Analytical homogenisation of transport processes in evolving porous media

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Abstract

Analytical homogenisation provides effective models for processes in multiscale media based on models at the microscale. For porous media, the pore geometry strongly affects the resulting effective models. We provide an analytical homogenisation method for complex porous media with non-periodic and evolving cavities. For this, we derive a generic framework based on coordinate transformations and homogenisation of the resulting replacement equations. We rigorously justify this approach by showing that the homogenisation of the replacement problems defined in periodically perforated domains is equivalent to the homogenisation of the original problems. A back-transformation of the homogenisation results completes the method and leads to homogenised equations taking into account the local microstructure.

We apply this method for the homogenisation of quasi-stationary and instationary Stokes flow in evolving porous media. This leads to a quasi-stationary Darcy law and a Darcy law with memory for evolving microstructure. Both translate the local microstructure into effective permeability tensors and provide an additional source term for the pressure resulting from the local change in porosity.

In addition, a reaction–diffusion equation with coupled pore evolution is homogenised. The resulting homogenised reactive transport system adjusts the diffusive flux by taking into account the local microstructure and scales the growth rate for the concentration with the changing porosity. The pore evolution and hence the effective transport properties are coupled to the unknown concentration by local upscaled microscopic processes.

Zusammenfassung

Analytische Homogenisierung liefert effektive Modelle für Prozesse in multiskalen Medien, die auf Modellen auf der Mikroskala basieren. In porösen Medien beeinflusst die Porengeometrie stark die resultierenden effektiven Modelle. Wir präsentieren eine analytische Charakterisierung für komplexe poröse Medien mit nicht-periodischer und sich verändernder Porenstruktur. Dazu leiten wir eine generische Methode her, die auf Koordinatentransformationen und der Homogenisierung der resultierenden Ersatzgleichungen beruht. Wir rechtfertigen dieses Vorgehen, indem wir zeigen, dass die Homogenisierung der Ersatzprobleme in den periodisch perforierten Gebieten äquivalent zur Homogenisierung der ursprünglichen Probleme ist. Mittels einer Rücktransformation der Homogenisierungsergebnisse vervollständigen wir diese Methode und erhalten homogenisierte Gleichungen, welche lokale Mikrostrukturen berücksichtigen.

Wir wenden diese Methode zur Homogenisierung von quasistationären und instationären Stokes-Strömungen in sich verändernden porösen Medien an. Dies führt zu einem quasistationären Darcy-Gesetz und einem Darcy-Gesetz mit Gedächtnis für sich verändernde Mikrostruktur. Beide Modelle übersetzen die lokale Mikrostruktur in effektive Permeabilitätstensoren und liefern einen zusätzlichen Quellterm für den Druck, der aus der lokalen Veränderung der Porosität resultiert.

Zudem wird eine Reaktions-Diffusions-Gleichung mit gekoppelter Porenentwicklung homogenisiert. Das resultierende homogenisierte reaktive Transportsystem passt den Diffusionsfluss unter Berücksichtigung der lokalen Mikrostruktur an und skaliert die Wachstumsrate für die Konzentration mit der sich ändernden Porosität. Die Porenevolution und damit die effektiven Transporteigenschaften sind durch lokale, hochskalierte mikroskopische Prozesse an die unbekannte Konzentration gekoppelt.

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Introduction

Motivation

Many processes in the geosciences or biology, such as groundwater flow or reactive transport, take place in the fine pore spaces of large porous media. Typically, these processes can be described by constitutive equations inside the pores on a small scale while the resulting physical effects are observed on a much larger scale. However, these are still strongly affected by the microscopic structure. At this point, homogenisation aims to transfer the models and pore geometries from the microscale into effective models on the macroscale.

From a mathematical point of view, such processes can be described by partial differential equations (PDEs), where the microscopic structure leads to oscillating coefficients or perforated domains. The period of the oscillations or perforations depends on the size of the microstructure and is related to the macroscopic size by means of a small parameter $\varepsilon > 0$. Since this parameter is very small, the limit $\varepsilon \to 0$ of the PDEs often provides a suitable approximation of the original problem for small positive ε. The advantage of this limit lies typically in the fact that the resulting PDEs have coefficients without microscopic oscillations and the limit processes averages the microstructure in a physically meaningful way.

In order to apply this limit process, one has to provide some ε-scaled version of the equations. This can be achieved by employing periodicity assumptions or the concepts of stationarity and ergodicity in a stochastic setting. However, these assumptions are too restrictive for many materials. In particular, if the process interacts with the heterogeneous structure local variations of the domain can occur, which have to be taken into account. In the case that the microstructure is prescribed by oscillating coefficients, tools like two-scale convergence can take into account local variations by weakening the assumption of periodicity to strong two-scale convergence while still allowing rigorous homogenisation. If the microscopic heterogeneity is given by the domain where the PDEs are defined, the homogenisation becomes far more complicated and there was no complete framework available that can handle this case. Such local geometries, as for instance, cavity constrictions in a porous medium, can have a significant effect. This can easily be observed for fluid flow in a porous medium, where local clogging along a cross section can even stop fluid flow completely. Furthermore, in many applications, the pore space undergoes an a-priori unknown evolution over time, which complicates the investigation. Typical examples of such coupled systems are reactive transport problems, where chemical reactions lead to dissolution and precipitation of the porous matrix, or transport processes in biological tissues, where biofilm formation affects the cavities.

Goal and main contribution of this work

This work provides an analytical method for the rigorous homogenisation of processes in locally differently perforated porous media. In particular, this method is capable of dealing with time-dependent microstructures. It is based on the transformation to periodic reference structures and is stated in a purely asymptotic framework. Therefore, it allows for pure compactness arguments, which even allow for the homogenisation of free boundary value problems on the microscale.

We use this method to study flow in porous media with an evolving microstructure. By homogenising the quasi-stationary and instationary Stokes equations in a locally evolving porous medium, we derive two Darcy laws for evolving microstructure. The limit results not only account for the locally varying microstructure through the permeability tensor, but also incorporate a source or sink term for the pressure resulting from the local change in porosity. Furthermore, we homogenise a reaction–diffusion equation coupled to the evolution of the microstructure. This provides an effective description of a reactive transport process that is coupled bidirectionally with the local pore structure.

From a mathematical point of view, the study of the locally different (evolving) microstructure is approached as follows: If the microscopic heterogeneity is given by some oscillating coefficient, tools such as two-scale convergence can weaken the assumption of strict periodicity in strong two-scale convergence, allowing local variations of the microstructure. We translate the local non-periodicity of the pore structure into this setting. Therefore, we use a periodically perforated reference domain and assume that it can be transformed into the locally periodically perforated domain by changing the coordinates. Transforming the PDEs from the actual domain into this surrogate domain leads to PDEs that includes transformation quantities. Thus, the non-periodicity of the geometry is translated into local periodicity of functions, which can be handled by two-scale convergence and allows the limit process $\varepsilon \to 0$. Homogenisation in the surrogate domain leads to a two-scale limit problem defined in a cylindrical two-scale domain. Transforming back the equations on the reference cell for each macroscopic point provides transformation-independent two-scale limit equations defined in a non-cylindrical two-scale domain and subsequently to a homogenised equation. This approach can be used for time-dependent microstructures by transforming the geometry for each point in time, and this is illustrated in Figure 1.

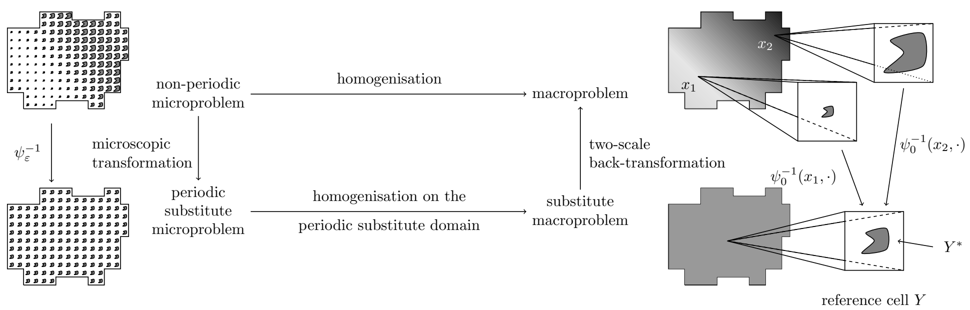

Figure 1.: Homogenisation by transformation on a periodic substitute domain

To make sense of this approach, it is essential not only that we can pass to the limit $\varepsilon \to 0$ in the substitute domain, but also that the limit process for homogenisation commutes with the ε-scaled transformations and the limit transformation for the upscaled microstructure. To formulate this commutativity property mathematically, let Ω_ε be a family of ε-scaled periodically perforated substitute domains and $\Omega_\varepsilon(t)$ be the family of actual ε-scaled domains with a family of coordinate transformations $\psi_\varepsilon(t,\cdot)\colon \Omega_\varepsilon \to \Omega_\varepsilon(t)$. Let $u_\varepsilon\colon \Omega_\varepsilon \to \mathbb{R}$ and $\hat{u}_\varepsilon\colon \Omega_\varepsilon(t) \to \mathbb{R}$ represent a function in the two different coordinate systems, i.e. $\hat{u}_\varepsilon(x) = u_\varepsilon(\psi_\varepsilon(t,x))$ for all $x \in \Omega_\varepsilon$. Then we have to make sure that the two-scale convergences of $\hat{u}_\varepsilon$ and u_ε are equivalent, i.e.

$$u_\varepsilon(x) \rightarrowtail u_0(x,y) \qquad \text{if and only if} \qquad \hat{u}_\varepsilon(x) \rightarrowtail \hat{u}_0(x,y)$$

and we have to identify the limits by $\hat{u}_0(x,y) = u_0(x,\psi_0(t,x,y))$ for a family of coordinate transformations $\psi_0(t,x,\cdot)\colon Y^* \to Y^*(t,x)$. The goal is to provide a framework, as general as possible, in which this commutativity is fulfilled and the transformation quantities resulting from the coordinate transformations are well-manageable in the limit process $\varepsilon \to 0$.

The derivation of such a framework requires not only the transformation of the functions themselves, but also of their gradients. These derivatives play a critical role in the homogenisation and require some additional correctors. This is also reflected in the two-scale transformation approach, where the two-scale limit of the gradients and the corresponding correctors do not follow the same transformation rules as the functions themselves, but require some additional corrections. Nevertheless, we can take this two-scale limit behaviour into account and formulate this transformation approach as a completely transformation-independent toolbox. We present this method by applying it to the homogenisation of a diffusion equation in a non-periodically perforated domain.

We then use this method to homogenise the quasi-stationary and instationary Stokes flow in an evolving perforated domain. Due to the high complexity of the Stokes equations and their homogenisation, we have to derive additional two-scale analysis results to make the transformation approach applicable. For example, the transformation of the ε-scaled Stokes equations leads to a symmetric gradient that is multiplied by transformation coefficients, which requires new Korn inequalities for this two-scale transformation framework. Furthermore, we derive transformation results for the divergence-type two-scale correctors which arise for the two-scale limit of the pressure. By providing solutions to all the difficulties arising from the local non-periodic microstructure, we are able to derive a quasi-stationary Darcy equation as the limit for the quasi-stationary Stokes flow as well as a Darcy equation with memory for the instationary Stokes flow. In particular, these results also hold for a locally periodic microstructure that does not evolve in time.

Moreover, we employ this transformation framework to homogenise a reaction–diffusion equation coupled to the evolution of the microstructure. Due to this a-priori unknown domain evolution, we have to apply a generic coordinate transformation which is coupled to the solution of the diffusion equation. Then, the non-linear reaction–diffusion process with free boundary becomes a highly non-linear system of equations in the transformed substitute domain. At this point, it becomes necessary that the method relies entirely on asymptotic properties, which can be verified purely by compactness results and do not require that the transformation can be written as ε-scaled power series. The homogenisation result is an effective reaction transport system, which is coupled to the local upscaled microstructure.

From an application point of view, these homogenisation results provide new insights into the effective flux in complex porous media. They enable better predictions for processes in locally periodic and evolving microstructures and also for reactive transport, which affects the pore structure.

From a mathematical perspective, we provide a powerful tool that allows homogenisation for complex microstructures. The generality and strength of the method is emphasised by showing its ability to homogenise the Stokes equations and free boundary value problems.

Overview of the literature

Analytical homogenisation methods

We provide only a brief overview of the general theory of periodic homogenisation and refer to [BLP78, ZKO94, Hor97, Pan97, CD99, MK06, Tar09] for more details.

In order to understand the asymptotic behaviour of parametrised PDEs with some fine-scale parameter ε tending to zero, several types of convergence have been introduced. In [DG75, Gio84], De Giorgi introduced Γ-convergence, which provides an abstract notion of convergence for functionals, which goes beyond the application of homogenisation. Spagnolo introduced the notion of G-convergence for the study of second order symmetric elliptic operators in [Spa68]. He defines the convergence of the operators in terms of the solutions of the corresponding PDEs and provides a compactness result. To overcome the restriction to symmetric operators, Murat and Tartar defined the convergence of the operators not only by the convergence of the solution but also by the convergence of the associated fluxes [MT77, MT97a, MT97b]. This so-called H-convergence avoids the instability that occurs for non-symmetric operators in the notion of G-convergence. Both notions of G- and H-convergence do not require any periodicity assumptions.

The energy method of Tartar [MT77], also known as the oscillating test function method, provides another approach to the homogenisation of partial differential equations. It uses oscillating test functions to pass to the limit. This approach corresponds closely to the compensated compactness results of Murat [Mur78] and Tartar [Tar79] and is presented in detail in [CD99].

The two-scale convergence method is devoted to periodic homogenisation. It was introduced by Nguetseng [Ngu89] and Allaire [All92a] and is very efficient due to its specific application. Roughly speaking, it rigorously justifies the first terms in the two-scale asymptotic expansion ansatz. This avoids the technical a-posteriori convergence analysis that is otherwise required to justify the asymptotic expansion approach. Two-scale convergence provides not only compactness results but also a simple approach to the derivation of the limit equations. It also has the advantage of allowing the consideration of systems of equations as well as slow diffusion processes, i.e. coefficients which degenerate for $\varepsilon \to 0$. The periodic unfolding method of [CDG02, CDG18] uses the so-called unfolding operator to transform the homogenisation problem into a convergence problem in a fixed space. It is not only a useful tool on its own, but also allows the translation of two-scale convergence into classical convergence in Lebesgue spaces [Vis06], which provides a powerful notion of strong two-scale convergence. Similar operators are also presented under the names of dilation operator [ADH90] and periodic modulation [BLM96] in the context of homogenisation. Furthermore, the concept of two-scale convergence is extended to a stochastic setting in [BMW94, ZP06, HNV22].

Homogenisation in locally periodically perforated and evolving domains

The above-mentioned analytical homogenisation tools strongly distinguish between microstructure given by some oscillating coefficients or represented by perforated domains. For instance, tools like the two-scale convergence can deal with locally periodic coefficients but so far require strict periodicity in the case of perforated domains. Nevertheless, such non-strictly periodic perforated domains are highly relevant for many applications. In particular, if local reaction processes affect the pore geometry, it becomes unreasonable to assume that the pore structure remains periodic over the whole domain. Since the pore geometry heavily affects the effective macroscopic behaviour, these local microscopic changes must be taken into account.

A locally periodic microstructure can be modelled in several ways. In [CP99], Chechkin and Piatnitski described the microstructure in terms of the level sets of a smooth function $\phi(x, \frac{x}{\varepsilon})$ and homogenised a Poisson equation by formal asymptotic expansion, which they subsequently justified by estimates on the residual. This description of the microstructure is extended to time-dependent level set functions in [vN08] and applied to the upscaling of further problems in [RvNFK12, SK17]. In this case of time-dependent level set functions, the level set functions were a-priori unknown and coupled to the process. However, these later works only consider a formal upscaling via the two-scale asymptotic expansion and without proving the convergence for $\varepsilon \to 0$. In [FY20], such a convergence proof was presented for a time-dependent microstructure, which is described by an a-priori given level set function of the form $\phi(t, x, \frac{x}{\varepsilon})$.

Blanc and Wolf [BW22, Wol23, Wol22] modelled a non-periodic microstructure by a local perturbation of periodically arranged isolated holes. For the limit $\varepsilon \to 0$, the perturbation is localised so that it does not affect the first order but only the second order of approximation. The limit equations are derived by two-scale asymptotic expansion and then justified by rigorous convergence estimates afterwards. In [MP94], the locally periodic microstructure was defined by means of a characteristic function on the reference cell which varies smoothly with respect to the macroscopic domain. The homogenisation of an elliptic problem was done by means of two-scale convergence. However, the rigorous homogenisation in all these approaches restricts the geometry to the case of isolated obstacles.

In [Pta13, Pta15], mesoscopically scaled patches with different but strictly periodic microstructures are used to model locally different microstructures. In particular, this approach allows for connected obstacles. The homogenisation is done by extending the concept of two-scale convergence and the periodic unfolding operator to this local structure.

Another approach to studying processes in time-dependent locally periodic microstructures was proposed by Peter in [Pet07b]. Instead of homogenising the actual problem, the equations are transformed into a periodically perforated substitute domain and homogenised there. This transformation translates the local periodicity of the domain into coefficients of the PDE, which can be handled by two-scale convergence. However, it remained open whether this approach is equivalent to the homogenisation of the actual problem, i.e. whether the transformation and the homogenisation commute in the sense of Figure 1. Furthermore, the question of how to back-transform the limit problem was only partially answered since the presented back-transformation of the limit equations yields transformation-dependent equations. Nevertheless, this approach allows the investigation of a new class of highly application-relevant problems and found applications in the homogenisation of thermoelasticity [EM17] or (advection–)reaction–diffusion processes [Pet07a, Pet09, Ede19, GNRP21] in the sense that the transformed equations were homogenised. This approach is based on the suitable transformation mappings ψ_ε and ψ_0 (see also Figure 1). A first example for explicitly constructed transformations leading to strongly two-scale convergent coefficients and allowing the homogenisation in the substitute domain is given in [Pet07a]. Moreover, in [Ede19], transformations ψ_ε for prescribed normal velocity of the microscopic interfaces are constructed. The case of two connected domains is presented in [Wie19].

In [Wie23], the transformation approach of [Pet07b] was rigorously justified by presenting a framework in which Figure 1 commutes. Indeed, this framework does not require substantially more assumptions about the transformations and domains than are already required for the homogenisation in the transformed coordinates. In this sense this framework is optimal. Moreover, in [Wie23], results are derived for the back-transformation of the homogenised correctors, which give new transformation-independent homogenisation results. These results are presented in Chapter 2. Furthermore, this approach was used to homogenise the quasi-stationary Stokes equations for evolving microstructure in [WP24]. The structure of the Stokes equations differs from those equations previously studied by this transformation approach. Therefore, several new results and extensions were necessary, which we elaborate in Chapter 2, following [WP24]. Moreover, this transformation is also able to deal with the instationary Stokes equations, which we will also see in Chapter 3.

In [GP23, WP23], this transformation approach was used for the homogenisation of two similar reaction–diffusion problems in porous media, where the evolution of the microstructure is a-priori not given but coupled to the unknown itself. The microstructure is modelled by spheres with evolving radii. The evolution of the radii is described by ordinary differential equations depending on the solution of the reaction–diffusion problem. This leads to a free boundary value problem at the microscale. We present the homogenisation of this coupled problem in Chapter 4 and provide a more detailed discussion of the differences between the approaches of [GP23] and [WP23].

Homogenisation of Stokes flow

Based on the results of experiments, Darcy presented a fundamental principle of fluid mechanics in porous media [Dar56]. Darcy's law states that the rate of flow through porous media is directly proportional to the the negative hydraulic gradient and the permeability coefficient, and inversely proportional to the viscosity of the fluid. It can be derived mathematically by means of homogenising the (Navier–)Stokes equations in a perforated domain. In particular, this mathematical approach provides a better understanding of the effects of the microscopic geometry on the permeability coefficient. First upscaling approaches used formal two-scale asymptotic expansion and are presented in [Kel80, Lio81, SP80].

The main difficulty in the rigorous homogenisation of the Stokes equations lies in the uniform a-priori estimate of the pressure. Tartar overcame this problem by constructing a restriction operator [Tar80] and provided a rigorous proof of the homogenisation. This operator was extended by Allaire to allow the homogenisation in the case where the solid space of the porous medium is also connected [All89]. A modification of this restriction operator [LA90] allowed the consideration of different boundary conditions at the pore interfaces. Furthermore, an extension of the restriction operator from H^1 to $W^{1,p}$ integrability enables the homogenisation of the Navier–Stokes equations [Mik91]. A different approach for the derivation of the a-priori estimates was presented by Zhikov in [Zhi94], who constructed a family of ε-scaled operators, which are right-inverses of the divergence operator. In particular, these operators enable a construction of a restriction operator in the sense of [Tar80] with weaker estimates, which are still sufficient in order to show the strong convergence of the pressure [Mik00]. This construction of these right-inverse divergence operators used the extension operators of [ACDP92]. A different construction for such operators, which does not require any extension result, is derived in [Wie19]. In particular, such ε-scaled right-inverse operators become useful for the homogenisation of the compressible (Navier–)Stokes equations [Mas02] or in our case, where the domain evolution motivates inhomogeneous Dirichlet boundary conditions leading to an inhomogeneous divergence condition. While these works considered Dirichlet or periodic boundary conditions at the boundary of the macroscopic domain, the case of normal stress boundary conditions is considered in [FMW17].

The upscaling of the instationary Stokes equation was first studied by formal two-scale asymptotic expansion in [Lio81] and rigorous homogenisation results are proven in [All92b] and [Mik94]. The result is a Darcy law with memory, which is an integro-differential equation and can be approximated for large times and constant force by the classical Darcy law [Mik94]. However, the ε-scaling of the viscosity becomes crucial and, for different scaling, the time derivative can vanish during the homogenisation leading directly to the stationary Darcy equation [Mik91].

The above-mentioned works considered the case where the porosity remains constant for $\varepsilon \to 0$. For the case of isolated obstacles it is possible to scale the obstacles asymptotically smaller than the periodicity size ε, i.e. the obstacles are of size ε^α for $\alpha > 1$ [All91b, All90b, All90a]. The homogenisation result depends on the exact value of α and leads for asymptotically small obstacles to the Stokes equation itself, for critically scaled obstacles to the Brinkman equation and for asymptotically large obstacles to a Darcy law. The permeability for the Darcy law differs from the strictly periodic case [All91a] and the additional Brinkman term, which arises for the critical scaling, corresponds to the “strange term coming from nowhere” of [CM97].

Homogenization of non-linear problems

For example, non-linearities occur in reaction processes. Strong two-scale compactness results are therefore useful for homogenisation. The derivation of such compactness results is fundamentally different for slow and fast processes occurring in highly heterogeneous media. For fast processes, spatial variations typically occur only on the macroscopic scale, giving rise to uniformly bounded gradients and making the Rellich–Kondrachov theorem applicable. Using an extension operator [ACDP92] or the unfolding method, this approach is also applicable to processes defined in periodically perforated domains and non-linear interface conditions [DN15]. For time-dependent functions, a uniform control with respect to time becomes helpful. Since spatially oscillating coefficients do not yield oscillations with respect to time, classical approaches such as the Aubin–Lions lemma [Aub63, Lio69] often become applicable. For perforated domains one can try again to use the extension to the whole domain. However, the extension operator of [ACDP92] controls only the L^p- and $W^{1,p}$-norm but not the $W^{1,p\,\prime}$-norm and, thus, the Aubin–Lions lemma is not directly applicable if the derivatives are only controlled in the $W^{1,p\,\prime}$-norm with respect to space. Nevertheless, in [MZ11] this problem is circumvented and a compactness result is given. A more elegant argument is given in [GNRK16b] (see also [GNRK16a, GNRK17, Gah23]), employing the Simon–Kolmogorov compactness result [Sim87]. In particular, this argument can be used even if the weak time derivative cannot be uniformly controlled in any space [WP23].

In the case of highly heterogeneous media, variations occur even at the microscopic scale. Thus, the macroscopic variable asymptotically becomes a parameter and only the microstructure can be controlled via the gradient. In the case that the non-linearity is given by the gradient of a λ-convex potential a rigorous homogenisation result was derived in [HJM94]. For a more general non-linearity a convergence result was derived by additional error estimates in [MRT14]. Under additional control of the macroscopic variation of the coefficients a generalisation of the Simon–Kolmogorov compactness result for $\mathbb{R}^n$ [GNR16] was applied to the homogenisation of non-linear boundary conditions [GNRP21].

Outline of the work

This thesis is structured as follows: In Chapter 1, we recall the notion of two-scale convergence and its fundamental compactness results. We also use the periodic unfolding method to obtain some additional results on two-scale calculus.

Chapter 2 presents an analytical framework for the homogenisation in locally periodically perforated domains, which is based on [Wie23, D. Wiedemann, *The two-scale-transformation method*, Asymptotic Analysis **131** (2023), 59–82]. The perforated domains under consideration are characterised by transformations onto periodically perforated domains in Section 2.1. In Section 2.2, we show that these ε-scaled transformations commute with the two-scale convergence. In Section 2.3, we employ these results to homogenise an elliptic problem in a non-periodically perforated domain. We formulate this transformation approach for time-dependent domains in Section 2.4.

Chapter 3 is devoted to the homogenisation of Stokes flow in locally periodically perforated evolving domains employing the transformation approach of Chapter 2. Section 3.1 is based on [WP24, D. Wiedemann and M. A. Peter, *Homogenisation of the Stokes equations for evolving microstructure*, Journal of Differential Equations, **396** (2024), 172–209] and considers the homogenisation of quasi-stationary Stokes flow leading to a Darcy law for locally periodically evolving microstructures. In Section 3.2, we consider the homogenisation of the instationary Stokes flow leading to memory effects in the resulting Darcy equation.

Chapter 4 is based on [WP23, D. Wiedemann and M. A. Peter *Homogenisation of local colloid evolution induced by reaction and diffusion*, Nonlinear Analysis **227** (2023), 113168] and deals with the homogenisation of a reaction–diffusion process with a free boundary, which is coupled with the unknown concentration. We present the microscopic model in Section 4.1. Then, we transform the problem by a generic transformation onto a periodically perforated reference domain in Section 4.2. In Section 4.3, we show the existence and uniqueness of a solution as well as uniform a-priori estimates. In Section 4.4, we pass to the homogenisation limit $\varepsilon \to 0$ in the substitute domain and transform the resulting limit equations back to a upscaled version of the actual and derive a transformation-independent homogenised equation.

In Chapter 5, we draw some conclusions and provide a brief outlook on possible future research.

Appendix A gives an existence result for time-dependent differential–algebraic equations, which we use in Section 3.2 for showing the existence and uniqueness of a solution of the instationary Stokes equations in the substitute coordinate system and for deriving a-priori estimates.

Chapter 1.

Two-scale convergence and periodic unfolding

In this chapter, we recap the notion of two-scale convergence and some of its fundamental properties. Moreover, we employ the unfolding operator $\mathcal{T}_\varepsilon$ in order to translate two-scale convergence into classical convergence in L^p spaces. This allows us to transfer several useful results from the L^p theory to the concept of two-scale convergence. The basic results are well-known and some extensions stem from [Wie23, D. Wiedemann, *The two-scale-transformation method*, Asymptotic Analysis **131** (2023), 59–82].

1.1. Basic results on two-scale convergence and the unfolding operator

Two-scale convergence is a functional analytical tool, which enables rigorous homogenisation for differential equations with periodic structures. The notion of *two-scale convergence* was introduced in [All92a] and is based on some fundamental convergence results for oscillating functionals in [Ngu89]. We refer also to [LNW02] for more detailed proofs and generalisations.

Two-scale convergence provides information on the asymptotic behaviour of a parameterised sequence of functions $(u_{\varepsilon_m})_{m\in\mathbb{N}}$ in $L^2(\Omega)$ for an open set $\Omega \subset \mathbb{R}^n$ for $n \geqslant 1$, where $(\varepsilon_m)_{m\in\mathbb{N}}$ is a sequence of strictly positive parameters which tends to zero. For the sake of simplifying notation, we omit the indices and write $\varepsilon = \varepsilon_n$ as well as $u_\varepsilon = u_{\varepsilon_n}$. We call $(\varepsilon_m)_{m\in\mathbb{N}}$ and $(u_{\varepsilon_m})_{m\in\mathbb{N}}$ a sequence ε and a sequence u_ε, respectively. We use the expression "for all $\varepsilon > 0$" in order to refer to all elements of the sequence ε. Moreover, for a subsequence of ε or u_ε, we use the same notation without adding any subscript.

In the following, let $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ be open and $Y = (0,1)^n$. Moreover, let C be a generic constant which is independent of ε.

Definition 1.1 (Distributional two-scale convergence)**.** *Let $1 \leqslant p < \infty$. A sequence u_ε in $L^p(\Omega)$ is said to two-scale converge distributionally to a limit $u_0 \in L^p(\Omega \times Y)$ if*

$$\lim_{\varepsilon\to 0} \int_\Omega u_\varepsilon(x)\varphi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x = \int_\Omega \int_Y u_0(x,y)\varphi(x,y)\,\mathrm{d}y\,\mathrm{d}x,$$

for any function $\varphi \in D(\Omega; C^\infty_\#(Y))$. We write $u_\varepsilon \xrightarrow{D} u_0$.

We write $u_\varepsilon(x) \xrightarrow{D} u_0(x,y)$, if we want to emphasize the functions' dependency on the variables.

Enlarging the space of test function yields the notion of weak two-scale convergence, which we often call two-scale convergence.

Definition 1.2 (Weak two-scale convergence)**.** *Let $1 \leqslant p < \infty$. A sequence u_ε in $L^p(\Omega)$ is said to two-scale converge weakly to a limit $u_0 \in L^p(\Omega \times Y)$ if*

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x) \varphi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x = \int_\Omega \int_Y u_0(x,y)\varphi(x,y)\,\mathrm{d}y\,\mathrm{d}x,$$

for any function $\varphi \in L^q(\Omega; C_\#(Y))$, where q is such that $\frac{1}{p} + \frac{1}{q} = 1$. We write $u_\varepsilon \xrightarrow{p} u_0$.

For bounded sequences and $p \in (1,\infty)$, the distributional two-scale convergence is equivalent to the weak two-scale convergence.

Lemma 1.3. *Let $p \in (1,\infty)$ and u_ε a bounded sequence in $L^p(\Omega)$ and $u_0 \in L^p(\Omega \times Y)$. Then, $u_\varepsilon \xrightarrow{D} u_0$ if and only if $u_\varepsilon \xrightarrow{p} u_0$.*

Proof. A proof is given in [LNW02, Proposition 1]. □

Typically, the two-scale convergence for a sequence is obtained by compactness arguments from the boundedness of the sequence. Due to the boundedness of the sequence, the distributional and weak two-scale convergence are equivalent, which causes some inconsistent usage of the term two-scale convergence in the literature, referring sometimes to distributional or weak two-scale convergence.

In the following lemma, we see that weak two-scale convergence implies also weak convergence.

Lemma 1.4. *Let $1 \leqslant p < \infty$ and let u_ε be a sequence in $L^p(\Omega)$, which two-scale converges to u_0. Then, u_ε is bounded and u_ε converges weakly in $L^p(\Omega)$ to u for $u(x) := \int_Y u_0(x,y)\,\mathrm{d}y$.*

Proof. Lemma 1.4 can be shown by choosing the test functions constant with respect to the y-variable. For a detailed proof see [LNW02, Theorem 6]. □

The notion of two-scale convergence is justified by the following compactness results.

Theorem 1.5. *Let $1 < p < \infty$. For every bounded sequence u_ε in $L^p(\Omega)$ there exist a subsequence u_ε and a $u_0 \in L^p(\Omega \times Y)$ such that this subsequence two-scale converges to u_0, i.e. $u_\varepsilon \xrightarrow{p} u_0$.*

Proof. For the case $p = 2$, the first proof of Theorem 1.5 was presented in [Ngu89] and a simpler proof is presented in [All92a]. The general case $1 < p < \infty$ is shown in [LNW02, Theorem 7]. □

Due to Lemma 1.4, it cannot be expected that Theorem 1.5 holds for $p = 1$. Nevertheless, the notion of two-scale convergence can be extended to measures [Ama98], in order to deal with the case $p = 1$ similarly as in classical L^p theory.

We remember that, in uniformly convex Banach spaces, the strong convergence of a sequence is equivalent to the weak convergence together with the convergence of the norms. This motivates the following definition for strong two-scale convergence.

Definition 1.6 (Strong two-scale convergence). *Let $1 < p < \infty$. A sequence u_ε in $L^p(\Omega)$ is said to two-scale converge strongly to a limit $u_0 \in L^p(\Omega \times Y)$ if $u_\varepsilon \xrightarrow{p} u_0$ and*

$$\|u_\varepsilon\|_{L^p(\Omega)} \to \|u_0\|_{L^p(\Omega\times Y)}.$$

We write $u_\varepsilon \overset{p}{\twoheadrightarrow} u_0$.

In particular, the two-scale test functions strongly two-scale converge.

Lemma 1.7. *Let $1 \leqslant p < \infty$ and $B_p(\Omega;Y)$ denote one of the spaces $L^p(\Omega;C_\#(Y))$, $L^p_\#(Y;C(\overline{\Omega}))$, $C(\overline{\Omega};C_\#(Y))$. Then, for every $u \in B_p(\Omega,Y)$, it holds $u(\cdot,\frac{\cdot}{\varepsilon}) \in L^p(\Omega)$ and*

$$\|u(\cdot,\tfrac{\cdot}{\varepsilon})\|_{L^p(\Omega)} \to \|u\|_{L^p(\Omega\times Y)}.$$

Moreover, for $1 < p < \infty$, it holds $u(\cdot,\frac{\cdot}{\varepsilon}) \overset{p}{\twoheadrightarrow} u$.

Proof. The measurability of $\varphi(\cdot,\frac{\cdot}{\varepsilon})$ and the convergence $\|\varphi(\cdot,\frac{\cdot}{\varepsilon})\|_{L^p(\Omega)} \to \|\varphi\|_{L^p(\Omega\times Y)}$ are shown in [LNW02, Theorem 3].

In remains to show the weak two-scale convergence of $\varphi(\cdot,\frac{\cdot}{\varepsilon})$ in order to conclude its strong two-scale convergence. Let $1 \leqslant p < \infty$, $u \in B_p(\Omega;Y)$ and $\varphi \in D(\Omega;C^\infty_\#(Y))$, then $u\varphi \in B_1(\Omega;Y)$. Therefore, we can pass to the limit in the distributional two-scale convergence for the positive and negative parts of $u\varphi$ and, thus, for the whole sequence. Then, Lemma 1.3 provides the weak two-scale convergence. ☐

If the two-scale limit does not depend on the y-variable, the strong two-scale convergence can be improved to the classical convergence in $L^p(\Omega)$.

Lemma 1.8. *Let $1 < p < \infty$ and u_ε a sequence in $L^p(\Omega)$ which two-scale converges strongly to $u_0 \in L^p(\Omega)$. Then, u_ε converges strongly to u_0 in $L^p(\Omega)$.*

Proof. First, we note that $\|u_\varepsilon\|_{L^p(\Omega)} \to \|u_0\|_{L^p(\Omega\times Y)} = \|u_0\|_{L^p(\Omega)}$. Moreover, by Lemma 1.4, we obtain the weak convergence $u_\varepsilon \rightharpoonup u_0$ in $L^p(\Omega)$. Since $L^p(\Omega)$ is uniformly convex for $1 < p < \infty$, this implies the strong convergence in $L^p(\Omega)$. ☐

Using the unfolding operator, we can translate the notion of two-scale convergence into convergence in L^p spaces. Thus, we can derive more subtle results on two-scale convergence. Nevertheless, the unfolding operator can be used as a homogenisation tool on its own [CDG18]. In order to define the unfolding operator, we introduce the following notation.

Notation 1.9. *Let* $\Omega \subset \mathbb{R}^n$ *and* $x = \sum_{i=1}^{n} x_i e_i \in \mathbb{R}^n$*, where* e_i *denotes the Euclidean unit vectors. We define*

$$[x]_Y := \sum_{i=1}^{n} \lfloor x_i \rfloor e_i \ , \quad \{x\}_Y := x - [x]_Y \ , \quad [x]_{\varepsilon,Y} := \varepsilon \left[\tfrac{x}{\varepsilon}\right]_Y \ , \quad \{x\}_{\varepsilon,Y} := \left\{\tfrac{x}{\varepsilon}\right\}_Y$$
$$I_\varepsilon := \{k \in \mathbb{Z}^n \mid \varepsilon k + \varepsilon Y \subset \overline{\Omega}\} \ , \quad \tilde{\Omega}_\varepsilon := \operatorname{int}\Big(\bigcup_{k \in I_\varepsilon} \varepsilon k + \varepsilon \overline{Y}\Big) \ , \quad \Lambda_\varepsilon = \Omega \backslash \tilde{\Omega}_\varepsilon.$$

Definition 1.10. *Let* $1 \leqslant p \leqslant \infty$*. The unfolding operator* $\mathcal{T}_\varepsilon : L^p(\Omega) \to L^p(\Omega \times Y)$ *is defined by*

$$\mathcal{T}_\varepsilon(\varphi)(x,y) := \begin{cases} \varphi([x]_{\varepsilon,Y} + \varepsilon y) & \text{for a.e. } (x,y) \in \tilde{\Omega}_\varepsilon \times Y, \\ \varphi(x) & \text{for a.e. } (x,y) \in \Lambda_\varepsilon \times Y. \end{cases}$$

Note that we define the unfolding operator by $\mathcal{T}_\varepsilon(\varphi)(x,y) = \varphi(x)$ on $\Lambda_\varepsilon \times Y$, i.e. on the cells that are not completely included in Ω, and not by $\mathcal{T}_\varepsilon(\varphi)(x,y) = 0$ as in [CDG08] or [CDG18]. By this slight modification, $\mathcal{T}_\varepsilon$ becomes isometric (cf. Theorem 1.11). Thus, we cannot only translate between the two-scale convergence of u_ε and the weak convergence of $\mathcal{T}_\varepsilon(u_\varepsilon)$ in $L^p(\Omega \times Y)$, as shown in [CDG08], but we can also translate between the strong two-scale convergence and the strong convergence in $L^p(\Omega \times Y)$.

Theorem 1.11. *Let* $1 \leqslant p \leqslant \infty$*. For every* $\varphi \in L^1(\Omega)$ *and* $\psi \in L^p(\Omega)$*, it holds*

$$\int_\Omega \int_Y \mathcal{T}_\varepsilon(\varphi)(x,y) \,\mathrm{d}y \,\mathrm{d}x = \int_\Omega \varphi(x) \,\mathrm{d}x, \tag{1.1}$$
$$\|\mathcal{T}_\varepsilon(\psi)\|_{L^p(\Omega \times Y)} = \|\psi\|_{L^p(\Omega)}. \tag{1.2}$$

Proof. First, we split the integral over Ω into $\tilde{\Omega}_\varepsilon$ and Λ_ε, i.e.

$$\int_\Omega \int_Y \mathcal{T}_\varepsilon(\varphi)(x,y) \,\mathrm{d}y \,\mathrm{d}x = \sum_{k \in I_\varepsilon} \int_{\varepsilon k + \varepsilon Y} \int_Y \varphi([x]_{\varepsilon,Y} + \varepsilon y) \,\mathrm{d}y \,\mathrm{d}x + \int_{\Lambda_\varepsilon \times Y} \varphi(x) \,\mathrm{d}y \,\mathrm{d}x.$$

Since $[x]_{\varepsilon,Y} = \varepsilon k$ on each cell $\varepsilon k + \varepsilon Y$, we obtain

$$\begin{aligned} \int_{\varepsilon k + \varepsilon Y} \int_Y \varphi([x]_{\varepsilon,Y} + \varepsilon y) \,\mathrm{d}y \,\mathrm{d}x &= \int_{\varepsilon k + \varepsilon Y} \int_Y \varphi(\varepsilon k + \varepsilon y) \,\mathrm{d}y \,\mathrm{d}x = |\varepsilon Y| \int_Y \varphi(\varepsilon k + \varepsilon y) \,\mathrm{d}y \\ &= \int_{\varepsilon k + \varepsilon Y} \varphi(x) \,\mathrm{d}x. \end{aligned}$$

Combining the previous two equations yields

$$\int_\Omega \int_Y \mathcal{T}_\varepsilon(\varphi)(x,y)\,\mathrm{d}y\,\mathrm{d}x = \sum_{k\in I_\varepsilon} \int_{\varepsilon k+\varepsilon Y} \varphi(x)\,\mathrm{d}x + \int_{\Lambda_\varepsilon} \varphi(x)\,\mathrm{d}x = \int_\Omega \varphi(x)\,\mathrm{d}x.$$

Since $|\mathcal{T}_\varepsilon(\varphi)|^p = \mathcal{T}_\varepsilon(|\varphi|^p)$, (1.2) follows for $p < \infty$ by applying (1.1) to $|\varphi|^p$. In order to show (1.2) for $p = \infty$, let $|\varphi| \geqslant C$ on an open set U with positive measure. Then, for every $\varepsilon > 0$, there exists $k \in I_\varepsilon$ such that $|(\varepsilon k + \varepsilon Y) \cap U| > 0$ or we have $|\Lambda_\varepsilon \cap U| > 0$. Then, $|\mathcal{T}_\varepsilon(\varphi)| \geqslant C$ on either $(\varepsilon k + \varepsilon Y) \times ((\varepsilon k + \varepsilon Y) \cap U)$ or $(\Lambda_\varepsilon \cap U) \times Y$. Reversely, we can deduce similarly from $|\mathcal{T}_\varepsilon(\varphi)| \geqslant C$ on an open set $U \subset \Omega \times Y$ with positive measure, that $|\varphi| \geqslant C$ on a set with positive measure. Thus, we obtain (1.2) for $p = \infty$. □

The following result translates two-scale convergence into classical L^p-convergence.

Proposition 1.12. *Let $1 < p < \infty$. Let u_ε be a sequence in $L^p(\Omega)$ and $u_0 \in L^p(\Omega \times Y)$. Then, the following statements hold:*

(1.) $u_\varepsilon \overset{p}{\rightharpoonup\!\!\rightharpoonup} u_0$ *if and only if* $\mathcal{T}_\varepsilon(u_\varepsilon) \rightharpoonup u_0$ *in* $L^p(\Omega \times Y)$,

(2.) $u_\varepsilon \overset{p}{\to\!\!\to} u_0$ *if and only if* $\mathcal{T}_\varepsilon(u_\varepsilon) \to u_0$ *in* $L^p(\Omega \times Y)$.

Proof. In order to prove (1.), we note that both convergences imply the boundedness of u_ε and $\mathcal{T}_\varepsilon(u_\varepsilon)$, respectively. The isometry of $\mathcal{T}_\varepsilon$ (see Theorem 1.11) transfers the boundedness of u_ε to the boundedness of $\mathcal{T}_\varepsilon(u_\varepsilon)$ and vice versa. Therefore, it suffices to test only with a dense subset of smooth test function (for the two-scale convergence see Lemma 1.3), i.e. (1.) follows if we show that

$$\begin{aligned}\lim_{\varepsilon\to 0} \int_\Omega u_\varepsilon(x)\varphi\left(x, \frac{x}{\varepsilon}\right)\mathrm{d}x &= \lim_{\varepsilon\to 0} \int_\Omega \int_Y \mathcal{T}_\varepsilon(u_\varepsilon)(x,y)\mathcal{T}_\varepsilon\left(\varphi\left(\cdot, \frac{\cdot}{\varepsilon}\right)\right)(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \lim_{\varepsilon\to 0} \int_\Omega \int_Y \mathcal{T}_\varepsilon(u_\varepsilon)(x,y)\varphi(x,y)\,\mathrm{d}y\,\mathrm{d}x \end{aligned} \tag{1.3}$$

for every smooth test function $\varphi \in D(\Omega; C^\infty_\#(Y))$. The first equality in (1.3) follows from (1.1) and the definition of $\mathcal{T}_\varepsilon$. For the second equality, we show that $\mathcal{T}_\varepsilon\big(\varphi\big(\cdot_x, \frac{\cdot_x}{\varepsilon}\big)\big)$ converges strongly to φ in $L^q(\Omega \times Y)$ for $\frac{1}{p} + \frac{1}{q} = 1$. Therefore, we note that for every $x \in \Omega$ there exists $\varepsilon_0(x) > 0$ small enough such that $x \in \tilde{\Omega}_\varepsilon$ for every $0 < \varepsilon < \varepsilon_0(x)$. Hence, we obtain the pointwise convergence

$$\mathcal{T}_\varepsilon\left(\varphi\left(\cdot_x, \frac{\cdot_x}{\varepsilon}\right)\right)(x,y) = \varphi\left([x]_{\varepsilon,Y} + \varepsilon y, \frac{[x]_{\varepsilon,Y} + \varepsilon y}{\varepsilon}\right) = \varphi([x]_{\varepsilon,Y} + \varepsilon y, y) \underset{\varepsilon\to 0}{\to} \varphi(x,y)$$

for every $(x,y) \in \Omega \times Y$. Since $|\mathcal{T}_\varepsilon\big(\varphi\big(\cdot, \frac{\cdot}{\varepsilon}\big)\big)(x,y)|$ is also pointwise bounded for a.e. $(x,y) \in \Omega \times Y$ and $\varepsilon < 1$ by $\chi_U \|\varphi\|_{L^\infty(\Omega\times Y)}$ for $U = \{x \in \mathbb{R}^n \mid \operatorname{dist}(\operatorname{supp}(\varphi), x) \leqslant C\}$ for some $C > 0$, we can apply Lebesgue's convergence theorem and obtain the strong convergence of $\mathcal{T}_\varepsilon(\varphi(\cdot_x, \frac{\cdot_x}{\varepsilon}))$ to φ in $L^q(\Omega \times Y)$ for $\frac{1}{p} + \frac{1}{q} = 1$, which implies (1.).

For the proof of (2.), we note that in uniformly convex Banach spaces, as $L^p(\Omega \times Y)$, the strong convergence is equivalent to the weak convergence together with the convergence of the norms. Hence, (2.) follows from (1.) and the isometry of $\mathcal{T}_\varepsilon$. □

For weakly differentiable functions, the unfolding operator and the weak derivative commute in the following sense.

Lemma 1.13. *Let $1 \leqslant p \leqslant \infty$ and $\varphi \in L^p(\Omega)$. Then, $\mathcal{T}_\varepsilon(\varphi) \in L^p(\tilde{\Omega}_\varepsilon; W^{1,p}(Y))$, with*

$$\mathcal{T}_\varepsilon(\nabla\varphi) = \varepsilon\nabla_y \mathcal{T}_\varepsilon(\varphi) \text{ in } \tilde{\Omega}_\varepsilon \times Y$$

Proof. Lemma can be shown by computations as in [CDG18, Proposition 1.35]. □

Two-scale convergence is also compatible with multiplication in the sense of the following two results. The first result extends the results of [All92a, Theorem 1.8] (for $p = 2$) and [LNW02, Theorem 11] (for $p \in [1, \infty)$), where it was shown that the product of a weakly with a strongly two-scale converging sequence converges in the distributional sense on Ω.

Lemma 1.14. *Let $1 \leqslant p, p_1, p_2 < \infty$ such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. Let u_ε be a sequence in $L^{p_1}(\Omega)$, which two-scale converges strongly to $u_0 \in L^{p_1}(\Omega \times Y)$ and v_ε a sequence in $L^{p_2}(\Omega)$, which two-scale converges weakly to $v_0 \in L^{p_2}(\Omega \times Y)$. Then, $u_\varepsilon v_\varepsilon \xrightarrow{p} u_0 v_0$.*

Proof. For the case $\frac{1}{p_1} + \frac{1}{p_2} = 1$, the distributional convergence (in $D(\Omega)'$) of the product was shown in [LNW02, Theorem 11]. This argumentation can be adapted for two scale test functions $\tau(\cdot, \frac{\cdot}{\varepsilon})$ for $\tau \in L^q(\Omega; C_\#(Y))$ with $\frac{1}{p} + \frac{1}{q} = 1$ instead of test functions $\tau \in D(\Omega)$. Then, for the first step, it has to be observed that $\phi\tau \in L^{q_2}(\Omega; C_\#(Y))$ for $\phi \in L^{p_1}(\Omega; C_\#(Y))$ and $\tau \in L^q(\Omega; C_\#(Y))$, where $\frac{1}{p_2} + \frac{1}{q_2} = 1$. Then, the proof can be adapted by repeating the approximation argument, which is given there.

For the case that $1 < p < \infty$ the unfolding operator $\mathcal{T}_\varepsilon$ (see Section 1.1) can be used for a simple alternative proof as follows. Lemma 1.12 implies $\mathcal{T}_\varepsilon(u_\varepsilon) \to u_0$ in $L^{p_1}(\Omega \times Y)$ as well as $\mathcal{T}_\varepsilon(v_\varepsilon) \rightharpoonup v_0$ in $L^{p_2}(\Omega \times Y)$. From classical L^p-theory, we obtain $\mathcal{T}_\varepsilon(u_\varepsilon)\mathcal{T}_\varepsilon(v_\varepsilon) \rightharpoonup u_0 v_0$ in $L^p(\Omega \times Y)$. After noting that $\mathcal{T}_\varepsilon(u_\varepsilon v_\varepsilon) = \mathcal{T}_\varepsilon(u_\varepsilon)\mathcal{T}_\varepsilon(v_\varepsilon)$, Lemma 1.12 translates this weak $L^p(\Omega \times Y)$-convergence back into $u_\varepsilon v_\varepsilon \xrightarrow{p} u_0 v_0$. □

For the case of two strongly two-scale converging sequences, we obtain the following analogous result.

Lemma 1.15. *Let $1 < p, p_1, p_2 < \infty$ such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. Let u_ε be a sequence in $L^{p_1}(\Omega)$, which two-scale converges strongly to $u_0 \in L^{p_1}(\Omega \times Y)$ and v_ε a sequence in $L^{p_2}(\Omega)$, which two-scale converges strongly $v_0 \in L^{p_2}(\Omega \times Y)$. Then, $u_\varepsilon v_\varepsilon \xrightarrow{p} u_0 v_0$.*

Proof. Similarly to the second proof of Lemma 1.14, Lemma 1.12 implies $\mathcal{T}_\varepsilon(u_\varepsilon) \to u_0$ in $L^{p_1}(\Omega \times Y)$ as well as $\mathcal{T}_\varepsilon(v_\varepsilon) \to v_0$. Thus, we obtain the convergence $\mathcal{T}_\varepsilon(u_\varepsilon v_\varepsilon) = \mathcal{T}_\varepsilon(u_\varepsilon)\mathcal{T}_\varepsilon(v_\varepsilon) \to u_0 v_0$ in $L^p(\Omega \times Y)$, which can be translated back by Lemma 1.12 into $u_\varepsilon v_\varepsilon \xrightarrow{p} u_0 v_0$. □

If u_ε is a sequence in $L^p(\Omega)$ for every $p \in (1, \infty)$ and $u_0 \in L^p(\Omega)$ for every $p \in (1, \infty)$ such that $u_\varepsilon \overset{p}{\twoheadrightarrow} u_0$ for every $p \in (1, \infty)$, we write

$$u_\varepsilon \xrightarrow{< \infty}\!\!\!\!\twoheadrightarrow u_0.$$

Having this notation, we can enhance Lemma 1.15.

Lemma 1.16. *Let $1 < p < \infty$, let v_ε be a sequence in $L^p(\Omega)$ and $v_0 \in L^p(\Omega \times Y)$ such that $v_\varepsilon \overset{p}{\twoheadrightarrow} v_0$ (resp. $v_\varepsilon \overset{p}{\rightharpoonup\!\!\!\rightharpoonup} v_0$). Let u_ε be a bounded sequence in $L^1(\Omega) \cap L^\infty(\Omega)$ and $u_0 \in L^1(\Omega \times Y) \cap L^\infty(\Omega \times Y)$ with $u_\varepsilon \xrightarrow{< \infty}\!\!\!\!\twoheadrightarrow u_0$. Then, $u_\varepsilon v_\varepsilon \overset{p}{\twoheadrightarrow} u_0 v_0$ (resp. $u_\varepsilon v_\varepsilon \overset{p}{\rightharpoonup\!\!\!\rightharpoonup} u_0 v_0$).*

Proof. First, we consider the case that $v_\varepsilon \overset{p}{\twoheadrightarrow} v_0$. Similarly to the proof of Lemma 1.15, we obtain $\mathcal{T}_\varepsilon(u_\varepsilon) \to u_0$ in $L^q(\Omega \times Y)$ for every $q \in (1, \infty)$ and $\mathcal{T}_\varepsilon(v_\varepsilon) \to v_0$ in $L^p(\Omega \times Y)$. Moreover, $\|\mathcal{T}_\varepsilon(u_\varepsilon)\|_{L^\infty(\Omega \times Y)} \leqslant C$ for some constant C. Then, we can pass to a subsequence such that the pointwise convergences $\mathcal{T}_\varepsilon(u_\varepsilon)(x, y) \to u_0(x, y)$ and $\mathcal{T}_\varepsilon(v_\varepsilon)(x, y) \to v_0(x, y)$ hold for a.e. $(x, y) \in \Omega \times Y$ and we have a pointwise majorant $h \in L^p(\Omega \times Y)$ almost everywhere, i.e. $|\mathcal{T}_\varepsilon(v_\varepsilon)(x, y)| \leqslant h(x, y)$ for a.e. $(x, y) \in \Omega \times Y$. Transferring the pointwise convergence and the dominating function onto the product, we obtain the pointwise convergence $\mathcal{T}_\varepsilon(u_\varepsilon v_\varepsilon)(x, y) \to u_0 v_0(x, y)$ for a.e. $(x, y) \in \Omega \times Y$ and $|\mathcal{T}_\varepsilon(u_\varepsilon v_\varepsilon)(x, y)| \leqslant C h(x, y)$ for $h \in L^p(\Omega \times Y)$. By applying the Lebesgue dominated convergence theorem, we obtain $\mathcal{T}_\varepsilon(u_\varepsilon v_\varepsilon) \to u_0 v_0$ in $L^p(\Omega \times Y)$. Since this argumentation holds for every subsequence, the convergence holds for the whole sequence. We translate this convergence in $L^p(\Omega \times Y)$ with Lemma 1.12 into $u_\varepsilon v_\varepsilon \overset{p}{\twoheadrightarrow} u_0 v_0$.

Next, we consider the case of $v_\varepsilon \overset{p}{\rightharpoonup\!\!\!\rightharpoonup} v_0$. From Lemma 1.12, we obtain $\mathcal{T}_\varepsilon(u_\varepsilon) \to u_0$ in $L^s(\Omega \times Y)$ for all $s \in (1, \infty)$, and Lemma 1.11 yields the boundedness of $\mathcal{T}_\varepsilon(u_\varepsilon)$ in $L^1(\Omega \times Y) \cap L^\infty(\Omega \times Y)$. By the same argumentation as above, we get $\mathcal{T}_\varepsilon(u_\varepsilon)\phi \to u_0\phi$ in $L^q(\Omega \times Y)$ for every $\phi \in L^q(\Omega \times Y)$, where $\frac{1}{p} + \frac{1}{q} = 1$. Using $\mathcal{T}_\varepsilon(v_\varepsilon) \rightharpoonup v_0$ in $L^p(\Omega \times Y)$, we can pass to the limit

$$\begin{aligned} \int\limits_{\Omega \times Y} \mathcal{T}_\varepsilon(u_\varepsilon v_\varepsilon)(x, y)\phi(x, y) \,\mathrm{d}y \,\mathrm{d}x &= \int\limits_{\Omega \times Y} \mathcal{T}_\varepsilon(v_\varepsilon)(x, y)(\mathcal{T}_\varepsilon(u_\varepsilon)\phi)(x, y) \,\mathrm{d}y \,\mathrm{d}x \\ &\to \int\limits_{\Omega \times Y} u_0(x, y) v_0(x, y)\phi(x, y) \,\mathrm{d}y \,\mathrm{d}x. \end{aligned}$$

Then, we translate this weak convergence with Lemma 1.12 back into $u_\varepsilon v_\varepsilon \overset{p}{\rightharpoonup\!\!\!\rightharpoonup} u_0 v_0$. □

In order to deal with non-linearities later, we note that the composition with a continuous function preserves the strong two-scale convergence.

Lemma 1.17. *Let $1 < p < \infty$ and $|\Omega| < 0$. Let u_ε be a sequence in $L^p(\Omega)$ and $u_0 \in L^p(\Omega \times Y)$ such that $u_\varepsilon \overset{p}{\twoheadrightarrow} u_0$ and let $f \in C(\mathbb{R})$ be bounded or globally Lipschitz continuous. Then, it holds $f(u_\varepsilon) \overset{p}{\twoheadrightarrow} f(u_0)$.*

Proof. From Lemma 1.12, we obtain $\mathcal{T}_\varepsilon(u_\varepsilon) \to u_0$ in $L^p(\Omega \times Y)$. Thus, we can pass to a subsequence and obtain $h \in L^p(\Omega \times Y)$ such that $\mathcal{T}_\varepsilon(u_\varepsilon)(x,y) \to u_0(x,y)$ and $|\mathcal{T}_\varepsilon(u_\varepsilon)| \leqslant h(x,y)$ for a.e. $(x,y) \in \Omega \times Y$. The pointwise convergence can be transferred via the continuity of f into $f(\mathcal{T}_\varepsilon(u_\varepsilon)(x,y)) \to f(u_0(x,y))$. For the case that f is bounded, i.e. $|f| \leqslant C < \infty$, one has $|f(\mathcal{T}_\varepsilon(u_\varepsilon)(x,y))| \leqslant C$ for a.e. $(x,y) \in \Omega \times Y$ and by the Lebesgue convergence theorem, we obtain $\mathcal{T}_\varepsilon(f(u_\varepsilon)) = f(\mathcal{T}_\varepsilon(u_\varepsilon)) \to f(u_0)$ in $L^p(\Omega \times Y)$. For the case that f is globally Lipschitz continuous with Lipschitz constant L_f, we obtain

$$|f(\mathcal{T}_\varepsilon(u_\varepsilon)(x,y))| \leqslant |f(0)| + L_f|\mathcal{T}_\varepsilon(u_\varepsilon)(x,y)| \leqslant C + L_f h(x,y)$$

for a.e. $(x,y) \in \Omega \times Y$. Then $C + L_f h$ can be used as majorant and we obtain $\mathcal{T}_\varepsilon(f(u_\varepsilon)) = f(\mathcal{T}_\varepsilon(u_\varepsilon)) \to f(u_0)$ in $L^p(\Omega \times Y)$ by the Lebesgue convergence theorem. Since this argumentation holds for every subsequence, it holds for the whole sequence. With Lemma 1.12, we translate this convergence back into $f(u_\varepsilon) \xrightarrow{p} f(u_0)$. □

The compactness result Theorem 1.5 can be improved for sequences of weakly differentiable functions by the following two well-known compactness results.

Theorem 1.18. *Let $p \in (1,\infty)$ and let u_ε be a bounded sequence in $W^{1,p}(\Omega)$ which converges weakly to $u_0 \in W^{1,p}(\Omega)$. Then, there exist $u_0 \in W^{1,p}(\Omega)$, $u_1 \in L^p(\Omega; W^{1,p}_\#(Y)/\mathbb{R})$ and a subsequence u_ε such that $u_\varepsilon \to u_0$ in $L^p(\Omega)$ and $\nabla u_\varepsilon \xrightarrow{p} \nabla_x u_0 + \nabla_y u_1$.*

Proof. A proof of Theorem 1.18 is given in [LNW02, Theorem 13]. □

Theorem 1.19. *Let $1 < p < \infty$. Let u_ε be a sequence in $W^{1,p}(\Omega)$ such that $\|u_\varepsilon\|_{L^p(\Omega)} \leqslant C$ and $\varepsilon\|\nabla u_\varepsilon\|_{L^p(\Omega)} \leqslant C$. Then, there exist a subsequence u_ε and $u_0 \in L^p(\Omega; W^{1,p}_\#(Y))$ such that $u_\varepsilon \xrightarrow{p} u_0$ and $\varepsilon\nabla u_\varepsilon \xrightarrow{p} \nabla_y u_0$.*

Proof. By applying Theorem 1.5 to u_ε and $\varepsilon\nabla u_\varepsilon$, we obtain the two-scale convergence for both sequences. Then, the two-scale limits of $\varepsilon\nabla u_\varepsilon$ can be identified with the y-gradient of the two-scale limits of u_ε using integration by parts. □

1.2. Two-scale convergence for periodically perforated domains and interfaces

In the following, we consider the concept of two-scale convergence for functions defined on periodically perforated domains. Let $Y^* \subset Y$ be open such that its periodic extension $Y^*_\# := \operatorname{int}\big(\bigcup_{k\in\mathbb{Z}^n} \varepsilon k + \varepsilon\overline{Y^*}\big)$ is a Lipschitz domain. We denote the characteristic functions of Y^* and $Y^*_\#$ by χ_{Y^*} and $\chi_{Y^*_\#}$, respectively. For an open Lipschitz domain $\Omega \subset \mathbb{R}^n$, we define the corresponding perforated domain by

$$\Omega_\varepsilon := \Omega \cap Y^*_\#.$$

In order to speak about the two-scale convergence for a sequence u_ε in $L^p(\Omega_\varepsilon)$, it is useful to extend it to functions in $L^p(\Omega)$. The simplest way is the extension by zero, which we denote by $\widetilde{\cdot}$ in the following. In Theorem 1.21, we will see that the two-scale limit for sequences of functions that are given via this extension (or functions that are zero on $\Omega\backslash\Omega_\varepsilon$) is zero on $Y\backslash Y^*$. Hence, it suffices to define the two-scale limit on Y^*. For the limit functions (or general functions defined on Y^*) we use $\widetilde{\cdot}$ as the extension by 0 from Y^* to Y.

Later we will partially omit $\widetilde{\cdot}$ and use the following notation for the sake of better readability.

Notation 1.20. *Let u_ε by a sequence in $L^p(\Omega_\varepsilon)$ and $u_0 \in L^p(\Omega \times Y^*)$. Then, we write*

$$u_\varepsilon \xrightarrow{p} u_0 \qquad \text{if} \qquad \widetilde{u_\varepsilon} \xrightarrow{p} \widetilde{u_0},$$

Analogously, we adept the notations $\xrightarrow{p}\!\!\!\!\twoheadrightarrow$ and $\xrightarrow{<\infty}\!\!\!\!\twoheadrightarrow$.

Theorem 1.21. *Let $1 < p < \infty$ and let u_ε be a bounded sequence in $L^p(\Omega_\varepsilon)$, i.e. $\|u_\varepsilon\|_{L^p(\Omega_\varepsilon)} \leqslant C$. Then, there exists a subsequence u_ε and $u_0 \in L^p(\Omega \times Y^*)$ such that $\widetilde{u_\varepsilon} \xrightarrow{p} \widetilde{u_0}$.*

Proof. We note that the boundedness of u_ε implies the boundedness of $\widetilde{u_\varepsilon}$. By applying Theorem 1.5, we obtain a two-scale limit u_0 for a subsequence. By choosing test functions φ that are zero in Y^* we obtain $u_0(x, y) = 0$ for a.e. $y \in Y\backslash Y^*$. □

Moreover, the compactness results Theorem 1.18 and Theorem 1.19, for weakly differentiable functions, can be extended to the case of perforated domains, too. However, the extension by zero does not necessarily preserve the weak differentiability and, thus, the original compactness results Theorem 1.18 and Theorem 1.19 cannot be applied. Nevertheless, the following two compactness results can be shown, where $W^{1,p}_\#(Y^*) := \{u \in W^{1,p}(Y^*) \mid u \text{ is } Y\text{-periodic}\}$.

Theorem 1.22. *Let $1 < p < \infty$ and $Y^*_\#$ be connected. Let u_ε be a bounded sequence in $W^{1,p}(\Omega_\varepsilon)$, i.e. $\|u_\varepsilon\|_{W^{1,p}(\Omega_\varepsilon)} \leqslant C$. Then, there exist a subsequence u_ε and $u_0 \in L^p(\Omega)$, $u_1 \in L^p(\Omega; W^{1,p}_\#(Y^*)/\mathbb{R})$ such that for this subsequence*

$$\widetilde{u_\varepsilon} \xrightarrow{p} \widetilde{u_0}, \qquad \widetilde{\nabla u_\varepsilon} \xrightarrow{p} \chi_{Y^*} \nabla_x u_0 + \widetilde{\nabla_y u_1}.$$

In the case of $u_\varepsilon|_{\partial\Omega\cap\partial\Omega_\varepsilon} = 0$, one has $u_0 \in H^1_0(\Omega)$.

Proof. For the case $p = 2$, a proof is given in [All92a, Theorem 2.9]. It can be improved to the case of arbitrary $p \in (1, \infty)$ by the results of [LNW02]. □

For the case of large gradients, we obtain the following result.

Theorem 1.23. *Let $1 < p < \infty$. Let u_ε be a sequence in $W^{1,p}(\Omega_\varepsilon)$ such that $\|u_\varepsilon\|_{L^p(\Omega_\varepsilon)} + \|\nabla u_\varepsilon\|_{L^p(\Omega_\varepsilon)} \leqslant C$. Then, there exist a subsequence u_ε and $u_0 \in L^p(\Omega; W^{1,p}_{\#}(Y^*)/\mathbb{R})$ such that for this subsequence*

$$\widetilde{u_\varepsilon} \xrightarrow{p} \chi_{Y^*} u_0, \qquad \varepsilon\widetilde{\nabla u_\varepsilon} \xrightarrow{p} \widetilde{\nabla_y u_0}.$$

Proof. For the case of $p = 2$, we refer to [All92a, Lemma 4.7]. It can be extended to the case of arbitrary $p \in (1, \infty)$ by the results of [LNW02]. □

In the case of a bounded sequence u_ε in $W^{1,p}(\Omega)$, the compact embedding into $L^p(\Omega)$ can be used for obtaining strong convergence in $L^p(\Omega)$, which becomes useful for non-linear problems. In the case of a perforated domain, we often only have a-priori the boundedness of u_ε in $W^{1,p}(\Omega_\varepsilon)$. Since the extension by zero does not preserve regularity, it cannot be used directly for the derivation of strong convergence. This can be solved using the following extension operator. For the following result, we assume that the domain Ω and the sequence ε are such that Ω consists of ε-scaled copies of Y, i.e. $\Omega = \operatorname{int}(\bigcup_{k \in I_\varepsilon} \varepsilon k + \varepsilon\overline{Y})$ for all ε.

Lemma 1.24. *Let $1 \leqslant p < \infty$. Then, there exists a family of extension operators*

$$E_\varepsilon : W^{1,p}(\Omega_\varepsilon) \to W^{1,p}(\Omega),$$

such that

$$\|E_\varepsilon(u_\varepsilon)\|_{L^p(\Omega)} \leqslant C\|u_\varepsilon\|_{L^p(\Omega_\varepsilon)}$$
$$\|\nabla E_\varepsilon(u_\varepsilon)\|_{L^p(\Omega)} \leqslant C\|\nabla u_\varepsilon\|_{L^p(\Omega_\varepsilon)},$$

for all $u_\varepsilon \in W^{1,p}(\Omega_\varepsilon)$. Moreover, if $u_\varepsilon = 0$ on $\partial\Omega_\varepsilon \cap \partial\Omega$, then $E_\varepsilon(u_\varepsilon) \in W^{1,p}_0(\Omega)$.

Proof. For the general construction of such a extension operator, we refer to [ACDP92]. This extension operator can be constructed for arbitrary domains Ω. However, the uniform continuity estimates hold only locally in the case of a non-zero boundary condition at $\partial\Omega_\varepsilon \cap \partial\Omega$. On the type of domain described above, which consists of entire ε-scaled cells only, this issue can be handled such that the estimates hold globally (see [Hö16]). □

Having this extension operator at hand, we can employ the compact Sobolev embedding from $W^{1,p}(\Omega)$ into $L^p(\Omega)$ for $E_\varepsilon(u_\varepsilon)$ and a bounded sequence u_ε in $W^{1,p}(\Omega_\varepsilon)$, in order to pass to a subsequence such that $E_\varepsilon(u_\varepsilon) \to v_0$ in $L^p(\Omega)$ for some $v_0 \in L^p(\Omega)$. Then, with the strong two-scale convergence of $\chi_{\Omega_\varepsilon} \xrightarrow{p} \chi_{Y^*}$ for every p (see Lemma 1.7) and the uniform essential boundedness of χ_{Y^*}, we can use Lemma 1.16 and get the strong two-scale convergence for the product $\widetilde{u_\varepsilon} = \chi_{\Omega_\varepsilon} E_\varepsilon(v_\varepsilon) \xrightarrow{p} \chi_{Y^*} v_0$. By the uniqueness of the two-scale limit, v_0 can be identified with the two-scale limit given by Theorem 1.22.

The homogenisation of processes in perforated domains often involves functions that are defined on the boundary of Ω_ε. In particular, the inner boundary $\Gamma_\varepsilon := \partial\Omega_\varepsilon \backslash \partial\Omega$ becomes interesting. We denote the interface in the reference cell by Γ, i.e. $\Gamma := \partial Y^* \backslash \partial Y$. In the following, we assume that Ω is bounded and (as already for the extension operator) that Ω consists of entire ε-scaled copies of the unit cell Y. The notion of two-scale convergence can be extended to functions defined on such interfaces as follows.

Definition 1.25. *Let $1 < p < \infty$. A sequence of functions $u_\varepsilon \in L^p(\Gamma_\varepsilon)$ is said to two-scale converge weakly on the surface Γ_ε to a limit $u_0 \in L^p(\Omega \times \Gamma)$, if $\varepsilon^{\frac{1}{p}} \|u_\varepsilon\|_{L^p(\Gamma_\varepsilon)} \leqslant C$ and*

$$\lim_{\varepsilon \to 0} \varepsilon \int_{\Gamma_\varepsilon} u_\varepsilon(t,x) \phi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}\sigma_x = \int_\Omega \int_\Gamma u_0(x,y) \phi(x,y) \,\mathrm{d}\sigma_y \,\mathrm{d}x$$

for every $\phi \in C(\overline{\Omega}, C_\#(\Gamma))$. We write $u_\varepsilon \overset{p}{\rightharpoonup\rightharpoonup} u_0$ on Γ_ε.

We say that a sequence u_ε two-scale converges strongly on Γ_ε, if additionally

$$\lim_{\varepsilon \to 0} \varepsilon^{\frac{1}{p}} \|u_\varepsilon\|_{L^p(\Gamma_\varepsilon)} = \|u_0\|_{L^p(\Omega \times \Gamma)},$$

and we write $u_\varepsilon \overset{p}{\twoheadrightarrow} u_0$ on Γ_ε.

For this notion, we obtain the following compactness result.

Theorem 1.26. *Let $1 < p < \infty$. Then, for every sequence $u_\varepsilon \in L^p(\Gamma_\varepsilon)$ with*

$$\varepsilon^{\frac{1}{p}} \|u_\varepsilon\|_{L^p(\Gamma_\varepsilon)} \leqslant C,$$

there exists $u_0 \in L^p(\Omega \times \Gamma)$ and a subsequence for which

$$u_\varepsilon \overset{p}{\twoheadrightarrow} u_0 \quad \text{on } \Gamma_\varepsilon.$$

Proof. For $p = 2$, a proof is given in [NR96], which can be generalised to arbitrary $p \in (1, \infty)$ by arguments as in [LNW02] for the two-scale convergence in Ω. □

Lemma 1.14 and Lemma 1.15 can be extended to the two-scale convergence on surfaces as follows.

Lemma 1.27. *Let $1 < p, p_1, p_2 < \infty$ such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. Let u_ε be a sequence in $L^{p_1}(\Gamma_\varepsilon)$, which two-scale converges strongly to $u_0 \in L^{p_1}(\Omega \times \Gamma)$ and v_ε a sequence in $L^{p_2}(\Gamma_\varepsilon)$, which two-scale converges strongly (resp. weakly) to $v_0 \in L^{p_2}(\Omega \times \Gamma)$. Then, $u_\varepsilon v_\varepsilon \overset{p}{\twoheadrightarrow} u_0 v_0$ (resp. $u_\varepsilon v_\varepsilon \overset{p}{\rightharpoonup\rightharpoonup} u_0 v_0$) on Γ_ε.*

Proof. The proof of Lemma 1.14 can be adapted by using the unfolding operator for surfaces (see Definition 1.29 below) instead of the unfolding operator for bulk terms. □

For functions in $W^{1,p}(\Omega_\varepsilon)$, the following ε-scaled trace inequality is useful.

Lemma 1.28. *Let $p \in [1, \infty)$. For every $\theta > 0$ there exists a constant $C(\theta) > 0$ independent of ε, such that for all $u_\varepsilon \in W^{1,p}(\Omega_\varepsilon)$*

$$\varepsilon^{\frac{1}{p}} \|u_\varepsilon\|_{L^p(\Gamma_\varepsilon)} \leqslant C(\theta) \|u_\varepsilon\|_{L^p(\Omega_\varepsilon)} + \theta\varepsilon \|\nabla u_\varepsilon\|_{L^p(\Omega_\varepsilon)}.$$

Proof. Lemma 1.28 can be shown by a classical argument, namely decomposing the domain Ω_ε into the ε-scaled cells, upscaling them and applying the trace inequality there. □

The concept of the unfolding operator can be extended to functions defined on the surface of perforated domains as follows, where we assume that $\Lambda_\varepsilon = 0$. This corresponds to the assumption that Ω consists of entire ε-scaled copies of Y. Moreover, let Ω_ε be given as above for the two-scale convergence for perforated domains.

Definition 1.29. *Let $1 \leqslant p \leqslant \infty$. The unfolding operator $\mathcal{T}_\varepsilon : L^p(\Gamma_\varepsilon) \to L^p(\Omega \times \Gamma)$ is defined by*

$$\mathcal{T}_\varepsilon(\varphi)(x, y) := \varphi([x]_{\varepsilon,Y} + \varepsilon y) \text{ for a.e. } (x, y) \in \Omega \times \Gamma.$$

The isometry result can be transferred to the unfolding operator for surfaces as follows.

Theorem 1.30. *Let $1 \leqslant p \leqslant \infty$. For every $\varphi \in L^1(\Gamma_\varepsilon)$ and $\psi \in L^p(\Gamma_\varepsilon)$, it holds*

$$\int_\Omega \int_\Gamma \mathcal{T}_\varepsilon(\varphi)(x, y) \, \mathrm{d}\sigma_y \, \mathrm{d}x = \varepsilon \int_{\Gamma_\varepsilon} \varphi(x) \, \mathrm{d}\sigma_x,$$
$$\|\mathcal{T}_\varepsilon(\psi)\|_{L^p(\Omega \times Y)} = \varepsilon^{1/p} \|\psi\|_{L^p(\Omega)}.$$

Proof. Theorem 1.30 can be proven by a similar computation as in the proof of Theorem 1.11. □

1.3. Time-dependent two-scale convergence

For processes over some time interval $(0, T)$ for $T > 0$, the following version of parameterised two-scale convergence becomes useful.

Definition 1.31 (Weak two-scale convergence)**.** *Let $1 \leqslant p, q < \infty$. A sequence u_ε in $L^p(0, T; L^q(\Omega))$ is said to two-scale converge weakly to a limit $u_0 \in L^p(0, T; L^q(\Omega \times Y))$ if*

$$\lim_{\varepsilon \to 0} \int_0^T \int_\Omega u_\varepsilon(t, x) \phi\left(t, x, \frac{x}{\varepsilon}\right) \mathrm{d}x \, \mathrm{d}t = \int_0^T \int_\Omega \int_Y u_0(t, x, y) \varphi(t, x, y) \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}t,$$

for any function $\varphi \in L^{p'}(0, T; L^{q'}(\Omega; C_\#(Y)))$, where p' and q' are such that $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. We write $u_\varepsilon \xrightarrow{p,q} u_0$.

