \quad I_n(m) := \int_0^1 \frac{\zeta^{2n}}{\sqrt{(1-\zeta^2)(1-m\zeta^2)}}\, d\zeta.$$

The main result of this section is the following proposition.

Proposition 3.4. *The quantities v and w satisfy the equations*

$$v = \frac{2}{4 + 3rw - 2rw^2} = \frac{I_1}{I_0}\left(-\frac{rw^2}{2}\right). \tag{33}$$

Proof. After a time reparametrization, we can assume without loss of generality that z attains at time 0 its global maximum

$$z_0 := \|z\|_0 = z(0).$$

Recall that there exists $\tau_0 \in S^1$ such that $z(\tau_0) = 0$. We further choose τ_0 as the smallest number in $(0, 1)$ with the property that $z(\tau_0) = 0$.

We first eliminate in the formula for b in (20) the variable $\|z'\|$. Using (21), we obtain via integration by parts

$$\|z'\|^2 = -\langle z, z''\rangle = b\|z\|^2 + 2a\|z^2\|^2.$$

Plugging this into the equation for b and using the equation for a, we get

$$b = \frac{1}{\|z\|^6} - b - \frac{r}{\|z\|^2 \cdot \|z^2\|^2} - \frac{r}{2\|z\|^2 \cdot \|z^2\|^2},$$

implying

$$b = \frac{1}{2\|z\|^6} - \frac{3r}{4\|z\|^2 \cdot \|z^2\|^2}. \tag{34}$$

From (21) we further infer that we have the conserved quantity

$$\frac{z'(\tau)^2}{2} + \frac{bz(\tau)^2}{2} + \frac{az(\tau)^4}{2} = \frac{c}{2} \tag{35}$$

for some constant c. Since $z(\tau_0) = 0$ and $z'(\tau_0) \neq 0$ we conclude that c is positive. Since z attains its global maximum z_0 at time $\tau = 0$, we necessarily have $z'(0) = 0$ and therefore

$$bz_0^2 + az_0^4 = c.$$

This means that z_0^2 is a root of the quadratic polynomial

$$p(x) = ax^2 + bx - c.$$

Moreover, positivity of c yields

$$az_0^2 + b > 0. \tag{36}$$

Case 1 ($r > 0$). In this case $a \neq 0$ as well and the second root of p is given by $-z_0^2 - b/a$. In particular, the quadratic polynomial factorizes as

$$p(x) = (x - z_0^2)(ax + az_0^2 + b).$$

Plugging this into (35) we obtain

$$z'(\tau)^2 = -p(z(\tau)) = (z_0^2 - z(\tau)^2)(az(\tau)^2 + az_0^2 + b), \quad \tau \in S^1. \tag{37}$$

Equation (21) is invariant under reflection at the origin and time reversal. Since $z(\tau_0) = 0$, we conclude that

$$z(\tau) = -z(2\tau_0 - \tau), \quad \tau \in [\tau_0, 2\tau_0].$$

In particular, we have

$$z(2\tau_0) = -z(0) \quad \text{and} \quad z'(2\tau_0) = 0.$$

Using once more invariance under time reversal, we conclude that

$$z(\tau) = -z(2\tau_0 + \tau), \quad \tau \in [0, 4\tau_0].$$

In particular, we have

$$z(4\tau_0) = z(0) \quad \text{and} \quad z'(4\tau_0) = z'(0) = 0.$$

We see that z is periodic of period $4\tau_0$. Since τ_0 was chosen as the first positive time at which z passes through the origin, we conclude that $4\tau_0$ is the minimal period of z. Since z is by assumption periodic of period 1, we conclude that there exists $n \in \mathbb{N}$ such that $4\tau_0 n = 1$, i.e.,

$$\tau_0 = \frac{1}{4n}.$$

If z is simple, then $n = 1$; otherwise it is multiply covered. Using (37) we therefore obtain

$$\begin{aligned}\frac{1}{4n} &= \int_0^{z_0} \frac{1}{\sqrt{(z_0^2 - z^2)(az^2 + az_0^2 + b)}}\, dz \\ &= \frac{1}{\sqrt{az_0^2 + b}} \int_0^1 \frac{1}{\sqrt{(1-\zeta^2)(1 + (az_0^2/(az_0^2+b))\zeta^2)}}\, d\zeta \\ &= \frac{1}{\sqrt{az_0^2 + b}} I_0\left(-\frac{az_0^2}{az_0^2 + b}\right). \end{aligned} \tag{38}$$

Similarly, we compute

$$\begin{aligned}\frac{\|z\|^2}{4n} &= \int_0^{z_0} \frac{z^2}{\sqrt{(z_0^2 - z^2)(az^2 + az_0^2 + b)}}\, dz \\ &= \frac{z_0^2}{\sqrt{az_0^2 + b}} \int_0^1 \frac{\zeta^2}{\sqrt{(1-\zeta^2)(1 + (az_0^2/(az_0^2+b))\zeta^2)}}\, d\zeta \\ &= \frac{z_0^2}{\sqrt{az_0^2 + b}} I_1\left(-\frac{az_0^2}{az_0^2 + b}\right) \end{aligned} \tag{39}$$

and

$$\begin{aligned}\frac{\|z^2\|^2}{4n} &= \int_0^{z_0} \frac{z^4}{\sqrt{(z_0^2-z^2)(az^2+az_0^2+b)}}\,dz \\ &= \frac{z_0^4}{\sqrt{az_0^2+b}} \int_0^1 \frac{\zeta^4}{\sqrt{(1-\zeta^2)(1+(az_0^2/(az_0^2+b))\zeta^2)}}\,d\zeta \\ &= \frac{z_0^4}{\sqrt{az_0^2+b}} I_2\left(-\frac{az_0^2}{az_0^2+b}\right). \end{aligned} \tag{40}$$

The elliptic function I_2 can be expressed using the elliptic functions I_0 and I_1 by

$$I_2(m) = \frac{2(m+1)I_1(m)}{3m} - \frac{I_0(m)}{3m}$$

as explained in (5) in the Appendix. Hence from (38), (39), and (40) we obtain the equality

$$\begin{aligned}\|z^2\|^2 &= \frac{4nz_0^4}{\sqrt{az_0^2+b}}\left(\frac{2(az_0^2/(az_0^2+b)-1)}{3\frac{az_0^2}{az_0^2+b}} I_1\left(-\frac{az_0^2}{az_0^2+b}\right)+\frac{az_0^2+b}{3az_0^2} I_0\left(-\frac{az_0^2}{az_0^2+b}\right)\right) \\ &= \frac{z_0^4}{\sqrt{az_0^2+b}}\left(\frac{az_0^2+b}{3az_0^2}\sqrt{az_0^2+b}-\frac{2b}{3az_0^2}\cdot\frac{\sqrt{az_0^2+b}}{z_0^2}\|z\|^2\right) \\ &= \frac{(az_0^2+b)z_0^2}{3a}-\frac{2b\|z\|^2}{3a} \\ &= \frac{z_0^4}{3}+\frac{b}{3a}(z_0^2-2\|z\|^2) \\ &= \frac{z_0^4}{3}+\left(\frac{\|z^2\|^4}{3r\|z\|^6}-\frac{\|z^2\|^2}{2\|z\|^2}\right)\cdot(z_0^2-2\|z\|^2) \\ &= \frac{z_0^4}{3}+\frac{\|z^2\|^4 z_0^2}{3r\|z\|^6}-\frac{\|z^2\|^2 z_0^2}{2\|z\|^2}-\frac{2\|z^2\|^4}{3r\|z\|^4}+\|z^2\|^2 \end{aligned}$$

where in the second to last equality we have used the equation for a from (20) and (34) for b. Removing $\|z^2\|^2$ on both sides and multiplying the remaining terms by $6r\|z\|^4/\|z^2\|^4$ we obtain the equation

$$0 = \frac{2r\|z\|^4 z_0^4}{\|z^2\|^4} + \frac{2z_0^2}{\|z\|^2} - \frac{3r\|z\|^2 z_0^2}{\|z^2\|^2} - 4.$$

By definition of v and w we can rewrite this as

$$0 = 2rw^2 + \frac{2}{v} - 3rw - 4$$

or equivalently

$$v = \frac{2}{4 + 3rw - 2rw^2}.$$

This proves the first equation in (33). We use this together with the equation for a in (20) and (34) for b to compute

$$\begin{aligned}
\frac{az_0^2}{az_0^2 + b} &= \frac{az_0^8}{az_0^8 + bz_0^6} \\
&= \frac{rz_0^8/(2\|z^2\|^4)}{rz_0^8/(2\|z^2\|^4) + z_0^6/(2\|z\|^6) - 3rz_0^6/(4\|z\|^2\|z^2\|^2)} \\
&= \frac{rw^2/(2v^2)}{rw^2/(2v^2) + 1/(2v^3) - 3rw/(4v^2)} \\
&= \frac{rw^2}{rw^2 + 1/v - 3rw/2} \\
&= \frac{rw^2}{2}.
\end{aligned} \tag{41}$$

Dividing (39) by (38) and combining the result with (41) we get

$$\|z\|^2 = z_0^2 \cdot \frac{I_1}{I_0}\left(-\frac{az_0^2}{az_0^2 + b}\right) = z_0^2 \cdot \frac{I_1}{I_0}\left(-\frac{rw^2}{2}\right).$$

Hence by definition of v this can be rephrased as

$$v = \frac{I_1}{I_0}\left(-\frac{rw^2}{2}\right).$$

This proves the second equation in (33) and completes the proof in the case $r \neq 0$.

Case 2 ($r = 0$). In the case $r = 0$ equation (33) becomes

$$v = \frac{1}{2} = \frac{I_1(0)}{I_0(0)}. \tag{42}$$

By (1) in the Appendix we have

$$I_n(0) = \frac{(2n-1)!!\pi}{2^{n+1}n!},$$

where $(2n-1)!!$ equals $(2n-1)(2n-3)\cdots 1$ for $n \geq 1$ and 1 for $n = 0$. In particular,

$$I_0(0) = \frac{\pi}{2}, \quad I_1(0) = \frac{\pi}{4}$$

and therefore the second equality in (42) follows. It remains to check the first equation in (42) which says that

$$\|z\|^2 = \frac{z(0)^2}{2}. \tag{43}$$

However, for $r = 0$ we have that $a = 0$ and therefore z is a solution of the ODE

$$z'' = -bz,$$

which implies that up to scaling and time-reparametrization z is given by the cosine function. Now (43) follows from

$$\int_0^1 \cos^2(2\pi nt)\, dt = \tfrac{1}{2}, \quad n \in \mathbb{N}.$$

This finishes the proof of (33) in the case $r = 0$ and the proposition follows. □

3C. ***Proof of nondegeneracy.*** In this section we prove Theorem 3.1. Assume first that $r > 0$. Let z be a critical point of $\mathcal{F}_r$ i.e., a solution of the problem

$$\begin{aligned} z'' &= -bz - 2az^3, \\ b &= \frac{1}{2\|z\|^6} - \frac{3r}{4\|z\|^2 \cdot \|z^2\|^2}, \\ a &= \frac{r}{2\|z^2\|^4}. \end{aligned} \tag{44}$$

As in the previous section, after a time shift we may assume that z attains its maximum $\|z\|_0 = z(0)$ at $\tau = 0$, hence $z'(0) = 0$. An element ξ in the kernel of the Hessian of $\mathcal{F}_r$ at z is a solution of the linearized problem

$$\xi'' = -b\xi - 6az^2\xi - db(\xi)z - 2da(\xi)z^3. \tag{45}$$

In order to prove nondegeneracy we need to show that ξ is a constant multiple of z'. Note that

$$da(\xi) = -\frac{4r\langle \xi, z^3\rangle}{\|z^2\|^6}, \quad db(\xi) = -\frac{3\langle \xi, z\rangle}{\|z\|^8} + \frac{3r\langle \xi, z\rangle}{2\|z\|^4 \cdot \|z^2\|^2} + \frac{3r\langle \xi, z^3\rangle}{\|z\|^2 \cdot \|z^2\|^4}.$$

Using (44) and (45) we obtain via integration by parts

$$\begin{aligned} -b\langle \xi, z\rangle - 2a\langle \xi, z^3\rangle &= \langle \xi, z''\rangle \\ &= \langle \xi'', z\rangle \\ &= -b\langle \xi, z\rangle - 6a\langle \xi, z^3\rangle - db(\xi)\|z\|^2 - 2da(\xi)\|z^2\|^2 \end{aligned}$$

and therefore

$$4a\langle \xi, z^3\rangle = -db(\xi)\|z\|^2 - 2da(\xi)\|z^2\|^2.$$

Plugging into this equation the formulas for a, da, and db, this becomes

$$\frac{2r\langle \xi, z^3\rangle}{\|z^2\|^4} = \frac{3\langle \xi, z\rangle}{\|z\|^6} - \frac{3r\langle \xi, z\rangle}{2\|z\|^2\cdot\|z^2\|^2} - \frac{3r\langle \xi, z^3\rangle}{\|z^2\|^4} + \frac{8r\langle \xi, z^3\rangle}{\|z^2\|^4},$$

which simplifies to

$$\frac{r\langle \xi, z\rangle}{2\|z\|^2\cdot\|z^2\|^2} - \frac{\langle \xi, z\rangle}{\|z\|^6} = \frac{r\langle \xi, z^3\rangle}{\|z^2\|^4}. \tag{46}$$

In particular (recall our assumption $r > 0$), we see from (46) that if $\langle \xi, z\rangle$ vanishes the same has to hold for $\langle \xi, z^3\rangle$.

We recall the quantities

$$v = \frac{\|z\|^2}{\|z\|_0^2}, \quad w = \frac{\|z\|^2\cdot\|z\|_0^2}{\|z^2\|^2}$$

where $\|z\|_0 = z(0)$. Their variations with respect to ξ are given by

$$\hat{v} = \frac{2\langle \xi, z\rangle}{\|z\|_0^2} - \frac{2\|z\|^2\xi_0}{\|z\|_0^3}, \quad \hat{w} = \frac{2\langle \xi, z\rangle\|z\|_0^2 + 2\|z\|^2\|z\|_0\xi_0}{\|z^2\|^2} - \frac{4\langle \xi, z^3\rangle\|z\|^2\|z\|_0^2}{\|z^2\|^4} \tag{47}$$

where $\xi_0 := \xi(0)$. From (33) we infer that

$$\hat{v} = \frac{d}{dw}\left(\frac{2}{4+3rw-2rw^2}\right)\hat{w} \tag{48}$$

and

$$0 = \frac{d}{dw}\left((4+3rw-2rw^2)\frac{I_1}{I_0}\left(-\frac{rw^2}{2}\right)\right)\hat{w}. \tag{49}$$

As explained in the Appendix in formula (8), the quotient I_1/I_0 of elliptic functions satisfies as a function of m the Riccati differential equation

$$\left(\frac{I_1}{I_0}\right)' = \frac{1}{2m(1-m)} - \frac{1}{m(1-m)}\frac{I_1}{I_0} + \frac{1}{2(1-m)}\left(\frac{I_1}{I_0}\right)^2.$$

Using this equation and (33) again we compute the derivative in (49) as follows:

$$\begin{aligned}
&\frac{d}{dw}\left((4+3rw-2rw^2)\frac{I_1}{I_0}\left(-\frac{rw^2}{2}\right)\right)\\
&=(3r-4rw)\frac{I_1}{I_0}\left(-\frac{rw^2}{2}\right)-rw(4+3rw-2rw^2)\left(\frac{I_1}{I_0}\right)'\left(-\frac{rw^2}{2}\right)\\
&=\frac{6r-8rw}{4+3rw-2rw^2}+\frac{2(4+3rw-2rw^2)}{w(2+rw^2)}-\frac{8}{w(2+rw^2)}-\frac{4rw}{(2+rw^2)(4+3rw-2rw^2)}\\
&=\frac{w(2+rw^2)(6r-8rw)+2(4+3rw-2rw^2)^2-8(4+3rw-2rw^2)-4rw^2}{w(2+rw^2)(4+3rw-2rw^2)}\\
&=\frac{36rw-36rw^2-18r^2w^3+18r^2w^2}{w(2+rw^2)(4+3rw-2rw^2)}\\
&=\frac{18r(1-w)(2+rw)}{(2+rw^2)(4+3rw-2rw^2)}.
\end{aligned}$$

Since r and w are positive, we see from this formula that this derivative vanishes only if $w=1$. In this case we obtain from (33) that

$$\frac{2}{4+r}=\frac{I_1}{I_0}\left(-\frac{r}{2}\right). \tag{50}$$

If we set $m=-r/2$ this amounts to the equation

$$1=(2-m)\frac{I_1}{I_0}(m).$$

By Lemma A.1 in the Appendix there are no solutions $m<0$ of this equation, and therefore there are no solutions $r>0$ of (50). Hence if $r>0$ we necessarily have $\hat{w}=0$, and therefore in view of (48) as well $\hat{v}=0$. From (47) we infer

$$\xi_0=\frac{\langle\xi,z\rangle\|z\|_0}{\|z\|^2},\quad \langle\xi,z^3\rangle=\frac{\|z^2\|^2}{2}\left(\frac{\langle\xi,z\rangle}{\|z\|^2}+\frac{\xi_0}{\|z\|_0}\right), \tag{51}$$

and therefore

$$\langle\xi,z^3\rangle=\frac{\|z^2\|^2}{\|z\|^2}\langle\xi,z\rangle.$$

Plugging this into (46) we obtain

$$\frac{r\langle\xi,z\rangle}{2\|z\|^2\cdot\|z^2\|^2}-\frac{\langle\xi,z\rangle}{\|z\|^6}=\frac{r\langle\xi,z\rangle}{\|z\|^2\cdot\|z^2\|^2},$$

or equivalently,

$$-\frac{\langle\xi,z\rangle}{\|z\|^6}=\frac{r\langle\xi,z\rangle}{2\|z\|^2\cdot\|z^2\|^2}.$$

Since $r > 0$, the two sides have opposite signs, therefore both sides have to be zero and we obtain

$$\langle \xi, z \rangle = 0.$$

Using (46) this implies

$$\langle \xi, z^3 \rangle = 0,$$

and therefore

$$da(\xi) = 0, \quad db(\xi) = 0.$$

From $\langle \xi, z \rangle = 0$ and the first equation in (51) we further conclude that

$$\xi(0) = \xi_0 = 0.$$

Hence from (45) we see that ξ is a solution of the ODE

$$\xi'' = -b\xi - 6az^2\xi \tag{52}$$

with $\xi(0) = 0$. Applying the same reasoning to z' in place of ξ, we conclude that z' also solves (52) with $z'(0) = 0$. From (44) and (36) we infer $z''(0) = -b\|z\|_0 - 2a\|z\|_0^3 < 0$, so we can define

$$c := \frac{\xi'(0)}{z''(0)} \in \mathbb{R}.$$

Then $\eta := \xi - cz'$ solves (52) with $\eta(0) = \eta'(0) = 0$, hence $\eta \equiv 0$ and $\xi = cz'$.