For $1 < p, q < \infty$, a sequence u_ε in $L^p(0,T;L^q(\Omega))$ is said to two-scale converge strongly to a limit $u_0 \in L^p(0,T;L^q(\Omega\times Y))$ if $u_\varepsilon \xrightarrow{p,q} u_0$ and additionally

$$\|u_\varepsilon\|_{L^p(0,T;L^q(\Omega))} \to \|u_0\|_{L^p(0,T;L^q(\Omega\times Y))}.$$

We write $u_\varepsilon \xrightarrow{p,q} u_0$.

If u_ε is a sequence in $L^p((0,T)\times\Omega)$ for every $p \in (1,\infty)$ and $u_0 \in L^p((0,T)\times\Omega)$ for every $p \in (1,\infty)$ such that $u_\varepsilon \xrightarrow{p,p} u_0$ for every $p \in (1,\infty)$, we write

$$u_\varepsilon \xrightarrow{<\infty, <\infty} u_0.$$

For this notion of parameterised two-scale convergence, we can transfer all of the above compactness results and further results accordingly. In case of bounded sequences, we can also reduce the space of test functions to functions that are also smooth with respect to time.

Lemma 1.32. *Let $1 < p, q < \infty$ and u_ε be a bounded sequence in $L^p(0,T;L^q(\Omega))$ and $u_0 \in L^p(0,T;L^q(\Omega\times Y))$ such that*

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega u_\varepsilon(x)\varphi(t)\phi\left(x,\frac{x}{\varepsilon}\right)\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega\int_Y u_0(t,x,y)\varphi(t)\phi(x,y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t,$$

for any function $\varphi \in D(0,T)$ and $\phi \in D(\Omega;C^\infty_\#(Y))$. Then, $u_\varepsilon \xrightarrow{p,q} u_0$.

Proof. Lemma 1.32 can be deduced by a density argument similar as in [LNW02, Proposition 1]. □

Lemma 1.33. *Let $1 < p, q < \infty$. Let u_ε be a bounded sequence in $W^p(0,T;L^q(\Omega))$. Then, there exists $u_0 \in W^{1,p}(0,T;L^q(\Omega\times Y))$ such that for a subsequence*

$$u_\varepsilon \xrightarrow{p,q} u_0, \qquad \partial_t u_\varepsilon \xrightarrow{p,q} \partial_t u_0, \qquad u_\varepsilon(0) \xrightarrow{p} u_0(0).$$

Proof. From the compactness result, we obtain the two scale convergence of u_ε and $\partial_t u_\varepsilon$. Then, we test with $\partial_t\varphi(t)\phi(x,\frac{x}{\varepsilon})$ for $\varphi \in D(0,T)$ and $\phi \in L^q(\Omega;C_\#(Y))$ in order to identify the limit of $\partial_t u_\varepsilon$ with $\partial_t u_0$.

Afterwards, we can conclude similarly the two-scale convergence of $u_\varepsilon(0)$ using test functions $\varphi \in C^\infty([0,T])$ with $\varphi(T) = 0$. □

Moreover, the unfolding operator for time-parameterised functions is defined as follows.

Definition 1.34. *Let $1 \leqslant p, q \leqslant \infty$. The unfolding operator $\mathcal{T}_\varepsilon : L^p(0,T;L^q(\Omega)) \to L^p(0,T;L^q(\Omega\times Y))$ is defined by*

$$\mathcal{T}_\varepsilon(\varphi)(t,x,y) := \begin{cases}\varphi(t,[x]_{\varepsilon,Y}+\varepsilon y) & \text{for a.e. } (t,x,y) \in (0,T)\times\tilde{\Omega}_\varepsilon\times Y,\\ \varphi(t,x) & \text{for a.e. } (t,x,y) \in (0,T)\times\Lambda_\varepsilon\times Y.\end{cases}$$

As for the two-scale convergence, results about the unfolding operator and its relation to two-scale convergence can be proved in the time-parameterised setting. Moreover, the unfolding operator for surfaces can be extended to the time-dependent setting accordingly.

Chapter 2.

Homogenisation for locally periodic domains

Substantial parts of this chapter are based on the publication [Wie23, D. Wiedemann, *The two-scale-transformation method*, Asymptotic Analysis **131** (2023), 59–82].

In this chapter, we derive a rigorous framework for the homogenisation in non-periodically perforated and evolving non-periodically perforated domains. The non-periodicity under consideration is on the scale ε and, thus, it will persist during the limit process. We consider the stationary case of a non-periodically perforated domain first. Afterwards, we add the time evolution as a parameter similarly as in the theory of two-scale convergence.

In order to do the homogenisation on the non-periodically perforated domain, we proceed as follows: first, we transform the domain and the equations onto a periodically perforated substitute domain. There, we pass to the homogenisation limit. Afterwards, we transform the resulting limit system back to the two-scale limit set for the non-periodically perforated domains. The two-scale limit set of the non-periodically perforated domains will have a non-cylindrical structure, i.e. for every macroscopic point the cell domain is different. We justify this approach by showing that it commutes with the homogenisation in the non-periodically perforated domain, i.e. the diagram in Figure 2.1 commutes.

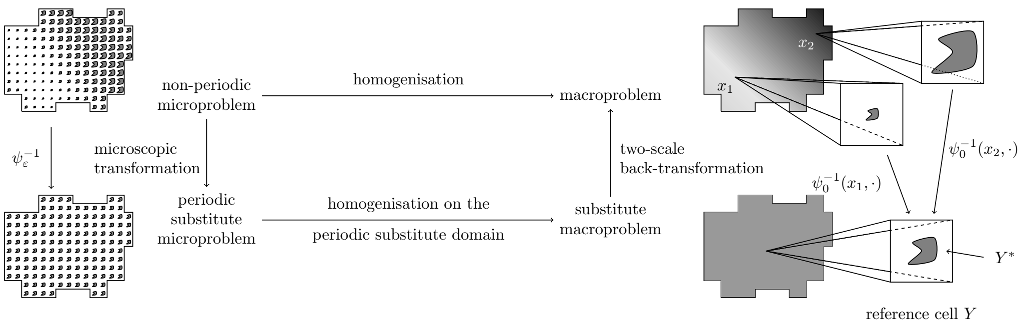

Figure 2.1.: Homogenisation by transformation on a periodic substitute domain

We consider this transformation approach in a qualitative and not quantitative way. That means, we will translate the convergence statement between the actual problem and the substitute problem, but no ε-scaled estimates. Thus, we can consider a wider range of non-periodic domains. In particular, we can deal with a-priori not given domains and it suffices to control these domains by compactness arguments.

This chapter is organised as follows. In Section 2.1, we define locally periodic domains Ω_ε

by means of a family of locally periodic transformation mappings $\psi_\varepsilon \colon \hat{\Omega}_\varepsilon \to \Omega_\varepsilon$, which are defined on a periodically perforated substitute domain and a two-scale limit transformation mapping $\psi_0(x, \cdot)$, which is given for almost every macroscopic point $x \in \Omega$ and acts on the unscaled reference cell. Then, we derive some uniform estimates on the Jacobian of ψ_ε and its determinant, which become useful for the homogenisation processes later.

In Section 2.2, we show that the two-scale convergence and the transformation mappings commute, i.e.

$$u_\varepsilon(\psi_\varepsilon(x)) \xrightarrow{p} u_0(x, \psi_0(x, y)) \qquad \text{if and only if} \qquad u_\varepsilon(x) \xrightarrow{p} u_0(x, y) \tag{2.1}$$

for $p \in (1, \infty)$. Moreover, we show that (2.1) holds also with respect to the strong two-scale convergence, which becomes useful for the transformation of oscillating coefficients. Afterwards, we extend the transformation results for the gradients of weakly differentiable functions. For this, we consider the two typical scalings of small (∇u_ε is bounded) and large gradients ($\varepsilon \nabla u_\varepsilon$ is bounded). In the case of large gradients, the transformation is similar to (2.1). For small gradients, it turns out that the corrector of the gradient cannot be transformed by (2.1) but requires an additional correction itself.

In Section 2.3, we present the homogenisation of an elliptic problem defined on a locally periodic domain by means of the transformation. We consider the cases of slow and of fast diffusion. After a transformation on the periodic substitute domain, we show uniform bounds for the solutions of the transformed equations. Then, we pass to the two-scale limit in the equations. We transform these two-scale limit equations back into transformation-independent two-scale limit equations, which are defined in the non-cylindrical two-scale limit set. Moreover, for the case of small gradients, we derive the homogenised equations by separating the macro- and microscopic variables in the two-scale limit equations. We do this for the transformed and back-transformed two-scale limit equations. For the sake of completeness, we transform the cell problems and show that the homogenised equations are equal, i.e. the homogenised tensors which arise from the cell problems are equal.

In Section 2.4, we extend this concept to locally evolving periodic domains, i.e. locally periodic domains with a time parameter. Moreover, we provide additional results for the time derivatives of the transformations, which become useful in the homogenisation of parabolic equations.

2.1. Locally periodic domains

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded Lipschitz domain and $\hat{Y}^* \subset Y = (0,1)^n$ be open. We assume that the Y-periodic extension of $\hat{Y}^*$, which we denote by $\hat{Y}^*_\# := \operatorname{int}\Big(\bigcup_{k \in \mathbb{Z}^n} k + \overline{\hat{Y}^*}\Big)$, is a Lipschitz domain. Let $\hat{\Omega}_\varepsilon := \Omega \cap \varepsilon \hat{Y}^*_\#$ denote the ε-scaled periodically perforated reference domains. Then, we define the locally periodic domains Ω_ε by a transformation of the periodic reference domains.

Definition 2.1. *A sequence of open domains $\Omega_\varepsilon \subset \mathbb{R}^n$ is locally periodic with two-scale limit set $\mathcal{Q} = \bigcup_{x \in \Omega} \{x\} \times Y^*(x) \subset \Omega \times Y$, where $Y^*(x) \subset Y$ is open for every $x \in \Omega$,* *if there exists a sequence of locally periodic transformations* ψ_ε *(see Definition 2.2) with limit transformation* ψ_0 *such that* $\Omega_\varepsilon = \psi_\varepsilon(\hat{\Omega}_\varepsilon)$ *for every* $\varepsilon > 0$ *and* $Y^*(x) = \psi_0(x, \hat{Y}^*)$ *for a.e.* $x \in \Omega$*.*

In order to define locally periodic transformations, we have to consider the two-scale convergence for sequences of functions defined on $\hat{\Omega}_\varepsilon$. Therefore, we extend them by 0 to Ω, which we denote by $\tilde{\cdot}$. Moreover, for functions defined on $\Omega \times Y^*$, we analogously denote their extension by 0 to $\Omega \times Y$ by $\tilde{\cdot}$.

Definition 2.2. *We say that a sequence of mappings* $\psi_\varepsilon \in C^2(\overline{\hat{\Omega}_\varepsilon})^n$ *is a sequence of locally periodic transformations with two-scale limit transformation* $\psi_0 \in L^\infty(\Omega; C^2(\overline{Y})^n)$ *if the following assumptions hold:*

1. *assumptions on* ψ_ε*:*
   a. $\psi_\varepsilon \colon \overline{\hat{\Omega}_\varepsilon} \to \overline{\Omega_\varepsilon} \subset \mathbb{R}^n$ *is bijective for every* $\varepsilon > 0$*,*
   b. *there exists* $c_J > 0$ *such that* $\det(\partial_x \psi_\varepsilon(x)) \geqslant c_J$ *for all* $x \in \overline{\hat{\Omega}_\varepsilon}$ *and every* $\varepsilon > 0$*,*
   c. *there exists a constant* C *such that*
   $$\varepsilon^{-1}\|\psi_\varepsilon - x\|_{C(\overline{\hat{\Omega}_\varepsilon})} + \|\partial_x \psi_\varepsilon\|_{C(\overline{\hat{\Omega}_\varepsilon})} + \varepsilon\|\partial_x\partial_x\psi_\varepsilon\|_{C(\overline{\hat{\Omega}_\varepsilon})} \leqslant C$$
   for every $\varepsilon > 0$*,*
2. *assumptions on* ψ_0*:*
   a. $\psi_0(x, \cdot) \colon \overline{Y} \to \overline{Y}$ *is bijective with* $Y^*(x) = \psi_0(x, \hat{Y}^*)$ *for a.e.* $x \in \Omega$*,*
   b. $\psi_0^{-1} \in L^\infty(\Omega; C^2(\overline{Y})^n)$*, where* $\psi_0^{-1}(x, \cdot)$ *is the inverse of* $\psi_0(x, \cdot)$*,*
   c. *the corresponding displacement mapping, defined by* $\breve{\psi}_0(x, y) := \psi_0(x, y) - y$ *for* $(x, y) \in \Omega \times Y$ *can be extended* Y*-periodically, i.e.* $\breve{\psi}_0 \in L^\infty(\Omega; C^2_\#(\overline{Y})^n)$*,*
3. *asymptotic behaviour*
   - $\varepsilon^{-1}(\psi_\varepsilon - x) \xrightarrow{<\infty}\!\!\!\!\rightarrow \chi_{\hat{Y}^*}(y)(\psi_0(x, y) - y)$
   - $\partial_x \psi_\varepsilon \xrightarrow{<\infty}\!\!\!\!\rightarrow \chi_{\hat{Y}^*}\partial_y \psi_0$*,*
   - $\varepsilon\partial_x\partial_x\psi_\varepsilon \xrightarrow{<\infty}\!\!\!\!\rightarrow \chi_{\hat{Y}^*}\partial_y\partial_y\psi_0$*.*

We have defined the limit transformation $\psi_0(x, \cdot)$ on entire $\overline{Y}$ and not only on $\overline{\hat{Y}^*}$ in order to ensure the measurability, when we use it as transformation. However, for the asymptotic behaviour in Definition 2.2 and the transformation results later, it suffices to have $\psi_0(x, \cdot)$ and $\psi_0^{-1}(x, \cdot)$ defined on $\hat{Y}^*$ and $Y^*(x)$, respectively. Then, we will implicitly restrict ψ_0 and ψ_0^{-1} accordingly and, where necessary, we use the implicit extension by 0.

Remark 2.3. *Note that we do not assume that* ψ_ε *maps each* ε*-scaled cell into the same* ε*-scaled cell. Moreover, the assumption that* $\psi_0(x, \cdot)$ *is bijective from* $\overline{Y}$ *onto* $\overline{Y}$ *can also be weakened and it suffices to require that* $\psi_0(x, \cdot)$ *is bijective from* $\mathbb{R}^n$ *to* $\mathbb{R}^n$ *and* Y*-periodic, which generalises the transformations at the cell boundaries.*

Notation 2.4. *Let ψ_ε and ψ_0 be given by Definition 2.2. We denote the inverse of ψ_ε by ψ_ε^{-1} and recap the notion $\psi_0^{-1}(x,\cdot)$ for the inverse of $\psi_0(x,\cdot)$ for a.e. $x \in \Omega$. We define the corresponding displacement mappings by*

$$\begin{aligned} \breve{\psi}_\varepsilon(x) &:= \psi_\varepsilon(x) - x, & \breve{\psi_\varepsilon^{-1}}(x) &:= \psi_\varepsilon^{-1}(x) - x \\ \breve{\psi}_0(x,y) &:= \psi_0(x,y) - y, & \breve{\psi_0^{-1}}(x,y) &:= \psi_0^{-1}(x,y) - y. \end{aligned}$$

We note that the displacement mappings can be identified with the displacement mappings of the inverse by

$$\begin{aligned} \breve{\psi_\varepsilon^{-1}}(x) &= \psi_\varepsilon^{-1}(x) - x = \psi_\varepsilon^{-1}(x) - \psi_\varepsilon(\psi_\varepsilon^{-1}(x)) = -\breve{\psi}_\varepsilon(\psi_\varepsilon^{-1}(x)), \\ \breve{\psi_0^{-1}}(x,y) &= \psi_0^{-1}(x,y) - y = \psi_0^{-1}(x,y) - \psi_0(x,\psi_0^{-1}(x,y)) = -\breve{\psi}_0(x,\psi_0^{-1}(x,y)). \end{aligned} \tag{2.2}$$

The Y-periodicity of $\breve{\psi}_0$ can be transferred via (2.2) to $\breve{\psi_0^{-1}}$. Thus, $\breve{\psi_0^{-1}} \in L^\infty(\Omega; C^2_\#(\overline{Y}))$.

Notation 2.5. *Let ψ_ε and ψ_0 be given by Definition 2.2. We use the following notation for the Jacobian matrix, its determinant and its adjugate matrix*

$$\Psi_\varepsilon(x) := \partial_x \psi_\varepsilon(x), \qquad J_\varepsilon(x) := \det(\Psi_\varepsilon(x)), \qquad A_\varepsilon(x) := \operatorname{Adj}(\Psi_\varepsilon(x))$$

for $x \in \overline{\hat{\Omega}_\varepsilon}$ and

$$\Psi_0(x,y) := \partial_y \psi_0(x,y), \qquad J_0(x,y) := \det(\Psi_0(x,y)), \qquad A_0(x,y) := \operatorname{Adj}(\Psi_0(x,y))$$

for a.e. $x \in \Omega$ and all $y \in \overline{Y}$.

We recall that $\operatorname{Adj}(B)B = B\operatorname{Adj}(B) = \det(B)\mathbb{1}$ for $B \in \mathbb{R}^{n\times n}$. Since $\det(\partial_x\psi_\varepsilon) \geqslant c_J$ (see Definition 2.2), Ψ_ε is invertible and in Lemma 2.9, we will see also $\det(\partial_y\psi_0) \geqslant c_J$ and, thus, Ψ_0 is invertible and we get

$$A_\varepsilon = J_\varepsilon \Psi_\varepsilon^{-1}, \qquad A_0 = J_0 \Psi_0^{-1}.$$

Remark 2.6. *For the homogenisation of the second-order elliptic problem in this chapter, we are not using the second derivative of ψ_ε if we transform only the weak and not the strong formulation of the equations. Therefore, it would suffice to define locally periodic domains by C^1-diffeomorphisms and, correspondingly, without any assumptions on their second derivatives. However, for the homogenisation of the Stokes equation, we will need the second derivatives. In order to avoid repetition, we formulate the transformation already with C^2-regularity here.*

The regularity can be even lowered to the case of bi-Lipschitz regular transformation, where the derivative is given only almost everywhere.

For the homogenisation of the transformed equation, we have to deal with the quantities Ψ_ε, J_ε, A_ε and their inverses as coefficients in the equations. Therefore, we need their uniform boundedness as well as their strong two-scale convergence. We obtain this by rewriting these terms as polynomials in J_ε^{-1} and the entries of Ψ_ε, which can be controlled by the assumptions from Definition 2.2.

Lemma 2.7. *Let $B \in \mathbb{R}^{n\times n}$ with $\det(B) \neq 0$. Then, there exist polynomials $p_{\det}$, $p_{\mathrm{Adj}_{ij}}$, $p_{\cdot_{ij}^{-1}}$, $p_{\mathrm{Adj}_{ij}^{-1}}$, for $i, j \in \{1,\dots,n\}$, such that*

$$\begin{aligned}
\det(B) &= p_{\det}(B_{11}, B_{12}, \dots, B_{nn}) =: P_{\det}(B),\\
(\mathrm{Adj}(B))_{ij} &= p_{\mathrm{Adj}_{ij}}(B_{11}, B_{12}, \dots, B_{nn}) =: P_{\mathrm{Adj}_{ij}}(B),\\
(B^{-1})_{ij} &= p_{\cdot_{ij}^{-1}}(\det(B)^{-1}, B_{11}, B_{12}, \dots, B_{nn}) =: P_{\cdot_{ij}^{-1}}(\det(B)^{-1}, B),\\
(\mathrm{Adj}(B)^{-1})_{ij} &= p_{A_{ij}^{-1}}(\det(B)^{-1}, B_{11}, B_{12}, \dots, B_{nn}) =: P_{\mathrm{Adj}_{ij}^{-1}}(\det(B)^{-1}, B),
\end{aligned} \tag{2.3}$$

Moreover, for space-dependent functions $B\colon \mathbb{R}^n \supset U \to \mathbb{R}^{n\times n}$ with $\det(B) \neq 0$ it holds

$$\begin{aligned}
\partial_{x_k} \det(B(x)) &= \sum_{m,l} \partial_{B_{ml}} P_{\det}(B(x)) \partial_{x_k} B_{ml}(x)\\
\partial_{x_k} \mathrm{Adj}(B(x))_{ij} &= \sum_{m,l} \partial_{B_{ml}} P_{\mathrm{Adj}_{ij}}(B(x)) \partial_{x_k} B_{ml}(x),\\
\partial_{x_k}(\det(B(x))^{-1}) &= -\det(B(x))^{-2} \partial_{x_k} \det(B(x)),\\
\partial_{x_k}(B^{-1})_{ij}(x) &= -\partial_{\det(B)^{-1}} P_{\cdot_{ij}^{-1}}(\det(B)^{-1}, B) \det(B(x))^{-2} \partial_{x_k} \det(B(x))\\
&\quad + \sum_{m,l} \partial_{B_{ml}} P_{\cdot_{ij}^{-1}}(\det(B)^{-1}(x), B(x)) \partial_{x_k} B_{ml}(x),\\
\partial_{x_k}(\mathrm{Adj}(B)^{-1})_{ij}(x) &= -\partial_{\det(B)^{-1}} P_{\mathrm{Adj}(B)_{ij}^{-1}}(\det(B)^{-1}(x), B(x)) \det(B(x))^{-2} \partial_{x_k} \det(B(x)),\\
&\quad + \sum_{m,l} \partial_{B_{ml}} P_{\mathrm{Adj}(B)_{ij}^{-1}}(\det(B)^{-1}(x), B(x)) \partial_{x_k} B_{ml}(x)
\end{aligned} \tag{2.4}$$

for all $k \in \{1,\dots,n\}$.

Proof. The existence of polynomials $p_{\det}$ and p_{Adj} follows directly from the definition of the determinant and the adjugate matrix. Noting that $B^{-1} = \det(B)^{-1}\mathrm{Adj}(B)$, $\mathrm{Adj}(B)^{-1} = \det(B)^{-1}B$, we obtain also $p_{\cdot_{ij}^{-1}}$ and $p_{\mathrm{Adj}_{ij}^{-1}}$. The second part of Lemma 2.7 can be derived from (2.3) and the chain rule. □

Lemma 2.8. *Let ψ_ε be locally periodic transformations in the sense of Definition 2.2. Then, there exists $C > 0$ such that*

$$\begin{aligned}
\|\Psi_\varepsilon\|_{C(\overline{\hat\Omega_\varepsilon})} + \|\Psi_\varepsilon^{-1}\|_{C(\overline{\hat\Omega_\varepsilon})} + \|J_\varepsilon\|_{C(\overline{\hat\Omega_\varepsilon})} + \|J_\varepsilon^{-1}\|_{C(\overline{\hat\Omega_\varepsilon})} + \|A_\varepsilon\|_{C(\overline{\hat\Omega_\varepsilon})} + \|A_\varepsilon^{-1}\|_{C(\overline{\hat\Omega_\varepsilon})} &\leqslant C,\\
\varepsilon\|\partial_x \Psi_\varepsilon\|_{C(\overline{\hat\Omega_\varepsilon})} + \varepsilon\|\partial_x \Psi_\varepsilon^{-1}\|_{C(\overline{\hat\Omega_\varepsilon})} + \varepsilon\|\partial_x J_\varepsilon\|_{C(\overline{\hat\Omega_\varepsilon})} + \varepsilon\|\partial_x J_\varepsilon^{-1}\|_{C(\overline{\hat\Omega_\varepsilon})} &\leqslant C,\\
\varepsilon\|\partial_x A_\varepsilon\|_{C(\overline{\hat\Omega_\varepsilon})} + \varepsilon\|\partial_x A_\varepsilon^{-1}\|_{C(\overline{\hat\Omega_\varepsilon})} &\leqslant C.
\end{aligned}$$

Proof. The estimates of Definition 2.2 give the uniform boundedness of Ψ_ε and $\varepsilon\partial_x\Psi_\varepsilon$. In order to estimate J_ε and A_ε, we rewrite them as polynomials with respect to the entries of Ψ_ε (see Lemma 2.7). Then, we can transfer the uniform boundedness and regularity of Ψ_ε onto J_ε and A_ε. From Definition 2.2, we obtain additionally the uniform boundedness of J_ε from below and, thus, with the regularity of J_ε, we obtain the regularity and uniform boundedness for J_ε^{-1}. Then, we write Ψ_ε^{-1} and A_ε^{-1} as polynomials J_ε^{-1} and the entries of Ψ_ε (see again Lemma 2.7). Afterwards, we can transfer the regularity and uniform boundedness onto Ψ_ε^{-1} and A_ε^{-1}.

Next, we rewrite the entries of $\partial_x J_\varepsilon$ as polynomials in the entries of Ψ_ε and $\partial_x\Psi_\varepsilon$ (see Lemma 2.7). Since every summand in this polynomial contains exactly one elementary factor which belongs to an entry of $\partial_x\Psi_\varepsilon$, we can rewrite $\varepsilon\partial_x J_\varepsilon$ as polynomial with respect to the entries of Ψ_ε and $\varepsilon\partial_x\Psi_\varepsilon$. Thus, the regularity and the uniform bounds of Ψ_ε and $\varepsilon\partial_x\Psi_\varepsilon$ induce those for $\varepsilon\partial_x J_\varepsilon$.

For the estimates of $\varepsilon\partial_x\Psi_\varepsilon^{-1}$, $\varepsilon\partial_x J_\varepsilon^{-1}$, $\varepsilon\partial_x A_\varepsilon$ and $\varepsilon\partial_x A_\varepsilon^{-1}$, we proceed analogously. The only difference is that we write them as polynomials in the entries of Ψ_ε and $\varepsilon\partial_x\Psi_\varepsilon$ and, now additionally, the entries of J_ε^{-1} and $\varepsilon\partial_x J_\varepsilon$. Again, we note that each summand of the polynomials contains only one factor belonging to a derivative of either J_ε^{-1} or Ψ_ε. Then, using the previous estimates, we obtain the regularity and uniform bounds for $\varepsilon\partial_x\Psi_\varepsilon^{-1}$, $\varepsilon\partial_x J_\varepsilon^{-1}$, $\varepsilon\partial_x A_\varepsilon$ and $\varepsilon\partial_x A_\varepsilon^{-1}$. ∎

In the following, we show the strong two-scale convergence for these coefficients. For this, we rewrite them again as polynomials with respect to quantities that we can control by Definition 2.2.

Lemma 2.9. *Let ψ_ε be locally periodic transformations with limit transformation ψ_0 in the sense of Definition 2.2. Then, there exist constants $c_J, C > 0$ such that*

$$\begin{aligned}\|\Psi_0\|_{L^\infty(\Omega;C(\overline{\hat{Y}^*}))} + \|\Psi_0^{-1}\|_{L^\infty(\Omega;C(\overline{\hat{Y}^*}))} + \|J_0\|_{L^\infty(\Omega;C(\overline{\hat{Y}^*}))} &\leqslant C,\\ \|A_0\|_{L^\infty(\Omega;C(\overline{\hat{Y}^*}))} + \|A_0^{-1}\|_{L^\infty(\Omega;C(\overline{\hat{Y}^*}))} &\leqslant C,\\ J_0(x,y) \geqslant c_J \text{ for a.e. } x \in \Omega \text{ and every } y \in Y^*.&\end{aligned}$$

Moreover, one has

$$\begin{gathered}\Psi_\varepsilon \xrightarrow{<\infty}\!\!\!\!\!\!\rightarrow \Psi_0, \quad \Psi_\varepsilon^{-1} \xrightarrow{<\infty}\!\!\!\!\!\!\rightarrow \Psi_0^{-1}, \quad J_\varepsilon \xrightarrow{<\infty}\!\!\!\!\!\!\rightarrow J_0, \quad J_\varepsilon^{-1} \xrightarrow{<\infty}\!\!\!\!\!\!\rightarrow J_0^{-1},\\ A_\varepsilon \xrightarrow{<\infty}\!\!\!\!\!\!\rightarrow A_0, \quad A_\varepsilon^{-1} \xrightarrow{<\infty}\!\!\!\!\!\!\rightarrow A_0^{-1}, \quad \varepsilon\partial_x\Psi_\varepsilon \xrightarrow{<\infty}\!\!\!\!\!\!\rightarrow \partial_y\Psi_0, \quad \varepsilon\partial_x\Psi_\varepsilon^{-1} \xrightarrow{<\infty}\!\!\!\!\!\!\rightarrow \partial_y\Psi_0^{-1},\\ \varepsilon\partial_x J_\varepsilon \xrightarrow{<\infty}\!\!\!\!\!\!\rightarrow \partial_y J_0, \quad \varepsilon\partial_x J_\varepsilon^{-1} \xrightarrow{<\infty}\!\!\!\!\!\!\rightarrow \partial_y J_0^{-1}, \quad \varepsilon\partial_x A_\varepsilon \xrightarrow{<\infty}\!\!\!\!\!\!\rightarrow \partial_y A_0, \quad \varepsilon\partial_x A_\varepsilon^{-1} \xrightarrow{<\infty}\!\!\!\!\!\!\rightarrow \partial_y A_0^{-1}.\end{gathered}$$

Proof. From Definition 2.2, we obtain the regularity of Ψ_0 and $\partial_y\Psi_0$ as well as the two-scale convergence of Ψ_ε and $\varepsilon\partial_x\Psi_\varepsilon$. Noting that J_0 is a polynomial in the entries of Ψ_0 (see Lemma 2.7), we can transfer the regularity onto J_0. Having the additional uniform boundedness of Ψ_ε and Ψ_0, we can apply Lemma 1.16 which yields the strong two-scale convergence for the polynomial and, thus, $J_\varepsilon \xrightarrow{<\infty}\!\!\!\!\!\!\rightarrow J_0$ (similarly we obtain $A_\varepsilon \xrightarrow{<\infty}\!\!\!\!\!\!\rightarrow A_0$).

Then, Proposition 1.12, gives the strong convergence $\mathcal{T}_\varepsilon(\widetilde{J_\varepsilon}) \to \widetilde{J_0}$ in $L^p(\Omega \times Y)$, which implies the pointwise convergence almost everywhere. Since $J_\varepsilon \geqslant c_J$, we get $\mathcal{T}_\varepsilon(\widetilde{J_\varepsilon}) \geqslant c_J$ almost everywhere in $\Omega \times Y^*$, which can be transferred via the pointwise convergence to $J_0 \geqslant c_J$ almost everywhere. With the y-regularity of J_0, we obtain $J_0 \geqslant c_J$ for a.e. $x \in \Omega$ and every $y \in Y$. Having this bound, we can estimate further

$$\begin{aligned}\|\mathcal{T}_\varepsilon(\widetilde{J_\varepsilon^{-1}}) - \widetilde{J_0^{-1}}\|_{L^p(\Omega\times Y)} &= \|(J_0 - \mathcal{T}_\varepsilon(\widetilde{J_\varepsilon}))/(J_0\mathcal{T}_\varepsilon(\widetilde{J_\varepsilon}))\|_{L^p(\Omega\times\hat{Y}^*)} \\ &\leqslant c_J^{-2}\|J_0 - \mathcal{T}_\varepsilon(\widetilde{J_\varepsilon})\|_{L^p(\Omega\times\hat{Y}^*)} = \frac{1}{c_J^2}\|\widetilde{J_0} - \mathcal{T}_\varepsilon(\widetilde{J_\varepsilon})\|_{L^p(\Omega\times Y)} \to 0,\end{aligned}$$

for every $p \in [1,\infty)$, which implies $J_\varepsilon^{-1} \xrightarrow{<\infty}\!\!\!\!\twoheadrightarrow J_0^{-1}$.

The strong two-scale convergence of Ψ_ε^{-1}, A_ε^{-1} follows likewise by rewriting them as polynomials, but now with the additional variable J_ε^{-1}, for which we have already shown the boundedness and the strong two-scale convergence.

Now we show the strong two-scale convergence of $\varepsilon\partial_x\Psi_\varepsilon^{-1}$, $\varepsilon\partial_x J_\varepsilon$, $\varepsilon\partial_x J_\varepsilon^{-1}$, $\varepsilon\partial_x A_\varepsilon$ and $\varepsilon\partial_x A_0$. We start with $\varepsilon\partial_x J_\varepsilon^{-1}$ and rewrite it as polynomial with respect to the entries of Ψ_ε and $\varepsilon\partial_x\Psi_\varepsilon$ (for the ε-scaling see also the proof of Lemma 2.8). Then, we obtain its strong two-scale convergence from the boundedness of its entries and the strong two-scale convergence of Ψ_ε and $\varepsilon\partial_x\Psi_\varepsilon$. Having the strong two-scale convergence of $\varepsilon\partial_x J_\varepsilon^{-1}$, we can likewise argue for the remaining terms. □

2.2. Two-scale transformation and two-scale convergence

We aim to translate between the two-scale convergence for a sequence u_ε in Ω_ε and the two-scale convergence of $\hat{u}_\varepsilon := u_\varepsilon \circ \psi_\varepsilon$ defined in the periodic substitute domain $\hat{\Omega}_\varepsilon$. We recap that the two-scale convergence for a sequence in $\hat{\Omega}_\varepsilon$ is given by its extension $\tilde{\cdot}$ to Ω by zero. We transfer this onto sequences in the non-periodically perforated domain Ω_ε. However, Definition 2.2 does not require $\Omega_\varepsilon \subset \Omega$ and, thus, the extension to Ω would not suffice. Nevertheless, due to the estimate $|\underbrace{\psi_\varepsilon}| \leqslant \varepsilon C$, we obtain $\Omega_\varepsilon \subset \Omega^{(\delta)}$ for some $\delta \geqslant 0$ and $\Omega^{(\delta)} = \{x \in \mathbb{R}^n \mid \operatorname{dist}(\Omega, x) \leqslant \delta\}$. Thus, we can extend functions defined on Ω_ε by 0 on $\Omega^{(d)}$ and use this macroscopic domain for the definition of its two-scale convergence. Moreover, the support of the two-scale limit function will be contained in $\Omega \times Y$ due to $|\widetilde{\psi_\varepsilon}| \leqslant \varepsilon C$. Thus, it is not too restrictive if we assume that $\Omega_\varepsilon \subset \Omega$, i.e. $\Omega^{(\delta)} = \Omega$, in the following. Nevertheless, the following results and argumentation can be carried out without this assumption for which we refer to [Wie23].

Having the extension $\tilde{\cdot}$ for a sequence in Ω_ε, we can define the two-scale limit as a function on $\Omega \times Y$. We will see that the two-scale limit will be zero for a.e. (x, y) with $y \in Y\backslash Y^*(x)$. Therefore, we will define the limit functions as element in $L^p(\Omega; L^p(Y^*(x)))$ and denote their extension by 0 to $\Omega \times Y$ by $\tilde{\cdot}$. We define the space $L^p(\Omega; L^q(Y^*(x)))$ by restriction of $L^p(\Omega; L^q(Y))$ to functions that are 0 for a.e. $(x, y) \in \Omega \times Y$ with $y \in Y\backslash Y^*(x)$, where the norm is defined by

$$\|u\|_{L^p(\Omega;L^q(Y^*(x)))} := \|\|u(x)\|_{L^q(Y^*(x))}\|_{L^p(\Omega)}$$

Later we will use also the space $L^p(\Omega; W^{1,p}_{\#}(Y^*(x)))$, which we define by

$$L^p(\Omega; W^{1,p}_{\#}(Y^*(x))) := \left\{ u \in L^p(\Omega; L^p(Y^*(x))) \,\middle|\, \begin{array}{l} u(x,\cdot) \in W^{1,p}_{\#}(Y^*(x)) \text{ for a.e. } x \in \Omega, \\ \partial_y u \in L^p(\Omega; L^p(Y^*(x))) \end{array} \right\}.$$

In Lemma 2.16, we will see that these spaces can be identified with $L^p(\Omega; L^q(\hat{Y}^*))$ and $L^p(\Omega; W^{1,p}_{\#}(\hat{Y}^*))$, respectively, and, thus they are well posed.

Analogously to the case of periodic perforation, we omit $\tilde{\cdot}$ in the notion of two-scale convergence.

Notation 2.10. *Let u_ε by a sequence in $L^p(\Omega_\varepsilon)$ and $u_0 \in L^p(\Omega; L^p(Y^*(x)))$. Then, we write*

$$u_\varepsilon \xrightarrow{p} u_0 \qquad \text{if} \qquad \widetilde{u_\varepsilon} \xrightarrow{p} \widetilde{u_0}.$$

Analogously, we adept the notion $\xrightarrow{p}$ for $p \in (1, \infty)$ and $\xrightarrow{< \infty}$.

2.2.1. Well-posedness of the transformation

Lemma 2.11. *Let $1 \leqslant p \leqslant \infty$ and $\hat{u}_\varepsilon = u_\varepsilon \circ \psi_\varepsilon$. Then, the following statements hold*

- *$u_\varepsilon \in L^p(\Omega_\varepsilon)$ if and only if $\hat{u}_\varepsilon \in L^p(\hat{\Omega}_\varepsilon)$. Moreover, there exist constants $c, C > 0$, which are independent of ε, such that*

$$c\|\hat{u}_\varepsilon\|_{L^p(\hat{\Omega}_\varepsilon)} \leqslant \|u_\varepsilon\|_{L^p(\Omega_\varepsilon)} \leqslant C\|\hat{u}_\varepsilon\|_{L^p(\hat{\Omega}_\varepsilon)}. \tag{2.5}$$

  In particular, u_ε is a bounded sequence in $L^p(\Omega_\varepsilon)$ if and only if $\hat{u}_\varepsilon$ is a bounded sequence in $L^p(\hat{\Omega}_\varepsilon)$.

- *$u_\varepsilon \in W^{1,p}(\Omega_\varepsilon)$ if and only if $\hat{u}_\varepsilon \in W^{1,p}(\hat{\Omega}_\varepsilon)$. Moreover, there exist constants $c, C > 0$, which are independent of ε, such that*

$$c\|\nabla\hat{u}_\varepsilon\|_{L^p(\hat{\Omega}_\varepsilon)} \leqslant \|\nabla u_\varepsilon\|_{L^p(\Omega_\varepsilon)} \leqslant C\|\nabla\hat{u}_\varepsilon\|_{L^p(\hat{\Omega}_\varepsilon)}. \tag{2.6}$$

  In particular, u_ε is a bounded sequence in $W^{1,p}(\Omega_\varepsilon)$ if and only if $\hat{u}_\varepsilon$ is a bounded sequence in $W^{1,p}(\hat{\Omega}_\varepsilon)$.

Proof. Since ψ_ε and ψ_ε^{-1} are Lipschitz continuous, the measurability of u_ε is transferred to $\hat{u}_\varepsilon$ and vice versa. Moreover, we obtain, with the uniform boundedness of J_ε from below and above

$$\|u_\varepsilon\|^p_{L^p(\Omega_\varepsilon)} = \int_{\Omega_\varepsilon} |u_\varepsilon(x)|^p \,\mathrm{d}x = \int_{\hat{\Omega}_\varepsilon} J_\varepsilon(x)|\hat{u}_\varepsilon(x)|^p \,\mathrm{d}x \leqslant C \int_{\hat{\Omega}_\varepsilon} |\hat{u}_\varepsilon(x)|^p \,\mathrm{d}x = C\|\hat{u}_\varepsilon\|^p_{L^p(\hat{\Omega}_\varepsilon)},$$

$$\|\hat{u}_\varepsilon\|^p_{L^p(\hat{\Omega}_\varepsilon)} = \int_{\Omega_\varepsilon} J_\varepsilon^{-1}(\psi_\varepsilon^{-1}(x))|u_\varepsilon(x)|^p \,\mathrm{d}x \leqslant c_J^{-1} \int_{\Omega_\varepsilon} |u_\varepsilon(x)|^p \,\mathrm{d}x = c_J^{-1}\|u_\varepsilon\|^p_{L^p(\Omega_\varepsilon)}.$$

Applying the chain rule to $\hat{u}_\varepsilon$ yields $\partial_x \hat{u}_\varepsilon(x) = \partial_x u_\varepsilon(\psi(x))\partial_x\psi(x)$ and, after rearranging, $(\nabla u_\varepsilon)(\psi_\varepsilon(x)) = \Psi_\varepsilon^{-\top}(x)\nabla\hat{u}_\varepsilon(x)$. Using the uniform estimates for J_ε from below and above as well as the uniform estimates of Ψ_ε and Ψ_ε^{-1} (see Definition 2.2 and Lemma 2.8), we obtain

$$\begin{aligned}
\|\nabla u_\varepsilon\|^p_{L^p(\Omega_\varepsilon)} &= \int\limits_{\Omega_\varepsilon} |\nabla u_\varepsilon(x)|^p \,\mathrm{d}x = \int\limits_{\hat{\Omega}_\varepsilon} J_\varepsilon(x)|\Psi_\varepsilon^{-\top}(x)\nabla\hat{u}_\varepsilon(x)|^p \,\mathrm{d}x \\
&\leqslant C \int\limits_{\hat{\Omega}_\varepsilon} \|\Psi_\varepsilon^{-\top}\|^p_{L^\infty(\hat{\Omega}_\varepsilon)} |\nabla\hat{u}_\varepsilon(x)|^p \,\mathrm{d}x \leqslant C \int\limits_{\hat{\Omega}_\varepsilon} |\nabla\hat{u}_\varepsilon(x)|^p \,\mathrm{d}x = C\|\nabla\hat{u}_\varepsilon\|^p_{L^p(\hat{\Omega}_\varepsilon)}, \\
\|\nabla\hat{u}_\varepsilon\|^p_{L^p(\hat{\Omega}_\varepsilon)} &= \int\limits_{\hat{\Omega}_\varepsilon} |\nabla\hat{u}_\varepsilon(x)|^p \,\mathrm{d}x = \int\limits_{\Omega_\varepsilon} J_\varepsilon^{-1}(\psi_\varepsilon^{-1}(x))|\Psi_\varepsilon^{\top}(\psi_\varepsilon^{-1}(x))\nabla u_\varepsilon(x)|^p \,\mathrm{d}x \\
&\leqslant c_J^{-1} \int\limits_{\Omega_\varepsilon} \|\Psi_\varepsilon^{\top}\|^p_{L^\infty(\hat{\Omega}_\varepsilon)} |\nabla u_\varepsilon(x)|^p \,\mathrm{d}x \leqslant C \int\limits_{\Omega_\varepsilon} |\nabla u_\varepsilon(x)|^p \,\mathrm{d}x = C\|\nabla u_\varepsilon\|^p_{L^p(\Omega_\varepsilon)},
\end{aligned}$$

which shows (2.6). □

In order to derive a similar result for the limit quantities, we have to discuss the measurability of the transformed quantities first. Since the mapping $(x, y) \mapsto \psi_0(x, y)$ is not bi-Lipschitz continuous with respect to x, it is a-priori not clear whether the composition of $(x, y) \mapsto u(x, \psi_0(x, y))$ is even measurable for measurable but not continuous u. Therefore, we have to analyse the measurability for such parameterised transformations. For the sake of clarity, in the following discussion, we call a set $A \subset \mathbb{R}^n$ measurable if it is Lebesgue measurable and we call it Borel if it is Borel measurable. Moreover, we write $\lambda_n(A)$ instead of $|A|$ in order to stress the Lebesgue measurability of A.

Definition 2.12. *Let $E \subset \mathbb{R}^n$ be a Lebesgue measurable set. A mapping $\varphi\colon E \to \mathbb{R}^l$ is said to satisfy Lusin's (N)-condition if for every $A \subset E$ with $\lambda_n(A) = 0$ it holds $\lambda_l(\phi(A)) = 0$.*

Lemma 2.13 (Lusin's Theorem)**.** *Let $A \subset \mathbb{R}^n$ be measurable with $\lambda_n(A) < \infty$ and let f be a real valued function on A. Then, for any $\delta > 0$, there is a compact set $K \subset A$ with $\lambda_n(A\backslash K) < \delta$ such that the restriction of f to K is continuous.*

Proof. See for instance [EG15, Theorem 1.14]. □

By means of Lusin's Theorem, we can derive the following lemma, by extending the proof of [Nau05, Theorem 1.3] from continuous to measurable functions.

Lemma 2.14. *Let $E \subset \mathbb{R}^n$ be Lebesgue measurable with $\lambda_n(E) < \infty$ and let $\varphi\colon E \to \mathbb{R}^l$ be a mapping. Then, the following statements hold.*

(1.) Let φ be a measurable function defined on E, which satisfies Lusin's (N)-condition. Then, for every measurable set $F \subset E$ it holds that $\varphi(F)$ is measurable.

(2.) Assume that

a. the set $\varphi(E)$ is measurable,

b. φ is injective,

c. $\varphi^{-1}\colon \varphi(E) \to E$ is measurable in $\varphi(E)$ and satisfies Lusin's (N)-condition,

d. $u\colon \varphi(E) \to \overline{\mathbb{R}}$ is measurable.

Then, $u \circ \varphi\colon E \to \overline{\mathbb{R}}$ is measurable.

Proof. (1.) The measurability of F is equivalent to the existence of closed sets F_i for $i \in \mathbb{N}$ and a set N with $\lambda_n(N) = 0$ such that $F = \Big(\bigcup_{i=0}^{\infty} F_i\Big) \cup N$. Moreover, from Lemma 2.13, we obtain a sequence of compact sets E_i for $i \in \mathbb{N}$ such that $\lambda_m(E\backslash E_i) < i^{-1}$ and φ is continuous on E_i. Without loss of generality, we can assume that $F_i \subset F_j$ and $E_i \subset E_j$ for $j \geqslant i$, by considering $\bigcup_{k=1}^{i} F_k$ and $\bigcup_{k=1}^{i} E_k$ instead of F_i and E_i. Consequently, one has $\lambda_n(F\backslash(E_i \cap F_i)) \leqslant \lambda_n(F\backslash F_i) + i^{-1}$ and, hence, there exists a null set $\tilde{N}$ such that

$$F = \Big(\bigcup_{i=0}^{\infty} (E_i \cap F_i)\Big) \cup \tilde{N}.$$

Since ϕ is continuous on E_i and $(F_i \cap E_i)$ is compact, it holds that $\phi(F_i \cap E_i)$ is compact and, thus, is a Borel set. Then, the countable union $\bigcup_{i=1}^{\infty} \phi(F_i \cap E_i)$ is a Borel set as well. Moreover, since ϕ fulfils Lusin's (N)-property, we have $\phi(\tilde{N}) = 0$ and, thus, $\phi(\tilde{N})$ is measurable. We infer that

$$\phi(F) = \phi\Big(\Big(\bigcup_{i=0}^{\infty} (F_i \cap E_i)\Big) \cup \tilde{N}\Big) = \Big(\bigcup_{i=0}^{\infty} \phi(F_i \cap E_i)\Big) \cup \phi(\tilde{N})$$

is measurable.

(2.) The measurabilty of $u\colon \varphi(E) \to \overline{\mathbb{R}}$ means that for every $a \in \mathbb{R}$, the set

$$\{y \in \varphi(E) \mid u(y) > a\}$$

is measurable. Since φ^{-1} is measurable and satisfies Lusin's (N)-condition, (1.) shows that, for every $a \in \mathbb{R}$, the set

$$\varphi^{-1}(\{y \in \varphi(E) \mid u(y) > a\})$$

is measurable. Now, we observe that

$$\{x \in E \mid (u \circ \varphi)(x) > a\} = \varphi^{-1}(\{y \in \varphi(E) \mid u(y) > a\})$$

and hence $u \circ \varphi$ is measurable.

□

In order to apply Lemma 2.14 for our purpose, we have to show that $(x, y) \mapsto (x, \psi_0(x, y))$ and $(x, y) \mapsto (x, \psi_0^{-1}(x, y))$ fulfil Lusin's (N)-condition. For instance, Lipschitz continuity implies Lusin's (N)-condition (see [Nau05]). However, $(x, y) \mapsto \psi_0(x, y)$ and $(x, y) \mapsto (x, \psi_0(x, y))$ are not Lipschitz continuous with respect to x. In fact $(x, y) \mapsto \psi_0(x, y)$ does not fulfil Lusins's (N)-condition. Nevertheless, with the following lemma, we can conclude that $(x, y) \mapsto (x, \psi_0(x, y))$ and $(x, y) \mapsto (x, \psi_0^{-1}(x, y))$ fulfil Lusin's (N)-condition even if they are not Lipschitz continuous with respect to x.

Lemma 2.15. *Let $m, n, l \in \mathbb{N}$ with $l \geqslant n$. Let $U \subset \mathbb{R}^m$ be measurable, $V \subset \mathbb{R}^n$ be closed and $\phi \in L^\infty(U; C(V)^l)$ be uniformly Lipschitz continuous with respect to the second argument, i.e. there exists $L > 0$ such that*

$$|\phi(x, y_1) - \phi(x, y_2)| \leqslant L|y_1 - y_2|$$

for a.e. $x \in U$ and every $y_1, y_2 \in V$. Then, $(x, y) \mapsto (x, \phi(x, y))$, satisfies Lusin's (N)-condition on $U \times V$.

Proof. Let $A \subset U \times V$ with $\lambda_{m+n}(A)$. Then, for every $\varepsilon > 0$, there exist cubes $C^{(k)} \subset \mathbb{R}^{m+n}$ for $k \in \mathbb{N}$ such that

$$A \subset \bigcup_{k=1}^{\infty} C^{(k)}, \qquad \sum_{k=1}^{\infty} \lambda(C^{(k)}) \leqslant \varepsilon$$

and, in particular, $2r_k \leqslant \varepsilon^{1/(n+m)}$ for all $k \in \mathbb{N}$. We identify these cubes by means of their centres $(x_k, y_k) \in \mathbb{R}^m \times \mathbb{R}^n$ and side lengths $2r_k$, i.e.

$$\begin{aligned} C^{(k)} &= Q_{r_k}^{m+n}((x_k, y_k)) := \{(x, y) \in \mathbb{R}^{m+n} \mid |x - x_k|_\infty < r_k, |y - y_k|_\infty < r_k\} \\ &= Q_{r_k}^m(x_k) \times Q_{r_k}^n(y_k) := \{x \in \mathbb{R}^m \mid |x - x_k|_\infty < r_k\} \times \{y \in \mathbb{R}^n \mid |y - y_k|_\infty < r_k\}. \end{aligned}$$

From the Lipschitz estimate, it follows

$$|\phi(x, y) - \phi(x, y_k)|_\infty \leqslant L|y - y_k|_\infty < Lr_k$$

for all $(x, y) \in C^{(k)}$ and, hence,

$$\phi(C^{(k)}) \subset \bigcup_{x \in Q_{r_k}^m(x_k)} \{x\} \times Q_{Lr_k}^l(\phi(x, y_k)) \subset U \times \mathbb{R}^m.$$

Moreover, we note that

$$\begin{aligned} \bigcup_{x \in Q_{r_k}^m(x_k)} \{x\} \times Q_{Lr_k}^l(\phi(x, y_k)) = &\bigcap_{i=1}^{l} \{(x, z) \in Q_{r_k}^m(x_k) \times \mathbb{R}^l \mid z_i - \phi(x, y_k)_i \leqslant Lr_k\} \\ &\cap \bigcap_{i=1}^{l} \{(x, y) \in Q_{r_k}^m(x_k) \times \mathbb{R}^l \mid z_i - \phi(x, y_k)_i \geqslant -Lr_k\}. \end{aligned} \tag{2.7}$$

By [LNW02, Theorem 1], we can fix the second argument of ϕ and get the measurability of $x \mapsto \phi(x, y_k)$ and, thus, $(x, z) \mapsto z_i - \phi(x, y_k)_i$ is measurable for all $i \in \{1, \ldots, l\}$. Therefore, all sets on the right-hand side of (2.7) are measurable and, hence, the whole right-hand side is measurable. This allows the application of Tonelli's theorem, which yields

$$\begin{aligned}\sum_{k=1}^{\infty} \lambda_{m+n}\Big(\bigcup_{x \in Q^m_{r_k}(x_k)} \{x\} \times Q^l_{Lr_k}(\phi(x, y_k))\Big) &= \sum_{k=1}^{\infty} \int_{x \in Q^m_{r_k}(x_k)} \lambda_l\big(Q^l_{Lr_k}(\phi(x, y_k))\big) \,\mathrm{d}x \\ &= \sum_{k=1}^{\infty} (2r_k)^m (2Lr_k)^l = \sum_{k=1}^{\infty} L^l (2r_k)^{l-n} \lambda\big(C^{(k)}\big) \leqslant L^l \varepsilon^{(l-n)/(n+m)} \sum_{k=1}^{\infty} \lambda\big(C^{(k)}\big) \\ &= L^l \varepsilon^{(l+m)/(n+m)-1} \sum_{k=1}^{\infty} \lambda\big(C^{(k)}\big) = L^l \varepsilon^{(l+m)/(n+m)}.\end{aligned}$$

By choosing ε arbitrarily small, we can conclude $\phi(A) = 0$, which shows Lusin's (N)-condition. ☐

Having Lemma 2.15 and Lemma 2.14, we can transform measurable function by means of the two-scale limit transformation.

Lemma 2.16. *Let $1 \leqslant p \leqslant \infty, 1 \leqslant q < \infty$ and $\hat{u}_0(x, y) = u_0(x, \psi_0(x, y))$ for a.e. $(x, y) \in \Omega \times \hat{Y}^*$, or equivalently $u_0(x, y) = \hat{u}_0(x, \psi_0^{-1}(x, y))$ for a.e. $(x, y) \in \mathcal{Q}$. Then, the following statements hold*

- *$u_0 \in L^p(\Omega; L^q(Y^*(x)))$ if and only if $\hat{u}_0 \in L^p(\Omega; L^q(\hat{Y}^*))$. Moreover, there exist constants $c, C > 0$, such that*

$$c\|\hat{u}_0\|_{L^p(\Omega;L^q(\hat{Y}^*))} \leqslant \|u_0\|_{L^p(\Omega;L^q(Y^*(x)))} \leqslant C\|\hat{u}_0\|_{L^p(\Omega;L^q(\hat{Y}^*))}. \tag{2.8}$$

- *$u_0 \in L^p(\Omega; W^{1,q}_{\#}(Y^*(x)))$ if and only if $\hat{u}_\varepsilon \in L^p(\Omega; W^{1,q}_{\#}(\hat{Y}^*))$. Moreover, there exist constants $c, C > 0$, which are independent of ε, such that*

$$c\|\nabla \hat{u}_\varepsilon\|_{L^p(\Omega;L^q(\hat{Y}^*))} \leqslant \|\nabla u_\varepsilon\|_{L^p(\Omega;L^q(Y^*(x)))} \leqslant C\|\nabla \hat{u}_\varepsilon\|_{L^p(\Omega;L^q(\hat{Y}^*))}. \tag{2.9}$$

Proof. Lemma 2.15 shows that $(x, y) \mapsto (x, \psi_0(x, y))$ and $(x, y) \mapsto (x, \psi_0^{-1}(x, y))$ fulfil Lusin's (N)-condition. Then, Lemma 2.14 shows that $\hat{u}_0$ is measurable if and only if u_0 is measurable.

Using the uniform boundedness of J_0 with respect to $(x, y) \in \Omega \times \hat{Y}^*$ from below and above, we obtain, for $p, q \in [1, \infty)$

$$\|u_0\|^p_{L^p(\Omega;L^q(Y^*(x)))} = \int_{\Omega} \|u_0(x)\|^p_{L^q(Y^*(x))} \,\mathrm{d}x = \int_{\Omega} \Big(\int_{Y^*(x)} |u_0(x, y)|^q \,\mathrm{d}y\Big)^{p/q} \mathrm{d}x$$

$$= \int_\Omega \Big(\int_{Y^*} J_0(x,y) |\hat{u}_0(x,y)|^q \,\mathrm{d}y \Big)^{p/q} \mathrm{d}x \leqslant C \int_\Omega \Big(\int_{Y^*} |\hat{u}_0(x,y)|^q \,\mathrm{d}y \Big)^{p/q} \mathrm{d}x$$
$$= C \|\hat{u}_0\|^p_{L^p(\Omega;L^q(\hat{Y}^*))}$$

and

$$\|\hat{u}_0\|^p_{L^p(\Omega;L^q(\hat{Y}^*))} = \int_\Omega \|\hat{u}_0(x)\|^p_{L^q(\hat{Y}^*)} \,\mathrm{d}x = \int_\Omega \Big(\int_{\hat{Y}^*} |\hat{u}_0(x,y)|^q \,\mathrm{d}y \Big)^{p/q} \mathrm{d}x$$
$$= \int_\Omega \Big(\int_{Y^*(x)} J_0^{-1}(x, \psi_0^{-1}(x,y)) |u_0(x,y)|^q \,\mathrm{d}y \Big)^{p/q} \mathrm{d}x \leqslant C \int_\Omega \Big(\int_{Y^*(x)} |u_0(x,y)|^q \,\mathrm{d}y \Big)^{p/q} \mathrm{d}x$$
$$= C \|u_0\|^p_{L^p(\Omega;L^q(Y^*(x)))}.$$

By similar argumentation, the equivalence can be shown if p is ∞. Thus, we obtain (2.8). Employing additionally the boundedness of Ψ_0 and Ψ_0^{-1}, we obtain (2.9) by a similar argumentation. □

Since $L^\infty(\hat{Y}^*)$ is not separable, it is not meaningful to consider Lemma 2.16 for $q = \infty$. Instead, we get the following result, which becomes useful for the transformation of coefficients.

Lemma 2.17. *Let $\hat{u}_0(x,y) = u_0(x, \psi_0(x,y))$ for a.e. $(x,y) \in \Omega \times \hat{Y}^*$, or equivalently $u_0(x,y) = \hat{u}_0(x, \psi_0^{-1}(x,y))$ for a.e. $(x,y) \in \mathcal{Q}$. Then, $\hat{u}_0 \in L^\infty(\Omega \times Y^*)$ if and only if $u_0 \in L^\infty(\mathcal{Q})$ and it holds*

$$\|\hat{u}_0\|_{L^\infty(\Omega \times \hat{Y}^*)} = \|u_0\|_{L^\infty(\mathcal{Q})}. \tag{2.10}$$

Proof. The measurability can be transferred as in Lemma 2.16. Since J_0 is essentially bounded from below and above, one has for every $A \subset \Omega \times \hat{Y}^*$ that $|\psi_0(A)| > 0$ if and only if $|A| > 0$, which shows (2.10). □

2.2.2. Equivalence of two-scale convergence

Now, we aim to show

$$u_\varepsilon \xrightarrow{p} u_0 \qquad \text{if and only if} \qquad \hat{u}_\varepsilon \xrightarrow{p} \hat{u}_0 \tag{2.11}$$

for $\hat{u}_\varepsilon(x) = u_\varepsilon(\psi_\varepsilon(x))$ and $\hat{u}_0(x,y) = \hat{u}_0(x, \psi_0(x,y))$ in Theorem 2.20. Moreover, we will show the same result for strong two-scale convergence in Theorem 2.21. Afterwards, we consider the transformation of small and large gradients in Theorem 2.23 and Theorem 2.24, respectively, which requires some additional correctors in the case of small gradients.

Since $\hat{u}_\varepsilon$ is bounded if and only if u_ε is bounded, it suffices to work with smooth two-scale test functions. By transforming the integral, we can shift the transformations from u_ε and $\hat{u}_\varepsilon$, respectively, onto the test functions. Thus, we need to show compatibility of transformations and two-scale convergence only for the test functions. We introduce the following notation.

Notation 2.18. *Let φ be a function which is defined on $\Omega \times Y$, then*

$$\varphi_{\varepsilon,\psi_\varepsilon}(x) := \varphi\left(\psi_\varepsilon(x), \frac{\psi_\varepsilon(x)}{\varepsilon}\right), \qquad \varphi_{\psi_0}(x,y) := \varphi(x, \psi_0(x,y)),$$

$$\varphi_{\varepsilon,\psi_\varepsilon^{-1}}(x) := \varphi\left(\psi_\varepsilon^{-1}(x), \frac{\psi_\varepsilon^{-1}(x)}{\varepsilon}\right), \qquad \varphi_{\psi_0^{-1}}(x,y) := \varphi(x, \psi_0^{-1}(x,y)),$$

where $\psi_0(x,\cdot)$ and $\psi_0^{-1}(x,\cdot)$ denote their restrictions to $\hat{Y}^*$ and $Y^*(x)$, respectively.

Lemma 2.19. *Let $\varphi \in D(\Omega; C_\#(Y))$. Then, $\varphi_{\varepsilon,\psi_\varepsilon} \xrightarrow{<\infty} \varphi_{\psi_0}$.*

Note that $\varphi_{\varepsilon,\psi_\varepsilon}$ is only defined on $\hat{\Omega}_\varepsilon$ and φ_{ψ_0} only on $\Omega \times Y^*$, hence the two-scale convergence has to be understood in the sense $\widetilde{\varphi_{\varepsilon,\psi_\varepsilon}} \xrightarrow{<\infty} \widetilde{\varphi_{\psi_0}}$.

Proof. Due to Proposition 1.12, it suffices to show that $\mathcal{T}_\varepsilon(\widetilde{\varphi_{\varepsilon,\psi_\varepsilon}})$ converges strongly to $\widetilde{\varphi_{\psi_0}}$ in $L^p(\Omega \times Y)$ for every $p \in (1,\infty)$. We show this convergence by the pointwise convergence and the Lebesgue convergence theorem. Let $(x,y) \in \Omega \times Y$. We choose $\varepsilon_0(x)$ small enough such that, for all $0 < \varepsilon < \varepsilon_0(x)$, x is contained in ε-scaled cells that is entirely in $\tilde{\Omega}_\varepsilon$, i.e. $[x]_{\varepsilon,Y} + \varepsilon Y \subset \tilde{\Omega}_\varepsilon$ (see Notation 1.9 for the definition of $\tilde{\Omega}_\varepsilon$).

For $y \in Y \backslash \hat{Y}^*$, we get $[x]_{\varepsilon,Y} + \varepsilon y \in \tilde{\Omega}_\varepsilon \cap (\Omega \backslash \Omega_\varepsilon)$ and, thus,

$$\mathcal{T}_\varepsilon(\widetilde{\varphi_{\varepsilon,\psi_\varepsilon}})(x,y) = 0 = \widetilde{\varphi_{\psi_0}}(x,y).$$

For $y \in Y^*$, we obtain $\mathcal{T}_\varepsilon(\widetilde{\varphi_{\varepsilon,\psi_\varepsilon}})(x,y) = \varphi\left(\psi_\varepsilon([x]_{\varepsilon,Y} + \varepsilon y), \frac{\psi_\varepsilon([x]_{\varepsilon,Y}+\varepsilon y)}{\varepsilon}\right)$, which can be rewritten using $\psi_\varepsilon(x) = x + \breve{\psi}_\varepsilon(x)$ and the Y-periodicity of φ,

$$\begin{aligned}
\mathcal{T}_\varepsilon(\widetilde{\varphi_{\varepsilon,\psi_\varepsilon}})(x,y) &= \varphi\left(\psi_\varepsilon([x]_{\varepsilon,Y} + \varepsilon y), \frac{\psi_\varepsilon([x]_{\varepsilon,Y} + \varepsilon y)}{\varepsilon}\right) \\
&= \varphi\left([x]_{\varepsilon,Y} + \varepsilon y + \breve{\psi}_\varepsilon([x]_{\varepsilon,Y} + \varepsilon y), \frac{[x]_{\varepsilon,Y} + \varepsilon y + \breve{\psi}_\varepsilon([x]_{\varepsilon,Y} + \varepsilon y)}{\varepsilon}\right) \\
&= \varphi\left([x]_{\varepsilon,Y} + \varepsilon y + \mathcal{T}_\varepsilon(\widetilde{\breve{\psi}_\varepsilon})(x,y), y + \frac{\mathcal{T}_\varepsilon(\widetilde{\breve{\psi}_\varepsilon})(x,y)}{\varepsilon}\right).
\end{aligned}$$

In order to pass to the limit $\varepsilon \to 0$, we note that the strong two-scale convergence of $\frac{1}{\varepsilon}\breve{\psi}_\varepsilon$ to $\breve{\psi}_0$, which is given by Definition 2.2, implies the strong convergence of $\frac{1}{\varepsilon}\mathcal{T}_\varepsilon(\widetilde{\breve{\psi}_\varepsilon})$ to $\widetilde{\breve{\psi}_0}$ in $L^p(\Omega \times Y)$. Hence, we can pass to a subsequence such that $\frac{1}{\varepsilon}\mathcal{T}_\varepsilon(\widetilde{\breve{\psi}_\varepsilon})(x,y) \to \widetilde{\breve{\psi}_0}(x,y)$ and $\mathcal{T}_\varepsilon(\widetilde{\breve{\psi}_\varepsilon})(x,y) \to 0$ for a.e. $(x,y) \in \Omega \times Y$. Moreover, $[x]_{\varepsilon,Y}$ converges to x and εy to 0. Since $\varphi \in D(\Omega; C_\#(Y))$, we can carry these pointwise convergences over to the pointwise convergence

$$\varphi\left([x]_{\varepsilon,Y} + \varepsilon y + \mathcal{T}_\varepsilon(\widetilde{\breve{\psi}_\varepsilon})(x,y), y + \frac{\mathcal{T}_\varepsilon(\widetilde{\breve{\psi}_\varepsilon})(x,y)}{\varepsilon}\right) \to \varphi(x, y + \breve{\psi}_0(x,y)) = \varphi(x, \psi_0(x,y))$$

for a.e. $(x,y) \in \Omega \times \hat{Y}^*$.

Since $\|\varphi\|_{C(\overline{\Omega \times Y})} \leqslant C$ we have $|\mathcal{T}_\varepsilon(\widetilde{\varphi_{\varepsilon,\psi_\varepsilon}}(x,y)| \leqslant C$ for a.e. $(x,y) \in \Omega \times Y$ and can apply Lebesgue's convergence theorem, which yields the strong convergence of $\mathcal{T}_\varepsilon(\widetilde{\varphi_{\varepsilon,\psi_\varepsilon}})$ to $\widetilde{\varphi_{\psi_0}}$ in $L^p(\Omega \times Y)$. Because this argumentation holds for every subsequence, it holds for the whole sequence. □

Now, we use the strong two-scale convergence of the transformed test functions in order to show the weak two-scale convergence for the transformed sequence of some arbitrary two-scale converging sequence. We thus obtain the equivalence of the weak two-scale convergence of sequences defined on Ω_ε and the corresponding sequences defined on $\hat{\Omega}_\varepsilon$.

Theorem 2.20. *Let $p \in (1, \infty)$. Let u_ε be a sequence in $L^p(\hat{\Omega}_\varepsilon)$ and $\hat{u}_\varepsilon = u_\varepsilon \circ \psi_\varepsilon$. Then,*

$$u_\varepsilon \xrightarrow{p} u_0 \qquad \text{if and only if} \qquad \hat{u}_\varepsilon \xrightarrow{p} \hat{u}_0 \tag{2.12}$$

for $u_0 \in L^p(\Omega; L^p(Y^*(x)))$ and $\hat{u}_0 \in L^p(\Omega \times \hat{Y}^*)$ and it holds

$$\begin{aligned} \hat{u}_0(x,y) &= u_0(x, \psi_0(x,y)) \quad \text{for a.e. } (x,y) \in \Omega \times \hat{Y}^*, \\ u_0(x,y) &= \hat{u}_0(x, \psi_0^{-1}(x,y)) \quad \text{for a.e. } (x,y) \in \mathcal{Q}. \end{aligned} \tag{2.13}$$

Proof. First, we assume that $\hat{u}_\varepsilon \xrightarrow{p} \hat{u}_0$ and show $u_\varepsilon \xrightarrow{p} u_0$. The two-scale convergence of $\hat{u}_\varepsilon$ implies the boundedness of $\|\hat{u}_\varepsilon\|_{L^p(\hat{\Omega}_\varepsilon)}$ (see Lemma 2.11) and also the boundedness of $\|\widetilde{u_\varepsilon}\|_{L^p(\Omega)} = \|u_\varepsilon\|_{L^p(\Omega_\varepsilon)}$. Since the limit $\widetilde{\hat{u}_{0,\psi_0^{-1}}}$ is also in $L^p(\Omega \times Y)$, it is sufficient to show the distributional two-scale convergence, i.e.

$$\lim_{\varepsilon \to 0} \int_\Omega \widetilde{u_\varepsilon}(x) \varphi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x = \int_\Omega \int_Y \widetilde{\hat{u}_{0,\psi_0^{-1}}}(x,y) \varphi(x,y) \,\mathrm{d}y \,\mathrm{d}x \tag{2.14}$$

for every smooth function $\varphi \in D(\Omega; C^\infty_\#(Y))$. For this, we transform the integrand of the left-hand side by ψ_ε

$$\begin{aligned} \int_\Omega \widetilde{u_\varepsilon}(x) \varphi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x &= \int_{\Omega_\varepsilon} u_\varepsilon(x) \varphi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x = \int_{\hat{\Omega}_\varepsilon} J_\varepsilon(x) \hat{u}_\varepsilon(x) \varphi_{\varepsilon,\psi_\varepsilon}(x) \,\mathrm{d}x \\ &= \int_\Omega \widetilde{J_\varepsilon}(x) \widetilde{\hat{u}_\varepsilon}(x) \widetilde{\varphi_{\varepsilon,\psi_\varepsilon}}(x) \,\mathrm{d}x. \end{aligned}$$

Having $\widetilde{\varphi}_{\varepsilon,\psi_\varepsilon} \xrightarrow{< \infty} \widetilde{\varphi_{\psi_0}}$ from Lemma 2.19 and $\widetilde{J}_\varepsilon \xrightarrow{< \infty} \widetilde{J}_0$ from Lemma 2.9, we can pass to the limit $\varepsilon \to 0$ and get

$$\lim_{\varepsilon\to 0}\int_\Omega \widetilde{J_\varepsilon}(x)\widetilde{\hat{u}_\varepsilon}(x)\widetilde{\varphi}_{\varepsilon,\psi_\varepsilon}(x)\,\mathrm{d}x = \int_\Omega\int_Y \widetilde{J_0}(x,y)\widetilde{\hat{u}_0}(x,y)\widetilde{\varphi_{\psi_0}}(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$

Then, we transform the Y-integral back with $\psi_0(x,\cdot)$

$$\begin{aligned}\int_\Omega\int_Y \widetilde{J_0}(x,y)\widetilde{\hat{u}_0}(x,y)\widetilde{\varphi_{\psi_0}}(x,y)\,\mathrm{d}y\,\mathrm{d}x &= \int_\Omega\int_{Y^*} J_0(x,y)\hat{u}_0(x,y)\varphi(x,\psi_0(x,y))\,\mathrm{d}y\,\mathrm{d}x\\ &= \int_\Omega\int_{Y^*(x)} \hat{u}_0(x,\psi_0^{-1}(x,y))\varphi(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_\Omega\int_Y \widetilde{\hat{u}_{0,\psi_0^{-1}}}(x,y)\varphi(x,y)\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

Combining these equations shows (2.14).

Now, we assume that $u_\varepsilon \xrightarrow{p} u_0$ and show $\hat{u}_\varepsilon \xrightarrow{p} \hat{u}_0$. Again the two-scale convergence of u_ε implies the boundedness of u_ε and Lemma 2.11 transfers the boundedness onto $\hat{u}_\varepsilon$. By Theorem 1.21, we can pass to a subsequence such that $\hat{u}_\varepsilon \xrightarrow{p} \hat{u}_0$ for $\hat{u}_0 \in L^p(\Omega\times\hat{Y}^*)$. By applying the previous argumentation on this subsequence, we can identify $\hat{u}_0 = u_{0,\psi_0}$. Since this argumentation holds for every subsequence, it holds for the whole sequence. □

The next theorem shows that also the strong two-scale convergence is compatible with the transformation. This becomes highly important in the homogenisation of problems where the microstructure arises not only from the domain but also from oscillating coefficients.

Theorem 2.21. *Let $p \in (1,\infty)$. Let u_ε be a sequence in $L^p(\hat{\Omega}_\varepsilon)$ and $\hat{u}_\varepsilon = u_\varepsilon \circ \psi_\varepsilon$. Then,*

$$u_\varepsilon \xrightarrow{p} u_0 \qquad \textit{if and only if} \qquad \hat{u}_\varepsilon \xrightarrow{p} \hat{u}_0$$

for $u_0 \in L^p(\Omega; L^p(Y^*(x)))$ and $\hat{u}_0 \in L^p(\Omega\times\hat{Y}^*)$. Moreover,

$$\begin{aligned}\hat{u}_0(x,y) &= u_0(x,\psi_0(x,y)) \quad \textit{for a.e. } (x,y)\in\Omega\times\hat{Y}^*,\\ u_0(x,y) &= \hat{u}_0(x,\psi_0^{-1}(x,y)) \quad \textit{for a.e. } (x,y)\in\mathcal{Q}.\end{aligned}$$

Proof. Assume that $u_\varepsilon \xrightarrow{p} u_0$. Because of Theorem 2.20, it is sufficient to show that $\lim\limits_{\varepsilon\to 0}\|\widetilde{\hat{u}_\varepsilon}\|_{L^p(\Omega)} = \|\widetilde{\hat{u}_0}\|_{L^p(\Omega\times Y)}$. By transforming via ψ_ε and ψ_0, respectively, we obtain

$$\begin{aligned}\|\widetilde{\hat{u}_\varepsilon}\|^p_{L^p(\Omega)} &= \int_\Omega |\widetilde{\hat{u}_\varepsilon}(x)|^p\,\mathrm{d}x = \int_\Omega \widetilde{J_\varepsilon^{-1}\circ\psi_\varepsilon^{-1}}(x)|\widetilde{u_\varepsilon}(x)|^p\,\mathrm{d}x\\ &= \int_\Omega\int_Y \mathcal{T}_\varepsilon(\widetilde{J_\varepsilon^{-1}\circ\psi_\varepsilon^{-1}})(x,y)|\mathcal{T}_\varepsilon(\widetilde{u_\varepsilon})(x,y)|^p\,\mathrm{d}y\,\mathrm{d}x\end{aligned} \tag{2.15}$$

and

$$\|\widetilde{\hat{u}_0}\|^p_{L^p(\Omega\times Y)} = \int_\Omega\int_Y |\widetilde{\hat{u}_0}(x,y)|^p \,\mathrm{d}y\,\mathrm{d}x = \int_\Omega\int_Y \widetilde{J^{-1}_{0,\psi_0^{-1}}}(x,y)|\widetilde{u_0}(x,y)|^p\,\mathrm{d}y\,\mathrm{d}x. \tag{2.16}$$

After subtracting (2.16) from (2.15), we obtain with the triangle inequality

$$\begin{aligned} &\Big|\|\widetilde{\hat{u}_\varepsilon}\|^p_{L^p(\Omega)} - \|\widetilde{\hat{u}_0}\|^p_{L^p(\Omega\times Y)}\Big| \\ &\leqslant \Big|\int_\Omega\int_Y \mathcal{T}_\varepsilon(\widetilde{J_\varepsilon^{-1}\circ\psi_\varepsilon^{-1}})(x,y)\Big(|\mathcal{T}_\varepsilon(\widetilde{u_\varepsilon})(x,y)|^p - |\widetilde{u_0}(x,y)|^p\Big)\,\mathrm{d}y\,\mathrm{d}x\Big| \\ &\quad + \Big|\int_\Omega\int_Y \Big(\mathcal{T}_\varepsilon(\widetilde{J_\varepsilon^{-1}\circ\psi_\varepsilon^{-1}})(x,y) - \widetilde{J^{-1}_{0,\psi_0^{-1}}}(x,y)\Big)|\widetilde{u_0}(x,y)|^p\,\mathrm{d}y\,\mathrm{d}x\Big|. \end{aligned} \tag{2.17}$$

Now, we show that both integrals on the right-hand side of (2.17) converge to zero. For the first integral, we note that $u_\varepsilon \xrightarrow{p}\!\!\!\!\to u_0$ gives the strong convergence $\mathcal{T}_\varepsilon(\widetilde{u_\varepsilon}) \to \widetilde{u_0}$ in $L^p(\Omega\times Y)$, which implies the strong convergence $|\mathcal{T}_\varepsilon(\widetilde{u_\varepsilon})|^p \to |\widetilde{u_0}|^p$ in $L^1(\Omega\times Y)$. Since $J_\varepsilon \geqslant c_J$, we obtain $\|\mathcal{T}_\varepsilon(\widetilde{J_\varepsilon^{-1}\circ\psi_\varepsilon^{-1}})\|_{L^\infty(\Omega\times Y)} \leqslant c_J^{-1}$. Then, with the Hölder inequality, we can deduce

$$\begin{aligned} &\Big|\int_\Omega\int_Y \mathcal{T}_\varepsilon(\widetilde{J_\varepsilon^{-1}\circ\psi_\varepsilon^{-1}})(x,y)\Big(|\mathcal{T}_\varepsilon(\widetilde{u_\varepsilon})(x,y)|^p - |\widetilde{u_0}(x,y)|^p\Big)\,\mathrm{d}y\,\mathrm{d}x\Big| \\ &\qquad \leqslant c_J^{-1}\int_\Omega\int_Y \Big||\mathcal{T}_\varepsilon(\widetilde{u_\varepsilon})(x,y)|^p - |\widetilde{u_0}(x,y)|^p\Big|\,\mathrm{d}y\,\mathrm{d}x \to 0. \end{aligned} \tag{2.18}$$

In order to estimate the second integral on the right-hand side of (2.17), we approximate $|\widetilde{u_0}|^p$, with respect to the $L^1(\Omega\times Y)$-norm, by a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of functions in $D(\Omega\times Y)$. Thus, we can estimate

$$\begin{aligned} &\Big|\int_\Omega\int_Y \Big(\mathcal{T}_\varepsilon(\widetilde{J_\varepsilon^{-1}\circ\psi_\varepsilon^{-1}})(x,y) - \widetilde{J^{-1}_{0,\psi_0^{-1}}}(x,y)\Big)|\widetilde{u_0}(x,y)|^p\,\mathrm{d}y\,\mathrm{d}x\Big| \\ &\quad = \Big|\int_\Omega\int_Y \Big(\mathcal{T}_\varepsilon(\widetilde{J_\varepsilon^{-1}\circ\psi_\varepsilon^{-1}})(x,y) - \widetilde{J^{-1}_{0,\psi_0^{-1}}}(x,y)\Big)\Big(|\widetilde{u_0}(x,y)|^p - \varphi_n(x,y)\Big)\,\mathrm{d}y\,\mathrm{d}x\Big| \\ &\qquad + \Big|\int_\Omega\int_Y \Big(\mathcal{T}_\varepsilon(\widetilde{J_\varepsilon^{-1}\circ\psi_\varepsilon^{-1}})(x,y) - \widetilde{J^{-1}_{0,\psi_0^{-1}}}(x,y)\Big)\varphi_n(x,y)\,\mathrm{d}y\,\mathrm{d}x\Big|. \end{aligned} \tag{2.19}$$

Since $\|\mathcal{T}_\varepsilon(\widetilde{J_\varepsilon^{-1}\circ\psi_\varepsilon^{-1}})\|_{L^\infty(\Omega\times Y)} \leqslant c_J^{-1}$ as well as $\|\widetilde{J^{-1}_{0,\psi_0^{-1}}}\|_{L^\infty(\Omega\times Y)} \leqslant c_J^{-1}$, which follows from

$J_0 \geqslant c_J$, we can estimate the first summand on the right-hand side of (2.19) by

$$\begin{aligned}\int_\Omega\int_Y &\Big|\Big(\mathcal{T}_\varepsilon(\widetilde{J_\varepsilon^{-1}\circ\psi_\varepsilon^{-1}})(x,y) - \widetilde{J^{-1}_{0,\psi_0^{-1}}}(x,y)\Big)\Big(|\widetilde{u_0}(x,y)|^p - \varphi_n(x,y)\Big)\Big|\,\mathrm{d}y\,\mathrm{d}x\\ &\leqslant c_J^{-1}\int_\Omega\int_Y\Big||\widetilde{u_0}(x,y)|^p - \varphi_n(x,y)\Big|\,\mathrm{d}y\,\mathrm{d}x.\end{aligned} \tag{2.20}$$

Then, choosing n large enough, this term becomes arbitrarily small.

Now, for fixed n, we show that the second summand on the right-hand side of (2.19) becomes arbitrarily small. We use that $J_\varepsilon^{-1}\xrightarrow{<\infty}\!\!\!\!\rightarrow J_0^{-1}$ (see Lemma 2.9) and, therefore, $J_\varepsilon^{-1}\xrightarrow{<\infty}\!\!\!\rightharpoonup J_0^{-1}$. Then, Theorem 2.20 implies $\widetilde{J_\varepsilon^{-1}\circ\psi_\varepsilon^{-1}}\xrightarrow{q}\!\!\!\rightharpoonup\widetilde{J^{-1}_{0,\psi_0^{-1}}}$ for all $q\in(1,\infty)$ and, accordingly, we obtain $\mathcal{T}_\varepsilon(\widetilde{J_\varepsilon^{-1}\circ\psi_\varepsilon^{-1}})\rightharpoonup\widetilde{J^{-1}_{0,\psi_0^{-1}}}$ in $L^q(\Omega\times Y)$, for all $q\in(1,\infty)$. Having $\varphi_n\in D(\Omega\times Y)\subset L^{q'}(\Omega\times Y)$, we can pass to the limit $\varepsilon\to 0$ for fixed n, i.e.

$$\Big|\int_\Omega\int_Y\Big(\mathcal{T}_\varepsilon(\widetilde{J_\varepsilon^{-1}\circ\psi_\varepsilon^{-1}})(x,y) - \widetilde{J^{-1}_{0,\psi_0^{-1}}}(x,y)\Big)\varphi_n(x,y)\,\mathrm{d}y\,\mathrm{d}x\Big|\to 0. \tag{2.21}$$

Thus, the left hand side of (2.19) converges to zero and together with (2.17) and (2.18), we can conclude $\lim_{\varepsilon\to0}\|\widetilde{\hat{u}_\varepsilon}\|_{L^p(\Omega)} = \|\widetilde{\hat{u}_0}\|_{L^p(\Omega\times Y)}$. Hence $\hat{u}_\varepsilon\xrightarrow{p}\!\!\!\!\rightarrow\hat{u}_0$.

The other direction, i.e. $u_\varepsilon\xrightarrow{p}\!\!\!\!\rightarrow u_0$ if $\hat{u}_\varepsilon\xrightarrow{p}\!\!\!\!\rightarrow\hat{u}_0$, follows by the same argumentation. □

Theorem 2.21 provides also the strong two-scale convergence of the set Ω_ε to $\mathcal{Q}$ in the sense of the characteristic functions.

Corollary 2.22. *Let Ω_ε be locally periodic domains with two-scale limit set $\mathcal{Q}$ in the sense of Definition 2.1. Then,*

$$\chi_{\Omega_\varepsilon}\xrightarrow{<\infty}\!\!\!\!\rightarrow\chi_{\mathcal{Q}},\qquad\quad (\textit{i.e.}\ \ \chi_{\Omega_\varepsilon}(x)\xrightarrow{<\infty}\!\!\!\!\rightarrow\chi_{\mathcal{Q}}(x,y) = \chi_{Y^*(x)}(y)\ ).$$

Proof. Since $\chi_{\hat{\Omega}_\varepsilon}$ is y-periodic, i.e. $\chi_{\hat{\Omega}_\varepsilon}(x) = \chi_{\hat{Y}^*_\#}(x/\varepsilon)$ for $x\in\Omega$, Lemma 1.7 provides the strong two-scale convergence $\chi_{\hat{\Omega}_\varepsilon}\xrightarrow{<\infty}\!\!\!\!\rightarrow\chi_{\hat{Y}^*}$. Then, Theorem 2.21 gives

$$\chi_{\Omega_\varepsilon}(x) = \chi_{\hat{\Omega}_\varepsilon}(\psi_\varepsilon^{-1}(x))\xrightarrow{<\infty}\!\!\!\!\rightarrow\chi_{\hat{Y}^*}(x,\psi_0^{-1}(x,y)) = \chi_{Y^*(x)}(y) = \chi_{\mathcal{Q}}(x,y).$$

□

In the next step, we show the transformation results for gradients.

Theorem 2.23. *Let $p\in(1,\infty)$. Let u_ε be a sequence in $L^p(\hat{\Omega}_\varepsilon)$ and $\hat{u}_\varepsilon = u_\varepsilon\circ\psi_\varepsilon$. Then,*

$$\nabla u_\varepsilon\xrightarrow{p}\!\!\!\rightharpoonup\chi_{\hat{Y}^*}\nabla_x u_0 + \nabla_y u_1\qquad \textit{if and only if}\qquad \nabla\hat{u}_\varepsilon\xrightarrow{p}\!\!\!\rightharpoonup\chi_{\hat{Y}^*}\nabla_x\hat{u}_0 + \nabla_y\hat{u}_1$$

for $u_0 \in W^{1,p}(\Omega)$, $\hat{u}_0 \in W^{1,p}(\Omega)$, $u_1 \in L^p(\Omega; W^{1,p}_{\#}(Y^*(x)))$ *and* $\hat{u}_1 \in L^p(\Omega; W^{1,p}_{\#}(\hat{Y}^*))$. *Moreover, it holds*

$$\begin{aligned} \hat{u}_0(x) &= u_0(x) && \textit{for a.e. } x \in \Omega, \\ \hat{u}_1(x,y) &= u_1(x, \psi_0(x,y)) + \breve{\psi}_0(x,y) \cdot \nabla_x u_0(x) && \textit{for a.e. } (x,y) \in \Omega \times \hat{Y}^*, \\ u_1(x,y) &= \hat{u}_1(x, \psi_0^{-1}(x,y)) + \breve{\psi}_0^{-1}(x,y) \cdot \nabla_x \hat{u}_0(x) && \textit{for a.e. } (x,y) \in \mathcal{Q}. \end{aligned} \tag{2.22}$$

Proof. First, assume that $\nabla \hat{u}_\varepsilon \xrightarrow{p} \chi_{\hat{Y}*} \nabla_x \hat{u}_0 + \nabla_y \hat{u}_1$. In order to show the two-scale convergence $\nabla u_\varepsilon \xrightarrow{p} \chi_{\hat{Y}*} \nabla_x u_0 + \nabla_y u_1$, we express these terms by $\chi_{\hat{Y}*}$, $\nabla \hat{u}_\varepsilon$, $\nabla_x \hat{u}_0$, $\nabla_y \hat{u}_1$. For this, we use the following identities, which arise from the chain rule

$$\begin{aligned} \partial_x u_\varepsilon(x) &= \partial_x(\hat{u}_\varepsilon(\psi_\varepsilon^{-1}(x))) = (\partial_x \hat{u}_\varepsilon)(\psi_\varepsilon^{-1}(x)) \partial_x(\psi_\varepsilon^{-1})(x), \\ \partial_x(\psi_\varepsilon^{-1})(x) &= (\partial_x \psi_\varepsilon)^{-1}(\psi_\varepsilon^{-1}(x)) = \Psi_\varepsilon^{-1}(\psi_\varepsilon^{-1}(x)), \end{aligned}$$

which yields

$$\nabla u_\varepsilon(x) = \Psi_\varepsilon^{-\top}(\psi_\varepsilon^{-1}(x))(\nabla \hat{u}_\varepsilon)(\psi_\varepsilon^{-1}(x)). \tag{2.23}$$

With Theorem 2.20, we can transfer the two-scale convergence of $\nabla \hat{u}_\varepsilon$ to

$$(\nabla \hat{u}_\varepsilon)(\psi_\varepsilon^{-1}(x)) \xrightarrow{p} \chi_{\hat{Y}*}(\psi_0^{-1}(x,y)) \nabla_x \hat{u}_0(x) + (\nabla_y \hat{u}_1)(x, \psi_0^{-1}(x,y))$$

and with Theorem 2.21, we can transfer the strong two-scale convergence of $\Psi_\varepsilon^{-\top}$ (see Lemma 2.9) to

$$\Psi_\varepsilon^{-\top}(\psi_\varepsilon^{-1}(x)) \xrightarrow{< \infty} \Psi_0^{-\top}(x, \psi_0^{-1}(x,y)).$$

Having additionally the uniform essential bound of $\Psi_\varepsilon^{-\top}$ (see Lemma 2.8), we can pass with Lemma 1.16 to the limit $\varepsilon \to 0$ for the product

$$\begin{aligned} \nabla u_\varepsilon(x) &= \Psi_\varepsilon^{-\top}(\psi_\varepsilon^{-1}(x))(\nabla \hat{u}_\varepsilon)(\psi_\varepsilon^{-1}(x)) \\ &\xrightarrow{p} \Psi_0^{-\top}(x, \psi_0^{-1}(x,y)) \big(\chi_{\hat{Y}*}(\psi_0^{-1}(x,y)) \nabla_x \hat{u}_0(x) + (\nabla_y \hat{u}_1)(x, \psi_0^{-1}(x,y)) \big) \\ &= \Psi_0^{-\top}(x, \psi_0^{-1}(x,y)) \chi_{Y^*(x)}(y) \nabla_x \hat{u}_0(x) + \Psi_0^{-\top}(x, \psi_0^{-1}(x,y))(\nabla_y \hat{u}_1)(x, \psi_0^{-1}(x,y)). \end{aligned} \tag{2.24}$$

After employing the chain rule, the coefficient $\Psi_0^{-\top}(x, \psi_0^{-1}(x,y))$ in front of the y-derivative of $\hat{u}_1$ will disappear as in (2.23). However, in the first summand on the right-hand side this substitution does not cancel $\Psi_0^{-\top}(x, \psi_0^{-1}(x,y))$, since we have no y-derivative and, thus, no chain rule to apply. Instead, we will separate $\Psi_0^{-\top}(x, \psi_0^{-1}(x,y)) \nabla_y \hat{u}_0(x)$ into a part which is constant in y and a part which can be written as a y-gradient, i.e.

$$\Psi_0^{-\top}(x, \psi_0^{-1}(x,y)) = (\nabla \psi_0)^{-1}(x, \psi_0^{-1}(x,y)) = \nabla(\psi_0^{-1})(x,y) = \mathbb{1} + \nabla \breve{\psi}_0^{-1}(x,y).$$

Then, we obtain

$$\begin{aligned}
&\Psi_0^{-\top}(x,\psi_0^{-1}(x,y))\chi_{Y^*(x)}(y)\nabla_x\hat{u}_0(x) + \Psi_0^{-\top}(x,\psi_0^{-1}(x,y))(\nabla_y\hat{u}_1)(x,\psi_0^{-1}(x,y))\\
&= (\mathbb{1} + \nabla_y\breve{\psi}_0^{-1}(x,y))\chi_{Y^*(x)}(y)\nabla_x\hat{u}_0(x) + \nabla_y(\hat{u}_1(x,\psi_0^{-1}(x,y)))\\
&= \chi_{Y^*(x)}(y)\nabla_x\hat{u}_0(x) + \nabla_y\breve{\psi}_0^{-1}(x,y)\nabla_x\hat{u}_0(x) + \nabla_y(\hat{u}_1(x,\psi_0^{-1}(x,y)))\\
&= \chi_{Y^*(x)}(y)\nabla_x\hat{u}_0(x) + \nabla_y\big(\breve{\psi}_0^{-1}(x,y)\cdot\nabla_x\hat{u}_0(x) + \hat{u}_1(x,\psi_0^{-1}(x,y))\big).
\end{aligned}\tag{2.25}$$

Combining (2.24) with (2.25) yields

$$\nabla u_\varepsilon \xrightarrow{p}\!\!\!\!\rightarrow \chi_{Y^*(x)}\nabla_x u_0 + \nabla_y u_1 \tag{2.26}$$

for $u_0(x) = \hat{u}_0(x)$ and $u_1(x,y) = \hat{u}_1(x,\psi_0^{-1}(x,y)) + \breve{\psi}_0^{-1}(x,y)\cdot\nabla_x\hat{u}_0(x)$.

For the other direction, we assume that $\nabla u_\varepsilon \xrightarrow{p}\!\!\!\!\rightarrow \chi_{\hat{Y}^*}\nabla_x u_0 + \nabla_y u_1$. From the chain rule, we obtain

$$\partial_x\hat{u}_\varepsilon(x) = \partial_x(u_\varepsilon(\psi_\varepsilon(x))) = (\partial_x u_\varepsilon)(\psi_\varepsilon(x))\partial_x\psi_\varepsilon(x).$$

Then, by the same argumentation as above, we can pass to the limit

$$\begin{aligned}
\nabla\hat{u}_\varepsilon(x) &= \Psi_\varepsilon^\top(x)(\nabla u_\varepsilon)(\psi_\varepsilon(x))\\
&\xrightarrow{p}\!\!\!\!\rightarrow \Psi_0^\top(x,y)\big(\chi_{Y^*(x)}(\psi_0(x,y))\nabla_x u_0(x) + (\nabla_y u_1)(x,\psi_0(x,y))\big).
\end{aligned}\tag{2.27}$$

The limit can be rewritten as

$$\begin{aligned}
&\Psi_0^\top(x,y)\big(\chi_{Y^*(x)}(\psi_0(x,y))\nabla_x u_0(x) + (\nabla_y u_1)(x,\psi_0(x,y))\big)\\
&= (\mathbb{1} + \nabla_y\breve{\psi}_0(x,y))\chi_{\hat{Y}^*}(y)\nabla_x u_0(x) + \Psi_0^\top(\nabla_y u_1)(x,\psi_0(x,y))\\
&= \chi_{\hat{Y}^*}(y)\nabla_x u_0(x) + \nabla_y\breve{\psi}_0(x,y)\nabla_x u_0(x) + \nabla_y(u_1(x,\psi_0(x,y)))\\
&= \chi_{\hat{Y}^*}(y)\nabla_x u_0(x) + \nabla_y\big(\breve{\psi}_0(x,y)\cdot\nabla_x u_0(x) + u_1(x,\psi_0(x,y))\big).
\end{aligned}\tag{2.28}$$

Combining (2.27) with (2.28) yields

$$\nabla\hat{u}_\varepsilon \xrightarrow{p}\!\!\!\!\rightarrow \chi_{\hat{Y}^*}\nabla_x\hat{u}_0 + \nabla_y\hat{u}_1$$

for $\hat{u}_0(x) = u_0$ and $\hat{u}_1(x,y) = \breve{\psi}_0(x,y)\cdot\nabla_x u_0(x) + \nabla_y(u_1(x,\psi_0(x,y)))$. □

We remember that $\breve{\psi}_0^{-1}(x,y) = -\breve{\psi}_0(x,\psi_0^{-1}(x,y))$ (see (2.2)) and, hence, we see that the two transformation rules between $\hat{u}_1$ and u_1 in (2.22) are consistent with each other

$$\begin{aligned}
\hat{u}_1(x,\psi_0^{-1}(x,y)) &= u_1(x,\psi_0(x,\psi_0^{-1}(x,y))) + \breve{\psi}_0(x,\psi_0^{-1}(x,y))\cdot\nabla_x u_0(x)\\
&= u_1(x,y) - \breve{\psi}_0^{-1}(x,y)\cdot\nabla_x\hat{u}_0(x).
\end{aligned}$$

In the next step we consider the transformation of the gradients in the case of large gradients.

Theorem 2.24. *Let $p \in (1,\infty)$. Let u_ε be a sequence in $W^{1,p}(\hat{\Omega}_\varepsilon)$ and $\hat{u}_\varepsilon = u_\varepsilon \circ \psi_\varepsilon$. Then,*

$$\varepsilon \nabla u_\varepsilon \xrightarrow{p} \nabla_y u_0 \qquad \text{if and only if} \qquad \varepsilon \nabla \hat{u}_\varepsilon \xrightarrow{p} \nabla_y \hat{u}_0 \tag{2.29}$$

for $u_0 \in L^p(\Omega; W^{1,p}_\#(Y^*(x)))$ and $\hat{u}_0 \in L^p(\Omega; W^{1,p}_\#(\hat{Y}^*))$ and

$$\begin{aligned} \hat{u}_0(x,y) &= u_0(x,\psi_0(x,y)) && \text{for a.e. } (x,y) \in \Omega \times \hat{Y}^*, \\ u_0(x,y) &= \hat{u}_0(x,\psi_0^{-1}(x,y)) && \text{for a.e. } (x,y) \in \mathcal{Q}. \end{aligned} \tag{2.30}$$

Proof. We assume that $\varepsilon \nabla \hat{u}_\varepsilon \xrightarrow{p} \nabla_y \hat{u}_0$. In order to show $\varepsilon \nabla u_\varepsilon(x) \xrightarrow{p} \nabla_y u_0(x, \psi_0(x,y))$, we rewrite it as in (2.23)

$$\varepsilon \nabla u_\varepsilon(x) = \varepsilon \Psi_\varepsilon^{-\top}(\psi_\varepsilon^{-1}(x))(\nabla \hat{u}_\varepsilon)(\psi_\varepsilon^{-1}(x)). \tag{2.31}$$

With Theorem 2.20, we can use the two-scale convergence of $\varepsilon \nabla \hat{u}_\varepsilon$ to show

$$\varepsilon (\nabla \hat{u}_\varepsilon)(\psi_\varepsilon^{-1}(x)) \xrightarrow{p} \nabla_y \hat{u}_0(x, \psi_0^{-1}(x,y))$$

and with Theorem 2.21, we can use the strong two-scale convergence of $\Psi_\varepsilon^{-\top}$ to show

$$\Psi_\varepsilon^{-\top}(\psi_\varepsilon^{-1}(x)) \xrightarrow{< \infty} \Psi_0^{-\top}(x, \psi_0^{-1}(x,y)).$$

Since $\Psi_\varepsilon^{-\top}$ is uniformly essentially bounded, we can pass with Lemma 1.16 to the limit $\varepsilon \to 0$ for the product in (2.31)

$$\begin{aligned} \varepsilon \nabla u_\varepsilon(x) &= \varepsilon \Psi_\varepsilon^{-\top}(\psi_\varepsilon^{-1}(x))(\nabla \hat{u}_\varepsilon)(\psi_\varepsilon^{-1}(x)) \\ &\xrightarrow{p} \Psi_0^{-\top}(x, \psi_0^{-1}(x,y))(\nabla_y \hat{u}_0)(x, \psi_0^{-1}(x,y)) = \nabla_y(\hat{u}_0(x, \psi_0^{-1}(x,y))), \end{aligned}$$

which gives

$$\varepsilon \nabla u_\varepsilon \xrightarrow{p} \nabla_y u_0(x,y)$$

for $u_0(x,y) = \hat{u}_0(x, \psi_0^{-1}(x,y))$.

For the other direction, we assume that $\varepsilon \nabla u_\varepsilon \xrightarrow{p} \nabla_y u_0$. Then, we obtain similarly

$$\varepsilon \nabla \hat{u}_\varepsilon(x) = \varepsilon \Psi_\varepsilon^{\top}(x)(\nabla u_\varepsilon)(\psi_\varepsilon(x)) \xrightarrow{p} \Psi_0^{\top}(x,y)(\nabla_y u_0)(\psi_0(x,y)) = \nabla_y(u_0(\psi_0(x,y))),$$

which gives

$$\varepsilon \nabla \hat{u}_\varepsilon \xrightarrow{p} \nabla_y \hat{u}_0$$

for $\hat{u}_0(x, y) = u_0(x, \psi_0(x, y))$. □

2.3. Homogenisation of an elliptic differential equation in locally periodic domains

Now, we employ the results of the previous section in order to homogenise an elliptical differential equation on a locally periodic domain. We consider the case of a fast diffusion (i.e. ε^0-scaling of the diffusion coefficient) and the case of a slow diffusion (i.e. ε^2-scaling of the diffusion coefficient). Let Ω_ε be a locally periodic domain with two-scale limit set $\mathcal{Q} \subset \Omega \times Y$ in the sense of Definition 2.1. For the case of the scaling ε^0, we additionally assume, in the following, that $Y^*_\#$ is connected and Lipschitz and that Ω is Lipschitz. Let a_ε be a bounded sequence in $L^\infty(\Omega_\varepsilon)^{n\times n}$ and b_ε be a bounded sequence in $L^\infty(\Omega_\varepsilon)$, which are uniformly elliptic and strongly two-scale converge to $a_0 \in L^\infty(\mathcal{Q})^{n\times n}$ and $b_0 \in L^\infty(\mathcal{Q})$, respectively, i.e. there exist constants $\alpha, C > 0$ such that, for all $\zeta \in \mathbb{R}^n$ and a.e. $x \in \Omega_\varepsilon$,

$$\begin{aligned} \|a_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} &\leqslant C, & \zeta^\top a_\varepsilon(x)\zeta &\geqslant \alpha\|\zeta\|^2, & a_\varepsilon &\xrightarrow{<\infty}\!\!\!\!\rightarrow a_0, \\ \|b_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} &\leqslant C, & b_\varepsilon(x) &> \alpha, & b_\varepsilon &\xrightarrow{<\infty}\!\!\!\!\rightarrow b_0. \end{aligned}$$

Let f_ε be sequence in $L^2(\Omega_\varepsilon)$ and $f_0 \in L^2(\mathcal{Q})$, such that $f_\varepsilon \xrightarrow{2} f_0$, and let $l \in \{0, 2\}$. We look for a solution u_ε of the problem

Microscopic elliptic problem

$$\begin{aligned} -\operatorname{div}(\varepsilon^l a_\varepsilon(x)\nabla u_\varepsilon(x)) + b_\varepsilon(x)u_\varepsilon(x) &= f_\varepsilon(x) && \text{in } \Omega_\varepsilon, \\ \varepsilon^l a_\varepsilon(x)\nabla u_\varepsilon(x)\cdot n(x) &= 0 && \text{on } \partial\Omega_\varepsilon, \end{aligned} \tag{2.32}$$

where n denotes the outer normal of Ω_ε. The weak formulation of (2.32) is given by:

Weak form of the microscopic elliptic problem

Find $u_\varepsilon \in H^1(\Omega_\varepsilon)$ such that

$$\int_{\Omega_\varepsilon} \varepsilon^l a_\varepsilon(x)\nabla u_\varepsilon(x)\cdot\nabla\varphi(x) + b_\varepsilon(x)u_\varepsilon(x)\varphi(x)\,\mathrm{d}x = \int_{\Omega_\varepsilon} f_\varepsilon(x)\varphi(x)\,\mathrm{d}x \tag{2.33}$$

for all $\varphi \in H^1(\Omega_\varepsilon)$.

We perform the homogenisation in two different ways. In the first approach, we transform the problem to the periodic reference domain, pass to the limit there and then transform the limit results back. In the second approach, we use the results of the previous section to derive two-scale compactness results for locally periodic domains. Having these, we can perform the homogenisation directly on the locally periodic domain without transformation.

As the previous theoretical discussion already showed, these two homogenisation approaches will lead to the same homogenisation result. Nevertheless, for the sake of clarity, we present both approaches.

Indeed, the second approach requires no transformation and is therefore shorter. However, for more complicated problems, such as the Stokes problem or parabolic problems, which we will see later, the derivation of the a-priori estimates becomes more complicated and the homogenisation of the interface terms would require additional discussion if we are not working mainly in the transformed setting.

2.3.1. Homogenisation by substitution to a periodic substitute domain

Since Ω_ε is locally periodic in the sense of Definition 2.1, we obtain a sequence ψ_ε of transformation mappings with $\Psi_\varepsilon(x) := \partial_x\psi_\varepsilon(x)$, $J_\varepsilon := \det(\Psi_\varepsilon(x))$ and $A_\varepsilon(x) := \operatorname{Adj}(\Psi_\varepsilon(x))$.

We define the transformed data

$$\hat{f}_\varepsilon(x) := f_\varepsilon(\psi_\varepsilon(x)), \qquad \hat{a}_\varepsilon(x) := a_\varepsilon(\psi_\varepsilon(x)), \qquad \hat{b}_\varepsilon(x) := b_\varepsilon(\psi_\varepsilon(x))$$

for a.e. $x \in \hat{\Omega}_\varepsilon$. Moreover, we note that

$$\Psi_\varepsilon^{-\top}(x)\hat{n}(x) = \|\Psi_\varepsilon^{-\top}(x)\hat{n}(x)\|\; n(\psi(x))$$

for a.e. $x \in \partial\hat{\Omega}_\varepsilon$, where $\hat{n}$ denotes the outer normal of $\hat{\Omega}_\varepsilon$ and n the outer normal of Ω_ε. By changing the coordinates for the unknown, i.e.

$$\hat{u}_\varepsilon(x) = u_\varepsilon(\psi_\varepsilon(x))$$

for $x \in \hat{\Omega}_\varepsilon$, we obtain the following strong formulation for the unknown $\hat{u}_\varepsilon$ after changing the coordinates in (2.32).

Microscopic elliptic problem in the reference coordinates

$$\begin{aligned} -J_\varepsilon^{-1}(x)\operatorname{div}(\varepsilon^l A_\varepsilon(x)\hat{a}_\varepsilon(x)\Psi_\varepsilon^{-\top}(x)\nabla\hat{u}_\varepsilon(x)) + \hat{b}_\varepsilon(x)\hat{u}_\varepsilon(x) &= \hat{f}_\varepsilon(x) && \text{in } \Omega_\varepsilon, \\ \varepsilon^l a_\varepsilon(x)\Psi_\varepsilon^{-\top}(x)\nabla\hat{u}_\varepsilon(x)\cdot\|\Psi_\varepsilon^{-\top}(x)\hat{n}(x)\|^{-1}\Psi_\varepsilon^{-\top}\hat{n}(x) &= 0 && \text{on } \partial\Omega_\varepsilon. \end{aligned} \tag{2.34}$$

In order to obtain the weak formulation of (2.34), we multiply the first equation of (2.34) by J_ε and the second by $J_\varepsilon\|\Psi_\varepsilon^{-\top}(x)\hat{n}(x)\|$, which yields

$$\begin{aligned} -\operatorname{div}(\varepsilon^l A_\varepsilon(x)\hat{a}_\varepsilon(x)\Psi_\varepsilon^{-\top}(x)\nabla\hat{u}_\varepsilon(x)) + J_\varepsilon(x)b_\varepsilon(x)\hat{u}_\varepsilon(x) &= J_\varepsilon(x)\hat{f}_\varepsilon(x) && \text{in } \Omega_\varepsilon, \\ \varepsilon^l A_\varepsilon(x)\hat{a}_\varepsilon(x)\Psi_\varepsilon^{-\top}(x)\nabla\hat{u}_\varepsilon(x)\cdot\hat{n}(x) &= 0 && \text{on } \partial\Omega_\varepsilon. \end{aligned} \tag{2.35}$$

Now, we obtain the corresponding weak formulation by standard procedure.

Weak form of the microscopic elliptic problem in the reference coordinates

Find $\hat{u}_\varepsilon \in H^1(\hat{\Omega}_\varepsilon)$ such that

$$\int\limits_{\Omega_\varepsilon} \varepsilon^l A_\varepsilon(x)\hat{a}_\varepsilon(x)\Psi_\varepsilon^{-\top}(x)\nabla\hat{u}_\varepsilon(x)\cdot\nabla\varphi(x) + J_\varepsilon(x)\hat{b}_\varepsilon(x)\hat{u}_\varepsilon(x)\varphi(x)\,\mathrm{d}x = \int\limits_{\Omega_\varepsilon} J_\varepsilon(x)\hat{f}_\varepsilon(x)\varphi(x)\,\mathrm{d}x \tag{2.36}$$

for all $\varphi \in H^1(\hat{\Omega}_\varepsilon)$.

The weak formulation can also be derived by directly transforming the weak formulation (2.33), which provides the following equivalence.

Lemma 2.25. *Let $u_\varepsilon \in H^1(\Omega_\varepsilon)$ and $\hat{u}_\varepsilon \in H^1(\hat{\Omega}_\varepsilon)$ with $\hat{u}_\varepsilon = u_\varepsilon \circ \psi_\varepsilon$. Then, u_ε solves* (2.33) *if and only if $\hat{u}_\varepsilon$ solves* (2.36)*.*

Proof. Lemma 2.25 follows from transforming (2.33) into (2.36) and the fact that $H^1(\Omega_\varepsilon)$ can be identified with $H^1(\hat{\Omega}_\varepsilon)$ via the coordinate transformation ψ_ε. □

Existence and uniqueness of a solution and a-priori estimates

In order to apply the compactness results, we have to derive some a-priori estimates first. Therefore, we will often use $A_\varepsilon = J_\varepsilon\Psi_\varepsilon^{-1}$, in the following, which yields

$$\int\limits_{\hat{\Omega}_\varepsilon} A_\varepsilon(x)\hat{a}_\varepsilon(x)\Psi_\varepsilon^{-\top}(x)\nabla\hat{u}_\varepsilon(x)\cdot\nabla\varphi(x)\,\mathrm{d}x = \int\limits_{\hat{\Omega}_\varepsilon} J_\varepsilon(x)\hat{a}_\varepsilon(x)\Psi_\varepsilon^{-\top}(x)\nabla\hat{u}_\varepsilon(x)\cdot\Psi_\varepsilon^{-\top}(x)\varphi(x)\,\mathrm{d}x.$$

Theorem 2.26. *There exists a unique solution $\hat{u}_\varepsilon \in H^1(\hat{\Omega}_\varepsilon)$ of* (2.36)*. Moreover,*

$$\|\hat{u}_\varepsilon\|_{L^2(\hat{\Omega}_\varepsilon)} + \varepsilon^{l/2}\|\nabla\hat{u}_\varepsilon\|_{L^2(\hat{\Omega}_\varepsilon)} \leqslant C. \tag{2.37}$$

Proof. We show the existence and uniqueness of a solution using the theorem of Lax–Milgram. Therefore, we show the coercivity and continuity of the left-hand side of (2.36). The coercivity of a_ε and b_ε and their uniform essential boundedness are pointwise properties and, hence, they are preserved under the transformation, i.e.

$$\|\hat{a}_\varepsilon\|_{L^\infty(\hat{\Omega}_\varepsilon)} \leqslant C, \quad \zeta^\top\hat{a}_\varepsilon(x)\zeta \geqslant \alpha\|\zeta\|^2, \quad \|\hat{b}_\varepsilon\|_{L^\infty(\hat{\Omega}_\varepsilon)} \leqslant C, \quad \hat{b}_\varepsilon(x) \geqslant \alpha$$

for a.e. $x \in \hat{\Omega}_\varepsilon$ all $\zeta \in \mathbb{R}^n$. Thus, we obtain

$$\begin{aligned} A_\varepsilon(x)\hat{a}_\varepsilon(x)\Psi_\varepsilon^{-\top}(x)\zeta\cdot\zeta &= \hat{a}_\varepsilon(x)\sqrt{J_\varepsilon}(x)\Psi_\varepsilon^{-\top}(x)\zeta\cdot\sqrt{J_\varepsilon}(x)\Psi_\varepsilon^{-\top}(x)\zeta \\ &\geqslant \alpha|\sqrt{J_\varepsilon}(x)\Psi_\varepsilon^{-\top}(x)\zeta|^2 \end{aligned}$$

for a.e. $x \in \hat{\Omega}_\varepsilon$ and every $\zeta \in \mathbb{R}^n$. With the uniform boundedness of $J_\varepsilon \geqslant c_J$ from below and the uniform essential boundedness of $\Psi_\varepsilon^\top$, we can conclude

$$\begin{aligned}|\zeta|^2 &= |(\sqrt{J_\varepsilon}(x)^{-1}\Psi_\varepsilon^\top(x))\sqrt{J_\varepsilon}(x)\Psi_\varepsilon^{-\top}(x)\zeta|^2 \leqslant |\sqrt{J_\varepsilon}(x)^{-1}\Psi_\varepsilon^\top(x)|^2|\sqrt{J_\varepsilon}(x)\Psi_\varepsilon^{-\top}(x)\zeta|^2 \\ &\leqslant c_J^{-1}C|\sqrt{J_\varepsilon}(x)\Psi_\varepsilon^{-\top}(x)\zeta|^2\end{aligned}$$

for a.e. $x \in \hat{\Omega}_\varepsilon$ and every $\zeta \in \mathbb{R}^n$. Combining the previous two equations shows

$$A_\varepsilon(x)\hat{a}_\varepsilon(x)\Psi_\varepsilon^{-\top}(x)\zeta \cdot \zeta \geqslant c|\zeta|^2 \tag{2.38}$$

for some $c > 0$. Then, we obtain with the uniform boundedness of J_ε and $\hat{b}_\varepsilon$ from below that

$$\begin{aligned}\int_{\Omega_\varepsilon} \varepsilon^l A_\varepsilon(x)\hat{a}_\varepsilon(x)\Psi_\varepsilon^{-\top}(x)\nabla u(x) \cdot \nabla u(x) + J_\varepsilon(x)\hat{b}_\varepsilon(x)u(x)u(x)\,\mathrm{d}x \\ \geqslant \varepsilon^l c\|\nabla u\|^2_{L^2(\hat{\Omega}_\varepsilon)} + \alpha c_J\|u\|^2_{L^2(\hat{\Omega}_\varepsilon)}\end{aligned} \tag{2.39}$$

for every $u \in H^1(\hat{\Omega}_\varepsilon)$. Now, we use the essential uniform boundedness of the coefficients $\hat{a}_\varepsilon, \hat{b}_\varepsilon$, A_ε, $\Psi_\varepsilon^{-\top}$ and J_ε (see Lemma 2.8), apply the Hölder inequality and transfer the uniform boundedness of $\|f_\varepsilon\|_{L^2(\Omega_\varepsilon)}$ via Lemma 2.11 on $\|\hat{f}_\varepsilon\|_{L^2(\hat{\Omega}_\varepsilon)}$, so that we obtain the continuity of the left- and the right-hand side of (2.36), i.e.

$$\begin{aligned}\int_{\hat{\Omega}_\varepsilon} \varepsilon^l A_\varepsilon(x)\hat{a}_\varepsilon(x)\Psi_\varepsilon^{-\top}(x)\nabla u(x) \cdot \nabla v(x) + J_\varepsilon(x)\hat{b}_\varepsilon(x)u(x)v(x)\,\mathrm{d}x \\ \leqslant \varepsilon^l C\|\nabla u\|_{L^2(\hat{\Omega}_\varepsilon)}\|\nabla v\|_{L^2(\hat{\Omega}_\varepsilon)} + C\|u\|_{L^2(\hat{\Omega}_\varepsilon)}\|v\|_{L^2(\hat{\Omega}_\varepsilon)}, \\ \int_{\hat{\Omega}_\varepsilon} J_\varepsilon(x)\hat{f}_\varepsilon(x)u(x)\,\mathrm{d}x \leqslant C\|\hat{f}_\varepsilon\|_{L^2(\hat{\Omega}_\varepsilon)}\|u\|_{L^2(\hat{\Omega}_\varepsilon)}.\end{aligned} \tag{2.40}$$

for all $u, v \in H^1(\hat{\Omega}_\varepsilon)$. Then, the Lax–Milgram theorem provides the existence and uniqueness of a solution $\hat{u}_\varepsilon$. By choosing u and v equal to $\hat{u}_\varepsilon$ in (2.39) and (2.40) and combining these estimates via (2.36), we obtain with the Young inequality

$$\begin{aligned}\varepsilon^l c\|\nabla\hat{u}_\varepsilon\|^2_{L^2(\hat{\Omega}_\varepsilon)} + \alpha c_J\|\hat{u}_\varepsilon\|^2_{L^2(\hat{\Omega}_\varepsilon)} &\leqslant C\|\hat{f}_\varepsilon\|_{L^2(\hat{\Omega}_\varepsilon)}\|\hat{u}_\varepsilon\|_{L^2(\hat{\Omega}_\varepsilon)} \\ &\leqslant \frac{1}{2\alpha c_J}C^2\|\hat{f}_\varepsilon\|^2_{L^2(\hat{\Omega}_\varepsilon)} + \frac{\alpha c_J}{2}\|\hat{u}_\varepsilon\|^2_{L^2(\hat{\Omega}_\varepsilon)},\end{aligned}$$

which provides the uniform a-priori estimate (2.37). □

Limit problem in the transformed coordinates

By passing to the limit $\varepsilon \to 0$ in (2.36), we derive the two-scale limit problems for $l \in \{0, 2\}$ and the homogenised problem for $l = 0$. For the case $l = 0$, the two-scale limit problem

Weak form of the two-scale limit problem in the reference coordinates (for $l = 0$)

Find $(\hat{u}_0, \hat{u}_1) \in H^1(\Omega) \times L^2(\Omega; H^1_\#(\hat{Y}^*)/\mathbb{R})$ such that

$$\int_\Omega \int_{\hat{Y}^*} A_0(x,y)\hat{a}_0(x,y)\Psi_0^{-\top}(x,y)\big(\nabla_x \hat{u}_0(x)+\nabla_y \hat{u}_1(x,y)\big)\cdot\big(\nabla_x\varphi_0(x) + \nabla_y\varphi_1(x,y)\big)\,\mathrm{d}y\,\mathrm{d}x$$

$$+\int_\Omega\int_{\hat{Y}^*} J_0(x,y)\hat{b}_0(x,y)\,\mathrm{d}y\;\hat{u}_0(x)\varphi_0(x)\,\mathrm{d}x = \int_\Omega\int_{\hat{Y}^*} J_0(x,y)\hat{f}_0(x,y)\,\mathrm{d}y\;\varphi_0(x)\,\mathrm{d}x \tag{2.41}$$

for every $(\varphi_0, \varphi_1) \in H^1(\Omega) \times L^2(\Omega; H^1_\#(\hat{Y}^*)/\mathbb{R})$.

By separating the x and y variables in (2.41), we get the homogenised equation:

Weak form of the homogenised problem in the reference coordinates (for $l = 0$)

Find $\hat{u}_0 \in H^1(\Omega)$, such that

$$\int_\Omega \hat{a}^*(x)\nabla_x\hat{u}_0(x)\nabla\varphi(x) + \hat{b}^*(x)\hat{u}_0(x)\varphi(x)\,\mathrm{d}x = \int_\Omega \hat{f}^*(x)\varphi(x)\,\mathrm{d}x, \tag{2.42}$$

for every $\varphi \in H^1(\Omega)$, where $\hat{a}^*$, $\hat{b}^*$ and $\hat{f}^*$ are given by

$$\hat{a}^*_{ij}(x) := \int_{\hat{Y}^*} A_0(x,y)\hat{a}_0(x,y)\Psi_0^{-\top}(x,y)(e_j + \nabla_y\hat{\zeta}_j(x,y))\cdot e_i\,\mathrm{d}y \tag{2.43}$$

$$\hat{b}^*(x) := \int_{\hat{Y}^*} J_0(x,y)\hat{b}_0(x,y)\,\mathrm{d}y \tag{2.44}$$

$$\hat{f}^*(x) := \int_{\hat{Y}^*} J_0(x,y)\hat{f}_0(x,y)\,\mathrm{d}y \tag{2.45}$$

for a.e. $x \in \Omega$ and every $i, j \in \{1, \dots, n\}$, where $\hat{\zeta}_j \in L^\infty(\Omega; H^1_\#(\hat{Y}^*))$ is the unique solution of

$$\int_{\hat{Y}^*} A_0(x,y)\hat{a}_0(x,y)\Psi_0^{-\top}(x,y)(\nabla_y\hat{\zeta}_j(x,y) + e_j)\cdot\nabla\varphi(y)\,\mathrm{d}x = 0 \tag{2.46}$$

for all $\varphi \in H^1_\#(\hat{Y}^*)$ and a.e. $x \in \Omega$.

For the case $l = 2$, we get the following two-scale limit problem:

Weak form of the two-scale limit problem in the reference coordinates (for $l = 2$)

Find $\hat{u}_0 \in L^2(\Omega; H^1_\#(\hat{Y}^*))$ such that

$$\begin{aligned}
&\int_\Omega \int_{\hat{Y}^*} A_0(x,y)\hat{a}_0(x,y)\Psi_0^{-\top}(x,y)\nabla_y \hat{u}_0(x,y) \cdot \nabla_y \varphi_0(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&\quad + \int_\Omega \int_{\hat{Y}^*} J_0(x,y)\hat{b}_0(x,y)\hat{u}_0(x,y)\varphi_0(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_\Omega \int_{\hat{Y}^*} J_0(x,y)\hat{f}_0(x,y)\varphi_0(x,y)\,\mathrm{d}y\,\mathrm{d}x
\end{aligned} \tag{2.47}$$

for every $\varphi_0 \in L^2(\Omega; H^1_\#(\hat{Y}^*))$.

By means of Theorem 2.21 and Theorem 2.20, respectively, we can infer the strong two-scale convergence of the coefficients of (2.36) and the weak two-scale convergence of the right-hand side, namely

$$\begin{aligned}
\hat{a}_\varepsilon(x) &\xrightarrow{<\infty} \hat{a}_0(x,y) := a_0(x, \psi_0(x,y)), \\
\hat{b}_\varepsilon(x) &\xrightarrow{<\infty} \hat{b}_0(x,y) := b_0(x, \psi_0(x,y)), \\
\hat{f}_\varepsilon(x) &\xrightarrow{2} \hat{f}_0(x,y) := f_0(x, \psi_0(x,y)).
\end{aligned}$$

Moreover, Lemma 2.17 transfers the essential boundedness of a_0 and b_0 to $\hat{a}_0$ and $\hat{b}_0$, respectively.

Then, the derivation of the two-scale limit problems (2.41) and (2.47) becomes a well known two-scale homogenisation task. We start with the case $l = 0$.

Theorem 2.27. *Let $l = 0$ and assume that $\hat{Y}^*_\#$ is connected. Then, for the sequence of solutions $\hat{u}_\varepsilon$ of* (2.36) *it holds*

$$\hat{u}_\varepsilon \xrightarrow{2} \chi_{\hat{Y}^*}\hat{u}_0, \qquad \nabla\hat{u}_\varepsilon \xrightarrow{2} \chi_{\hat{Y}^*}\nabla_x \hat{u}_0 + \nabla_y \hat{u}_1$$

where $(\hat{u}_0, \hat{u}_1) \in H^1(\Omega) \times L^2(\Omega; H^1_\#(\hat{Y}^*)/\mathbb{R})$ is the unique solution of the two-scale limit problem (2.41).

Proof. From Lemma 2.9, we obtain the strong two-scale convergence of A_ε, $\Psi_\varepsilon^{-\top}$, J_ε (with respect to every L^p for $p \in (1,\infty)$ and with Lemma 2.8 their uniform essential boundedness. Together with the strong two-scale convergence of $\hat{a}_\varepsilon$ and $\hat{b}_\varepsilon$, their uniform essential boundedness and the two-scale convergence of $\hat{f}_\varepsilon$, we can deduce from Lemma 1.16 the strong two-scale convergence of $A_\varepsilon \hat{a}_\varepsilon \Psi_\varepsilon^{-\top}$ and $J_\varepsilon \hat{b}_\varepsilon$ as well as the weak two-scale convergence of $J_\varepsilon \hat{f}_\varepsilon$.

Having the a-priori estimates (2.37) for $\hat{u}_\varepsilon$, the uniform essential estimates for the coefficients and the two-scale convergences for the coefficients and the right-hand side, we can pass to the limit $\varepsilon \to 0$ by classical two-scale argumentation. With the a-priori estimates (2.37) and the compactness result Theorem 1.18, we obtain $(\hat{u}_0, \hat{u}_1) \in H^1(\Omega) \times L^2(\Omega; H^1_\#(\hat{Y}^*)/\mathbb{R})$ such that for a subsequence $\hat{u}_\varepsilon$ one has $\hat{u}_\varepsilon \overset{2}{\rightharpoonup} \chi_{\hat{Y}^*}\hat{u}_0$ and $\nabla \hat{u}_\varepsilon \overset{2}{\rightharpoonup} \chi_{\hat{Y}^*}\nabla_x \hat{u}_0 + \nabla_y \hat{u}_1$. Then, we test (2.36) with $\varphi_0 + \varphi_1(\cdot, \frac{\cdot}{\varepsilon})$ for $\varphi_0 \in H^1(\Omega)$ and $\varphi_1 \in L^2(\Omega; C_\#(\hat{Y}^*))$. After passing to the limit, we obtain (2.41) for test functions $\varphi_0 \in H^1(\Omega)$ and $\varphi_1 \in L^2(\Omega; C_\#(\hat{Y}^*))$ and by a density argument for test functions $(\varphi_0, \varphi_1) \in H^1(\Omega) \times L^2(\Omega; H^1_\#(\hat{Y}^*)/\mathbb{R})$. Since the argumentation holds for every subsequence, we obtain the convergence for the whole sequence by showing the uniqueness of a solution of (2.41).

The existence of a solution of (2.41) is already ensured by the homogenisation process, while the uniqueness follows from the theorem of Lax–Milgram. We use the solution space $H^1(\Omega) \times L^2(\Omega; H^1_\#(\hat{Y}^*)/\mathbb{R})$ and focus on showing the coercivity of the left-hand side. Analogously to the derivation of the coercivity estimate (2.39) for the ε-scaled problem, we obtain the uniform coercivity of $A_0\hat{a}_0\Psi_0^{-\top}$, which yields

$$\begin{aligned}
&\int_\Omega \int_{\hat{Y}*} A_0(x,y)\hat{a}_0(x,y)\Psi_0^{-\top}(x,y)\big(\nabla_x\varphi_0(x) + \nabla_y\varphi_1(x,y)\big) \cdot \big(\nabla_x\varphi_0(x) + \nabla_y\varphi_1(x,y)\big)\,\mathrm{d}y\,\mathrm{d}x \\
&\qquad + \int_\Omega \int_{\hat{Y}*} J_0(x,y)\hat{b}_0(x,y)\,\mathrm{d}y\ \hat{u}_0(x)\varphi_0(x)\,\mathrm{d}x \\
&\qquad \geqslant c\int_\Omega \int_{\hat{Y}*} \|\nabla_x\varphi_0(x) + \nabla_y\varphi_1(x,y)\|^2\,\mathrm{d}y\,\mathrm{d}x + \alpha c_J \int_\Omega \int_{\hat{Y}*} \|\varphi_0(x)\|^2 \\
&\qquad \geqslant c\|\nabla_x\varphi_0\|^2_{L^2(\Omega)} + c\|\nabla_y\varphi_1\|^2_{L^2(\Omega\times\hat{Y}^*)}\,\mathrm{d}x + \alpha c_J|\hat{Y}^*|\|\nabla_x\varphi_0\|_{L^2(\Omega)},
\end{aligned}$$

for every $(\varphi_0, \varphi_1) \in H^1(\Omega) \times L^2(\Omega; H^1_\#(\hat{Y}^*)/\mathbb{R})$, where we refer to [CDG18, Lemma 5.4] for the last inequality. This shows the coercivity of the left-hand side of (2.41). The continuity of the left- and the right-hand side follow easily. Afterwards, the theorem of Lax–Milgram provides the uniqueness of the solution of (2.41). ☐

Theorem 2.28. *Let $(\hat{u}_0, \hat{u}_1)$ be the solution of the two-scale limit problem (2.41). Then $\hat{u}_0$ solves the homogenised problem (2.42) and it holds*

$$\hat{u}_1 = \sum_{j=1}^{n} \partial_{x_j}\hat{u}_0\hat{\zeta}_j.$$

Proof. Theorem 2.28 follows by classical separation of the x and y variables. ☐

Now, we consider the case $l = 2$.

Theorem 2.29. *Let $l = 2$. Then, for the sequence of solutions $\hat{u}_\varepsilon$ of (2.36) it holds*

$$\hat{u}_\varepsilon \xrightarrow{2} \hat{u}_0, \qquad \varepsilon \nabla \hat{u}_\varepsilon \xrightarrow{2} \nabla_y \hat{u}_0$$

where $\hat{u}_0 \in L^2(\Omega; H^1_\#(\hat{Y}^*))$ is the unique solution of the two-scale limit problem (2.47).

Proof. Using the same argumentation as in the proof of Theorem 2.27, we obtain the strong and weak two-scale convergence, for the coefficients and data, respectively. Then, the derivation of (2.47) becomes a classical two-scale homogenisation task. ☐

Back-transformation

For the case $l = 0$, the back-transformation of the two-scale limit problem (2.41) results in:

Weak form of the two-scale limit equations (for $l = 0$)

Find $(u_0, u_1) \in H^1(\Omega) \times L^2(\Omega; H^1_\#(Y^*(x))/\mathbb{R})$ such that

$$\begin{aligned} &\int_\Omega \int_{Y^*(x)} a_0(x,y)\big(\nabla_x u_0(x) + \nabla_y u_1(x,y)\big) \cdot \big(\nabla_x \varphi_0(x) + \nabla_y \varphi_1(x,y)\big)\,\mathrm{d}y\,\mathrm{d}x \\ &\qquad + \int_\Omega \int_{Y^*(x)} b_0(x,y)\,\mathrm{d}y\; u_0(x)\varphi_0(x)\,\mathrm{d}x = \int_\Omega \int_{Y^*(x)} f_0(x,y)\,\mathrm{d}y\; \varphi_0(x)\,\mathrm{d}x \end{aligned} \tag{2.48}$$

for every $(\varphi_0, \varphi_1) \in H^1(\Omega) \times L^2(\Omega; H^1_\#(Y^*(x))/\mathbb{R})$.

By separating the x and y variables in (2.48), we obtain the following homogenised equation. Later we show also how this homogenised equation can be derived by some algebraic manipulations of the cell problems and the effective tensors, which are based on the transformation rules for the gradients (2.22).

Weak form of the homogenised equations (for $l = 0$)

Find $u_0 \in H^1(\Omega)$, such that

$$\int\limits_{\Omega} a^*(x)\nabla_x u_0(x)\nabla_x\varphi(x) + b^*(x)u_0(x)\varphi(x)\,\mathrm{d}x = \int\limits_{\Omega} f^*(x)\varphi(x)\,\mathrm{d}x, \tag{2.49}$$

for every $\varphi \in H^1(\Omega)$, where the effective coefficients and data are given by

$$a^*_{ij}(x) := \int\limits_{Y^*(x)} a_0(x,y)(e_j + \nabla_y\zeta_j(x,y)) \cdot e_i\,\mathrm{d}y \tag{2.50}$$

$$b^*(x) := \int\limits_{Y^*(x)} b_0(x,y)\,\mathrm{d}y \tag{2.51}$$

$$f^*(x) := \int\limits_{Y^*(x)} f_0(x,y)\,\mathrm{d}y \tag{2.52}$$

for a.e. $x \in \Omega$ and every $i, j \in \{1, \dots, n\}$, where $\zeta_j \in L^\infty(\Omega; H^1_\#(Y^*(x)))$ is the unique solution of

$$\int\limits_{\hat{Y}^*} a_0(x,y)(\nabla_y\zeta_j(x,y) + e_j) \cdot \nabla_y\varphi(y) = 0 \tag{2.53}$$

all $\varphi \in H^1_\#(Y^*(x))$ and a.e. $x \in \Omega$.

The strong form of (2.49) is given by:

Homogenised limit problem (for $l = 0$)

$$\mathrm{div}_x(a^*\nabla_x u_0) + b^* u_0 = f^*, \tag{2.54}$$

where a^*, b^* and f^* are given by (2.50)–(2.52)

For the case $l = 2$, the back-transformation of the two-scale limit problem (2.47) leads to:

Weak form of the two-scale limit equations (for $l = 2$)

Find $u_0 \in L^2(\Omega; H^1_\#(Y^*(x)))$ such that

$$\begin{aligned}\int_\Omega \int_{Y^*(x)} & a_0(x,y)\nabla_y u_0(x,y) \cdot \nabla_y \varphi_0(x,y) \,\mathrm{d}y \,\mathrm{d}x \\ + \int_\Omega \int_{Y^*(x)} & b_0(x,y) u_0(x,y)\varphi_0(x,y) \,\mathrm{d}y \,\mathrm{d}x = \int_\Omega \int_{Y^*(x)} f_0(x,y)\varphi_0(x,y) \,\mathrm{d}y \,\mathrm{d}x\end{aligned} \tag{2.55}$$

for every $\varphi_0 \in L^2(\Omega; H^1_\#(Y^*(x)))$.

By means of the preliminary work on two-scale transformation, we can also transfer the convergence results and, thus, identify the above equations as the limit equations.

Theorem 2.30. *Let $l = 0$ and assume that $\hat{Y}^*_\#$ is connected. Then, for the sequence of solutions $u_\varepsilon \in H^1(\Omega_\varepsilon)$ of* (2.33) *it holds*

$$u_\varepsilon \xrightarrow{2} \chi_{\hat{Y}^*} u_0, \qquad \nabla u_\varepsilon \xrightarrow{2} \chi_{\hat{Y}^*} \nabla_x u_0 + \nabla_y u_1 \tag{2.56}$$

where $(u_0, u_1) \in H^1(\Omega) \times L^2(\Omega; H^1_\#(Y^*(x))/\mathbb{R})$ is the unique solution of the two-scale limit problem (2.48).

Proof. From the relation $\hat{u}_\varepsilon = u_\varepsilon \circ \psi_\varepsilon$, the two-scale convergence of $\hat{u}_\varepsilon$ (see Theorem 2.27) and the transformation rules from Theorem 2.23, we obtain the two-scale convergence (2.56), where $u_0 = \hat{u}_0$ and $u_1(x,y) = \hat{u}_1(x, \psi_0^{-1}(x,y)) + \overset{\smile}{\psi_0^{-1}}(x,y) \cdot \nabla_x \hat{u}_0(x)$ and $(\hat{u}_0, \hat{u}_1)$ is the solution of (2.41). From Lemma 2.31 below, it follows that (u_0, u_1) solves (2.48). □

Lemma 2.31. *Let $\hat{u}_0 = u_0 \in H^1(\Omega)$, $\hat{u}_1 \in L^2(\Omega; H^1_\#(\hat{Y}^*)/\mathbb{R})$, $u_1 \in L^2(\Omega; H^1_\#(Y^*(x))/\mathbb{R})$ with*

$$\hat{u}_1(x,y) = u_1(x, \psi_0(x,y)) + \breve{\psi}_0(x,y) \cdot \nabla_x u_0(x) \tag{2.57}$$

for a.e. $(x,y) \in \Omega \times \hat{Y}^*$. Then, $(\hat{u}_0, \hat{u}_1)$ solves (2.41) *if and only if (u_0, u_1) solves* (2.48).

Proof. Let $(\hat{u}_0, \hat{u}_1)$ be the solution of (2.41). We test (2.41) with $(\varphi_0, \hat{\varphi}_1) \in H^1(\Omega) \times L^2(\Omega; H^1_\#(\hat{Y}^*)/\mathbb{R})$ for $\varphi_0 \in H^1(\Omega)$ and $\hat{\varphi}_1(x,y) = \varphi_1(x, \psi_0(x,y)) + \breve{\psi}_0(x,y) \cdot \nabla_x \varphi_0(x)$ for $\varphi_1 \in L^2(\Omega; H^1_\#(Y^*(x))/\mathbb{R})$. We note that $\breve{\psi}_0 \cdot \nabla_x u_0 \in L^2(\Omega; H^1_\#(\hat{Y}^*)/\mathbb{R})$ since $\breve{\psi}_0$ is

Y-periodic. Then, we transform the integral in (2.41) via ψ_0^{-1}, which yields

$$\int_\Omega \int_{Y^*(x)} \hat{a}_0(x, \psi_0^{-1}(x,y)) \Psi_0^{-\top}(x, \psi_0^{-1}(x,y)) \big(\nabla_x \hat{u}_0(x) + (\nabla_y \hat{u}_1)(x, \psi_0^{-1}(x,y))\big) \\ \cdot \Psi_0^{-\top}(x, \psi_0^{-1}(x,y)) \big(\nabla_x \varphi_0(x) + (\nabla_y \hat{\varphi}_1)(x, \psi_0^{-1}(x,y))\big) \,\mathrm{d}y \,\mathrm{d}x \\ + \int_\Omega \int_{Y^*(x)} \hat{b}_0(x, \psi_0^{-1}(x,y)) \,\mathrm{d}y \; \hat{u}_0(x) \varphi_0(x) \,\mathrm{d}x = \int_\Omega \int_{Y^*(x)} \hat{f}_0(x, \psi_0^{-1}(x,y)) \,\mathrm{d}y \; \hat{u}_0(x) \varphi_0(x) \,\mathrm{d}x. \tag{2.58}$$

With the definitions of $\hat{a}_0$, $\hat{b}_0$, $\hat{f}_0$ and computations as in (2.25) applied for the unknowns $(\hat{u}_0, \hat{u}_1)$ and analogously for the test functions $(\varphi_0, \hat{\varphi}_1)$, we can simplify (2.58) to (2.48) for u_0 and u_1 given by (2.57). Hence, (u_0, u_1) solves (2.48).

The other implication can be shown similarly. □

Now, we separate the macro- and microscopic variables in (2.48) in order to derive the homogenised equation (2.49). Moreover, we show that the homogenised equations (2.42), (2.49) coincide.

Theorem 2.32. *Let* (u_0, u_1) *be the solution of the two-scale limit problem* (2.48)*. Then* u_0 *solves the homogenised problem* (2.49) *and it holds*

$$u_1 = \sum_{j=1}^n \partial_{x_j} u_0 \zeta_j.$$

Moreover, we have the following relation between the solutions of the cell problems (2.46) *and* (2.53)

$$\hat{\zeta}_j(x,y) = \zeta_j(x, \psi_0(x,y)) + \breve{\psi}_0(x,y) \cdot e_j = \zeta_j(x, \psi_0(x,y)) + (\breve{\psi}_0)_j(x,y) \tag{2.59}$$

for every $j \in \{1, \dots, n\}$*. The coefficients of* (2.42)*, which are given in* (2.43)*,* (2.44)*,* (2.45) *and the coefficients of* (2.49)*, which are given in* (2.50)*,* (2.51)*,* (2.52) *are equal, i.e.*

$$\hat{a}^* = a^*, \qquad \hat{b}^* = b^*, \qquad \hat{f}^* = f^*. \tag{2.60}$$

Proof. The first part of Theorem 2.32 follows by a standard separation of the x and y variables.

The relation (2.59) between $\hat{\zeta}_j$ and ζ_j follows analogously to Lemma 2.31, by replacing $u_0 = \hat{u}_0$ by x_j, u_1 by ζ_j and $\hat{u}_1$ by $\hat{\zeta}_j$.

To show the equality $\hat{a}^* = a^*$, we rewrite the right-hand sides of (2.43) and (2.50) using the cell problems (2.46) and (2.53), respectively

$$\hat{a}^*_{ij}(x) = \int_{\hat{Y}*} A_0(x,y)\hat{a}_0(x,y)\Psi_0^{-\top}(x,y)(e_j + \nabla_y\hat{\zeta}_j(x,y)) \cdot (e_i + \nabla_y\hat{\zeta}_i(x,y))\,\mathrm{d}y, \tag{2.61}$$

$$a^*_{ij}(x) = \int_{Y*(x)} a_0(x,y)(e_j + \nabla_y\zeta_j(x,y)) \cdot (e_i + \nabla_y\zeta_i(x,y))\,\mathrm{d}y. \tag{2.62}$$

Using computations as in Lemma 2.31 and with (2.59) we get

$$\Psi_0^{-\top}(x,y)(e_j + \nabla_y\hat{\zeta}_j(x,y)) = (e_j + (\nabla_y\zeta_j)(x,\psi_0(x,y)). \tag{2.63}$$

for $j \in \{1,\dots,n\}$. Then, the right-hand sides of (2.61) can be transformed into the right-hand side of (2.62). This provides the identity $\hat{a}^* = a^*$.

The identities $\hat{b}^* = b^*$ and $\hat{f}^* = f^*$ follow directly from the transformation of the integrals. □

Now, we consider the case $l = 2$.

Theorem 2.33. *Let $l = 2$. Then, for the sequence of solutions u_ε of* (2.36) *it holds*

$$u_\varepsilon \xrightarrow{2} u_0, \qquad \varepsilon\nabla u_\varepsilon \xrightarrow{2} \nabla_y u_0 \tag{2.64}$$

where $u_0 \in L^2(\Omega; H^1_\#(Y^*(x)))$ is the unique solution of the two-scale limit problem (2.55).

Proof. Theorem 2.33 can be shown analogously to Theorem 2.30 using Lemma 2.34 below. □

Lemma 2.34. *Let $\hat{u}_0 \in L^2(\Omega; H^1_\#(\hat{Y}^*))$ and $u_0 \in L^2(\Omega; H^1_\#(Y^*(x)))$ with*

$$\hat{u}_0(x,y) = u_0(x,\psi_0(x,y)) \tag{2.65}$$

for a.e. $(x,y) \in \Omega \times \hat{Y}^*$. Then, $\hat{u}_0$ solves (2.47) *if and only if u_0 solves* (2.55).

Proof. Lemma 2.34 follows directly from transforming the integrals. □

2.3.2. Direct homogenisation in the locally periodic domain

For simple problems as for instance (2.32), the limit process $\varepsilon \to 0$ can be done also in the non-periodic domain directly. Therefore, we use the following compactness results for functions defined on the locally periodic domains Ω_ε, which follow from the two-scale transformation results Theorem 2.20, Theorem 2.23 and Theorem 2.24.

Theorem 2.35. *Let $1 < p < \infty$ and let Ω_ε be locally periodic domains in the sense of Definition 2.1. Then, for every bounded sequence u_ε in $L^p(\Omega_\varepsilon)$, there exists a subsequence u_ε and $u_0 \in L^p(\Omega; L^p(Y^*(x)))$ such that*

$$u_\varepsilon \xrightarrow{p} u_0.$$

Moreover,

- *if* Ω_ε *(or* $\hat{Y}^*_\#$*) is connected and* u_ε *is a bounded sequence in* $W^{1,p}(\Omega_\varepsilon)$*, then, there exists a subsequence* u_ε *and* $u_0 \in W^{1,p}(\Omega)$*,* $u_1 \in L^p(\Omega; W^{1,p}_\#(Y^*(x))/\mathbb{R})$ *such that*

$$u_\varepsilon \xrightarrow{p} u_0, \qquad \nabla u_\varepsilon(x) \xrightarrow{p} \chi_{Y^*(x)}(y)\nabla_x u_0(x) + \nabla_y u_1(x,y).$$

  Additionally, if u_ε *is zero on* $\partial\Omega$*, then* $u_0 \in W^{1,p}_0(\Omega)$*.*

- *if* u_ε *is a sequence in* $W^{1,p}(\Omega_\varepsilon)$ *such that* $\|u_\varepsilon\|_{L^p(\Omega_\varepsilon)} + \varepsilon\|\nabla u_\varepsilon\|_{L^p(\Omega_\varepsilon)} \leqslant C$*. Then, there exists a subsequence* u_ε *and* $u_0 \in L^p(\Omega; W^{1,p}_\#(Y^*(x)))$ *such that*

$$u_\varepsilon \xrightarrow{p} u_0, \qquad \varepsilon\nabla u_\varepsilon \xrightarrow{p} \nabla_y u_0.$$

Proof. Let $\hat{u}_\varepsilon = u_\varepsilon \circ \psi_\varepsilon$. Then, Lemma 2.11 transforms the boundedness of u_ε into the boundedness of $\hat{u}_\varepsilon$. Then, we apply the compactness result Theorem 1.21, which gives a subsequence u_ε and $\hat{u}_0 \in L^p(\Omega; L^p(\hat{Y}^*))$ such that $\hat{u}_\varepsilon \xrightarrow{p} \hat{u}_0$. Transforming this two-scale convergence back onto u_ε via Theorem 2.20 leads $u_\varepsilon \xrightarrow{p} u_0$.

The compactness results for weakly differentiable functions can be derived analogously, by employing the compactness results Theorem 1.22 and Theorem 1.23, respectively, and subsequently employing the transformation results of Theorem 2.23 and Theorem 2.24, respectively. ☐

Having these compactness results, the homogenisation can be done as follows: using the theorem of Lax–Milgram it can easily be shown that (2.32) has a unique solution, and by energy estimates, we obtain

$$\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \varepsilon^l\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leqslant C.$$

Then, we can apply the compactness result Theorem 2.35 and can pass to the homogenisation limit and obtain the two-scale limit equation (2.48) for $l = 0$ with the corresponding homogenised limit equation (2.49) and the two-scale limit equation (2.55) for $l = 2$.

Indeed, this approach is by far faster, however, it benefits heavily from the simple structure of the problem (2.32). If the equations involve boundary terms or algebraic constraints, as for instance the Stokes equation, the derivation of uniform a-priori estimates and the passage to the homogenisation limit becomes far more complicated in the locally periodic domain. Hence, we will work mainly with the transformed equations in the following.

2.4. Locally evolving periodic domains

Now, we consider locally periodic domains, which are evolving on a time interval $(0,T)$. Let $\Omega \subset \mathbb{R}^n$ be an open Lipschitz domain and $Y = (0,1)^n$ with an open subset $Y^* \subset Y$.

We denote the Y-periodic extension of Y^* by $Y^*_{\#} := \operatorname{int}\Big(\bigcup_{k\in\mathbb{Z}^n} k+\overline{Y^*}\Big)$ and assume that it is a Lipschitz domain.

Let $\Omega_\varepsilon := \Omega\cap\varepsilon Y^*_{\#}$ be the ε-scaled periodic reference domains. Then, we define the locally evolving periodic domains Ω_ε by a transformation of the periodic reference domains.

Definition 2.36. *A sequence of measurable sets $\mathcal{Q}^T_\varepsilon = \bigcup_{t\in(0,T)} \{t\}\times\{\Omega_\varepsilon(t)\} \subset (0,T)\times\mathbb{R}^n$ for open sets $\Omega_\varepsilon(t)\subset\mathbb{R}^n$ is locally evolving periodic with two-scale limit set*

$$\mathcal{Q}(t) = \{(x,y)\in\Omega\times Y \mid y\in Y^*(t,x)\} \quad \text{for } t\in(0,T),$$
$$\mathcal{Q}^T = \{(t,x,y)\in(0,T)\times\Omega\times Y \mid y\in Y^*(t,x)\},$$

where $Y^*(t,x)\subset Y$ is open for every $(t,x)\in(0,T)\times\Omega$, if there exists a sequence of locally evolving periodic transformations ψ_ε (see Definition 2.37) with a limit transformation ψ_0 such that $\Omega_\varepsilon(t)=\psi_\varepsilon(t,\Omega_\varepsilon)$ and $Y^*(t,x)=\psi_0(t,x,Y^*)$ for a.e. $(t,x)\in(0,T)\times\Omega$.

In the time-dependent case, we denote the periodically perforated reference domain without $\hat{\cdot}$, since the transformed domains are already indicated by (t).

Definition 2.37. *We say that a sequence of mappings $\psi_\varepsilon\in L^\infty(0,T;C^2(\Omega)^n)$ is a sequence of locally evolving periodic transformations with two-scale limit transformation $\psi_0\in L^\infty((0,T)\times\Omega;C^2(\overline{Y})^n)$ if the following assumptions hold:*

1. *assumptions on ψ_ε*
   a. *$\psi_\varepsilon(t,\cdot)\colon\overline{\Omega}\to\overline{\Omega}$ is bijective for a.e. $t\in(0,T)$ and every $\varepsilon>0$,*
   b. *$\psi_\varepsilon^{-1}\in L^\infty(0,T;C^2(\overline{\Omega})^n)$, where $\psi_\varepsilon^{-1}(t,\cdot)$ is the inverse of $\psi_\varepsilon^{-1}(t,\cdot)$,*
   c. *$\Omega_\varepsilon(t)=\psi(t,\Omega_\varepsilon(t))$ for a.e. $t\in(0,T)$,*
   d. *there exists $c_J>0$ such that $\det(\partial_x\psi_\varepsilon(t,x))\geqslant c_J$ for a.e. $t\in(0,T)$ and all $x\in\overline{\Omega}$ and every $\varepsilon>0$,*
   e. *there exists a constant C such that*
   $$\varepsilon^{-1}\|\psi_\varepsilon - x\|_{C(\overline{\Omega_\varepsilon})} + \|\partial_x\psi_\varepsilon\|_{C(\overline{\Omega_\varepsilon})} + \varepsilon\|\partial_x\partial_x\psi_\varepsilon\|_{C(\overline{\Omega_\varepsilon})} \leqslant C$$
   for every $\varepsilon>0$.
2. *assumptions on ψ_0:*
   a. *$\psi_0(t,x,\cdot)\colon Y\to Y$ is bijective, with $Y^*(t,x)=\psi_0(t,x,Y^*)$ for a.e. $(t,x)\in(0,T)\times\Omega$,*
   b. *$\psi_0^{-1}\in L^\infty((0,T)\times\Omega;C^2(\overline{Y})^n)$, where $\psi_0^{-1}(t,x,\cdot)$ is the inverse of $\psi_0(t,x,\cdot)$,*
   c. *the corresponding displacement mapping, defined by $\breve{\psi}_0(t,x,y):=\psi_0(t,x,y)-y$ for $(t,x,y)\in(0,T)\times\Omega\times Y$ can be extended Y-periodically, i.e. $\breve{\psi}_0\in L^\infty((0,T)\times\Omega;C^2_{\#}(\overline{Y^*(x)})^n)$,*

3. *asymptotic behaviour:*

- $\chi_{\Omega_\varepsilon}\varepsilon^{-1}(\psi_\varepsilon - x) \xrightarrow{<\infty,<\infty} \chi_{Y^*}(\psi_0 - y)$
- $\chi_{\Omega_\varepsilon}\partial_x\psi_\varepsilon \xrightarrow{<\infty,<\infty} \chi_{Y^*}\partial_y\psi_0$,
- $\varepsilon\chi_{\Omega_\varepsilon}\partial_x\partial_x\psi_\varepsilon \xrightarrow{<\infty,<\infty} \chi_{Y^*}\partial_y\partial_y\psi_0$.

Similarly to the limit transformation for the stationary case, we define ψ_ε on all of Ω and $\psi_0(t,x,\cdot)$ on all of $\overline{Y}$ in order to ensure the measurability when we use it as transformation. However, for the asymptotic behaviour in Definition 2.2 and the transformation results later, it suffices to consider $\psi_\varepsilon(t,\cdot)$ and $\psi_\varepsilon^{-1}(t,\cdot)$ on Ω_ε and $\Omega_\varepsilon(t)$, respectively, and $\psi_0(t,x,\cdot)$ and $\psi_0^{-1}(t,x,\cdot)$ on Y^* and $Y^*(t,x)$, respectively. Then, we will implicitly restrict $\psi_\varepsilon(t,\cdot)$, $\psi_\varepsilon^{-1}(t,\cdot)$, ψ_0 and ψ_0^{-1}, accordingly, and where necessary we use the implicit extension $\tilde{\cdot}$ by 0.

We define the displacement mappings and the Jacobi matrix with its determinant and adjugate matrix as in the stationary case.