This proves nondegeneracy for the case $r > 0$. In the case $r = 0$ the functional $\mathcal{F}_0$ is just the functional for the regularized free fall for which nondegeneracy can be checked directly; see [11, Lemma 3.6]. This finishes the proof of Theorem 3.1.

3D. ***Uniqueness of symmetric critical points.*** In this subsection we prove a uniqueness result for critical points of $\mathcal{F}_r$, $r \geq 0$. To formulate the result, we introduce some terminology from [4]. A *symmetric critical point* of $\mathcal{F}_r$ is a smooth map $z : \mathbb{R}/2\mathbb{Z} \to \mathbb{R}$ satisfying the critical point (21) and the symmetry conditions

$$z(1+\tau) = -z(\tau) \quad \text{and} \quad z(\tau) = z(1-\tau) \text{ for all } \tau. \tag{53}$$

By the discussion at the beginning of this section and Lemma 2.2, z has an odd number of zeroes in the interval $[0, 1)$ all of which are nondegenerate. Its Levi–Civita transform $q : S^1 \to \mathbb{R}$ has an odd number of zeroes and satisfies $q(1-t) = q(t)$. Note that (53) implies

$$z(0) = z'\left(\tfrac{1}{2}\right) = 0.$$

In particular, restriction to symmetric critical points removes the translation invariance of the functional $\mathcal{F}_r$.

Symmetric critical points of $\mathcal{F}_r$ may still be nonunique because they may be multiply covered. Therefore, we restrict to symmetric critical points z which are *simple*, i.e., of minimal period 2. This is equivalent to z having zeroes precisely at integer points $\tau \in \mathbb{Z}$, and critical points at $\tau \in \frac{1}{2} + \mathbb{Z}$. The remaining ambiguity $z \mapsto -z$ can be removed by requiring z to be *normalized* by $z(\tau) > 0$ for all $\tau \in (0, 1)$. Using Theorem 3.1 we will prove

Corollary 3.5. *For each $r \in [0, \infty)$, the functional $\mathcal{F}_r$ has a unique normalized simple symmetric critical point. This critical point is nondegenerate of index zero.*

The proof is based on the following lemma.

Lemma 3.6. *Let z be a simple symmetric critical point of the functional $\mathcal{F}_r$. Then its C^0-norm $\|z\|_0$ satisfies*

$$1 \le \|z\|_0 \tag{54}$$

and

$$\|z\|_0 \le \sqrt{2 + (2r)^{1/3}}. \tag{55}$$

Proof. After a suitable time reparametrization, the Levi–Civita transformation q of z satisfies the hypotheses of the loop q_2 in [4, Lemma D.2]. This gives us the estimates

$$1 \le \|q\|_0, \tag{56}$$

$$\|q\|_0 \le \min\left\{2\bar{q}, 2 + \frac{2r}{(2\bar{q})^2}\right\}, \tag{57}$$

where $\bar{q} > 0$ is the average of q. Inequality (56) via Levi–Civita transformation implies that $1 \le \|z\|_0$, proving (54).

To get an estimate on the norm observe that for $\omega \ge 0$ and $x > 0$ we have

$$\min\left\{x, \frac{\omega}{x^2}\right\} \le \omega^{1/3}.$$

We use this for $\omega := 2r$ and $x := 2\bar{q}$ together with (57) to get

$$\|q\|_0 \le 2 + (2r)^{1/3}.$$

In view of $\|q\|_0 = \|z\|_0^2$ this proves (55) completing the proof of the lemma. □

Proof of Corollary 3.5. Following Sections 6.2 and 6.3 of [4], we introduce the Hilbert space of *symmetric loops*

$$H^2_{\mathrm{sym}}(S^1, \mathbb{R}) := \{z \in H^2(\mathbb{R}/2\mathbb{Z}, \mathbb{R}) \mid -z(1+\tau) = z(\tau) = z(1-\tau) \text{ for all } \tau\}$$

and its open subset

$$X := \{z \in H^2_{\mathrm{sym}}(S^1, \mathbb{R}) \mid z'(0) > 0,\ z(\tau) > 0 \text{ for all } \tau \in (0, 1)\}.$$

We consider on $H^2_{\mathrm{sym}}(S^1, \mathbb{R})$ the L^2-inner product $\langle z, w\rangle = \int_0^1 z(\tau)w(\tau)d\tau$ (which is an inner product in view of the condition $z(1+\tau) = -z(\tau)$). By the discussion at the beginning of this section, $\mathcal{F}_r : X \to \mathbb{R}$ has an L^2-gradient $\nabla\mathcal{F}_r$ of class C^1. For $R > 0$ consider the set

$$\mathcal{Z}_R = \{(r, z) \mid \nabla\mathcal{F}_r(z) = 0\} \subset [0, R] \times X$$

with its induced topology. By the preceding discussion, $\mathcal{Z}_R$ consists of pairs (r, z) such that z is a normalized simple symmetric critical point of $\mathcal{F}_r$.

Next we show that $\mathcal{Z}_R$ is compact. Recall that for $(r, z) \in \mathcal{Z}_R$ the loop z satisfies the ODE (21) with constants a, b depending on (r, z) given in (20). Thus the C^0-bound from Lemma 3.6 and (21) give a uniform C^3-bound on z for $(r, z) \in \mathcal{Z}_R$, and thus compactness of $\mathcal{Z}_R$ by the Arzelà–Ascoli theorem, provided we have uniform bounds on the constants a, b. Note that by definition $a \geq 0$. We will use the quantities $v, w > 0$ defined in (32).

Case 1 ($r = 0$). In this case by definition $a = 0$. Since $v = \|z\|^2/\|z\|_0^2$, (33) yields $2\|z\|^2 = \|z\|_0^2$. Now (34) and (54) imply boundedness of b.

Case 2 ($r > 0$). In this case let us denote $\mathcal{A} := rw^2 > 0$. Using this notation we can write

$$\frac{1}{w} = \frac{\sqrt{r}}{\sqrt{\mathcal{A}}} \leq \frac{\sqrt{R}}{\sqrt{\mathcal{A}}}.$$

Equation (33) together with $v > 0$ implies

$$4 + 3\frac{1}{w}\mathcal{A} - 2\mathcal{A} > 0.$$

Together with the above upper bound on $\frac{1}{w}$ this gives us

$$4 + 3\sqrt{R}\sqrt{\mathcal{A}} - 2\mathcal{A} > 0,$$

which implies a uniform upper bound on $\mathcal{A}$. Therefore, (41) implies

$$0 \leq \frac{a\|z\|_0^2}{a\|z\|_0^2 + b} \leq \tilde{C} \tag{58}$$

for some constant $\tilde{C}$ independent of r. Hence the definition of I_0 yields

$$0 < I_0(-\tilde{C}) \leq I_0\left(-\frac{a\|z\|_0^2}{a\|z\|_0^2 + b}\right) \leq I_0(0) < \infty \tag{59}$$

for all $r \in [0, R]$. Equation (38) (with $n = 1$ since z is a simple orbit) then gives us constants $c, C > 0$ with

$$c \leq a\|z\|_0^2 + b \leq C \tag{60}$$

for all $r \in [0, R]$. The second inequality in (60) together with (58) gives us an upper bound on $a\|z\|_0^2$. The latter together with (54) gives an upper bound on a. The bounds are uniform with respect to $r \in [0, R]$. The double inequality (60) together with an upper bound on $a\|z\|_0^2$ gives us an upper bound on $|b|$. This concludes the proof of compactness of $\mathcal{Z}_R$.

On the other hand, according Theorem 3.1 each critical point of $\mathcal{F}_r$ is nondegenerate. This implies that $\mathcal{Z}_R$ is (as a preimage of a regular value) a compact 1-dimensional submanifold transverse to the slices $\{r = \text{const}\}$. To see transversality, assume by contradiction that $\mathcal{Z}_R$ is not transverse to the slice $\{r = \text{const}\}$ at $(r^*, z^*) \in \mathcal{Z}_R$. Then there exists a local parametrization

$$(-\varepsilon, \varepsilon) \ni s \mapsto (r(s), z(s)) \in \mathcal{Z}_R$$

with $(r(0), z(0)) = (r^*, z^*)$ and $\frac{d}{ds}|_{s=0} r(s) = 0$, whereas $v := \frac{d}{ds}|_{s=0} z(s) \neq 0$. We differentiate the critical point equation

$$\nabla \mathcal{F}_{r(s)}(z(s)) = 0$$

at $s = 0$ and use the chain rule to get

$$D\nabla \mathcal{F}_{r^*}(z^*)v = 0,$$

contradicting nondegeneracy of z^*. Now it is easy to see that for $r = 0$ (describing the free fall) there exists a unique normalized simple symmetric critical point z_0. By the proof of Proposition D.4 in [4], the Hessian of $\mathcal{F}_0$ at z_0 is positive definite, so z_0 is nondegenerate of index zero. Therefore, $\mathcal{Z}_R$ intersects each slice $\{r = \text{const}\}$ in a single point z_r, which is nondegenerate of index zero as a critical point of $\mathcal{F}_r$. Since R was arbitrary, this proves the corollary. □

4. Frozen planet orbits for mean interaction

Let us recall the variational approach for frozen planet orbits in helium from [4]. For the mean interaction between the two electrons, one considers the space

$$\mathcal{H}_{av} = \left\{ (z_1, z_2) \in H^2(S^1, \mathbb{R}^2) \;\middle|\; \|z_1\| > 0, \|z_2\| > 0, \frac{\|z_1^2\|^2}{\|z_1\|^2} > \frac{\|z_2^2\|^2}{\|z_2\|^2} \right\}$$

and defines on it the functional $\mathcal{B}_{av} \colon \mathcal{H}_{av} \to \mathbb{R}$ by

$$\mathcal{B}_{av}(z_1, z_2) := 2 \sum_{i=1}^{2} \left(\|z_i\|^2 \cdot \|z_i'\|^2 + \frac{1}{\|z_i\|^2} \right) - \frac{\|z_1\|^2 \cdot \|z_2\|^2}{\|z_1^2\|^2 \cdot \|z_2\|^2 - \|z_2^2\|^2 \cdot \|z_1\|^2}.$$

Here z_1 and z_2 correspond to the Levi–Civita regularizations of the outer and inner electron, respectively. We restrict to Sobolev class H^2 right away because on this

space the L^2-gradient of $\mathcal{B}_{av}$ will be of class C^1. This L^2-gradient is given by

$$\nabla\mathcal{B}_{av}(z_1,z_2)=(-4\|z_1\|^2\mathcal{V}_1(z_1,z_2),-4\|z_2\|^2\mathcal{V}_2(z_1,z_2)), \tag{61}$$

$$\mathcal{V}_1(z_1,z_2)=-z_1''+a_1z_1+b_1z_1^3,\quad \mathcal{V}_2(z_1,z_2)=-z_2''+a_2z_2+b_2z_2^3,$$

$$a_1=\frac{\|z_1'\|^2}{\|z_1\|^2}-\frac{1}{\|z_1\|^6}-\frac{\|z_2\|^4\cdot\|z_1^2\|^2}{2\|z_1\|^2\cdot(\|z_1^2\|^2\cdot\|z_2\|^2-\|z_2^2\|^2\cdot\|z_1\|^2)^2},$$

$$b_1=\frac{\|z_2\|^4}{(\|z_1^2\|^2\cdot\|z_2\|^2-\|z_2^2\|^2\cdot\|z_1\|^2)^2},$$

$$a_2=\frac{\|z_2'\|^2}{\|z_2\|^2}-\frac{1}{\|z_2\|^6}+\frac{\|z_1\|^4\cdot\|z_2^2\|^2}{2\|z_2\|^2\cdot(\|z_1^2\|^2\cdot\|z_2\|^2-\|z_2^2\|^2\cdot\|z_1\|^2)^2},$$

$$b_2=-\frac{\|z_1\|^4}{(\|z_1^2\|^2\cdot\|z_2\|^2-\|z_2^2\|^2\cdot\|z_1\|^2)^2}.$$

It was shown in [4] that $\nabla\mathcal{B}_{av}$ is of class C^1; we refer to its derivative as the *Hessian* of $\mathcal{B}_{av}$. According to Lemma 3.1 in [9] or Lemma D2 of [4], for each critical point (z_1,z_2) of $\mathcal{B}_{av}$ the first component z_1 is constant. Thus the S^1-action by time shift acts trivially on the first component and (by a similar argument as for $\mathcal{F}_r$) nontrivially on the second one, leading to a 1-dimensional subspace of the kernel of the Hessian. By analogy with $\mathcal{F}_r$, we say that a critical point of $\mathcal{B}_{av}$ is *nondegenerate* if the nullity of its Hessian is precisely 1. Now we can formulate the main result of this section, whose proof will occupy the rest of this section.

Theorem 4.1. *All critical points of $\mathcal{B}_{av}$ are nondegenerate.*

4A. ***Relating frozen planet orbits to critical points of $\mathcal{F}_\rho$.*** We introduce the numerical parameters

$$\rho:=(\sqrt{2}-1)^2,\quad \alpha:=\frac{\sqrt{2}-1}{\sqrt{2}}$$

and observe that

$$0<\alpha<1.$$

We introduce the following constant for any $z\in H^2_*(S^1,\mathbb{R})$:

$$c(z):=\alpha^{-1/2}\frac{\|z^2\|}{\|z\|}.$$

Lemma 4.2. *The frozen functional $\mathcal{F}_\rho$ is related to the functional $\mathcal{B}_{av}$ by*

$$\mathcal{F}_\rho(z)=\mathcal{B}_{av}(c(z),z). \tag{62}$$

Proof. This follows from the following computation:

$$\begin{aligned}
&\mathcal{B}_{av}\left(\frac{2^{1/4}\cdot\|z^2\|}{\sqrt{\sqrt{2}-1}\cdot\|z\|}, z\right)\\
&\quad= 2\|z\|^2\|z'\|^2+\frac{2}{\|z\|^2}+\frac{2(\sqrt{2}-1)\cdot\|z\|^2}{\sqrt{2}\cdot\|z^2\|^2}\\
&\qquad\qquad -\frac{\|z\|^2}{\big((\sqrt{2}\|z^2\|^2)/((\sqrt{2}-1)\|z\|^2)\big)\|z\|^2-\|z^2\|^2}\\
&\quad= 2\|z\|^2\|z'\|^2+\frac{2}{\|z\|^2}+\frac{\sqrt{2}(\sqrt{2}-1)\cdot\|z\|^2}{\|z^2\|^2}-\frac{\|z\|^2}{(\sqrt{2}/(\sqrt{2}-1)-1)\|z^2\|^2}\\
&\quad= 2\|z\|^2\|z'\|^2+\frac{2}{\|z\|^2}+\frac{\sqrt{2}(\sqrt{2}-1)\cdot\|z\|^2}{\|z^2\|^2}-\frac{(\sqrt{2}-1)\|z\|^2}{\|z^2\|^2}\\
&\quad= 2\|z\|^2\|z'\|^2+\frac{2}{\|z\|^2}+(\sqrt{2}-1)^2\frac{\|z\|^2}{\|z^2\|^2}\\
&\quad= \mathcal{F}_\rho(z).
\end{aligned}$$