Notation 2.38. *Let ψ_ε and ψ_0 be given by Definition 2.37. We denote the corresponding displacement mappings by*

$$\begin{aligned} \breve{\psi}_\varepsilon(t,x) &:= \psi_\varepsilon(t,x) - x, & \breve{\psi_\varepsilon^{-1}}(t,x) &:= \psi_\varepsilon^{-1}(t,x) - x \\ \breve{\psi}_0(t,x,y) &:= \psi_0(t,x,y) - y, & \breve{\psi_0^{-1}}(t,x,y) &:= \psi_0^{-1}(t,x,y) - y. \end{aligned}$$

Analogously to (2.2)*, we obtain*

$$\begin{aligned} \breve{\psi_\varepsilon^{-1}}(t,x) &= -\breve{\psi}_\varepsilon(t,\psi_\varepsilon^{-1}(t,x)), \\ \breve{\psi_0^{-1}}(t,x,y) &= -\breve{\psi}_0(t,x,\psi_0^{-1}(t,x,y)) \end{aligned} \tag{2.66}$$

The Y-periodicity of $\breve{\psi}_0$ can be transferred via (2.2) on $\breve{\psi_0^{-1}}$. Thus, $\breve{\psi_0^{-1}} \in L^\infty((0,T)\times\Omega; C^2_\#(\overline{Y}))$.

Notation 2.39. *Let ψ_ε and ψ_0 be given by Definition 2.2. Then, we use the following notation for the Jacobian matrix, its determinant and its adjugate matrix*

$$\Psi_\varepsilon(t,x) := \partial_x\psi_\varepsilon(t,x), \quad J_\varepsilon(t,x) := \det(\Psi_\varepsilon(t,x)), \quad A_\varepsilon(t,x) := \mathrm{Adj}(\Psi_\varepsilon(t,x))$$

for a.e. $t \in (0,T)$ and every $x \in \overline{\Omega_\varepsilon}$ and

$$\Psi_0(t,x,y) := \partial_y\psi_0(t,x,y), \quad J_0(t,x,y) := \det(\Psi_0(t,x,y)), \quad A_0(t,x,y) := \mathrm{Adj}(\Psi_0(t,x,y))$$

for a.e. $(t,x) \in (0,T)\times\Omega$ and all $y \in \overline{Y^*}$.

Lemma 2.40. *Let ψ_ε be locally evolving periodic transformations with limit transformation ψ_0 in the sense of Definition 2.37. Then, there exists a constant $C > 0$ such that*

$$\begin{aligned}
\|\Psi_\varepsilon(t)\|_{C(\overline{\Omega_\varepsilon})} + \|\Psi_\varepsilon^{-1}(t)\|_{C(\overline{\Omega_\varepsilon})} + \|J_\varepsilon(t)\|_{C(\overline{\Omega_\varepsilon})} + \|J_\varepsilon^{-1}(t)\|_{C(\overline{\Omega_\varepsilon})} &\leqslant C,\\
\|A_\varepsilon(t)\|_{C(\overline{\Omega_\varepsilon})} + \|A_\varepsilon^{-1}(t)\|_{C(\overline{\Omega_\varepsilon})} &\leqslant C,\\
\varepsilon\|\partial_x\Psi_\varepsilon(t)\|_{C(\overline{\Omega_\varepsilon})} + \varepsilon\|\partial_x\Psi_\varepsilon^{-1}(t)\|_{C(\overline{\Omega_\varepsilon})} + \varepsilon\|\partial_x J_\varepsilon(t)\|_{C(\overline{\Omega_\varepsilon})} + \varepsilon\|\partial_x J_\varepsilon^{-1}(t)\|_{C(\overline{\Omega_\varepsilon})} &\leqslant C,\\
\varepsilon\|\partial_x A_\varepsilon(t)\|_{C(\overline{\Omega_\varepsilon})} + \varepsilon\|\partial_x A_\varepsilon^{-1}(t)\|_{C(\overline{\Omega_\varepsilon})} &\leqslant C
\end{aligned} \tag{2.67}$$

for a.e. $t \in (0,T)$. These estimates hold for every $t \in [0,T]$ if, additionally, Assumption 2.41 is fulfilled.

Proof. Analogously to the proof of Lemma 2.8, we obtain the bounds for a.e. $t \in (0,T)$. From Assumption 2.41, it follows that $\partial_t\psi_\varepsilon$, $\partial_x\partial_t\psi_\varepsilon$, $\partial_x\partial_x\partial_t\psi_\varepsilon$ are in $L^\infty((0,T)\times\Omega)^n$, $L^\infty((0,T)\times\Omega)^{n\times n}$ and $L^\infty((0,T)\times\Omega)^{n\times n\times n}$, respectively. Thus, ψ_ε, $\partial_x\psi_\varepsilon$, $\partial_x\partial_x\psi_\varepsilon$ are continuous with respect to time, which is transferred to Ψ_ε, Ψ_ε^{-1}, J_ε, J_ε^{-1}, A_ε, A_ε^{-1} and their derivatives with respect to space ($\partial_x\Psi_\varepsilon$, $\partial_x\Psi_\varepsilon^{-1}$, $\partial_x J_\varepsilon$, $\partial_x J_\varepsilon^{-1}$, $\partial_x A_\varepsilon$, $\partial_x A_\varepsilon^{-1}$). Then, the continuity extends the estimate to every $t \in [0,T]$. □

The following assumption becomes useful when we work with instationary processes, i.e. if time is not only a parameter but the time derivative is involved in the differential equation itself as for instance in parabolic partial differential equations.

Assumption 2.41. *Let ψ_ε be locally evolving periodic transformations in the sense of Definition 2.37. We assume that ψ_ε and ψ_0 are weakly differentiable with respect to the time variable, i.e.*

$$\begin{aligned}
&\partial_t\psi_\varepsilon \in L^\infty((0,T)\times\Omega_\varepsilon)^n, \quad \partial_x\partial_t\psi_\varepsilon \in L^\infty((0,T)\times\Omega_\varepsilon)^{n\times n},\\
&\partial_x\partial_x\partial_t\psi_\varepsilon \in L^\infty((0,T)\times\Omega_\varepsilon)^{n\times n\times n},\\
&\partial_t\psi_0 \in L^\infty((0,T)\times\Omega\times Y^*)^n, \quad \partial_y\partial_t\psi_0 \in L^\infty((0,T)\times\Omega\times Y^*)^{n\times n},\\
&\partial_y\partial_y\partial_t\psi_0 \in L^\infty((0,T)\times\Omega\times Y^*)^{n\times n\times n}.
\end{aligned}$$

Moreover, we assume that

$$\varepsilon^{-1}\partial_t\psi_\varepsilon \xrightarrow{<\infty}\!\!\!\!\!\!\rightarrow \partial_t\psi_0, \qquad \partial_x\partial_t\psi_\varepsilon \xrightarrow{<\infty}\!\!\!\!\!\!\rightarrow \partial_y\partial_t\psi_0, \qquad \varepsilon\partial_x\partial_x\partial_t\psi_\varepsilon \xrightarrow{<\infty}\!\!\!\!\!\!\rightarrow \partial_y\partial_y\partial_t\psi_0.$$

In order to transfer the uniform bounds and time- and space derivatives from ψ_ε onto J_ε, Ψ_ε and A_ε, we rewrite them again as polynomials with respect to time.

Lemma 2.42. *Let $U \subset \mathbb{R}^n$, $(0,T) \subset \mathbb{R}$, $B\colon (0,T)\times U \to \mathbb{R}^{n\times n}$ with $\det(B) \neq 0$. Then, we obtain the same polynomials as in (2.7) in Lemma 2.7. Moreover, (2.4) holds also for ∂_{x_k} replaced by ∂_t.*

Proof. Lemma 2.42 can be shown in the same way as Lemma 2.7. □

Lemma 2.43. *Let ψ_ε be locally evolving periodic domains in the sense of Definition 2.37 and assume that Assumption 2.41 holds. Then, there exists a constant C, which is independent of ε, such that*

$$\begin{aligned}
\|\partial_t \Psi_\varepsilon\|_{L^\infty((0,T)\times\Omega_\varepsilon)} + \|\partial_t \Psi_\varepsilon^{-1}\|_{L^\infty((0,T)\times\Omega_\varepsilon)} &\leqslant C, \\
\|\partial_t J_\varepsilon\|_{L^\infty((0,T)\times\Omega_\varepsilon)} + \|\partial_t J_\varepsilon^{-1}\|_{L^\infty((0,T)\times\Omega_\varepsilon)} &\leqslant C, \\
\|\partial_t A_\varepsilon\|_{L^\infty((0,T)\times\Omega_\varepsilon)} + \|\partial_t A_\varepsilon^{-1}\|_{L^\infty((0,T)\times\Omega_\varepsilon)} &\leqslant C
\end{aligned} \tag{2.68}$$

for all $\varepsilon > 0$.

Proof. Arguing as in the proof of Lemma 2.8, we can rewrite all the terms of (2.68) as polynomials. However, compared to Lemma 2.8, the time derivative of ψ_ε is even bounded without the factor ε, and thus we obtain (2.68). □

Lemma 2.44. *Let ψ_ε be locally evolving periodic transformations with limit transformation ψ_0 in the sense of Definition 2.37. Then, there exist constants c_J, C such that*

$$\begin{aligned}
\|\Psi_0\|_{L^\infty((0,T)\times\Omega;C(\overline{Y^*}))} + \|\Psi_0^{-1}\|_{L^\infty((0,T)\times\Omega;C(\overline{Y^*}))} + \|J_0\|_{L^\infty((0,T)\times\Omega;C(\overline{Y^*}))} &\leqslant C, \\
\|A_0\|_{L^\infty((0,T)\times\Omega;C(\overline{Y^*}))} + \|A_0^{-1}\|_{L^\infty((0,T)\times\Omega;C(\overline{Y^*}))} &\leqslant C, \\
J_0(t,x,y) \geqslant c_J \text{ for a.e. } (t,x) \in (0,T)\times\Omega \text{ and every } y \in Y^*&.
\end{aligned}$$

Moreover, it holds

$$\begin{aligned}
&\Psi_\varepsilon \xrightarrow{<\infty,<\infty} \Psi_0, && \Psi_\varepsilon^{-1} \xrightarrow{<\infty,<\infty} \Psi_0^{-1}, && J_\varepsilon \xrightarrow{<\infty,<\infty} J_0, \\
&J_\varepsilon^{-1} \xrightarrow{<\infty,<\infty} J_0^{-1} && A_\varepsilon \xrightarrow{<\infty,<\infty} A_0, && A_\varepsilon^{-1} \xrightarrow{<\infty,<\infty} A_0^{-1}, \\
&\varepsilon\partial_x \Psi_\varepsilon \xrightarrow{<\infty,<\infty} \partial_y \Psi_0, && \varepsilon\partial_x \Psi_\varepsilon^{-1} \xrightarrow{<\infty,<\infty} \partial_y \Psi_0^{-1}, && \varepsilon\partial_x J_\varepsilon \xrightarrow{<\infty,<\infty} \partial_y J_0, \\
&\varepsilon\partial_x J_\varepsilon^{-1} \xrightarrow{<\infty,<\infty} \partial_y J_0^{-1}, && \varepsilon\partial_x A_\varepsilon \xrightarrow{<\infty,<\infty} \partial_y A_0, && \varepsilon\partial_x A_\varepsilon^{-1} \xrightarrow{<\infty,<\infty} \partial_y A_0^{-1}.
\end{aligned}$$

Proof. Lemma 2.44 can be shown by the same argumentation as Lemma 2.9. □

For $1 \leqslant p \leqslant \infty$, $1 \leqslant q < \infty$, we define $L^p(0,T;L^q(\Omega_\varepsilon(t)))$ and $L^p(0,T;W^{1,q}(\Omega_\varepsilon(t)))$ via restriction from Ω to $\Omega_\varepsilon(t)$, analogously to the definition of $L^p(\Omega;L^q(Y^*(x)))$ and $L^p(\Omega;W^{1,p}_\#(Y^*(x)))$. In particular, due to the following lemma these spaces are well-posed.

Lemma 2.45. *Let $1 \leqslant p \leqslant \infty$, $1 \leqslant q < \infty$ and $\hat{u}_\varepsilon(t,x) = u_\varepsilon(t,\psi_\varepsilon(t,x))$ for a.e. $(t,x) \in (0,T)\times\Omega_\varepsilon)$. Then, the following statements hold*

- *$u_\varepsilon \in L^p(0,T;L^q(\Omega_\varepsilon(t)))$ if and only if $\hat{u}_\varepsilon \in L^p(0,T;L^q(\Omega_\varepsilon))$. Moreover, there exist constants $c, C > 0$, which are independent of ε, such that*

$$c\|\hat{u}_\varepsilon\|_{L^p(0,T;L^q(\Omega_\varepsilon))} \leqslant \|u_\varepsilon\|_{L^p(0,T;L^q(\Omega_\varepsilon(t)))} \leqslant C\|\hat{u}_\varepsilon\|_{L^p(0,T;L^q(\Omega_\varepsilon))}. \tag{2.69}$$

  In particular, u_ε is a bounded sequence in $L^p(0,T;L^q(\Omega_\varepsilon(t)))$ if and only if $\hat{u}_\varepsilon$ is a bounded sequence in $L^p(0,T;L^q(\Omega_\varepsilon))$.

- *$u_\varepsilon \in L^p(0,T;W^{1,q}(\Omega_\varepsilon(t)))$ if and only if $\hat{u}_\varepsilon \in L^p(0,T;W^{1,q}(\Omega_\varepsilon))$. Moreover, there exist constants $c, C > 0$, which are independent of ε, such that*

$$c\|\nabla \hat{u}_\varepsilon\|_{L^p(0,T;L^q(\Omega_\varepsilon))} \leqslant \|\nabla u_\varepsilon\|_{L^p(0,T;L^q(\Omega_\varepsilon(t)))} \leqslant C\|\nabla \hat{u}_\varepsilon\|_{L^p(0,T;L^q(\Omega_\varepsilon))}. \tag{2.70}$$

In particular, u_ε is a bounded sequence in $L^p(0,T;W^{1,q}(\Omega_\varepsilon(t)))$ if and only if $\hat{u}_\varepsilon$ is a bounded sequence in $L^p(0,T;W^{1,q}(\Omega_\varepsilon))$.

Proof. Lemma 2.15 shows that $(t,x) \mapsto (t,\psi_\varepsilon(t,x))$ and $(t,x) \mapsto (t,\psi_\varepsilon^{-1}(t,x))$ fulfil Lusin's (N)-condition. Then, Lemma 2.14 shows that $\hat{u}_\varepsilon$ is measurable if and only if u_ε is measurable. By similar computations as in the proof of Lemma 2.11, we obtain (2.69). Analogously, we obtain the measurability of the gradients and (2.70). □

Lemma 2.46. *Let $\hat{u}_\varepsilon(t,x) = u_\varepsilon(t,\psi_\varepsilon(t,x))$ for a.e. $(t,x) \in (0,T)\times\Omega_\varepsilon$. Then, $u_\varepsilon \in L^\infty(\mathcal{Q}_\varepsilon^T)$ if and only if $\hat{u}_\varepsilon \in L^\infty((0,T)\times\Omega_\varepsilon)$ and it holds*

$$\|\hat{u}_\varepsilon\|_{L^\infty((0,T)\times\Omega_\varepsilon)} = \|u_\varepsilon\|_{L^\infty(\mathcal{Q}_\varepsilon^T)}. \tag{2.71}$$

Proof. The measurability can be shown as in Lemma 2.45. Since J_ε is essentially bounded from below and above it holds for every $A \subset (0,T)\times\Omega_\varepsilon$ that $|\psi_\varepsilon(A)| > 0$ if and only if $|A| > 0$, which shows (2.71). □

For $1 \leqslant p \leqslant \infty$, $1 \leqslant q, r < \infty$, we define the spaces $L^p(0,T;L^q(\Omega;L^r(Y^*(t,x))))$ and $L^p(0,T;L^q(\Omega;W^{1,r}_\#(Y^*(t,x))))$ via restriction from Y to $Y^*(t,x)$, analogously to the definition of $L^p(\Omega;L^q(Y^*(x)))$ and $L^p(\Omega;W^{1,p}_\#(Y^*(x)))$. In particular, due to the following lemma, these spaces are well-posed.

Lemma 2.47. *Let $1 \leqslant p \leqslant \infty$, $1 \leqslant q, r < \infty$ and $\hat{u}_0(t,x,y) = u_0(t,x,\psi_0(t,x,y))$ for a.e. $(t,x,y) \in (0,T)\times\Omega\times Y^*$, or equivalently $u_0(t,x,y) = \hat{u}_0(t,x,\psi_0^{-1}(t,x,y))$ for a.e $(t,x,y) \in \mathcal{Q}^T$. Then, the following statements hold:*

- *$u_0 \in L^p(0,T;L^q(\Omega;L^r(Y^*(t,x))))$ if and only if $\hat{u}_0 \in L^p(0,T;L^q(\Omega;L^r(Y^*)))$. Moreover, there exist constants $c, C > 0$, such that*

$$c\|\hat{u}_0\|_{L^p(0,T;L^q(\Omega;L^r(Y^*)))} \leqslant \|u_0\|_{L^p(0,T;L^q(\Omega;L^r(Y^*(t,x))))} \leqslant C\|\hat{u}_0\|_{L^p(0,T;L^q(\Omega;L^r(Y^*)))}. \tag{2.72}$$

- *$u_0 \in L^p(0,T;L^q(\Omega;W^{1,r}_\#(Y^*(t,x))))$ if and only if $\hat{u}_\varepsilon \in L^p(0,T;L^q(\Omega;W^{1,r}_\#(Y^*)))$. Moreover, there exist constants $c, C > 0$, which are independent of ε, such that*

$$\begin{aligned} c\|\nabla \hat{u}_\varepsilon\|_{L^p(0,T;L^q(\Omega;L^r(Y^*)))} &\leqslant \|\nabla u_\varepsilon\|_{L^p(0,T;L^q(\Omega;L^r(Y^*(t,x))))} \\ &\leqslant C\|\nabla \hat{u}_\varepsilon\|_{L^p(0,T;L^q(\Omega;L^r(Y^*)))}. \end{aligned} \tag{2.73}$$

Proof. Lemma 2.47 can be proven by similar computations as in Lemma 2.16. □

Lemma 2.48. *Let $\hat{u}_0(t,x,y) = u_0(t,x,\psi_0(t,x,y))$ for a.e. $(t,x,y) \in (0,T) \times \Omega \times Y^*$. Then, $u_0 \in L^\infty(\mathcal{Q}^T)$ if and only if $\hat{u}_0 \in L^\infty((0,T) \times \Omega \times Y^*)$, and one has*

$$\|\hat{u}_0\|_{L^\infty((0,T)\times\Omega\times Y^*)} = \|u_0\|_{L^\infty(\mathcal{Q}^T)}. \tag{2.74}$$

Proof. The measurability can be transferred between u_0 and $\hat{u}_0$ as in Lemma (2.45). Since J_0 is essentially bounded from below and above it holds for every $A \subset (0,T) \times \Omega \times Y^*$ that $|\psi_\varepsilon(A)| > 0$ if and only if $|A| > 0$, which shows (2.74). □

Now, we state the transformation of weak and strong two-scale convergence for functions as well as for their gradients. These are parameterised versions of the stationary case and can be shown by the same argumentations. Therefore, we only state the results for the sake of completeness without repeating the proofs from the time-independent case.

Theorem 2.49. *Let $p,q \in (1,\infty)$. Let u_ε be a sequence in $L^p(0,T;L^q(\Omega_\varepsilon))$ and $\hat{u}_\varepsilon(t,x) = u_\varepsilon(t,\psi_\varepsilon(t,x))$ for a.e. $(t,x) \in (0,T) \times \Omega_\varepsilon$. Then,*

$$u_\varepsilon \xrightarrow{p,q} u_0 \qquad \text{if and only if} \qquad \hat{u}_\varepsilon \xrightarrow{p,q} \hat{u}_0$$

for $u_0 \in L^p(0,T;L^q(\Omega;L^q(Y^*)))$ and $\hat{u}_0 \in L^p(0,T;L^q(\Omega;L^q(Y^*(t,x))))$, and one has

$$\begin{aligned} \hat{u}_0(t,x,y) &= u_0(t,x,\psi_0(t,x,y)) && \text{for a.e. } (t,x,y) \in (0,T) \times \Omega \times Y^*, \\ u_0(t,x,y) &= \hat{u}_0(t,x,\psi_0^{-1}(t,x,y)) && \text{for a.e. } (t,x,y) \in \mathcal{Q}^T. \end{aligned}$$

Theorem 2.50. *Let $p,q \in (1,\infty)$. Let u_ε be a sequence in $L^p(0,T;L^q(\Omega_\varepsilon))$ and $\hat{u}_\varepsilon(t,x) = u_\varepsilon(t,\psi_\varepsilon(t,x))$ for a.e. $(t,x) \in (0,T) \times \Omega_\varepsilon$. Then,*

$$u_\varepsilon \xrightarrow{p,q} u_0 \qquad \text{if and only if} \qquad \hat{u}_\varepsilon \xrightarrow{p,q} \hat{u}_0$$

for $u_0 \in L^p(0,T;L^q(\Omega;L^q(Y^*)))$ and $\hat{u}_0 \in L^p(0,T;L^q(\Omega;L^q(Y^*(t,x))))$, and one has

$$\begin{aligned} \hat{u}_0(t,x,y) &= u_0(t,x,\psi_0(t,x,y)) && \text{for a.e. } (t,x,y) \in (0,T) \times \Omega \times Y^*, \\ u_0(t,x,y) &= \hat{u}_0(t,x,\psi_0^{-1}(t,x,y)) && \text{for a.e. } (t,x,y) \in \mathcal{Q}^T. \end{aligned}$$

Corollary 2.51. *Let $\mathcal{Q}_\varepsilon^T$ be locally evolving periodic domains with two-scale limit set $\mathcal{Q}^T$ in the sense of Definition 2.36. Then,*

$$\chi_{\mathcal{Q}_\varepsilon^T} \xrightarrow{<\infty,<\infty} \chi_{\mathcal{Q}^T} \qquad (\text{i.e. } \chi_{\mathcal{Q}_\varepsilon^T}(t,x) \xrightarrow{<\infty,<\infty} \chi_{\mathcal{Q}^T}(t,x,y) = \chi_{Y^*(t,x)}(y)\,).$$

Theorem 2.52. *Let $p,q \in (1,\infty)$. Let u_ε be a sequence in $L^p(0,T;L^q(\Omega_\varepsilon))$ and $\hat{u}_\varepsilon(t,x) = u_\varepsilon(t,\psi_\varepsilon(t,x))$ for a.e. $(t,x) \in (0,T) \times \Omega_\varepsilon$. Then,*

$$\nabla u_\varepsilon \xrightarrow{p,q} \chi_{\mathcal{Q}^T} \nabla_x u_0 + \nabla_y u_1 \qquad \text{if and only if} \qquad \nabla \hat{u}_\varepsilon \xrightarrow{p,q} \chi_{Y^*} \nabla_x \hat{u}_0 + \nabla_y \hat{u}_1$$

for $u_0 \in L^p(0,T;W^{1,q}(\Omega))$, $\hat{u}_0 \in L^p(0,T;W^{1,q}(\Omega))$, $u_1 \in L^p(0,T;L^q(\Omega;W^{1,q}_\#(Y^*(t,x))))$

and $\hat{u}_1 \in L^p(0,T;L^q(\Omega;W^{1,q}_{\#}(Y^*)))$. *Moreover,*

$$\begin{aligned}
\hat{u}_0(t,x) &= u_0(t,x) && \text{for a.e. } (t,x) \in (0,T)\times\Omega,\\
\hat{u}_1(t,x,y) &= u_1(t,x,\psi_0(t,x,y)) + \breve{\psi}_0(t,x,y)\cdot\nabla_x u_0(t,x)\\
&\qquad \text{for a.e. } (t,x,y) \in (0,T)\times\Omega\times Y^*,\\
u_1(t,x,y) &= \hat{u}_1(t,x,\psi_0^{-1}(t,x,y)) + \breve{\psi_0^{-1}}(t,x,y)\cdot\nabla_x \hat{u}_0(t,x)\\
&\qquad \text{for a.e. } (t,x,y) \in \mathcal{Q}^T.
\end{aligned}$$

Theorem 2.53. *Let* $p,q \in (1,\infty)$. *Let* u_ε *be a sequence in* $L^p(0,T;L^q(\Omega_\varepsilon(t)))$ *and* $\hat{u}_\varepsilon(t,x) = u_\varepsilon(t,\psi_\varepsilon(t,x))$ *for a.e.* $(t,x) \in (0,T)\times\Omega_\varepsilon$. *Then,*

$$\varepsilon\nabla u_\varepsilon \xrightarrow{p,q} \nabla_y u_0 \qquad \text{if and only if} \qquad \varepsilon\nabla\hat{u}_\varepsilon \xrightarrow{p,q} \nabla_y \hat{u}_0$$

for $u_0 \in L^p(0,T;L^q(\Omega;W^{1,q}_{\#}(Y^*(t,x))))$ *and* $\hat{u}_0 \in L^p(0,T;L^q(\Omega;W^{1,q}_{\#}(Y^*)))$, *and it holds*

$$\begin{aligned}
\hat{u}_0(t,x,y) &= u_0(t,x,\psi_0(t,x,y)) && \text{for a.e. } (t,x,y) \in (0,T)\times\Omega\times Y^*,\\
u_0(t,x,y) &= \hat{u}_0(t,x,\psi_0^{-1}(t,x,y)) && \text{for a.e. } (t,x,y) \in \mathcal{Q}^T.
\end{aligned}$$

Chapter 3.

Stokes flow in porous media with evolving microstructure

In this chapter, we consider the homogenisation of Stokes flow in a porous medium for given evolving microstructure. First, we consider the case of the quasi-stationary Stokes equations (3.1) and afterwards the case of the instationary Stokes equations (3.87). At the pore interfaces $\Gamma_\varepsilon(t)$, we assume inhomogeneous Dirichlet boundary values v_{Γ_ε} for the fluid velocity. In particular, this models a no-slip boundary condition for the moving interface if we choose v_{Γ_ε} equal to the velocity of the interface. Moreover, we will not restrict to the case that the volume of the total pore space remains constant. Therefore, the fluid's incompressibility condition requires enabling fluid in- and out-flow. We model this by a normal stress boundary condition at the outer boundary of the porous medium.

In order to pass to the homogenisation limit, we transform the Stokes equations to a periodic substitute domain. There, we pass to the homogenisation limit. Afterwards, we transform the limit equations back. In the case of quasi-stationary Stokes flow, this leads to a *Darcy law for evolving microstructure* and in the case of the instationary Stokes equation, we derive a *Darcy law with memory for evolving microstructure*. Compared with the classical Darcy law, which can be derived by homogenisation of the Stokes flow for fixed microstructure, these resulting Darcy laws take the local cell geometry into account and, thus, yield a time- and space-dependent permeability. Moreover, the moving domain causes an inhomogeneous divergence condition for the resulting effective fluid velocity, via the inhomogeneous Dirichlet boundary conditions. This becomes a source or sink term for the pressure via the Darcy law.

This chapter is organised as follows: In Section 3.1, we consider the homogenisation of the quasi-stationary Stokes equations in an evolving porous domain. First, we present a strong and weak formulation of the problem in Section 3.1.1. Due to the quasi-stationarity, it can be formulated pointwisely with respect to time and, thus, becomes a homogenisation task in a locally periodic domain. Moreover, we formulate the assumptions on the domain and the data there. Afterwards, we transform the equations onto a periodic substitute domain in Section 3.1.2. In Section 3.1.3, we show the existence and uniqueness of the solution of the ε-scaled Stokes problem. By a subtle ε-scaling of the involved spaces, the existence result provides directly uniform a-priori estimates, which are sufficient for the compactness result. However, the transformation of the symmetric gradient in the Stokes equations causes some transformation coefficient in the symmetric gradient, which requires the derivation of an ε-scaled Korn-type inequality for two-scale transformations.

In Section 3.1.4, we pass to the limit $\varepsilon \to 0$ by means of two-scale compactness arguments which are based on the previously shown a-priori estimates. This results in a two-pressure Stokes equations in the cylindrical two-scale substitute set. We separate the micro- and macroscopic variable in the two-pressure Stokes equation in Section 3.1.5. Due to the transformation quantities, we obtain two different cell problems. After identifying these cell problems up to a perturbation of the microscopic pressure, we obtain a Darcy law formulated with respect to the transformed coordinates. In Section 3.1.6, we transform the two-pressure Stokes equations back onto the non-cylindrical two-scale limit set. For this, we extend the transformation results for the two-scale gradients of Chapter 2 to the divergence operator and employ the identification of the two different cell problems in the transformed setting. Afterwards, we separate the micro- and macroscopic variable and, finally, obtain a Darcy law for evolving microstructure.

In Section 3.2, we consider the homogenisation for the instationary Stokes flow. The basic procedure follows the stationary case. We formulate the microscopic equations in Section 3.2.1. Then, we transform them onto the periodically perforated substitute domain in Section 3.2.2. In Section 3.2.3, we show the existence of a solution for the microscopic problem and derive uniform a-priori estimates. By a previous substitution, we obtain a time-independent divergence constraint, which allows us to to apply an existence result on generic time-dependent differential-algebraic equations, which we derive in Appendix A. This substitution requires an extension of the Korn-type inequality for the two-scale transformation method, where the gradient is multiplied on both sides by transformation dependent matrices. Given these a-priori estimates, we identify the limit problem in Section 3.2.4. Compared to the quasi-stationary case, we have to modify the compactness argumentation for the pressure, which leads to only weak convergence. Afterwards, we transform the resulting limit equations back in Section 3.2.5 and, finally, derive a Darcy law with memory for evolving microstructure.

3.1. Homogenisation of quasi-stationary Stokes flow

This section is heavily based on the publication [WP24, D. Wiedemann and M. A. Peter, *Homogenisation of the Stokes equations for evolving microstructure*, Journal of Differential Equations, **396** (2024), 172–209]. Some preliminary work on the homogenisation of the quasi-stationary Stokes equation is presented in [Wie19, D. Wiedemann, *Homogenization of Stokes flow with evolving microstructure*, Master's thesis, Technical University of Munich (2019)]. In the following points, the approach and results of [Wie19] differ substantially from those of [WP24] and the ones presented here: In [Wie19], the Stokes equations are considered without symmetrising part for the gradient. Therefore, it does not require the Korn-type inequality for two-sale transformations, which is derived in [WP24]. In [Wie19], the family of operators $\operatorname{div}_\varepsilon^{-1}$, which is used for the a priori estimates are derived directly, while we argue here as in [WP24] and deduce the operators from the restriction operators of [Tar80, All89]. The explicit construction of [Wie19] provides additional understanding due to its physical motivation but is very technical. In [Wie19], only the weak two-scale convergence for the pressure is derived compared to the strong convergence here.

Moreover, we pass to the homogenisation limit $\varepsilon \to 0$ in a different way than in [Wie19]. In particular, we do not require any more that ψ_ε can be expressed as an ε-scaled version of ψ_0 but rely only on the asymptotic behaviour of ψ_ε similarly to Chapter 2. Moreover, [Wie19] concludes with the two-pressure Stokes equations in the substitute coordinates, while [WP24] presents also a back-transformation to transformation-independent two-scale limit equations on the actual two-scale limit domain as well as a Darcy law without transformation quantities.

3.1.1. The microscopic equations

Let $\Omega \subset \mathbb{R}^n$ be an an open bounded set, representing the domain of the porous medium, and let $(0,T)$ for $T > 0$ be the time interval. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a monotone positive sequence, which converges to 0 and scales the microstructure. We write $\varepsilon = \varepsilon_n$ in the following. We assume that Ω consists of whole ε-scaled copies of the unit cell $Y = (0,1)^n$, i.e. $\Omega = \operatorname{int}\big(\bigcup_{k\in I_\varepsilon} \varepsilon k + \varepsilon \overline{Y}\big)$ for some sets $I_\varepsilon \subset \mathbb{Z}^n$. We assume that for every ε and every $t \in [0,T]$, there exists an open set $\Omega_\varepsilon(t) \subset \Omega$, which represents the pore space. The complementary solid space is given by $\Omega^s_\varepsilon(t) = \operatorname{int}(\Omega \backslash \overline{\Omega_\varepsilon(t)})$. We denote the interface of the pore and the solid phase at time $t \in [0,T]$ by $\Gamma_\varepsilon(t) := \partial\Omega_\varepsilon(t) \cap \partial\Omega^s_\varepsilon(t)$ and the remaining boundary of the pore space at the boundary of the porous medium by $\Xi_\varepsilon(t) := \partial\Omega_\varepsilon(t) \backslash \Gamma_\varepsilon(t)$. Having the domains defined for every $t \in [0,T]$, we can define the evolving domain and its boundary by

$$Q^T_\varepsilon := \bigcup_{t\in[0,T]} \{t\} \times \Omega_\varepsilon(t), \qquad G^T_\varepsilon := \bigcup_{t\in[0,T]} \{t\} \times \Gamma_\varepsilon(t), \qquad H^T_\varepsilon := \bigcup_{t\in[0,T]} \{t\} \times \Xi_\varepsilon(t).$$

In this domain, we consider the quasi-stationary Stokes equation for the unknown fluid velocity v_ε and pressure p_ε:

Quasi-stationary Stokes equations in an evolving perforated domain

$$\begin{aligned} -\operatorname{div}\big(\mu\varepsilon^2\big(\nabla v_\varepsilon + (\nabla v_\varepsilon)^\top\big)\big) + \nabla p_\varepsilon &= f_\varepsilon && \text{in } Q^T_\varepsilon, \\ \operatorname{div}(v_\varepsilon) &= 0 && \text{in } Q^T_\varepsilon, \\ v_\varepsilon &= v_{\Gamma_\varepsilon} && \text{on } G^T_\varepsilon, \\ \big(-\varepsilon^2\mu\big(\nabla v_\varepsilon + (\nabla v_\varepsilon)^\top\big) + p_\varepsilon \mathbb{1}\big)\, n &= p_{b,\varepsilon} n && \text{on } H^T_\varepsilon, \end{aligned} \tag{3.1}$$

where $\mu > 0$ is the fluid's viscosity, f_ε the source term, v_{Γ_ε} the fluid velocity at the interface, $p_{b,\varepsilon}$ the normal stress and n the outer normal of $\Omega_\varepsilon(t)$.

The scaling of the viscosity by the factor ε^2 causes the velocity to have a non-trivial limit. From a physical point of view, it balances the friction of the fluid at the interface, which arises from the no-slip boundary condition (see also [Hor97, Chapter 3]).

Since the unknowns of (3.1) do not contain any time derivative, the time becomes a parameter and we can consider the equations as stationary problem for every point $t \in [0, T]$ separately. Therefore, we fix a point $t \in [0, T]$ in the following and omit indicating this parameter t at the unknown function and data as well as at the transformations ψ_ε later. We only state the parameter t at the domains $\Omega_\varepsilon(t)$ and its boundaries $\Gamma_\varepsilon(t)$ and $\Xi_\varepsilon(t)$ in order to distinguish them from the periodically perforated substitute domain, which we denote by Ω_ε, Γ_ε and Ξ_ε, respectively.

In order to derive the weak formulation, we assume that the Dirichlet boundary values v_{Γ_ε} and the normal stress $p_{b,\varepsilon}$ can be extended to $\Omega_\varepsilon(t)$. Then, we subtract these extensions from the fluid velocity v_ε and the pressure p_ε, i.e.

$$w_\varepsilon = v_\varepsilon - v_{\Gamma_\varepsilon}, \qquad q_\varepsilon = p_\varepsilon - p_{b,\varepsilon}.$$

which gives

$$\begin{aligned} -\operatorname{div}(\mu\varepsilon^2 2e(w_\varepsilon)) + \nabla q_\varepsilon &= f_\varepsilon + \operatorname{div}(\mu\varepsilon^2 2e(v_{\Gamma_\varepsilon})) - \nabla p_{b,\varepsilon} && \text{in } \Omega_\varepsilon(t), \\ \operatorname{div}(w_\varepsilon) &= -\operatorname{div}(v_{\Gamma_\varepsilon}) && \text{in } \Omega_\varepsilon(t), \\ w_\varepsilon &= 0 && \text{on } \Gamma_\varepsilon(t), \\ \left(-\varepsilon^2\mu 2e(w_\varepsilon) + q_\varepsilon \mathbb{1}\right) n &= \varepsilon^2 \mu 2e(v_{\Gamma_\varepsilon}) n && \text{on } \Xi_\varepsilon(t), \end{aligned} \tag{3.2}$$

where $e(w_\varepsilon)$ denotes the symmetric gradient, i.e.

$$e(v) := (\nabla v + (\nabla v)^\top)/2.$$

We multiply the first equation of (3.2) by a test function $\varphi \in H^1_{\Gamma_\varepsilon(t)}(\Omega_\varepsilon(t))^n$, where $H^1_{\Gamma_\varepsilon(t)}(\Omega_\varepsilon(t)) := \{v \in H^1(\Omega_\varepsilon(t)) \mid v_{\Gamma_\varepsilon(t)} = 0\}$. Then, we integrate over $\Omega_\varepsilon(t)$ and subsequently integrate the left-hand side by parts. By employing the two boundary conditions, we obtain the first equation of (3.3). Moreover, we multiply the second equation by $\eta \in L^2(\Omega_\varepsilon(t))$ and integrate over $\Omega_\varepsilon(t)$. In total, we obtain the following weak form:

Weak form of the quasi-stationary Stokes equations in an evolving perforated domain

Find $(w_\varepsilon, q_\varepsilon) \in H^1_{\Gamma_\varepsilon(t)}(\Omega_\varepsilon(t))^n \times L^2(\Omega_\varepsilon(t))$ such that

$$\begin{aligned}\int_{\Omega_\varepsilon(t)} \varepsilon^2 \mu 2e(w_\varepsilon(x)) : \nabla\varphi(x)\,\mathrm{d}x - \int_{\Omega_\varepsilon(t)} q_\varepsilon(x)\operatorname{div}(\varphi(x))\,\mathrm{d}x \\ = \int_{\Omega_\varepsilon(t)} (f_\varepsilon(x) - \nabla p_{b,\varepsilon}(x))\cdot\varphi(x)\,\mathrm{d}x - \int_{\Omega_\varepsilon(t)} \varepsilon^2\mu 2e(v_{\Gamma_\varepsilon})(x) : \nabla\varphi(x)\,\mathrm{d}x \\ \int_{\Omega_\varepsilon(t)} \operatorname{div}(w_\varepsilon(x))\eta(x)\,\mathrm{d}x = -\int_{\Omega_\varepsilon(t)} \operatorname{div}(v_{\Gamma_\varepsilon}(x))\eta(x)\,\mathrm{d}x\end{aligned} \tag{3.3}$$

for every $(\varphi, \eta) \in H^1_{\Gamma_\varepsilon(t)}(\Omega_\varepsilon(t))^n \times L^2(\Omega_\varepsilon(t))$.

We make the following assumptions on the data and the domain.

Assumption 3.1. *We assume that*

- $\Omega_\varepsilon(t)$ *is a sequence of locally periodic domains in the sense of Definition 2.1, with two-scale limit domain*

  $$\mathcal{Q}(t) = \{(x,y) \in \Omega \times Y \mid y \in Y^*(t,x)\}$$

  and interfaces $\Gamma(t,x) := \partial Y^*(t,x)\backslash\partial\Omega$ *for* $(t,x) \in [0,T]\times\Omega$. *We denote the periodic substitute domain by* Ω_ε*, the pore space of the reference cell by* $Y^* \subset Y = (0,1)^n$ *and the solid space by* $Y^{\mathrm{s}} = \operatorname{int}(Y\backslash Y^*)$. *For the periodic substitute domain, we assume that*

  - $0 < |Y^*| < 1$,
  - $Y^*_\# := \operatorname{int}\big(\bigcup_{k\in\mathbb{Z}^n} \varepsilon k + \varepsilon\overline{Y^*}\big)$ *and* $\operatorname{int}(\mathbb{R}^n\backslash Y^*_\#)$ *are open sets with* C^1*-boundary, which are locally located on one side of their boundary. Moreover,* $Y^*_\#$ *is connected,*
  - Y^* *is an open connected set with a locally Lipschitz boundary.*

  For a detailed discussion of the assumptions on the periodic substitute domain, see [All89].

  Furthermore, we assume that there exists a constant $c > 0$ *such that*

  $$|(\varepsilon k + \varepsilon Y) \cap \Omega_\varepsilon(t)| \geqslant \varepsilon^n c$$

  for every $k \in I_\varepsilon$ *and* $\varepsilon > 0$.

- f_ε *is a sequence in* $L^2(\Omega_\varepsilon(t))^n$ *and* $f \in L^2(\Omega)^n$ *such that*

  $$f_\varepsilon \xrightarrow{2} \chi_{\mathcal{Q}(t)} f.$$

- v_{Γ_ε} *is a sequence in* $H^1(\Omega_\varepsilon(t))^n$ *and* $v_\Gamma \in L^2(\Omega; H^1(Y^*(t,x))^n)$ *such that*

$$\varepsilon^{-1} v_{\Gamma_\varepsilon} \xrightarrow{2} v_\Gamma, \qquad \nabla v_{\Gamma_\varepsilon} \xrightarrow{2} \nabla_y v_\Gamma.$$

- $p_{b,\varepsilon}$ *is a sequence in* $H^1(\Omega_\varepsilon(t))$ *and* $(p_{b,0}, p_{b,1}) \in H^1(\Omega) \times L^2(\Omega; H^1(Y^*(t,x)))$, *such that*

$$\nabla p_{b,\varepsilon} \xrightarrow{2} \chi_{\mathcal{Q}(t)} \nabla_x p_{b,0} + \nabla_y p_{b,1}.$$

3.1.2. Transformation to a periodic substitute domain

We transform the Stokes equations (3.1) as well as the weak formulation (3.3) onto the reference domain Ω_ε. We denote the transformed data by

$$\hat{f}_\varepsilon = f_\varepsilon \circ \psi_\varepsilon, \qquad \hat{v}_{\Gamma_\varepsilon} = v_{\Gamma_\varepsilon} \circ \psi_\varepsilon, \qquad \hat{p}_{b,\varepsilon} = \hat{p}_\varepsilon \circ \psi_\varepsilon, \tag{3.4}$$

where $\psi_\varepsilon \colon \Omega_\varepsilon \to \Omega_\varepsilon(t)$ are the locally periodic transformations in the sense of Definition 2.2. We define the boundaries Γ_ε and Ξ_ε by $\Gamma_\varepsilon = \psi_\varepsilon^{-1}(\Gamma_\varepsilon(t))$ and $\Xi_\varepsilon = \psi_\varepsilon^{-1}(\Xi_\varepsilon(t))$, respectively, and recap the notation $\Psi_\varepsilon := \partial_x \psi_\varepsilon$, $J_\varepsilon := \det(\Psi_\varepsilon)$ and $A_\varepsilon := \operatorname{Adj}(\Psi_\varepsilon)$. Then, we obtain for

$$\hat{v}_\varepsilon = v_\varepsilon \circ \psi_\varepsilon, \qquad \hat{p}_\varepsilon = p_\varepsilon \circ \psi_\varepsilon,$$

the transformed strong formulation:

Quasi-stationary Stokes equations in an evolving perforated domain in the reference coordinates

$$\begin{aligned}
-J_\varepsilon^{-1} \operatorname{div}\left(\mu\varepsilon^2 A_\varepsilon 2 e_\varepsilon(\hat{v}_\varepsilon)\right) + \Psi_\varepsilon^{-\top} \nabla \hat{p}_\varepsilon &= \hat{f}_\varepsilon && \text{in } \Omega_\varepsilon, \\
J_\varepsilon^{-1} \operatorname{div}(A_\varepsilon \hat{v}_\varepsilon) &= 0 && \text{in } \Omega_\varepsilon, \\
\hat{v}_\varepsilon &= \hat{v}_{\Gamma_\varepsilon} && \text{on } \Gamma_\varepsilon, \\
\left(-\varepsilon^2 \mu 2 e_\varepsilon(\hat{v}_\varepsilon) + \hat{p}_\varepsilon \mathbb{1}\right) \|\Psi_\varepsilon^{-\top}\hat{n}\|^{-1} \Psi_\varepsilon^{-\top}\hat{n} &= \hat{p}_{b,\varepsilon} \|\Psi_\varepsilon^{-\top}\hat{n}\|^{-1} \Psi_\varepsilon^{-\top}\hat{n} && \text{on } \Xi_\varepsilon,
\end{aligned} \tag{3.5}$$

where e_ε denotes the transformed symmetric gradient, i.e.

$$e_\varepsilon(\varphi) := \left(\Psi_\varepsilon^{-\top} \nabla \varphi + (\Psi_\varepsilon^{-\top} \nabla \varphi)^\top\right)/2$$

and $\hat{n}$ denotes the outer normal of Ω_ε.

Moreover, we obtain the transformed weak formulation for

$$\hat{w}_\varepsilon = w_\varepsilon \circ \psi_\varepsilon, \qquad \hat{q}_\varepsilon = q_\varepsilon \circ \psi_\varepsilon,$$

where we use the function space $H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^n := \{v \in H^1(\Omega_\varepsilon)^n \mid v|_{\Gamma_\varepsilon} = 0\}$ as solution space.

Weak form of the quasi-stationary Stokes equations in an evolving perforated domain in the reference coordinates

Find $(\hat{w}_\varepsilon, \hat{q}_\varepsilon) \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^n \times L^2(\Omega_\varepsilon)$ such that

$$\begin{aligned}
&\int_{\Omega_\varepsilon} \varepsilon^2 \mu A_\varepsilon(x) 2 e_\varepsilon(\hat{w}_\varepsilon)(x) : \nabla\varphi(x)\,\mathrm{d}x - \int_{\Omega_\varepsilon} \hat{q}_\varepsilon(x) \operatorname{div}(A_\varepsilon(x)\varphi(x))\,\mathrm{d}x \\
&= \int_{\Omega_\varepsilon} (J_\varepsilon(x)\hat{f}_\varepsilon(x) - A_\varepsilon^\top(x)\nabla\hat{p}_{b,\varepsilon}(x))\cdot\varphi(x)\,\mathrm{d}x - \int_{\Omega_\varepsilon} \varepsilon^2\mu A_\varepsilon(x) 2 e_\varepsilon(\hat{v}_{\Gamma_\varepsilon})(x) : \nabla\varphi(x)\,\mathrm{d}x \\
&\int_{\Omega_\varepsilon} \operatorname{div}(A_\varepsilon(x)\hat{w}_\varepsilon(x))\eta(x)\,\mathrm{d}x = -\int_{\Omega_\varepsilon} \operatorname{div}(A_\varepsilon(x)\hat{v}_{\Gamma_\varepsilon}(x))\eta(x)\,\mathrm{d}x
\end{aligned} \tag{3.6}$$

for every $(\varphi, \eta) \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^n \times L^2(\Omega_\varepsilon)$.

Remark 3.2. *The Piola identity says that*

$$\operatorname{div}(\operatorname{Adj}(\partial_x\varphi)) = 0 \tag{3.7}$$

for every $\varphi \in C^2(U;\mathbb{R})^n$ on an open set $U \subset \mathbb{R}^n$. In Lipschitz domains, (3.7) *can be shown for $\varphi \in W^{1,\infty}(U;\mathbb{R})^n$ by a density argument. Together with the Leibniz rule, it allows the simplification of the divergence terms, i.e.* $\operatorname{div}(A_\varepsilon\hat{v}_\varepsilon) = \operatorname{div}(A_\varepsilon)\cdot\hat{v}_\varepsilon + A_\varepsilon : \nabla\hat{v}_\varepsilon = A_\varepsilon : \nabla\hat{v}_\varepsilon$.

For the transformed data, we can transfer the a-priori estimates and convergence assumption onto the reference domain by means of the results of Chapter 2.

Lemma 3.3. *Let f_ε, $p_{b,\varepsilon}$ and v_{Γ_ε} be given by Assumption 3.1 and let $\hat{f}_\varepsilon, \hat{p}_{b,\varepsilon}, \hat{v}_{\Gamma_\varepsilon}$ be given by* (3.4)*. Then, it holds*

$$\hat{f}_\varepsilon \xrightarrow{2} \chi_{Y^*}\hat{f}, \quad \varepsilon^{-1}\hat{v}_{\Gamma_\varepsilon} \xrightarrow{2} v_\Gamma, \quad \nabla\hat{v}_{\Gamma_\varepsilon} \xrightarrow{2} \nabla_y\hat{v}_\Gamma, \quad \nabla\hat{p}_{b,\varepsilon} \xrightarrow{2} \chi_{Y^*}\nabla_x\hat{p}_{b,0} + \nabla_y\hat{p}_{b,1}$$

for

$$\begin{aligned}
\hat{f}(x) &= f(x), & \hat{v}_\Gamma(x,y) &= v_\Gamma(x,\psi_0(x,y)), \\
\hat{p}_{b,0}(x) &= p_{b,0}(x), & \hat{p}_{b,1}(x,y) &= p_{b,1}(x,\psi_0(x,y)) + \nabla_x\hat{p}_{b,0}(x)\cdot\breve{\psi}_0(x,y).
\end{aligned}$$

In particular, there exists a constant C such that

$$\|\hat{f}_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|\hat{p}_{b,\varepsilon}\|_{L^2(\Omega_\varepsilon)} + \|\nabla\hat{p}_{b,\varepsilon}\|_{L^2(\Omega_\varepsilon)} + \varepsilon^{-1}\|\hat{v}_{\Gamma_\varepsilon}\|_{L^2(\Omega_\varepsilon)} + \|\nabla\hat{v}_{\Gamma_\varepsilon}\|_{L^2(\Omega_\varepsilon)} \leqslant C.$$

Proof. The two-scale convergence can be transferred from f_ε, $\varepsilon^{-1}v_{\Gamma_\varepsilon}$, $\nabla v_{\Gamma_\varepsilon}$ and $\nabla p_{b,\varepsilon}$ to the transformed quantities by means of Theorem 2.20, Theorem 2.24 and Theorem 2.23, respectively. Afterwards, the two-scale convergence implies the uniform boundedness. □

3.1.3. Existence, uniqueness and a-priori estimates

In this section, we show the following existence and uniqueness result for the solution of the Stokes equations (3.6). It provides also the a-priori estimates which we will use for the two-scale compactness arguments later.

Theorem 3.4. *For every $\varepsilon > 0$, there exists a unique solution $(\hat{w}_\varepsilon, \hat{q}_\varepsilon) \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^n \times L^2(\Omega_\varepsilon)$ of* (3.6)*. Moreover, there exists a constant C such that*

$$\|\hat{w}_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \varepsilon\|\nabla \hat{w}_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|\hat{q}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leqslant C \tag{3.8}$$

for every $\varepsilon > 0$.

We prove Theorem 3.4 by means of the following generic existence and uniqueness result for saddle point problems. By using a subtle scaling of the involved norms, it provides also the uniform a-priori estimates (3.8). For Banach spaces V, W and $a \in \mathcal{L}(V, W')$, we write $a(v, w) = a(v)(w)$ for $v \in V$ and $w \in W$.

Proposition 3.5. *Let V, Q be Hilbert spaces, $a \in \mathcal{L}(V, V')$, $b \in \mathcal{L}(V, Q')$, $f \in V'$, $g \in Q'$, with constants $\alpha, \beta > 0$ such that*

$$a(v, v) \geqslant \alpha\|v\|_V \qquad \text{for all } v \in V,$$
$$\inf_{q \in Q\setminus\{0\}} \sup_{v \in V\setminus\{0\}} \frac{b(q, v)}{\|q\|_Q \|v\|_V} \geqslant \beta.$$

Then, there exists a unique solution $(v, p) \in V \times Q$ such that

$$\begin{aligned} a(v, w) + b(w, p) &= f(w), \\ b(v, q) &= g(q) \end{aligned}$$

for every $(w, q) \in V \times Q$. This solution $(v, p) \in V \times Q$ is bounded by

$$\begin{aligned} \|v\|_V &\leqslant \frac{1}{\alpha}\|f\|_{V'} + \frac{2\|a\|_{\mathcal{L}(V,V')}}{\alpha\beta}\|g\|_{Q'}, \\ \|p\|_Q &\leqslant \frac{2\|a\|_{\mathcal{L}(V,V')}}{\alpha\beta}\|f\|_{V'} + \frac{2\|a\|_{\mathcal{L}(V,V')}}{\alpha\beta^2}\|g\|_{Q'}. \end{aligned} \tag{3.9}$$

If, moreover, $a(\cdot, \cdot)$ is symmetric, the estimates are improved to

$$\begin{aligned} \|v\|_V &\leqslant \frac{1}{\alpha}\|f\|_{V'} + \frac{2\|a\|^{1/2}_{\mathcal{L}(V,V')}}{\alpha^{1/2}\beta}\|g\|_{Q'}, \\ \|p\|_Q &\leqslant \frac{2\|a\|^{1/2}_{\mathcal{L}(V,V')}}{\alpha^{1/2}\beta}\|f\|_{V'} + \frac{\|a\|_{\mathcal{L}(V,V')}}{\beta^2}\|g\|_{Q'}. \end{aligned} \tag{3.10}$$

Proof. Proposition 3.5 is shown, for example, in [BBF13, Theorem 4.2.3]. □

As (3.10) already suggests, it is crucial for the ε-independent a-priori estimate that we obtain an ε-independent coercivity constant for a and an ε-independent inf–sup constant for b. For the uniform estimate of the coercivity constant of a, we derive a Korn-type inequality for $e_\varepsilon(\varphi) = \Psi_\varepsilon^{-\top}\nabla\varphi + (\Psi_\varepsilon^{-\top}\nabla\varphi)^\top$. In order to derive a uniform estimate for the inf–sup constant of b, we employ a family of continuous linear operators $\operatorname{div}_\varepsilon^{-1}$: $L^2(\Omega_\varepsilon) \to H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)$, which are right-inverse to the divergence.

Right-inverse divergence operator

By means of the following restriction operator, we can trace the construction of the operator $\operatorname{div}_\varepsilon^{-1}\colon L^2(\Omega_\varepsilon) \to H^1(\Omega_\varepsilon)^n$ back to the construction of $\operatorname{div}^{-1}\colon L^2(\Omega) \to H^1(\Omega)^n$, which is defined ε-independently. This restriction operator was originally presented in [Tar80] and extended to the case of connected solid domains in [All89, Theorem 2.3].

Lemma 3.6. *There exists a family of linear and continuous operators*

$$R_\varepsilon : H^1(\Omega)^n \to H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^n$$

such that

- $u \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)$ *implies* $R_\varepsilon(u) = u$ *in* Ω_ε,
- $\operatorname{div}(R_\varepsilon(u)) = \operatorname{div}(u) + \sum\limits_{k\in I_\varepsilon} \frac{1}{|\varepsilon Y^*|}\chi_{\varepsilon k+\varepsilon Y^*} \int\limits_{\varepsilon k+\varepsilon Y^{\mathrm{s}}} \operatorname{div}(u)\,\mathrm{d}x$,
- *there exists a constant* C *such that*

  $$\|R_\varepsilon u\|_{L^2(\Omega_\varepsilon)} + \varepsilon\|\nabla R_\varepsilon u\|_{L^2(\Omega_\varepsilon)} \leqslant C\big(\|u\|_{L^2(\Omega)} + \varepsilon\|\nabla u\|_{L^2(\Omega)}\big)$$

  for every $u \in H^1(\Omega)$.

Proof. In [All89], this restriction operator was explicitly constructed as an operator from $H^1_0(\Omega)^n$ to $H^1_0(\Omega_\varepsilon)^n$. Indeed, the construction is done locally and, thus, the same construction provides also an operator $R_\varepsilon\colon H^1(\Omega) \to H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^n$, which does not incorporate the zero values at $\partial\Omega$. □

Lemma 3.7. *Let* $U \subset \mathbb{R}^n$ *be a bounded domain. Then, there exists a linear and continuous operator* $\operatorname{div}^{-1}\colon L^2(U) \to H^1(U)^n$ *such that* $\operatorname{div}\circ\operatorname{div}^{-1} = \mathrm{id}_{L^2(U)}$.

Proof. See for instance [Gal11, Exercise III.3.1]. □

By combining the previous two results, we obtain the following right-inverse ε-scaled divergence operator.

Lemma 3.8. *There exists a family of linear continuous operators*

$$\operatorname{div}_\varepsilon^{-1}\colon L^2(\Omega_\varepsilon) \to H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^n,$$

such that

$$\operatorname{div}(\operatorname{div}_\varepsilon^{-1}(f)) = f,$$
$$\|\operatorname{div}_\varepsilon^{-1}(f)\|_{L^2(\Omega_\varepsilon)} + \varepsilon\|\nabla \operatorname{div}_\varepsilon^{-1}(f)\|_{L^2(\Omega_\varepsilon)} \leqslant C\|f\|_{L^2(\Omega_\varepsilon)}$$

for every $f \in L^2(\Omega_\varepsilon)$ and a constant C which is independent of ε.

Proof. Lemma 3.7 provides a linear continuous operator $\operatorname{div}^{-1}\colon L^2(\Omega) \to H^1(\Omega)^n$ such that $\operatorname{div} \circ \operatorname{div}^{-1} = \mathrm{id}_{L^2(\Omega)}$. With this operator and the restriction operator R_ε of Lemma 3.6, we define for $f \in L^2(\Omega_\varepsilon)$

$$\operatorname{div}_\varepsilon^{-1}(f) := R_\varepsilon(\operatorname{div}^{-1}(\widetilde{f})),$$

where $\widetilde{f}$ denotes the extension of f by 0 to $\Omega \backslash \Omega_\varepsilon$.

Then, the explicit formula for $\operatorname{div} \circ R_\varepsilon$ from Lemma 3.6 leads to

$$\begin{aligned}\operatorname{div}(\operatorname{div}_\varepsilon^{-1}(f)) &= \operatorname{div}(R_\varepsilon(\operatorname{div}^{-1}(\widetilde{f}))) = \\ &= \operatorname{div}(\operatorname{div}^{-1}(\widetilde{f})) + \frac{1}{|\varepsilon Y^*|}\sum_{k\in I_\varepsilon} \chi_{\varepsilon k+\varepsilon Y^*} \int\limits_{\varepsilon k+\varepsilon Y^{\mathrm{s}}} \operatorname{div}(\operatorname{div}^{-1}(\widetilde{f}(x)))\,\mathrm{d}x \\ &= \widetilde{f} + \frac{1}{|\varepsilon Y^*|}\sum_{k\in I_\varepsilon} \chi_{\varepsilon k+\varepsilon Y^*} \int\limits_{\varepsilon k+\varepsilon Y^{\mathrm{s}}} \widetilde{f}(x)\,\mathrm{d}x = f.\end{aligned}$$

Moreover, with the continuity estimates for the restriction operator and for div^{-1}, we obtain for $\varepsilon > 0$

$$\begin{aligned}&\|\operatorname{div}_\varepsilon^{-1}(f)\|_{L^2(\Omega_\varepsilon)} + \varepsilon\|\nabla \operatorname{div}_\varepsilon^{-1}(f)\|_{L^2(\Omega_\varepsilon)} \\ &\qquad\leqslant C\Big(\|\operatorname{div}^{-1}(\widetilde{f})\|_{L^2(\Omega)} + \varepsilon\|\nabla \operatorname{div}^{-1}(\widetilde{f})\|_{L^2(\Omega)}\Big) \\ &\qquad\leqslant C(1+\varepsilon)\|\operatorname{div}^{-1}(\widetilde{f})\|_{H^1(\Omega)} \leqslant C(1+\varepsilon)\|\widetilde{f}\|_{L^2(\Omega)} \leqslant C\|f\|_{L^2(\Omega_\varepsilon)}.\end{aligned}$$

□

Korn-type inequality for two-scale transformation

The aim of this section is the derivation of the following Korn-type inequality.

Proposition 3.9. *There exists a constant α such that*

$$\|\Psi_\varepsilon^{-\top}\nabla v + (\Psi_\varepsilon^{-\top}\nabla v)^\top\|^2_{L^2(\Omega_\varepsilon)} \geqslant \alpha\|\nabla v\|^2_{L^2(\Omega_\varepsilon)} \tag{3.11}$$

for all $v \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^n$ and every $\varepsilon > 0$.

The proof of Proposition 3.9 is a consequence of the following lemmas which break down all the arising difficulties. The first difficulty arises already from the multiplication of the gradient by a fixed space-dependent matrix. For $n = 3$, this was solved in [Nef02] first. In [Pom03], the regularity assumptions on the matrix were reduced, which leads to the following result.

Lemma 3.10. *Let $1 < p < \infty$ and U be an open and bounded domain in $\mathbb{R}^n$ for $n \geqslant 2$ with Lipschitz boundary ∂U. Let S be an open subset of $\partial\Omega$ with $|S| > 0$. Let $A \in C(\overline{U})^{n\times n}$ with $\det(A(x)) \geqslant c > 0$. Then, there exists a constant $\alpha > 0$ such that*

$$\int_U |A(x)\nabla u(x) + (A(x)\nabla u(x))^\top|^p \,\mathrm{d}x \geqslant \alpha \int_U |\nabla u(x)|^p \,\mathrm{d}x \tag{3.12}$$

for every $u \in W^{1,p}_S(U)^n := \{v \in W^{1,p}(U)^n \mid v|_S = 0\}$.

Proof. See [Pom03, Corollary 4.1]. □

For the special case that A arises as gradient of an C^1-diffeomorphism, (3.12) can be shown by transforming the integral with this diffeomorphism and applying the standard Korn inequality afterwards.

The constant α in (3.12), depends on the matrix A. However, in (3.11), we have to deal with a family of matrices instead of one fixed matrix. By the following continuity argument of [WP24], we can uniformly choose α for a compact set of matrices. A similar result was provided independently in [MR20].

Lemma 3.11. *Let $1 < p < \infty$ and U be an open, bounded domain in $\mathbb{R}^n$ for $n \geqslant 2$ with Lipschitz boundary ∂U. Let S be an open subset of ∂U with $|S| > 0$. Let $\mathcal{A} \subset C(\overline{U})^{n\times n}$ with $\det(A(x)) \geqslant c > 0$ for every $A \in \mathcal{A}$. Then, there exists a constant $\alpha > 0$ such that*

$$\|A\nabla u + (A\nabla u)^\top\|^p_{L^p(U)} \geqslant \alpha \|\nabla u(x)\|^p_{L^p(u)}$$

for every $u \in W^{1,p}_S(U)^n$ and every $A \in \mathcal{A}$.

Proof. We define the family of mappings $\{\lambda_v : \mathcal{A} \to \mathbb{R} \mid v \in W^{1,p}_S(U)^n \backslash \{0\}\}$, by

$$\lambda_v(A) := \frac{\|A\nabla v + (A\nabla v)^\top\|_{L^p(U)}}{\|\nabla v\|_{L^p(U)}}.$$

Using the triangle inequality and the Hölder inequality, we obtain for $A, B \in \mathcal{A}$

$$\begin{aligned}
&\big|\|A\nabla v + (A\nabla v)^\top\|_{L^p(U)} - \|B\nabla v + (B\nabla v)^\top\|_{L^p(U)}\big| \\
&\leqslant \|A\nabla v - B\nabla v + (A\nabla v - B\nabla v)^\top\|_{L^p(U)} \leqslant 2\|A\nabla v - B\nabla v\|_{L^p(U)} \\
&\leqslant 2\|A - B\|_{C(\overline{U})}\|\nabla v\|_{L^p(U)}.
\end{aligned}$$

Thus,

$$|\lambda_v(A) - \lambda_v(B)| \leqslant 2\|A - B\|_{C(\overline{U})}$$

for every $A, B \in \mathcal{A}$ and $v \in W^{1,p}_V(U)^n$, which shows that the family $\{\lambda_v \mid v \in W^{1,p}_S(U)^n \backslash \{0\}\}$ is equicontinuous. Due to the equicontinuity, the pointwise infimum of this family is continuous as well, i.e. $\lambda\colon \mathcal{A} \to \mathbb{R}$ defined by

$$\lambda(A) := \inf_{v \in W^{1,p}_S(U)^n \backslash \{0\}} \lambda_v(A) = \inf_{v \in W^{1,p}_S(U)^n \backslash \{0\}} \frac{\|A\nabla v + (A\nabla v)^\top\|_{L^p(U)}}{\|\nabla v\|_{L^p(U)}},$$

is continuous. Therefore, λ attains its minimum over the compact set $\mathcal{A}$ at a point $A_0 \in \mathcal{A}$. Due to Lemma 3.10, $\lambda(A) > 0$ for all $A \in \mathcal{A}$ and, in particular, $\lambda(A_0) > 0$. Hence, there exists a constant $\alpha = \lambda(A_0) > 0$ such that

$$\inf_{v \in W^{1,p}_S(U)^n \backslash \{0\}} \frac{\|A\nabla v + (A\nabla v)^\top\|_{L^p(U)}}{\|\nabla v\|_{L^p(U)}} = \lambda(A) \geqslant \lambda(A_0) = \alpha > 0$$

for all $A \in \mathcal{A}$, which proves Lemma 3.11. □

In order to show (3.11), we not only have to deal with a family of coefficients $\Psi_\varepsilon^{-\top}$ but also a family of domains Ω_ε. Since every cell $\varepsilon k + \varepsilon Y^*$ contains a subset of the boundary Γ_ε, on which $v \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^n$ is zero, we can upscale these cells and apply Lemma 3.11 for every cell. In order to formulate the compactness of the matrices across the upscaling process, we quantify the compactness using Hölder continuity.

Lemma 3.12. *Let $1 < p < \infty$. Then, for every $c, C > 0$ and $\lambda \in (0,1]$, there exists an ε-independent constant $\alpha > 0$ such that*

$$\alpha\|\nabla v\|^p_{L^p(\Omega_\varepsilon)} \leqslant \|A_\varepsilon \nabla v + (A_\varepsilon \nabla v)^\top\|^p_{L^p(\Omega_\varepsilon)}$$

for all $v \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^n$ and every $A_\varepsilon \in C^{0,\lambda}(\overline{\Omega_\varepsilon})^{n\times n}$ with

$$\begin{aligned}
&\|A_\varepsilon\|_{C(\overline{\Omega_\varepsilon})} \leqslant C, \\
&\varepsilon^\lambda |A_\varepsilon(x_1) - A_\varepsilon(x_2)| \leqslant C|x_1 - x_2|^\lambda && \text{for all } x_1, x_2 \in \overline{\Omega_\varepsilon}, \\
&\det(A_\varepsilon(x)) \geqslant c && \text{for all } x \in \overline{\Omega_\varepsilon}.
\end{aligned}$$

Proof. Since

$$\begin{aligned}
\|A_\varepsilon \nabla v + (A_\varepsilon \nabla v)^\top\|^p_{L^p(\Omega_\varepsilon)} &= \sum_{k \in I_\varepsilon} \|A_\varepsilon \nabla v + (A_\varepsilon \nabla v)^\top\|^p_{L^p(\varepsilon k + \varepsilon Y^*)}, \\
\|\nabla v\|^p_{L^p(\Omega_\varepsilon)} &= \sum_{k \in I_\varepsilon} \|\nabla v\|^p_{L^p(\varepsilon k + \varepsilon Y^*)},
\end{aligned}$$

it suffices to show the estimate for every cell $\varepsilon k + \varepsilon Y^*$ separately. Thus, after scaling and shifting $\varepsilon k + \varepsilon Y^*$ to Y^*, it suffices to show

$$\alpha\|\nabla v\|^p_{L^p(Y^*)} \leqslant \|A_{\varepsilon,k} \nabla v + (A_{\varepsilon,k} \nabla v)^\top\|^p_{L^p(Y^*)} \tag{3.13}$$

for all $v \in W^{1,p}_{\Gamma}(Y^*)^n$ and $k \in I_\varepsilon$, where $A_{\varepsilon,k}(x) = A_\varepsilon(\varepsilon k + \varepsilon x)$. This scaling yields

$$\begin{aligned}|A_{\varepsilon,k}(x_1) - A_{\varepsilon,k}(x_2)| &= |A_\varepsilon(\varepsilon k + \varepsilon x_1) - A_\varepsilon(\varepsilon k + \varepsilon x_2)| \leqslant \varepsilon^{-\lambda} C|(\varepsilon k + \varepsilon x_1) - (\varepsilon k + \varepsilon x_2)|^\lambda \\ &= C|x_1 - x_2|^\lambda\end{aligned}$$

for all $x_1, x_2 \in \overline{Y^*}$ and $\det(A_{\varepsilon,k}(x)) = \det(A_\varepsilon(\varepsilon k + \varepsilon x)) \geqslant c$. Thus, one has $A_{\varepsilon,k} \in \mathcal{A}$ for every $\varepsilon > 0$ and $k \in I_\varepsilon$, where

$$\mathcal{A} := \left\{ A \in C^{0,\lambda}(\overline{Y^*})^{n\times n} \,\middle|\, \begin{array}{l} \|A\|_{C(\overline{Y^*})} \leqslant C, \det(A) \geqslant c, \\ |A(x_1) - A(x_2)| \leqslant C|x_1 - x_2|^\lambda \text{ for all } x_1, x_2 \in \overline{Y^*} \end{array} \right\}.$$

The uniform Hölder continuity implies the equicontinuity of $\mathcal{A}$, and since the functions in $\mathcal{A}$ are also pointwise bounded the theorem of Arzelà–Ascoli shows that $\mathcal{A}$ is relatively compact in $C^{0,\lambda}(\overline{Y^*})^{n\times n}$. Finally, Lemma 3.11 applied on the closure of $\mathcal{A}$ provides $\alpha > 0$ for (3.13) and, hence, proves Lemma 3.12. Note that the determinant is a continuous function and, therefore, it holds $\det(A) \geqslant c$ also for every A in the closure of $\mathcal{A}$. □

Having done this preliminary work, it suffices to show that $\Psi_\varepsilon^{-\top}$ fulfills the assumptions of Lemma 3.12, in order to prove Proposition 3.9.

Proof of Proposition 3.9. From Lemma 2.8, we obtain constants $c, C > 0$ such that

$$\begin{aligned} &\|\Psi_\varepsilon^{-\top}\|_{C(\overline{\Omega_\varepsilon})} \leqslant C, \\ &\det(\Psi_\varepsilon^{-\top}(x)) = J_\varepsilon^{-1}(x) \geqslant c && \text{for all } x \in \overline{\Omega_\varepsilon}, \\ &\varepsilon|\Psi_\varepsilon^{-\top}(x_1) - \Psi_\varepsilon^{-\top}(x_2)| \leqslant \varepsilon C\|\partial_x \Psi_\varepsilon^{-\top}\|_{C(\overline{\Omega_\varepsilon})}|x_1 - x_2| \leqslant C|x_1 - x_2| && \text{for all } x_1, x_2 \in \overline{\Omega_\varepsilon}. \end{aligned}$$

Then, we can apply Lemma 3.12, which provides the constant α. □

Before we can finally show the existence and uniqueness of the solution for the Stokes equations, we recap the following ε-scaled Poincaré inequality.

Lemma 3.13. *There exists a constant $C > 0$ such that*

$$\|v\|_{L^2(\Omega_\varepsilon)} \leqslant \varepsilon C\|\nabla v\|_{L^2(\Omega_\varepsilon)} \tag{3.14}$$

for all $v \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)$.

Proof. Lemma 3.13 can be proven by decomposing Ω_ε in ε-scaled reference cells εY^* and applying the Poincaré inequality there, see for instance [Hor97, Chapter 3, Lemma 1.6]. □

Having done all the preliminary work, we can finally show the existence and uniqueness of the solution for the ε-scaled transformed Stokes equations.

Proof of Theorem 3.4. Let $V_\varepsilon = H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)$ and $Q_\varepsilon = L^2(\Omega_\varepsilon)$, with the scalar products and norms defined by

$$\|v\|^2_{V_\varepsilon} := (v, v)_{V_\varepsilon} := \varepsilon^2(\nabla v, \nabla v)_{L^2(\Omega_\varepsilon)}, \qquad \|q\|^2_{V_\varepsilon} := (q, q)_{Q_\varepsilon} := (q, q)_{L^2(\Omega_\varepsilon)}$$

for $v \in V_\varepsilon$ and $q \in Q_\varepsilon$. Due to the Poincaré inequality (3.14), $\|\cdot\|_{V_\varepsilon}$ actually defines a norm on V_ε and $(\cdot,\cdot)_{V_\varepsilon}$ is a scalar product. We define a_ε in $\mathcal{L}(V_\varepsilon, V_\varepsilon')$ and b_ε in $\mathcal{L}(Q_\varepsilon, V_\varepsilon')$ by

$$a_\varepsilon(v,w) = \mu\varepsilon^2\big(A_\varepsilon 2e_\varepsilon(v), \nabla w\big)_{L^2(\Omega_\varepsilon)} \qquad \text{for } v,w \in V_\varepsilon, \tag{3.15}$$

$$b_\varepsilon(q,v) = (q, \operatorname{div}(A_\varepsilon v))_{L^2(\Omega_\varepsilon)} \qquad \text{for } q \in Q_\varepsilon, v \in V_\varepsilon. \tag{3.16}$$

We define the right-hand sides $h_\varepsilon \in V_\varepsilon'$ and $g_\varepsilon \in Q_\varepsilon'$ by

$$h_\varepsilon(w) = \int_{\Omega_\varepsilon} (J_\varepsilon(x)\hat{f}_\varepsilon(x) - A_\varepsilon^\top(x)\nabla\hat{p}_{b,\varepsilon}(x)) \cdot w(x)\,\mathrm{d}x - a_\varepsilon(\hat{v}_{\Gamma_\varepsilon}, w) \qquad \text{for } w \in V_\varepsilon,$$

$$g_\varepsilon(q) = -b_\varepsilon(q, \hat{v}_{\Gamma_\varepsilon}) \qquad \text{for } q \in Q_\varepsilon.$$

Thus, we have embedded the weak formulation of the Stokes equations (3.6) into the generic framework of Proposition 3.5.

Now, we show the uniform coercivity of a_ε and the uniform inf–sup estimate for b_ε as well as the continuity estimates for a_ε, b_ε, h_ε, g_ε.

- Coercivity of a_ε: Let $v \in V_\varepsilon$. First, we rewrite $a_\varepsilon(v,v)$, by shifting the factor Ψ_ε^{-1} from $A_\varepsilon = J_\varepsilon\Psi_\varepsilon^{-1}$ to the second argument of the scalar product of a_ε. Then, we use the fact that for matrices $A, B \in \mathbb{R}^{n\times n}$, one has

$$\begin{aligned}(A + A^\top, B) = (A + A^\top) : B = \operatorname{tr}((A + A^\top)^\top B) = \operatorname{tr}((A + A^\top)B^\top)\\ = \tfrac{1}{2}\operatorname{tr}((A + A^\top)^\top(B + B^\top)) = \tfrac{1}{2}(A + A^\top, B + B^\top),\end{aligned}$$

  which gives

$$\begin{aligned}a_\varepsilon(v,v) &= \mu\varepsilon^2(J_\varepsilon\big(\Psi_\varepsilon^{-\top}\nabla v + (\Psi_\varepsilon^{-\top}\nabla v)^\top\big), \Psi_\varepsilon^{-\top}\nabla v)_{L^2(\Omega_\varepsilon)}\\ &= \tfrac{1}{2}\mu\varepsilon^2\big(J_\varepsilon\big(\Psi_\varepsilon^{-\top}\nabla v + (\Psi_\varepsilon^{-\top}\nabla v)^\top\big), \big(\Psi_\varepsilon^{-\top}\nabla v + \Psi_\varepsilon^{-\top}\nabla v)^\top\big)_{L^2(\Omega_\varepsilon)}\\ &= \tfrac{1}{2}\mu\varepsilon^2\|\sqrt{J_\varepsilon}\big(\Psi_\varepsilon^{-\top}\nabla v + (\Psi_\varepsilon^{-\top}\nabla v)^\top\big)\|_{L^2(\Omega_\varepsilon)}^2.\end{aligned}$$

  With the boundedness of $J_\varepsilon \geqslant c_J$ from below and the Korn-type inequality given in Proposition 3.9, we can estimate further and obtain $\alpha > 0$ such that

$$\begin{aligned}a_\varepsilon(v,v) &\geqslant \tfrac{1}{2}\mu\varepsilon^2 c_J\|\big(\Psi_\varepsilon^{-\top}\nabla v + (\Psi_\varepsilon^{-\top}\nabla v)^\top\big)\|_{L^2(\Omega_\varepsilon)}^2\\ &\geqslant \varepsilon^2\alpha\|\nabla v\|_{L^2(\Omega_\varepsilon)}^2 = \alpha\|v\|_{V_\varepsilon}^2\end{aligned} \tag{3.17}$$

  for every $v \in V_\varepsilon$.