□

Let us illustrate the significance of the constant $c(z)$ from another angle. Let

$$H^2_{\text{const}} := \{(z_1, z_2) \in H^2(S^1, \mathbb{R}^2) \mid z_1 = \text{const}\}$$

be the subspace of functions with constant first component and set

$$\begin{aligned}
\mathcal{H}^{\text{const}}_{av} &:= \mathcal{H}_{av} \cap H^2_{\text{const}}\\
&= \left\{(z_1, z_2) \in H^2(S^1, \mathbb{R}^2) \;\middle|\; z_1 = \text{const}, \|z_2\| > 0, z_1^2 > \frac{\|z_2^2\|^2}{\|z_2\|^2}\right\}. \qquad (63)
\end{aligned}$$

Consider the following codimension 1 Hilbert submanifold of H^2_{const}:

$$\text{graph}(c) := \{(c(z), z) \in H^2_{\text{const}} \mid z \in H^2_*(S^1, \mathbb{R})\}.$$

Then the definition of $c(z)$ and $\alpha < 1$ imply

$$\text{graph}(c) \subset \mathcal{H}^{\text{const}}_{av}.$$

Lemma 4.3. *Assume that* $(z_1, z_2) \in \mathcal{H}^{\text{const}}_{av}$. *Then the equation*

$$\mathcal{V}_1(z_1, z_2) = 0$$

is equivalent to the equation

$$z_1 = \pm c(z_2).$$

Moreover, if this is the case, then

$$a_1 = -\frac{2}{z_1^6}, b_1 = \frac{2}{z_1^8}.$$

Proof. For constant z_1, the functions a_1 and b_1 that enter the formula for $\mathcal{V}_1$ simplify to

$$a_1 = -\frac{1}{z_1^6} - \frac{\|z_2\|^4 z_1^2}{2(\|z_2\|^2 z_1^4 - \|z_2^2\|^2 z_1^2)^2}, \qquad b_1 = \frac{\|z_2\|^4}{(\|z_2\|^2 z_1^4 - \|z_2^2\|^2 z_1^2)^2}. \tag{64}$$

Therefore, we rewrite $\mathcal{V}_1$ as follows:

$$\mathcal{V}_1(z_1, z_2) = a_1 z_1 + b_1 z_1^3 = -\frac{1}{z_1^5} + \frac{\|z_2\|^4 z_1^3}{2(\|z_2\|^2 z_1^4 - \|z_2^2\|^2 z_1^2)^2}.$$

Thus, $\mathcal{V}_1(z_1, z_2) = 0$ is equivalent to

$$2(\|z_2\|^2 z_1^4 - \|z_2^2\|^2 z_1^2)^2 = \|z_2\|^4 z_1^8. \tag{65}$$

The defining inequality in (63) implies that (65) is equivalent to

$$\sqrt{2}(\|z_2\|^2 z_1^4 - \|z_2^2\|^2 z_1^2) = \|z_2\|^2 z_1^4.$$

Since $z_1 \neq 0$ by (63), the latter is equivalent to

$$\|z_2^2\|^2 = \frac{\sqrt{2}-1}{\sqrt{2}} \|z_2\|^2 z_1^2,$$

in other words to

$$z_1 = \pm \frac{2^{1/4} \cdot \|z_2^2\|}{\sqrt{\sqrt{2}-1} \cdot \|z_2\|}.$$

This shows the first statement. The second statement follows by substituting (65) in (64). □

Taking the derivative of (62) with respect to z yields

$$D\mathcal{F}_\rho(z) = D_1\mathcal{B}_{av}(c(z), z)Dc(z) + D_2\mathcal{B}_{av}(c(z), z) = D_2\mathcal{B}_{av}(c(z), z).$$

Here D_1 and D_2 denotes the derivative with respect to the first and second component, respectively, and $D_1\mathcal{B}_{av}(c(z), z) = 0$ by Lemma 4.3. As a consequence, we get the following relation between the L^2-gradients of $\mathcal{F}_\rho(z)$ and $\mathcal{B}_{av}$:

$$\nabla\mathcal{F}_\rho(z) = \mathcal{V}_2(c(z), z). \tag{66}$$

Here we have dropped the factor $-4\|z\|^2$ from $\nabla\mathcal{F}_\rho(z)$ which has no relevance for the subsequent discussion. Taking another derivative with respect to z, we get

$$D\nabla\mathcal{F}_\rho(z)\xi = D_1\mathcal{V}_2(c(z), z)Dc(z)\xi + D_2\mathcal{V}_2(c(z), z)\xi \tag{67}$$

for any $\xi \in H^2(S^1, \mathbb{R})$.

4B. ***Proof of Theorem 4.1 modulo two key lemmas.*** Let (z_1, z_2) be a critical point of $\mathcal{B}_{av}$. Recall from [4; 9] that z_1 is constant. Therefore, Lemma 4.3 implies that $z_1 = \pm c(z_2)$. In the following we assume $z_1 > 0$ (the case $z_1 < 0$ being analogous), so that

$$(z_1, z_2) \in \operatorname{graph}(c).$$

Equation (66) with $z = z_2$ implies that z_2 is a critical point of $\mathcal{F}_\rho$.

Let $\xi = (\xi_1, \xi_2) \in H^2(S^1, \mathbb{R}^2)$ lie in the kernel of the Hessian at (z_1, z_2), that is

$$D_1\mathcal{V}_1(z_1, z_2)\xi_1 + D_2\mathcal{V}_1(z_1, z_2)\xi_2 = 0 \tag{68}$$

and

$$D_1\mathcal{V}_2(z_1, z_2)\xi_1 + D_2\mathcal{V}_2(z_1, z_2)\xi_2 = 0. \tag{69}$$

We need the following two lemmas.

Lemma 4.4. *Assume that* $(z_1, z_2) \in \operatorname{graph}(c)$ *and* $(\xi_1, \xi_2) \in H^2(S^1, \mathbb{R}^2)$ *are such that* (68) *is satisfied. Then* ξ_1 *is constant, that is* $(\xi_1, \xi_2) \in H^2_{\text{const}}$.

Lemma 4.5. *Assume that* $(z_1, z_2) \in \operatorname{graph}(c)$ *and* $(\xi_1, \xi_2) \in H^2_{\text{const}}$ *are such that* (68) *is satisfied. Then*

$$\xi_1 = Dc(z_2)\xi_2,$$

that is $(\xi_1, \xi_2) \in T_{(z_1,z_2)} \operatorname{graph}(c)$.

Assuming these two lemmas, set $z = z_2$, $c(z) = z_1$ and $\xi = \xi_2$ in (67) and solve it for $D_2\mathcal{V}_2(z_1, z_2)\xi_2$ to get

$$D_2\mathcal{V}_2(z_1, z_2)\xi_2 = D_2\nabla\mathcal{F}_\rho(z_2)\xi_2 - D_1\mathcal{V}_2(z_1, z_2)Dc(z_2)\xi_2.$$

The latter allows us to rewrite (69) as

$$D\nabla\mathcal{F}_\rho(z_2)\xi_2 = D_1\mathcal{V}_2(z_1, z_2)(Dc(z_2)\xi_2 - \xi_1). \tag{70}$$

Apply Lemma 4.5 to get

$$D\nabla\mathcal{F}_\rho(z_2)\xi_2 = 0.$$

Since z_2 is a critical point of $\mathcal{F}_\rho$, it is nondegenerate by Theorem 3.1, so ξ_2 belongs to a subspace of dimension 1. This together with Lemma 4.5 shows that (ξ_1, ξ_2) belongs to a subspace of dimension 1, giving us nondegeneracy of (z_1, z_2) as a critical point of $\mathcal{B}_{av}$. This proves Theorem 4.1 modulo the two lemmas above.

4C. ***Proof of Lemma 4.4.*** Observe that (68) is equivalent to

$$\xi_1'' = a_1\xi_1 + 3b_1 z_1^2\xi_1 + (da_1(\xi_1,\xi_2)z_1 + db_1(\xi_1,\xi_2)z_1^3).$$

Note that the term in the round brackets is a function constant in time, which we abbreviate as *const*. So, we can rewrite it as

$$\xi_1'' = (a_1 + 3b_1 z_1^2)\xi_1 + \text{const}\,.$$

Inserting for a_1, b_1 the simplified expressions from Lemma 4.3, we see that

$$a_1 + 3b_1 z_1^2 = -\frac{2}{z_1^6} + \frac{6z_1^2}{z_1^8} = \frac{4}{z_1^6} > 0$$

is a positive constant. Now we argue by contradiction.

Suppose that ξ_1 is not constant and let (τ_1, τ_2) be a maximal interval on which $\xi_2' > 0$. Then $\xi_1'(\tau_1) = \xi_1'(\tau_2) = 0$, $\xi_1''(\tau_1) \geq 0$, and $\xi_1''(\tau_2) \leq 0$. Since the coefficient in front of ξ_1 is constant in time and positive, we get the following chain of inequalities for $\tau_1 \leq \tau \leq \tau_2$:

$$0 \leq \xi_1''(\tau_1) \leq \xi_1''(\tau) \leq \xi_1''(\tau_2) \leq 0.$$

So we must have equalities everywhere, $\xi''(\tau) = 0$, and therefore $\xi'(\tau)$ for all $\tau \in (\tau_1, \tau_2)$, contradicting our hypothesis.

4D. ***Proof of Lemma 4.5.*** **Step 1.** Let $z = (z_1, z_2) \in \text{graph}(c)$ and $\xi = (\xi_1, \xi_2) \in H^2_{\text{const}}$ satisfy (68). It follows from (61) that the restriction

$$\mathcal{W} := \mathcal{V}_1|_{H^2_{\text{const}}}$$

lands in the space of constant functions and can, therefore, be considered as a map

$$H^2_{\text{const}} \to \mathbb{R}.$$

Since both z_1 and ξ_1 are constant in t, we can assume that we vary z_1 within the space of constant functions when computing the derivative $D_1\mathcal{V}_1$ with respect to z_1. Therefore, (68) is equivalent to

$$D\mathcal{W}(z)\xi = D_1\mathcal{W}(z_1, z_2)\xi_1 + D_2\mathcal{W}(z_1, z_2)\xi_2 = 0. \tag{71}$$

In this terminology, Lemma 4.3 is equivalent to

$$\mathcal{H}_{av} \cap \{\mathcal{W} = 0\} = \text{graph}(c).$$

Assume for the moment that

$$D\mathcal{W}(z) \neq 0. \tag{72}$$

Then

$$T_z \,\text{graph}(c) = \ker D\mathcal{W}(z),$$

since the inclusion $T_z \operatorname{graph}(c) \subset \ker DW(z)$ is obvious and both spaces have codimension 1 in H^2_{const}. This shows the lemma modulo (72), which is an immediate consequence of the next step.

Step 2. For any $z = (z_1, z_2) \in \operatorname{graph}(c)$, we have

$$D_1 W(z_1, z_2) \neq 0. \tag{73}$$

This is achieved by a brute force computation. Since z_1 is constant in t, we set $z_1'' = 0$ in formula (61) to get

$$W(z_1, z_2) = a_1 z_1 + b_1 z_1^3, \tag{74}$$

where a_1, b_1 are given by (64) to be

$$a_1 = -\frac{1}{z_1^6} - \frac{z_1^2}{2} b_1, \quad b_1 = \frac{\|z_2\|^4}{P(z_1)^2}$$

with the polynomial

$$P(z_1) := \|z_2\|^2 z_1^4 - \|z_2^2\|^2 z_1^2.$$

We rewrite (74) as

$$W(z_1, z_2) = -\left(\frac{1}{z_1^6} + \frac{z_1^2}{2} b_1\right) z_1 + b_1 z_1^3 = -\frac{1}{z_1^5} + \frac{z_1^3}{2} b_1 \tag{75}$$

and compute the derivative with respect to z_1:

$$D_1 W(z_1, z_2) = \tfrac{5}{z_1^6} + \tfrac{3}{2} z_1^2 b_1 + \tfrac{1}{2} z_1^3 D_1 b_1(z_1).$$

Now we differentiate b_1:

$$D_1 b_1(z_1) = -2\|z_2\|^4 \frac{P'(z_1)}{P(z_2)^3} = -2 \frac{P'(z_1)}{P(z_1)} b_1.$$

Therefore,

$$D_1 W(z_1, z_2) = \frac{5}{z_1^6} + \left(\tfrac{3}{2} z_1^2 - z_1^3 \frac{P'(z_1)}{P(z_1)}\right) b_1. \tag{76}$$

Now we manipulate $P(z_1)$ and $P'(z_1)$:

$$\begin{aligned} P(z_1) &= \|z_2\|^2 z_1^4 - \|z_2^2\|^2 z_1^2 \\ &\overset{(*)}{=} \|z_2\|^2 z_1^4 - \alpha \|z_2\|^2 z_1^4 \\ &= (1-\alpha)\|z_2\|^2 z_1^4. \\ P'(z_1) &= 4\|z_2\|^2 z_1^3 - 2\|z_2^2\|^2 z_1 \\ &= z_1(4\|z_2\|^2 z_1^2 - 2\|z_2^2\|^2) \\ &\overset{(*)}{=} z_1(4\|z_2\|^2 z_1^2 - 2\alpha\|z_2\|^2 z_1^2) \\ &= 2(2-\alpha)\|z_2\|^2 z_1^3. \end{aligned}$$

Here the equalities marked with $(*)$ use $z_1 = c(z_2)$ in the form

$$\|z_2^2\|^2 = \alpha \|z_2\|^2 z_1^2.$$

Now we set

$$X := P(z_1)^3 z_1^6 D_1 \mathcal{W}(z_1, z_2)$$

and continue from (76):

$$\begin{aligned} X &= 5P(z_1)^3 + \tfrac{3}{2}\|z_2\|^4 z_1^8 P(z_1) - \|z_2\|^4 z_1^9 P'(z_1) \\ &= \big(5(1-\alpha)^3 + \tfrac{3}{2}(1-\alpha) - 2(2-\alpha)\big)\|z_2\|^6 z_1^{12}. \end{aligned}$$

We compute the numerical coefficient

$$K := 5(1-\alpha)^3 + \tfrac{3}{2}(1-\alpha) - 2(2-\alpha)$$

in front of $\|z_2\|^6 z_1^{12}\xi_1$. With

$$k := 1 - \alpha = \tfrac{1}{\sqrt{2}}$$

we compute

$$K + 2\alpha = 5(1-\alpha)^3 + \tfrac{3}{2}(1-\alpha) - 2(2-2\alpha) = 5k^3 + \tfrac{3}{2}k - 4k = 5k\big(k^2 - \tfrac{1}{2}\big) = 0.$$

This shows $K = -2\alpha \neq 0$, which implies $X \neq 0$ and therefore proves (73).

4E. ***Uniqueness of symmetric frozen planet orbits.*** The discussion in the preceding subsections shows that critical points of $\mathcal{B}_0 = \mathcal{B}_{av}$ (i.e., frozen planet orbits for the mean interaction) are in one-to-one correspondence with critical points of $\mathcal{F}_\rho$ for $\rho = (\sqrt{2}-1)^2$. Moreover, this correspondence preserves the indices and nullities of the critical points. By Corollary 3.5, the functional $\mathcal{F}_\rho$ has a unique normalized simple symmetric critical point, which is nondegenerate of index zero. Hence the same holds for the corresponding normalized simple symmetric critical point of

$\mathcal{B}_{av}$ and we conclude the following result, which corresponds to Theorem A in the introduction.

Corollary 4.6. *The unique normalized simple symmetric frozen planet orbit for the mean intersection functional $\mathcal{B}_{av}$ is nondegenerate of Morse index* 0. □

5. Determinant lines of self-adjoint Fredholm operators

The goal of this section is to show that under certain conditions the restriction of the determinant bundle to the space of symmetric index zero Fredholm operators bounded from below (or above) is trivial. We begin by describing the general Fredholm setting. Then we will describe the more specific Hilbert space setting and state the main result.