- Continuity of a_ε: With the Cauchy–Schwarz inequality, the triangle inequality and the uniform essential boundedness of A_ε and Ψ_ε^{-1} (see Lemma 2.8), we can estimate

$$\begin{aligned} |a_\varepsilon(v,w)| &\leqslant \mu\varepsilon^2 \|A_\varepsilon(\Psi_\varepsilon^{-\top}\nabla v + (\Psi_\varepsilon^{-\top}\nabla v)^\top)\|_{L^2(\Omega_\varepsilon)} \|\nabla w\|_{L^2(\Omega_\varepsilon)} \\ &\leqslant \varepsilon^2 C \|\Psi_\varepsilon^{-\top}\nabla v + (\Psi_\varepsilon^{-\top}\nabla v)^\top\|_{L^2(\Omega_\varepsilon)} \|\nabla w\|_{L^2(\Omega_\varepsilon)} \\ &\leqslant \varepsilon^2 C \|\Psi_\varepsilon^{-\top}\nabla v\|_{L^2(\Omega_\varepsilon)} \|\nabla w\|_{L^2(\Omega_\varepsilon)} \\ &\leqslant \varepsilon^2 C \|\nabla v\|_{L^2(\Omega_\varepsilon)} \|\nabla w\|_{L^2(\Omega_\varepsilon)} \leqslant C\|v\|_{V_\varepsilon}\|w\|_{V_\varepsilon} \end{aligned} \tag{3.18}$$

for every $v, w \in V_\varepsilon$.

- Inf–sup estimate of b_ε: From Lemma 3.8, we obtain, for every $q \in Q_\varepsilon$, a function $v_0 \in V_\varepsilon$ such that

$$\begin{aligned} \operatorname{div}(v_0) &= q, \\ \|v_0\|_{L^2(\Omega_\varepsilon)} + \varepsilon\|\nabla v_0\|_{L^2(\Omega_\varepsilon)} &\leqslant C\|q\|_{L^2(\Omega_\varepsilon)}. \end{aligned}$$

  Then, together with the estimates for the coefficients from Lemma 2.8 and the Poincaré inequality (3.14), we obtain a constant $\beta > 0$ such that

$$\begin{aligned} \varepsilon\|\nabla(A_\varepsilon^{-1}v_0)\|_{L^2(\Omega_\varepsilon)} &\leqslant \varepsilon C\|\nabla A_\varepsilon^{-1}\|_{L^\infty(\Omega_\varepsilon)}\|v_0\|_{L^2(\Omega_\varepsilon)} + \varepsilon C\|A_\varepsilon^{-1}\|_{L^\infty(\Omega_\varepsilon)}\|\nabla v_0\|_{L^2(\Omega_\varepsilon)} \\ &\leqslant C\|v_0\|_{L^2(\Omega_\varepsilon)} + \varepsilon C\|\nabla v_0\|_{L^2(\Omega_\varepsilon)} \leqslant \beta^{-1}\|q\|_{L^2(\Omega_\varepsilon)} \end{aligned}$$

  By choosing $v = A_\varepsilon^{-1}v_0 \in V_\varepsilon$, we get

$$\begin{aligned} \sup_{v\in V_\varepsilon\setminus\{0\}} \frac{b(q,v)}{\|q\|_{Q_\varepsilon}\|v\|_{V_\varepsilon}} = \sup_{v\in V_\varepsilon\setminus\{0\}} \frac{(q,\operatorname{div}(A_\varepsilon v))_{L^2(\Omega_\varepsilon)}}{\|q\|_{Q_\varepsilon}\|v\|_{V_\varepsilon}} &\geqslant \frac{(q,\operatorname{div}(v_0))_{L^2(\Omega_\varepsilon)}}{\|q\|_{Q_\varepsilon}\|A_\varepsilon^{-1}v_0\|_{V_\varepsilon}} \\ \geqslant \frac{\|q\|^2_{L^2(\Omega_\varepsilon)}}{\|q\|_{L^2(\Omega_\varepsilon)}\beta^{-1}\|q\|_{L^2(\Omega_\varepsilon)}} \geqslant \beta, \end{aligned} \tag{3.19}$$

  which provides an ε-independent inf–sup constant β.

- Continuity estimate of b_ε: Using the Piola identity (3.7), the Hölder inequality and the estimates for A_ε from Lemma 2.8, we obtain

$$\begin{aligned} |b_\varepsilon(v,q)| &= (q,\operatorname{div}(A_\varepsilon v))_{L^2(\Omega_\varepsilon)} = (q,\operatorname{div}(A_\varepsilon)\cdot v + A_\varepsilon : \nabla v))_{L^2(\Omega_\varepsilon)} = (q, A_\varepsilon : \nabla v)_{L^2(\Omega_\varepsilon)} \\ &\leqslant C\|q\|_{L^2(\Omega_\varepsilon)}\|\nabla v\|_{L^2(\Omega_\varepsilon)} \leqslant \varepsilon^{-1}C\|q\|_{Q_\varepsilon}\|v\|_{V_\varepsilon} \end{aligned} \tag{3.20}$$

  for every $q \in Q_\varepsilon$ and every $v \in V_\varepsilon$.

- Continuity estimate of h_ε: In order to estimate h_ε, we use the Hölder inequality and the continuity estimate of a_ε. Then, we employ the boundedness of the transformation coefficients (see Lemma 2.8), the boundedness of the data $\hat{f}_\varepsilon$, $\nabla\hat{p}_{b,\varepsilon}$ and $\hat{v}_{\Gamma_\varepsilon}$ (see

Lemma 3.3) and the ε-scaled Poincaré inequality (3.14). Accordingly, we obtain

$$
\begin{aligned}
|f_\varepsilon(w)| = & \int_{\Omega_\varepsilon} (J_\varepsilon(x)\hat{f}_\varepsilon(x) - A_\varepsilon^\top(x)\nabla\hat{p}_{b,\varepsilon}(x)) \cdot w(x)\,\mathrm{d}x - a_\varepsilon(\hat{v}_{\Gamma_\varepsilon}, w) \\
\leqslant & (\|J_\varepsilon\|_{L^\infty(\Omega_\varepsilon)}\|\hat{f}_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|A_\varepsilon^\top\|_{L^\infty(\Omega_\varepsilon)}\|\nabla\hat{p}_{b,\varepsilon}\|_{L^2(\Omega_\varepsilon)})\|w\|_{L^2(\Omega_\varepsilon)} + C\|\hat{v}_{\Gamma_\varepsilon}\|_{V_\varepsilon}\|w\|_{V_\varepsilon} \\
\leqslant & C\|w\|_{L^2(\Omega_\varepsilon)} + \varepsilon C\|w\|_{V_\varepsilon} \leqslant C\|w\|_{V_\varepsilon}
\end{aligned} \tag{3.21}
$$

for every $w \in V_\varepsilon$. Note that in the derivation of (3.21) the term $a_\varepsilon(\hat{v}_{\Gamma_\varepsilon}, w)$ is of order ε. Therefore, it will also vanish during the homogenisation process later.

- Continuity estimate of g_ε: We use the continuity estimate for b_ε (3.21) and the boundedness of $\hat{v}_{\Gamma_\varepsilon}$ (see Lemma 3.3) in order to estimate g_ε. The estimate of b_ε (3.21) provides a factor ε^{-1}, which is canceled by the estimate for $\hat{v}_{\Gamma_\varepsilon}$, i.e.

$$
|g_\varepsilon(q)| = |b_\varepsilon(q, \hat{v}_{\Gamma_\varepsilon})| \leqslant \varepsilon^{-1}C\|q\|_{Q_\varepsilon}\|\hat{v}_{\Gamma_\varepsilon}\|_{V_\varepsilon} = C\|q\|_{Q_\varepsilon}\|\nabla\hat{v}_{\Gamma_\varepsilon}\|_{L^2(\Omega_\varepsilon)} \leqslant C\|q\|_{Q_\varepsilon} \tag{3.22}
$$

for every $q \in Q_\varepsilon$.

Now, we employ Proposition 3.5, which gives a unique solution $(\hat{w}_\varepsilon, \hat{q}_\varepsilon) \in V_\varepsilon \times Q_\varepsilon = H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^n \times L^2(\Omega_\varepsilon)$ of the weak formulation of the Stokes equations (3.6).

Moreover, from (3.18), (3.21) and (3.22), we obtain

$$
\|a_\varepsilon\|_{\mathcal{L}(V,V')} + \|f_\varepsilon\|_{V'} + \|g_\varepsilon\|_{Q'} \leqslant C \tag{3.23}
$$

and together with the uniform coercivity estimate (3.17) and the uniform inf–sup estimate (3.19) all terms of the right-hand side of (3.9) are ε-independently bounded and, thus, we obtain

$$
\varepsilon\|\nabla\hat{w}_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|\hat{q}_\varepsilon\|_{L^2(\Omega_\varepsilon)} = \|\hat{w}_\varepsilon\|_{V_\varepsilon} + \|\hat{q}_\varepsilon\|_{Q_\varepsilon} \leqslant C.
$$

With the Poincaré inequality (3.14), we can estimate $\|\hat{w}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leqslant C$ afterwards, which finally shows the desired a-priori estimate (3.8). □

Remark 3.14. *We note that the estimates* (3.9) *and* (3.10) *do not depend on* $\|b\|_{\mathcal{L}(V,Q')}$. *This becomes crucial in our derivation of the a-priori estimates for the solution of the Stokes equations, since we can not bound* $\|b\|_{\mathcal{L}(V_\varepsilon,Q'_\varepsilon)}$ *ε-uniformly.*

3.1.4. Identification of the two-scale limit problem

Now, we pass to the homogenisation limit $\varepsilon \to 0$ and derive the following two-pressure Stokes equation as two-scale limit equation in the cylindrical substitute two-scale domain for unknowns $\hat{w}_0, \hat{p}, \hat{p}_1$, where $\hat{w}_0$ is the two-scale limit of $\hat{w}_\varepsilon$ and $\hat{v}_\varepsilon$, and $\hat{p} = \hat{q} + \hat{p}_{b,0}$ where $\hat{q}$ is the limit of some extension of $\hat{q}_\varepsilon$ on Ω.

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$$
\begin{aligned}
J_0^{-1} \operatorname{div}_y(\mu A_0^{-1}\Psi_0^{-\top}\nabla_y \hat{w}_0) + \Psi_0^{-\top}\nabla_x \hat{p} + \Psi_0^{-\top}\nabla_y \hat{p}_1 &= f && \text{in } \Omega \times Y^*,\\
J_0^{-1} \operatorname{div}_y(A_0 \hat{w}_0) &= 0 && \text{in } \Omega \times Y^*,\\
\hat{w}_0 &= 0 && \text{on } \Omega \times \Gamma,\\
y \mapsto \hat{w}_0, \hat{p}_1 & && Y-\text{periodic},\\
\operatorname{div}_x\Big(\int_{Y*} A_0 \hat{w}_0 \,\mathrm{d}y\Big) &= \int_\Omega\int_Y \operatorname{div}(A_0 \hat{v}_\Gamma)\,\mathrm{d}y && \text{in } \Omega,\\
\hat{p} &= \hat{p}_{b,0} && \text{on } \partial\Omega.
\end{aligned}
\tag{3.24}
$$

The transposed velocity gradient vanishes in (3.24). This, is a consequence of the microscopic incompressibility condition for $\hat{w}_0$ and the boundary values of $\hat{w}_0$.

Compactness results

Before we can identify the limit equations, we have to derive compactness results for the velocity and pressure. We start with the strong convergence of the pressure. For this, we follow the argumentation of [Tar80], which was extended in [All89], [LA90], [Mik91] and [FMW17], and adapt it to our setting, where we have to deal with the coefficients as well as different function spaces, due to the different boundary condition at the outer boundary $\partial\Omega \cap \partial\Omega_\varepsilon$. We extend $\hat{q}_\varepsilon$ on Ω by

$$
\hat{Q}_\varepsilon(x) := \begin{cases} \hat{q}_\varepsilon(x) & \text{if } x \in \Omega_\varepsilon,\\ \fint_{\varepsilon k+\varepsilon Y*} \hat{q}_\varepsilon(z)\,\mathrm{d}z & \text{if } x \in \varepsilon k + \varepsilon Y^{\mathrm{s}} \text{ for } k \in I_\varepsilon. \end{cases} \tag{3.25}
$$

Lemma 3.15. *Let $\hat{q}_\varepsilon \in L^2(\Omega_\varepsilon)$ be given by the solution of* (3.6) *and let $\hat{Q}_\varepsilon$ be its extension defined in* (3.25). *Then, there exists $\hat{q} \in L^2(\Omega)$ and a subsequence $\hat{Q}_\varepsilon$ that converges strongly to $\hat{q}$ in $L^2(\Omega)$.*

Proof. We define $F_\varepsilon \in (H^1(\Omega)^n)'$ by

$$
\langle F_\varepsilon, \varphi\rangle_{H^1(\Omega)', H^1(\Omega)} := \int_{\Omega_\varepsilon} \hat{q}_\varepsilon(x) \operatorname{div}(R_\varepsilon \varphi(x))\,\mathrm{d}x.
$$

By testing (3.6) with $A_\varepsilon^{-1} R_\varepsilon \varphi(x)$, we can rewrite this functional by

$$
\int_\Omega \hat{q}_\varepsilon(t) \operatorname{div}(R_\varepsilon\varphi)\,\mathrm{d}x = (\varepsilon^2 \mu A_\varepsilon 2 e_\varepsilon(\hat{w}_\varepsilon), \nabla(A_\varepsilon^{-1} R_\varepsilon \varphi))_{L^2(\Omega_\varepsilon)} - (\Psi_\varepsilon^\top \hat{f}_\varepsilon - \nabla \hat{p}_{b,\varepsilon}, R_\varepsilon\varphi)_{L^2(\Omega_\varepsilon)}
$$

$$+ (\varepsilon^2 \mu A_\varepsilon 2 e_\varepsilon(\hat{v}_{\Gamma_\varepsilon}), \nabla(A_\varepsilon^{-1} R_\varepsilon \varphi))_{L^2(\Omega_\varepsilon)}.$$

Using the uniform boundedness of $\varepsilon \nabla \hat{w}_\varepsilon$ (see (3.8)), the coefficients (see Lemma 2.8) and the data (see Lemma 3.3), we obtain, after applying the Hölder inequality and the Leibniz rule,

$$\begin{aligned}
|\langle F_\varepsilon, \varphi \rangle_{H^1(\Omega)', H^1(\Omega)}| &\leqslant C\varepsilon \|\nabla(A_\varepsilon^{-1} R_\varepsilon \varphi)\|_{L^2(\Omega_\varepsilon)} + C\|R_\varepsilon \varphi\|_{L^2(\Omega_\varepsilon)} + \varepsilon^2 C \|\nabla(A_\varepsilon^{-1} R_\varepsilon \varphi)\|_{L^2(\Omega_\varepsilon)} \\
&\leqslant C\varepsilon(1+\varepsilon)\|\nabla A_\varepsilon^{-1} R_\varepsilon \varphi + \nabla(R_\varepsilon \varphi) A_\varepsilon^\top\|_{L^2(\Omega_\varepsilon)} + C\|R_\varepsilon \varphi\|_{L^2(\Omega_\varepsilon)} \\
&\leqslant C\varepsilon \|\nabla A_\varepsilon^{-1}\|_{L^\infty(\Omega_\varepsilon)} \|R_\varepsilon \varphi\|_{L^2(\Omega_\varepsilon)} + C\varepsilon \|\nabla R_\varepsilon \varphi\|_{L^2(\Omega_\varepsilon)} \|A_\varepsilon^\top\|_{L^\infty(\Omega_\varepsilon)} + C\|R_\varepsilon \varphi\|_{L^2(\Omega_\varepsilon)} \\
&\leqslant C\|R_\varepsilon \varphi\|_{L^2(\Omega_\varepsilon)} + \varepsilon C \|\nabla R_\varepsilon \varphi\|_{L^2(\Omega_\varepsilon)} \\
&\leqslant C\|\varphi\|_{L^2(\Omega)} + \varepsilon C\|\nabla \varphi\|_{L^2(\Omega)}
\end{aligned} \tag{3.26}$$

for every $\varphi \in H^1(\Omega)^n$. Thus, F_ε is uniformly bounded in $(H^1(\Omega)^n)'$, i.e. $\|F_\varepsilon\|_{H^1(\Omega)'} \leqslant C$.

If $\operatorname{div}(\varphi) = 0$, one has $\operatorname{div}(R_\varepsilon \varphi) = 0$ and, thus, we obtain

$$\langle F_\varepsilon, \varphi \rangle_{H^1(\Omega)', H^1(\Omega)} = \int_{\Omega_\varepsilon} \hat{q}_\varepsilon(x) \operatorname{div}(R_\varepsilon \varphi(x)) \, \mathrm{d}x = 0$$

for every $\varphi \in H^1(\Omega)^n$ with $\operatorname{div}(\varphi) = 0$, which yields $F_\varepsilon \in \ker(\operatorname{div})^\perp$.

Since div is surjective and, in particular, has a closed range (see Lemma 3.7), we can apply the closed-range theorem, which provides $\hat{Q}_\varepsilon \in L^2(\Omega)$ such that

$$\int_\Omega \hat{Q}_\varepsilon(x) \operatorname{div}(\varphi(x)) \, \mathrm{d}x = \langle F_\varepsilon, \varphi \rangle_{H^1(\Omega)', H^1(\Omega)} = \int_{\Omega_\varepsilon} \hat{q}_\varepsilon(x) \operatorname{div}(R_\varepsilon \varphi(x)) \, \mathrm{d}x \tag{3.27}$$

for all $\varphi \in H^1(\Omega)^n$. Furthermore, we can bound $\|\hat{Q}_\varepsilon\|_{L^2(\Omega)}$ uniformly using the right-inverse of the divergence (see Lemma 3.7) and (3.26)

$$\begin{aligned}
\|\hat{Q}_\varepsilon\|_{L^2(\Omega)}^2 &= \int_\Omega \hat{Q}_\varepsilon(x) \operatorname{div}(\operatorname{div}^{-1}(\hat{Q}_\varepsilon))(x) \, \mathrm{d}x = |\langle F_\varepsilon, \operatorname{div}^{-1}(\hat{Q}_\varepsilon) \rangle_{H^1(\Omega)', H^1(\Omega)}| \\
&\leqslant C\Big(\|\operatorname{div}^{-1}(\hat{Q}_\varepsilon)\|_{L^2(\Omega)} + \varepsilon \|\nabla \operatorname{div}^{-1}(\hat{Q}_\varepsilon)\|_{L^2(\Omega}\Big) \leqslant C\|\hat{Q}_\varepsilon\|_{L^2(\Omega)}.
\end{aligned} \tag{3.28}$$

In order to identify $\hat{Q}_\varepsilon$ with $\hat{q}_\varepsilon$ on Ω_ε, we note that $R_\varepsilon(\widetilde{\varphi}) = \varphi$ for every $\varphi \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^n$, where $\widetilde{\varphi}$ is the extension by 0 of φ and, thus, the right-hand side of (3.27) can be simplified and we obtain

$$\int_\Omega \hat{Q}_\varepsilon \operatorname{div}(\widetilde{\varphi}) \, \mathrm{d}x = \int_{\Omega_\varepsilon} \hat{Q}_\varepsilon \operatorname{div}(\varphi) \, \mathrm{d}x = \int_{\Omega_\varepsilon} \hat{q}_\varepsilon \operatorname{div}(\varphi) \, \mathrm{d}x.$$

Then, Lemma 3.7, provides $\varphi \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^n$ with $\operatorname{div}(\varphi) = \hat{Q}_\varepsilon - \hat{q}_\varepsilon$, and by testing the previous equation with this φ, we obtain $\hat{Q}_\varepsilon = \hat{q}_\varepsilon$ in Ω_ε. This identifies $\hat{Q}_\varepsilon$ with the explicit formula (3.25) in Ω_ε.

In order to show the strong convergence of $\hat{Q}_\varepsilon$, we note that the boundedness of $\hat{Q}_\varepsilon$ in $L^2(\Omega)$ allows us to pass to a subsequence $\hat{Q}_\varepsilon$, which converges weakly to a function $\hat{q} \in L^2(\Omega)$. Since weak convergence is preserved under linear continuous operations, we obtain for the same subsequence that $\varphi_\varepsilon := \operatorname{div}^{-1}(\hat{Q}_\varepsilon)$ converges weakly to $\varphi = \operatorname{div}^{-1}(\hat{q})$ in $H^1(\Omega)$, where div^{-1} is given by Lemma 3.7. Moreover, we obtain from (3.27) and (3.26)

$$|(\hat{Q}_\varepsilon, \operatorname{div}(\varphi_\varepsilon - \varphi))_{L^2(\Omega)}| \leqslant C\big(\|\varphi_\varepsilon - \varphi\|_{L^2(\Omega)} + \varepsilon\|\nabla(\varphi_\varepsilon - \varphi)\|_{L^2(\Omega)}\big). \tag{3.29}$$

Now, we show that the right-hand side of (3.29) tends to zero. From the weak convergence $\varphi_\varepsilon \rightharpoonup \varphi$, we deduce the boundedness of φ_ε in $H^1(\Omega)$ and, thus, with the factor ε the second term on the right-hand side of (3.29) tends to zero. Moreover, the compact embedding of $H^1(\Omega)$ into $L^2(\Omega)$ implies the strong convergence of φ_ε to φ in $L^2(\Omega_\varepsilon)$ (after passing to a further subsequence and identifying the strong limit in $L^2(\Omega)$ with the weak limit in $H^1(\Omega)$). With this strong convergence, the first term on the right-hand side of (3.29) tends to zero, too. Hence,

$$(\hat{Q}_\varepsilon, \hat{Q}_\varepsilon - \hat{q}) = (\hat{Q}_\varepsilon, \operatorname{div}(\varphi_\varepsilon - \varphi))_{L^2(\Omega_\varepsilon)} \to 0.$$

Employing additionally the weak convergence of $\hat{Q}_\varepsilon$ to $\hat{q}$, we obtain in total

$$\|\hat{Q}_\varepsilon - \hat{q}\|^2_{L^2(\Omega_\varepsilon)} = (\hat{Q}_\varepsilon, \hat{Q}_\varepsilon - \hat{q})_{L^2(\Omega_\varepsilon)} - (\hat{q}, \hat{Q}_\varepsilon - \hat{q})_{L^2(\Omega_\varepsilon)} \to 0,$$

which shows the strong convergence of $\hat{Q}_\varepsilon$ to $\hat{q}$.

The identification of $\hat{Q}_\varepsilon$ in $\Omega \backslash \Omega_\varepsilon$ with the explicit formula (3.25) can be shown as in [All89]. □

Lemma 3.16. *Let $\hat{w}_\varepsilon \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^n$ be the first part of the solution of* (3.6) *and $\widetilde{\hat{w}_0}$ be its extension by* 0 *to* Ω*. Then, there exists $\hat{w}_0 \in L^2(\Omega; H^1_\#(Y)^n)$ and a subsequence $\widetilde{\hat{w}_\varepsilon}$ such that, for this subsequence,*

$$\widetilde{\hat{w}_\varepsilon} \xrightarrow{2} \hat{w}_0, \qquad \varepsilon\widetilde{\nabla \hat{w}_\varepsilon} \xrightarrow{2} \nabla_y \hat{w}_0, \tag{3.30}$$

Furthermore, $\hat{w}_0$ satisfies

$$\hat{w}_0 = 0 \qquad \text{in } \Omega \times Y^{\mathrm{s}}, \tag{3.31}$$

$$\int_\Omega \operatorname{div}_x\Big(\int_{Y^*} A_0(x,y)\hat{w}_0(x,y)\,\mathrm{d}y\Big)\eta_0(x)\,\mathrm{d}x = \int_\Omega\int_{Y^*} \operatorname{div}_y(A_0(x,y)\hat{v}_\Gamma(x,y))\,\mathrm{d}y\,\mathrm{d}x, \tag{3.32}$$

$$\int_\Omega\int_{Y^*} \operatorname{div}_y(A_0(x,y)\hat{w}_0(x,y))\eta_1(x,y)\,\mathrm{d}y\,\mathrm{d}x = 0 \tag{3.33}$$

for all $\eta_0 \in L^2(\Omega)$ and all $\eta_1 \in L^2(\Omega; L^2_0(Y^*))$ for $L^2_0(Y^*) := \big\{v \in L^2(Y^*) \mid \int_{Y^*} f(y)\,\mathrm{d}y = 0\big\}$.

Proof. Given the a-priori estimate (3.8), the two-scale compactness result of Theorem 1.19 provides $\hat{w}_0 \in L^2(\Omega; H^1_\#(Y^*)^n)$ and a subsequence such that (3.30) holds. Moreover, from the compactness theorem for perforated domains (Theorem 1.21), we obtain (3.31).

- Macroscopic divergence condition (3.32): We test the divergence equation of (3.6) with $\eta_0 \in D(\Omega)$ and pass twice to the limit $\varepsilon \to 0$ but once we integrate by parts beforehand. For the limit processes, we use the strong two-scale convergence of A_ε, $\varepsilon\nabla A_\varepsilon$, $\varepsilon^{-1}\hat{v}_{\Gamma_\varepsilon}$ and $\nabla\hat{v}_{\Gamma_\varepsilon}$ (see Lemma 2.9), which yields

$$
\begin{aligned}
&\int_\Omega \operatorname{div}_x \Big(\int_{Y^*} A_0(x,y)\hat{w}_0(x,y)\,\mathrm{d}y \Big)\, \eta_0(x)\,\mathrm{d}x \\
&= -\int_\Omega \int_{Y^*} A_0(x,y)\hat{w}_0(x,y)\,\mathrm{d}y \cdot \nabla_x \eta_0(x)\,\mathrm{d}x \\
&= -\lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} A_\varepsilon(x)\hat{w}_\varepsilon(x) \cdot \nabla_x \eta_0(x)\,\mathrm{d}x \\
&= \lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} \operatorname{div}(A_\varepsilon(x)\hat{w}_\varepsilon(x))\eta_0(x)\,\mathrm{d}x \\
&= -\lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} \operatorname{div}(A_\varepsilon(x)\hat{v}_{\Gamma_\varepsilon}(x))\eta_0(x)\,\mathrm{d}x \\
&= -\lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} \varepsilon \operatorname{div}(A_\varepsilon(x)) \cdot \varepsilon^{-1}\hat{v}_{\Gamma_\varepsilon}(x) + A_\varepsilon(x) : \nabla \hat{v}_{\Gamma_\varepsilon}(x))\eta_0(x)\,\mathrm{d}x \\
&= -\int_\Omega \int_{Y^*} \operatorname{div}_y(A_0(x,y)) \cdot \hat{v}_\Gamma(x,y) + A_0(x,y) : \nabla_y \hat{v}_\Gamma(x,y)\,\mathrm{d}y\ \eta_0(x)\,\mathrm{d}x. \\
&= -\int_\Omega \int_{Y^*} \operatorname{div}_y \big(A_0(x,y)\hat{v}_{\Gamma_\varepsilon}(x,y)\big)\,\mathrm{d}y\ \eta_0(x)\,\mathrm{d}x.
\end{aligned}
$$

  The boundary integral of the first integration by parts vanishes on $\partial Y^* \cap \partial Y$ since A_0 and $\hat{w}_0$ are Y-periodic and on Γ since $\hat{w}_0$ is zero on Γ. During the second integration by parts, the boundary integral vanishes on Γ_ε since $\hat{w}_\varepsilon$ is zero there and on $\partial\Omega_\varepsilon \cap \partial\Omega$ since η_0 is zero on $\partial\Omega$. Afterwards, due to the density of $D(\Omega)$ in $L^2(\Omega)$, we obtain the macroscopic divergence condition (3.32).

- Microscopic divergence condition (3.33): We choose $\eta_1 \in D(\Omega; C^\infty_\#(Y))$ in (3.33). Then, we integrate by parts, apply the two-scale convergence and again integrate by parts

$$
\begin{aligned}
&\int_{\Omega}\int_{Y^*} \operatorname{div}_y(A_0(x,y)\hat{w}_0(x,y))\eta_1(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&\qquad = -\int_{\Omega}\int_{Y^*} A_0(x,y)\hat{w}_0(x,y)\cdot\nabla_y\eta_1(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&\qquad = -\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} A_\varepsilon(x)\hat{w}_\varepsilon(x)\cdot\left(\varepsilon\nabla_x\eta_1\left(x,\frac{x}{\varepsilon}\right)+\nabla_y\eta_1\left(x,\frac{x}{\varepsilon}\right)\right)\mathrm{d}x \\
&\qquad = \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} \varepsilon\operatorname{div}(A_\varepsilon(x)\hat{w}_\varepsilon(t,x))\eta_1\left(x,\frac{x}{\varepsilon}\right)\mathrm{d}x.
\end{aligned}
\tag{3.34}
$$

In (3.34), the boundary integrals vanish during the integration by parts by the same argumentation as above.

With the boundedness of A_ε (see Lemma 2.8) and of $\nabla\hat{v}_{\Gamma_\varepsilon}$ (see Lemma 3.3) as well as with $\operatorname{div}(A_\varepsilon) = 0$, we can estimate

$$
\begin{aligned}
\|\operatorname{div}(A_\varepsilon\hat{w}_\varepsilon)\|_{L^2(\Omega_\varepsilon)} &= \|\operatorname{div}(A_\varepsilon\hat{v}_{\Gamma_\varepsilon})\|_{L^2(\Omega_\varepsilon))} = \|\operatorname{div}(A_\varepsilon)\cdot\hat{v}_{\Gamma_\varepsilon} + A_\varepsilon : \nabla\hat{v}_{\Gamma_\varepsilon}\|_{L^2(\Omega_\varepsilon))} \\
&= \|A_\varepsilon : \nabla\hat{v}_{\Gamma_\varepsilon}\|_{L^2(\Omega_\varepsilon))} = C\|\nabla\hat{v}_{\Gamma_\varepsilon}\|_{L^2(\Omega_\varepsilon))}\|\nabla A_\varepsilon^\top\|_{L^\infty(\Omega_\varepsilon))} \leqslant C
\end{aligned}
$$

Thus, the right-hand side of (3.34) is zero, and we obtain (3.33). By a density argument, it holds for arbitrary $\eta_1 \in L^2(\Omega; L^2(Y^*))$.

□

Identification of the limit of the momentum equation

Using these compactness results, namely Lemma 3.25 for $\hat{q}_\varepsilon$ and Lemma 3.16 for $\hat{w}_\varepsilon$, we can pass to the limit $\varepsilon \to 0$ in (3.6). This results in the following weak form for the two-pressure Stokes equations (3.24), where we use the function space $H^1_{\Gamma\#}(Y^*) := \{v \in H^1(Y^*) \mid v|_\Gamma = 0 \text{ and } v \text{ is } Y\text{-periodic}\}$

Weak form of the two-pressure Stokes equation in the reference coordinates

Find $(u_0, \hat{q}, \hat{q}_1) \in L^2(\Omega; H^1_{\Gamma\#}(Y^*)^n) \times H^1_0(\Omega) \times L^2(\Omega; L^2_0(Y^*))$ such that

$$\int_\Omega \int_{Y^*} \mu A_0(x,y)\Psi_0^{-\top}(x,y)\nabla_y \hat{w}_0(x,y) : \nabla_y \varphi(x,y)\,\mathrm{d}y\,\mathrm{d}x$$
$$+ \int_\Omega \int_{Y^*} A_0^\top(x,y)\nabla_x \hat{q}(x)\cdot\varphi(x,y) - \hat{q}_1(x,y)\,\mathrm{div}_y(A_0(x,y)\varphi(x,y))\,\mathrm{d}y\,\mathrm{d}x$$
$$= \int_\Omega \int_{Y^*} (J_0(x,y)\hat{f}(x) - A_0^\top(x,y)(\nabla_x \hat{p}_{b,0}(x) + \nabla_y \hat{p}_{b,1}(x,y))\cdot\varphi(x,y)\,\mathrm{d}y\,\mathrm{d}x,$$
$$\int_\Omega \mathrm{div}_x\Big(\int_{Y^*} A_0(x,y)\hat{w}_0(x,y)\,\mathrm{d}y\Big)\,\eta_0(x)\,\mathrm{d}x$$
$$= -\int_\Omega \int_{Y^*} \mathrm{div}_y\big(A_0(x,y)\hat{v}_\Gamma(x,y)\big)\,\mathrm{d}y\;\eta_0(x)\,\mathrm{d}x,$$
$$\int_\Omega \int_{Y^*} \mathrm{div}_y(A_0(x,y)\hat{w}_0(x,y))\eta_1(x,y)\,\mathrm{d}y\,\mathrm{d}x = 0 \tag{3.35}$$

for all $(\varphi, \eta_0, \eta_1) \in L^2(\Omega; H^1_{\Gamma\#}(Y^*)^n) \times H^1_0(\Omega) \times L^2(\Omega; L^2(Y^*))$.

We can equivalently choose $\eta_1 \in L^2(\Omega; L^2_0(Y^*))$, since $\mathrm{div}_y\big(A_0(x,y)\hat{v}_\Gamma(x,y)\big) \in L^2_0(Y^*)$, which can be shown by means of the Theorem of Gauß

$$\int_{Y^*} \mathrm{div}_y(A_0(x,y)\hat{w}_0(x,y))\,\mathrm{d}y = \int_{\partial Y^*} A_0(x,y)\hat{w}_0(x,y)\cdot\hat{n}\,\mathrm{d}\sigma_y$$
$$= \int_\Gamma A_0(x,y)\hat{w}_0(x,y)\cdot\hat{n}\,\mathrm{d}\sigma_y + \int_{\partial Y^*\cap\partial Y} A_0(x,y)\hat{w}_0(x,y)\cdot\hat{n}\,\mathrm{d}\sigma_y = 0,$$

where the integral over Γ vanishes since $\hat{w}_0$ is zero on Γ and the integral over $\partial Y^* \cap \partial Y$ vanishes due to the Y-periodicity of A_0 and $\hat{w}_0$.

Theorem 3.17. *Let $(\hat{w}_\varepsilon, \hat{q}_\varepsilon)$ be the solution of* (3.6) *and $\hat{q}$ be the extension of $\hat{q}_\varepsilon$ as defined in* (3.25)*. Then,*

$$\hat{w}_\varepsilon \xrightarrow{2} \hat{w}_0, \tag{3.36}$$
$$\varepsilon\nabla\hat{w}_\varepsilon \xrightarrow{2} \nabla_y \hat{w}_0, \tag{3.37}$$
$$\hat{Q}_\varepsilon \to \hat{q} \qquad \text{in } L^2(\Omega), \tag{3.38}$$

where $(\hat{w}_0, \hat{q}) \in L^2(\Omega; H^1_{\Gamma\#}(Y^*)^n) \times H^1_0(\Omega)$ *are the first two components of the solution of* (3.35).

Proof of Theorem 3.17. By means of Lemma 3.15, we can pass to a subsequence $\hat{w}_\varepsilon$ and obtain $\hat{w}_0 \in L^2(\Omega; H^1_{\Gamma\#}(Y^*)^n)$ such that the convergences $\hat{w}_\varepsilon \xrightarrow{2} \hat{w}_0$ and $\varepsilon\nabla\hat{w}_\varepsilon \to \nabla_y \hat{w}_0$ hold and $\hat{w}_0$ fulfils the two divergence conditions in (3.35). By passing to a further subsequence, Lemma 3.16 provides $\hat{q} \in L^2(\Omega)$ such that $\hat{Q}_\varepsilon \to \hat{q}$ in $L^2(\Omega)$.

It remains to show the first equation of (3.35). Let $\varphi \in C^\infty(\overline{\Omega}; H^1_{\Gamma\#}(Y^*)^n)$ such that $\operatorname{div}_y(\varphi) = 0$. Testing the first equation of (3.6) with $A_\varepsilon^{-1}(x)\varphi(x, \frac{x}{\varepsilon})$ gives

$$\begin{aligned}
&\int_{\Omega_\varepsilon} \varepsilon^2 \mu A_\varepsilon(x) 2e_\varepsilon(\hat{w}_\varepsilon)(x) : \nabla(A_\varepsilon^{-1}(x)\varphi(x, \tfrac{x}{\varepsilon}))\,\mathrm{d}x - \int_{\Omega_\varepsilon} \hat{q}_\varepsilon(x) \operatorname{div}(\varphi(x, \tfrac{x}{\varepsilon}))\,\mathrm{d}x \\
&= \int_{\Omega_\varepsilon} (J_\varepsilon(x)\hat{f}_\varepsilon(x) - A_\varepsilon^\top(x)\hat{p}_{b,\varepsilon}(x)) \cdot A_\varepsilon^{-1}(x)\varphi(x, \tfrac{x}{\varepsilon})\,\mathrm{d}x \\
&\quad - \int_{\Omega_\varepsilon} \varepsilon^2 \mu A_\varepsilon(x) 2e_\varepsilon(\hat{v}_{\Gamma_\varepsilon})(x) : \nabla(A_\varepsilon^{-1}(x)\varphi(x, \tfrac{x}{\varepsilon}))\,\mathrm{d}x.
\end{aligned} \tag{3.39}$$

In order to pass to the limit $\varepsilon \to 0$ in (3.39), we note that Lemma 1.16 implies the strong two-scale convergence for the product, i.e.

$$\begin{aligned}
\varepsilon\nabla(A_\varepsilon^{-1}(x)\varphi(x, \tfrac{x}{\varepsilon})) &= \varepsilon\nabla A_\varepsilon^{-1}(x)\varphi(x, \tfrac{x}{\varepsilon}) + \varepsilon\nabla_x\varphi(x, \tfrac{x}{\varepsilon})A_\varepsilon^{-\top}(x) + \nabla_y\varphi(x, \tfrac{x}{\varepsilon})A_\varepsilon^{-\top}(x) \\
&\xrightarrow{2} \nabla_y A_0^{-1}(x)\varphi(x,y) + 0 + \nabla_y\varphi(x,y)A_0^\top(x,y) = \nabla_y(A_0^{-1}(x,y)\varphi(x,y)).
\end{aligned}$$

For the second integral in (3.39), we use that $\operatorname{div}_y(\varphi) = 0$, which gives

$$\begin{aligned}
\int_{\Omega_\varepsilon} \hat{q}_\varepsilon(x) \operatorname{div}\left(\varphi\big(x, \tfrac{x}{\varepsilon}\big)\right)\mathrm{d}x &= \int_{\Omega_\varepsilon} \hat{q}_\varepsilon(x) \operatorname{div}_x\left(\varphi\big(x, \tfrac{x}{\varepsilon}\big)\right) + \hat{q}_\varepsilon(x)\varepsilon^{-1}\operatorname{div}_y\left(\varphi\big(x, \tfrac{x}{\varepsilon}\big)\right)\mathrm{d}x \\
&= \int_{\Omega_\varepsilon} \hat{q}_\varepsilon(x) \operatorname{div}_x\left(\varphi\big(x, \tfrac{x}{\varepsilon}\big)\right)\mathrm{d}x \to \int_\Omega\int_{Y*} \hat{q}(x)\operatorname{div}_x(\varphi(x,y))\,\mathrm{d}y\,\mathrm{d}x.
\end{aligned}$$

For the last integral in (3.39), we note that $\nabla\hat{v}_{\Gamma_\varepsilon}$ and $\varepsilon\nabla(A_\varepsilon^{-1}\varphi(\cdot, \frac{\cdot}{\varepsilon}))$ are bounded in $L^2(\Omega_\varepsilon)$. Together with the boundedness of the transformation coefficients from Lemma 2.8, one factor ε remains and, thus, the term vanishes in the limit, i.e.

$$\begin{aligned}
&\Big|\int_{\Omega_\varepsilon} \varepsilon^2\mu A_\varepsilon(x)\big(\Psi_\varepsilon^{-\top}(x)\nabla\hat{v}_{\Gamma_\varepsilon}(x) + (\Psi_\varepsilon^{-\top}(x)\nabla\hat{v}_{\Gamma_\varepsilon}(x))^\top\big) : \nabla(A_\varepsilon^{-1}(x)\varphi(x, \tfrac{x}{\varepsilon}))\,\mathrm{d}x\Big| \\
&\quad\leqslant \varepsilon C\|A_\varepsilon\|_{L^\infty(\Omega_\varepsilon)}\|\Psi_\varepsilon^{-\top}\|_{L^\infty(\Omega_\varepsilon)}\|\nabla\hat{v}_{\Gamma_\varepsilon}\|_{L^2(\Omega_\varepsilon)}\varepsilon\|\nabla(A_\varepsilon^{-1}\varphi(\cdot, \tfrac{\cdot}{\varepsilon}))\|_{L^2(\Omega_\varepsilon)} \leqslant \varepsilon C \to 0.
\end{aligned}$$

Having done these computations, we can pass to the limit $\varepsilon \to 0$ in (3.39), which gives

$$\begin{aligned}\int_\Omega \int_{Y^*} &\mu A_0(x,y)(\Psi_\varepsilon^{-\top}(x,y)\nabla_y \hat{w}_0(x,y)) + (\Psi_\varepsilon^{-\top}(x,y)\nabla_y \hat{w}_0(x,y))^\top) \\ &: \nabla_y (A_0^{-1}(x,y)\varphi(x,y)) \,\mathrm{d}y\,\mathrm{d}x - \int_\Omega \int_{Y^*} \hat{q}(x) \operatorname{div}_x(\varphi(x,y)) \,\mathrm{d}y\,\mathrm{d}x \\ &= \int_\Omega \int_{Y^*} \big(J_0(x,y)\hat{f}(x) - A_0^\top(x,y)(\nabla_x \hat{p}_{b,0}(x) + \nabla_y \hat{p}_{b,1}(x,y))\big) \cdot A_0^{-1}(x,y)\varphi(x,y) \,\mathrm{d}y\,\mathrm{d}x\end{aligned} \tag{3.40}$$

for any $\varphi \in C^\infty(\overline{\Omega}; H^1_{\Gamma\#}(Y^*)^n)$ with $\operatorname{div}_y(\varphi) = 0$. By a density argument, (3.40) holds for every $\varphi \in L^2(\Omega; H^1_{\Gamma\#}(Y^*)^n)$ with $\operatorname{div}_y(\varphi) = 0$.

In the next step, we identify $\hat{q}$ with an element in $H^1_0(\Omega)$. Therefore, let $\varphi_i \in H^1_{\Gamma\#}(Y^*)^n$ with $\operatorname{div}_y(\varphi_i) = 0$ and $\int_{Y^*} \varphi_i(y)\,\mathrm{d}y = e_i$ for $i \in \{1, \dots, n\}$ (for instance φ_i can be constructed by the Stokes operator similar to the proof of [All92a, Lemma 2.10]). Now, we test (3.40) by $\varphi\varphi_i$ for $\varphi \in C^\infty(\overline{\Omega})$ and obtain

$$\int_\Omega -\hat{q}(x)\partial_{x_i}\varphi(x)\,\mathrm{d}x = \int_\Omega G_i(x)\varphi(x)\,\mathrm{d}x = 0$$

for

$$\begin{aligned}G_i(x) = &\int_{Y^*} \big(J_0(x,y)\hat{f}(x) - A_0^\top(x,y)(\nabla_x \hat{p}_{b,0}(x) + \nabla_y \hat{p}_{b,1}(x,y))\big) \cdot A_0^{-1}(x,y)\varphi_i(y) \,\mathrm{d}y\,\mathrm{d}x \\ &- \int_{Y^*} \mu A_0(x,y)(\Psi_\varepsilon^{-\top}(x,y)\nabla_y \hat{w}_0(x,y)) + (\Psi_\varepsilon^{-\top}(x,y)\nabla_y \hat{w}_0(x,y))^\top) \\ &\qquad : \nabla_y (A_0^{-1}(x,y)\varphi_i(x,y)) \,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

Since $G_i \in L^2(\Omega)$ for all $i \in \{1, \dots, n\}$, we obtain $\hat{q} \in H^1_0(\Omega)$.

In order to increase the set of test functions to non solenoidal functions, we reconstruct some microscopic pressure. The Bogovskiĭ-operator, applied on the domain Y^*, provides the surjectivity of $\operatorname{div}_y \colon L^2(\Omega; H^1_0(Y^*)) \subset L^2(\Omega; H^1_{\Gamma\#}(Y^*)) \to L^2(\Omega; L^2_0(Y^*))$. Thus, we can apply the closed-range theorem, which gives $\hat{q}_1 \in L^2(\Omega; L^2_0(Y^*))$ such that

$$\begin{aligned}
&\int_\Omega \int_{Y^*} \mu A_0(x,y)(\Psi_0^{-\top}(x,y)\nabla_y \hat{w}_0(x,y) + (\Psi_0^{-\top}(x,y)\nabla_y \hat{w}_0(x,y))^\top) : \\
&\qquad\qquad \nabla_y (A_0^{-1}(x,y)\varphi(x,y)) \,\mathrm{d}y\,\mathrm{d}x \\
&\qquad + \int_\Omega \int_{Y^*} \nabla_x \hat{q}(x) \cdot \varphi(x,y) \,\mathrm{d}y\,\mathrm{d}x - \int_\Omega \int_{Y^*} \hat{q}_1(x,y) \operatorname{div}_y(\varphi(x,y)) \,\mathrm{d}y\,\mathrm{d}x \\
&\quad = \int_\Omega \int_{Y^*} \big(J_0(x,y)\hat{f}(x) - A_0^\top(x,y)(\nabla_x \hat{p}_{b,0}(x) + \nabla_y \hat{p}_{b,1}(x,y))\big) \cdot A_0^{-1}(x,y)\varphi(x,y) \,\mathrm{d}y\,\mathrm{d}x
\end{aligned} \tag{3.41}$$

for all $\varphi \in L^2(\Omega; H^1_{\Gamma\#}(Y^*))$. By testing (3.41) with $A_0\varphi$, we can remove the factor A_0^{-1} in front of the test functions, i.e. we obtain

$$\begin{aligned}
&\int_\Omega \int_{Y^*} \nu A_0(x,y)(\Psi_0^{-\top}(x,y)\nabla_y \hat{w}_0(x,y) + (\Psi_0^{-\top}(x,y)\nabla_y \hat{w}_0(x,y))^\top) : \nabla_y \varphi(x,y) \,\mathrm{d}y\,\mathrm{d}x \\
&\qquad + \int_\Omega \int_{Y^*} A_0^\top(x,y)\nabla_x \hat{q}(x) \cdot \varphi(x,y) \,\mathrm{d}y\,\mathrm{d}x - \int_\Omega \int_{Y^*} \hat{q}_1(x,y) \operatorname{div}_y(A_0(x,y)\varphi(x,y)) \,\mathrm{d}y\,\mathrm{d}x \\
&\quad = \int_\Omega \int_{Y^*} \big(J_0(x,y)\hat{f}(x) - A_0^\top(x,y)(\nabla_x \hat{p}_{b,0}(x) + \nabla_y \hat{p}_{b,1}(x,y))\big) \cdot \varphi(x,y) \,\mathrm{d}y\,\mathrm{d}x
\end{aligned} \tag{3.42}$$

for all $\varphi \in L^2(\Omega; H^1_{\Gamma\#}(Y^*)^n)$.

It remains to show, for a.e. $x \in \Omega$, that

$$\int_{Y^*} A_0(x,y)(\Psi_0^{-\top}(x,y)\nabla_y \hat{w}_0(x,y))^\top : \nabla_y \hat{\varphi}(x,y) \,\mathrm{d}y\,\mathrm{d}x = 0 \tag{3.43}$$

for all $\hat{\varphi} \in H^1_{\Gamma\#}(Y^*)^n$, which simplifies (3.42) to the first equation of (3.35). In order to simplify the computations, we transform the left-hand side of (3.43) by ψ_0, which gives

$$\begin{aligned}
&\int_{Y^*} A_0(x,y)(\Psi_0^{-\top}(x,y)\nabla_y \hat{w}_0(x,y))^\top : \nabla_y \hat{\varphi}(x,y) \,\mathrm{d}y\,\mathrm{d}x \\
&\quad = \int_{Y^*(t,x)} (\nabla_y w_0(x,y))^\top : \nabla_y \varphi(x,y) \,\mathrm{d}y\,\mathrm{d}x
\end{aligned} \tag{3.44}$$

for $w_0(x,y) = \hat{w}_0(x, \psi_0^{-1}(x,y))$ and $\varphi(x,y) = \hat{\varphi}(x, \psi_0^{-1}(x,y))$, where one has for a.e. $x \in \Omega$, $w_0(x,\cdot), \varphi(x,\cdot) \in H^1_{\Gamma(t,x)\#}(Y^*(t,x))^n$. Moreover, we test the microscopic incompressibility condition (3.35) with $\eta_1(x, \psi_0(x, y))$ for $\eta_1 \in L^2(\Omega; L^2(Y^*(t, x)))$, which gives

$$\begin{aligned} 0 &= \int\limits_{\Omega} \int\limits_{Y^*} \operatorname{div}_y(A_0(x, y)\hat{w}_0(x, y))\eta_1(x, \psi_0(x, y)) \,\mathrm{d}y \,\mathrm{d}x \\ &= \int\limits_{\Omega} \int\limits_{Y^*(t,x)} \operatorname{div}_y(w_0(x, y))\eta_1(x, y) \,\mathrm{d}y \,\mathrm{d}x \end{aligned} \tag{3.45}$$

and, thus, $\operatorname{div}_y(w_0) = 0$. Next, we approximate w_0 by smooth functions. For this, we note that the solenoidal smooth functions $\{u \in C^\infty_{\Gamma(t,x)\#}(Y^*(x))^n \mid \operatorname{div}_y(u) = 0\}$ are dense in the solenoidal H^1-functions $\{u \in H^1_{\Gamma(t,x)\#}(Y^*(x))^n \mid \operatorname{div}_y u = 0\}$ with respect to the H^1-norm (see [Gal11, Chapter III.4]). Thus, we can choose a sequence $(u_n(x, \cdot))_{n\in\mathbb{N}}$ in $C^\infty_{\Gamma(t,x)\#}(Y^*(x))^n$ with $\operatorname{div}_y(u_n(x, \cdot)) = 0$, which converges to w_0 with respect to the H^1-norm. Then, we obtain, after integration by parts,

$$\begin{aligned} \int\limits_{Y^*(t,x)} (\nabla_y w_0(x, y))^\top : \nabla_y \varphi(x, y) \,\mathrm{d}y &= \lim_{n\to\infty} \int\limits_{Y^*(t,x)} \nabla_y (u_n(x, y))^\top : \nabla_y \varphi(x, y) \,\mathrm{d}y \\ &= -\lim_{n\to\infty} \int\limits_{Y^*(t,x)} \operatorname{div}_y(\nabla_y(u_n(x, y))^\top) \cdot \varphi(x, y) \,\mathrm{d}y = 0, \end{aligned} \tag{3.46}$$

where the last equality of (3.46) follows from

$$\begin{aligned} (\operatorname{div}_y(\nabla_y u_n(x, y))^\top)_i &= \sum_{j=1}^n \partial_{y_j}((\nabla u_n(x, y))^\top)_{ji} = \sum_{j=1}^n \partial_{y_j}\partial_{y_i}(u_n)_j(x, y) \\ &= \partial_{y_i} \sum_{j=1}^n \partial_{y_j}(u_n)_j(x, y) = \partial_{y_i} \operatorname{div}_y(u_n(x, y)) = 0. \end{aligned}$$

Combining (3.44) with (3.46) shows (3.43) and, thus, (3.42) can be simplified to the first equation of (3.35).

Finally, from Lemma 3.19, we obtain the uniqueness of the solution $(\hat{w}_0, \hat{q}, \hat{q}_1)$ of (3.35) and, since the argumentation holds for every arbitrary subsequence, it holds for the whole sequence. □

We remember that we have subtracted the Dirichlet boundary values from $\hat{v}_\varepsilon$, i.e. $\hat{w}_\varepsilon = \hat{v}_\varepsilon - \hat{v}_{\Gamma_\varepsilon}$. Since $\hat{v}_{\Gamma_\varepsilon}$ is of order ε, the two-scale convergence of $\hat{v}_\varepsilon$ and $\hat{w}_\varepsilon$ are equivalent and their two-scale limits coincide.

Corollary 3.18. *Let $\hat{v}_\varepsilon = \hat{w}_\varepsilon + \hat{v}_{\Gamma_\varepsilon}$ for $\hat{w}_\varepsilon$ given as the solution of* (3.6)*. Then,*

$$\hat{v}_\varepsilon \xrightarrow{2} \hat{w}_0, \qquad \varepsilon\nabla\hat{v}_\varepsilon \xrightarrow{2} \nabla_y \hat{w}_0,$$

where $\hat{w}_0$ is given in Theorem 3.17.

Proof. The two-scale convergence of $\hat{v}_\varepsilon$ and $\varepsilon\nabla\hat{v}_\varepsilon$ follows directly from the uniform estimates $\|\hat{v}_{\Gamma_\varepsilon}\|_{L^2(\Omega_\varepsilon)} + \varepsilon\|\nabla\hat{v}_{\Gamma_\varepsilon}\|_{L^2(\Omega_\varepsilon)} \leqslant \varepsilon C$, which is given in Lemma 3.3, and the two-scale convergence of $\hat{w}_\varepsilon$ and $\varepsilon\nabla\hat{w}_\varepsilon$ from Theorem 3.17. □

Lemma 3.19. *The two-pressure Stokes problem* (3.35) *has a unique solution* $(\hat{w}_0, \hat{q}, \hat{q}_1) \in L^2(\Omega; H^1_{\Gamma\#}(Y^*)^n) \times H^1_0(\Omega) \times L^2(\Omega; L^2_0(Y^*))$.

Proof. The existence of a solution is already secured by the homogenisation process and it remains to show only the uniqueness. We rewrite (3.35) in the setting of the generic saddle-point formulation of Proposition 3.5. Therefore, we define the linear operators $a_0 \in \mathcal{L}(L^2(\Omega; H^1_{\Gamma\#}(\Omega)^n), L^2(\Omega; H^1_{\Gamma\#}(\Omega)^n)')$ and $b_0 \in \mathcal{L}\big(L^2(\Omega; H^1_{\Gamma\#}(\Omega)^n), \big(H^1_0(\Omega) \times L^2(\Omega; L^2_0(Y^*))\big)'\big)$ by

$$\begin{aligned} a_0(v, w) &= \int_\Omega \int_{Y^*} \mu A_0(x, y)\Psi_0^{-\top}(x, y)\nabla_y v(x, y) : \nabla_y w(x, y) \,\mathrm{d}y \,\mathrm{d}x, \\ b_0(v, (p_0, p_1)) &= \int_\Omega \int_{Y^*} A_0^\top(x, y)\nabla_x p_0(x) \cdot v(x, y) \,\mathrm{d}y \,\mathrm{d}x \\ &\quad - \int_\Omega \int_{Y^*} p_1(x, y) \operatorname{div}_y(A_0(x, y)v(x, y)) \,\mathrm{d}y \,\mathrm{d}x. \end{aligned} \tag{3.47}$$

Now, we verify the assumptions of Proposition 3.5.

- Coercivity of a_0: We use the boundedness of J_0 from below and the boundedness of $\Psi_0^\top$ from above, in order to estimate a_0 from below as we did in (2.38). Then, we apply the Poincaré inequality of $H^1_{\Gamma\#}(Y^*)$ and obtain the coercivity of a_0, i.e. we obtain $c > 0$ such that
$$a_0(v, v) \geqslant \mu c_J \|\Psi_0^\top\|^{-2}_{L^\infty(\Omega\times Y^*)} \|\nabla_y v\|^2_{L^2(\Omega\times Y^*)} \geqslant c\|v\|^2_{L^2(\Omega; H^1_{\Gamma\#}(Y^*))}, \tag{3.48}$$
for all $v \in L^2(\Omega; H^1_{\Gamma\#}(Y^*)^n)$.

- Continuity of a_0: With the Hölder inequality, we obtain the continuity of a_0, namely, we obtain a constant C such that
$$\begin{aligned} a_0(v, w) &\leqslant \|\sqrt{J_0}\Psi_0^{-\top}\|^2_{L^\infty(\Omega\times Y^*)} \|\nabla_y v\|_{L^2(\Omega\times Y^*)} \|\nabla_y w\|_{L^2(\Omega\times Y^*)} \\ &\leqslant C\|w\|_{L^2(\Omega; H^1_{\Gamma\#}(Y^*))} \|v\|_{L^2(\Omega; H^1_{\Gamma\#}(Y^*))} \end{aligned} \tag{3.49}$$
for any $v, w \in L^2(\Omega; H^1_{\Gamma\#}(Y^*)^n)$.

- Inf–sup estimate for b_0: From Lemma 3.20, we get a uniform positive inf–sup constant for b_0.

- Continuity of b_0: Let $(v, (p_0, p_1)) \in L^2(\Omega; H^1_{\Gamma\#}(Y^*)^n) \times (H^1_0(\Omega) \times L^2(\Omega; L^2_0(Y^*)))$. Using the Leibniz rule, the Piola identity (3.7) ($\operatorname{div}_y(A_0) = 0$) and the Poincaré inequalities for $H_0^1(\Omega)$ and $H_{\Gamma\#}^1(Y^*)$, we can infer

$$\begin{aligned}
b_0(v,(p_0,p_1)) &= (A_0\nabla_x p_0, v)_{L^2(\Omega\times Y^*)} - (p_1, \operatorname{div}_y(A_0 v))_{L^2(\Omega\times Y^*)} \\
&= (A_0\nabla_x p_0, v)_{L^2(\Omega\times Y^*)} - (p_1, \operatorname{div}_y(A_0)\cdot v + A_0 : \nabla_y v)_{L^2(\Omega\times Y^*)} \\
&\leqslant C\|\nabla p_0\|_{L^2(\Omega)}\|v\|_{L^2(\Omega\times Y^*)} + C\|p_1\|_{L^2(\Omega\times Y^*)}\|\nabla_y v\|_{L^2(\Omega\times Y^*)} \\
&\leqslant C\big(\|p_0\|_{H_0^1(\Omega)} + \|p_1\|_{L^2(\Omega\times Y^*)}\big)\|v\|_{L^2(\Omega;H_{\Gamma\#}^1(Y^*))}.
\end{aligned}$$

Having these estimates, and using the linearity and continuity of the right-hand sides of (3.35), we obtain a unique solution $(\hat{w},\hat{q},\hat{q}_1) \in L^2(\Omega; H_{\Gamma\#}^1(Y^*)^n) \times H_0^1(\Omega) \times L^2(\Omega; L_0^2(Y^*))$ of (3.35) from Proposition 3.5. ☐

Lemma 3.20. *Let b_0 be given by* (3.47)*. Then, there exists a constant $\beta \in \mathbb{R}$ such that*

$$\sup_{v\in L^2(\Omega;H_{\Gamma\#}^1(Y^*)^n)\setminus\{0\}} \frac{|b_0(v,(\phi_0,\phi_1))|}{\|v\|_{L^2(\Omega;H_{\Gamma\#}^1(Y^*))}\|(\phi_0,\phi_1)\|_{H_0^1(\Omega)\times L^2(\Omega;L_0^2(Y^*))}} \geqslant \beta \tag{3.50}$$

for any $(\phi_0,\phi_1) \in H_0^1(\Omega) \times L^2(\Omega; L_0^2(Y^*))$.

Proof. Let $(\phi_0,\phi_1) \in H_0^1(\Omega) \times L^2(\Omega; L_0^2(Y^*))$. First, we apply the Bogovskiĭ-operator from [Bog79] and [Bog80] for the domain Y^* on ϕ_1, which gives $u \in L^2(\Omega; H_{\Gamma\#}^1(Y^*)^n)$ such that

$$\operatorname{div}_y(u) = \phi_1, \qquad \|u\|_{L^2(\Omega;H_0^1(Y^*))} \leqslant C\|\phi_1\|_{L^2(\Omega;L_0^2(Y^*))} \tag{3.51}$$

for a constant C which depends only on Y^* and not on ϕ_1.

Now, we define the functions $v_1,\dots,v_n \in H_{\Gamma\#}^1(Y^*)^n$ as the solutions of the following Stokes problems:

Find $(v_i,p_i) \in H_{\Gamma\#}^1(Y^*)^n \times L_0^2(Y^*)$ such that

$$\begin{aligned}
(\nabla v_i, \nabla\varphi)_{L^2(Y^*)} - (p_i, \operatorname{div}(\varphi))_{L^2(Y^*)} &= (e_i,\varphi)_{L^2(Y^*)}, \\
(\operatorname{div}(v_i),\eta)_{L^2(Y^*)} &= 0
\end{aligned}$$

for any $(\varphi,\eta) \in H_{\Gamma\#}^1(Y^*)^n \times L_0^2(\Omega)$.

Choosing $\varphi = v_j$ shows

$$A := \begin{pmatrix} \vdots & & \vdots \\ \int_{Y^*} v_1(y)\,\mathrm{d}y & \cdots & \int_{Y^*} v_n(y)\,\mathrm{d}y \\ \vdots & & \vdots \end{pmatrix} = \begin{pmatrix} (\nabla v_1,\nabla v_1)_{L^2(Y^*)} & \cdots & (\nabla v_1,\nabla v_n)_{L^2(Y^*)} \\ \vdots & & \vdots \\ (\nabla v_n,\nabla v_1)_{L^2(Y^*)} & \cdots & (\nabla v_n,\nabla v_n)_{L^2(Y^*)} \end{pmatrix}.$$

Since A is the permeability tensor from the usual Darcy law, it is symmetric and positive definite (see for instance [SP80, Chapter 7, Proposition 2.2]). Therefore, the following boundary-value problem is well-posed:

Find a solution $w \in H_0^1(\Omega)$ such that

$$(A\nabla w, \nabla\varphi)_{L^2(\Omega)} = (\nabla\phi_0, \nabla\varphi)_{L^2(\Omega)} + \Big(\int_{Y*} u(\cdot, y)\,\mathrm{d}y, \nabla\varphi\Big)_{L^2(\Omega)}$$

for all $\varphi \in H_0^1(\Omega)$. The Theorem of Lax–Milgram provides a unique solutions $w \in H_0^1(\Omega)$, which can be estimated by

$$\|w\|_{H^1(\Omega)} \leqslant C\big(\|\phi_0\|_{H^1(\Omega)} + \|u\|_{L^2(\Omega;L^1(Y*))}\big) \leqslant C\big(\|\phi_0\|_{H^1(\Omega)} + \|u\|_{L^2(\Omega\times Y*)}\big).$$

Then, we define $v(x,y) := A_0^{-1}(x,y)\,(\sum_{i=1}^n v_i(y)\ \partial_{x_i} w(x) - u(x,y))$ and estimate with the essential boundedness of A_0^{-1} and (3.51)

$$\begin{aligned}\|v\|_{L^2(\Omega;H^1_{\Gamma\#}(Y*))} &\leqslant C\big(\|w\|_{H_0^1(\Omega)} + \|u\|_{L^2(\Omega;H^1_{\Gamma\#}(Y*))}\big)\\ &\leqslant C\big(\|\phi_0\|_{H_0^1(\Omega)} + \|\phi_1\|_{L^2(\Omega;L_0^2(Y*))}\big).\end{aligned}$$

From the construction of v, we obtain

$$(A_0 v, \nabla\phi_0)_{L^2(\Omega\times Y*)} = \Big(A\nabla w - \int_{Y*} u(\cdot,y)\,\mathrm{d}y, \nabla\phi_0\Big)_{L^2(\Omega)} = (\nabla\phi_0, \nabla\phi_0)_{L^2(\Omega)},$$

$$\operatorname{div}_y(A_0 v) = \sum_{i=1}^n \operatorname{div}_y(v_i(y))\partial_{x_i} w(x) - \operatorname{div}_y(u(x,y)) = -\phi_1(x).$$

Using this explicitly constructed v, we can deduce (3.50). □

3.1.5. Separation of the microscopic and macroscopic variables

Now, we separate the micro-and macroscopic variables in the two-pressure Stokes equations (3.35). The result is the following Darcy law for evolving microstructure, for the unknowns

$$\hat{w}(x) := \int_{Y*} J_0(x,y)\hat{w}_0(x,y)\,\mathrm{d}y, \qquad \hat{p} := \hat{q} + \hat{p}_{b,0}, \tag{3.52}$$

where $(\hat{w}_0, \hat{q})$ are the solution of (3.35).

Strong form of the quasi-stationary Darcy law for evolving microstructure in the reference coordinates

$$\begin{aligned} \hat{w}(x) &= \frac{1}{\mu}\hat{K}(x)(f(x) - \nabla_x \hat{p}(x)) && \text{in } \Omega, \\ \operatorname{div}_x(\hat{w}(x)) &= \int_\Gamma A_0(x,y)\hat{v}_\Gamma(x,y)\,\mathrm{d}\sigma_y && \text{in } \Omega, \\ \hat{p}(x) &= \hat{p}_{b,0}(x) && \text{on } \partial\Omega \end{aligned} \tag{3.53}$$

The permeability tensor $\hat{K} \in L^\infty(\Omega)^{n\times n}$ is given by

$$\hat{K}_{ij}(x) = \int\limits_{Y*} J_0(x,y)\hat{\zeta}_j(x,y)\cdot e_i\,\mathrm{d}y = \int\limits_{Y*} A_0(x,y)\Psi_0^{-\top}(x,y)\nabla_y\hat{\zeta}_j(x,y) : \nabla_y\hat{\zeta}_i(x,y)\,\mathrm{d}y, \tag{3.54}$$

where $(\hat{\zeta}_i, \hat{\pi}_i)$, for $i \in \{1,\ldots,n\}$, are the solutions of the cell problems

$$\begin{aligned} -J_0^{-1}\operatorname{div}_y(A_0\Psi_0^{-\top}\nabla\hat{\zeta}_i) + \Psi_0^{-\top}\nabla\hat{\pi}_i &= e_i && \text{in } Y^*, \\ J_0^{-1}\operatorname{div}(A_0^\top\hat{\zeta}_i) &= 0 && \text{in } Y^*, \\ \hat{\zeta}_i &= 0 && \text{on } Y^*, \\ y \mapsto \hat{\zeta}(y), \hat{\pi}_i(y) & && Y\text{-periodic.} \end{aligned} \tag{3.55}$$

By taking the divergence on both sides in the first equation of (3.53) and combining it with the second equation, we can eliminate $\hat{w}$ and obtain an elliptic Dirichlet boundary value problem for $\hat{p}$. Afterwards $\hat{w}$ can be computed explicitly.

In order to derive this Darcy law, we rewrite $\hat{w}_0$ by means of two cell problems. After identifying these cell problems, we obtain the first equation of the Darcy law. The reason why we have to deal with a second cell problem is the factor $\Psi_0^{-\top}$ in the coefficient $A_0 = J_0\Psi_0^{-\top}$, which appears in front of the gradient of the macroscopic pressure $\hat{q}$ in the two-pressure Stokes formulation (3.35). The same coefficient $A_0 = J_0\Psi_0^{-\top}$ appears also in the macroscopic divergence condition of the two-pressure Stokes equation, where we have to remove $\Psi_0^{-\top}$ in order to derive the macroscopic divergence condition of (3.53). These two tasks are closely related to the transformation rules for gradients, which we have shown in the previous chapter in Theorem 2.23. We shift the y-dependency of the coefficient $\Psi_0^{-\top}$ in front of the macroscopic pressure into the microscopic pressure. Then, we identify the two cell problems. In order to remove it from the macroscopic divergence condition, we employ additionally the microscopic incompressibility condition.

Identification of the two cell problems

The solution $(\hat{w}_0, \hat{q}, \hat{q}_1)$ of (3.35) can be expressed by means of the solution of the following two cell problems via

$$\begin{aligned}\hat{w}_0(x,y) &= \frac{1}{\mu}\sum_{i=1}^n \hat{\zeta}_i(x,y) f_i(x) - \frac{1}{\mu}\sum_{i=1}^n \hat{\zeta}'_i(x,y)\partial_{x_i}(\hat{q}+\hat{p}_{b,0})(x),\\ \hat{q}_1(x,y)+\hat{p}_{b,1}(x,y) &= \frac{1}{\mu}\sum_{i=1}^n \hat{\pi}_i(x,y) f_i(x) - \frac{1}{\mu}\sum_{i=1}^n \hat{\pi}'_i(x,y)\partial_{x_i}(\hat{q}+\hat{p}_{b,0})(x),\end{aligned} \tag{3.56}$$

where $(\hat{\zeta}, \hat{\pi}_i) \in L^\infty(\Omega; H^1_{\Gamma\#}(Y^*)^n) \times L^\infty(\Omega; L^2_0(Y^*))$ solves the weak form of (3.55), i.e.

$$\begin{aligned}\int_{Y^*} A_0(x,y)\Psi_0^{-\top}(x,y)\nabla\hat{\zeta}_i(x,y) : \nabla\varphi(y)\,\mathrm{d}y - \int_{Y^*} \hat{\pi}_i(x,y)\operatorname{div}(A_0(x,y)\varphi(y))\,\mathrm{d}y\\ = \int_{Y^*} J_0(x,y)e_i\cdot\varphi(y)\,\mathrm{d}y,\\ \operatorname{div}_y(A_0(x,y)\hat{\zeta}_i(x,y)) = 0\end{aligned} \tag{3.57}$$

for all $\varphi \in H^1_{\Gamma\#}(Y^*)^n$ and a.e. $x \in \Omega$.

The second cell problem is given by:

Find $(\hat{\zeta}', \hat{\pi}'_i) \in L^\infty(\Omega; H^1_{\Gamma\#}(Y^*)^n) \times L^\infty(\Omega; L^2_0(Y^*))$ such that

$$\begin{aligned}\int_{Y^*} A_0(x,y)\Psi_0^{-\top}(x,y)\nabla\hat{\zeta}'_i(x,y) : \nabla\varphi(y)\,\mathrm{d}y - \int_{Y^*} \hat{\pi}'_i(x,y)\operatorname{div}(A_0(x,y)\varphi(y))\,\mathrm{d}y\\ = \int_{Y^*} A_0(x,y)e_i\cdot\varphi(y)\,\mathrm{d}y,\\ \operatorname{div}_y(A_0^\top(x,y)\hat{\zeta}'_i(x,y)) = 0\end{aligned} \tag{3.58}$$

for all $\varphi \in H^1_{\Gamma\#}(Y^*)^n$ and a.e. $x \in \Omega$.

In order to identify these two cell problems, we note that

$$\begin{aligned}A_0^\top\xi &= J_0\Psi_0^{-\top}\xi = J_0\xi + J_0(\Psi_0^{-\top} - \mathbb{1})\xi\\ &= J_0\xi + J_0\Psi_0^{-\top}(\mathbb{1} - \Psi_0^\top)\xi = J_0\xi + A_0^\top\nabla_y((y-\psi_0)\cdot\xi)\end{aligned} \tag{3.59}$$

for all $\xi \in \mathbb{R}^n$. From integration by parts, we obtain

$$\begin{aligned}\int_{Y^*} A_0^\top(x,y)e_i\cdot\varphi(y)\,\mathrm{d}y &= \int_{Y^*} J_0(x,y)e_i\cdot\varphi(y)\,\mathrm{d}y + \int_{Y^*} A_0^\top(x,y)\nabla_y((y-\psi_0)\cdot e_i)\cdot\varphi(y)\,\mathrm{d}y\\ &= \int_{Y^*} J_0(x,y)e_i\cdot\varphi(y)\,\mathrm{d}y - \int_{Y^*} ((y-\psi_0)\cdot e_i)\operatorname{div}_y(A_0(x,y)\varphi(y))\,\mathrm{d}y\end{aligned} \tag{3.60}$$

for every $\varphi \in H^1_{\Gamma\#}(Y^*)^n$. The boundary integral over ∂Y^* that arises in the integration by parts in (3.60) vanishes on Γ since φ is zero on Γ and vanishes on $\partial Y^* \cap \partial Y$ since A_0, φ and $y - \psi_0$ are Y-periodic.

By inserting (3.60) into (3.58), we can identify the solution $(\hat{\zeta}_i', \hat{\pi}_i')$ of (3.58) with the solution $(\hat{\zeta}_i, \hat{\pi}_i)$ of (3.57) via

$$\begin{aligned} \hat{\zeta}_i(x,y) &= \hat{\zeta}_i(x,y), \\ \hat{\pi}_i(x,y) &= \hat{\pi}_i'(x,y) + (\psi_0(x,y) - y) \cdot e_i. \end{aligned} \tag{3.61}$$

Thus, we can simplify (3.56) to

$$\begin{aligned} \hat{w}_0(x,y) &= \frac{1}{\mu} \sum_{i=1}^n \hat{\zeta}_i(x,y)(f_i(x) - \partial_{x_i}(\hat{q} + \hat{p}_{b,0})(x)), \\ \hat{q}_1(x,y) + \hat{p}_{b,1}(x,y) &= \frac{1}{\mu} \sum_{i=1}^n \hat{\pi}_i(x,y)(f_i(x) - \partial_{x_i}(\hat{q} + \hat{p}_{b,0})(x)) \\ &\quad + \frac{1}{\mu}(\psi_0(x,y) - y) \cdot (\hat{q} + \hat{p}_{b,0})(x), \end{aligned} \tag{3.62}$$

which requires only the solution of the cell problem (3.57).

Lemma 3.21. *Let $\hat{w}$ be given by* (3.52)*, then*

$$\begin{aligned} \hat{w}(x) &= \frac{1}{\mu} \sum_{i=1}^n \int_{Y^*} J_0(x,y)\hat{\zeta}_i(x,y) \,\mathrm{d}y \; (f_i(x) - \partial_{x_i}(\hat{q} + \hat{p}_{b,0})(x)) \\ &= \frac{1}{\mu} \hat{K}(x)(f(x) - \nabla_x(\hat{q} + \hat{p}_{b,0})(x)), \end{aligned}$$

where $\hat{\zeta}_i$ is the first part of the solution of (3.56)*.*

Proof. Lemma 3.21 follows from inserting (3.62) into (3.52). □

Macroscopic divergence condition

Lemma 3.22. *Let $u \in L^2(\Omega; H^1_{\Gamma\#}(Y^*)^n)$ with*

$$\mathrm{div}_y(A_0(x,y)u(x,y)) \,\mathrm{d}y = 0 \tag{3.63}$$

for a.e. $x \in \Omega$. Then,

$$\int_{Y^*} A_0(x,y)u(x,y) \,\mathrm{d}y = \int_{Y^*} J_0(x,y)u(x,y) \,\mathrm{d}y \tag{3.64}$$

for a.e. $x \in \Omega$. In particular, for the solution $\hat{w}_0 \in L^2(\Omega; H^1_{\Gamma\#}(Y^*)^n)$ of (3.35) *and $\hat{w}$*

given by (3.52)*, it holds*

$$\begin{aligned}\operatorname{div}_x(\hat{w}(x)) &= \operatorname{div}_x\Big(\int_{Y^*} J_0(x,y)\hat{w}_0(x,y)\,\mathrm{d}y\Big) = \operatorname{div}_x\Big(\int_{Y^*} A_0(x,y)\hat{w}_0(x,y)\,\mathrm{d}y\Big)\\ &= \int_{Y^*} \operatorname{div}_y\big(A_0(x,y)\hat{v}_\Gamma(x,y)\big)\,\mathrm{d}y\end{aligned} \tag{3.65}$$

for a.e. $x \in \Omega$.