5A. ***The determinant line bundle.*** For two Banach spaces X and Y we denote the space of continuous linear maps from X to Y by $\mathcal{L}(X, Y)$, the subspace of Fredholm operators from X to Y by $\mathcal{F}(X, Y)$, and the set of surjective Fredholm operators by $\mathcal{F}^*(X, Y)$. Recall from [12] that the determinants

$$\det(D) = \Lambda^{\max}(\ker D^*) \otimes \Lambda^{\max}(\ker D)$$

for any $D \in \mathcal{F}(X, Y)$ give rise to a real line bundle, the *determinant line bundle*

$$\det \to \mathcal{F}(X, Y).$$

We describe the bundle structure for det following [12]. For this recall the bundle of kernels over $\mathcal{F}^*(X, Y)$ and note that the restriction $\det|_{\mathcal{F}^*(X,Y)}$ is just the top exterior power of the kernel bundle over $\mathcal{F}^*(X, Y)$. Let now $T \in \mathcal{F}(X, Y)$ be a Fredholm operator and set $N := \dim \operatorname{coker} T$. Let

$$\Phi : \mathbb{R}^N \to Y$$

be an isomorphism onto a direct complement to $\operatorname{im} T$. Then the stabilized operator

$$D \oplus \Phi : X \oplus \mathbb{R}^N \to Y, \quad (x, \zeta) \mapsto D(x) + \Phi(\zeta)$$

is surjective for $D := T$ by construction. Therefore, it is surjective for any D in a small open neighborhood $U_{T,\Phi}$ of T. The idea is to construct a certain fiberwise linear bijection ι_Φ between the restriction of the determinant bundle to $U_{T,\Phi}$ and its restriction to the image of $U_{T,\Phi}$ under stabilization. We then declare that ι_Φ be a bundle isomorphism. To define ι_Φ pick any $D \in U_{T,\Phi}$ and set $k := \dim \ker D$, $l := \dim \operatorname{coker} D$. Then

$$\dim \ker(D \oplus \Phi) = \operatorname{ind}(D \oplus \Phi) = (k - l) + N$$

and

$$\ker D \times \{0\} \subset \ker(D \oplus \Phi)$$

is a subspace of dimension k. In particular,

$$k = \dim\ker D \le \dim\ker(D \oplus \Phi) = N + (k - l). \tag{77}$$

A complement to $\ker D \times \{0\}$ in $\ker(D \oplus \Phi)$ can be described as follows. Pick $N - l$ linearly independent vectors $\{\zeta_j\}_{j=l+1}^N$ in $\mathbb{R}^N$ and $N - l$ vectors $\{\xi_j\}_{j=l+1}^N$ in X subject to

$$D\xi_j + \Phi\zeta_j = 0, \quad j = l+1, \dots, N.$$

Then the collection $\{(\xi_j, \zeta_j)\}_{j=l+1}^N$ spans the desired complement. We complete $\{\zeta_j\}_{j=l+1}^N$ to a basis $\{\zeta_j\}_{j=1}^N$ of $\mathbb{R}^N$. Now for any

$$\theta = (y_l^* \wedge \cdots \wedge y_1^*) \otimes (x_1 \wedge \cdots \wedge x_k) \in \det(D)$$

define

$$\iota_\Phi(D, \theta) := (-1)^{kl} \frac{\det\langle y_j^*, \Phi\zeta_i\rangle_{i,j=1,\dots,l}}{\det(\zeta_1, \dots, \zeta_N)} (x_1, 0) \wedge \cdots \wedge (x_k, 0) \wedge (\xi_{l+1}, \zeta_{l+1}) \wedge \cdots \wedge (\xi_N, \zeta_N).$$

It is shown in [12] that the map ι_Φ is independent of the choices made and the collection of all maps ι_Φ does indeed define a bundle structure on the determinant bundle over $\mathcal{F}(X, Y)$.

5B. ***Self-adjoint Fredholm operators.*** Let now $(F, \langle\cdot,\cdot\rangle_F)$ be a real Hilbert space and $E \subset F$ a dense linear subspace. Recall that a linear map

$$T : E \to F$$

is called *symmetric* if

$$\langle Tx, y\rangle_F = \langle x, Ty\rangle_F \quad \text{for all } x, y \in E,$$

and *self-adjoint* if in addition for each $y \in F$ the existence of a constant C_y such that

$$\langle Tx, y\rangle_F \le C_y \|x\|_F \tag{78}$$

for all $x \in E$ implies that $y \in E$.

Assume now that $(E, \langle\cdot,\cdot\rangle_E)$ is itself a Hilbert space, so we can talk about the space of Fredholm operators $\mathcal{F}(E, F)$.

Lemma 5.1. *In the setting above, a Fredholm operator $T \in \mathcal{F}(E, F)$ is self-adjoint if and only if it is symmetric and of index zero. Moreover, in this case we have the equalities*

$$\ker T = (\operatorname{im} T)^{\perp_F}, \quad (\ker T)^{\perp_F} = \operatorname{im} T. \tag{79}$$

Proof. Suppose first that T is self-adjoint. Then it is in particular symmetric and we obtain the inclusion $\ker T \subset (\operatorname{im} T)^{\perp_F}$. On the other hand, consider $y \in (\operatorname{im} T)^{\perp_F}$. Then $\langle Tx, y\rangle = 0$ for all $x \in E$, so y satisfies (78) with $C_y = 0$. Since T is self-adjoint, this implies $y \in E$, and symmetry of T gives $\langle x, Ty\rangle = 0$ for all $x \in E$. By density of $E \subset F$ this implies $Ty = 0$, so we have shown $\ker T = (\operatorname{im} T)^{\perp_F}$. This proves $\operatorname{ind} T = 0$ and the first equality, and the second equality follows from the first one by taking orthogonal complements.

Suppose now that T is symmetric and of index zero. Then the inclusion $\ker T \subset (\operatorname{im} T)^{\perp_F}$ has to be an equality because both spaces have the same finite dimension, so the equalities (79) hold. To prove self-adjointness, assume first that T is surjective. Let $y \in F$ satisfy (78). By the Riesz representation theorem in F for the functional $x \mapsto \langle Tx, y\rangle_F$ and surjectivity of T, there exists $z \in E$ with

$$\langle Tx, y\rangle_F = \langle x, Tz\rangle_F$$

for all $x \in E$. We apply symmetry of T to the right-hand side of the last displayed equation to get

$$\langle Tx, y - z\rangle_F = 0$$

for all $x \in E$. Since $\operatorname{im} T = F$, this implies $y - z = 0$ and thus $y = z \in E$. This proves self-adjointness in the case that T is surjective. If T is not surjective, then we replace the triple (F, E, T) by the triple

$$(F_1 := \operatorname{im} T,\ E_1 := \operatorname{im} T \cap E,\ T_1 := T|_{\operatorname{im} T \cap E}).$$

Since the codimension of $\operatorname{im} T$ is finite, the linear subspace E_1 is dense in F_1 (because $\operatorname{im} T$ possesses a complement which is contained in E). Equations (79) imply that T_1 is surjective, so by the discussion above T_1 is self-adjoint. Therefore, the original operator T is self-adjoint. □

We denote the space of operators as in Lemma 5.1 by

$$\mathcal{F}_s(E, F) := \{T \in \mathcal{F}(E, F) \mid \operatorname{ind} T = 0 \text{ and } T \text{ is symmetric}\}.$$

Let $T \in \mathcal{F}_s(E, f)$. By Lemma 5.1 and its proof, $\operatorname{im} T \cap E$ is a complement to $\ker T$ in E. So the restriction of T to $\operatorname{im} T \cap E$ defines an isomorphism

$$\widetilde{T} := T|_{\operatorname{im} T \cap E} : (\operatorname{im} T \cap E, \langle \cdot, \cdot \rangle_E) \xrightarrow{\cong} (\operatorname{im} T, \langle , \rangle_F).$$

Assume now in addition that the inclusion $E \hookrightarrow F$ is compact. Then the inverse of $\widetilde{T}$ can be viewed as a compact operator

$$\widetilde{T}^{-1} : (\operatorname{im} T, \langle \cdot, \cdot \rangle_F) \to (\operatorname{im} T \cap E, \langle \cdot, \cdot \rangle_E) \hookrightarrow (\operatorname{im} T, \langle \cdot, \cdot \rangle_F).$$

As such, the spectrum of $\widetilde{T}^{-1}$ consists only of eigenvalues which are uniformly bounded and can accumulate only at zero, and all nonzero eigenvalues have finite

multiplicity; see e.g., [5]. Since the eigenvalue zero of T has only finite multiplicity by the Fredholm hypothesis, we obtain:

Lemma 5.2. *Consider the setting above and assume that the inclusion $E \hookrightarrow F$ is compact. Then for $T \in \mathcal{F}_s(E, F)$ the spectrum $\sigma(T) \subset \mathbb{R}$ is discrete and consists only of eigenvalues of finite multiplicity.* □

For T as in the preceding lemma and a measurable subset $Z \subset \mathbb{R}$ let

$$E_Z^T : F \to F$$

denote the projection onto the sum of eigenspaces to eigenvalues in Z (this is a very special case of a *spectral measure* [5]), so we get the spectral decomposition

$$T = \bigoplus_{\lambda \in \sigma(A)} \lambda E_\lambda^T.$$

We conclude this subsection with an important lemma about the continuity of the spectrum and of the spectral measure.

Lemma 5.3. *Consider the setting as in Lemma 5.2 and $T \in \mathcal{F}_s(E, F)$:*

(a) Let $I \subset \mathbb{R}$ be a compact interval such that T has no eigenvalues in I. Then there exists an open neighborhood W_T of T in $\mathcal{F}_s(E, F)$ such that the spectrum of any $T' \in W_T$ is disjoint from I.

(b) Let $J \subset \mathbb{R}$ be a finite interval whose boundary points are no eigenvalues of T. Then there exists an open neighborhood W_T of T in $\mathcal{F}_s(E, F)$ such that the assignment

$$W_T \to \mathcal{L}(F, F), \quad T' \mapsto E_J^{T'} \tag{80}$$

is continuous.

Proof. (a) The family $\{T_t := T - t\,\mathrm{Id}\}_{t \in I} \subset \mathcal{F}_s(E, F)$ consists of invertible operators, and every T_t has an open neighborhood U_t which consists of invertible operators. Since the family $\{T_t\}_{t \in I}$ is compact, it is covered by finitely many U_{t_j}, $j = 1, \ldots, l$ and we set

$$W_T := \bigcap_{j=1}^{l} (U_{t_j} + t_j\,\mathrm{Id}).$$

(b) By part (a), we find an open neighborhood W_T such that no $T' \in W_T$ has eigenvalues at the boundary points of J. The desired continuity can now be deduced e.g., from Dunford calculus [6] as follows. Let $\Gamma \subset \mathbb{C}$ be a smooth simple closed curve enclosing J and disjoint from the spectrum of T' for all $T' \in W_T$. We orient Γ as the boundary of the connected component of $\mathbb{C} \setminus \Gamma$ containing J. Then

the spectral measure in question can be expressed by the operator valued Cauchy integral formula

$$E_J^{T'} = \frac{1}{2\pi i} \int_\Gamma (\lambda \operatorname{Id} - T')^{-1}\, d\lambda; \tag{81}$$

see, e.g., [6, Section X.1 Formula (i)]. In [6] it is stated for bounded operators, but it immediately generalizes to our case as follows. Note that the integral itself on the right-hand side of (81) still makes sense. Let now e be an eigenvector for an eigenvalue μ of T'. Then

$$(\lambda \operatorname{Id} - T')^{-1} e = (\lambda - \mu)^{-1} e.$$

Since the spectrum of T' consists of eigenvalues, the usual Cauchy formula from complex analysis implies that the right-hand side of (81) does indeed define the desired spectral measure. The continuous dependence of $E_J^{T'}$ on T' now follows from the continuous dependence of the Cauchy integral on its integrand. □

5C. ***The main result and strategy of proof.*** We introduce the following notation for any real number $\mathfrak{R}$, where $\sigma(T)$ denotes the spectrum of T:

$$\begin{aligned}
\mathcal{F}_s^{>\mathfrak{R}}(E, F) &:= \{T \in \mathcal{F}_s(E, F) \mid \sigma(T) \subset (\mathfrak{R}, +\infty)\},\\
\mathcal{F}_s^{>\mathfrak{R}}(E, F)^* &:= \{T \in \mathcal{F}_s^{>\mathfrak{R}}(E, F) \mid \ker T = \operatorname{coker} T = 0\},\\
\mathcal{F}_s^{<\mathfrak{R}}(E, F) &:= \{T \in \mathcal{F}_s(E, F) \mid \sigma(T) \subset (-\infty, \mathfrak{R})\}.
\end{aligned}$$

Now we can state the main result of this section.

Theorem 5.4. *Consider the setting above and assume that the inclusion $\iota : E \hookrightarrow F$ is compact. Let $\mathfrak{R}$ be any real number. Then the equalities $(\operatorname{im} T)^{\perp_F} = \ker T$ give rise to a canonical orientation of the restrictions*

$$\det|_{\mathcal{F}_s^{>\mathfrak{R}}(E,F)} \quad \textit{and} \quad \det|_{\mathcal{F}_s^{<\mathfrak{R}}(E,F)}$$

of the determinant line bundle to the subspaces of index zero symmetric Fredholm operators bounded from below or above.

The proof will occupy the rest of this section. We begin by explaining the main ideas. Observe that the involution

$$\mathcal{F}_s^{>\mathfrak{R}}(E, F) \to \mathcal{F}_s^{<-\mathfrak{R}}(E, F), \quad A \mapsto -A$$

preserves the determinant line bundle. Therefore, it is enough to prove our statement for $\mathcal{F}_s^{>\mathfrak{R}}(E, F)$.

For $T \in \mathcal{F}_s(E, F)$ let $v_1, \dots, v_k$ be an orthonormal basis (with respect to $\langle \cdot, \cdot \rangle_F$) of $(\operatorname{im} T)^{\perp_F} = \ker T$. We identify $\ker T$ with its dual using the Hilbert space structure. The element

$$t(T) := v_k \wedge \cdots \wedge v_1 \otimes v_1 \wedge \cdots \wedge v_k \in \Lambda^k(\ker T) \otimes \Lambda^k(\ker T) = \det(T) \tag{82}$$

does not depend on the choice of the orthonormal basis, so these elements define a canonical section t of $\det|_{\mathcal{F}_s(E,F)}$. Unfortunately, this section will turn out to be discontinuous — it has singularities at operators that have more kernel than some of their neighbors. Observe, however, that the restriction of the determinant line bundle to the subset $\mathcal{F}_s(E, F)^*$ of invertible operators is a trivial bundle $\mathcal{F}_s(E, F)^* \times \mathbb{R}$. Moreover, the section t over $\mathcal{F}_s(E, F)^*$ is the constant $1 \in \mathbb{R}$. The section t can be thought of as a tautological section and its restriction to $\mathcal{F}_s(E, F)^*$ is tautologically continuous.

The idea now is to compensate the above discontinuity using a certain spectral count. Recall from Lemma 5.2 that the spectrum $\sigma(T) \subset \mathbb{R}$ of $T \in \mathcal{F}_s(E, F)$ is discrete and consists only of eigenvalues of finite multiplicity. Let

$$\rho : \mathbb{R} \to (-\infty, 1]$$

be a nondecreasing smooth function which restricts on the interval $(-\infty, a]$ as the identity $\lambda \mapsto \lambda$ and which is constant equal to 1 on the interval $[b, \infty)$ for some $0 < a < b$. We define our spectral count

$$\mu : \mathcal{F}_s^{>\mathfrak{R}}(E, F) \to \mathbb{R}$$

by the formula

$$\mu(T) := \prod_{\lambda \in \sigma(T)\setminus\{0\}} \rho(\lambda),$$

where each eigenvalue appears with its multiplicity. Note that the product above contains only finitely many factors different from 1 because all eigenvalues of T are $> \mathfrak{R}$ and $\rho(\lambda) = 1$ for large λ. The function μ is nowhere zero. Observe, however, that part of the spectrum may converge to 0 in a family of operators. On such a family the function μ converges to 0 and compensates the singularity in the section t above. We define a modified section of the determinant line bundle over $\mathcal{F}_s^{>\mathfrak{R}}(E, F)$ by

$$s(T) := \mu(T) v_k \wedge \cdots \wedge v_1 \otimes v_1 \wedge \cdots \wedge v_k.$$

We will show that s is continuous. Note that different choices of the function ρ lead to positively proportional sections s, therefore the induced orientation is canonical.

Let $T \in \mathcal{F}_s^{>\mathfrak{R}}(E, F)$ be given. We specify the general Fredholm setting above to the present situation. Namely, set $N := \dim \ker T$ and pick a linear isomorphism

$$\Phi : \mathbb{R}^N \to \ker T = (\operatorname{im} T)^{\perp F} \subset F$$

respecting scalar products. Let $U_{T,\Phi}$ be a neighborhood of T in $\mathcal{F}(E, F)$ as above and consider its restriction to $U_T := U_{T,\Phi} \cap \mathcal{F}_s^{>\mathfrak{R}}(E, F)$. Let $\{T_n\}_{n\in\mathbb{N}} \subset U_T$ be any sequence with

$$\lim_{n\to\infty} T_n = T.$$

We need to show

$$\lim_{n\to\infty} s(T_n) = s(T). \tag{83}$$

By a standard trick, it suffices to show that for every subsequence (T_{n_k}) there exists a subsequence $(T_{n_{k_j}})$ such that $\lim_{j\to\infty} s(T_{n_{k_j}}) = s(T)$. Using this, in the proof of (83) we will repeatedly pass to subsequences of (T_n), always renaming them back to T_n.

Inequality (77) with $k = l$ (because of index zero) implies that after passing to a subsequence we can assume

$$\dim\ker T_n = k \le \dim\ker T = N$$

for some constant k. We understand the desired convergence (83) in terms of the above isomorphism ι_Φ. That is, we have to prove that

$$\lim_{n\to\infty} \iota_\Phi(T_n, s(T_n)) = \iota_\Phi(T, s(T)). \tag{84}$$

Let us recall the choices we have to make in order to define the necessary objects entering the last assertion:

(i) An orthonormal basis $v_1, \dots, v_N$ of $\ker T$.

(ii) An orthonormal basis $v_1^n, \dots, v_k^n$ of $\ker T_n$ for each $n \in \mathbb{N}$.

(iii) A basis $\zeta_1, \dots, \zeta_N$ of $\mathbb{R}^N$ (to define $\iota_\Phi(T, \cdot)$).

(iv) A basis $\zeta_1^n, \dots, \zeta_N^n$ of $\mathbb{R}^N$ for each $n \in \mathbb{N}$ (to define $\iota_\Phi(T_n, \cdot)$).

(v) A collection $\{\xi_j^n\}_{j=k+1}^N \subset E$ subject to

$$T_n\xi_j^n = -\Phi\zeta_j^n, \quad j = k+1, \dots, N \tag{85}$$

(to define $\iota_\Phi(T_n, \cdot)$).

This allows us to write out

$$\iota_\Phi(T_n, s(T_n)) = (-1)^k C_n \mu(T_n)(v_1^n, 0) \wedge \cdots \wedge (v_k^n, 0) \wedge (\xi_{k+1}^n, \zeta_{k+1}^n) \wedge \cdots \wedge (\xi_N^n, \zeta_N^n)$$

and

$$\iota_\Phi(T, s(T)) = (-1)^N C \mu(T)(v_1, 0) \wedge \cdots \wedge (v_k, 0) \wedge (v_{k+1}, 0) \wedge \cdots \wedge (v_N, 0)$$

with the abbreviations

$$C_n := \frac{\det\langle v_i^n, \Phi\zeta_j^n\rangle_{i,j=1,\dots,k}}{\det(\zeta_1^n, \dots, \zeta_N^n)}, \quad C := \frac{\det\langle v_i, \Phi\zeta_j\rangle_{i,j=1,\dots,N}}{\det(\zeta_1, \dots, \zeta_N)}.$$

The desired convergence (84) thus follows from the following three assertions (for suitable choices (i)–(v)):

Assertion (A) $\lim_{n\to\infty} v_1^n \wedge \cdots \wedge v_k^n = v_1 \wedge \cdots \wedge v_k.$

Assertion (B)

$$\lim_{n\to\infty} \mu(T_n)(-\xi_{k+1}^n, \zeta_{k+1}^n) \wedge \cdots \wedge (-\xi_N^n, \zeta_N^n) = \mu(T)(v_{k+1}, 0) \wedge \cdots \wedge (v_N, 0).$$

Assertion (C)

$$\lim_{n\to\infty} \det\langle v_i^n, \Phi\zeta_j^n\rangle_{i,j=1,\ldots,k} = \det\langle v_i, \Phi\zeta_j\rangle_{i,j=1,\ldots,N}, \tag{86}$$

$$\lim_{n\to\infty} \det(\zeta_1^n, \ldots, \zeta_N^n) = \det(\zeta_1, \ldots, \zeta_N). \tag{87}$$

Let us describe the strategy for making the choices above. We are going to show that in a certain sense $\ker T_n$ converges to a subspace $\ker_{\ker} T$ of $\ker T$ as $n \to \infty$. Note that each $\ker T_n$ is a member of an infinite dimensional Grassmannian. Therefore, we need an additional construction ("parametrizing the kernels") relating $\ker T_n$ to a subspace of $\ker T$. This will give us the key orthogonal splitting

$$\ker T = \ker_{\ker} T \oplus \ker_{\mathrm{im}} T,$$

where $\ker_{\mathrm{im}} T$ is the orthogonal complement to $\ker_{\ker} T$ in $\ker T$. Alternatively, $\ker_{\mathrm{im}} T$ is the limit of $\mathrm{im}\, T_n \cap \ker T$ as $n \to \infty$. We let $v_1, \ldots, v_k$ be any orthonormal basis of $\ker_{\ker} T$ and $v_{k+1}, \ldots, v_N$ be any orthonormal basis of $\ker_{\mathrm{im}} T$. We define $\{\zeta_j\}_{j=1}^N$ to be the preimage of $\{v_j\}_{j=1}^N$ under Φ and $\zeta_j^n := \zeta^j$ for $j = 1, \ldots, k$. We then define $\{v_j^n\}_{j=1}^k$ as the orthonormalization of a suitable projection of $\{v_j\}_{j=1}^k$ to $\ker T_n \cap \ker T$, and $\{\Phi\zeta_j^n\}_{j=k+1}^N$ as the orthogonal projection of $\{v_j\}_{j=k+1}^N$ to $\mathrm{im}\, T_n$. This will allow us to define $\{\xi_j\}_{j=k+1}^N$ using (85) and it remains to verify the three assertions above (see Section 5E).

5D. ***Parametrizing the kernels.*** Let X and Y be two Banach spaces and $\mathcal{F}^*(X, Y)$ be the space of surjective Fredholm operators from X to Y. Pick any $D^0 \in \mathcal{F}^*(X, Y)$ and let $U_{D^0} \subset \mathcal{F}^*(X, Y)$ be a connected open neighborhood of D^0. We describe one possible trivialization of the kernel bundle over U_{D^0}. Consider the direct sum $Y \oplus \ker D^0$ and let $\mathcal{C}_0 \subset X$ be a closed direct complement of $\ker D^0$ in X. Let

$$P_{\ker D^0} : X \to \ker D^0$$

denote the projection onto $\ker D^0$ along $\mathcal{C}_0$.

To every operator $D \in U_{D^0}$ we associate an operator $\tilde{D} \in L(X, Y \oplus \ker D^0)$ defined by $\tilde{D}(x) := (D(x), P_{\ker D^0}(x))$. By construction the operator $\tilde{D}^0$ is bijective. Therefore, by shrinking U_{D^0} if necessary, we may assume that $\tilde{D}$ is bijective for every $D \in U_{D^0}$. As a consequence, each $\tilde{D}$ admits continuous inverse depending continuously on D. Given any $x \in X$, let $\tilde{x} \in X$ be the unique solution of the equation

$$\tilde{D}(\tilde{x}) = (D(x), 0). \tag{88}$$

Writing out (88) in components gives us

$$D(\tilde{x}) = D(x), \quad P_{\ker D^0}(\tilde{x}) = 0.$$

In other words,

$$D(x - \tilde{x}) = 0, \quad \tilde{x} \in \mathcal{C}_0.$$

Observe that $\tilde{x}$ depends continuously on $(D, x) \in U_{D^0} \times X$, and $\tilde{x} = 0$ for $D = D^0$ and any $x \in \ker D^0$. So the isomorphisms

$$\mathcal{Q}_D : \ker D^0 \xrightarrow{\cong} \ker D, \quad \mathcal{Q}_D(x) := x - \tilde{x}$$

depend continuously on $D \in U_{D^0}$ and

$$\mathcal{Q}_{D^0} = \mathrm{Id}_{\ker D^0}.$$

To apply this in our situation set $X := E \oplus \mathbb{R}^N$, $Y := F$, $D^0 := T \oplus \Phi$, $D := T_n \oplus \Phi$. Furthermore, in our case $\ker T \cong \ker T \times \{0\} = \ker(T \oplus \Phi)$ and we can take $\mathcal{C}_0 := \operatorname{im} T \cap E$. Then the preceding construction gives us isomorphisms

$$\mathcal{Q}_n := \mathcal{Q}_{T_n \oplus \Phi} : \ker T \xrightarrow{\cong} \ker(T_n \oplus \Phi)$$

such that

$$\lim_{n \to \infty} \mathcal{Q}_n = \mathrm{Id}_{\ker T}. \tag{89}$$

Equation (89) implies in particular

$$\|(\mathcal{Q}_n)_1 - \mathrm{Id}_{\ker T}\| \to 0, \quad \|\mathcal{Q}_n^{-1}|_{\ker T_n \times \{0\}} - \mathrm{Id}_{\ker T_n \times \{0\}}\| \to 0 \tag{90}$$

as $n \to \infty$, where $(\mathcal{Q}_n)_1$ denotes the first component of $\mathcal{Q}_n$.

Using the isomorphisms $\mathcal{Q}_n$ we can now establish the orthogonal splitting $\ker T = \ker_{\ker} T \oplus \ker_{\mathrm{im}} T$. For two subspaces H_1, H_2 of a finite dimensional Hilbert space $(H, \langle \cdot, \cdot \rangle_H)$ we define

$$\langle H_1, H_2 \rangle := \sup\{\langle u, v \rangle_H \mid (u, v) \in H_1 \times H_2, \|u\| = \|v\| = 1\}.$$

Note that $\langle H_1, H_2 \rangle = 0$ if and only if H_1 and H_2 are orthogonal to each other. We consider $\ker T$ and $\ker(T_n \oplus \Phi)$ as finite dimensional Hilbert spaces with the scalar products induced from F. Let "$\perp$" denote "orthogonal complement in $\ker(T_n \oplus \Phi)$" for the discussion below. For a subspace K of $\ker(T_n \oplus \Phi)$ let $(K)_2 \subset \mathbb{R}^N$ denote the projection of K on the second component. Observe that

$$(\ker T_n \times \{0\})^\perp = \{(\xi, \zeta) \in (\operatorname{im} T_n \cap E) \times \mathbb{R}^N \mid T_n \xi = -\Phi \zeta\}.$$

Since T_n restricts to an isomorphism from $\operatorname{im} T_n \cap E$ to $\operatorname{im} T_n$, we get

$$\Phi((\ker T_n \times \{0\})^\perp)_2 = \operatorname{im} T_n \cap \ker T.$$

Since the Grassmannian of k-dimensional subspaces of $\ker T$ is compact, we get (after passing to a subsequence if necessary) the following limits:

$$\ker_{\ker} T := \lim_{n\to\infty} Q_n^{-1}(\ker T_n \times \{0\}), \quad \ker_{\operatorname{im}} T := \lim_{n\to\infty} (\operatorname{im} T_n \cap \ker T). \tag{91}$$

For a closed subspace $C \subset F$ we denote by

$$P_C : F \to F$$

the orthogonal projection onto C. In terms of orthogonal projections, the preceding equations mean

$$P_{\ker_{\ker} T} = \lim_{n\to\infty} P_{Q_n^{-1}(\ker T_n \times \{0\})}, \quad P_{\ker_{\operatorname{im}} T} = \lim_{n\to\infty} P_{\operatorname{im} T_n \cap \ker T}. \tag{92}$$

Our next goal is to show orthogonality of $\ker_{\ker} T$ and $\ker_{\operatorname{im}} T$. Note that $\ker T_n \times \{0\}$ and $\operatorname{im} T_n \cap \ker T$ are orthogonal to each other as subspaces of F. This together with (90) implies that

$$\langle \operatorname{im} T_n \cap \ker T,\, Q_n^{-1}(\ker T_n \times \{0\})\rangle \to 0 \quad \text{as } n \to \infty.$$

The desired orthogonality now follows from (91). Since

$$\dim \ker_{\ker} T = k \quad \text{and} \quad \dim \ker_{\operatorname{im}} T = N - k,$$

we get the orthogonal splitting

$$\ker T = \ker_{\ker} T \oplus \ker_{\operatorname{im}} T. \tag{93}$$

5E. ***Proof of Theorem 5.4.*** **The choices (i)–(v).** Now we are ready to make the choices (i)–(v) in Section 5C. For (i) let $v_1, \dots, v_N$ be an orthonormal basis of $\ker T$ such that $v_1, \dots, v_k$ form a basis of $\ker_{\ker} T$, and $v_{k+1}, \dots, v_N$ form a basis of $\ker_{\operatorname{im}} T$. For (ii), let GS_n denote the Gram–Schmidt retraction from the space of all bases of $\ker T_n$ to the space of orthonormal bases of $\ker T_n$ and set

$$\{v_j^n\}_{j=1}^k := GS_n(P_{\ker T_n \times \{0\}} \mathcal{Q}_n(\{v_j\}_{j=1}^k)). \tag{94}$$

For (iii) we define

$$\zeta_j := \Phi^{-1} v_j, \quad j = 1, \dots, N. \tag{95}$$

For (iv) we define

$$\zeta_j^n := \begin{cases} \zeta_j & 1 \le j \le k, \\ \Phi^{-1} P_{\operatorname{im} T_n \cap \ker T} v_j & k+1 \le j \le N. \end{cases} \tag{96}$$

The second part of this definition rewrites as

$$\Phi \zeta_j^n = P_{\operatorname{im} T_n \cap \ker T} v_j, \quad j = k+1, \dots, N. \tag{97}$$

In view of the second equation in (92) and $v_j \in \ker_{\mathrm{im}} T$ for $j = k+1, \dots, N$ this implies

$$\Phi\zeta_j^n \to P_{\ker_{\mathrm{im}} T} v_j = v_j \quad \text{as } n \to \infty,\ j = k+1, \dots, N. \tag{98}$$

Applying Φ^{-1} to both sides of the last equation, we get

$$\zeta_j^n \to \zeta_j \quad \text{as } n \to \infty,\ j = k+1, \dots N. \tag{99}$$

In particular, we see that $\{\zeta_j^n\}_{j=1}^N$ form a basis of $\mathbb{R}^N$ for large n. For (v), we use the isomorphism

$$\widetilde{T}_n := T_n|_{\mathrm{im}\, T_n \cap E} : \mathrm{im}\, T_n \cap E \xrightarrow{\cong} \mathrm{im}\, T_n \tag{100}$$

to define

$$\xi_j^n := -\widetilde{T}_n^{-1} \Phi \zeta_j^n, \quad j = 1, \dots, N. \tag{101}$$

With these choices, we will now verify assertions (A)–(C) from Section 5C.

Proof of Assertion (A). We will use the following simple lemma.

Lemma 5.5. *Let $\{A_n\}_{n\in\mathbb{N}}$ and $\{B_n\}_{n\in\mathbb{N}}$ be two bounded sequences of operators $A_n \in \mathcal{L}(Y, Z)$, $B_n \in \mathcal{L}(X, Y)$ between Banach spaces. Assume convergence $B := \lim_{n\to\infty} B_n$ and the equality $A_n B_n = 0$ for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} A_n B = 0$.*

Proof. Since the sequence $\{A_n\}_{n\in\mathbb{N}}$ is bounded, we get

$$\|A_m(B - B_n)\| \to 0 \quad \text{as } n \to \infty$$

uniformly with respect to m. Set $m := n$ and use $A_n B_n = 0$ to conclude. □

We apply the lemma with $X = Y = Z = \ker T$ and the projections

$$A_n := \mathcal{Q}_n^{-1} P_{(\ker T_n \times \{0\})^\perp} \mathcal{Q}_n, \quad B_n := P_{\mathcal{Q}_n^{-1}(\ker T_n \times \{0\})}.$$

Since obviously $A_n B_n = 0$, and $\lim_{n\to\infty} B_n = P_{\ker_{\ker} T}$ by (92), Lemma 5.5 yields

$$\lim_{n\to\infty} \mathcal{Q}_n^{-1} P_{\ker T_n \times \{0\}^\perp} \mathcal{Q}_n P_{\ker_{\ker} T} = 0. \tag{102}$$

In view of $P_{\ker T_n \times \{0\}^\perp} + P_{\ker T_n \times \{0\}} = \mathrm{Id}_{\ker T}$ and (89), this implies

$$\lim_{n\to\infty} P_{\ker T_n \times \{0\}} \mathcal{Q}_n P_{\ker_{\ker} T} = P_{\ker_{\ker} T}. \tag{103}$$

This implies

$$P_{\ker T_n \times \{0\}} \mathcal{Q}_n(\{v_j\}_{j=1}^k) \to \{v_j\}_{j=1}^k \quad \text{as } n \to \infty, \tag{104}$$

and since $\{v_j\}_{j=1}^k$ is orthonormal we get

$$\{v_j^n\}_{j=1}^k \to \{v_j\}_{j=1}^k \quad \text{as } n \to \infty. \tag{105}$$

This proves Assertion (A). □

Proof of Assertion (C). The choice (95) of $\{\zeta_j\}_{j=1,\dots,N}$ implies that the right-hand side of (86) equals 1. The choice (96) of $\{\zeta_j^n\}_{j=1,\dots,k}$ and (105) implies that the left-hand side of (86) equals 1. Equation (87) follows from (96) (for $j \le k$) and (99) (for $j > k$). □

Proof of Assertion (B). First, we prepare several convergence statements. Let $\varepsilon > 0$ be such that 0 is the only eigenvalue of T in $(-2\varepsilon, 2\varepsilon)$ and the function ρ equals the identity on $(-\varepsilon, \varepsilon)$. Then by continuity (80) of the spectral measure we get

$$\lim_{n\to\infty} E^{T_n}_{(-\varepsilon,\varepsilon)} = E^{T}_{(-\varepsilon,\varepsilon)} = P_{\ker T}, \quad \lim_{n\to\infty} E^{T_n}_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} = \mathrm{Id} - P_{\ker T}. \tag{106}$$

Let $k+1 \le j \le N$. We apply the operations in (106) to both sides of (98) to get

$$\lim_{n\to\infty} E^{T_n}_{(-\varepsilon,\varepsilon)} \Phi\zeta_j^n = v_j, \quad \lim_{n\to\infty} E^{T_n}_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \Phi\zeta_j^n = 0. \tag{107}$$

Observe that $P_{\ker T_n} \Phi\zeta_j^n = 0$ and define

$$z_j^n := E^{T_n}_{(-\varepsilon,\varepsilon)\setminus\{0\}} \Phi\zeta_j^n = E^{T_n}_{(-\varepsilon,\varepsilon)} \zeta_j^n, \quad \tilde{z}_j^n := E^{T_n}_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \Phi\zeta_j^n.$$