Proof. For $\xi \in \mathbb{R}^n$, we note that

$$\begin{aligned}A_0\xi &= J_0\Psi_0^{-1}\xi = J_0\xi + (\mathbb{1} - \Psi_0)J_0\Psi_0^{-1}\xi = J_0\xi + \partial_y(y-\psi_0)A_0\xi\\ &= J_0\xi + \begin{pmatrix}\nabla_y((y-\psi_0)_1)\cdot A_0\xi\\ \vdots\\ \nabla_y((y-\psi_0)_n)\cdot A_0\xi\end{pmatrix}.\end{aligned} \tag{3.66}$$

We set $\xi = u$ for $u \in L^2(\Omega; H^1_{\Gamma\#}(Y^*)^n)$ with $\operatorname{div}_y(A_0(x,y)u(x,y)) = 0$. Then, we integrate the second summand on the right-hand side of (3.66) over Y^*, subsequently, integrate by parts and use the microscopic incompressibility condition (3.63). This shows

$$\begin{aligned}&\int_{Y^*} \nabla_y(y_i - \psi_0(x,y)_i)\cdot A_0(x,y)u(x,y)\,\mathrm{d}y\\ &\qquad = -\int_{Y^*} (y_i - \psi_0(x,y)_i)\cdot \operatorname{div}_y(A_0(x,y)u(x,y))\,\mathrm{d}y = 0\end{aligned} \tag{3.67}$$

for every $i \in \{1,\dots,n\}$, where the boundary integral of the integration by parts vanishes on Γ since $\hat{w}_0$ is zero and vanishes on $\partial Y \cap \partial Y^*$ since $y - \psi_0$, A_0 and u are Y-periodic. Therefore, the second summand on the right hand side of (3.66) has mean value zero and vanishes after integrating over Y^*, which yields (3.64).

Since the solution $\hat{w}_0$ of (3.35) satisfies the microscopic incompressibility condition, we can rewrite the macroscopic incompressibility condition of (3.35) into (3.65). □

The weak form of the Darcy law for evolving microstructure

By combining Lemma 3.21 and Lemma 3.22, we obtain the following weak form of the Darcy law (3.53):

Weak form of the quasi-stationary Darcy law

Find $\hat{q} \in H_0^1(\Omega)$ such that

$$\begin{aligned} \int_\Omega \hat{w}(x)\nabla\varphi(x)\,\mathrm{d}x &= \int_\Omega\int_\Gamma A_0(x,y)\hat{v}_\Gamma(x,y)\,\mathrm{d}\sigma_y\,\mathrm{d}x, \\ \hat{w}(x) &= \frac{1}{\mu}\hat{K}(x)(f(x) - \nabla_x(\hat{q} + \hat{p}_{b,0})(x)) \end{aligned} \tag{3.68}$$

for every $\varphi \in H_0^1(\Omega)$.

Corollary 3.23. *Let $\hat{q} \in H_0^1(\Omega)$ be given by the solution of* (3.35)*. Then, $\hat{q}$ solves* (3.68)*.*

Proof. Corollary 3.23 follows directly from Lemma 3.21 and Lemma 3.22. □

3.1.6. Back-transformation – a Darcy law for evolving microstructure

Having derived the convergence for the solution $(\hat{w}_0, \hat{q})$ of the transformed Stokes equation, we can use the results of Chapter 2 in order to transfer the convergence to the solution $(w_\varepsilon, q_\varepsilon)$ of the Stokes equation (3.3) in the non-periodic (time-dependent) domain. The resulting two-pressure Stokes equation is defined on the non-cylindrical two-scale limit set

$$\mathcal{Q}(t) := \{(x,y) \in \Omega \times Y \mid y \in Y^*(t,x)\}$$

with interfaces $G(t) := \{(x,y) \in \Omega \times Y \mid y \in \Gamma(t,x)\}$

Two-pressure Stokes equation

$$\begin{aligned} \operatorname{div}_y(\mu\nabla_y v_0) + \nabla_x p + \nabla_y p_1 &= f && \text{in } Q(t), \\ \operatorname{div}_y(v_0) &= 0 && \text{in } Q(t), \\ v_0 &= 0 && \text{on } G(t), \\ y &\mapsto v_0, q_1 && Y\text{-periodic}, \\ \operatorname{div}_x\Big(\int_{Y^*(t,x)} v_0\,\mathrm{d}y\Big) &= -\int_\Omega\int_{Y^*(t,x)} \operatorname{div}_y(v_\Gamma)\,\mathrm{d}y && \text{in } \Omega, \\ p &= p_{b,0} && \text{on } \partial\Omega. \end{aligned} \tag{3.69}$$

The corresponding weak formulation is given by:

Weak form of the two-pressure Stokes equation

Find $(u_0, q, q_1) \in L^2(\Omega; H^1_{\Gamma\#}(Y^*)^n) \times H^1_0(\Omega) \times L^2(\Omega; L^2_0(Y^*))$

$$\int_\Omega \int_{Y^*(t,x)} \mu \nabla_y w_0(x,y) : \nabla_y \varphi(x,y) \,\mathrm{d}y\,\mathrm{d}x$$

$$+ \int_\Omega \int_{Y^*(t,x)} \nabla_x q(x) \cdot \varphi(x,y) - q_1(x,y) \operatorname{div}_y(\varphi(x,y)) \,\mathrm{d}y\,\mathrm{d}x$$

$$= \int_\Omega \int_{Y^*(t,x)} (f(x) - (\nabla_x p_{b,0}(x) + \nabla_x p_{b,1}(x,y))) \cdot \varphi(x,y) \,\mathrm{d}y\,\mathrm{d}x,$$

$$\int_\Omega \operatorname{div}_x \Big(\int_{Y^*(x)} w_0(x,y) \,\mathrm{d}y \Big) \eta_0(x) \,\mathrm{d}x = - \int_\Omega \int_{Y^*(t,x)} \operatorname{div}_y(v_\Gamma(x,y)) \,\mathrm{d}y\ \eta_0(x) \,\mathrm{d}x,$$

$$\int_\Omega \int_{Y^*(t,x)} \operatorname{div}_y(w_0(x,y)) \eta_1(x,y) \,\mathrm{d}y\,\mathrm{d}x = 0 \tag{3.70}$$

for all $(\varphi, \eta_0, \eta_1) \in L^2(\Omega; H^1_{\Gamma\#}(Y^*(t,x))^n) \times H^1_0(\Omega) \times L^2(\Omega; L^2(Y^*(t,x)))$.

In order to derive the strong convergence for the pressure q_ε to q, we have to extend q_ε on Ω. If we extended q_ε by means of $\hat{Q}_\varepsilon$, i.e. by transforming $\hat{Q}_\varepsilon$ back, the extension would be transformation-dependent due to the average in every cell. Instead, we define the extension directly on $\Omega_\varepsilon(t)$ by

$$Q_\varepsilon(x) := \begin{cases} q_\varepsilon(x) & \text{if } x \in \Omega_\varepsilon(t), \\ \displaystyle\fint_{(\varepsilon k + \varepsilon Y) \cap \Omega_\varepsilon(t)} q_\varepsilon(x) & \text{if } x \in (\varepsilon k + \varepsilon Y) \backslash \Omega_\varepsilon(t) \text{ for } k \in I_\varepsilon, \end{cases} \tag{3.71}$$

which is transformation-independent. Then, we obtain the following limit result.

Theorem 3.24. *Let $(w_\varepsilon, q_\varepsilon) \in H^1_{\Gamma_\varepsilon(t)}(\Omega_\varepsilon(t))^n \times L^2(\Omega_\varepsilon(t))$ be the solution of (3.3) and Q_ε be defined via (3.71). Then,*

$$w_\varepsilon \xrightarrow{2} w_0, \tag{3.72}$$

$$\varepsilon \nabla w_\varepsilon \xrightarrow{2} \nabla_y w_0, \tag{3.73}$$

$$Q_\varepsilon \to q \qquad \text{in } L^2(\Omega), \tag{3.74}$$

where $(w_0, q) \in L^2(\Omega; H^1_{\Gamma(t,x)\#}(Y^*(t,x))^n) \times H^1_0(\Omega)$ are the first two components of the solution of (3.70).

Proof. With Theorem 2.20 and Theorem 2.24, we can translate the two-scale convergence of $\hat{w}_0$ and $\varepsilon\nabla\hat{w}_0$, which is given in Theorem 3.17, into

$$w_\varepsilon(x) \xrightarrow{2} \hat{w}_0(x, \psi_0^{-1}(x,y)), \qquad \varepsilon\nabla w_\varepsilon(x) \xrightarrow{2} \nabla_y \hat{w}_0(x, \psi_0^{-1}(x,y)). \tag{3.75}$$

In order to show the strong convergence of Q_ε, we decompose Q_ε additively

$$Q_\varepsilon = \widetilde{q}_\varepsilon + \chi_{\Omega\setminus\Omega_\varepsilon(t)} \frac{1}{m_\varepsilon} q'_\varepsilon, \tag{3.76}$$

where $\widetilde{q}_\varepsilon$ is the extension of q_ε by zero on Ω and

$$q'_\varepsilon(x) := \varepsilon^{-n} \int_{(\varepsilon k + \varepsilon Y)\cap\Omega_\varepsilon(t)} q_\varepsilon(z)\,\mathrm{d}z \qquad \text{for } x \in \varepsilon k + \varepsilon Y \text{ with } k \in I_\varepsilon,$$

$$m_\varepsilon(x) := \varepsilon^{-n} |(\varepsilon k + \varepsilon Y) \cap \Omega_\varepsilon(t)| \qquad \text{for } x \in \varepsilon k + \varepsilon Y \text{ with } k \in I_\varepsilon.$$

The strong convergence of $\hat{Q}_\varepsilon$ to $\hat{q}$ implies the strong two-scale convergence of $\hat{Q}_\varepsilon$ to $\hat{q}$ and, with Lemma 1.16, we can infer

$$\widetilde{\hat{q}}_\varepsilon = \chi_{\Omega_\varepsilon} \hat{Q}_\varepsilon \xrightarrow{2} \chi_{Y^*} \hat{q}$$

and, afterwards, we transfer this convergence with Lemma 2.21 into

$$\widetilde{q}_\varepsilon \xrightarrow{2} \chi_{Y^*(t,x)} q. \tag{3.77}$$

for $q = \hat{q}$. Then, we translate (3.77) into the strong convergence of $\mathcal{T}_\varepsilon(\widetilde{q}_\varepsilon)$ to $\chi_{Y^*(t,x)} q$ in $L^2(\Omega \times Y)$. By applying the Hölder inequality on the Y-integral, we can deduce the strong convergence for the average over the cells, i.e.

$$\mathcal{T}_\varepsilon(q'_\varepsilon) = \int_Y \mathcal{T}_\varepsilon(\widetilde{q}_\varepsilon)(x,y)\,\mathrm{d}y \to \int_Y \chi_{Y^*(t,x)}(y)\,\mathrm{d}y \; q = |Y^*(t,x)| q$$

in $L^2(\Omega)$, which can be translated back into the two-scale convergence

$$q'_\varepsilon \xrightarrow{2} |Y^*(t,x)| q.$$

Moreover, we transfer the strong two-scale convergence of $\chi_{\Omega_\varepsilon(t)}$ similarly via the unfolding operator and the Hölder inequality into

$$m_\varepsilon = \int_Y \mathcal{T}_\varepsilon\big(\chi_{\Omega_\varepsilon(t)}\big)(x,y)\,\mathrm{d}y \to \int_Y \chi_{Y^*(t,x)}(y)\,\mathrm{d}y = |Y^*(t,x)| \qquad \text{in } L^p(\Omega)$$

for every $p \in [1, \infty)$. Since $m_\varepsilon(x)$ is uniformly bounded from below, i.e. $m_\varepsilon(x) \geqslant c > 0$, it holds also $m_\varepsilon^{-1} \to |Y^*(t,x)|^{-1}$ in $L^p(\Omega)$ for every $p \in [1,\infty)$, which implies

$$m_\varepsilon^{-1} \xrightarrow{<\infty}\!\!\!\!\!\rightarrow |Y^*(t,x)|^{-1} \tag{3.78}$$

Finally, we insert the strong two-scale convergence (3.77), (3.78) as well as the strong two-scale convergence of $\chi_{\Omega\setminus\Omega_\varepsilon(t)}$ into (3.76) and obtain

$$Q_\varepsilon = \widetilde{q}_\varepsilon + \chi_{\Omega\setminus\Omega_\varepsilon(t)} \frac{1}{m_\varepsilon} q'_\varepsilon \xrightarrow{2}\!\!\!\!\!\rightarrow \chi_{Y^*(t,x)} q + \chi_{Y\setminus Y^*(t,x)} |Y^*(t,x)|^{-1} |Y^*(t,x)| q = q.$$

Since the two-scale limit function q is independent of y, this implies the strong convergence in $L^2(\Omega)$.

Now, it remains to identify $\hat{w}_0(x, \psi_0^{-1}(x,y))$ and $q = \hat{q}$ with the first two arguments of the solution of (3.70). By arguing as in (3.60), we can rewrite $A_0^\top \nabla_x \hat{q}$ into $J_0 \nabla_x \hat{q}$ plus an additional term which can be included in the microscopic pressure. Then, one can easily transform the first equation of (3.35) into the first equation of (3.70).

The microscopic incompressibility condition (3.35) was already derived in (3.45).

In order to transform the macroscopic divergence condition of (3.35), we rewrite its left-hand side using (3.64) and obtain

$$\operatorname{div}_x \Big(\int_{Y^*} J_0(x,y) \hat{w}_0(x,y) \,\mathrm{d}y \Big) = \int_{Y^*} \operatorname{div}_y \big(A_0(x,y) \hat{v}_{\Gamma_\varepsilon}(x,y) \big) \,\mathrm{d}y.$$

Subsequently, we transform the Y^* integrals which gives the macroscopic divergence condition of (3.70). □

A Darcy law for evolving microstructure

Now, we separate the micro- and macroscopic variable in the two-pressure Stokes equations, which yields the following Darcy law for the unknowns

$$w(x) := \int_{Y^*} w_0(x,y) \,\mathrm{d}y, \qquad p := q + p_{b,0}, \tag{3.79}$$

where (w_0, q) are the first two components of the solution of (3.70).

Quasi-stationary Darcy law for evolving microstructure

$$
\begin{aligned}
w(x) &= \frac{1}{\mu} K(x)(f(x) - \nabla_x p(x)) && \text{in } \Omega, \\
\operatorname{div}_x(w(x)) &= \int_{\Gamma(t,x)} v_\Gamma(x,y)\, \mathrm{d}\sigma_y && \text{in } \Omega, \\
p(x) &= p_{b,0}(x) && \text{on } \partial\Omega.
\end{aligned}
\tag{3.80}
$$

The permeability tensor $K \in L^\infty(\Omega)^{n \times n}$ is given by

$$
K_{ij}(x) = \int_{Y^*(t,x)} \zeta_j(x,y) \cdot e_i \,\mathrm{d}y = \int_{Y^*(t,x)} \nabla \zeta_j(x,y) : \nabla \zeta_i(x,y)\, \mathrm{d}y, \tag{3.81}
$$

where (ζ_i, π_i), for $i \in \{1, \ldots, n\}$, are the solution of the cell problems

$$
\begin{aligned}
-\operatorname{div}_y(\nabla_y \zeta_i) + \nabla_y \pi_i &= e_i && \text{in } Y^*(t,x), \\
\operatorname{div}(\zeta_i) &= 0 && \text{in } Y^*(t,x), \\
\zeta_i &= 0 && \text{on } \Gamma(t,x), \\
y \mapsto \zeta(y), \pi_i(y) & && Y\text{-periodic.}
\end{aligned}
\tag{3.82}
$$

The permeability, the effective fluid velocity and the pressure in (3.80) coincide with the one in the transformed Darcy equation (3.53), i.e.

$$
K = \hat{K}, \qquad w = \hat{w}, \qquad q = \hat{q}
$$

and the solutions of the cell problems (3.82) can be identified with the solution of (3.55) by

$$
\zeta_i(x,y) = \hat{\zeta}_i(x, \psi_0^{-1}(x,y)), \qquad \pi_i(x,y) = \hat{\pi}_i(x, \psi_0^{-1}(x,y)).
$$

The weak form of the cell problems is given by:

Find $(\zeta_i, \pi_i) \in L^\infty(\Omega; H^1_{\Gamma\#}(Y^*)^n) \times L^\infty(\Omega; L^2_0(Y^*(t,x)))$ such that, for all $i \in \{1, \ldots, n\}$ and a.e. $x \in \Omega$,

$$
\begin{aligned}
\int_{Y^*(t,x)} \nabla \zeta_i(x,y) : \nabla \varphi(y)\, \mathrm{d}y - \int_{Y^*} \pi_i(x,y) \operatorname{div}(\varphi(y))\, \mathrm{d}y &= \int_{Y^*(t,x)} e_i \cdot \varphi(y)\, \mathrm{d}y, \\
\operatorname{div}_y(\zeta_i(x,y)) &= 0
\end{aligned}
\tag{3.83}
$$

for all $\varphi \in H^1_{\Gamma(t,x)\#}(Y^*(t,x))^n$.

The weak form of this Darcy law is given by:

Weak form for the quasi-stationary Darcy law for evolving microstructure

Find $q \in H_0^1(\Omega)$ such that

$$\int_\Omega w(x)\nabla\varphi(x)\,\mathrm{d}x = \int_\Omega\int_\Gamma A_0(x,y)v_\Gamma(x,y)\,\mathrm{d}\sigma_y\,\mathrm{d}x, \qquad w(x) = \frac{1}{\mu}K(x)(f(x) - \nabla_x(q + p_{b,0})(x)) \tag{3.84}$$

for every $\varphi \in H_0^1(\Omega)$.

Theorem 3.25. *Let (w_0, q_ε) be the solution of (3.3) and Q_ε be given (3.71). Then*

$$\widetilde{w_\varepsilon} \rightharpoonup w \qquad \text{in } L^2(\Omega),$$
$$Q_\varepsilon \to q \qquad \text{in } L^2(\Omega),$$

where $(w,q) \in L^2(\Omega)^n \times H_0^1(\Omega)$ are given as the solution of (3.84).

Proof. After separating the micro- and macroscopic variable in (3.70), we obtain (3.84) for $w = \int_{Y^*(t,x)} w_0\,\mathrm{d}y$. From Theorem 3.24, we obtain the strong convergence of $Q_\varepsilon \to q$ and the weak two-scale convergence of w_ε to w_0, which yields the weak convergence $\widetilde{w_\varepsilon} \rightharpoonup w$. □

The case of no-slip boundary conditions

In the case of no-slip boundary conditions, i.e.

$$v_{\Gamma_\varepsilon}(t,x) = v_{\Gamma_\varepsilon}(t,x,y) = (\partial_t\psi_\varepsilon)(t,\psi_\varepsilon^{-1}(t,x)),$$

the right-hand side of the macroscopic divergence condition can be expressed by means of the time-derivative of the porosity. In order to derive this result, we restore the time dependency in the transformations, i.e. $\psi_\varepsilon(x) = \psi_\varepsilon(t,x)$ and $\psi_0(x,y) = \psi_0(t,x,y)$, $J_0(t,x,y) = J_0(t,x,y)$ and $A_0(x,y) = A_0(t,x,y)$.

Lemma 3.26. *Let $v_{\Gamma_\varepsilon}(x) = v_{\Gamma_\varepsilon}(t,x) = \partial_t\psi_\varepsilon(t,\psi_\varepsilon(t,x))$. Then,*

$$\int_{Y^*(t,x)} \mathrm{div}_y(v_{\Gamma_\varepsilon}(t,x))\,\mathrm{d}y = \int_{Y^*} \mathrm{div}_y(A_0(t,x,y)\hat{v}_{\Gamma_\varepsilon}(t,x,y))\,\mathrm{d}y = \partial_t\Theta(t,x) \tag{3.85}$$

for $\Theta(t,x) := |Y^*(t,x)|$.

Proof. The Jacobi formula says that almost everywhere

$$\partial_t \det(A(t)) = \mathrm{tr}(\mathrm{adj}(A(t))\partial_t A(t)) = \det(A(t))A^{-1}(t) : \partial_t A^\top(t)$$

for every $A \in W^{1,\infty}(0,T)^{n\times n}$

With the Leibniz rule, the Jacobi formula applied to $\partial_y \psi_0$ and the Piola identity (3.7), we infer

$$\begin{aligned}\operatorname{div}_y(A_0(t,x,y)\partial_t\psi_0(t,x,y)) &= A_0(t,x,y) : \nabla\partial_t\psi_0(t,x,y) + \operatorname{div}_y(A_0(t,x,y))\partial_t\psi_0(t,x,y)\\ &= \partial_t J_0(t,x,y) + 0\partial_t\psi_0(t,x,y) = \partial_t J_0(t,x,y).\end{aligned}$$

Hence, we obtain

$$\int_{Y^*} \operatorname{div}_y(A_0(t,x,y)\hat{v}_{\Gamma_\varepsilon}(t,x,y))\,\mathrm{d}y = \int_{Y^*} \partial_t J_0(t,x,y)\,\mathrm{d}y = \partial_t \int_{Y^*} J_0(t,x,y)\,\mathrm{d}y = \partial_t\Theta(t,x).$$

□

Consequently, we can simplify the Darcy equation:

Quasi-stationary Darcy law for no-slip boundary condition

$$\begin{aligned} w(t,x) &= K(t,x)(f(t,x) - \nabla p(t,x)) && \text{in } (0,T)\times\Omega,\\ \operatorname{div}(w(t,x)) &= -\partial_t\Theta(t,x) && \text{in } (0,T)\times\Omega, \\ p(t,x) &= p_{b,0}(t,x) && \text{on } (0,T)\times\partial\Omega.\end{aligned} \tag{3.86}$$

In (3.86), we can observe that the change of the local porosity yields some inhomogeneous divergence condition. Together with the first equation of (3.86), the local change of the porosity becomes a source and sink term for the pressure.

3.2. Homogenisation of instationary Stokes flow

3.2.1. The microscopic equations

Now, we consider the homogenisation for the instationary Stokes flow. For sake of completeness, we recap the geometric setting, which we have also used for the quasi-stationary Stokes flow. Let $\Omega \subset \mathbb{R}^n$ be an an open set, representing the domain of the porous medium, and let $(0,T)$ for $T > 0$ be the time interval. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a decreasing positive sequence which converges to 0, and scales the microstructure. We write $\varepsilon = \varepsilon_n$ in the following. We assume that Ω is such that it consists of entire ε-scaled copies of the unit cell $Y = (0,1)^n$, i.e. $\Omega = \operatorname{int}\big(\bigcup_{\varepsilon k\in I_\varepsilon} k + \varepsilon\overline{Y}\big)$ for some $I_\varepsilon \subset \mathbb{Z}^n$. Now, we assume that for every ε and every $t \in [0,T]$, there exists an open set $\Omega_\varepsilon(t) \subset \Omega$, which represents the pore space, and complementary solid space $\Omega^{\mathrm{s}}_\varepsilon(t) = \operatorname{int}(\Omega\backslash\Omega_\varepsilon(t))$. We denote the interface of the pore and the solid phase at time t by $\Gamma_\varepsilon(t) := \partial\Omega_\varepsilon(t) \cap \partial\Omega^{\mathrm{s}}_\varepsilon(t)$ and the remaining boundary of the pore space by $\Xi_\varepsilon(t) := \partial\Omega_\varepsilon(t)\backslash\Gamma_\varepsilon(t)$. Then, we define the evolving domain with its boundary by

$$\mathcal{Q}_\varepsilon^T := \bigcup_{t\in[0,T]} \{t\} \times \Omega_\varepsilon(t), \quad G_\varepsilon^T := \bigcup_{t\in[0,T]} \{t\} \times \Gamma_\varepsilon(t), \quad H_\varepsilon^T := \bigcup_{t\in[0,T]} \{t\} \times \Xi_\varepsilon(t)$$

The instationary Stokes equation for the unknown fluid velocity v_ε and pressure p_ε is given by:

Instationary Stokes equations in an evolving perforated domain

$$\begin{aligned}
\partial_t v_\varepsilon - \operatorname{div}\left(\varepsilon^2 \mu 2e(v_\varepsilon)\right) + \nabla p_\varepsilon &= f_\varepsilon && \text{in } \mathcal{Q}_\varepsilon^T,\\
\operatorname{div}(v_\varepsilon) &= 0 && \text{in } \mathcal{Q}_\varepsilon^T,\\
v_\varepsilon &= v_{\Gamma_\varepsilon} && \text{on } G_\varepsilon^T,\\
\left(-\varepsilon^2 \mu 2e(v_\varepsilon) + p_\varepsilon \mathbb{1}\right) n &= p_{b,\varepsilon} n && \text{on } H_\varepsilon^T,\\
v_\varepsilon(0) &= v_\varepsilon^{\text{in}} && \text{in } \Omega_\varepsilon(0),
\end{aligned} \tag{3.87}$$

where $e(\varphi)$ denotes the symmetric gradient

$$e(\varphi) := \left(\nabla\varphi + (\nabla\varphi)^\top\right)/2.$$

$\mu > 0$ is the fluid's viscosity, f_ε the source term, $v_\varepsilon^{\text{in}}$ the fluid's initial velocity, v_{Γ_ε} the fluid velocity at the interface, $p_{b,\varepsilon}$ the normal stress and n the outer normal of $\Omega_\varepsilon(t)$.

In order to derive the weak formulation, we assume that the the Dirichlet boundary values v_{Γ_ε} and the normal stress $p_{b,\varepsilon}$ can be extended into $\Omega_\varepsilon(t)$. Then, we subtract these extensions from the fluid velocity v_ε and the pressure p_ε, i.e. we set

$$w_\varepsilon = v_\varepsilon - v_{\Gamma_\varepsilon}, \qquad w_\varepsilon^{\text{in}} = v_\varepsilon^{\text{in}} - v_{\Gamma_\varepsilon}(0), \qquad q_\varepsilon = p_\varepsilon - p_{b,\varepsilon},$$

which gives

$$\begin{aligned}
\partial_t w_\varepsilon - \operatorname{div}\left(\mu\varepsilon^2 2e(w_\varepsilon)\right) + \nabla q_\varepsilon &= f_\varepsilon - \nabla p_{b,\varepsilon} - \partial_t v_{\Gamma_\varepsilon} + \operatorname{div}\left(\varepsilon^2\mu 2e(v_{\Gamma_\varepsilon})\right) && \text{in } \mathcal{Q}_\varepsilon^T,\\
\operatorname{div}(w_\varepsilon) &= -\operatorname{div}(v_{\Gamma_\varepsilon}) && \text{in } \mathcal{Q}_\varepsilon^T,\\
w_\varepsilon &= 0 && \text{on } G_\varepsilon^T,\\
\left(-\varepsilon^2\mu 2e(w_\varepsilon) + q_\varepsilon I\right) n &= \varepsilon^2 \mu 2e(v_{\Gamma_\varepsilon}) n && \text{on } H_\varepsilon^T,\\
w_\varepsilon(0) &= w_\varepsilon^{\text{in}} && \text{in } \Omega_\varepsilon(0)
\end{aligned} \tag{3.88}$$

for $w_\varepsilon^{\text{in}} = v_\varepsilon^{\text{in}} - v_{\Gamma_\varepsilon}(0)$. We multiply the first equation of (3.88) by a test function $\varphi \in H^1_{\Gamma_\varepsilon(t)}(\Omega_\varepsilon(t))^n$, integrate over $\Omega_\varepsilon(t)$ and subsequently integrate the left-hand side by parts. By employing the two boundary conditions we obtain the first equation of (3.89). Moreover, we multiply the second equation by $\eta \in L^2(\Omega_\varepsilon(t))$ and integrate over

$\Omega_\varepsilon(t)$. Then, we obtain the weak form:

Weak form of the instationary Stokes equations in an evolving perforated domain

Find $(w_\varepsilon, q_\varepsilon) \in L^2(0,T; H^1_{\Gamma_\varepsilon(t)}(\Omega_\varepsilon(t))^n) \times L^2(\Omega_\varepsilon(t))$ with $\partial_t w_\varepsilon \in L^2(0,T; L^2(\Omega_\varepsilon(t))^n)$ such that, for a.e. $t \in (0,T)$,

$$\begin{aligned}
&\int\limits_{\Omega_\varepsilon(t)} \partial_t w_\varepsilon(t,x) \cdot \varphi(x)\,\mathrm{d}x + \int\limits_{\Omega_\varepsilon(t)} \varepsilon^2 \mu 2e(w_\varepsilon)(x) : \nabla\varphi(x)\,\mathrm{d}x - \int\limits_{\Omega_\varepsilon(t)} q_\varepsilon(x)\operatorname{div}(\varphi(x))\,\mathrm{d}x \\
&= \int\limits_{\Omega_\varepsilon(t)} (f_\varepsilon(x) - \nabla p_{b,\varepsilon}(x)) \cdot \varphi(x)\,\mathrm{d}x \\
&\quad - \int\limits_{\Omega_\varepsilon(t)} \partial_t v_{\Gamma_\varepsilon}(t,x) \cdot \varphi(x)\,\mathrm{d}x - \varepsilon^2 \mu 2e(v_{\Gamma_\varepsilon}(x)) : \nabla\varphi(x)\,\mathrm{d}x \\
&\int\limits_{\Omega_\varepsilon(t)} \operatorname{div}(w_\varepsilon(x))\eta(x)\,\mathrm{d}x = -\int\limits_{\Omega_\varepsilon(t)} \operatorname{div}(v_{\Gamma_\varepsilon}(x))\eta(x)\,\mathrm{d}x
\end{aligned} \tag{3.89}$$

for every $(\varphi, \eta) \in H^1_{\Gamma_\varepsilon(t)}(\Omega_\varepsilon(t))^n \times L^2(\Omega_\varepsilon(t))$ and $w_\varepsilon(0) = w_\varepsilon^{\mathrm{in}}$.

The weak differentiability of w_ε with respect to time has to be understood in the sense that the extension of w_ε by zero is in $H^1(0,T; L^2(\Omega)^n)$ and the time derivative is zero outside of $\Omega_\varepsilon(t)$ i.e. $\partial_t w_\varepsilon \in L^2(0,T; L^2(\Omega_\varepsilon(t))^n)$. Thus, the initial condition is also well-posed.

We make the following assumptions on the data and the domain.

Assumption 3.27. *We assume that:*

- $\Omega_\varepsilon(t)$ *is a sequence of locally evolving periodic domains over a time interval* $[0,T]$ *in the sense of Definition 2.36, with two-scale limit domains*

$$\begin{aligned}
\mathcal{Q}(t) &= \{(x,y) \in \Omega \times Y \mid y \in Y^*(t,x)\} \\
\mathcal{Q}^T &= \{(x,y,t) \in [0,T] \times \Omega \times Y \mid (x,y) \in \mathcal{Q}(t)\},
\end{aligned}$$

  for $t \in [0,T]$. *We denote the periodic substitute domain by* Ω_ε *and the reference cell by* Y^*.

  Moreover, we assume that the sequence of locally evolving periodic transformation has improved time regularity, namely, $\psi_\varepsilon \in C^{1,1}([0,T]; C^2(\Omega_\varepsilon)^n)$ *and there exists a constant* C *such that*

$$\varepsilon^{l-1} \|\partial_t \psi_\varepsilon(t_1) - \partial_t \psi_\varepsilon(t_2)\|_{C^l(\overline{\Omega_\varepsilon})} \leqslant C|t_1 - t_2|, \tag{3.90}$$

for $l \in \{0,1,2\}$. *Furthermore,* ψ_ε *and its derivatives satisfy Assumption 2.41.*

For the periodic substitute domain, we assume that

- $0 < |Y^*| < 1$,
- $Y^*_\# := \operatorname{int}\big(\bigcup_{k\in\mathbb{Z}^n} \varepsilon k + \varepsilon\overline{Y^*}\big)$ *and* $\operatorname{int}(\mathbb{R}^n\backslash Y^*_\#)$ *are open sets with* C^1*-boundary, which are locally located on one side of their boundary and* $Y^*_\#$ *is connected,*
- Y^* *is an open connected set with a locally Lipschitz boundary.*

For a detailed discussion of the assumptions on the periodic substitute domain see [All89].

- f_ε *is a sequence in* $L^2(Q^T_\varepsilon)^n$ *and* $f \in L^2((0,T)\times\Omega)^n$*, such that*

$$f_\varepsilon \xrightarrow{2,2} \chi_{Q^T} f.$$

- $v^{\text{in}}_\varepsilon$ *is a sequence in* $H^1(\Omega_\varepsilon(0))^n$ *with* $\operatorname{div}(v^{\text{in}}_\varepsilon) = 0$ *and there exists* $v^{\text{in}}_0 \in L^2(\Omega; H^1(Y^*(0,x))^n))$ *with* $\operatorname{div}(v^{\text{in}}_0) = 0$ *such that*

$$\|v^{\text{in}}_\varepsilon\|_{H^1(\Omega_\varepsilon(0))} \leqslant C,$$
$$v^{\text{in}}_\varepsilon \xrightarrow{2} v^{\text{in}}_0.$$

- v_{Γ_ε} *is a sequence in* $H^1(0,T;H^2(\Omega)^n)$ *and there exists* $v_\Gamma \in H^1(0,T;L^2(\Omega;H^1_\#(Y)^n))$ *such that*

$$\begin{aligned}
&v_{\Gamma_\varepsilon}(0) = v^{\text{in}}_\varepsilon \text{ on } \Gamma_\varepsilon(0), \qquad v_\Gamma(0) = v^{\text{in}}_0(0,x) \text{ on } \partial Y^*(t,x) \text{ for a.e. } x\in\Omega,\\
&\varepsilon^{-1} v_{\Gamma_\varepsilon} \xrightarrow{2,2} v_\Gamma, \qquad \nabla v_{\Gamma_\varepsilon} \xrightarrow{2} \nabla_y v_\Gamma, \qquad \partial_t v_{\Gamma_\varepsilon} \xrightarrow{2} 0,\\
&\varepsilon^{-1} v_{\Gamma_\varepsilon}(0) \xrightarrow{2,2} v_\Gamma(0), \qquad \nabla v_{\Gamma_\varepsilon}(0) \xrightarrow{2} \nabla_y v_\Gamma(0),\\
&\|\partial_t \nabla v_{\Gamma_\varepsilon}\|_{L^2((0,T)\times\Omega)} + \|\nabla\nabla v_{\Gamma_\varepsilon}\|_{L^2((0,T)\times\Omega)} \leqslant C.
\end{aligned}$$

- $p_{b,\varepsilon}$ *is a sequence in* $L^2(0,T;H^1(\Omega_\varepsilon(t)))$ *and* $(p_{b,0},p_{b,1}) \in L^2(0,T;H^1(\Omega))\times L^2((0,T)\times\Omega;H^1_\#(Y^*(t,x)))$*, such that*

$$\nabla p_{b,\varepsilon} \xrightarrow{2,2} \chi_{Q^T}\nabla_x p_{b,0} + \nabla_y p_{b,1}.$$

The Lipschitz regularity with respect to time of $\partial_t\psi_\varepsilon$ can be transferred to the Jacobians.

Lemma 3.28. *Let* ψ_ε *satisfy Assumption 3.27. Then, there exists a constant* C *such that*

$$\begin{aligned}
\|\Psi_\varepsilon(t_1) - \Psi_\varepsilon(t_2)\|_{L^\infty(\Omega_\varepsilon)} + \|\Psi_\varepsilon^{-1}(t_1) - \Psi_\varepsilon^{-1}(t_2)\|_{L^\infty(\Omega_\varepsilon)} &\leqslant C|t_1 - t_2|,\\
\|J_\varepsilon(t_1) - J_\varepsilon(t_2)\|_{L^\infty(\Omega_\varepsilon)} + \|J_\varepsilon^{-1}(t_1) - J_\varepsilon^{-1}(t_2)\|_{L^\infty(\Omega_\varepsilon)} &\leqslant C|t_1 - t_2|,\\
\|A_\varepsilon(t_1) - A_\varepsilon(t_2)\|_{L^\infty(\Omega_\varepsilon)} + \|A_\varepsilon^{-1}(t_1) - A_\varepsilon^{-1}(t_2)\|_{L^\infty(\Omega_\varepsilon)} &\leqslant C|t_1 - t_2|,
\end{aligned}$$

$$
\begin{aligned}
\varepsilon\|\partial_x A_\varepsilon^{-1}(t_1) - \partial_x A_\varepsilon^{-1}(t_2)\|_{L^\infty(\Omega_\varepsilon)} &\leqslant C|t_1 - t_2|, \\
\|\partial_t \Psi_\varepsilon(t_1) - \partial_t \Psi_\varepsilon(t_2)\|_{C(\overline{\Omega_\varepsilon})} + \|\partial_t \Psi_\varepsilon^{-1}(t_1) - \partial_t \Psi_\varepsilon^{-1}(t_2)\|_{C(\overline{\Omega_\varepsilon})} &\leqslant C|t_1 - t_2|, \\
\|\partial_t J_\varepsilon(t_1) - \partial_t J_\varepsilon(t_2)\|_{C(\overline{\Omega_\varepsilon})} + \|\partial_t J_\varepsilon^{-1}(t_1) - \partial_t J_\varepsilon^{-1}(t_2)\|_{C(\overline{\Omega_\varepsilon})} &\leqslant C|t_1 - t_2|, \\
\|\partial_t A_\varepsilon(t_1) - \partial_t A_\varepsilon(t_2)\|_{C(\overline{\Omega_\varepsilon})} + \|\partial_t A_\varepsilon^{-1}(t_1) - \partial_t A_\varepsilon^{-1}(t_2)\|_{C(\overline{\Omega_\varepsilon})} &\leqslant C|t_1 - t_2|
\end{aligned}
$$

for every $t_1, t_2 \in [0,T]$ and all $\varepsilon > 0$. Moreover, the estimate of Lemma 2.43 holds also pointwise in time for every $t \in [0,T]$.

Proof. The Lipschitz estimates for $\Psi_\varepsilon, \Psi_\varepsilon^{-1}, J_\varepsilon, J_\varepsilon^{-1}, A_\varepsilon, A_\varepsilon^{-1}$ follow from the uniform estimates of the time derivatives, which are provided by Lemma 2.43. The Lipschitz estimate for $\partial_t \Psi_\varepsilon = \partial_t \partial_x \psi_\varepsilon$ is given in Assumption 3.27. By means of Lemma 2.42, the entries of $\partial_t A_\varepsilon$ and $\partial_t J_\varepsilon$ are polynomials in the entries of Ψ_ε and $\partial_t \Psi_\varepsilon$. Then, we obtain the Lipschitz estimate for $\partial_t A_\varepsilon$ and $\partial_t J_\varepsilon$ from the uniform boundedness and Lipschitz regularity of Ψ_ε and $\partial_t \Psi_\varepsilon$.

The uniform Lipschitz estimate for J_ε can be transferred to J_ε^{-1} since $J_\varepsilon \geqslant c_J$. Then, Lemma 2.42 shows that $\partial_t J_\varepsilon^{-1}$ and the entries of $\partial_t \Psi_\varepsilon^{-1}$ and $\partial_t A_\varepsilon^{-1}$ are polynomials in the entries of Ψ_ε, $\partial_t \Psi_\varepsilon$ and J_ε^{-1}, for which we have already shown the uniform boundedness and Lipschitz estimate.

□

Remark 3.29. *While it is natural to state the assumptions on the right-hand sides h_ε and $p_{b,\varepsilon}$ in Eulerian coordinates, it can be natural, depending on the application, to state the assumptions on v_{Γ_ε} in some fixed reference coordinates, i.e. set the assumptions for $\hat{v}_{\Gamma_\varepsilon}$. From an analytical point of view, we will also work with the properties of $\hat{v}_{\Gamma_\varepsilon}$. Therefore, we note that the assumptions on v_{Γ_ε} in Assumption 3.27 can be replaced by the following assumptions on $\hat{v}_{\Gamma_\varepsilon}$, where $\hat{v}_{\Gamma_\varepsilon}(t,x) = v_{\Gamma_\varepsilon}(t, \psi_\varepsilon(t,x))$, $\hat{v}_\Gamma(t,x,y) = v_{\Gamma_\varepsilon}(t,x,\psi_\varepsilon(t,x,y))$. We assume that $\hat{v}_{\Gamma_\varepsilon} \in H^1(0,T;H^1(\Omega_\varepsilon)^n)$ and there exists $\hat{v}_\Gamma \in H^1(0,T;L^2(\Omega;H^1_\#(Y^*)^n))$ such that*

$$
\begin{aligned}
&\varepsilon^{-1}\hat{v}_{\Gamma_\varepsilon} \xrightarrow{2,2} \hat{v}_\Gamma, && \nabla \hat{v}_{\Gamma_\varepsilon} \xrightarrow{2} \nabla_y \hat{v}_\Gamma, \\
&\varepsilon^{-1}\hat{v}_{\Gamma_\varepsilon}(0) \xrightarrow{2,2} \hat{v}_\Gamma(0), && \nabla \hat{v}_{\Gamma_\varepsilon}(0) \xrightarrow{2} \nabla_y \hat{v}_\Gamma(0) \\
&\partial_t \hat{v}_{\Gamma_\varepsilon} \xrightarrow{2,2} 0.
\end{aligned}
$$

In particular, this implies that there exists a constant $C > 0$ such that

$$
\begin{aligned}
\varepsilon^{-1}\|\hat{v}_{\Gamma_\varepsilon}\|_{L^2((0,T)\times\Omega_\varepsilon)} + \|\nabla \hat{v}_{\Gamma_\varepsilon}\|_{L^2((0,T)\times\Omega_\varepsilon)} + \|\partial_t \hat{v}_{\Gamma_\varepsilon}\|_{L^2((0,T)\times\Omega_\varepsilon)} &\leqslant C, \\
\varepsilon^{-1}\|\hat{v}_{\Gamma_\varepsilon}(0)\|_{L^2(\Omega_\varepsilon)} + \|\nabla \hat{v}_{\Gamma_\varepsilon}(0)\|_{L^2(\Omega_\varepsilon)} &\leqslant C.
\end{aligned}
$$

Moreover, we assume that there exists a constant $C > 0$ such that

$$
\|\partial_t \nabla \hat{v}_{\Gamma_\varepsilon}\|_{L^2((0,T)\times\Omega_\varepsilon)} \leqslant C.
$$

Remark 3.30. *In the case of no-slip boundary conditions, i.e.* $\hat{v}_{\Gamma_\varepsilon} = \partial_t \psi_\varepsilon$*, the two-scale convergence of* $\varepsilon^{-1}\hat{v}_{\Gamma_\varepsilon}$ *and* $\nabla \hat{v}_{\Gamma_\varepsilon}$ *is given by Assumption 2.41. However, the two-scale convergence of* $\hat{v}_{\Gamma_\varepsilon}(0)$ *and the uniform estimate of* $\partial_t \nabla \hat{v}_{\Gamma_\varepsilon}$ *lead to a higher time regularity for* $\partial_t \psi_\varepsilon$ *and* $\partial_t \psi_0$*.*

3.2.2. Transformation to a periodic substitute domain

We transform the Stokes equations (3.87) as well as the weak formulation (3.89) onto the reference domain Ω_ε, where we denote the transformed data by

$$\begin{aligned} &\hat{f}_\varepsilon(t,x) := f_\varepsilon(t,\psi_\varepsilon(t,x)), \qquad \hat{v}_{\Gamma_\varepsilon}(t,x) := v_{\Gamma_\varepsilon}(t,\psi_\varepsilon(t,x)), \qquad \hat{p}_{b,\varepsilon}(t,x) := \hat{p}_\varepsilon(t,\psi_\varepsilon(t,x)) \\ &\hat{v}_\varepsilon^{\text{in}}(t,x) := v_\varepsilon^{\text{in}}(t,\psi_\varepsilon(t,x)) \qquad \hat{w}_\varepsilon^{\text{in}}(t,x) := w_\varepsilon^{\text{in}}(t,\psi_\varepsilon(t,x)), \end{aligned} \tag{3.91}$$

where ψ_ε are the locally periodic transformations in the sense of Definition 2.37. We define the boundaries Γ_ε and Ξ_ε by $\Gamma_\varepsilon = \psi_\varepsilon^{-1}(t,\Gamma_\varepsilon(t))$ and $\Xi_\varepsilon = \psi_\varepsilon^{-1}(t,\Xi_\varepsilon(t))$, respectively, and recap the notation $\Psi_\varepsilon := \partial_x \psi_\varepsilon$, $J_\varepsilon := \det(\Psi_\varepsilon)$ and $A_\varepsilon := \operatorname{Adj}(\Psi_\varepsilon)$. Then, we obtain for

$$\hat{v}_\varepsilon(t,x) = v_\varepsilon(t,\psi_\varepsilon(t,x)), \qquad \hat{p}_\varepsilon(t,x) = p_\varepsilon(t,\psi_\varepsilon(t,x))$$

the transformed strong formulation:

Instationary Stokes equations in an evolving perforated domain in the reference coordinates

$$\begin{aligned} \partial_t \hat{v}_\varepsilon - \nabla \hat{v}_\varepsilon^\top \Psi_\varepsilon^{-1} \partial_t \psi_\varepsilon - J_\varepsilon^{-1} \operatorname{div}\left(\varepsilon^2 \mu A_\varepsilon 2 e_\varepsilon(\hat{v}_\varepsilon)\right) + \Psi_\varepsilon^{-\top} \nabla \hat{p}_\varepsilon &= \hat{f}_\varepsilon && \text{in } (0,T) \times \Omega_\varepsilon, \\ J_\varepsilon^{-1} \operatorname{div}(A_\varepsilon \hat{v}_\varepsilon) &= 0 && \text{in } (0,T) \times \Omega_\varepsilon, \\ \hat{v}_\varepsilon &= \hat{v}_{\Gamma_\varepsilon} && \text{on } (0,T) \times \Gamma_\varepsilon, \\ \left(-\varepsilon^2 \mu 2 e_\varepsilon(\hat{v}_\varepsilon) + \hat{p}_\varepsilon \mathbb{1}\right) \|\Psi_\varepsilon^{-\top} \hat{n}\|^{-1} \Psi_\varepsilon^{-\top} \hat{n} &= \hat{p}_{b,\varepsilon} \|\Psi_\varepsilon^{-\top} \hat{n}\|^{-1} \Psi_\varepsilon^{-\top} \hat{n} && \text{on } (0,T) \times \Xi_\varepsilon, \\ \hat{v}_\varepsilon(0) &= \hat{v}_\varepsilon^{\text{in}} && \text{in } \Omega_\varepsilon, \end{aligned} \tag{3.92}$$

where $e_\varepsilon(v_\varepsilon) := \left(\Psi_\varepsilon^{-\top} \nabla \hat{v}_\varepsilon + (\Psi_\varepsilon^{-\top} \nabla \hat{v}_\varepsilon)^\top\right)/2$, denotes the transformed symmetric gradient and $\hat{n}$ the outer normal of Ω_ε.

For

$$\hat{w}_\varepsilon(t,x) = w_\varepsilon(t,\psi_\varepsilon(t,x)), \qquad \hat{q}_\varepsilon(t,x) = q_\varepsilon(t,\psi_\varepsilon(t,x)) \tag{3.93}$$

we obtain the transformed weak formulation, where we drop the t- and x-dependency of the functions for the sake of better readability, which we continue in the following when useful.

Weak form of the instationary Stokes equations in an evolving perforated domain in the reference coordinates

Find $(\hat{w}_\varepsilon, \hat{q}_\varepsilon) \in L^2(0,T;H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^n) \times L^2((0,T)\times\Omega_\varepsilon)$ with $\partial_t w_\varepsilon \in L^2(0,T;L^2(\Omega_\varepsilon)^n)$ such that, for a.e. $t \in (0,T)$,

$$\begin{aligned}
&\int\limits_{\Omega_\varepsilon} J_\varepsilon \partial_t \hat{w}_\varepsilon \cdot \varphi \,\mathrm{d}x - \int\limits_{\Omega_\varepsilon} (\nabla \hat{w}_\varepsilon)^\top A_\varepsilon \partial_t \psi_\varepsilon \cdot \varphi \,\mathrm{d}x + \int\limits_{\Omega_\varepsilon} \varepsilon^2 \mu A_\varepsilon 2 e_\varepsilon(\hat{w}_\varepsilon) : \nabla\varphi \,\mathrm{d}x \\
&\quad - \int\limits_{\Omega_\varepsilon} \hat{q}_\varepsilon \operatorname{div}(A_\varepsilon \varphi)\,\mathrm{d}x = \int\limits_{\Omega_\varepsilon} \big(J_\varepsilon \hat{f}_\varepsilon - A_\varepsilon^\top \hat{p}_{b,\varepsilon} - J_\varepsilon \partial_t \hat{v}_{\Gamma_\varepsilon} + (\nabla \hat{v}_{\Gamma_\varepsilon})^\top A_\varepsilon \partial_t \psi_\varepsilon\big) \cdot \varphi \,\mathrm{d}x \\
&\quad - \int\limits_{\Omega_\varepsilon} \varepsilon^2 \mu A_\varepsilon 2 e_\varepsilon(\hat{v}_{\Gamma_\varepsilon}) \cdot \nabla\varphi \,\mathrm{d}x \\
&\quad \int\limits_{\Omega_\varepsilon} \operatorname{div}(A_\varepsilon \hat{w}_\varepsilon)\eta \,\mathrm{d}x = - \int\limits_{\Omega_\varepsilon} \operatorname{div}(A_\varepsilon \hat{v}_{\Gamma_\varepsilon})\eta \,\mathrm{d}x,
\end{aligned} \tag{3.94}$$

for every $(\varphi, \eta) \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^n \times L^2(\Omega_\varepsilon)$ and $\hat{w}_\varepsilon(0) = \hat{w}_\varepsilon^{\mathrm{in}}$.

For the transformed data, we obtain the following estimates and convergence results.

Lemma 3.31. *Assume that $v_\varepsilon^{\mathrm{in}}$, f_ε, $p_{b,\varepsilon}$ and v_{Γ_ε} satisfy Assumption 3.27 and let $\hat{w}_\varepsilon^{\mathrm{in}}$, $\hat{f}_\varepsilon$, $\hat{p}_{b,\varepsilon}$, $\hat{v}_{\Gamma_\varepsilon}$ be given by* (3.91)*. Then,*

$$\begin{aligned}
&\hat{f}_\varepsilon \xrightarrow{2,2} \chi_{Y^*} \hat{f}, && \nabla \hat{p}_{b,\varepsilon} \xrightarrow{2,2} \chi_{Y^*} \nabla_x \hat{p}_{b,0} + \nabla_y \hat{p}_{b,1} && \hat{w}_\varepsilon^{\mathrm{in}} \xrightarrow{2} \hat{w}_0^{\mathrm{in}}, \\
&\chi_{\Omega_\varepsilon} \varepsilon^{-1} \hat{v}_{\Gamma_\varepsilon} \xrightarrow{2,2} \chi_{Y^*} v_\Gamma, && \chi_{\Omega_\varepsilon} \nabla \hat{v}_{\Gamma_\varepsilon} \xrightarrow{2,2} \chi_{Y^*} \nabla_y \hat{v}_\Gamma, && \chi_{\Omega_\varepsilon} \partial_t \hat{v}_{\Gamma_\varepsilon} \xrightarrow{2,2} 0
\end{aligned}$$

where

$$\begin{aligned}
&\hat{f}(t,x) = f(t,x), && \hat{v}_\Gamma(t,x,y) = v_\Gamma(t,x,\psi_0(t,x,y)), \quad \hat{w}_0^{\mathrm{in}}(x,y) = w_0^{\mathrm{in}}(x,\psi_0(0,t,x)) \\
&\hat{p}_{b,0}(t,x) = p_{b,0}(t,x), && \hat{p}_{b,1}(t,x,y) = p_{b,1}(t,x,\psi_0(t,x,y)) + \nabla_x \hat{p}_{b,0}(t,x) \cdot \breve{\psi}_0(t,x,y)
\end{aligned}$$

for a.e. $(t,x,y) \in (0,T)\times\Omega\times Y^*$. In particular, there exists a constant $C > 0$ such that

$$\begin{aligned}
\varepsilon^{-1}\|\hat{v}_{\Gamma_\varepsilon}\|_{L^2((0,T)\times\Omega_\varepsilon)} + \|\nabla \hat{v}_{\Gamma_\varepsilon}\|_{L^2((0,T)\times\Omega_\varepsilon)} + \|\partial_t \hat{v}_{\Gamma_\varepsilon}\|_{L^2((0,T)\times\Omega_\varepsilon)} &\leqslant C, \\
\|\hat{f}_\varepsilon\|_{L^2((0,T)\times\Omega_\varepsilon)} + \|\hat{w}_\varepsilon^{\mathrm{in}}\|_{L^2(\Omega_\varepsilon)} + \|\hat{p}_{b,\varepsilon}\|_{L^2((0,T)\times\Omega_\varepsilon)} + \|\nabla \hat{p}_{b,\varepsilon}\|_{L^2((0,T)\times\Omega_\varepsilon)} &\leqslant C.
\end{aligned}$$

Moreover, there exists a constant $C > 0$

$$\|\hat{w}_\varepsilon^{\mathrm{in}}\|_{H^1(\Omega_\varepsilon)} + \|\nabla \hat{v}_{\Gamma_\varepsilon}(0)\|_{L^2(\Omega_\varepsilon)} + \|\partial_t \nabla \hat{v}_{\Gamma_\varepsilon}\|_{L^2((0,T)\times\Omega_\varepsilon)} \leqslant C.$$

Proof. By means of the results of Chapter 2, we can transfer the two-scale convergences and uniform bounds from f_ε, $\nabla p_{b,\varepsilon}$, $\varepsilon^{-1} v_{\Gamma_\varepsilon}$, $\nabla v_{\Gamma_\varepsilon}$, which are given in Assumption 3.27, to $\hat{f}_\varepsilon$, $\nabla \hat{p}_{b,\varepsilon}$, $\varepsilon^{-1}\hat{v}_{\Gamma_\varepsilon}$, $\nabla \hat{v}_{\Gamma_\varepsilon}$,

The two-scale convergence of $\partial_t \hat{v}_{\Gamma_\varepsilon}$ can be deduced with the identity $\partial_t \hat{v}_{\Gamma_\varepsilon}(t,x) = \partial_t v_{\Gamma_\varepsilon}(t,\psi_\varepsilon(t,x)) + \nabla \hat{v}_{\Gamma_\varepsilon}^\top(t,x)\Psi_\varepsilon^{-1}(t,x)\partial_t\psi_\varepsilon(t,x)$ and the two-scale convergences of v_{Γ_ε} and $\partial_t \psi_\varepsilon \xrightarrow{2,2} 0$ as well as the boundedness of $\nabla \hat{v}_{\Gamma_\varepsilon}$ and $\Psi_\varepsilon^{-\top}$. Similarly, the uniform bounds of $\|\partial_t \nabla \hat{v}_{\Gamma_\varepsilon}\|_{L^2((0,T)\times\Omega_\varepsilon)}$ can be deduced by applying the chain rule and estimating the resulting summands.

In order to transform the initial values $w_\varepsilon^{\text{in}}$, we note that, for a sequence of locally evolving periodic transformations in the sense of Definition 2.37, Assumption 2.41 gives the continuity of the transformations with respect to time. Then, for every point in time the transformations are locally periodic transformations in the sense of Definition 2.2.

□

3.2.3. Existence, uniqueness and a-priori estimates

In this section, we show the following existence and uniqueness result for the solution of the Stokes equations (3.6). It provides also the a-priori estimates, which we will use later for the two-scale compactness arguments.

Theorem 3.32. *For every $\varepsilon > 0$, there exists a unique solution $(\hat{w}_\varepsilon, \hat{q}_\varepsilon) \in L^2(0,T;H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^n) \times L^2((0,T)\times\Omega_\varepsilon)$ with $\partial_t \hat{w}_\varepsilon \in L^2((0,T)\times\Omega_\varepsilon)$ of (3.94). Moreover, there exists a constant C such that*

$$\|\hat{w}_\varepsilon\|_{L^2((0,T)\times\Omega_\varepsilon)} + \varepsilon\|\nabla\hat{w}_\varepsilon\|_{L^2((0,T)\times\Omega_\varepsilon)} + \|\hat{q}_\varepsilon\|_{L^2((0,T)\times\Omega_\varepsilon)} \leqslant C \tag{3.95}$$

for every $\varepsilon > 0$.

We prove Theorem 3.32 by means of the following generic existence and uniqueness result for time-dependent differential equations with algebraic constraints. Employing a subtle scaling of the involved norms, it will provide directly the a-priori estimates (3.95).

For Banach spaces V, W and $a \in \mathcal{L}(V, W')$, we write $a(v,w) := a(v)(w)$ for $v \in V$ and $w \in W$.

Theorem 3.33. *Let V, H be separable Hilbert spaces such that V is densely embedded into H with continuity constant $C_{V\to H}$, i.e. $\|v\|_H \leqslant C_{V\to H}\|v\|_V$ for all $v \in V$, and let Q be a Banach space. Assume that $a \in C^1([0,T];\mathcal{L}(H,H'))$, $b \in C^{0,1}([0,T];\mathcal{L}(V,V'))$ and $c \in \mathcal{L}(V,Q')$ satisfy Assumption 3.34 (1.), Assumption 3.34 (2.) and Assumption 3.34 (3.), respectively. Then, for every $g \in H^1(0,T;Q')$, $f_1 \in L^2(0,T;H')$, $f_2 \in H^1(0,T;V')$ and $v^{\text{in}} \in V$ with $cv^{\text{in}} = g(0)$, there exists a unique solution $(v,q) \in (H^1(0,T;H)\cap L^2(0,T;V)) \times L^2(0,T;Q)$ such that, for a.e. $t \in (0,T)$,*

$$\begin{aligned} a(t)\partial_t v(t) + b(t)v(t) + c^* p(t) &= f_1(t) + f_2(t) && \text{in } V', \\ cv(t) &= g(t) && \text{in } Q', \\ v(0) &= v^{\text{in}}. \end{aligned} \tag{3.96}$$

Moreover, there exists a constant C such that

$$\|\partial_t v\|_{L^2(0,T;H)} + \|v\|_{L^\infty(0,T;H)} + \|v\|_{L^2(0,T;V)} + \|p\|_{L^2(0,T;Q)} \leqslant C,$$

where C depends only on $T, C_{V\to H}, C_a, C_b, C_{b^1}, C_{b^2}, C_{b^3}, C_{b^4}, \alpha, \beta, \gamma, L_a, L_{b^1}, L_{b^3}, \|v^{\mathrm{in}}\|_V$, $\|g\|_{H^1(0,T;Q')}$, $\|f_1\|_{L^2(0,T;H')}$, $\|f_2\|_{H^1(0,T;V')}$, which are given in Assumption 3.34, but does not depend on $\|c\|_{\mathcal{L}(V,Q')}$.

Proof. A proof of Theorem 3.33 is given in Appendix A. □

Assumption 3.34. *(1.) Let $a \in C^1([0,T];\mathcal{L}(H,H'))$ be Lipschitz continuous, uniformly coercive and symmetric, i.e. there exist constants L_a, α, C_a such that*

$$\|a(t_1) - a(t_2)\|_{\mathcal{L}(H,H')} \leqslant L_a|t_1 - t_2| \quad \text{for all } t_1, t_2 \in [0,T], \tag{3.97}$$

$$a(t)(v,v) \geqslant \alpha\|v\|_H^2 \quad \text{for all } t \in [0,T] \text{ and all } v \in H, \tag{3.98}$$

$$a(t)(v,w) = a(t)(w,v) \quad \text{for all } t \in [0,T] \text{ and all } v,w \in H, \tag{3.99}$$

We write $C_a := \|a\|_{C([0,T];\mathcal{L}(H,H'))}$.

(2.) Let $b \in C^{0,1}([0,T];\mathcal{L}(V,V'))$. Assume that b can be decomposed into

$$b = b^1 + b^2 + b^3 + b^4 \tag{3.100}$$

for

$$b^1 \in C^{0,1}([0,T];\mathcal{L}(V,V')),$$
$$b^2 \in C^{0,1}([0,T];\mathcal{L}(V,H')),$$
$$b^3 \in C^{0,1}([0,T];\mathcal{L}(H,V')),$$
$$b^4 \in C^{0,1}([0,T];\mathcal{L}(H,H')),$$

with Lipschitz constants $L_b, L_{b^1}, L_{b^2}, L_{b^3}, L_{b^4}$, i.e.

$$\|b^1(t_1) - b^1(t_2)\|_{\mathcal{L}(V,V')} \leqslant L_{b^1}|t_1 - t_2|,$$
$$\|b^2(t_1) - b^2(t_2)\|_{\mathcal{L}(V,H')} \leqslant L_{b^2}|t_1 - t_2|,$$
$$\|b^3(t_1) - b^3(t_2)\|_{\mathcal{L}(H,V')} \leqslant L_{b^3}|t_1 - t_2|,$$
$$\|b^4(t_1) - b^4(t_2)\|_{\mathcal{L}(H,H')} \leqslant L_{b^4}|t_1 - t_2|.$$

Moreover, we assume that b^1 is symmetric and uniformly coercive, i.e. there exists a constant $\beta > 0$ such that

$$b^1(t)(v,v) \geqslant \beta\|v\|_V^2,$$
$$b^1(t)(v,w) = b^1(t)(w,v)$$

for every $t \in [0,T]$ and all $v, w \in V$.

We write $C_b := \|b\|_{C([0,T];\mathcal{L}(V,V'))}$, $C_{b^1} := \|b^1\|_{C([0,T];\mathcal{L}(V,V'))}$, $C_{b^2} := \|b^2\|_{C([0,T];\mathcal{L}(V,H'))}$, $C_{b^3} := \|b^3\|_{C([0,T];\mathcal{L}(H,V'))}$ *and* $C_{b^4} := \|b^4\|_{C([0,T];\mathcal{L}(H,H'))}$.

(3.) Let $c \in \mathcal{L}(Q, V')$ *fulfil a uniform inf–sup condition, i.e. there exists* $\gamma > 0$*, such that*

$$\inf_{p\in Q\setminus\{0\}} \sup_{v\in V\setminus\{0\}} \frac{c(v,p)}{\|p\|_Q\|v\|_V} \geqslant \gamma.$$

Now, we use this generic existence results in order to show the existence, uniqueness and a-priori estimates for the transformed instationary Stokes equation.

Proof of Theorem 3.32. To apply Theorem 3.33 for (3.94), we need a time-independent algebraic constraint, which we obtain by substituting

$$u_\varepsilon = A_\varepsilon \hat{w}_\varepsilon, \qquad \text{i.e.} \quad \hat{w}_\varepsilon = A_\varepsilon^{-1} u_\varepsilon = J_\varepsilon^{-1}\Psi_\varepsilon u_\varepsilon$$

in (3.94) and using test functions $A_\varepsilon^{-1}\varphi$. For the substitution in the time-derivative term, we note that

$$\begin{aligned} J_\varepsilon \partial_t (A_\varepsilon^{-1} u_\varepsilon) \cdot A_\varepsilon^{-1}\varphi &= J_\varepsilon \partial_t A_\varepsilon^{-1} u_\varepsilon \cdot A_\varepsilon^{-1}\varphi + J_\varepsilon A_\varepsilon^{-1} \partial_t u_\varepsilon \cdot A_\varepsilon^{-1}\varphi \\ &= \partial_t A_\varepsilon^{-1} u_\varepsilon \cdot \Psi_\varepsilon \varphi + J_\varepsilon A_\varepsilon^{-1}\partial_t u_\varepsilon \cdot A_\varepsilon^{-1}\varphi \end{aligned}$$

and for the spatial derivatives, we use the Leibniz rule

$$\nabla(A_\varepsilon^{-1}\varphi) = \nabla(A_\varepsilon^{-1})\varphi + \nabla\varphi A_\varepsilon^{-\top}.$$

Then, we obtain the following weak form, which is equivalent to (3.94):

Find $u_\varepsilon \in L^2(0,T; H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^n)$ with $\partial_t u_\varepsilon$ in $L^2(0,T;L^2(\Omega_\varepsilon)^n)$ and $\hat{q}_\varepsilon \in L^2(0,T;L^2(\Omega_\varepsilon))$ such that, for a.e. $t \in (0,T)$,

$$\begin{aligned} a_\varepsilon(t)(\partial_t u_\varepsilon(t),\varphi) + b_\varepsilon(t)(u_\varepsilon(t),\varphi) + c_\varepsilon(q_\varepsilon(t),\varphi) &= f_{1,\varepsilon}(t)(\varphi) + f_{2,\varepsilon}(t)(\varphi), \\ c_\varepsilon(q_\varepsilon(t),\theta) &= g_\varepsilon(t)(\theta) \end{aligned} \tag{3.101}$$

for every $(\varphi,\theta) \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^n \times L^2(\Omega_\varepsilon)$, where

$$\begin{aligned} a_\varepsilon(t)(u,v) &:= (J_\varepsilon(t) A_\varepsilon^{-1}(t)u, A_\varepsilon^{-1}(t)v)_{L^2(\Omega_\varepsilon)} = (\Psi_\varepsilon(t)u, A_\varepsilon^{-1}(t)v)_{L^2(\Omega_\varepsilon)}, \\ b_\varepsilon(t)(u,v) &:= \sum_{i=1}^4 b_\varepsilon^i(t)(u,v), \\ b_\varepsilon^1(t)(u,v) &:= \big(\varepsilon^2\mu 2e'_\varepsilon(u), A_\varepsilon^\top(t)\nabla v A_\varepsilon^{-\top}(t)\big)_{L^2(\Omega_\varepsilon)}, \\ b_\varepsilon^2(t)(u,v) &:= \big(\varepsilon^2\mu 2e'_\varepsilon(u)(t), A_\varepsilon^\top(t)\nabla(A_\varepsilon^{-1}(t))v\big)_{L^2(\Omega_\varepsilon)}, \\ b_\varepsilon^3(t)(u,v) &:= \big(\varepsilon^2\mu\big(\Psi_\varepsilon^{-\top}(t)\nabla(A_\varepsilon^{-1}(t))u + \big(\Psi_\varepsilon^{-\top}(t)\nabla(A_\varepsilon^{-1}(t))u\big)^\top\big), A_\varepsilon^\top(t)\nabla v A_\varepsilon^{-\top}(t)\big)_{L^2(\Omega_\varepsilon)} \\ &\quad + \big(A_\varepsilon^{-1}(t)\nabla u^\top A_\varepsilon(t)\partial_t\psi_\varepsilon(t), A_\varepsilon^{-1}v\big)_{L^2(\Omega_\varepsilon)}, \end{aligned}$$

$$
\begin{aligned}
b^4_\varepsilon(t)(u,v) :=& \big(\varepsilon^2\mu\big(\Psi_\varepsilon^{-\top}(t)\nabla(A_\varepsilon^{-1}(t))u + \big(\Psi_\varepsilon^{-\top}(t)\nabla(A_\varepsilon^{-1}(t))u\big)^\top\big), A_\varepsilon^\top(t)\nabla(A_\varepsilon^{-1})(t)v\big)_{L^2(\Omega_\varepsilon)} \\
&+ (\partial_t(A_\varepsilon^{-1})(t)u, \Psi_\varepsilon(t)v)_{L^2(\Omega_\varepsilon)} + \big((\nabla(A_\varepsilon^{-1})(t)u)^\top A_\varepsilon(t)\partial_t\psi_\varepsilon(t), A_\varepsilon^{-1}v\big)_{L^2(\Omega_\varepsilon)}, \\
c_\varepsilon(p,v) :=& (\operatorname{div}(v), p)_{L^2(\Omega_\varepsilon)}, \\
f_{1,\varepsilon}(t)(u) :=& (\hat{f}_\varepsilon(t), \Psi_\varepsilon(t)\varphi)_{\Omega_\varepsilon} - (\nabla\hat{p}_{b,\varepsilon}(t), u)_{L^2(\Omega_\varepsilon)} - (J_\varepsilon(t)\partial_t\hat{v}_{\Gamma_\varepsilon}, A_\varepsilon^{-1}(t)u)_{L^2(\Omega_\varepsilon)} \\
&+ ((\nabla\hat{v}_{\Gamma_\varepsilon}(t))^\top A_\varepsilon(t)\partial_t\psi_\varepsilon(t), A_\varepsilon^{-1}(t)u)_{L^2(\Omega_\varepsilon)}, \\
f_{2,\varepsilon}(t)(v) :=& -(\varepsilon^2\mu A_\varepsilon(t)2e_\varepsilon(\hat{v}_{\Gamma_\varepsilon})(t), A_\varepsilon^{-1}(t)\nabla u)_{L^2(\Omega_\varepsilon)}, \\
g_\varepsilon(t)(p) :=& -(\operatorname{div}(A_\varepsilon(t)\hat{v}_{\Gamma_\varepsilon}(t)), p)_{L^2(\Omega_\varepsilon)} = -(A_\varepsilon(t) : \nabla\hat{v}_{\Gamma_\varepsilon}(t), p)_{L^2(\Omega_\varepsilon)}
\end{aligned}
$$

for

$$
e'_\varepsilon(u)(t,x) = \big(\Psi_\varepsilon^{-\top}(t,x)\nabla u(x)A_\varepsilon^{-\top}(t,x) + \big(\Psi_\varepsilon^{-\top}(t,x)\nabla u(x)A_\varepsilon^{-\top}(t,x)\big)^\top\big)/2
$$

We set $V_\varepsilon = H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^n$, $H_\varepsilon = L^2(\Omega_\varepsilon)^n$ and $Q_\varepsilon = L^2(\Omega_\varepsilon)$ with the scalar products and norms defined by

$$
\begin{aligned}
&\|v\|^2_{V_\varepsilon} := (v,v)_{V_\varepsilon} && (v,w)_{V_\varepsilon} := \varepsilon^2(\nabla v, \nabla w)_{L^2(\Omega_\varepsilon)} && \text{for } v,w \in V_\varepsilon, \\
&\|w\|^2_{H_\varepsilon} := (w,w)_{H_\varepsilon} && (v,w)_{V_\varepsilon} := (v,w)_{L^2(\Omega_\varepsilon)} && \text{for } v,w \in H_\varepsilon, \\
&\|q\|^2_{V_\varepsilon} := (q,q)_{Q_\varepsilon} && (q,p)_{Q_\varepsilon} := (q,p)_{L^2(\Omega_\varepsilon)} && \text{for } p,q \in Q_\varepsilon.
\end{aligned}
$$

Due to the Poincaré inequality (3.14), $\|\cdot\|_{V_\varepsilon}$ actually defines a norm on V_ε and $(\cdot,\cdot)_{V_\varepsilon}$ is a scalar product. Now, we show that these bilinear forms and right-hand sides fulfil the assumptions of Theorem 3.33. Moreover, we show that all constants which appear in the estimate of Theorem 3.33 can be chosen ε-independently.

- embedding constant: From Lemma 3.13, we obtain a constant C such that

  $$
  \|v\|_{H_\varepsilon} = \|v\|_{L^2(\Omega_\varepsilon)} \leqslant \varepsilon C\|\nabla v\|_{L^2(\Omega_\varepsilon)} = C\|v\|_{V_\varepsilon}
  $$

  and, thus, the embedding constant of V_ε into H_ε is independent of ε.

- bilinear form a_ε:
  - continuity of $a_\varepsilon(t)$: Using the Hölder inequality and the boundedness of Ψ_ε and A_ε^{-1} given by Lemma 2.40, we obtain a constant C_a such that

    $$
    \begin{aligned}
    |a_\varepsilon(t)(u,v)| &= |(\Psi_\varepsilon(t)u, A_\varepsilon^{-1}(t)v)_{L^2(\Omega_\varepsilon)}| \\
    &\leqslant \|\Psi_\varepsilon(t)\|_{L^\infty(\Omega_\varepsilon)}\|A_\varepsilon^{-1}(t)\|_{L^\infty(\Omega_\varepsilon)}\|u\|_{H_\varepsilon}\|v\|_{H_\varepsilon} \leqslant C_a\|u\|_{H_\varepsilon}\|v\|_{H_\varepsilon}
    \end{aligned}
    $$

    for all $t \in [0,T]$ and all $u, v \in H_\varepsilon$.
  - Lipschitz regularity with respect to time: Using again the uniform bounds of the coefficients from Lemma 2.40 and their uniform Lipschitz estimates with respect to time, which are provided by Lemma 3.28, we obtain a constant

$L_a > 0$ such that

$$
\begin{aligned}
&|(a_\varepsilon(t_1) - a_\varepsilon(t_2))(u, v)| \\
&=|(\Psi_\varepsilon(t_1)u, A_\varepsilon^{-1}(t_1)v)_{L^2(\Omega_\varepsilon)} - (\Psi_\varepsilon(t_2)u, A_\varepsilon^{-1}(t_2)v)_{L^2(\Omega_\varepsilon)}| \\
&\leqslant|(\Psi_\varepsilon(t_1) - \Psi_\varepsilon(t_2)u, A_\varepsilon^{-1}(t_1)v)_{L^2(\Omega_\varepsilon)}| + |(\Psi_\varepsilon(t_2)u, A_\varepsilon^{-1}(t_1) - A_\varepsilon^{-1}(t_2)v)_{L^2(\Omega_\varepsilon)}| \\
&\leqslant\|\Psi_\varepsilon(t_1) - \Psi_\varepsilon(t_2)\|_{L^\infty(\Omega_\varepsilon)}\|A_\varepsilon^{-1}(t_1)\|_{L^\infty(\Omega_\varepsilon)}\|u\|_{H_\varepsilon}\|v\|_{H_\varepsilon} \\
&\quad + \|\Psi_\varepsilon(t_2)\|_{L^\infty(\Omega_\varepsilon)}\|A_\varepsilon^{-1}(t_1) - A_\varepsilon^{-1}(t_2)\|_{L^\infty(\Omega_\varepsilon)}\|u\|_{H_\varepsilon}\|v\|_{H_\varepsilon} \\
&\leqslant L_a|t_1 - t_2|\|u\|_{H_\varepsilon}\|v\|_{H_\varepsilon}.
\end{aligned}
\tag{3.102}
$$

for every $t_1, t_2 \in [0, T]$ and all $u, v \in H_\varepsilon$.

- coercivity of $a_\varepsilon(t)$: Employing the essential boundedness of the coefficient given by Lemma 2.40, we obtain a constant α such that

$$
\begin{aligned}
\|v\|_{H_\varepsilon}^2 &= \|J_\varepsilon^{-1/2}(t)A_\varepsilon(t)J_\varepsilon^{1/2}(t)A_\varepsilon^{-1}(t)v\|_{L^2(\Omega_\varepsilon)}^2 \\
&\leqslant \|J_\varepsilon^{-1/2}(t)A_\varepsilon(t)\|_{L^\infty(\Omega_\varepsilon)}^2\|J_\varepsilon^{-1/2}(t)A_\varepsilon^{-1}(t)v\|_{L^2(\Omega_\varepsilon)}^2 \\
&\leqslant \frac{1}{\alpha}\|J_\varepsilon^{-1/2}(t)\Psi_\varepsilon(t)v\|_{L^2(\Omega_\varepsilon)}^2 = \frac{1}{\alpha}(J_\varepsilon(t)A_\varepsilon^{-1}(t)v, A_\varepsilon^{-1}(t)v)_{\Omega_\varepsilon} \\
&= \frac{1}{\alpha}a_\varepsilon(t)(v, v)
\end{aligned}
$$

every $t \in [0, T]$ and all $u, v \in H_\varepsilon$.