Note that

$$\Phi\zeta_j^n = z_j^n + \tilde{z}_j^n.$$

Recall the isomorphism $\widetilde{T}_n$ from (100). The eigenvalues of $\widetilde{T}_{n,\varepsilon}^{-1} := \widetilde{T}_n^{-1}|_{\mathrm{im}\, E^{T_n}_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)}}$ are bounded above in absolute value by ε^{-1}. Since the spectral radius of a compact self-adjoint operator equals its norm, this implies

$$\|\widetilde{T}_n^{-1}\tilde{z}_j^n\| = \|\widetilde{T}_{n,\varepsilon}^{-1}\tilde{z}_j^n\| \le \|\widetilde{T}_{n,\varepsilon}^{-1}\|\,\|\tilde{z}_j^n\| \le \varepsilon^{-1}\|\tilde{z}_j^n\|. \tag{108}$$

The preceding discussion can be summarized as follows for $j = k+1, \dots, N$:

$$\Phi\zeta_j^n = z_j^n + \tilde{z}_j^n, \quad \lim_{n\to\infty} z_j^n = v_j, \quad \lim_{n\to\infty} \tilde{z}_j^n = \lim_{n\to\infty} \widetilde{T}_n^{-1}\tilde{z}_j^n = 0. \tag{109}$$

Second, we manipulate the spectral count μ and determinants of certain finite dimensional operators. Set

$$\mu_n := \prod_{\lambda\in\sigma(T_n)\cap(\mathbb{R}\setminus(-\varepsilon,\varepsilon))} \rho(\lambda).$$

Since $\rho(\lambda) = \lambda$ for $\lambda \in (-\varepsilon, \varepsilon)$, we get

$$\mu(T_n) = \mu_n \prod_{\lambda\in\sigma(T_n)\cap((-\varepsilon,\varepsilon)\setminus\{0\})} \lambda = \mu_n \det(\widetilde{T}_n|_{\mathrm{im}\, E^{T_n}_{(-\varepsilon,+\varepsilon)\setminus\{0\}}}) \tag{110}$$

and

$$\lim_{n\to\infty} \mu_n = \mu(T). \tag{111}$$

Now we attend to the main part of the argument. Using the notation $\Lambda_{j=a}^{b} w_j = w_a \wedge \cdots \wedge w_b$, Assertion (B) reads

$$\lim_{n\to\infty} \mu(T_n)\Lambda^N_{j=k+1}(-\xi^n_j, \zeta^n_j) = \mu(T)\Lambda^N_{j=k+1}(v_j, 0).$$

Identifying vectors $\xi \in E$ and $\zeta \in \mathbb{R}^N$ with their images $(\xi, 0)$ and $(0, \zeta)$ in $E \oplus \mathbb{R}^N$ and using (101) and (109), this reads

$$\lim_{n\to\infty} \mu(T_n)\Lambda^N_{j=k+1}(\widetilde{T}_n^{-1} z^n_j + \widetilde{T}_n^{-1}\tilde{z}^n_j + \zeta^n_j) = \mu(T)\Lambda^N_{j=k+1} v_j. \tag{112}$$

We split the wedge product on the left-hand side as the leading term plus the rest,

$$\Lambda^N_{j=k+1}(\widetilde{T}_n^{-1} z^n_j + \widetilde{T}_n^{-1}\tilde{z}^n_j + \zeta^n_j) = \Lambda^N_{j=k+1}\widetilde{T}_n^{-1} z^n_j + \tilde{R}_n. \tag{113}$$

We first discuss the rest term $\tilde{R}_n$. Modulo signs a typical summand is

$$(\Lambda^l_{i=k+1}\widetilde{T}_n^{-1} z^n_{j_i}) \wedge (\Lambda^m_{i=l+1}\widetilde{T}_n^{-1}\tilde{z}^n_{j_i}) \wedge (\Lambda^N_{i=m+1}\zeta^n_{j_i})$$

for some $k \le l \le m \le N$ with $l < N$. Equation (98) and the third equation in (109) imply that the second and third factors in the last displayed equation remain bounded as $n \to \infty$. For the first factor, we pick orthonormal bases $e^n_{k+1}, \dots, e^n_N$ of the spaces $\operatorname{im} E^{T_n}_{(-\varepsilon,+\varepsilon)\setminus\{0\}}$ consisting of eigenvectors of T_n and converging to an orthonormal system $e_{k+1}, \dots, e_N$ as $n \to \infty$. We write $z^n_j = \sum_{i=k+1}^N c^i_{jn} e^n_i$ in these bases, with coefficients $c^i_{jn} \in \mathbb{R}$. Since $z^n_j \to v_j$ by the second part of (109), the sequence $\{c^i_{jn}\}_{n\in\mathbb{N}}$ converges and thus in particular remains bounded as $n \to \infty$, for all $i, j = k+1, \dots, N$. This way the first factor $\widetilde{T}_n^{-1} z^n_{j_1} \wedge \cdots \wedge \widetilde{T}_n^{-1} z^n_{j_l}$ can be viewed as a homogeneous polynomial in $\lambda_{k+1}^{-1}, \dots, \lambda_N^{-1}$ of degree $l-k$ strictly less than $N-k$. Moreover, this polynomial has bounded coefficients in $\Lambda^{l-k} \operatorname{im} E^{T_n}_{(-\varepsilon,+\varepsilon)\setminus\{0\}}$, and each variable λ_j^{-1} for $j = k+1, \dots, N$ enters every monomial with power 0 or 1. This implies

$$\lim_{n\to\infty} \mu(T_n)(\Lambda^l_{i=k+1}\widetilde{T}_n^{-1} z^n_{j_i}) = 0, \tag{114}$$

and therefore

$$\lim_{n\to\infty} \mu(T_n)\tilde{R}_n = 0. \tag{115}$$

Now we consider the leading term in (113) which we rewrite as

$$\mu(T_n)\Lambda^N_{j=k+1}\widetilde{T}_n^{-1} z^n_j = \mu(T_n)\det(\widetilde{T}_n^{-1}|_{\operatorname{im} E^{T_n}_{(-\varepsilon,+\varepsilon)\setminus\{0\}}})\Lambda^N_{j=k+1} z^n_j = \mu_n \Lambda^N_{j=k+1} z^n_j,$$

where the first equality follows from the definition of the determinant of a finite dimensional operator and the second equality follows from (110). In view of (111) and the second equation in (109), this implies

$$\lim_{n\to\infty} \mu(T_n)\Lambda^N_{j=k+1} T_n^{-1} z^n_j = \mu(T)\Lambda^N_{j=k+1} v_j,$$

which together with (115) proves Assertion (B) and thus Theorem 5.4. □

Remark 5.6. Recall that on the subset $\mathcal{F}_s^{>\mathfrak{R}}(E,F)^*$ consisting of invertible operators we have another orientation of the determinant bundle given by the section t; see (82) and the discussion following it. Observe that for $T \in \mathcal{F}_s^{>\mathfrak{R}}(E,F)^*$ the values $s(T)$ and $t(T)$ differ by $\mu(T)$, where $\mu(T)$ is the product of all negative eigenvalues of T and some positive factor resulting from the restriction $\rho|_{(0,+\infty)}$. We rephrase this in terms of the respective orientations. Let us call the orientation of $\det|_{\mathcal{F}_s^{>\mathfrak{R}}(E,F)^*}$ induced by t the *tautological orientation*. Any other orientation of $\det|_{\mathcal{F}_s^{>\mathfrak{R}}(E,F)^*}$ differs from the tautological one by a sign. For $T \in \mathcal{F}_s^{>\mathfrak{R}}(E,F)$ let $i(T)$ denote the (finite) number of negative eigenvalues of T. Then the restriction of the orientation s to $\mathcal{F}_s^{>\mathfrak{R}}(E,F)^*$ is related to the tautological orientation by

$$s(T) = (-1)^{i(T)} t(T) \quad \text{for } T \in \mathcal{F}_s^{>\mathfrak{R}}(E,F)^*.$$

5F. ***A counterexample.*** In this subsection we construct a loop of self-adjoint Fredholm operators with unbounded spectrum and nonorientable determinant bundle, thus showing that the boundedness of the spectrum from below or above in Theorem 5.4 is essential.

We consider

$$l^2 := \left\{ x = (x_n)_{n\in\mathbb{Z}} \;\middle|\; x_n \in \mathbb{R}, \sum_{n\in\mathbb{Z}} x_n^2 < \infty \right\}$$

as a real Hilbert space with scalar product

$$\langle x, y\rangle := \sum_{n\in\mathbb{Z}} x_n y_n.$$

Similarly, we consider

$$h^1 := \left\{ x = (x_n)_{n\in\mathbb{Z}} \;\middle|\; x_n \in \mathbb{R}, \sum_{n\in\mathbb{Z}} n^2 x_n^2 < \infty \right\}$$

as a real Hilbert space with scalar product

$$\langle x, y\rangle_{h^1} := \sum_{n\in\mathbb{Z}} x_n y_n + \sum_{n\in\mathbb{Z}} n^2 x_n y_n.$$

Since $h^1 \subset l^2$ is dense and the inclusion $h^1 \hookrightarrow l^2$ is compact, the pair (h^1, l^2) satisfies the hypotheses of Theorem 5.4. Our goal is to prove

Proposition 5.7. *The restriction* $\det|_{\mathcal{F}_s(h^1,l^2)}$ *of the determinant line bundle to the space of self-adjoint Fredholm operators* $l^2 \supset h^1 \to l^2$ *is nonorientable.*

The proof uses the following lemma. Recall the standard Hilbert space basis $\{e_n := (\dots, 0, 1, 0, \dots)\}_{n\in\mathbb{Z}}$ of l^2, where the 1 occupies position n.

Lemma 5.8. *There exists a continuous path $\{U_\tau\}_{\tau\in[1,2]}$ of unitary isomorphisms of l^2 that restrict to (nonunitary) isomorphisms of h^1 such that*:

(i) *U_1 maps e_n to e_{n-1} for all $n \in \mathbb{Z}$.*

(ii) *$U_2 = \mathrm{Id}$.*

(iii) *$U_\tau^{-1}(e_0)$ is a positive linear combination of e_0 and e_1 for all $\tau \in (1, 2)$.*

Proof. The following proof was found by Bernd Schmidt. The idea is to concatenate the following two rotational isotopies defined in terms of the standard basis of l^2. For $\tau \in \left[0, \frac{1}{2}\right]$ we define a unitary operator V_τ^1 of l^2 by requiring that it sends

$$\begin{cases} e_{-n} & \mapsto \cos(\pi\tau)e_{-n} - \sin(\pi\tau)e_{n+1}, \\ e_{n+1} & \mapsto \sin(\pi\tau)e_{-n} + \cos(\pi\tau)e_{n+1}, \end{cases}$$

for all $n \geq 0$. Similarly, for $\tau \in \left[0, \frac{1}{2}\right]$ we define a unitary operator V_τ^2 of l^2 by requiring that it sends

$$\begin{cases} e_0 & \mapsto e_0, \\ e_{-n} & \mapsto \cos(\pi\tau)e_{-n} + \sin(\pi\tau)e_n, \\ e_n & \mapsto -\sin(\pi\tau)e_{-n} + \cos(\pi\tau)e_n, \end{cases}$$

for all $n > 0$. It is clear that both operators V_τ^1 and V_τ^2 restrict to isomorphisms of h^1. Define

$$V_\tau := \begin{cases} V_\tau^1 & \tau \in \left[0, \frac{1}{2}\right], \\ V_{\tau-1/2}^2 \circ V_{1/2}^1 & \tau \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

For $\tau = 0$ we have $V_0 = \mathrm{Id}$. For $\tau = 1$ and $n \geq 0$

$$\begin{aligned} V_1 e_{-n} &= V_{1/2}^2 V_{1/2}^1 e_{-n} = -V_{1/2}^2 e_{n+1} = e_{-(n+1)}, \\ V_1 e_{n+1} &= V_{1/2}^2 V_{1/2}^1 e_{n+1} = V_{1/2}^2 e_{-n} = e_n. \end{aligned}$$

To prove the statement about the preimage of e_0, we compute for $\tau \in \left[0, \frac{1}{2}\right]$:

$$\begin{aligned} &V_\tau(\cos(\pi\tau)e_0 + \sin(\pi\tau)e_1) \\ &\quad = V_\tau^1(\cos(\pi\tau)e_0 + \sin(\pi\tau)e_1) \\ &\quad = \cos(\pi\tau)(\cos(\pi\tau)e_0 - \sin(\pi\tau)e_1) + \sin(\pi\tau)(\sin(\pi\tau)e_0 + \cos(\pi\tau)e_1) \\ &\quad = e_0. \end{aligned}$$

Since V_τ^2 preserves e_0 for all $\tau \in \left[0, \frac{1}{2}\right]$, it follows that $V_\tau^{-1}(e_0)$ is a positive linear combination of e_0 and e_1 for all $\tau \in (0, 1)$. Therefore, backward reparametrized family $U_\tau := V_{2-\tau}$ for $\tau \in [1, 2]$ has the desired properties. □

Proof of Proposition 5.7. To construct our counterexample, we consider $L^2([0,1],\mathbb{C})$ as a real Hilbert space with scalar product

$$\langle f, g\rangle := \operatorname{Re}\int_0^1 f(t)\bar{g}(t)\,dt.$$

Similarly, we consider $H^1([0,1],\mathbb{C})$ as a real Hilbert space with scalar product

$$\langle f, g\rangle_{H^1} := \operatorname{Re}\int_0^1 f(t)\bar{g}(t)\,dt + \operatorname{Re}\int_0^1 f'(t)\bar{g}'(t)\,dt.$$

We abbreviate from now on

$$L^2 := L^2([0,1],\mathbb{C}) \quad \text{and} \quad H^1 := H^1([0,1],\mathbb{C}).$$

Consider the following densely defined operator in L^2:

$$L := -i\frac{d}{dt} : L^2 \supset H^1 \to L^2$$

and the following loop of dense subspaces of L^2:

$$E_\tau := \{f \in H^1 \mid f(0) \in \mathbb{R}, e^{i\pi\tau} f(1) \in \mathbb{R}\}, \quad \tau \in [0,1].$$

Note that $E_0 = H^1$ and the inclusion $E_\tau \hookrightarrow L^2$ is compact for all $\tau \in [0,1]$. We define the restrictions

$$L_\tau := L|_{E_\tau} : L^2 \supset E_\tau \to L^2.$$

We claim that L_τ is symmetric. Indeed,

$$\langle L_\tau f, g\rangle = \operatorname{Re}\int_0^1 -if'(t)\bar{g}(t)\,dt = \operatorname{Re}(-if\bar{g}|_0^1) - \operatorname{Re}\int_0^1 -if'(t)\bar{g}(t)'\,dt.$$

Now $f(0)\bar{g}(0) \in \mathbb{R}$ and $f(1)\bar{g}(1) = e^{i\pi\tau} f(1)\overline{e^{i\pi\tau} g(1)} \in \mathbb{R}$ implies $\operatorname{Re}(-if\bar{g}|_0^1) = 0$, which in view of $if\bar{g}' = -f\overline{ig'}$ yields

$$\langle L_\tau f, g\rangle = \langle f, L_\tau g\rangle.$$

The spectrum of L_τ consists of eigenvalues

$$\lambda_n^\tau = \pi(n-\tau), \quad n \in \mathbb{Z}$$

with 1-dimensional eigenspaces spanned by the eigenvectors

$$e_n^\tau(t) = e^{i\pi(n-\tau)t}, \quad n \in \mathbb{Z}.$$

In particular, the spectrum of L_τ is unbounded from both sides. Note that $\ker L_\tau = 0$ for $\tau \in (0,1)$ and $\ker L_0 = \mathbb{R}e_0^0$. It is easy to see that $(\operatorname{im} L_\tau)^\perp = \ker L_\tau$, so the operator L_τ is Fredholm of index 0, and therefore self-adjoint by Lemma 5.1, for

all $\tau \in [0,1]$. Since the inclusion $E_\tau \hookrightarrow L^2$ is compact, the eigenvectors e_n^τ of L_τ form a Hilbert space basis of L^2 for all $\tau \in [0,1]$. In particular, for $\tau = 0$ we get a unitary isomorphism

$$l^2 \xrightarrow{\cong} L^2, \quad e_n \mapsto e_n^0 \tag{116}$$

that restricts to an isomorphism $h^1 \xrightarrow{\cong} E_0$. In the sequel we will use this isomorphism to identify l^2 with L^2 and e_n with e_n^0, so we can view the U_τ from Lemma 5.8 as unitary isomorphisms of L^2 restricting to isomorphisms of E_0.

The loop $\{L_\tau\}_{\tau\in[0,1]}$ of self-adjoint Fredholm operators is the heart of the counterexample. Unfortunately, these operators have varying domains of definition. Therefore, we will conjugate $\{L_\tau\}_{\tau\in[0,1]}$ with a suitable loop of unitary operators to bring all the operators from the desired loop on the same footing. Let $\{U_\tau\}_{\tau\in[1,2]}$ be the path of unitary operators in L^2 obtained from the one in Lemma 5.8 via identification (116).

We define a modified loop $\{\tilde{E}_\tau\}_{\tau\in[0,2]}$ of subspaces of L^2 by

$$\tilde{E}_\tau := \begin{cases} E_\tau & \tau \in [0,1], \\ E_0 = E_1 & \tau \in [1,2]. \end{cases}$$

We define a loop $\{\Phi_\tau\}_{\tau\in[0,2]}$ of isomorphisms of L^2 that restrict to isomorphisms

$$\Phi_\tau|_{E_0} : E_0 \to \tilde{E}_\tau$$

by

$$\Phi_\tau f := \begin{cases} e^{-i\pi\tau t} f & \tau \in [0,1], \\ U_\tau f & \tau \in [1,2]. \end{cases}$$

Note that $e^{-i\pi t} e_n = e^{-i\pi t} e^{i\pi n t} = e_{n-1} = U_1 e_n$ and $U_2 = \mathrm{Id}$, so $\{\Phi_\tau\}_{\tau\in[0,2]}$ is indeed a continuous loop. Finally, we define

$$\tilde{L}_\tau := L|_{\tilde{E}_\tau} \quad \text{and} \quad T_\tau := \Phi_\tau^{-1} \tilde{L}_\tau \Phi_\tau$$

for $\tau \in [0,2]$. The definition of Φ_τ and Lemma 5.8 imply that $\{T_\tau\}_{\tau\in[0,2]}$ is a loop of self-adjoint operators in L^2 with domain of definition E_0. This puts us in the setting of Section 5 with $F := L^2$, $E := E_0$ and $\{T_\tau\}_{\tau\in[0,2]} \subset \mathcal{F}_s(E,F)$. Note that

$$\ker T_\tau = \begin{cases} \mathbb{R} e_0 & \tau = 0, \\ 0 & \tau \in (0,1), \\ \mathbb{R} U_\tau^{-1} e_0 & \tau \in [1,2]. \end{cases}$$

To stabilize the loop $\{T_\tau\}_{\tau\in[0,2]}$, we fix some $a, b > 0$ and consider the vector

$$G := a e_0 + b e_1.$$

The positivity in Lemma 5.8(iii) and the positivity of a and b imply that

$$\langle G, U_\tau^{-1} E_0 \rangle > 0$$

for all $\tau \in [1,2]$. Therefore, the vector G is transverse to $\operatorname{im} T_\tau = (\ker T_\tau)^\perp$ for all $\tau \in [1,2]$, so the stabilized operator

$$\hat{T}_\tau : E_0 \oplus \mathbb{R} \to L^2, \quad (f,\zeta) \mapsto T_\tau f + \zeta G$$

is surjective for all $\tau \in [0,2]$. The discussion at the beginning of Section 5 implies that the determinant bundle of $\{T_\tau\}_{\tau\in[0,2]}$ is isomorphic to the kernel bundle of $\{\hat{T}_\tau\}_{\tau\in[0,2]}$. This loop splits naturally into two parts:

The part $\tau \in [0,1]$: Using $-i\frac{d}{dt}e_n = \pi n e_n$, we compute for $f = \sum_{n\in\mathbb{Z}} c_n e_n \in E_0$

$$\begin{aligned} T_\tau f &= e^{i\pi n\tau t}\Big(-i\frac{d}{dt}\Big)e^{-i\pi n\tau t} f \\ &= e^{i\pi n\tau t}\Big(-\pi\tau e^{-i\pi\tau t} f - e^{-i\pi\tau t} i\frac{d}{dt} f\Big) \\ &= -\pi\tau f - i\frac{d}{dt} f \\ &= \sum_{n\in\mathbb{Z}} \pi(n-\tau) c_n e_n. \end{aligned}$$

Hence the equation

$$\hat{T}_\tau(f,\zeta) = \sum_{n\in\mathbb{Z}} \pi(n-\tau)c_n e_n + \zeta(a e_0 + b e_1) = 0$$

is equivalent to $c_n = 0$ for all $n \neq 0, 1$ and

$$\pi(-\tau)c_0 + \zeta a = \pi(1-\tau)c_1 + \zeta b = 0.$$

Thus $\ker \hat{T}_\tau$ is described by the relations

$$\zeta = \tfrac{1}{a}\pi\tau c_0 = \tfrac{1}{b}\pi(\tau-1)c_1 \quad \text{and} \quad (c_0, c_1) \in \mathbb{R}(a(\tau-1), b\tau).$$

The part $\tau \in [1,2]$: Here we write $U_\tau f = \sum_{n\in\mathbb{Z}} c_n e_n$. A computation similar to the one above shows that

$$\hat{T}_\tau(f,\zeta) = \sum_{n\in\mathbb{Z}} \pi n c_n e_n + \zeta(a e_0 + b e_1) = 0$$

is equivalent to $c_n = 0$ for all $n \neq 0, 1$ and

$$0 + \zeta a = \pi c_1 + \zeta b = 0.$$

The last equations mean that $\zeta = c_1 = 0$, and therefore

$$\ker \hat{T}_\tau = \mathbb{R} U_\tau^{-1} e_0 \oplus 0 = \mathbb{R}(c_0(\tau)e_0 + c_1(\tau)e_1) \oplus 0$$

with coefficients $c_i(\tau) \in \mathbb{R}$ which according to Lemma 5.8(iii) satisfy $c_i(\tau) > 0$ for $\tau \in (1,2)$ and $i = 0, 1$.

After rescaling we may assume $c_1(1) = b$ and $c_0(2) = a$. Then the kernels $\{\ker \hat{T}_\tau\}_{\tau\in[0,2]}$ are spanned by the continuous section $(f(\tau), \zeta(\tau))$ with

$$f(\tau) = \begin{cases} a(\tau-1)e_0 + b\tau e_1 & \tau \in [0,1], \\ c_0(\tau)e_0 + c_1(\tau)e_1 & \tau \in [1,2], \end{cases} \qquad \zeta(\tau) = \begin{cases} \pi\tau(\tau-1) & \tau \in [0,1], \\ 0 & \tau \in [1,2]. \end{cases}$$

Since $f(0) = -ae_0$ and $f(2) = ae_0$, this shows nontriviality of the kernel bundle $\{\ker \hat{T}_\tau\}_{\tau\in[0,2]}$ over the circle, and thus of the determinant bundle of $\{T_\tau\}_{\tau\in[0,2]}$. In view of isomorphism (116), this concludes the proof of Proposition 5.7. □

Remark 5.9. For E, F as in Theorem 5.4 consider the spaces of operators

$$\mathcal{F}_s^{>\mathfrak{R}}(E,F)^* \subset \mathcal{F}_s^{>\mathfrak{R}}(E,F) \subset \mathcal{F}_s(E,F) \subset \mathcal{F}_0(E,F), \tag{117}$$

where $\mathcal{F}_0(E,F)$ denotes the space of Fredholm operators of index zero. The space $\mathcal{F}_0(E,F)$ is connected and $\pi_1\mathcal{F}_0(E,F) = \mathbb{Z}/2\mathbb{Z}$; see, e.g., [7, Proposition 1.3.5 and Corollary 1.5.10] (here and in the following π_1 is the fundamental group based at the identity). Proposition 5.7 shows that $\pi_1\mathcal{F}_s(E,F)$ and the map $\pi_1\mathcal{F}_s(E,F) \to \pi_1\mathcal{F}_0(E,F)$ induced by the inclusion are nontrivial. On the other hand, $\mathcal{F}_s^{>\mathfrak{R}}(E,F)^*$ has infinitely many connected components given by the numbers $i(T)$ of negative eigenvalues, see Remark 5.6. It would be interesting to determine other homotopy groups of the spaces in (117). For example, if one could show that $\mathcal{F}_s^{>\mathfrak{R}}(E,F)$ is simply connected, then this would provide an alternative proof of the orientability of the determinant bundles in Theorem 5.4.

Remark 5.10. The nonorientability phenomenon described in Proposition 5.7 arises for example in the algebraic count of intersections of nonorientable Lagrangians, or of Reeb chords between nonorientable Legendrians.

5G. ***The Euler number of gradient vector fields.*** Consider now a Hilbert manifold X (an open subset of a Hilbert space will suffice for our purposes) and a Hilbert space bundle $E \to X$ with a continuous bundle inclusion $TX \subset E$ such that $T_xX \subset E_x$ is dense and the inclusion $T_xX \hookrightarrow E_x$ is compact for each $x \in X$. Let E be equipped with a fiberwise Hilbert space inner product $\langle\cdot,\cdot\rangle_E$. In our application, X and E will be maps of Sobolev class H^2 and L^2, respectively.

Let $f : X \to \mathbb{R}$ be a C^1-map which has an *E-gradient of class C^1*, i.e., a C^1-section $\nabla_E f : X \to E$ satisfying

$$Df(x)w = \langle \nabla_E f(x), w\rangle_E \quad \text{for all } w \in T_xX, x \in X.$$

Note that this condition uniquely determines $\nabla_E f$. Taking a derivative of this equation at $x \in \mathrm{Crit}(f)$ we obtain

$$D^2 f(x)(v,w) = \langle D\nabla_E f(x)v, w\rangle_E \quad \text{for all } v, w \in T_xX, x \in \mathrm{Crit}(f),$$

where $D\nabla_E f(x) : T_x X \to E_x$ is the linearization of the section $\nabla_E f$ at x. Hence the second derivative $D^2 f(x)$ is the composition of the continuous linear maps

$$T_x X \otimes T_x X \hookrightarrow T_x X \otimes E_x \xrightarrow{D\nabla_E f(x)\otimes \mathrm{id}} E_x \otimes E_x \xrightarrow{\langle\cdot,\cdot\rangle_E} \mathbb{R}.$$

This implies that $f : X \to \mathbb{R}$ is of class C^2, so its second derivative $D^2 f(x)(v, w)$ is symmetric in v, w (this follows from the usual finite dimensional result applied to the map $(s, t) \mapsto f \circ \exp_x(sv + tw)$). Therefore, the linear map

$$D\nabla_E f(x) : T_x X \to E_x$$

is symmetric in the sense of Section 5. For $\mathfrak{R} \in \mathbb{R}$ we denote by

$$\mathrm{Func}_s^{>\mathfrak{R}}(X) \subset C^2(X, \mathbb{R})$$

the space of C^1-functions $f : X \to \mathbb{R}$ with E-gradient of class C^1 such that $D\nabla_E f(x) \in \mathcal{F}_s^{>\mathfrak{R}}(T_x X, E_x)$ for each $x \in \mathrm{Crit}(f)$. Then Theorem 5.4 implies

Corollary 5.11. *The real line bundle*

$$\mathcal{L} \to \{(f, x) \in \mathrm{Func}_s^{>\mathfrak{R}}(X) \times X \mid Df(x) = 0\}$$

associating to (f, x) *the determinant line* $\det(D\nabla_E f(x))$ *has a canonical orientation.* □

For a critical point x of $f \in \mathrm{Func}_s^{>\mathfrak{R}}(X)$ we denote by $\mathrm{ind}(x) \in \mathbb{N}_0$ the maximal dimension of a subspace of $T_x X$ on which $D^2 f(x)$ is negative definite, or equivalently, the number of negative eigenvalues (with multiplicity) of $D\nabla_E f(x)$. A critical point x is called *nondegenerate* if $D\nabla_E f(x) : T_x X \to E_x$ is invertible.

Theorem 5.12. *To each* $f \in \mathrm{Func}_s^{>\mathfrak{R}}(X)$ *with compact critical point set we can associate its* ***Euler number*** $\chi(\nabla_E f) \in \mathbb{Z}$ *which is uniquely characterized by the following axioms*:

(*Transversality*) *If all critical points of* f *are nondegenerate, then there are only finitely many critical points and*

$$\chi(\nabla_E f) = \sum_{x \in \mathrm{Crit}(f)} (-1)^{\mathrm{ind}(x)}.$$

(*Excision*) *For any open neighborhood* $\widetilde{X} \subset X$ *of* $\mathrm{Crit}(f)$ *we have* $\chi(\nabla_E f) = \chi(\nabla_E f|_{\widetilde{X}})$.

(*Homotopy*) *If* $F : [0, 1] \times X \to \mathbb{R}$ *is a* C^1*-map with* E*-gradient of class* C^1 *and compact critical point set such that* $f_t = F(t, \cdot) \in \mathrm{Func}_s^{>\mathfrak{R}}(X)$ *for each* $t \in [0, 1]$, *then* $\chi(\nabla_E f_0) = \chi(\nabla_E f_1)$.

Proof. The proof consists in applying the proof of [4, Theorem C.1] to the E-gradients, and upgrading it to integer coefficients using the orientations of the determinant bundles. Here are the details.

Consider $f \in \mathrm{Func}_s^{>\mathfrak{R}}(X)$ with compact critical point set. Its E-gradient defines a C^1-Fredholm section $\nabla_E f : X \to E$ of index zero with compact zero set $(\nabla_E f)^{-1}(0) = \mathrm{Crit}(f)$. By Kuiper's theorem, we can trivialize the Hilbert space bundle $E \cong X \times Y$ and thus view $\nabla_E f$ as a C^1-Fredholm map $X \to Y$ to a Hilbert space Y. This map satisfies the hypotheses of [4, Theorem C.1], so it has a mod 2 Euler number uniquely characterized by the analogues of the (Transversality), (Excision) and (Homotopy) axioms. By Corollary 5.11, the determinant line bundle $\det(D\nabla_E f) \to \mathrm{Crit}(f)$ is canonically oriented. This implies that the transversely cut out 0- and 1-dimensional submanifolds in the proof of [4, Theorem C.1] inherit canonical orientations and we get a well-defined integer valued Euler number $\chi(\nabla_E f) \in \mathbb{Z}$; see [3]. The formula in the (Transversality) axiom follows from Remark 5.6. □

Remark 5.13. We have stated Corollary 5.11 and Theorem 5.12 only for gradient vector fields because this covers their applications in this paper. Their proofs actually give more general statements for C^1-sections $S : X \to E$ such that $DS(x) \in \mathcal{F}_s^{>\mathfrak{R}}(T_x X, E_x)$ for each $x \in S^{-1}(0)$. These should be compared to results on orientability of Fredholm maps and their mapping degree in the literature; see, e.g., [8; 1].

6. Frozen planet orbits with instantaneous interaction

In this section we consider the real helium atom with the *instantaneous interaction* between the electrons according to their Coulomb repulsion. *Frozen planet orbits* of this system are smooth maps $q = (q_1, q_2) : S^1 \to \mathbb{R}^2$ satisfying

$$\begin{cases} \ddot{q}_1(t) = -\frac{2}{q_1(t)^2} + \frac{1}{(q_1(t)-q_2(t))^2}, \\ \ddot{q}_2(t) = -\frac{2}{q_2(t)^2} - \frac{1}{(q_1(t)-q_2(t))^2}, \end{cases} \tag{118}$$

as well as the condition

$$q_1(t) > q_2(t) \geq 0 \quad \text{for all } t \in S^1. \tag{119}$$

Thus q_1 describes the outer electron and q_2 the inner electron, where the latter undergoes collisions with the nucleus at the origin. To regularize these collisions, the following setup was introduced in [4]. One considers the space

$$\mathcal{H}_{in} := \{z = (z_1, z_2) \in H^2(S^1, \mathbb{R}^2) \mid \|z_1\| > 0, \|z_2\| > 0, \\ z_1^2(\tau_{z_1}(t)) - z_2^2(\tau_{z_2}(t)) > 0 \text{ for all } t \in S^1\}, \tag{120}$$

where z_i corresponds to the Levi–Civita regularization of q_i and $\tau_{z_i} = t_{z_i}^{-1}$ are the time reparametrizations defined in Section 2. Note that $\mathcal{H}_{in}$ is an open subset of the Hilbert space $H^2(S^1, \mathbb{R}^2)$ and the last condition in its definition corresponds to condition (119). Integrating this condition we see that $\mathcal{H}_{in} \subset \mathcal{H}_{av}$. The *instantaneous interaction functional* $\mathcal{B}_{in} : \mathcal{H}_{in} \to \mathbb{R}$ is defined by

$$\mathcal{B}_{in}(z_1, z_2) := \sum_{i=1}^{2} \left(2\|z_i\|^2 \|z_i'\|^2 + \frac{2}{\|z_i\|^2} \right) - \int_0^1 \frac{1}{z_1^2(\tau_{z_1}(t)) - z_2^2(\tau_{z_2}(t))}\, dt. \quad (121)$$

It is proved in [4] that under the Levi–Civita transformations $q_i(t) = z_i(\tau_i(t))$ critical points of $\mathcal{B}_{in}$ correspond to solutions of (118) and (119).

We interpolate between the mean and instantaneous interaction functionals by

$$\mathcal{B}_r := r\mathcal{B}_{in} + (1-r)\mathcal{B}_{av} : \mathcal{H}_{in} \to \mathbb{R}, \quad r \in [0, 1].$$

It is proved in [4] that for each $r \in [0, 1]$ the L^2-gradient

$$\nabla\mathcal{B}_r = r\nabla\mathcal{B}_{in} + (1-r)\nabla\mathcal{B}_{av} : \mathcal{H}_{in} \to L^2(S^1, \mathbb{R}^2)$$

is a C^1-Fredholm map of index zero.