- symmetry of $a_\varepsilon(t)$:

$$a_\varepsilon(t)(u, v) = (J_\varepsilon(t)A_\varepsilon^{-1}(t)u, A_\varepsilon^{-1}(t)v)_{\Omega_\varepsilon} = a_\varepsilon(t)(v, u)$$

for every $t \in [0, T]$ and all $u, v \in H_\varepsilon$.

- bilinear form $b_\varepsilon(t)$: We note that $b_\varepsilon^1(t) \in \mathcal{L}(V_\varepsilon, V_\varepsilon')$, $b_\varepsilon^2(t) \in \mathcal{L}(V_\varepsilon, H_\varepsilon')$, $b_\varepsilon^3(t) \in \mathcal{L}(H_\varepsilon, V_\varepsilon')$, $b_\varepsilon^4(t) \in \mathcal{L}(H_\varepsilon, H_\varepsilon')$, for every $t \in [0, T]$. Hence, by the embedding of V_ε into H_ε, we obtain $b_\varepsilon(t) = \sum_{i=1}^4 b_\varepsilon^4(t) \in \mathcal{L}(V_\varepsilon, V_\varepsilon')$.
  - continuity of $b_{\varepsilon,i}(t)$: Lemma 2.40 provides uniform essential bounds for the coefficients $\Psi_\varepsilon^{-\top}$, $A_\varepsilon^{-\top}$ and $\varepsilon\nabla A_\varepsilon^{-1}$. Moreover, having the ε-scaling in the norm $\|\cdot\|_{V_\varepsilon}$, we observe that every gradient, no matter if it belongs to a coefficient or to u or v, requires one factor ε for the ε-uniform estimates.

    We separate the symmetric gradients and write $b_\varepsilon^1(t)$ and $b_\varepsilon^2(t)$ as two sums each. Then, we observe that each summand contains two gradient terms, which get exactly balanced with the ε^2 term. Thus, we obtain with the Hölder inequality constants $C_{b^1}, C_{b^2} > 0$ such that

$$
\begin{aligned}
|b_\varepsilon^1(t)(u, v)| \leqslant&\varepsilon^2\mu 2\|\Psi_\varepsilon^{-\top}(t)\|_{L^\infty(\Omega_\varepsilon)}\|\nabla u\|_{L^2(\Omega_\varepsilon)}\|A_\varepsilon^{-\top}(t)\|_{L^\infty(\Omega_\varepsilon)} \\
&\|A_\varepsilon^{\top}(t)\|_{L^\infty(\Omega_\varepsilon)}\|\nabla v\|_{L^2(\Omega_\varepsilon)}\|A_\varepsilon^{-\top}(t)\|_{L^\infty(\Omega_\varepsilon)} \\
\leqslant&C_{b^1}\varepsilon\|\nabla u\|_{L^2(\Omega_\varepsilon)}\varepsilon\|\nabla v\|_{L^2(\Omega_\varepsilon)} \leqslant C_{b^1}\|u\|_{V_\varepsilon}\|v\|_{V_\varepsilon}
\end{aligned}
$$

for every $t \in [0,T]$ and all $u, v \in V_\varepsilon$ and

$$|b_\varepsilon^2(t)(u,v)| \leqslant C_{b^2}\varepsilon\|\nabla u\|_{L^2(\Omega_\varepsilon)}\|v\|_{L^2(\Omega_\varepsilon)} \leqslant C_{b^1}\|u\|_{V_\varepsilon}\|v\|_{H_\varepsilon}$$

for every $t \in [0,T]$ and all $u \in V_\varepsilon$ and $v \in H_\varepsilon$.

In order to estimate $b_\varepsilon^3(t)$ and $b_\varepsilon^4(t)$, we use the same argument. However, the second summand of $b_\varepsilon^3(t)$ and the last summand $b_\varepsilon^4(t)$ have one gradient term but no explicit ε-factor. Nevertheless, we can estimate these terms uniformly with respect to ε since the essential bound for $\varepsilon^{-1}\partial_t\psi_\varepsilon$ generates the missing ε-factor. Moreover, for the second summand of $b_\varepsilon^4(t)$, we note that $\partial_t A_\varepsilon^{-1}$ is uniformly essentially bounded (see Lemma 2.40). Hence, these bilinear forms can be estimated with the Hölder inequality and we obtain constants $C_{b^3}, C_{b^4} > 0$ such that

$$|b_\varepsilon^3(t)(u,v)| \leqslant C_{b^3}\|u\|_{L^2(\Omega_\varepsilon)}\varepsilon\|\nabla v\|_{L^2(\Omega_\varepsilon)} \leqslant C_{b^3}\|u\|_{H_\varepsilon}\|v\|_{V_\varepsilon}$$

for every $t \in [0,T]$ and all $u \in V_\varepsilon$ and $v \in H_\varepsilon$ and

$$|b_\varepsilon^4(t)(u,v)| \leqslant C_{b^4}\|u\|_{L^2(\Omega_\varepsilon)}\|v\|_{L^2(\Omega_\varepsilon)} \leqslant C_{b^4}\|u\|_{H_\varepsilon}\|v\|_{H_\varepsilon}$$

for every $t \in [0,T]$ and all $u, v \in H_\varepsilon$.

- Lipschitz regularity of b_ε: Using the Lipschitz estimates and the uniform bounds for the coefficients, we can follow the argumentation of (3.102) in order to derive the following Lipschitz regularities. The same argumentation that we have used for the derivation of the uniform bounds for $b_{\varepsilon,i}(t)$ shows that these Lipschitz constants are independent of ε. Thus, we obtain constants $L_{b^1}, L_{b^2}, L_{b^3}, L_{b^4} > 0$ such that

$$\begin{aligned}
&|(b_\varepsilon^1(t_1) - b_\varepsilon^1(t_2))(u,v)| \leqslant L_{b^1}|t_1 - t_2|\|u\|_{V_\varepsilon}\|v\|_{V_\varepsilon} && \text{for all } u, v \in V_\varepsilon,\\
&|(b_\varepsilon^2(t_1) - b_\varepsilon^2(t_2))(u,v)| \leqslant L_{b^2}|t_1 - t_2|\|u\|_{V_\varepsilon}\|v\|_{H_\varepsilon} && \text{for all } u \in V_\varepsilon, v \in H_\varepsilon,\\
&|(b_\varepsilon^3(t_1) - b_\varepsilon^3(t_2))(u,v)| \leqslant L_{b^3}|t_1 - t_2|\|u\|_{H_\varepsilon}\|v\|_{V_\varepsilon} && \text{for all } u \in H_\varepsilon, v \in V_\varepsilon,\\
&|(b_\varepsilon^4(t_1) - b_\varepsilon^4(t_2))(u,v)| \leqslant L_{b^4}|t_1 - t_2|\|u\|_{H_\varepsilon}\|v\|_{H_\varepsilon} && \text{for all } u, v \in H_\varepsilon
\end{aligned}$$

for all $t_1, t_2 \in [0,T]$.

- coercivity of b_ε^1: First, we rewrite $b_\varepsilon^1(t)$. Then, we use the essential boundedness of $J_\varepsilon^{-1}(t) \leqslant c_J^{-1}$, which provides

$$\|e_\varepsilon'(w)(t)\|_{L^2(\Omega_\varepsilon)} = \|J_\varepsilon^{-1/2}(t)J_\varepsilon^{1/2}(t)e_\varepsilon'(w)(t)\|_{L^2(\Omega_\varepsilon)} \leqslant c_J^{-1}\|J_\varepsilon^{1/2}(t)e_\varepsilon'(w)(t)\|_{L^2(\Omega_\varepsilon)}.$$

Afterwards, we employ the Korn-inequality of Lemma 3.35 from below, which provides a constant $\beta > 0$ such that

$$\begin{aligned} b^1_\varepsilon(t)(u,v) =& \tfrac{1}{2}\big(\varepsilon^2\mu 2e'_\varepsilon(v)(t), A^\top_\varepsilon(t)\nabla v A^{-\top}_\varepsilon(t) + \big(A^\top_\varepsilon(t)\nabla v A^{-\top}_\varepsilon(t)\big)^\top\big)_{L^2(\Omega_\varepsilon)} \\ =& \varepsilon^2\mu\big(e'_\varepsilon(v)(t), J_\varepsilon(t)e'_\varepsilon(w)(t)\big)_{L^2(\Omega_\varepsilon)} = \varepsilon^2\mu\|\sqrt{J_\varepsilon(t)}e'_\varepsilon(w)(t)\|_{L^2(\Omega_\varepsilon)} \\ \geqslant& \varepsilon^2\mu c_J\|e'_\varepsilon(w)(t)\|_{L^2(\Omega_\varepsilon)} \geqslant \tfrac{1}{\beta}\varepsilon^2\|\nabla v\|^2_{L^2(\Omega_\varepsilon)} \geqslant \tfrac{1}{\beta}\|v\|^2_{V_\varepsilon} \end{aligned}$$

for every $t \in [0,T]$ and all $v \in V_\varepsilon$.

- symmetry of b^1_ε: The symmetry follows by rewriting b^1_ε as we did for showing the coercivity of b^1_ε.

- bilinear form c_ε:
  - continuity: From the Hölder inequality, we obtain

$$c_\varepsilon(v,q) = (p, \operatorname{div}(v))_{L^2(\Omega_\varepsilon)} \leqslant C\|p\|_{L^2(\Omega_\varepsilon)}\|\nabla v_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leqslant \varepsilon^{-1}C\|p\|_{Q_\varepsilon}\|v_\varepsilon\|_{V_\varepsilon} \tag{3.103}$$

  for every $p \in P_\varepsilon$ and $v \in V_\varepsilon$. We note that c_ε is not uniformly bounded with respect to ε. However, c_ε is not incorporated in the estimates of the solution that is provided by Theorem 3.33.
  - inf–sup estimate: with the operator $\operatorname{div}^{-1}_\varepsilon : L^2(\Omega_\varepsilon) \to H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^n$ from Lemma 3.8, we obtain $\gamma > 0$ such that

$$\begin{aligned} &\inf_{p\in Q_\varepsilon\setminus\{0\}} \sup_{v\in V_\varepsilon\setminus\{0\}} \frac{|c_\varepsilon(v,p)|}{\|p\|_{Q_\varepsilon}\|v\|_{V_\varepsilon}} \geqslant \inf_{p\in Q_\varepsilon\setminus\{0\}} \frac{|c_\varepsilon(\operatorname{div}(\operatorname{div}^{-1}_\varepsilon p, v))|}{\|p\|_{Q_\varepsilon}\|\operatorname{div}^{-1}_\varepsilon p\|_{V_\varepsilon}} \\ &= \inf_{p\in Q_\varepsilon\setminus\{0\}} \frac{\|p\|^2_{L^2(\Omega_\varepsilon)}}{\|p\|_{Q_\varepsilon}\varepsilon\|\nabla \operatorname{div}^{-1}_\varepsilon(p)\|_{L^2(\Omega_\varepsilon)}} \geqslant \inf_{p\in Q_\varepsilon\setminus\{0\}} \frac{\|p\|^2_{Q_\varepsilon}}{\|p\|_{Q_\varepsilon}\gamma^{-1}\|p\|_{Q_\varepsilon}} \geqslant \gamma. \end{aligned}$$

- estimates on the data (right-hand sides):
  - boundedness of $f_{1,\varepsilon}$: Employing the Hölder inequality, the estimates on the coefficients, which are given by Lemma 2.40 and the bounds on the data from Lemma 3.31, we can estimate the first two summands of $f_{1,\varepsilon}(t)$ by

$$\begin{aligned} |(J_\varepsilon(t)\hat{f}_\varepsilon(t), A^{-1}_\varepsilon(t)u)_{L^2(\Omega_\varepsilon)}| &\leqslant C\|\hat{f}_\varepsilon(t)\|\|u\|_{L^2(\Omega_\varepsilon)} \leqslant K_\varepsilon(t)\|u\|_{H_\varepsilon}, \\ |(A^\top_\varepsilon(t)\nabla\hat{p}_{b,\varepsilon}(t), A^{-1}_\varepsilon(t)u)_{L^2(\Omega_\varepsilon)}| &\leqslant C\|\nabla\hat{p}_{b,\varepsilon}(t)\|_{L^2(\Omega_\varepsilon)}\|u\|_{L^2(\Omega_\varepsilon)} \leqslant K_\varepsilon(t)\|u\|_{H_\varepsilon} \end{aligned}$$

  for some $K_\varepsilon \in L^2(0,T)$, which is ε-independently bounded in $L^2(0,T)$, i.e. there exists a constant C such that $\|K_\varepsilon\|_{L^2(0,T)} \leqslant C$. The remaining summands of $f_{1,\varepsilon}(t)$ can be uniformly estimated with respect to time. For this, we use the Hölder inequality and the boundedness of the coefficients and the data $\partial_t\hat{v}_{\Gamma_\varepsilon}$ and $\hat{v}_{\Gamma_\varepsilon}$

$$|(J_\varepsilon(t)\partial_t\hat{v}_{\Gamma_\varepsilon}(t), A^{-1}_\varepsilon(t)u)_{L^2(\Omega_\varepsilon)}| \leqslant C\|\partial_t\hat{v}_{\Gamma_\varepsilon}(t)\|_{L^2(\Omega_\varepsilon)}\|u\|_{H_\varepsilon} \leqslant C\|u\|_{H_\varepsilon}$$

$$|((\nabla \hat{v}_{\Gamma_\varepsilon}(t))^\top A_\varepsilon(t)\partial_t \psi_\varepsilon(t), A_\varepsilon^{-1}(t)u)_{L^2(\Omega_\varepsilon)}|,$$
$$\leqslant C\|\nabla \hat{v}_{\Gamma_\varepsilon}(t)\|_{L^2(\Omega_\varepsilon)}\|\partial_t \psi_\varepsilon(t)\|_{L^\infty(\Omega_\varepsilon)}\|u\|_{H_\varepsilon} \leqslant \varepsilon C\|u\|_{H_\varepsilon}.$$

After integrating over $(0,T)$, we obtain

$$\|f_{1,\varepsilon}\|_{L^2(0,T;H_\varepsilon')}\| \leqslant C.$$

These calculations show that the summands of $f_{1,\varepsilon}$, which belong to $\hat{v}_{\Gamma_\varepsilon}$ and $\nabla\hat{v}_{\Gamma_\varepsilon}$ are of order ε. Therefore, they will also vanish in the homogenisation process.

– boundedness of $f_{2,\varepsilon}$: By similar estimates as used for $f_{1,\varepsilon}$, we obtain

$$|f_{2,\varepsilon}(t)(u)| \leqslant \varepsilon C\|u\|_{V_\varepsilon}$$

for every $t \in [0,T]$. Since $f_{2,\varepsilon}$ is of order ε, it will also vanish during the homogenisation process later.

Moreover, using the Leibniz rule, the Hölder inequality and the boundedness of the coefficients as well as the bounds of their derivatives (see Lemma 2.8 and Lemma 2.43), we can estimate the time derivative of $f_{2,\varepsilon}$ by

$$\begin{aligned}|\partial_t f_{2,\varepsilon}(t)(u)| &\leqslant \varepsilon^2\mu|(\partial_t A_\varepsilon(t)e_\varepsilon(\hat{v}_{\Gamma_\varepsilon})(t), \nabla u)_{L^2(\Omega_\varepsilon)}|\\
&+ \varepsilon^2\mu|(A_\varepsilon(t)\big(\partial_t\Psi_\varepsilon^{-\top}(t)\nabla\hat{v}_{\Gamma_\varepsilon}(t) + (\partial_t\Psi_\varepsilon^{-\top}(t)\nabla\hat{v}_{\Gamma_\varepsilon}(t))^\top\big), \nabla u)_{L^2(\Omega_\varepsilon)}|\\
&+ \varepsilon^2\mu|(A_\varepsilon(t)e_\varepsilon(\partial_t\hat{v}_{\Gamma_\varepsilon})(t), \nabla u)_{L^2(\Omega_\varepsilon)}|\\
&\leqslant \varepsilon C\big(\|\nabla\hat{v}_{\Gamma_\varepsilon}(t)\|_{L^2(\Omega_\varepsilon)} + \|\partial_t\nabla\hat{v}_{\Gamma_\varepsilon}(t)\|_{L^2(\Omega_\varepsilon)}\big)\varepsilon\|u\|_{V_\varepsilon}\end{aligned}$$

After integrating over $(0,T)$ an applying the boundedness of $\nabla\hat{v}_{\Gamma_\varepsilon}$ and $\varepsilon\partial_t\nabla\hat{v}_{\Gamma_\varepsilon}$, we get

$$\|\partial_t f_{2,\varepsilon}\|_{L^2(0,T;V_\varepsilon')} \leqslant \big(\varepsilon C + C\|\partial_t\nabla\hat{v}_{\Gamma_\varepsilon}\|_{L^2((0,T)\times\Omega_\varepsilon)}\big) \leqslant C.$$

– boundedness of g_ε: With the Hölder inequality and the uniform bound of $A_\varepsilon(t)$, $\nabla\hat{v}_{\Gamma_\varepsilon}(t)$ and $\partial_t\nabla\hat{v}_{\Gamma_\varepsilon}$, we get

$$\begin{aligned}\|g_\varepsilon\|_{L^2 0,T;Q_\varepsilon')} =&\|A_\varepsilon : \nabla\hat{v}_{\Gamma_\varepsilon}\|_{L^2((0,T)\times\Omega_\varepsilon)}\\
&\leqslant C\|A_\varepsilon(t)\|_{L^\infty((0,T)\times\Omega_\varepsilon)}\|\nabla\hat{v}_{\Gamma_\varepsilon}\|_{L^\infty((0,T)\times\Omega_\varepsilon)} \leqslant C\\
\|\partial_t g_\varepsilon\|_{L^2 0,T;Q_\varepsilon')} =&\|\partial_t(A_\varepsilon : \nabla\hat{v}_{\Gamma_\varepsilon})\|_{L^2((0,T)\times\Omega_\varepsilon)}\\
\leqslant& C\|\partial_t A_\varepsilon\|_{L^\infty((0,T)\times\Omega_\varepsilon)}\|\nabla\hat{v}_{\Gamma_\varepsilon}\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))}\\
&+ C\|A_\varepsilon\|_{L^\infty((0,T)\times\Omega_\varepsilon)}\|\partial_t\nabla\hat{v}_{\Gamma_\varepsilon}\|_{L^2((0,T)\times\Omega_\varepsilon)} \leqslant C.\end{aligned}$$

Having shown all the assumptions of Theorem 3.33 with constants independent of ε, we obtain a unique solution $(u_\varepsilon, \hat{q}_\varepsilon) \in L^2(0,T;V_\varepsilon) \times L^2(0,T;Q_\varepsilon)$ with $\partial_t u_\varepsilon \in L^2(0,T;H_\varepsilon)$ of

(3.101) and a constant C independent of ε such that

$$\|u_\varepsilon\|_{L^\infty(0,T;H_\varepsilon)} + \varepsilon\|\nabla u_\varepsilon\|_{L^2(0,T;V_\varepsilon)} + \|\hat{q}_\varepsilon\|_{L^2(0,T;Q_\varepsilon)} \leqslant C. \tag{3.104}$$

Since the weak form (3.101) is equivalent to the weak form (3.94), we obtain a unique solution $(\hat{w}_\varepsilon, \hat{q}_\varepsilon) \in L^2(0,T; H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^n) \times L^2(0,T; L^2(\Omega_\varepsilon))$ with $\partial_t u_\varepsilon \in L^2(0,T; L^2(\Omega_\varepsilon)^n)$. Moreover, we can transfer the uniform estimate (3.104) onto $(\hat{w}_\varepsilon, \hat{q}_\varepsilon)$ via

$$\hat{w}_\varepsilon = A_\varepsilon^{-1} u_\varepsilon, \qquad \partial_t \hat{w}_\varepsilon = \partial_t A_\varepsilon^{-1} u_\varepsilon + A_\varepsilon^{-1}\partial_t \hat{w}_\varepsilon, \qquad \varepsilon\nabla\hat{w}_\varepsilon = \varepsilon\nabla A_\varepsilon^{-1} u_\varepsilon + \varepsilon \nabla u_\varepsilon A_\varepsilon^{-\top}$$

using the uniform essential boundedness of $\partial_t A_\varepsilon^{-1}$, $\varepsilon\nabla A_\varepsilon^{-1}$, $A_\varepsilon^{-\top}$. □

Korn-type inequality

We have used the following Korn-type inequality in order to show the coercivity of b_ε^1 in the proof of Theorem 3.32.

Proposition 3.35. *There exists a constant β such that*

$$\|\Psi_\varepsilon^{-\top}(t)\nabla v A_\varepsilon^{-1}(t) + (\Psi_\varepsilon^{-\top}(t)\nabla v A_\varepsilon^{-1}(t))^\top\|^2_{L^2(\Omega_\varepsilon)} \geqslant \beta\|\nabla v\|^2_{L^2(\Omega_\varepsilon)} \tag{3.105}$$

for all $v \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^n$ and every $\varepsilon > 0$.

Compared to the Korn-type inequality from Proposition 3.9, which we have used for the quasi-stationary case, the gradient is now multiplied not only from one but from both sides by matrices. Nevertheless, it can be shown by a similar argumentation and we point out the main differences. Again, we reduce it to a Korn-type inequality for a fixed domain and fixed space-dependent coefficient.

Lemma 3.36. *Let $1 < p < \infty$ and U be an open, bounded domain in $\mathbb{R}^n$ for $n \geqslant 2$ with Lipschitz boundary ∂U. Let S be an open subset of $\partial\Omega$ with $|S| > 0$. Let $A, B \in C(\overline{U})^{n\times n}$ with $\det(A(x)), \det(B(x)) \geqslant c > 0$. Then, there exists a constant $\alpha > 0$ such that*

$$\int_U |A(x)\nabla u(x)B(x) + (A(x)\nabla u(x)B(x))^\top|^p \,\mathrm{d}x \geqslant \alpha \int_U |\nabla u(x)|^p \,\mathrm{d}x \tag{3.106}$$

for every $u \in W^{1,p}_S(U)^n$.

In order to prove Lemma 3.36, we reduce it to the following generic Korn-type inequality, which was shown in [Pom03]. Therefore, we need the following definition.

Definition 3.37. *Let $m, n, r \in \mathbb{N}$. A mapping $A : \mathbb{R}^{m\times n} \to \mathbb{R}^r$ with $r \geqslant m$ is called elliptic if $A(\eta\xi^\top) \neq 0$ for all $\eta \in \mathbb{R}^m$, $\xi \in \mathbb{R}^n$ with $\eta \neq 0$ and $\xi \neq 0$.*

Lemma 3.38. *Let $1 < p < \infty$ and $U \subset \mathbb{R}^n$ $(n \geqslant 2)$ be a connected, open, bounded Lipschitz domain with $V \subset \partial U$. Let $m, r \in \mathbb{N}$ with $r \geqslant m$ and let $A(x) : \mathbb{R}^{m\times n} \to \mathbb{R}^r$ be a* *family of linear elliptic mappings whose coefficients* a_k^{ij} *are continuous on* $\overline{U}$*. Then, there exists a constant* $c > 0$ *such that*

$$\left(\int_U |A(x)\nabla v(x)|^p \,\mathrm{d}x\right)^{1/p} + \left(\int_V |v(x)|^p \,\mathrm{d}\sigma_x\right)^{1/p} \geqslant c\|v\|_{W^{1,p}(U)}$$

for all $v \in W^{1,p}(U)^m$*.*

Proof. See [Pom03, Theorem 2.4]. □

Extending the argumentation of [Pom03, Corollary 4.1], we can prove Lemma 3.36.

Proof of Lemma 3.36. Due to Lemma 3.38 it suffcies to show that

$$F \mapsto AFB + (AFB)^\top \in \mathbb{R}^{n\times n} = \mathbb{R}^{n^2}$$

is elliptic, i.e.

$$A(\eta\xi^\top)B + (A(\eta\xi^\top)B)^\top \neq 0 \tag{3.107}$$

for all $\eta, \xi \in \mathbb{R}^n\backslash\{0\}$. Let $\eta, \xi \in \mathbb{R}^n\backslash\{0\}$ and assume that (3.107) does not hold, then $A(\eta\xi^\top)B$ is skew-symmetric, which implies that $\operatorname{rank}(A(\eta\xi^\top)B) = 0$ or $\operatorname{rank} A(\eta\xi^\top)B \geqslant 2$. Since $\det(A), \det(B) \neq 0$, we obtain $\operatorname{rank}(A(\eta\xi^\top)B) = \operatorname{rank}(\eta\xi^\top) = 1$, which is a contradiction. Thus, $F \mapsto AFB + (AFB)^\top$ is elliptic. □

Proof of Proposition 3.35. Proposition 3.35 can be traced back to Lemma 3.36 by the same arguments that we used in order to trace back Proposition 3.9 to Lemma 3.10. □

3.2.4. Identification of the two-scale limit problem

Now, we derive the following instationary two-pressure Stokes equations as the two-scale limit problem of (3.92).

Instationary two-pressure Stokes equations in the reference coordinates

$$\begin{aligned}
\partial_t \hat{w}_0 - \nabla_y \hat{w}_0^\top \Psi_0^{-1} \partial_t \psi_0 - J_0^{-1} \operatorname{div}_y \left(\mu A_0 \Psi_0^{-\top} \nabla_y \hat{w}_0\right) & & \\
+\Psi_0^{-\top} \nabla_x \hat{p} + \Psi_0^{-\top} \nabla_y \hat{p}_1 &= f && \text{in } (0,T) \times \Omega \times Y^*, \\
J_0^{-1} \operatorname{div}_y (A_0 \hat{w}_0) &= 0 && \text{in } (0,T) \times \Omega \times Y^*, \\
\hat{w}_0 &= 0 && \text{on } (0,T) \times \Omega \times \Gamma, \\
\hat{w}_0(0) &= \hat{v}^{\text{in}} && \text{in } \Omega \times Y^*, \\
y \mapsto \hat{w}_0, \hat{q}_1 & && Y\text{-periodic}, \\
\operatorname{div}_x \Big(\int_{Y^*} A_0 \hat{w}_0 \,\mathrm{d}y \Big) &= \int_{Y^*} \operatorname{div}(A_0 \hat{v}_\Gamma) \,\mathrm{d}y && \text{in } (0,T) \times \Omega, \\
\hat{p} &= \hat{p}_{b,0} && \text{on } (0,T) \times \partial\Omega,
\end{aligned} \tag{3.108}$$

The weak formulation of this instationary two-pressure Stokes equation is given by:

Weak form of the instationary two-pressure Stokes equations in the reference coordinates

Find $(\hat{w}_0, \hat{q}, \hat{q}_1) \in L^2((0,T) \times \Omega; H^1_{\Gamma\#}(Y^*)^n) \times L^2(0,T; H^1_0(\Omega)) \times L^2((0,T) \times \Omega; L^2_0(Y^*))$ with $\partial_t \hat{w}_0 \in L^2((0,T) \times \Omega; L^2(Y^*)^n)$ such that, for a.e. $t \in (0,T)$,

$$\begin{aligned}
&\int_\Omega \int_{Y^*} J_0 \partial_t \hat{w}_0 \cdot \varphi \,\mathrm{d}y \,\mathrm{d}x + \int_{Y^*} (\nabla \hat{w}_0)^\top A_0 \partial_t \psi_0 \cdot \varphi \,\mathrm{d}y \,\mathrm{d}x + \int_\Omega \int_{Y^*} \mu A_0 \Psi_0^\top \nabla_y \hat{w}_0 : \nabla_y \varphi \,\mathrm{d}y \,\mathrm{d}x \\
&+ \int_\Omega \int_{Y^*} A_0^\top \nabla_x \hat{q} \cdot \varphi - \hat{q}_1 \operatorname{div}_y (A_0 \varphi) \,\mathrm{d}y \,\mathrm{d}x = \int_\Omega \int_{Y^*} (J_0 f - A_0^\top (\nabla_x \hat{p}_{b,0} + \nabla_y \hat{p}_{b,1})) \cdot \varphi \,\mathrm{d}y \,\mathrm{d}x, \\
&\qquad \int_\Omega \operatorname{div}_x \Big(\int_{Y^*} A_0 \hat{w}_0 \,\mathrm{d}y \Big) \eta_0 \,\mathrm{d}x = - \int_\Omega \int_{Y^*} \operatorname{div}_y \left(A_0 \hat{v}_\Gamma\right) \mathrm{d}y \; \eta_0 \,\mathrm{d}x, \\
&\qquad \int_\Omega \int_{Y^*} \operatorname{div}_y (A_0 \hat{w}_0) \eta_1 \,\mathrm{d}y \,\mathrm{d}x = 0
\end{aligned} \tag{3.109}$$

for all $(\varphi, \eta_0, \eta_1) \in L^2(\Omega; H^1_{\Gamma\#}(Y^*)^n) \times H^1_0(\Omega) \times L^2(\Omega; L^2(Y^*))$ and $\hat{w}_0(0) = \hat{w}_0^{\text{in}}$.

For the identification of the limit problem, we can partially follow the argumentation for the stationary case. However, we cannot derive a strong L^2-compactness result for the pressure $\hat{q}_\varepsilon$ or some extension since we have no additional time regularity for the pressure, which would be needed for compactness arguments. Nevertheless, we can show the weak two-scale convergence of $\hat{q}_\varepsilon$ and that its limit is constant with respect to y, which is sufficient for the identification of the limit equations.

Theorem 3.39. *Let* $(\hat{w}_\varepsilon, \hat{q}_\varepsilon)$ *be the solution of* (3.92). *Then,*

$$\widetilde{\hat{w}_\varepsilon} \xrightarrow{2,2} \widetilde{\hat{w}_0}, \tag{3.110}$$

$$\varepsilon\widetilde{\nabla \hat{w}_\varepsilon} \xrightarrow{2,2} \widetilde{\nabla_y \hat{w}_0}, \tag{3.111}$$

$$\varepsilon\widetilde{\partial_t \hat{w}_\varepsilon} \xrightarrow{2,2} \widetilde{\partial_t \hat{w}_0}, \tag{3.112}$$

$$\hat{q}_\varepsilon \xrightarrow{2,2} \chi_{Y^*}\hat{q} \tag{3.113}$$

where $(\hat{w}_0, \hat{q}) \in L^2((0,T)\times\Omega; H^1_{\Gamma\#}(Y^*)^n) \times L^2(0,T; H^1_0(\Omega))$ *are the first two components of the solution of* (3.109).

Proof. Due to the a-priori estimates (3.95), we can apply two-scale compactness results and obtain $\hat{w}_0$ such that (3.110)–(3.112) and $\hat{w}_\varepsilon(0) \xrightarrow{2} \hat{w}_0(0)$ hold for a subsequence. Moreover, from $\hat{w}_\varepsilon(0) = \hat{w}_\varepsilon^{\text{in}} \xrightarrow{2} \hat{w}_0^{\text{in}}$ it follows that $\hat{w}_0(0) \xrightarrow{2} \hat{w}_0^{\text{in}}$.

By arguing as in Lemma 3.16 for the quasi-stationary case, it can be shown that $\hat{w}_0$ fulfils the microscopic incompressibility and the macroscopic compressibility condition of (3.109), as well as that $\hat{w}_0$ is zero on Γ.

Due to the a-priori estimate (3.95) for the pressure $\hat{q}_\varepsilon$, we obtain $\hat{q} \in L^2((0,T)\times\Omega\times Y^*)$ such that $\hat{q}_\varepsilon \xrightarrow{2,2} \hat{q}$ for a further subsequence. In order to show that $\hat{q}$ is independent of y, i.e. $\hat{q} = \chi_{Y^*}\hat{q}$ for $\hat{q} \in L^2((0,T)\times\Omega)$, we test (3.94) by $\varepsilon\phi(t)A_\varepsilon^{-1}(t,x)\varphi(x,\frac{x}{\varepsilon})$ for $\varphi \in C^\infty(\overline{\Omega}; H^1_{\Gamma\#}(Y^*)^n)$ with $\operatorname{div}_y(\varphi) = 0$ and $\phi \in C([0,T])$ and integrate over $(0,T)$. Due to the factor ε in the test function, all terms in (3.94) besides the pressure term are of order ε at least (for the second summand on the left-hand side, we note that the factor $\partial_t\psi_\varepsilon$ is of order ε which compensates $\nabla\hat{w}_0$, which is of order ε^{-1}). Therefore, these terms vanish in the limit $\varepsilon \to 0$ and we obtain

$$\begin{aligned}
0 &= \lim_{\varepsilon\to 0}\int_0^T\int_{\Omega_\varepsilon} \hat{q}_\varepsilon(t,x)\operatorname{div}\left(\varepsilon\varphi\left(x,\tfrac{x}{\varepsilon}\right)\right)\phi(t)\,\mathrm{d}x\,\mathrm{d}t\\
&= \lim_{\varepsilon\to 0}\int_0^T\int_{\Omega_\varepsilon} \hat{q}_\varepsilon(t,x)\phi(t)\left(\varepsilon\operatorname{div}_x\left(\varphi\left(x,\tfrac{x}{\varepsilon}\right)\right) + \operatorname{div}_y\left(\varphi\left(x,\tfrac{x}{\varepsilon}\right)\right)\right)\mathrm{d}x\,\mathrm{d}t\\
&= \int_0^T\int_\Omega\int_{Y^p} \hat{q}(t,x,y)\phi(t)\operatorname{div}_y(\varphi(x,y))\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

which shows that $\hat{q}$ is constant on Y^*, i.e. $\hat{q}(t,x,y) = \chi_{Y^*}(y)\hat{q}(t,x)$.

Now, we can identify the limit equation as in the quasi-stationary case, which we only sketch here. First, we test (3.94) by $\phi(t)A_\varepsilon^{-1}(t,x)\varphi(x,\frac{x}{\varepsilon})$ for $\varphi \in C^\infty(\overline{\Omega}; H^1_{\Gamma\#}(Y^*)^n)$ with $\operatorname{div}_y(\varphi) = 0$, $\phi \in C([0,T])$ and integrate over $(0,T)$. Then, we pass to the limit $\varepsilon \to 0$ and increase the set of test function by a density argument. Then, we show that $\hat{q}$ is weakly differentiable with respect to space, i.e. $\hat{q} \in L^2(0,T;H_0^1(\Omega))$. Afterwards, we reconstruct the microscopic pressure $\hat{q}_1$ such that the limit equation holds for arbitrary test functions which are not divergence-free. Finally, we show that $\int_{Y^*} A_0 \Psi_0^{-\top} \nabla \hat{w}_0^\top : \nabla \varphi \,\mathrm{d}y = 0$, which gives (3.109).

The uniqueness of the solution of the limit problem can be shown by means of Theorem 3.33 after a substitution of the solution and the test function as in the ε-scaled problem. The inf–sup estimate for the algebraic divergence constraints can be shown as in Lemma 3.20.

Due the uniqueness of the solution of the limit equation, the convergence holds for the whole sequence. □

Back-transformation of the instationary two-pressure Stokes equations

After a back-transformation of (3.109), we obtain the following instationary two-pressure Stokes equations

Weak form of the instationary two-pressure Stokes equations

Find $w_0 \in L^2((0,T) \times \Omega; H^1_{\Gamma(t,x)\#}(Y^*(t,x))^n)$ with $\partial_t \hat{w}_0 \in L^2((0,T) \times \Omega; L^2(Y^*(t,x))^n)$ and $q \in L^2(0,T;H_0^1(\Omega))$, $q_1 \in L^2((0,T) \times \Omega; L_0^2(Y^*(t,x)))$ such that, for a.e. $t \in (0,T)$,

$$\begin{aligned}
&\int_\Omega \int_{Y^*(t,x)} \partial_t w_0 \cdot \varphi \,\mathrm{d}y \,\mathrm{d}x + \int_\Omega \int_{Y^*(t,x)} \mu \nabla_y w_0 : \nabla_y \varphi \,\mathrm{d}y \,\mathrm{d}x \\
&+ \int_\Omega \int_{Y^*(t,x)} \nabla_x q \cdot \varphi - q_1 \operatorname{div}_y(\varphi) \,\mathrm{d}y \,\mathrm{d}x = \int_\Omega \int_{Y^*(t,x)} (f - \nabla_x p_{b,0} - \nabla_y p_{b,1}) \cdot \varphi \,\mathrm{d}y \,\mathrm{d}x, \\
&\int_\Omega \operatorname{div}_x \Big(\int_{Y^*(t,x)} w_0 \,\mathrm{d}y \Big) \eta_0 \,\mathrm{d}x = - \int_\Omega \int_{Y^*(t,x)} \operatorname{div}_y \left(v_\Gamma \right) \mathrm{d}y \; \eta_0 \,\mathrm{d}x, \\
&\int_\Omega \int_{Y^*(t,x)} \operatorname{div}_y(w_0) \eta_1 \,\mathrm{d}y \,\mathrm{d}x = 0
\end{aligned} \tag{3.114}$$

for all $(\varphi, \eta_0, \eta_1) \in L^2(\Omega; H^1_{\Gamma(t,x)\#}(Y^*(t,x))^n) \times H_0^1(\Omega) \times L^2(\Omega; L^2(Y^*(t,x)))$ and $w_0(0) = w_0^{\mathrm{in}}$.

The time derivative $\partial_t w_0$ has to be understood in the sense of the extension by zero as already in the ε-scaled case, i.e. $\widetilde{w_0} \in H^1(0,T;L^2(\Omega \times Y))$. The strong form of (3.114) is given by:

Instationary two-pressure Stokes equations

$$
\begin{aligned}
\partial_t w_0 - \operatorname{div}_y \big(\mu \nabla_y w_0\big) + \nabla_x p + \nabla_y p_1 &= f && \text{in } Q^T,\\
\operatorname{div}_y(w_0) &= 0 && \text{in } Q^T,\\
v_0 &= 0 && \text{on } G^T,\\
v_0(0) &= v^{\text{in}} && \text{on } Q(0),\\
y \mapsto w_0, p_1 & && Y\text{-periodic},\\
\operatorname{div}_x\Big(\int_{Y^*} w_0 \,\mathrm{d}y\Big) &= \int_\Omega \int_{Y^*(t,x)} \operatorname{div}(v_\Gamma)\,\mathrm{d}y && \text{in } (0,T)\times\Omega,\\
p &= p_{b,0} && \text{on } (0,T)\times\partial\Omega,
\end{aligned}
\tag{3.115}
$$

Theorem 3.40. *Let* $(w_\varepsilon, q_\varepsilon)$ *be the solution of* (3.89)*. Then, it holds*

$$\widetilde{w_\varepsilon} \xrightarrow{2} \widetilde{w_0}, \tag{3.116}$$

$$\varepsilon \widetilde{\nabla w_\varepsilon} \xrightarrow{2} \widetilde{\nabla_y w_0}, \tag{3.117}$$

$$\varepsilon \widetilde{\partial_t w_\varepsilon} \xrightarrow{2} \widetilde{\partial_t w_0}, \tag{3.118}$$

$$q_\varepsilon \xrightarrow{2} \chi_{Y^*} q \tag{3.119}$$

where $(w_0, q) \in L^2((0,T)\times\Omega; H^1_{\Gamma(t,x)\#}(Y^*(t,x))^n) \times L^2(0,T; H^1_0(\Omega))$ *are the first two components of the solution of* (3.114)*.*

Proof. The back-transformation can be done analogously to the quasi-stationary case. □

3.2.5. A Darcy law with memory for evolving microstructure

Now, we separate the micro- and macroscopic variables in (3.108) and, thus, derive effective equations, namely a Darcy law with memory for evolving microstructure. In order to shorten the writing, we consider in the following only the case of no-slip boundary condition, i.e. $\hat{v}_{\Gamma_\varepsilon} = \partial_t \psi_\varepsilon$, which yields

$$\int_{Y^*} \operatorname{div}_y(A_0(t,x,y)\partial_t \psi_0(t,x,y))\,\mathrm{d}y = \partial_t \Theta(t,x) \tag{3.120}$$

for $\Theta(t,x) = |Y^*(t,x)|$. The general case can be done as in the quasi-stationary case.

As in the quasi-stationary case, we aim to express the macroscopic quantities

$$w(t,x) := \int_{Y^*} J_0(t,x,y)\hat{w}_0(t,x,y)\,\mathrm{d}y = \int_{Y^*} w_0(t,x,y)\,\mathrm{d}y, \qquad p := \hat{q} + \hat{p}_{b,0} = q + p_{b,0}. \tag{3.121}$$

In order to express w explicitly by means of cell problems, we have to use two different types of cell problems. The first cell problems take into account the source terms of the momentum equation of the instationary two-pressure Stokes equation, namely, the macroscopic pressure and the force term. The solutions $(\hat{\zeta}_i(s,x;t,y), \hat{\pi}_i(s,x;t,y))$ of the cell problems depend on the parameters $s \in (0,T)$ and $x \in \Omega$, which model the initial time and the macroscopic position, respectively, and on the variables $t \in (s,T)$ and $y \in Y^*$, which represent the time and microscopic position.

Cell problems for the permeability coefficient in the reference coordinates

$$\begin{aligned}
\partial_t \hat{\zeta}_i - \nabla_y \hat{\zeta}_i^\top \Psi_0^{-1} \partial_t \psi_0 - J_0^{-1} \operatorname{div}_y(A_0 \Psi_0^{-\top} \nabla_y \hat{\zeta}_i) + \Psi_0^{-\top} \nabla \hat{\pi}_i &= 0 && \text{in } (s,T) \times Y^*,\\
J_0 \operatorname{div}_y(A_0 \hat{\zeta}_i) &= 0 && \text{in } (s,T) \times Y^*,\\
\hat{\zeta}_i &= 0 && \text{on } (s,T) \times \Gamma,\\
y \mapsto \hat{\zeta}_i, \hat{\pi}_i & && Y\text{-periodic},\\
\hat{\zeta}_i(t=s) &= e_i && \text{in } Y^*,
\end{aligned} \tag{3.122}$$

for parameters $(s,x) \in (0,T) \times \Omega$.

The second cell problem is (3.123) it has a solution $\hat{\zeta}^{\text{in}}(t,x,y), \hat{\pi}^{\text{in}}(t,x,y)$ and takes into account the initial condition of the instationary two-pressure Stokes equation.

Cell problems for the initial values in the reference coordinates

$$\begin{aligned}
\partial_t \hat{\zeta}^{\text{in}} - (\nabla_y \hat{\zeta}^{\text{in}})^\top \Psi_0^{-1} \partial_t \psi_0 - J_0^{-1} \operatorname{div}_y(\mu A_0 \Psi_0^{-\top} \nabla_y \hat{\zeta}^{\text{in}}) + \Psi_0^{-\top} \nabla \hat{\pi}^{\text{in}} &= 0 && \text{in } (0,T) \times Y^*,\\
J_0 \operatorname{div}_y(A_0 \hat{\zeta}^{\text{in}}) &= 0 && \text{in } (0,T) \times Y^*,\\
\hat{\zeta}^{\text{in}} &= 0 && \text{on } (0,T) \times \Gamma,\\
y \mapsto \hat{\zeta}^{\text{in}}, \hat{\pi}^{\text{in}} & && Y\text{-periodic},\\
\hat{\zeta}^{\text{in}}(t=0) &= \hat{v}_0^{\text{in}} && \text{in } Y^*,
\end{aligned} \tag{3.123}$$

for parameter $x \in \Omega$.

Using these cell problems, we can separate the micro- and macroscopic variable in (3.108) and obtain the following Darcy law with memory for evolving microstructure.

Darcy law with memory for evolving microstructure

$$\begin{aligned} v(t,x) &= v^{\text{in}}(t,x) + \frac{1}{\mu}\int_0^t K(s,x,t)(f - \nabla p)(s,x)\,\mathrm{d}s && \text{in } (0,T)\times\Omega, \\ \operatorname{div}(v(t,x)) &= -\partial_t\Theta && \text{in } (0,T)\times\Omega, \\ p(t,x) &= p_{b,0}(t,x) && \text{on } (0,T)\times\partial\Omega, \end{aligned} \tag{3.124}$$

where the permeability coefficient K is given by

$$K_{ji}(t,s,x) := \int_{Y^*} J_0(t,x,y)\hat{\zeta}_i(s,x;t,y)\cdot e_j\,\mathrm{d}y, \tag{3.125}$$

for $i,j \in \{1,\dots,n\}$ and the contribution v^{in} of the initial values by

$$v^{\text{in}}(t,x) := \int_{Y^*} J_0(t,x,y)\hat{\zeta}^{\text{in}}(t,x,y)\,\mathrm{d}y.$$

Since the initial values in (3.122) are not compatible with the boundary condition, we cannot expect that the cell problems have solutions with $\partial_t\zeta_t(s,x,\cdot,\cdot) \in L^2((s,T);L^2(Y^*)^n)$ and $\hat{\pi}_i(s,x,\cdot,\cdot) \in L^2_0((s,T);L^2_0(Y^*))$. Thus, we cannot use the same solution concept as we have used for the instationary two-pressure Stokes equation in (3.109). Instead, we look for a solution of (3.122) with less regular time-derivative and only distributional pressures. After a substitution of $\hat{\zeta}_i$ by $\hat{u}_i = A_0\hat{\zeta}_i$ in (3.122) and multiplication of the first equation by $\Psi_0^\top$, we obtain a time independent divergence condition as already in the ε-scaled Stokes equation. Then, the Leibniz rule can be employed to fit (3.122) in the setting of [Zim21, Chapter 7.1], which provides a well-posed weak formulation with a unique solution.

Then, one can rewrite

$$\hat{w}_0(t,x,y) = \int_0^t \sum_{i=1}^n \hat{\zeta}_i(s,x;t,y)(f_i(s,x) - \partial_{x_i}p(s,x)\,\mathrm{d}s, \tag{3.126}$$

which leads to (3.124)–(3.125).

Back-transformation of the cell problems

We transform the cell problems back to the moving cell domains, i.e.

$$\hat{\zeta}_i(s,x;t,y) = \zeta_i(s,x;t,\psi_0(t,x,y)), \qquad \hat{\pi}(s,x;t,y) = \pi_i(s,x;t,\psi_0(t,x,y))$$

solve

Cell problems for the permeability coefficient

$$
\begin{aligned}
\partial_t \zeta_i - \operatorname{div}_y(\nabla_y \zeta_i) + \nabla \pi_i &= 0 && \text{for } t \in (s,T), y \in Y^*(t,x),\\
\operatorname{div}_y(\zeta_i) &= 0 && \text{for } t \in (s,T), y \in Y^*(t,x),\\
\zeta_i &= 0 && \text{for } t \in (s,T), y \in \partial\Gamma(t,x),\\
y \mapsto \zeta_i, \pi_i & && Y\text{-periodic},\\
\zeta_i(t=s) &= e_i && \text{in } Y^*.
\end{aligned}
\tag{3.127}
$$

Then, the permeability tensor can be equivalently written by

$$K_{ji}(t,s,x) := \int_{Y^*} \zeta_i(s,x;t,y) \cdot e_j \,\mathrm{d}y.$$

The transformation of the cell problems for the initial values shows that

$$\hat{\zeta}_i(s,x;t,y) = \zeta_i(s,x;t,\psi_0(t,x,y)), \qquad \hat{\pi}(s,x;t,y) = \pi_i(s,x;t,\psi_0(t,x,y))$$

solve

Cell problems for the initial values

$$
\begin{aligned}
\partial_t \zeta^{\text{in}} - \operatorname{div}_y(\mu \nabla_y \zeta^{\text{in}}) + \nabla \pi^{\text{in}} &= 0 && \text{for } t \in (0,T), y \in Y^*(t,x),\\
\operatorname{div}_y(\zeta^{\text{in}}) &= 0 && \text{for } t \in (0,T), y \in Y^*(t,x),\\
\zeta_i &= 0 && \text{for } t \in (0,T), y \in \partial\Gamma(t,x),\\
y \mapsto \zeta_i, \pi_i & && Y-\text{periodic},\\
\zeta_i^{\text{in}}(t=0) &= v_0^{\text{in}} && \text{in } Y^*,
\end{aligned}
\tag{3.128}
$$

which gives

$$v^{\text{in}}(t,x) := \int_{Y^*(t,x)} \zeta^{\text{in}}(t,x,y) \,\mathrm{d}y.$$

Further discussion of the Darcy law with memory

In the instationary two-pressure Stokes equation, the force term and the macroscopic pressure contribute as source terms in the momentum equation. Due to the memory structure of the resulting Darcy law, i.e. the integration over the time interval $(0,t)$, these source terms become initial values in the cell problems. Correspondingly, the cell problems do not model the fluid velocity but its acceleration. One should be aware of this if one wants to formulate the equations with unites, since the Stokes equations structure of the cell equations could be misleading otherwise.

In the case of a stationary domain the cell problems can be integrated with respect to time, which translates the initial condition into a source term for the momentum equation. This integration changes the units accordingly and one has to define the permeability tensor K by means of the time derivative of the solution of these new cell problems instead of by the solution itself in the definition. The resulting equations are presented for instance in [Hor97, Chapter 3.2]. This approach can not be used directly for the case of a time-dependent domain since the integration and differentiation of differential equations with time-dependent coefficients leads to additional terms.

Chapter 4.

Reaction–diffusion problem with coupled evolving microstructure

This chapter is based on [WP23, D. Wiedemann and M. A. Peter *Homogenisation of local colloid evolution induced by reaction and diffusion*, Nonlinear Analysis **227** (2023), 113168] and is devoted to the homogenisation of a reaction–diffusion process with coupled microstructure evolution. In contrast to the previous chapter, we do not consider an a-priori given evolution of the microstructure. Instead, the evolution of the domain is coupled to the solution of the reaction–diffusion equation, leading to a free boundary problem. Nevertheless, we can apply the transformation approach of Chapter 2. The microscopic domain is given by ε-scaled periodically distributed spherical obstacles with evolving radii. Concentration-dependent reactions model a precipitation and dissolution process at the interface of the obstacles and couple the domain evolution with the unknown of a reaction–diffusion equation.

To account for the unknown evolving microstructure, we transform the equations by a generic coordinate transformation. This results in a highly non-linear system including a partial differential equation for the unknown concentration $\hat{u}_\varepsilon$ and ordinary differential equations for the radii r_ε, which describe the spherical obstacles. We show the existence of a solution by means of Schauder's fixed-point theorem and the uniqueness by energy estimates. Due to the non-linear structure of the problem, strong compactness results become necessary for the homogenisation. For the unknown $\hat{u}_\varepsilon$ and its spatial gradient $\nabla \hat{u}_\varepsilon$, we obtain uniform a-priori bounds by energy estimates. However, we do not obtain a uniform bound for the time-derivative $\partial_t \hat{u}_\varepsilon$ and, therefore, cannot apply the Aubin–Lions lemma. Instead, we use a Steklov average in order to provide additional control over $\hat{u}_\varepsilon$ with respect to time. Then, we can use Simon–Kolmogorov's compactness argument to infer strong convergence for the unknown $\hat{u}_\varepsilon$. With the strong convergence of $\hat{u}_\varepsilon$, we deduce strong convergence for the radii by comparing the ordinary differential equations for the radii with the limit equation. Having the strong convergence of the radii, we can can show the convergence of the transformation mappings in the sense of Chapter 4.

Having all necessary two-scale compactness results, we can pass to the homogenisation limit in the transformed coordinates. By separating the micro- and macroscopic variables, we arrive at a homogenised reactive transport problem. We also translate the equations back to the locally upscaled microstructure, which leads to a transformation-independent homogenisation result. This limit reactive transport system consists of a reaction–diffusion equation and an ordinary differential equation at each macroscopic point modelling the local upscaled microstructure evolution. On the one hand the growth rate for the concentration of the reaction–diffusion equation is scaled by the evolution of the local upscaled porosity and the local effective diffusivity is adapted to the local microstructure using cell problems. On the other hand, the ordinary differential equations that describe the evolution of the microstructure and the porosity depend on the local microstructure. Thus, the resulting reactive transport system couples the micro- and macroscopic processes.

This chapter is based on the results of [WP23] but differs from it in the microscopic existence result. In [WP23], Banach's fixed-point theorem was used for the existence result, which also provides the uniqueness of the solution. However, the contraction property requires delicate estimates, so we present a different more elegant argument here. In particular, the contraction property was shown for small and ε-dependent time intervals. Thus, the existence proof for the whole time interval required the concatenation of solutions for small intervals and additional estimates to ensure that the solution does not blow up in finite time.

The homogenisation of a similar problem was also considered in [GP23] using also the transformation approach. There, the existence of a solution for the microscopic problem was shown by Rothe's method and similar a-priori estimates were derived. In order to prove strong compactness results, uniform estimates for the shifts of the radii with respect to space and time were shown. This leads to the strong convergence of the radii and the strong two-scale convergence of the corresponding transformation coefficient, which were used to infer the strong convergence of the unknown for the reaction–diffusion equation.

This chapter is structured as follows: in Section 4.1, we present the microsopic evolving domain and derive a coupling with the reaction–diffusion process based on the law of conservation of mass. In Section 4.2, we construct a generic parameterisable transformation to the upscaled reference cell. Using this transformation, we transform the ε-scaled perforated domain to a periodically perforated domain. For this system, we show the existence and uniqueness of a solution by means of a fixed-point argument in Section 4.3. From the existence proof, we extract uniform a-priori estimates on the solution of the reaction–diffusion problem and its spatial derivative. In Section 4.4, we pass to the limit $\varepsilon \to 0$ in the system of equations. Then, we separate the micro- and macroscopic variables and transform the equations back leading to a transformation-independent reactive transport system.

4.1. The microscopic model in the evolving domain

Microscopic geometry

Let $T > 0$, $\Omega \subset \mathbb{R}^n$ for $n \in \mathbb{N}$ be a domain such that Ω consists of entire ε-scaled cells $Y = (0,1)^n$, i.e. $\Omega = \operatorname{int}(\bigcup\limits_{k \in I_\varepsilon} \varepsilon k + \varepsilon \overline{Y})$ for all $\varepsilon > 0$, where $I_\varepsilon \subset \mathbb{Z}^n$ and ε is a positive monotone sequence $\varepsilon = (\varepsilon_n)_{n\in\mathbb{N}}$ which converges to 0.

The pore structure is assumed to be given by removing one spherical obstacle in each ε-scaled cell $\varepsilon k + \varepsilon Y$ for $k \in I_\varepsilon$. The obstacles are centred in the ε-scaled cell and have radii of order ε, which depend on the time. Thus, the ε-scaled perforated domain is defined by

$$\Omega_\varepsilon(t) := \Omega \backslash \varepsilon \bigcup_{k \in I_\varepsilon} \overline{B_{r_{\varepsilon,k}(t)}}(k + \mathfrak{m}) \tag{4.1}$$

where $\mathfrak{m} := (0.5, \dots, 0.5)^\top$ is the center of the reference cell and $r_{\varepsilon,k}(t)$ is the ε^{-1}-scaled radius of the solid obstacle located in the cell $\varepsilon k + \varepsilon Y$ at time $t \in [0, T]$. In order to avoid topological changes, the radii can only grow or shrink between given bounds, i.e.

$$r_{\min} \leqslant r_{\varepsilon,k}(t) \leqslant r_{\max} \qquad \text{for all } k \in I_\varepsilon, t \in [0, T]$$

and $0 < r_{\min} < r_{\max} < 0.5$. Thus, the interface of the pore space with the obstacle in each cell $k \in I_\varepsilon$ is given for each point in time $t \in [0, T]$ by

$$\Gamma_{\varepsilon,k}(t) = \partial \varepsilon B_{r_{\varepsilon,k}(t)}(k + \mathfrak{m}) = \partial B_{\varepsilon r_{\varepsilon,k}(t)}(\varepsilon(k + \mathfrak{m})).$$

We denote the union of these interfaces for a point in time $t \in [0, T]$ by $\Gamma_\varepsilon(t)$, i.e.

$$\Gamma_\varepsilon(t) := \partial\Omega_\varepsilon(t) \backslash \partial\Omega = \bigcup_{k \in I_\varepsilon} \Gamma_{\varepsilon,k}(t).$$

Then, the in time non-cylindrical pore space and its interface are given by

$$\mathcal{Q}_\varepsilon^T := \bigcup_{t \in [0,T]} \{t\} \times \Omega_\varepsilon(t), \qquad H_\varepsilon^T := \bigcup_{t \in [0,T]} \{t\} \times \Gamma_\varepsilon(t), \quad H_{\varepsilon,k}^T := \bigcup_{t \in [0,T]} \{t\} \times \Gamma_{\varepsilon,k}(t). \tag{4.2}$$

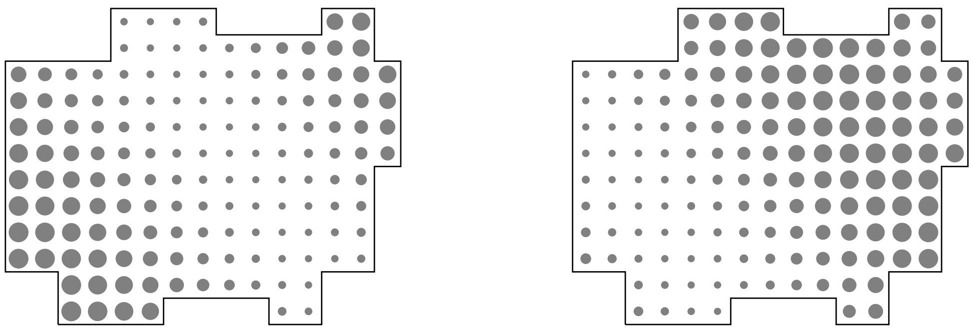

Figure 4.1.: Microscopic domain $\Omega_\varepsilon(t)$ in white for two different values of t.

Evolution equations

The evolution of the obstacles is motivated by concentration-dependent reaction kinetics at their interface. It models the dissolution and precipitation of the solid obstacle. The reaction rate $\varepsilon g(u_\varepsilon, r_{\varepsilon,k})$ depends on the concentration u_ε and the radius $r_{\varepsilon,k}$ and corresponds to the flux $j_{\varepsilon,k}(t,x)$ in normal direction through $\Gamma_{\varepsilon,k}(t)$, i.e.

$$j_\varepsilon(t,x)\cdot n(t,x) = \varepsilon g(u_\varepsilon(t,x), r_{\varepsilon,k}(t)) \qquad \text{for } t\in[0,T],\ k\in I_\varepsilon,\ x\in\Gamma_{\varepsilon,k}(t), \tag{4.3}$$

where n denotes the outer normal of $\Omega_\varepsilon(t)$. This flux leads to a growth or shrinkage of the solid obstacle. Under the assumption that the solid has a constant density c_s and remains spherical, the conservation of mass implies

$$\frac{\mathrm{d}}{\mathrm{d}t}|B_{\varepsilon r_{\varepsilon,k}(t)}(\varepsilon(k+\mathfrak{m}))|c_s = \int\limits_{\Gamma_{\varepsilon,k}(t)} j_\varepsilon(t,x)\cdot n(t,x)\,\mathrm{d}\sigma_x. \tag{4.4}$$

From elementary calculus, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}|B_{\varepsilon r_{\varepsilon,k}(t)}(\varepsilon(k+\mathfrak{m}))| = \partial_t V_n(\varepsilon r_{\varepsilon,k}(t)) = S_{n-1}(\varepsilon r_{\varepsilon,k}(t))\varepsilon\partial_t r_{\varepsilon,k}(t),$$

where $V_n(r)$ denotes the volume of the n-ball with radius r and $S_{n-1}(r)$ denotes the surface volume of the $(n-1)$-sphere with radius r. Combining the last three equations leads to the following ordinary differential equation for the radii

$$\partial_t r_{\varepsilon,k}(t) = \frac{1}{c_s} \fint\limits_{\Gamma_{\varepsilon,k}(t)} g(u_\varepsilon(t,x), r_{\varepsilon,k}(t,x))\,\mathrm{d}\sigma_x. \tag{4.5}$$

The loss or gain of mass in the solid region is accompanied by the opposed loss or gain of dissolved solute concentration in the pore region. This process is modeled by setting the flux $j_\varepsilon(t,x)$ at the moving interface $\Gamma_{\varepsilon,k}(t)$ equal to the sum of the diffusive and advective flow. The diffusive flux is modeled by Fick's law, i.e. $j_{\varepsilon,\mathrm{D}}(t,x) = -D\nabla u_\varepsilon(t,x)$. The advective flux is induced by the moving interface and is given by $j_{\varepsilon,\mathrm{A}}(t,x) = -v_{\Gamma_{\varepsilon,k}}(t,x)u(t,x)$, where $v_{\Gamma_{\varepsilon,k}}$ denotes the velocity of the interface. This advective flux can be understood in the following sense: When the carrier medium becomes solid and contains a higher concentration than the density c_s of the solid medium, then any excess dissolved concentration is pushed into the pore space. Thus, the total flux at the interface is $\Gamma_{\varepsilon,k}(t)$ for $t\in[0,T]$ and $k\in I_\varepsilon$ is given by

$$j_\varepsilon(t,x) = j_{\varepsilon,\mathrm{D}}(t,x) + j_{\varepsilon,\mathrm{A}}(t,x) = -D\nabla u_\varepsilon(t,x) - v_{\Gamma_{\varepsilon,k}}(t,x)u(t,x).$$

Together with the identification of the normal flux with the reaction rate in (4.3), we obtain the following boundary condition

$$(-D\nabla u_\varepsilon(t,x) - v_{\Gamma_{\varepsilon,k}}(t,x)u(t,x))\cdot n(t,x) = j_\varepsilon(t,x)\cdot n(t,x) = g(u_\varepsilon(t,x), r_{\varepsilon,k}(t,x)). \tag{4.6}$$

Using the fact that $v_{\Gamma_{\varepsilon,k}}(t,x) = -\varepsilon\partial_t r_{\varepsilon,k}(t)n(t,x)$, we can simplify

$$-v_{\Gamma_{\varepsilon,k}}(t,x)u_\varepsilon(t,x)\cdot n(t,x) = \varepsilon\partial_t r_{\varepsilon,k}(t)n(t,x)u_\varepsilon(t,x)\cdot n(t,x) = \partial_t r_{\varepsilon,k}(t)u_\varepsilon(t,x).$$

In the pore space $\Omega_\varepsilon(t)$, we model the transport with a reaction–diffusion equation and assume a homogeneous Neumann boundary condition at the outer boundary $\partial\Omega$. We complete the system with the boundary condition (4.6) at the interfaces $\Gamma_{\varepsilon,k}(t)$, the ordinary differential equation (4.5) for the radii and initial values $u_\varepsilon^{\text{in}}$ and $r_\varepsilon^{\text{in}}$ for the concentration and the radii, respectively. This leads to the following system with unknowns u_ε and $r_{\varepsilon,k}$ for $k \in I_\varepsilon$.

Microscopic reaction–diffusion equation with coupled domain evolution

$$\begin{aligned} \partial_t u_\varepsilon(t,x) - \operatorname{div}(D\nabla u_\varepsilon(t,x)) &= f(t,x) && \text{in } \mathcal{Q}_\varepsilon^T, \\ -D\nabla u_\varepsilon(t,x)\cdot n(t,x) + \varepsilon\partial_t r_{\varepsilon,k}(t) &= \varepsilon g(u_\varepsilon(t,x), r_{\varepsilon,k}(t,x)) && \text{on } H_{\varepsilon,k}^T, k \in I_\varepsilon, \\ -D\nabla u_\varepsilon(t,x)\cdot n(t,x) &= 0 && \text{on } (0,T)\times\partial\Omega, \\ u_\varepsilon(0,x) &= u_\varepsilon^{\text{in}}(x) && \text{in } \Omega_\varepsilon, \\ \partial_t r_{\varepsilon,k}(t) = \frac{1}{c_s} &\fint_{\Gamma_{\varepsilon,k}(t)} g(u_\varepsilon(t,z), r_{\varepsilon,k}(t,z))\,\mathrm{d}\sigma_z && \text{for } t\in(0,T), k\in I_\varepsilon, \\ r_{\varepsilon,k}(0) &= r_{\varepsilon,k}^{\text{in}} && \text{for } k \in I_\varepsilon, \end{aligned} \tag{4.7}$$

where $n(t,x)$ denotes the outer normal of $\Omega_\varepsilon(t)$ at $x \in \partial\Omega_\varepsilon(t)$ and $D > 0$ the diffusion coefficient. Moreover, the domain $\Omega_\varepsilon(t)$ and, thus, $\mathcal{Q}_\varepsilon^T$ as well as the interfaces $\Gamma_{\varepsilon,k}(t)$ in (4.7) are coupled with r_ε via (4.1)–(4.2).

We note that r_ε is a vector-valued function, i.e. $r_\varepsilon = (r_{\varepsilon,k})_{k\in I_\varepsilon} : [0,T] \to \mathbb{R}^{|I_\varepsilon|}$. For the sake of simplifying the notation and for stating the convergence results on r_ε, we abuse its notation. Namely, we identify r_ε with the piecewise constant function $r_\varepsilon : [0,T]\times\Omega \to [r_{\min}, r_{\max}]$, $r_\varepsilon(t,x) := r_{\varepsilon,k}(t)$ for $x \in \varepsilon(k+Y)$ with $k \in I_\varepsilon$. Similarly, we write for the initial values $r_\varepsilon^{\text{in}}(x) = r_{\varepsilon,k}^{\text{in}}$ for $x \in \varepsilon(k+Y)$ with $k \in I_\varepsilon$. We switch between these interpretations wherever it is more convenient.

Assumption 4.1. *We assume that:*

- *$f \in C([0,T]\times\overline{\Omega})$ is uniformly Lipschitz continuous with respect to x, i.e. there exists $L_f > 0$ such that*

$$|f(t,x_1) - f(t,x_2)| \leqslant L_f|x_1 - x_2| \tag{4.8}$$

  for all $x_1, x_2 \in \Omega$ and all $t \in [0,T]$.

- *$g : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is bounded and uniformly Lipschitz, i.e. there exist constants $C_g, L_g > 0$ such that*

$$\|g\|_{C(\mathbb{R}^2)} \leqslant C_g < \infty,$$

$$|g(u_1, r_1) - g(u_1, r_1)| \leqslant L_g(|u_1 - u_2| + |r_1 - r_2|)$$

for all $u_1, u_2, r_1, r_2 \in \mathbb{R}$. *Moreover, we assume that*

$$g(u, r) \geqslant 0 \quad \text{for } r \leqslant r_{min} \qquad \text{and} \qquad g(u, r) \leqslant 0 \quad \text{for } r \geqslant r_{max} \tag{4.9}$$

holds for all $u \in \mathbb{R}$.

- $r_\varepsilon^{\text{in}} \subset [r_{min}, r_{max}]^{|I_\varepsilon|}$, *i.e.* $r_\varepsilon^{\text{in}}$ *can be identified with a sequence of piecewise constant functions on* Ω *with values in* $[r_{min}, r_{max}]$. *Moreover, there exists* $r_0^{\text{in}} \in L^2(\Omega)$ *such that* $r_\varepsilon^{\text{in}} \to r_0^{\text{in}}$ *in* $L^2(\Omega)$.
- $u_\varepsilon^{\text{in}}$ *is a sequence in* $L^2(\Omega_\varepsilon(0))$ *such that* $u_\varepsilon^{\text{in}}(x) \xrightarrow{2} \chi_{Y^*_{r_0^{\text{in}}(x)}}(y) u_0(x)$ *for some* $u_0 \in L^2(\Omega)$ *for* $Y^*_{r_0^{\text{in}}(x)} := Y \backslash B_{r_0^{\text{in}}}(\mathfrak{m})$.

The assumption (4.9) ensures that the solution of the ordinary differential equation from (4.7) stays between the bounds $r_{\min}$ and $r_{\max}$, i.e. $r_{\varepsilon,k}(t) \subset [r_{\min}, r_{\max}]$ for every $t \in [0, T]$ and every $k \in I_\varepsilon$.

4.2. Transformation to a periodic reference domain

In order to transform $\Omega_\varepsilon(t)$ to a periodic reference domain Ω_ε, we use some generic diffeomorphism $\psi(r; \cdot) : \overline{Y} \mapsto \overline{Y}$ which is defined on the reference cell $Y = [0, 1]^n$ and is parameterised by $r \in [r_{\min}, r_{\max}]$. It maps the reference cell with obstacle of radius $R \in [r_{\min}, r_{\max}]$ onto the cell with obstacle radius r and it satisfies the following properties:

$$\begin{aligned} &\psi(r; \cdot) : \overline{Y} \to \overline{Y} && \text{is bijective,} \\ &\psi(r; \overline{B_R(\mathfrak{m})}) = \overline{B_r(\mathfrak{m})}, && \\ &\psi(r; y) = y && \text{for } \in Y \backslash B_{r_{\max}+\delta}(\mathfrak{m}), \end{aligned} \tag{4.10}$$

for all $r \in [r_{\min}, r_{\max}]$ and a safety constant $0 < \delta < \max\{r_{\min}/2, (0.5 - r_{\max})/2\}$, which is necessary in order to achieve a smooth transition between the deformed region and the region where $\psi(r; \cdot)$ is the identity. The last condition ensures that ψ_r is the identity close to the boundary Y and allows us to glue transformations $\psi(r; \cdot)$ next to each other when we use it in order to define the ε-scaled transformations ψ_ε. Moreover, we assume that $\psi(r; \cdot)$ is also smooth with respect to the parameter r, i.e. $((r; y) \mapsto \psi(r; y)) \in C^\infty([r_{\min}, r_{\max}] \times \overline{Y})^n$ and satisfies the following uniform bounds

$$\begin{aligned} |\partial_r \psi(r; y)| + |\partial_y \psi(r; y)| + |\partial_r \partial_y \psi(r; y)| + |\partial_r \partial_r \partial_y \psi(r; y)| &\leqslant C, \\ \det(\partial_y \psi(r; y)) &\geqslant c_J > 0 \end{aligned} \tag{4.11}$$

for all $(r, y) \in [r_{\min}, r_{\max}] \times \overline{Y}$.

Employing the radial symmetry, we can construct the generic cell transformation $\psi(r;\cdot)$ by means of a mapping $\Phi(r;\cdot) : [0,\infty) \to [0,\infty)$, via

$$\psi(r;\cdot)(y) = \begin{cases} \mathfrak{m} + \Phi(r;\cdot)(\|y-\mathfrak{m}\|)\frac{y-\mathfrak{m}}{\|y-\mathfrak{m}\|} & \text{for } y \neq \mathfrak{m}, \\ \mathfrak{m} & \text{for } y = \mathfrak{m}, \end{cases} \tag{4.12}$$

where (4.10) can be reduced to

$$\begin{aligned} &\Phi(r;\cdot) : [0,\infty) \to [0,\infty) && \text{is bijective,} \\ &\Phi(r;[0,R]) = [0,r], && \\ &\Phi(r;y) = y && \text{for } y \geqslant r_{\max} + \delta, \end{aligned} \tag{4.13}$$

for all $r \in [r_{\min}, r_{\max}]$. Furthermore, the assumptions on the regularity for ψ can be reduced to the regularity of

$$\begin{aligned} &((r,y) \mapsto \psi(r;y)) \in C^\infty([r_{\min}, r_{\max}] \times [0,\infty)), \\ &\Phi(r;y) = y \qquad \text{for } y \leqslant r_{\min} - \delta, r \in [r_{\min}, r_{\max}], \end{aligned}$$

where the latter condition yields $\psi(r;y) = y$ for $y \in B_{r_{\min}-\delta}(\mathfrak{m})$ and, thus, ensures the regularity of $\psi(r;\cdot)$ at $\mathfrak{m}$ and also the uniform bounds in a neighbourhood of $\mathfrak{m}$.

Moreover, outside this neighbourhood of $\mathfrak{m}$ the uniform bounds (4.11) can be deduced from

$$\begin{aligned} |\partial_r\Phi(r;y)| + |\partial_y\Phi(r;y)| + |\partial_r\partial_y\Phi(r;y)| + |\partial_r\partial_r\partial_y\Phi(r;y)| &\leqslant C, \\ \partial_y\Phi(r;y) &\geqslant c \end{aligned} \tag{4.14}$$

for all $(r,y) \in [r_{\min}, r_{\max}] \times \overline{Y}$ and some $c > 0$ after a change to the spherical coordinates. Note that the translation of (4.13) into (4.14) leads to an a-priori y-dependent constant on the right-hand side. However, since we consider $\Phi(r;\cdot)$ only in a bounded region with positive distance to the origin $\mathfrak{m}$, we can omit this y-dependence.

Indeed the precise choice of Φ is not important for the arguments later and it can be constructed in several ways. Nevertheless, for sake of completeness, we present one construction. For this, we construct the following family of continuous and piecewise affine functions $\tilde{\Phi}(r;\cdot) : \mathbb{R} \to \mathbb{R}$ first:

$$\tilde{\Phi}(r;y) := \begin{cases} y & \text{for } y \leqslant r_{\min} - 2\tilde{\delta}, \\ r_{\min} - 2\tilde{\delta} + c_1(r)(y - (r_{\min} - 2\tilde{\delta})) & \text{for } r_{\min} - 2\tilde{\delta} \leqslant y \leqslant R - \tilde{\delta}, \\ r + (y - R) & \text{for } R - \tilde{\delta} \leqslant y \leqslant R + \tilde{\delta}, \\ r_{\max} + 2\tilde{\delta} + c_2(r)(y - (r_{\max} + 2\tilde{\delta})) & \text{for } R + \tilde{\delta} \leqslant y \leqslant r_{\max} + 2\tilde{\delta}, \\ y & \text{for } r_{\max} + 2\tilde{\delta} \leqslant y, \end{cases} \tag{4.15}$$

for $r \in [r_{\min}, r_{\max}]$ and $\tilde{\delta} := \delta/3$ with slopes

$$c_1(r) = \frac{r - \tilde{\delta} - (r_{\min} - 2\tilde{\delta})}{R - \tilde{\delta} - (r_{\min} - 2\tilde{\delta})} = \frac{(r - r_{\min}) + \tilde{\delta}}{(R - r_{\min}) + \tilde{\delta}},$$
$$c_2(r) = \frac{r_{\max} + 2\tilde{\delta} - (r + \tilde{\delta})}{r_{\max} + 2\tilde{\delta} - (R + \tilde{\delta})} = \frac{(r_{\max} - r) + \tilde{\delta}}{(r_{\max} - R) + \tilde{\delta}}.$$

An illustration of $\tilde{\Phi}(r; \cdot)$ is given in Figure 4.2.

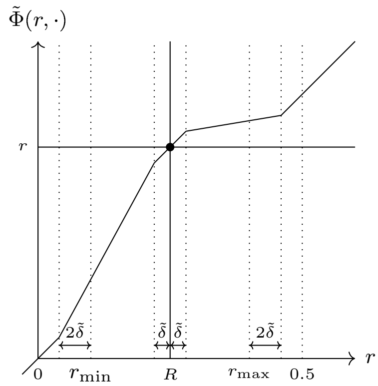

Figure 4.2.: $\tilde{\Phi}(r; \cdot)$

We note that

$$\partial_y \tilde{\Phi}_r(y) \in \{1, c_1(r), c_2(r)\},$$
$$\frac{\tilde{\delta}}{(r_{\max} - r_{\min}) + \tilde{\delta}} \leqslant 1, c_1(r), c_2(r) \leqslant \frac{(r_{\max} - r_{\min}) + \tilde{\delta}}{\tilde{\delta}}$$

for all $r, R \in [r_{\min}, r_{\max}]$ and a.e. $y \in \mathbb{R}$, which bounds the derivatives from below and above. Moreover, we observe that, for every fixed $y \in \mathbb{R}$, $r \mapsto \tilde{\Phi}(r; y) \in C^\infty([r_{\min}, r_{\max}]; \mathbb{R})$ and the derivatives are uniformly bounded with respect to y, i.e. for every $l \in \mathbb{N}$, there exists a constant C_l such that

$$|\partial_r^l \tilde{\Phi}(r; y)| \leqslant C_l$$

for all $r \in [r_{\min}, r_{\max}]$ and all $y \in \mathbb{R}$.