As in [4, Section 6] and Section 3D above, we remove the symmetries of $\mathcal{B}_r$ by restricting it to a suitable subspace. For $k \in \mathbb{N}_0$ we introduce the Hilbert space of *symmetric loops*

$$\begin{aligned} &H^k_{\mathrm{sym}}(S^1, \mathbb{R}^2) \\ &:= \big\{ z = (z_1, z_2) \in H^k(\mathbb{R}/2\mathbb{Z}, \mathbb{R}^2) \mid z_1(1+\tau) = z_1(\tau) = z_1(1-\tau) \\ &\qquad\qquad \text{and } -z_2(1+\tau) = z_2(\tau) = z_2(1-\tau) \text{ for all } \tau \big\}. \end{aligned}$$

We consider on $H^2_{\mathrm{sym}}(S^1, \mathbb{R}^2)$ the L^2-inner product $\langle z, w\rangle = \sum_{i=1}^2 \int_0^1 z_i(\tau) w_i(\tau) d\tau$ and define the open subset

$$\begin{aligned} X := \big\{ z = (z_1, z_2) \in H^2_{\mathrm{sym}}(S^1, \mathbb{R}^2) \\ \mid \|z_1\| > 0, \|z_2\| > 0, z_1^2(\tau_{z_1}(t)) - z_2^2(\tau_{z_2}(t)) > 0 \text{ for all } t, \\ z_2'(0) > 0,\ z_i(\tau) > 0 \text{ for all } \tau \in (0, 1) \text{ and } i = 1, 2 \big\}. \end{aligned}$$

Note that the first two lines in the definition of X correspond to the conditions for $\mathcal{H}_{in}$, and the third line implies that z_2 is simple and normalized in the sense of Section 3D. We will refer to critical points of $\mathcal{B}_r$ on X and their Levi–Civita transforms as *normalized simple symmetric frozen planet orbits* for $\mathcal{B}_r$.

It is proved in [4] that for each $r \in [0, 1]$ the L^2-gradient $\nabla\mathcal{B}_r : X \to H^0_{\mathrm{sym}}(S^1, \mathbb{R}^2)$ is a C^1-Fredholm map of index zero, and the critical point set $\mathcal{Z}$ of the C^2-map

$$[0, 1] \times X \to \mathbb{R}, \quad (r, z) \mapsto \mathcal{B}_r(z) \quad (122)$$

is compact. We wish to apply Theorem 5.12 to this map. For this, we need the following lemma.

Lemma 6.1. *There exists a constant $\mathfrak{R} < 0$ such that for each $(r, z) \in \mathcal{Z}$ the spectrum of the Hessian $D\nabla\mathcal{B}_r(z)$ is contained in $(\mathfrak{R}, \infty)$.*

Proof. We claim that the Hessian at $(r, z) \in \mathcal{Z}$ has the form

$$D\nabla\mathcal{B}_r(z) = \mathfrak{P}_z + \mathfrak{K}_{r,z} : H^2_{\mathrm{sym}}(S^1, \mathbb{R}^2) \to H^0_{\mathrm{sym}}(S^1, \mathbb{R}^2),$$

where the leading order term is given on $v = (v_1, v_2) \in H^2_{\mathrm{sym}}(S^1, \mathbb{R}^2)$ by

$$\mathfrak{P}_z v = (-4\|z_1\|^2 v_1'', -4\|z_2\|^2 v_2''),$$

and the lower order term $\mathfrak{K}_{r,z}$ extends to a bounded operator $H^1_{\mathrm{sym}}(S^1, \mathbb{R}^2) \to H^0_{\mathrm{sym}}(S^1, \mathbb{R}^2)$ depending continuously on (r, z). In [4] it is shown that the lower order term is compact, but a more careful inspection of the compactness argument reveals that it actually shows the extension to H^1. It is enough to see this for $\mathcal{B}_{av}$ and $\mathcal{B}_{in}$ separately. Below we abbreviate $H^1_{\mathrm{sym}}(S^1, \mathbb{R}^2)$ as H^1 etc.

For $\mathcal{B}_{av}$ this follows from equations (93) and (94) of [4]. For $\mathcal{B}_{in}$ it follows from the analysis of the Hessian of the instantaneous interactions term carried out in Sections A3 to A5 of [4]. The main additions on top of what is already done in [4] are the following:

- It is written in [4] that the map $v \mapsto D_{z_1} v$ defined by formula (99) is a bounded linear operator $L^2 \mapsto H^1$. This map should be used as such when referred to in Sections A3 to A5 of [4].
- The map $v \mapsto Xv$ defined by formula (109) extends as a bounded linear operator $H^1 \mapsto H^1$.

This shows the claim above. Now compactness of $\mathcal{Z}$ implies the existence of uniform constants $\delta, C > 0$ (independent of r, z) such that

$$\langle \mathfrak{P}_z v, v\rangle = \sum_{i=1}^{2} \langle -4\|z_i\|^2 v_i'', v_i\rangle = \sum_{i=1}^{2} 4\|z_i\|^2 \|v_i'\|^2 \geq \delta \|v'\|^2$$

and

$$\|\mathfrak{K}_{r,z} v\| \leq C\|v\|_{H^1} = C(\|v\| + \|v'\|).$$

It follows that

$$\begin{aligned}
\langle D\nabla\mathcal{B}_r(z)v, v\rangle &\geq \delta\|v'\|^2 - C(\|v\| + \|v'\|)\|v\| \\
&= \delta\left(\|v'\| - \frac{C}{2\delta}\|v\|\right)^2 - \left(C + \frac{C^2}{4\delta}\right)\|v\|^2 \\
&\geq -\left(C + \frac{C^2}{4\delta}\right)\|v\|^2,
\end{aligned}$$

and thus

$$\left\langle \left(D\nabla\mathcal{B}_r(z) + C + \frac{C^2}{4\delta} \right) v, v \right\rangle \geq 0.$$

This shows that the spectrum of $D\nabla\mathcal{B}_r(z)$ is contained in $(\mathfrak{R}, \infty)$ for any $\mathfrak{R} < -C - \frac{C^2}{4\delta}$. □

By the preceding discussion and Lemma 6.1, the map (122) satisfies the hypotheses of Theorem 5.12. It follows that the integral Euler number $\chi(\nabla\mathcal{B}_r) \in \mathbb{Z}$ is defined for each $r \in [0, 1]$ and independent of r. Now by Corollary 4.6, the functional $\mathcal{B}_0 = \mathcal{B}_{av}$ has a unique normalized simple symmetric critical point, which is nondegenerate of index zero. Therefore, Theorem 5.12 implies the following result which corresponds to Corollary B from the introduction.

Corollary 6.2. *The integral count of normalized simple symmetric frozen planet orbits equals*

$$\chi(\nabla\mathcal{B}_{in}) = \chi(\nabla\mathcal{B}_{av}) = 1 \in \mathbb{Z}. \qquad \square$$

Appendix: Elliptic integrals

For $n \in \mathbb{N}_0$ we consider the elliptic integrals

$$I_n \colon (-\infty, 1) \to \mathbb{R}, \quad m \mapsto \int_0^1 \frac{\zeta^{2n}}{\sqrt{(1-\zeta^2)(1-m\zeta^2)}}\, d\zeta.$$

At $m = 0$ these integrals can be computed in terms of Euler's beta function and evaluated elementarily, namely

$$\begin{aligned} I_n(0) &= \int_0^1 \frac{\zeta^{2n}}{\sqrt{1-\zeta^2}}\, d\zeta \\ &= \tfrac{1}{2} \int_0^1 \xi^{n-1/2}(1-\xi)^{-1/2}\, d\xi \\ &= \frac{B\left(n+\frac{1}{2}, \frac{1}{2}\right)}{2} \\ &= \frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(n+1)} \\ &= \frac{(2n-1)!!\Gamma\left(\frac{1}{2}\right)^2}{2^{n+1}n!} \\ &= \frac{(2n-1)!!\pi}{2^{n+1}n!}. \end{aligned} \tag{1}$$

Here $(2n-1)!!$ equals $(2n-1)(2n-3)\cdots 1$ for $n \geq 1$ and 1 for $n = 0$. For m different from zero these elliptic integrals can be expressed via elliptic integrals of

the first and second kind. These are defined for $m \in (-\infty, 1)$ by

$$K(m) = \int_0^1 \frac{1}{\sqrt{(1-\zeta^2)(1-m\zeta^2)}}\, d\zeta, \qquad E(m) = \int_0^1 \frac{\sqrt{1-m\zeta^2}}{\sqrt{1-\zeta^2}}\, d\zeta.$$

For n equal to zero or one this is obvious. Indeed, we have just

$$I_0(m) = K(m) \tag{2}$$

for any $m \in (-\infty, 1)$ and if $m \neq 0$, then

$$I_1(m) = \frac{K(m) - E(m)}{m}. \tag{3}$$

For larger n this follows from the recursion formula

$$I_{n+2}(m) = \frac{2(n+1)(m+1)I_{n+1}(m)}{(2n+3)m} - \frac{(2n+1)I_n(m)}{(2n+3)m} \tag{4}$$

which allows to express I_n with the help I_0 and I_1 and consequently in terms of K and E using (2) and (3). The recursion formula (4) follows from

$$\begin{aligned}
0 &= \int_0^1 \frac{d}{d\xi}(\xi^{n+1/2}\sqrt{1-(m+1)\xi+m\xi^2})\, d\xi \\
&= \int_0^1 \left(\left(n+\tfrac{1}{2}\right)\xi^{n-1/2}\sqrt{1-(m+1)\xi+m\xi^2} + \frac{\xi^{n+1/2}(2m\xi-m-1)}{2\sqrt{1-(m+1)\xi+m\xi^2}} \right) d\xi \\
&= \int_0^1 \frac{(2n+1)\xi^{n-1/2}(1-(m+1)\xi+m\xi^2) + \xi^{n+1/2}(2m\xi-m-1)}{2\sqrt{1-(m+1)\xi+m\xi^2}}\, d\xi \\
&= \int_0^1 \frac{(2n+3)m\xi^{n+3/2} - (2n+2)(m+1)\xi^{n+1/2} + (2n+1)\xi^{n-1/2}}{2\sqrt{(1-\xi)(1-m\xi)}}\, d\xi \\
&= \int_0^1 \frac{(2n+3)m\zeta^{2n+3} - (2n+2)(m+1)\zeta^{2n+1} + (2n+1)\zeta^{2n-1}}{\sqrt{(1-\zeta^2)(1-m\zeta^2)}}\, \zeta\, d\zeta \\
&= \int_0^1 \frac{(2n+3)m\zeta^{2(n+2)} - (2n+2)(m+1)\zeta^{2(n+1)} + (2n+1)\zeta^{2n}}{\sqrt{(1-\zeta^2)(1-m\zeta^2)}}\, d\zeta \\
&= (2n+3)mI_{n+2}(m) - 2(n+1)(m+1)I_{n+1}(m) + (2n+1)I_n(m).
\end{aligned}$$

In particular, we obtain

$$I_2(m) = \frac{2(m+1)I_1(m)}{3m} - \frac{I_0(m)}{3m}, \tag{5}$$

which with the help of (2) and (3) we can alternatively write as

$$I_2(m) = \frac{(m+2)K(m) - 2(m+1)E(m)}{3m^2}.$$

The derivatives of the elliptic integrals of the first and second kind can be expressed as a linear combination of them with m-dependent coefficients as follows

$$K'(m) = \frac{E(m) - (1-m)K(m)}{2m(1-m)}, \quad E'(m) = \frac{E(m) - K(m)}{2m}. \tag{6}$$

For E this is a straightforward application of (3). Indeed,

$$E'(m) = -\tfrac{1}{2}\int_0^1 \frac{\zeta^2}{\sqrt{(1-\zeta^2)(1-m\zeta^2)}}\,d\zeta = -\frac{I_1(m)}{2} = \frac{E(m)-K(m)}{2m}.$$

For K this is more involved and follows from the following computation

$$\begin{aligned}
0 &= -m\int_0^1 \frac{d}{d\zeta}\frac{\zeta\sqrt{1-\zeta^2}}{\sqrt{1-m\zeta^2}}\,d\zeta \\
&= -m\int_0^1 \frac{(1-\zeta^2)(1-m\zeta^2) - \zeta^2(1-m\zeta^2) + m\zeta^2(1-\zeta^2)}{\sqrt{(1-\zeta^2)(1-m\zeta^2)^3}}\,d\zeta \\
&= -m\int_0^1 \frac{1-2\zeta^2+m\zeta^4}{\sqrt{(1-\zeta^2)(1-m\zeta^2)^3}}\,d\zeta \\
&= \int_0^1 \frac{m(1-m)\zeta^2 - (1-m\zeta^2)^2 + (1-m)(1-m\zeta^2)}{\sqrt{(1-\zeta^2)(1-m\zeta^2)^3}}\,d\zeta \\
&= 2m(1-m)K'(m) - E(m) + (1-m)K(m).
\end{aligned}$$

It follows from (6) that the quotient of the elliptic integrals E and K satisfies the Riccati differential equation

$$\left(\frac{E}{K}\right)' = -\frac{1}{2m} + \frac{1}{m}\frac{E}{K} - \frac{1}{2m(1-m)}\left(\frac{E}{K}\right)^2, \tag{7}$$

as the following computation shows:

$$\begin{aligned}
\left(\frac{E}{K}\right)' &= \frac{E'K - EK'}{K^2} \\
&= \frac{1}{2m}\frac{E}{K} - \frac{1}{2m} - \frac{1}{2m(1-m)}\left(\frac{E}{K}\right)^2 + \frac{1}{2m}\frac{E}{K} \\
&= -\frac{1}{2m} + \frac{1}{m}\frac{E}{K} - \frac{1}{2m(1-m)}\left(\frac{E}{K}\right)^2.
\end{aligned}$$

Combining (2) and (3) we get

$$\frac{I_1}{I_0} = \frac{1}{m}\left(1 - \frac{E}{K}\right).$$

Differentiating this expression and using (7) we compute

$$\begin{aligned}\left(\frac{I_1}{I_0}\right)' &= \frac{1}{m^2}\frac{E}{K} - \frac{1}{m^2} - \frac{1}{m}\left(\frac{E}{K}\right)' \\ &= \frac{1}{m^2}\frac{E}{K} - \frac{1}{m^2} + \frac{1}{2m^2} - \frac{1}{m^2}\frac{E}{K} + \frac{1}{2m^2(1-m)}\left(\frac{E}{K}\right)^2 \\ &= -\frac{1}{2m^2} + \frac{1}{2m^2(1-m)}\left(1 - m\frac{I_1}{I_0}\right)^2\end{aligned}$$

so that we obtain for the quotient of I_1 and I_0 the Riccati differential equation

$$\left(\frac{I_1}{I_0}\right)' = \frac{1}{2m(1-m)} - \frac{1}{m(1-m)}\frac{I_1}{I_0} + \frac{1}{2(1-m)}\left(\frac{I_1}{I_0}\right)^2. \tag{8}$$

We end this appendix with a technical lemma about the function

$$F\colon (-\infty, 1) \to \mathbb{R}, \quad m \mapsto (2-m)\frac{I_1}{I_0}(m)$$

which we need to prove our nondegeneracy of critical points of collision type for the frozen functional for positive parameters r. From (1) we have

$$I_0(0) = \frac{\pi}{2}, \quad I_1(0) = \frac{\pi}{4},$$

and therefore

$$F(0) = 1.$$

Our technical lemma is the following.

Lemma A.1. *For $m < 0$ we have $F(m) > 1$.*

Proof. We consider the function

$$G\colon (-\infty, 1) \to \mathbb{R}, \quad m \mapsto (2-m)I_1(m)$$

and show that it is strictly decreasing for negative m. For that purpose we compute its derivative

$$\begin{aligned}G'(m) &= -\int_0^1 \frac{\zeta^2}{\sqrt{(1-\zeta^2)(1-m\zeta^2)}}\,d\zeta + \tfrac{1}{2}(2-m)\int_0^1 \frac{\zeta^4}{\sqrt{(1-\zeta^2)(1-m\zeta^2)^3}}\,d\zeta \\ &= \tfrac{1}{2}\int_0^1 \frac{(2-m)\zeta^4 - 2\zeta^2(1-m\zeta^2)}{\sqrt{(1-\zeta^2)(1-m\zeta^2)^3}}\,d\zeta \\ &= \tfrac{1}{2}\int_0^1 \frac{\zeta^2((2+m)\zeta^2 - 2)}{\sqrt{(1-\zeta^2)(1-m\zeta^2)^3}}\,d\zeta.\end{aligned}$$

If m is negative the enumerator is nonpositive, and strictly negative for $\zeta \in (0, 1)$. This shows that

$$G'(m) < 0 \quad \text{for } m < 0.$$

Since $I_0 = K$ is strictly increasing, we see that

$$F = \frac{G}{I_0}$$

as the quotient of a positive strictly decreasing function and a positive strictly increasing function is strictly decreasing for negative values of m. Since $F(0) = 1$, we conclude that

$$F(m) > 1 \quad \text{for } m < 0,$$

which proves the lemma. □

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