However, $\tilde{\Phi}$ is only Lipschitz continuous with respect to y while $r \mapsto \partial_y \tilde{\Phi}(r; y)$ is not even continuous. By means of a convolution with a standard mollifier with respect to the second argument of $\tilde{\Phi}$, the function becomes smooth, while other desired properties remain preserved. Namely, we define

$$\Phi(r; y) := \int_{\mathbb{R}} \tilde{\Phi}(r; x)\, \eta(y - x)\, \mathrm{d}x,$$

where

$$\eta(x) := \begin{cases} \left(\int\limits_{\mathbb{R}} \exp\left(\frac{-1}{1-|y/\tilde{\delta}|^2} \right) \mathrm{d}y \right)^{-1} \exp\left(\frac{-1}{1-|x/\tilde{\delta}|^2} \right) & \text{for } |x| < \tilde{\delta}, \\ 0 & |x| \geqslant \tilde{\delta}. \end{cases}$$

First, we note that

$$\Phi(r; R) := \int\limits_{\mathbb{R}} \tilde{\Phi}(r; x)\, \eta(R - x)\, \mathrm{d}x = \int\limits_{R-\tilde{\delta}}^{R+\tilde{\delta}} (r + (x - R))\, \eta(R - x)\, \mathrm{d}x = r,$$

$$\Phi(r; y) = \int\limits_{y-\tilde{\delta}}^{y+\tilde{\delta}} x\, \eta(r - x)\, \mathrm{d}x = y \qquad \text{for } y \leqslant r_{\min} - 3\tilde{\delta} = r_{\min} - \delta,$$

$$\Phi(r; y) = \int\limits_{y-\tilde{\delta}}^{y+\tilde{\delta}} x\, \eta(r - x)\, \mathrm{d}x = y \qquad \text{for } y \geqslant r_{\max} + 3\tilde{\delta} = r_{\max} - \delta,$$

which shows the equalities in (4.13) as well as the equality thereafter. Moreover, the essential bound of $\partial_y \Phi_r \geqslant \frac{\tilde{\delta}}{(r_{\max} - r_{\min}) + \tilde{\delta}}$ from below, is preserved during the convolution since

$$\partial_y \Phi(r; y) = \int\limits_{\mathbb{R}} \partial_y \tilde{\Phi}(r; x)\, \eta(y - x)\, \mathrm{d}x \geqslant \int\limits_{\mathbb{R}} \frac{\tilde{\delta}}{(r_{\max} - r_{\min}) + \tilde{\delta}}\, \eta(y - x)\, \mathrm{d}x$$
$$\geqslant \frac{\tilde{\delta}}{(r_{\max} - r_{\min}) + \tilde{\delta}}.$$

Hence, Φ is strictly monotonically increasing and, in particular, bijective. It remains to show the regularity and the uniform bounds for the derivatives. For this, we deduce iteratively

$$\partial_r^k \partial_y^l \Phi(r; y) = \int\limits_{\mathbb{R}} \partial_r^k \Phi(r; x) \partial_y^l \eta(y - x)\, \mathrm{d}x$$

for all $k, l \in \mathbb{N}$. Since $\partial_r^k \Phi(r; y)$ and $\partial_y^l \eta$ are bounded in $[r_{\min}, r_{\max}] \times \mathbb{R}$ for every $k, l \in \mathbb{N}$, we obtain the first estimate of (4.14) and can iteratively infer the continuity of the derivatives leading to the regularity $\psi \in C^\infty([r_{\min}, r_{\max}] \times \overline{Y})^n$.

The cell displacement mapping

Having the cell transformation ψ, we define the corresponding displacement mapping by

$$\check{\psi}(r; y) = \psi(r; y) - y,$$

which is zero in $\overline{Y}\backslash B_{r_{\max}+\delta}(\mathfrak{m})$ due to (4.10). Hence, the Y-periodic extension of $\breve{\psi}$ is smooth. We identify the displacement mapping $\breve{\psi}$ with its Y-periodic extension in the following. Moreover, the uniform upper bound on the derivative of ψ from (4.11) can be transferred onto $\breve{\psi}$, i.e.

$$\begin{aligned} |\breve{\psi}(r;y)| + |\partial_r\breve{\psi}(r;y)| + |\partial_r\partial_r\breve{\psi}(r;y)| + |\partial_y\breve{\psi}(r;y)| + |\partial_r\partial_y\breve{\psi}(r;y)| \leqslant C, \\ |\partial_r\partial_r\partial_y\breve{\psi}(r;y)| + |\partial_y\partial_y\breve{\psi}(r;y)| + |\partial_r\partial_y\partial_y\breve{\psi}(r;y)| + |\partial_r\partial_r\partial_y\partial_y\breve{\psi}(r;y)| \leqslant C \end{aligned} \tag{4.16}$$

for every $(r, y) \in [r_{\min}, r_{\max}] \times \mathbb{R}^n$.

Further properties of the transformation

For a time-dependent radius $r : [0,T] \mapsto [r_{\min}, r_{\max}]$ and $y \in \partial B_R(\mathfrak{m})$, we note that

$$\begin{aligned} \partial_t\psi(r(t), y) &= \partial_t\Phi(r(t); \|y-\mathfrak{m}\|)\frac{y-\mathfrak{m}}{\|y-\mathfrak{m}\|} = \partial_t\Phi(r(t); R)\frac{y-\mathfrak{m}}{\|y-\mathfrak{m}\|} = \partial_t r(t)\frac{y-\mathfrak{m}}{\|y-\mathfrak{m}\|} \\ &= \partial_t r(t)\hat{n}(y), \end{aligned}$$

where $\hat{n}(y)$ denotes the outer normal of $\overline{Y}\backslash B_R(\mathfrak{m})$ at $y \in \partial B_R(\mathfrak{m})$.

Moreover, by transforming the surface integral

$$S_{n-1}(r) = \int_{\partial B_r(\mathfrak{m})} 1 \,\mathrm{d}\sigma_y = \int_{\partial B_R(\mathfrak{m})} \|\operatorname{Adj}(\partial_y\psi(r;y))\hat{n}\| 1 \,\mathrm{d}\sigma_y$$

and using the radial symmetry of $\psi(r;\cdot)$, we obtain

$$|\operatorname{Adj}(\partial_y\psi(r;y))\hat{n} \;= \frac{S_{n-1}(r)}{S_{n-1}(R)} = \frac{r^{n-1}}{R^{n-1}} = \left(\frac{r}{R}\right)^{n-1} \tag{4.17}$$

for $y \in \partial B_R(\mathfrak{m})$.

ε-scaling of the transformation

For a fixed reference radius $R \in [r_{\min}, r_{\max}]$, we define the periodic reference domain by

$$\Omega_\varepsilon := \Omega\backslash\varepsilon \bigcup_{k\in I_\varepsilon} \overline{B_R(k+\mathfrak{m})}$$

with interfaces

$$\Gamma_{\varepsilon,k} := \partial\varepsilon B_R(k+\mathfrak{m}) = \partial B_{\varepsilon R}(\varepsilon(k+\mathfrak{m})), \qquad \Gamma_\varepsilon := \bigcup_{k\in I_\varepsilon} \Gamma_{\varepsilon,k}.$$

Let $\Omega_\varepsilon(t)$ be given by radii $r_\varepsilon(t)$, then we map Ω_ε onto $\Omega_\varepsilon(t)$ by scaling and shifting the mapping $\psi(r;\cdot)$ for every ε-scaled cell. Since $\psi(r;\cdot)$ is the identity near the boundary of

Y, we can glue these mappings smoothly together. Hence, we define

$$\psi_\varepsilon(t,x) := [x]_{\varepsilon,Y} + \varepsilon\psi(r_\varepsilon(t,x), \{x\}_{\varepsilon,Y}), \tag{4.18}$$

where $[x]_{\varepsilon,Y} = \sum_{i=1}^{n} \varepsilon\lfloor \frac{x_i}{\varepsilon} \rfloor e_i$ and $\{x\}_{\varepsilon,Y} = \varepsilon^{-1}(x - [x]_{\varepsilon,Y})$ for the Euclidian unit vectors e_i.

The resulting displacement $\check{\psi}_\varepsilon(t,x) := \psi_\varepsilon(t,x) - x$ can be also expressed by the displacement $\check{\psi}(r;y) = \psi(r;y) - y$ of the generic cell transformation

$$\begin{aligned} \psi_\varepsilon(t,x) - x &= [x]_{\varepsilon,Y} + \varepsilon\psi(r_\varepsilon(t,x), \{x\}_{\varepsilon,Y}) - x \\ &= [x]_{\varepsilon,Y} + \varepsilon\check{\psi}(r_\varepsilon(t,x), \{x\}_{\varepsilon,Y}) + \varepsilon\{x\}_{\varepsilon,Y} - x \\ &= \varepsilon\check{\psi}(r_\varepsilon(t,x), \{x\}_{\varepsilon,Y}) = \varepsilon\check{\psi}(r_\varepsilon(t,x), x/\varepsilon). \end{aligned} \tag{4.19}$$

Since $\check{\psi} = 0$ in a neighbourhood of ∂Y, no jumps arise at $\varepsilon k + \varepsilon\partial Y$, for $k \in I_\varepsilon$, although r_ε and $\{x\}_{\varepsilon,Y}$ are not continuous there. Hence, $\check{\psi}_\varepsilon$ and ψ_ε are smooth with respect to x.

Uniform a-priori estimates for ψ_ε

In the following, we derive estimates for ψ_ε and its derivatives, which are independent of the radii r_ε as long as the radii stay between certain bounds. These bounds are ensured by the ordinary differential equations that define r_ε. Thus, we can derive uniform estimates on the a-priori unknown coefficients, which arise due to the coordinate transformation of (4.7). Moreover, we show that these coefficients depend Lipschitz regularly on the radii, which becomes useful for proving the uniqueness of the solution of the system.

Lemma 4.2. *Let $r_\varepsilon \in C^{0,1}([0,T];[r_{min}, r_{max}])^{|I_\varepsilon|}$ with $\|\partial_t r_{\varepsilon,k}\|_{L^\infty(0,T)} \leqslant C_g c_s^{-1}$ for every $k \in I_\varepsilon$. Let ψ_ε be given by (4.18). Then, $\psi_\varepsilon \in C^{0,1}([0,T];C^2(\overline{\Omega}))$ and $\psi_\varepsilon(t,\cdot)$ is bijective from $\overline{\Omega}$ onto $\overline{\Omega}$ with $\psi_\varepsilon(t,\Omega_\varepsilon) = \Omega_\varepsilon(t)$. Moreover, there exist constants C, c_J, which are independent of r_ε such that*

$$\begin{aligned} \varepsilon^{-1}\|\psi_\varepsilon - x\|_{C([0,T]\times\overline{\Omega})} + \|\partial_x\psi_\varepsilon\|_{C([0,T]\times\overline{\Omega})} + \varepsilon\|\partial_x\partial_x\psi_\varepsilon\|_{C([0,T]\times\overline{\Omega})} &\leqslant C, \\ J_\varepsilon(t,x) &\geqslant c_J, \\ \varepsilon^{-1}\|\partial_t\psi_\varepsilon\|_{L^\infty(0,T;C(\overline{\Omega}))} + \|\partial_t\partial_x\psi_\varepsilon\|_{L^\infty(0,T;C(\overline{\Omega}))} + \varepsilon\|\partial_t\partial_x\partial_x\psi_\varepsilon\|_{L^\infty(0,T;C(\overline{\Omega}))} &\leqslant C. \end{aligned}$$

Proof. In order to derive the estimates, we employ the identity $\check{\psi}_\varepsilon(t,x) = \psi_\varepsilon(t,x) - x$ and the identification $\check{\psi}(r;y) = \psi_\varepsilon(r;y) - y$ given in (4.19). Then, we can compute with the chain rule and the uniform bounds on $\check{\psi}$ and its derivatives

$$\begin{aligned} &\|\check{\psi}_\varepsilon\|_{C([0,T]\times\overline{\Omega})} \leqslant \varepsilon\|\check{\psi}\|_{C([r_{\min},r_{\max}]\times Y)} \leqslant \varepsilon C, \\ &\|\partial_t\check{\psi}_\varepsilon\|_{L^\infty(0,T;C(\overline{\Omega}))} \leqslant C\|\partial_r\check{\psi}\|_{C([r_{\min},r_{\max}]\times Y)}\|\partial_t r_\varepsilon\|_{L^\infty((0,T)\times\Omega)} \leqslant C, \\ &\|\partial_x\check{\psi}_\varepsilon\|_{C([0,T]\times\overline{\Omega})} \leqslant \|\partial_y\check{\psi}\|_{C([r_{\min},r_{\max}]\times Y)} \leqslant C, \\ &\|\partial_t\partial_x\check{\psi}_\varepsilon\|_{L^\infty(0,T;C(\overline{\Omega}))} \leqslant C\|\partial_r\partial_y\check{\psi}\|_{C([r_{\min},r_{\max}]\times Y)}\|\partial_t r_\varepsilon\|_{L^\infty((0,T)\times\Omega)} \leqslant C, \end{aligned}$$

$$\|\partial_x\partial_x\breve{\psi_\varepsilon}\|_{C([0,T]\times\overline{\Omega})} \leqslant \varepsilon^{-1}\|\partial_y\partial_y\breve{\psi}\|_{C([r_{\min},r_{\max}]\times Y)} \leqslant \varepsilon^{-1}C,$$
$$\|\partial_t\partial_x\partial_x\breve{\psi_\varepsilon}\|_{L^\infty(0,T;C(\overline{\Omega}))} \leqslant C\|\partial_r\partial_y\partial_y\breve{\psi}\|_{C([r_{\min},r_{\max}]\times Y)}\|\partial_t r_\varepsilon\|_{L^\infty((0,T)\times\Omega)} \leqslant \varepsilon^{-1}C.$$

These estimates can be transferred to ψ_ε via

$$\partial_t\psi_\varepsilon = \partial_t\breve{\psi_\varepsilon}, \quad \partial_x\psi_\varepsilon = \partial_x\breve{\psi_\varepsilon} - \mathbb{1}, \quad \partial_t\partial_x\psi_\varepsilon = \partial_x\partial_t\breve{\psi}$$
$$\partial_x\partial_x\psi_\varepsilon = \partial_x\partial_x\breve{\psi_\varepsilon}, \quad \partial_t\partial_x\partial_x\psi_\varepsilon = \partial_x\partial_x\partial_t\breve{\psi_\varepsilon}$$

The estimate for J_ε follows from (4.11) via the pointwise estimate

$$\begin{aligned} J_\varepsilon(t,x) &= \det(\partial_x\psi_\varepsilon(t,x)) = \det(\partial_x\breve{\psi_\varepsilon}(t,x) + \mathbb{1}) = \det(\partial_y\breve{\psi}(r(t,x),x/\varepsilon) + \mathbb{1}) \\ &= \det(\partial_y\psi(r(t,x),x/\varepsilon)) \geqslant c_J \end{aligned}$$

for every $(t,x) \in [0,T]\times\overline{\Omega}$. □

We recap the notions for the Jacobian matrix of ψ_ε, its determinant and adjugate matrix from Chapter 2

$$\begin{aligned} &\Psi_\varepsilon(t,x) := \partial_x\psi(t,x), \quad J_\varepsilon(t,x) = \det(\partial_x\psi_\varepsilon(t,x)), \\ &A_\varepsilon(t,x) = \operatorname{Adj}(\Psi_\varepsilon(t,x)) = J_\varepsilon(t,x)\Psi_\varepsilon^{-1}(t,x). \end{aligned} \tag{4.20}$$

Lemma 4.3. *Let $r_\varepsilon \in C^{0,1}([0,T];[r_{min},r_{max}])^{|I_\varepsilon|}$ with $\|\partial_t r_{\varepsilon,k}\|_{L^\infty(0,T)} \leqslant C_g c_s^{-1}$ for every $k \in I_\varepsilon$. Let ψ_ε be given by (4.18) and Ψ_ε, J_ε and A_ε by (4.20). Then, there exists a constant C, which is independent of ε and r_ε such that*

$$\begin{aligned} \|\Psi_\varepsilon(t)\|_{C(\overline{\Omega_\varepsilon})} + \|\Psi_\varepsilon^{-1}(t)\|_{C(\overline{\Omega_\varepsilon})} + \|J_\varepsilon(t)\|_{C(\overline{\Omega_\varepsilon})} + \|J_\varepsilon^{-1}(t)\|_{C(\overline{\Omega_\varepsilon})} &\leqslant C, \\ \|A_\varepsilon(t)\|_{C(\overline{\Omega_\varepsilon})} + \|A_\varepsilon^{-1}(t)\|_{C(\overline{\Omega_\varepsilon})} + \varepsilon\|\nabla J_\varepsilon^{-1}(t)\|_{C(\overline{\Omega_\varepsilon})} &\leqslant C \end{aligned}$$

for every $t \in [0,T]$ and

$$\begin{aligned} \|\partial_t\Psi_\varepsilon(t)\|_{L^\infty(\Omega_\varepsilon)} + \|\partial_t\Psi_\varepsilon^{-1}(t)\|_{L^\infty(\Omega_\varepsilon)} &\leqslant C, \\ \|\partial_t J_\varepsilon(t)\|_{L^\infty(\Omega_\varepsilon)} + \|\partial_t J_\varepsilon^{-1}(t)\|_{L^\infty(\Omega_\varepsilon)} &\leqslant C, \\ \|\partial_t A_\varepsilon(t)\|_{L^\infty(\Omega_\varepsilon)} + \|\partial_t A_\varepsilon^{-1}(t)\|_{L^\infty(\Omega_\varepsilon)} &\leqslant C \end{aligned}$$

for a.e. $t \in (0,T)$.

Moreover, $A_\varepsilon D\Psi_\varepsilon^{-\top}$ is uniformly coercive, i.e. there exists a constant α, which does not depend on ε or r_ε, such that

$$A_\varepsilon(t,x)D\Psi_\varepsilon^{-\top}(t,x)\xi\cdot\xi \geqslant \alpha|\xi|^2$$

for all $(t,x) \in [0,T]\times\Omega_\varepsilon$ and all $\xi \in \mathbb{R}^n$.

Proof. Arguing as in the proof of Lemma 2.8 and Lemma 2.43, we can infer the uniform a-priori estimate from the uniform estimates given in Lemma 4.2. The uniform coercivity of $A_\varepsilon D\Psi_\varepsilon^{-\top}$ can be deduced from these uniform bounds as in (2.38). $\square$

In order to obtain the uniqueness of the solution of the microscopic problem, the following Lipschitz estimates become useful.

Lemma 4.4. *Let $r_{\varepsilon,i} \in C^{0,1}([0,T];[r_{min}, r_{max}])^{|I_\varepsilon|}$ with $\|\partial_t r_{\varepsilon,i,k}\|_{L^\infty(0,T)} \leqslant C_g c_s^{-1}$ for every $k \in I_\varepsilon$ for $i \in \{1,2\}$ and $\psi_{\varepsilon,i}$ be given by (4.18) for $r_\varepsilon = r_{\varepsilon,i}$. Let $\Psi_{\varepsilon,i}, J_{\varepsilon,i}, A_{\varepsilon,i}$ be defined accordingly. Then, there exist constants C which are independent of $r_{\varepsilon,i}$ such that for a.e. $t \in (0,T)$*

$$\begin{aligned}
\|\Psi_{\varepsilon,1} - \Psi_{\varepsilon,2}\|_{C([0,t]\times\overline{\Omega})} + \|J_{\varepsilon,1} - J_{\varepsilon,2}\|_{C([0,t]\times\overline{\Omega})} &\leqslant C\|r_{\varepsilon,1} - r_{\varepsilon,2}\|_{C([0,t];L^\infty(\Omega))},\\
\|\partial_x\Psi_{\varepsilon,1} - \partial_x\Psi_{\varepsilon,2}\|_{C([0,t]\times\overline{\Omega})} + \|\partial_x J_{\varepsilon,1} - \partial_x J_{\varepsilon,2}\|_{C([0,t]\times\overline{\Omega})} &\leqslant \varepsilon^{-1}C\|r_{\varepsilon,1} - r_{\varepsilon,2}\|_{C([0,t];L^\infty(\Omega))},\\
\|A_{\varepsilon,1} - A_{\varepsilon,2}\|_{C([0,t]\times\overline{\Omega})} + \|\Psi_{\varepsilon,1}^{-1} - \Psi_{\varepsilon,2}^{-1}\|_{C([0,t]\times\overline{\Omega})} &\leqslant C\|r_{\varepsilon,1} - r_{\varepsilon,2}\|_{C([0,t];L^\infty(\Omega))},
\end{aligned}$$

and

$$\begin{aligned}
\|\partial_t\psi_{\varepsilon,1} - \partial_t\psi_{\varepsilon,2}\|_{L^\infty(\Omega;L^2(0,t))} &\leqslant \varepsilon C\big(\|\partial_t r_{\varepsilon,1} - \partial_t r_{\varepsilon,2}\|_{L^\infty(\Omega;L^2(0,t))} + \|r_{\varepsilon,1} - r_{\varepsilon,2}\|_{L^\infty((0,t)\times\Omega)}\big),\\
\|\partial_t J_{\varepsilon,1} - \partial_t J_{\varepsilon,2}\|_{L^\infty(\Omega;L^2(0,t))} &\leqslant C\big(\|\partial_t r_{\varepsilon,1} - \partial_t r_{\varepsilon,2}\|_{L^\infty(\Omega;L^2(0,t))} + \|r_{\varepsilon,1} - r_{\varepsilon,2}\|_{L^\infty((0,t)\times\Omega)}\big).
\end{aligned}$$

This implies in particular

$$\begin{aligned}
\|A_{\varepsilon,1}D\Psi_{\varepsilon,1}^{-\top} - A_{\varepsilon,2}D\Psi_{\varepsilon,2}^{-\top}\|_{L^\infty(\Omega;L^2(0,t))} &\leqslant C\|r_{\varepsilon,1} - r_{\varepsilon,2}\|_{L^\infty((0,t)\times\Omega)},\\
\|A_{\varepsilon,1}\partial_t\psi_{\varepsilon,1} - A_{\varepsilon,2}\partial_t\psi_{\varepsilon,2}\|_{L^\infty(\Omega;L^2(0,t))} &\leqslant \varepsilon C\big(\|\partial_t r_{\varepsilon,1} - \partial_t r_{\varepsilon,2}\|_{L^\infty(\Omega;L^2(0,t))}\\
&\qquad + \|r_{\varepsilon,1} - r_{\varepsilon,2}\|_{L^\infty((0,t)\times\Omega)}\big).
\end{aligned}$$

Proof. Similarly to the proof of Lemma 4.2, we use $\psi_\varepsilon(t,x) = \breve{\psi}_\varepsilon(t,x) + x$ and $\breve{\psi}_\varepsilon = \varepsilon\breve{\psi}(r_\varepsilon(t,x), x/\varepsilon)$. Then, we can estimate

$$\begin{aligned}
\|\Psi_{\varepsilon,1} - \Psi_{\varepsilon,2}\|_{C([0,t]\times\overline{\Omega})} = \|\partial_x\breve{\psi}_{\varepsilon,1} - \partial_x\breve{\psi}_{\varepsilon,1}\|_{C([0,t]\times\overline{\Omega})} &= \|\partial_y\breve{\psi}(r_{\varepsilon,1},\cdot/\varepsilon) - \partial_y\breve{\psi}(r_{\varepsilon,2},\cdot/\varepsilon)\|_{C([0,t]\times\overline{\Omega})}\\
&\leqslant \|\partial_r\partial_y\breve{\psi}\|_{C([r_{\min},r_{\max}]\times\overline{Y})}\|r_{\varepsilon,1} - r_{\varepsilon,2}\|_{C([0,t];L^\infty(\Omega))}\\
&\leqslant C\|r_{\varepsilon,1} - r_{\varepsilon,2}\|_{C([0,t];L^\infty(\Omega))}.
\end{aligned}$$

The estimate for $\|J_{\varepsilon,1} - J_{\varepsilon,2}\|_{C([0,T]\times\overline{\Omega})}$ follows from the fact that $J_{\varepsilon,i}$ are polynomials in the entries of $\Psi_{\varepsilon,i}$, and that $\Psi_{\varepsilon,i}$ as well as $r_{\varepsilon,i}$ are uniformly bounded. Since $J_{\varepsilon,i}$ is uniformly bounded from below, we can transfer this estimate to

$$\|J_{\varepsilon,1}^{-1} - J_{\varepsilon,2}^{-1}\|_{C([0,T]\times\overline{\Omega})} \leqslant C\|r_{\varepsilon,1} - r_{\varepsilon,2}\|_{C([0,t];L^\infty(\Omega))}.$$

Similarly, due to the polynomial structure of $\Psi_{\varepsilon,i}^{-1}$ and $A_{\varepsilon,i}^{-1}$ with respect to the entries of $\Psi_{\varepsilon,i}$ and $J_{\varepsilon,i}^{-1}$, we obtain the estimate

$$\|A_{\varepsilon,1} - A_{\varepsilon,2}\|_{C([0,T]\times\overline{\Omega})} + \|\Psi_{\varepsilon,1}^{-1} - \Psi_{\varepsilon,2}^{-1}\|_{C([0,T]\times\overline{\Omega})} \leqslant C\|r_{\varepsilon,1} - r_{\varepsilon,2}\|_{C([0,T];L^\infty(\Omega))}.$$

The estimate of the spatial derivatives can be inferred similarly, by

$$\begin{aligned}
\|\partial_x \Psi_{\varepsilon,1} - \partial_x \Psi_{\varepsilon,2}\|_{C([0,t]\times\overline{\Omega})} &= \|\partial_x \partial_x \check{\psi}_{\varepsilon,1} - \partial_x \partial_x \check{\psi}_{\varepsilon,1}\|_{C([0,t]\times\overline{\Omega})} \\
&= \varepsilon^{-1} \|\partial_y \partial_y \check{\psi}(r_{\varepsilon,1}, \cdot/\varepsilon) - \partial_y \partial_y \check{\psi}(r_{\varepsilon,2}, \cdot/\varepsilon)\|_{C([0,t]\times\overline{\Omega})} \\
&\leqslant \varepsilon^{-1} C \|\partial_r \partial_y \partial_y \check{\psi}\|_{C([r_{\min}, r_{\max}]\times\overline{Y})} \|r_{\varepsilon,1} - r_{\varepsilon,2}\|_{C([0,t];L^\infty(\Omega))} \\
&\leqslant \varepsilon^{-1} C \|r_{\varepsilon,1} - r_{\varepsilon,2}\|_{C([0,t]\times\overline{\Omega})}.
\end{aligned}$$

Due to the polynomial structure of J_ε and the uniform boundedness of the entries of Ψ_ε and $\varepsilon \partial_x \Psi_\varepsilon$, we can transfer this estimate to

$$\|\partial_x J_{\varepsilon,1} - \partial_x J_{\varepsilon,2}\|_{C([0,t]\times\overline{\Omega})} \leqslant \varepsilon^{-1} C \|r_{\varepsilon,1} - r_{\varepsilon,2}\|_{C([0,t]\times\overline{\Omega})}.$$

The estimate on the time-derivatives follows with the uniform essential boundedness of $\partial_r \partial_r \breve{\psi_\varepsilon}$ and $\partial_t r_\varepsilon$ by

$$\begin{aligned}
&\|\partial_t \psi_{\varepsilon,1} - \partial_t \psi_{\varepsilon,2}\|_{L^\infty(\Omega;L^2(0,t))} = \varepsilon \|\partial_r \check{\psi}(r_{\varepsilon,1}, \cdot/\varepsilon) \partial_t r_{\varepsilon,1} - \partial_r \check{\psi}(r_{\varepsilon,2}, \cdot/\varepsilon) \partial_t r_{\varepsilon,2}\|_{L^\infty(\Omega;L^2(0,t))} \\
&\quad \leqslant \varepsilon \|\partial_r \check{\psi}(r_{\varepsilon,1}, \cdot/\varepsilon) - \partial_r \check{\psi}(r_{\varepsilon,2}, \cdot/\varepsilon)\|_{L^\infty((0,t)\times\Omega)} \|\partial_t r_{\varepsilon,1}\|_{L^\infty(\Omega;L^2(0,t))} \\
&\qquad + \varepsilon \|\partial_r \check{\psi}(r_{\varepsilon,2}, \cdot/\varepsilon)\|_{L^\infty((0,t)\times\Omega)} \|\partial_t r_{\varepsilon,1} - \partial_t r_{\varepsilon,2}\|_{L^\infty(\Omega;L^2(0,t))} \\
&\quad \leqslant \varepsilon \|\partial_r \partial_r \check{\psi}\|_{L^\infty((0,t)\times Y)} \|r_{\varepsilon,1} - r_{\varepsilon,2}\|_{L^\infty((0,t)\times\Omega)} \|\partial_t r_{\varepsilon,1}\|_{L^\infty(\Omega;L^2(0,t))} \\
&\qquad + \varepsilon \|\partial_r \check{\psi}\|_{L^\infty((0,t)\times Y)} \|\partial_t r_{\varepsilon,1} - \partial_t r_{\varepsilon,2}\|_{L^\infty(\Omega;L^2(0,t))} \\
&\quad \leqslant \varepsilon C \big(\|r_{\varepsilon,1} - r_{\varepsilon,2}\|_{L^\infty((0,t)\times\Omega)} + \|\partial_t r_{\varepsilon,1} - \partial_t r_{\varepsilon,2}\|_{L^\infty(\Omega;L^2(0,t))} \big).
\end{aligned}$$

An easy extension of this argument shows the estimate for $\|\partial_t \Psi_{\varepsilon,1} - \partial_t \Psi_{\varepsilon,2}\|_{L^\infty(\Omega;L^2(0,t))}$. Then, we use the polynomial structure of J_ε in order to deduce

$$\|\partial_t J_{\varepsilon,1} - \partial_t J_{\varepsilon,2}\|_{L^\infty(\Omega;L^2(0,t))} \leqslant C \big(\|\partial_t r_{\varepsilon,1} - \partial_t r_{\varepsilon,2}\|_{L^\infty(\Omega;L^2(0,t))} + \|r_{\varepsilon,1} - r_{\varepsilon,2}\|_{L^\infty((0,t)\times\Omega_\varepsilon)} \big)$$

from the previous Lipschitz estimates.

Finally, the last two Lipschitz estimates of Lemma 4.4 follow from the triangle inequality, the previous Lipschitz estimates and the essential boundedness of the involved terms. □

Lemma 4.5. *Let $r_\varepsilon \in C^{0,1}([0,T]; [r_{min}, r_{max}])^{|I_\varepsilon|}$ with $\|\partial_t r_{\varepsilon,k}\|_{L^\infty(0,T)} \leqslant C_g c_s^{-1}$ for every $k \in I_\varepsilon$. Let ψ_ε be given by (4.18), f by Assumption 4.1 and $\hat{f}_\varepsilon(t,x) := f(t, \psi_\varepsilon(t,x))$ for a.e. $(t,x) \in (0,T) \times \Omega_\varepsilon$. Then, $\hat{f}_\varepsilon \in L^2((0,T) \times \Omega_\varepsilon)$ and, in particular,*

$$\|\hat{f}_\varepsilon\|_{L^2((0,T)\times\Omega_\varepsilon)} \leqslant C$$

for a constant C which does not depend on ε and r_ε.

Proof. Due to Lemma 4.2, J_ε is uniformly bounded from below. Then, the uniform estimate on $\hat{f}_\varepsilon$ can be deduced by computations as in Lemma 2.11. □

Further properties of the coordinate transformation

The ε-scaling of (4.17),yields

$$J_\varepsilon(t,x)\|\Psi_\varepsilon^{-\top}(t,x)\hat{n}(t,x)\| = \frac{S_{n-1}(\varepsilon r_\varepsilon(t,x))}{S_{n-1}(\varepsilon R)} = \left(\frac{r_\varepsilon(t,x)}{R}\right)^{n-1} \tag{4.21}$$

for $x \in \Gamma_\varepsilon$. Furthermore, we note the identity

$$J_\varepsilon \Psi_\varepsilon^{-\top}(t,x)\hat{n}\cdot\hat{n} = |J_\varepsilon(t,x)\Psi_\varepsilon^{-\top}(t,x)\hat{n}| \text{ for } (t,x) \in (0,T)\times\partial\Omega_\varepsilon,$$

which holds for general coordinate transformations and can be derived by means of the theorem of Gauß and the integral transformation for bulk and surface integrals. Since $J_\varepsilon > 0$, we obtain in particular

$$\Psi_\varepsilon^{-\top}(t,x)\hat{n}\cdot\hat{n} = |\Psi_\varepsilon^{-\top}(t,x)\hat{n}| \qquad \text{for } (t,x) \in (0,T)\times\partial\Omega_\varepsilon. \tag{4.22}$$

Strong form in reference coordinates

With the transformation ψ_ε from (4.18), we transform the data and the unknown u_ε by

$$\hat{f}_\varepsilon(t,x) := f(t,\psi_\varepsilon(t,x)), \qquad \hat{u}_\varepsilon^{\text{in}}(0) := u_\varepsilon(\psi_\varepsilon(0,x)), \qquad \hat{u}_\varepsilon(t,x) := u_\varepsilon(t,\psi_\varepsilon(t,x)).$$

We define

$$\Gamma_{\varepsilon,k} := \partial\varepsilon B_R(k+\mathfrak{m}) \text{ for } k \in I_\varepsilon, \qquad \Gamma_\varepsilon := \bigcup_{k\in I_\varepsilon}\Gamma_{\varepsilon,k}.$$

Then, transforming (4.7) onto the periodic reference domain leads to:

Microscopic reaction–diffusion equation with coupled domain evolution in the reference coordinates

$$\begin{aligned}
\partial_t\hat{u}_\varepsilon - \partial_x\hat{u}_\varepsilon\Psi_\varepsilon^{-1}\partial_t\psi_\varepsilon - J_\varepsilon^{-1}\operatorname{div}(A_\varepsilon D\Psi_\varepsilon^{-\top}\nabla\hat{u}_\varepsilon) &= \hat{f}_\varepsilon && \text{in } (0,T)\times\Omega_\varepsilon,\\
-D\Psi_\varepsilon^{-\top}\nabla\hat{u}_\varepsilon\cdot\|\Psi_\varepsilon^{-\top}\hat{n}\|^{-1}\Psi_\varepsilon^{-\top}\hat{n} + \varepsilon\partial_t r_{\varepsilon,k} &= \varepsilon g(\hat{u}_\varepsilon, r_{\varepsilon,k}) && \text{on } (0,T)\times\Gamma_{\varepsilon,k}, k\in I_\varepsilon,\\
-D\Psi_\varepsilon^{-\top}\nabla\hat{u}_\varepsilon\cdot\|\Psi_\varepsilon^{-\top}(t,x)\hat{n}\|^{-1}\Psi_\varepsilon^{-\top}\hat{n} &= 0 && \text{on } (0,T)\times\partial\Omega,\\
\hat{u}_\varepsilon(t=0) &= \hat{u}_\varepsilon^{\text{in}} && \text{in } \Omega,\\
\partial_t r_{\varepsilon,k}(t) &= \frac{1}{c_s}\fint_{\Gamma_{\varepsilon,k}(t)} g(\hat{u}_\varepsilon(t,z), r_{\varepsilon,k}(t,z))\,\mathrm{d}\sigma_z && \text{for } t\in(0,T), k\in I_\varepsilon,\\
r_{\varepsilon,k}(0) &= r_{\varepsilon,k}^{\text{in}} && \text{for } k\in I_\varepsilon,
\end{aligned} \tag{4.23}$$

where $\hat{n}$ denotes now the outer normal vector of Ω_ε.

Weak form in reference coordinates

In order to derive the weak formulation, we multiply the first equation of (4.23) by $J_\varepsilon\varphi$ for $\varphi \in H^1(\Omega_\varepsilon)$. Then, we obtain with the Leibniz rule and the fact that $\operatorname{div}_y(A_\varepsilon\partial_t\psi_\varepsilon) = \partial_t J_\varepsilon$ (cf. also the proof of Lemma 3.26) that

$$\begin{aligned} J_\varepsilon\partial_t\hat{u}_\varepsilon - J_\varepsilon\partial_x\hat{u}_\varepsilon\Psi_\varepsilon^{-1}\partial_t\psi_\varepsilon &= \partial_t(J_\varepsilon\hat{u}_\varepsilon) - \partial_t J_\varepsilon\hat{u}_\varepsilon - J_\varepsilon\Psi_\varepsilon^{-1}\partial_t\psi_\varepsilon\cdot\nabla\hat{u}_\varepsilon \\ &= \partial_t(J_\varepsilon\hat{u}_\varepsilon) - \operatorname{div}(A_\varepsilon\partial_t\psi_\varepsilon)\hat{u}_\varepsilon - J_\varepsilon\Psi_\varepsilon^{-1}\partial_t\psi_\varepsilon\cdot\nabla\hat{u}_\varepsilon \\ &= \partial_t(J_\varepsilon\hat{u}_\varepsilon) - \operatorname{div}(A_\varepsilon\partial_t\psi_\varepsilon\hat{u}_\varepsilon). \end{aligned} \tag{4.24}$$

After this substitution, we integrate over Ω_ε, which yields

$$\int_{\Omega_\varepsilon} \partial_t(J_\varepsilon\hat{u}_\varepsilon)\varphi\,\mathrm{d}x - \operatorname{div}(A_\varepsilon D\Psi_\varepsilon^{-\top}\nabla\hat{u}_\varepsilon + A_\varepsilon\partial_t\psi_\varepsilon\hat{u}_\varepsilon)\varphi\,\mathrm{d}x = \int_{\Omega_\varepsilon} J_\varepsilon\hat{f}_\varepsilon\varphi\,\mathrm{d}x.$$

We integrate the divergence term by parts, which leads to

$$\begin{aligned} &-\int_{\Omega_\varepsilon} \operatorname{div}(A_\varepsilon D\Psi_\varepsilon^{-\top}\nabla\hat{u}_\varepsilon + A_\varepsilon\partial_t\psi_\varepsilon\hat{u}_\varepsilon)\varphi\,\mathrm{d}x \\ &= \int_{\Omega_\varepsilon} (A_\varepsilon D\Psi_\varepsilon^{-\top}\nabla\hat{u}_\varepsilon + A_\varepsilon\partial_t\psi_\varepsilon\hat{u}_\varepsilon) : \nabla\varphi\,\mathrm{d}x - \int_{\partial\Omega_\varepsilon} (A_\varepsilon D\Psi_\varepsilon^{-\top}\nabla\hat{u}_\varepsilon + A_\varepsilon\partial_t\psi_\varepsilon\hat{u}_\varepsilon)\varphi\cdot\hat{n}\,\mathrm{d}x. \end{aligned}$$

We rewrite this boundary integral using the fact that $\partial_t\psi_\varepsilon = 0$ on $\partial\Omega$ and $\partial_t\psi_\varepsilon = -\varepsilon\partial_t r_\varepsilon n$ on Γ_ε, (4.22), (4.21) and as well as the boundary conditions from (4.23),

$$\begin{aligned} &-\int_{\partial\Omega_\varepsilon} A_\varepsilon D\Psi_\varepsilon^{-\top}\nabla\hat{u}_\varepsilon\cdot\varphi\hat{n} + A_\varepsilon\partial_t\psi_\varepsilon\hat{u}_\varepsilon\cdot\varphi\hat{n}\,\mathrm{d}\sigma_x \\ &= -\sum_{k\in I_\varepsilon}\int_{\Gamma_{\varepsilon,k}} A_\varepsilon D\Psi_\varepsilon^{-\top}\nabla\hat{u}_\varepsilon\cdot\varphi\hat{n} - \varphi\partial_t r_\varepsilon\hat{u}_\varepsilon J_\varepsilon\Psi_\varepsilon^{-\top}\hat{n}\cdot\hat{n}\,\mathrm{d}\sigma_x - \int_{\partial\Omega} A_\varepsilon D\Psi_\varepsilon^{-\top}\nabla\hat{u}_\varepsilon\cdot\varphi\hat{n}\,\mathrm{d}\sigma_x \\ &= -\sum_{k\in I_\varepsilon}\int_{\Gamma_{\varepsilon,k}} A_\varepsilon D\Psi_\varepsilon^{-\top}\nabla\hat{u}_\varepsilon\cdot\varphi\hat{n} - J_\varepsilon\|\Psi_\varepsilon^{-\top}\hat{n}\|\partial_t r_{\varepsilon,k}\varphi\,\mathrm{d}\sigma_x \\ &= -\sum_{k\in I_\varepsilon}\int_{\Gamma_{\varepsilon,k}} \varepsilon J_\varepsilon\|\Psi_\varepsilon^{-\top}\hat{n}\| g(\hat{u}_\varepsilon, r_{\varepsilon,k})\varphi\,\mathrm{d}\sigma_x \\ &= -\sum_{k\in I_\varepsilon}\left(\frac{r_\varepsilon}{R}\right)^{n-1}\int_{\Gamma_{\varepsilon,k}} \varepsilon g(\hat{u}_\varepsilon, r_{\varepsilon,k})\varphi\,\mathrm{d}\sigma_x. \end{aligned}$$

Moreover, we use (4.21) in order to transform the boundary integral of the right-hand side of the ordinary differential equation of (4.23). Then, we obtain the following weak form:

Weak form for the transformed microscopic reactive transport problem

Find $(\hat{u}_\varepsilon, r_\varepsilon) \in L^2(0,T;H^1(\Omega_\varepsilon)) \times C^{0,1}([0,T];[r_{\min}, r_{\max}])^{|I_\varepsilon|}$ with $\partial_t(J_\varepsilon \hat{u}_\varepsilon) \in L^2(0,T;H^1(\Omega_\varepsilon)')$ such that, for a.e. $t \in (0,T)$,

$$
\begin{aligned}
&\langle \partial_t(J_\varepsilon \hat{u}_\varepsilon)(t), \varphi \rangle_{H^1(\Omega_\varepsilon)', H^1(\Omega_\varepsilon)} + \int_{\Omega_\varepsilon} A_\varepsilon(t,x) D\Psi_\varepsilon^{-\top}(t,x) \nabla \hat{u}_\varepsilon(t,x) \cdot \nabla \varphi(x) \,\mathrm{d}x \\
&\quad + \int_{\Omega_\varepsilon} A_\varepsilon(t,x) \partial_t \psi_\varepsilon(t,x) \hat{u}_\varepsilon(t,x) \cdot \nabla \varphi(x) \,\mathrm{d}x = \int_{\Omega_\varepsilon} J_\varepsilon(t,x) \hat{f}_\varepsilon(t,x) \varphi(x) \,\mathrm{d}x \\
&\quad - \sum_{k \in I_\varepsilon} \left(\frac{r_{\varepsilon,k}(t)}{R} \right)^{n-1} \int_{\Gamma_{\varepsilon,k}} \varepsilon g(\hat{u}_\varepsilon(t,x), r_{\varepsilon,k}(t)) \varphi \,\mathrm{d}\sigma_x, \\
&\partial_t r_{\varepsilon,k}(t) = \frac{1}{c_s} \fint_{\Gamma_{\varepsilon,k}} \varepsilon g(\hat{u}_\varepsilon(t,x), r_{\varepsilon,k}(t)) \,\mathrm{d}\sigma_x \qquad \text{for all } k \in I_\varepsilon, \\
&r_\varepsilon(0) = r_\varepsilon^{\mathrm{in}}, \qquad \hat{u}_\varepsilon(0) = \hat{u}_\varepsilon^{\mathrm{in}}
\end{aligned}
\tag{4.25}
$$

for every $\varphi \in H^1(\Omega_\varepsilon)$ and $J_\varepsilon, \Psi_\varepsilon, A_\varepsilon$ depending on r_ε as described above.

We note that $r_\varepsilon \in C^{0,1}([0,T];[r_{\min}, r_{\max}])^{|I_\varepsilon|}$ yields $J_\varepsilon \in C^{0,1}([0,T]; C^\infty(\overline{\Omega_\varepsilon}))$ and with

$$
\begin{aligned}
\langle \partial_t(J_\varepsilon \hat{u}_\varepsilon)(t), \varphi \rangle_{H^1(\Omega_\varepsilon)', H^1(\Omega_\varepsilon)} =& \langle \partial_t \hat{u}_\varepsilon(t), J_\varepsilon(t)\varphi \rangle_{H^1(\Omega_\varepsilon)', H^1(\Omega_\varepsilon)} \\
&+ \langle \hat{u}_\varepsilon(t), \partial_t J_\varepsilon(t) \varphi \rangle_{H^1(\Omega_\varepsilon)', H^1(\Omega_\varepsilon)},
\end{aligned}
$$

we obtain $\partial_t \hat{u}_\varepsilon \in L^2(0,T;H^1(\Omega_\varepsilon))$ and, thus, the initial condition $\hat{u}_\varepsilon(0) = \hat{u}_\varepsilon^{\mathrm{in}}$ is well-posed.

4.3. Existence, uniqueness and a-priori estimates

We show the existence of a solution of (4.25) with a fixed-point argument and the uniqueness by means of energy estimates. From the existence result, we can directly deduce uniform a-priori estimates for $\hat{u}_\varepsilon$, $\nabla \hat{u}_\varepsilon$ and $\partial_t(J_\varepsilon \hat{u}_\varepsilon)$.

Theorem 4.6. *For every $\varepsilon > 0$, there exists a unique solution $(\hat{u}_\varepsilon, r_\varepsilon) \in L^2(0,T;H^1(\Omega_\varepsilon)) \times C^{0,1}([0,T];[r_{min}, r_{max}])^{|I_\varepsilon|}$ with $\partial_t(J_\varepsilon \hat{u}_\varepsilon) \in L^2(0,T;H^1(\Omega_\varepsilon)')$ of (4.25). Moreover, there exists a constant C such that*

$$
\begin{aligned}
\varepsilon^{-1} \|\partial_t \hat{u}_\varepsilon\|_{L^2(0,T;H^1(\Omega_\varepsilon)')} + \|\partial_t(J_\varepsilon \hat{u}_\varepsilon)\|_{L^2(0,T;H^1(\Omega_\varepsilon)')} &\leqslant C, \\
\|\hat{u}_\varepsilon\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} + \|\hat{u}_\varepsilon\|_{L^2(0,T;H^1(\Omega_\varepsilon))} &\leqslant C.
\end{aligned}
\tag{4.26}
$$

Proof. We reformulate (4.25) as fixed-point problem in $L^2(S;H^\beta(\Omega_\varepsilon))$ for $\beta \in \left(\frac{1}{2}, 1\right)$ with fixed-point operator

$$\begin{aligned}\mathcal{L} : L^2(0,T;H^\beta(\Omega_\varepsilon)) &\to L^2(0,T;H^\beta(\Omega_\varepsilon)),\\ u &\mapsto \mathcal{L}(u) := \mathcal{L}_2(\mathcal{L}_1(u),u),\end{aligned}$$

where $\mathcal{L}_1$ is the solution operator of the ordinary differential equations in (4.25), i.e.

$$\begin{aligned}\mathcal{L}_1(u) : L^2(S;H^\beta(\Omega_\varepsilon)) &\to R_\varepsilon,\\ u &\mapsto \mathcal{L}_1(u) := r_\varepsilon,\end{aligned}$$

where r_ε is the solution of

$$\begin{aligned}\partial_t r_{\varepsilon,k} &= \frac{1}{c_s} \fint_{\Gamma_{\varepsilon,k}} g(u, r_{\varepsilon,k})\,\mathrm{d}\sigma_x \qquad \text{for all } k \in I_\varepsilon,\\ r_\varepsilon(0) &= r_\varepsilon^{\mathrm{in}},\end{aligned} \tag{4.27}$$

and

$$R_\varepsilon := \{r_\varepsilon \in C^{0,1}([0,T];[r_{\min}, r_{\max}])^{|I_\varepsilon|} \mid |r_{\varepsilon,k}(t_1) - r_{\varepsilon,k}(t_2)| \leqslant C_g c_s^{-1}, k \in I_\varepsilon, t_1, t_2 \in [0,T]\},$$

where we endow R_ε with the $W^{1,2}(0,T)^{|I_\varepsilon|}$-norm. The operator $\mathcal{L}_2$ is the solution operator of the reaction–diffusion equation (4.25) for given right-hand side and transformation quantities, i.e.

$$\begin{aligned}\mathcal{L}_2 : R_\varepsilon \times L^2(0,T;H^\beta(\Omega_\varepsilon)) &\to L^2(0,T;H^\beta(\Omega_\varepsilon)),\\ (r_\varepsilon, u) &\mapsto L^2(r_\varepsilon, u) := \hat{u}_\varepsilon,\end{aligned}$$

where $\hat{u}_\varepsilon \in L^2(0,T;H^1(\Omega_\varepsilon))$ with $\partial_t(J_\varepsilon \hat{u}_\varepsilon) \in L^2(0,T;H^1(\Omega_\varepsilon)')$ solves

$$\begin{aligned}&\langle \partial_t(J_\varepsilon \hat{u}_\varepsilon)(t), \varphi\rangle_{H^1(\Omega_\varepsilon)',H^1(\Omega_\varepsilon)} + \big(A_\varepsilon(t)\big(D\Psi_\varepsilon^{-\top}(t)\nabla \hat{u}_\varepsilon(t) + \partial_t\psi_\varepsilon(t)\hat{u}_\varepsilon(t)\big), \nabla\varphi\big)_{L^2(\Omega_\varepsilon)}\\ &\quad = (J_\varepsilon(t)\hat{f}_\varepsilon(t)), \nabla\varphi)_{L^2(\Omega_\varepsilon)} - \sum_{k\in I_\varepsilon} \left(\frac{r_{\varepsilon,k}(t)}{R}\right)^{n-1} \varepsilon(g(u(t), r_{\varepsilon,k}(t)), \varphi)_{L^2(\Gamma_{\varepsilon,k})},\\ &\hat{u}_\varepsilon(0) = \hat{u}_\varepsilon^{\mathrm{in}}\end{aligned} \tag{4.28}$$

for almost every $t \in (0,T)$ and every $\varphi \in H^1(\Omega_\varepsilon)$.

In Lemma 4.7 and Lemma 4.12, we will show that the operators $\mathcal{L}_1$ and $\mathcal{L}_2$ are well-posed and with Lemma 4.8 and Lemma 4.13, we can infer their continuity. Thus, $\mathcal{L}$ is well-posed and continuous. Moreover, Lemma 4.12 provides a constant C_ε such that

$$\|\partial_t \hat{u}_\varepsilon\|_{L^2(0,T;H^1(\Omega_\varepsilon)')} + \|\hat{u}_\varepsilon\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} + \|\hat{u}_\varepsilon\|_{L^2(0,T;H^1(\Omega_\varepsilon))} \leqslant C_\varepsilon \tag{4.29}$$

for all $(r,u) \in R_\varepsilon \times L^2(0,T;H^\beta(\Omega_\varepsilon))$, where $\hat{u}_\varepsilon = \mathcal{L}_2(r,u)$. We fix this constant C_ε and denote the set of all functions $u_\varepsilon \in L^2(0,T;H^\beta(\Omega_\varepsilon))$ that satisfy (4.29) by $K_\varepsilon \subset$

$L^2(0,T;H^\beta(\Omega_\varepsilon))$. From (4.29) and the construction of $\mathcal{L}$, we obtain

$$\|\partial_t \mathcal{L}(u)\|_{L^2(0,T;H^1(\Omega_\varepsilon)')} + \|\mathcal{L}(u)\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} + \|\mathcal{L}(u)\|_{L^2(0,T;H^1(\Omega_\varepsilon))} \leqslant C_\varepsilon.$$

for all $u \in L^2(0,T;H^\beta(\Omega_\varepsilon))$ and, in particular, $\mathcal{L}$ maps K_ε into K_ε. It can be easily observed that K_ε is convex. Moreover, the lemma of Aubin–Lions shows that K_ε is compact in $L^2(0,T;H^\beta(\Omega_\varepsilon))$. Thus, Schauder's fixed-point theorem provides a fixed-point of $\mathcal{L}$ in K_ε and by the construction of $\mathcal{L}$ and $\mathcal{L}_1$, we can infer that $(\hat{u}_\varepsilon, \mathcal{L}_1(\hat{u}_\varepsilon))$ solves (4.25).

The uniqueness of the solution will be shown in Lemma 4.14. □

In the following lemmas, we show that the operators $\mathcal{L}_1$ and $\mathcal{L}_2$ are well-posed and satisfy some Lipschitz type estimate.

Lemma 4.7. *Let $\mathcal{L}_1$ be defined as in the proof of Theorem 4.6. Then, $\mathcal{L}_1$ is well-posed.*

Proof. Since $(t,r) \mapsto \frac{1}{c_s} \fint_{\Gamma_{\varepsilon,k}} g(u(t,x),r)\,\mathrm{d}\sigma_x$ is globally Lipschitz continuous with respect to r and measurable with respect to t for $u \in L^2(0,T;H^\beta(\Omega_\varepsilon))$, Carathéodory's existence theorem gives a unique solution $r_{\varepsilon,k} \in W^{1,1}(0,T)$ of the ordinary differential equation (4.27) for every $k \in I_\varepsilon$. Moreover, Assumption (4.9) ensures that $r_{\varepsilon,k} \in [r_{\min}, r_{\max}]$ and the boundedness of g implies that

$$|\partial_t r_{\varepsilon,k}(t)| = \frac{1}{c_s}\Big| \fint_{\Gamma_{\varepsilon,k}} g(u(t,x), r_{\varepsilon,k}(t))\,\mathrm{d}\sigma_x \Big| \leqslant \frac{C_g}{c_s}$$

for a.e. $t \in (0,T)$. Hence, $r_\varepsilon \in R_\varepsilon$ and $\mathcal{L}_1$ is well-posed. □

Lemma 4.8. *Let $\mathcal{L}_1$ be defined as in the proof of Theorem 4.6. Then, there exists a constant C_ε such that, for every $t \in [0,T]$*

$$\|r_{\varepsilon,1} - r_{\varepsilon,1}\|_{L^\infty((0,t)\times\Omega)} + \|\partial_t r_{\varepsilon,1} - \partial_t r_{\varepsilon,2}\|_{L^\infty(\Omega;L^2(0,t))} \leqslant C_\varepsilon \|u_1 - u_2\|_{L^2((0,t)\times\Gamma_\varepsilon)} \tag{4.30}$$

where $r_{\varepsilon,i} = \mathcal{L}_1(u_i)$ for arbitrary $u_1, u_2 \in L^2(S;H^\beta(\Omega_\varepsilon))$ and $i \in \{1,2\}$. In particular $\mathcal{L}_1$ is continuous.

Proof. Let $t \in (0,T)$ and $u_i \in L^2(0,t;H^\beta(\Omega_\varepsilon))$ for $i \in \{1,2\}$ and $\delta u = u_1 - u_2$. Let $r_{\varepsilon,i,k}$ be the solution of (4.27) for data u_i and we write $\delta r_{\varepsilon,k} = r_{\varepsilon,1,k} - r_{\varepsilon,2,k}$.

Multiplying (4.27) with $\delta r_{\varepsilon,k}$ and integrating over $(0,t)$ yields

$$\begin{aligned}\frac{1}{2}|\delta r_{\varepsilon,k}(t)|^2 &= \int_0^t \partial_t \delta r_{\varepsilon,k}(\tau)\delta r_{\varepsilon,k}(\tau)\,\mathrm{d}\tau \\ &= \int_0^t \frac{1}{c_s} \fint_{\Gamma_{\varepsilon,k}} \big(g(u_1(\tau,x), r_{\varepsilon,1,k}(\tau)) - g(u_1(\tau,x), r_{\varepsilon,1,k}(\tau))\big)\,\mathrm{d}\sigma_x \delta r_{\varepsilon,k}(\tau)\,\mathrm{d}\tau.\end{aligned}$$

Then, we obtain with the Lipschitz estimate for g, the Hölder and Young inequalities

$$\begin{aligned}\frac{1}{2}|\delta r_{\varepsilon,k}(t)|^2 &\leqslant \varepsilon C\Big(\int_0^t \fint_{\Gamma_{\varepsilon,k}} |\delta u(\tau,x)|\,\mathrm{d}\sigma_x + |\delta r_{\varepsilon,k}(\tau,x)|\Big)\delta r_{\varepsilon,k}(\tau)\,\mathrm{d}\tau\\ &\leqslant C_\varepsilon\big(\|\delta u\|_{L^2(0,t;L^1(\Gamma_{\varepsilon,k}))} + \|\delta r_{\varepsilon,k}\|_{L^2(0,t)}\big)\|\delta r_{\varepsilon,k}\|_{L^2(0,t)}\\ &\leqslant C_\varepsilon\big(\|\delta u\|^2_{L^2((0,t)\times\Gamma_{\varepsilon,k})} + \|\delta r_{\varepsilon,k}\|^2_{L^2(0,t)}\big)\end{aligned}$$

Then, the Lemma of Gronwall leads to

$$\|\delta r_{\varepsilon,k}\|^2_{L^\infty(0,t)} \leqslant C_\varepsilon\|\delta u\|^2_{L^2((0,t)\times\Gamma_{\varepsilon,k})}. \tag{4.31}$$

In order to estimate the time-derivative, we multiply (4.27) with $\partial_t\delta r_{\varepsilon,k}$, integrate over $(0,t)$, proceed as above and use (4.31), which gives

$$\begin{aligned}\|\partial_t\delta r_{\varepsilon,k}\|^2_{L^2(0,t)} &= \varepsilon\int_0^t \frac{1}{c_s}\fint_{\Gamma_{\varepsilon,k}} \big(g(u_1(\tau,x),r_{\varepsilon,1,k}(\tau)) - g(u_2(\tau,x),r_{\varepsilon,2,k}(\tau))\big)\,\mathrm{d}\sigma_x\partial_t\delta r_{\varepsilon,k}(\tau)\,\mathrm{d}t\\ &\leqslant C_\varepsilon\big(\|\delta u\|_{L^2((0,t)\times\Gamma_{\varepsilon,k})} + \|\delta r_{\varepsilon,k}\|_{L^2(0,T)}\big)\|\partial_t\delta r_{\varepsilon,k}\|_{L^2(0,t)}\\ &\leqslant C_\varepsilon\|\delta u\|_{L^2((0,t)\times\Gamma_{\varepsilon,k})}\|\partial_t\delta r_{\varepsilon,k}\|_{L^2(0,t)}.\end{aligned} \tag{4.32}$$

By noting that

$$\begin{aligned}\|\delta r_\varepsilon\|^2_{L^\infty((0,t)\times\Omega)} &= \max_{k\in I_\varepsilon}\|\delta r_{\varepsilon,k}\|^2_{L^2(0,t)} \leqslant \max_{k\in I_\varepsilon} C_\varepsilon\|\delta u\|^2_{L^2((0,t)\times\Gamma_{\varepsilon,k})} \leqslant C_\varepsilon\|\delta u\|^2_{L^2((0,t)\times\Gamma_\varepsilon)},\\ \|\partial_t\delta r_\varepsilon\|^2_{L^\infty(\Omega;L^2(0,t))} &= \max_{k\in I_\varepsilon}\|\partial_t\delta r_{\varepsilon,k}\|^2_{L^2(0,t)} \leqslant \max_{k\in I_\varepsilon} C_\varepsilon\|\delta u\|^2_{L^2((0,t)\times\Gamma_{\varepsilon,k})} \leqslant C_\varepsilon\|\delta u\|^2_{L^2((0,t)\times\Gamma_\varepsilon)},\end{aligned}$$

we obtain the desired result. □

In order to show that $\mathcal{L}_2$ is well-posed, we use the theory of monotone operators from [Sho97].

Definition 4.9. *Let V be a Banach space. A function $\mathcal{A} : V \to V'$ is monotone if $\langle\mathcal{A}(u) - \mathcal{A}(v), u - v\rangle_{V',V} \geqslant 0$ for all $u, v \in V$.*

Definition 4.10. *Let W be a separable Hilbert space. A family of operators $\{\mathcal{B}(t) \in \mathcal{L}(W;W') \mid t \in [0,T]\}$ with $B(\cdot)(u,v) \in L^\infty(0,T)$ for all $u, v \in W$ is called regular if for each pair $u, v \in W$ the function $B(\cdot)(u,v)$ is absolutely continuous on $[0,T]$ and there exists $K \in L^1(0,T)$ such that*

$$|\partial_t(B(t)(u,v))| \leqslant K(t)\|u\|_W\|v\|_W \text{ for all } u,v \in W \text{ and a.e. } t \in (0,T). \tag{4.33}$$

Proposition 4.11. *Let V, W be separable Hilbert spaces such that V can be embedded continuously and densely in W. Let $\mathcal{A}(t) \in \mathcal{L}(V,V')$ and $\mathcal{B}(t) \in \mathcal{L}(W,W')$ for every*

$t \in [0,T]$ and assume that $\mathcal{A}(\cdot)(u,v) \in L^\infty(0,T)$ for all $u,v \in V$ and $\mathcal{B}(\cdot)(u,v) \in L^\infty(0,T)$ for all $u,v \in W$. Moreover, we assume that $\{B(t) \in \mathcal{L}(W,W') \mid t \in [0,T]\}$ is a regular family of self-adjoint operators with $\mathcal{B}(0)$ monotone and we assume that there are constants $\lambda, c > 0$ such that

$$2\mathcal{A}(t)(v,v) + \lambda\mathcal{B}(t)(v,v) + \mathcal{B}'(t)(v,v) \geqslant c\|v\|_V$$

for all $v \in V$ and a.e. $t \in (0,T)$. Then, for given $u^{\mathrm{in}} \in W$ and $f \in L^2(0,T;V')$ there exists $u \in L^2(0,T;V)$ such that

$$\partial_t(\mathcal{B}u)(t) + \mathcal{A}(t)u(t) = f(t) \text{ in } L^2(0,T;V') \tag{4.34}$$

with $(\mathcal{B}u)(0) = \mathcal{B}(0)u^{\mathrm{in}}$.

We note that $\mathcal{B}'(t) \in \mathcal{L}(W,W')$ is defined by $\mathcal{B}'(t)u := \partial_t(\mathcal{B}(t)(u))$ for $u \in V$.

Proof. See [Sho97, Chapter III.3, Proposition 3.2]. □

Lemma 4.12. *Let $\mathcal{L}_2$ be defined as in the proof of Theorem 4.6. Then, $\mathcal{L}_2$ is well-posed and there exists a constant C such that*

$$\varepsilon\|\partial_t\hat{u}_\varepsilon\|_{L^2(0,T;H^1(\Omega_\varepsilon)')} + \|\partial_t(J_\varepsilon\hat{u}_\varepsilon)\|_{L^2(0,T;H^1(\Omega_\varepsilon)')} \leqslant C, \tag{4.35}$$

$$\|\hat{u}_\varepsilon\|_{L^\infty((0,T)\times\Omega_\varepsilon)} + \|\nabla\hat{u}_\varepsilon\|_{L^2((0,T)\times\Omega_\varepsilon)} \leqslant C \tag{4.36}$$

for every $(r_\varepsilon, u) \in R_\varepsilon \times L^2(0,T;H^\beta(\Omega_\varepsilon))$ where $\hat{u}_\varepsilon = \mathcal{L}_2(r_\varepsilon,u)$. In particular, $\partial_t(J_\varepsilon\hat{u}_\varepsilon), \partial_t\hat{u}_\varepsilon \in L^2(0,T;H^1(\Omega_\varepsilon)')$ and $\hat{u}_\varepsilon \in L^2(0,T;H^1(\Omega_\varepsilon))$.

Proof. First, we show the existence of a solution of (4.28) by means of Proposition 4.11. Then, we show the uniqueness of the solution and the uniform a-priori estimates by energy estimates. We choose $V = H^1(\Omega_\varepsilon)$, $W = L^2(\Omega_\varepsilon)$ and define

$$\begin{aligned} \mathcal{A}(t)(u,v) &:= (A_\varepsilon(t)(D\Psi_\varepsilon^{-\top}(t)\nabla u - \partial_t\psi_\varepsilon(t)u, \nabla v)_{L^2(\Omega_\varepsilon)} && \text{for } u,v \in V,\\ \mathcal{B}(t)(u,v) &:= (J_\varepsilon(t)u,v)_{L^2(\Omega_\varepsilon)} && \text{for } u,v \in W, \end{aligned}$$

and

$$f(u) = (J_\varepsilon\hat{f}_\varepsilon, \varphi)_{L^2((0,T)\times\Omega_\varepsilon)} - \sum_{k\in I_\varepsilon} \left(\varepsilon\left(\tfrac{r_{\varepsilon,k}}{R}\right)^{n-1} g(u, r_{\varepsilon,k}), \varphi\right)_{L^2((0,T)\times\Gamma_{\varepsilon,k})}$$

for $u \in L^2(0,T;H^\beta(\Omega_\varepsilon))$.

Now, we have to verify the assumptions of Proposition 4.11. We transfer the essential boundedness of r_ε and $\partial_t r_\varepsilon$ to the uniform a-priori estimates on ψ_ε and its derivatives via Lemma 4.2. These estimates can be transferred further to the coefficients via Lemma 4.3. Then, we can analyse the operators $\mathcal{A}$ and $\mathcal{B}$ as well as the right-hand side f.

- $\mathcal{A}(\cdot)(v,w) \in L^\infty(0,T)$ for all $u,v \in V$: From the Hölder inequality, we get

  $$\begin{aligned}\mathcal{A}(t)(u,v) &:= (A_\varepsilon(t)(D\Psi_\varepsilon^{-\top}(t)\nabla u - \partial_t\psi_\varepsilon(t)u, \nabla v)_{L^2(\Omega_\varepsilon)} \\ &\leqslant \big(\|A_\varepsilon(t)D\Psi_\varepsilon^{-\top}(t)\|_{L^\infty(\Omega_\varepsilon)}\|\nabla u\|_{L^2(\Omega_\varepsilon)} + \|A_\varepsilon\|_{L^\infty(\Omega_\varepsilon)}\|\partial_t\psi_\varepsilon\|_{L^\infty(\Omega_\varepsilon)}\|u\|_{L^2(\Omega_\varepsilon)}\big)\|\nabla v\|_{L^2(\Omega_\varepsilon)} \\ &\leqslant C\|u\|_V\|v\|_V\end{aligned}$$

  for all $u,v \in V$ and every $t \in (0,T)$.

- $\mathcal{B}(\cdot)(v,w) \in L^\infty(0,T)$ for all $u,v \in W$: from the Hölder inequality, we get

  $$\mathcal{B}(t)(u,v) := (J_\varepsilon(t)u,v)_{L^2(\Omega_\varepsilon)} = \|J_\varepsilon(t)\|_{L^\infty(\Omega_\varepsilon)}\|u\|_{L^2(\Omega_\varepsilon)}\|v\|_{L^2(\Omega_\varepsilon)} \leqslant C\|u\|_W\|v\|_W.$$

  for all $u,v \in V$ and every $t \in (0,T)$.

- $\mathcal{B}(t)$ is a regular family of self-adjoint operators and $\mathcal{B}(0)$ is monotone: using the uniform essential boundedness of $\partial_t J_\varepsilon$, we obtain with the Hölder inequality

  $$|\partial_t(B(t)(u,v))| = |(\partial_t J_\varepsilon(t)u,v)_{L^2(\Omega_\varepsilon)}| \leqslant C\|u\|_{L^2(\Omega_\varepsilon)}\|v\|_{L^2(\Omega_\varepsilon)}$$

  for all $u,v \in W$ and a.e. $t \in (0,T)$, which shows that B is a family of regular operators. From the structure of $\mathcal{B}(t)$ it is clear that it is self-adjoint for every $t \in [0,T]$ and that $\mathcal{B}(0)$ is monotone.

- coercivity: the uniform coercivity of $A_\varepsilon D\Psi_\varepsilon^{-\top}$ and J_ε is given by Lemma 4.3 and Lemma 4.2, respectively. Together with the essential bounds of the coefficients, we obtain constants $C_1, C_2 > 0$ such that

  $$\begin{aligned}\alpha\|\nabla u\|_{L^2(\Omega_\varepsilon)} &\leqslant (A_\varepsilon(t)D\Psi_\varepsilon^{-\top}(t)\nabla u, \nabla u)_{L^2(\Omega_\varepsilon)}, \\ c_J\|u\|_{L^2(\Omega_\varepsilon)} &\leqslant (J_\varepsilon(t)u,u)_{L^2(\Omega_\varepsilon)} = \mathcal{B}(t)(u,u), \\ (A_\varepsilon(t)\partial_t\psi_\varepsilon(t)u, \nabla v)_{L^2(\Omega_\varepsilon)} &\leqslant \varepsilon C\|u\|_{L^2(\Omega_\varepsilon)}\|\nabla u\|_{L^2(\Omega_\varepsilon)} \leqslant C_1\|u\|^2_{L^2(\Omega_\varepsilon)} + \alpha/2\|\nabla u\|^2_{L^2(\Omega_\varepsilon)}, \\ -\mathcal{B}'(t)(u,u) &= -(\partial_t J_\varepsilon(t)u,u)_{L^2(\Omega_\varepsilon)} \leqslant C_2\|u\|_{L^2(\Omega_\varepsilon)}\end{aligned} \tag{4.37}$$

  for a.e. $t \in (0,T)$. Combining these estimates, yields

  $$\begin{aligned}2\mathcal{A}(t)(v,v) + \lambda B(t)(v,v) + \mathcal{B}'(t)v(v) &\geqslant \alpha\|\nabla v\|_{L^2(\Omega_\varepsilon)} + (\lambda c_J - 2C_1 - C_2)\|v\|^2_{L^2(\Omega_\varepsilon)} \\ &\geqslant \alpha\|v\|^2_{H^1(\Omega_\varepsilon)}\end{aligned}$$

  for $\lambda = (\alpha + 2C_1 + C_2)/c_J$.

- right-hand side f: from Lemma 4.5, we obtain $\hat{f}_\varepsilon \in L^2((0,T)\times\Omega_\varepsilon)$ and with the embedding of $H^\beta(\Omega_\varepsilon)$ into $L^2(\Gamma_{\varepsilon,k})$, the essential boundedness of J_ε and g, we can infer $f \in L^2(0,T;H^1(\Omega_\varepsilon)')$. Moreover, ε-independent estimates for f are presented below.

With all requirements of Proposition 4.11 shown, we obtain a solution $\hat{u}_\varepsilon \in L^2(0,T;H^1(\Omega_\varepsilon))$ with $\partial_t(J_\varepsilon \hat{u}_\varepsilon) \in L^2(0,T;H^1(\Omega_\varepsilon)')$ of (4.28) such that $(J_\varepsilon \hat{u}_\varepsilon)(0) = J_\varepsilon(0)\hat{u}_\varepsilon^{\text{in}}$.

Since $J_\varepsilon \in C^{0,1}([0,T];C^1(\overline{\Omega_\varepsilon}))$, we obtain $\partial_t \hat{u}_\varepsilon \in L^2(0,T;H^1(\Omega_\varepsilon)')$ with

$$\begin{aligned} &\int_0^T \langle \partial_t(J_\varepsilon \hat{u}_\varepsilon)(t), \varphi(t)\rangle_{H^1(\Omega_\varepsilon)',H^1(\Omega_\varepsilon)} \,\mathrm{d}t \\ &= \int_0^T \langle \partial_t \hat{u}_\varepsilon(t), J_\varepsilon(t)\varphi(t)\rangle_{H^1(\Omega_\varepsilon)',H^1(\Omega_\varepsilon)} \,\mathrm{d}t + (\hat{u}_\varepsilon, \partial_t J_\varepsilon \varphi)_{L^2((0,T)\times\Omega_\varepsilon)} \end{aligned} \tag{4.38}$$

for every $\varphi \in L^2(0,T;H^1(\Omega_\varepsilon))$. Thus, we can embed $\hat{u}_\varepsilon$ in $C([0,T];L^2(\Omega_\varepsilon))$ and with $J_\varepsilon \in C([0,T];C(\overline{\Omega_\varepsilon}))$, we can transfer the initial condition to $\hat{u}_\varepsilon(0) = \hat{u}_\varepsilon^{\text{in}}$.

In the following, we derive the uniform a-priori estimates (4.35)–(4.36) and show the uniqueness of the solution. Therefore, we test (4.28) with the solution itself and integrate over $(0,t)$ for $t \in (0,T)$, which gives

$$\begin{aligned} &\int_0^t \langle \partial_t(J_\varepsilon \hat{u}_\varepsilon)(\tau), \hat{u}_\varepsilon(\tau)\rangle_{H^1(\Omega_\varepsilon)',H^1(\Omega_\varepsilon)} \,\mathrm{d}\tau + (A_\varepsilon D\Psi_\varepsilon^{-\top}\nabla \hat{u}_\varepsilon, \nabla \hat{u}_\varepsilon)_{L^2((0,t)\times\Omega_\varepsilon)} \\ &\quad + (A_\varepsilon \partial_t \psi_\varepsilon \hat{u}_\varepsilon, \nabla \hat{u}_\varepsilon)_{L^2((0,t)\times\Omega_\varepsilon)} = (J_\varepsilon \hat{f}_\varepsilon, \nabla \hat{u}_\varepsilon)_{L^2((0,t)\times\Omega_\varepsilon)} \\ &\quad - \sum_{k\in I_\varepsilon} \Big(\big(\tfrac{r_{\varepsilon,k}}{R}\big)^{n-1} \varepsilon g(u, r_{\varepsilon,k}), \hat{u}_\varepsilon \Big)_{L^2((0,t)\times\Gamma_{\varepsilon,k})}. \end{aligned} \tag{4.39}$$

With computations as in (4.38), we can rewrite the first term of the left-hand side of (4.39) by

$$\begin{aligned} &\int_0^t \langle \partial_t(J_\varepsilon \hat{u}_\varepsilon)(\tau), \hat{u}_\varepsilon(\tau)\rangle_{H^1(\Omega_\varepsilon)',H^1(\Omega_\varepsilon)} \,\mathrm{d}\tau \\ &\quad = \tfrac{1}{2}\|J_\varepsilon^{1/2}(t)\hat{u}_\varepsilon(t)\|_{L^2(\Omega_\varepsilon)} - \tfrac{1}{2}\|J_\varepsilon^{1/2}(0)\hat{u}_\varepsilon^{\text{in}}\|_{L^2(\Omega_\varepsilon)} + \tfrac{1}{2}((\partial_t J_\varepsilon)\hat{u}_\varepsilon, \hat{u}_\varepsilon)_{L^2((0,t)\times\Omega_\varepsilon)} \end{aligned}$$

where the last term can be estimated with the uniform essential bound for $\partial_t J_\varepsilon$ by

$$|((\partial_t J_\varepsilon)\hat{u}_\varepsilon, \hat{u}_\varepsilon)_{L^2((0,t)\times\Omega_\varepsilon)}| = \|\partial_t J_\varepsilon\|_{L^\infty((0,T)\times\Omega_\varepsilon)} \|\hat{u}_\varepsilon\|^2_{L^2((0,t)\times\Omega_\varepsilon)} \leqslant C\|\hat{u}_\varepsilon\|^2_{L^2((0,t)\times\Omega_\varepsilon)}.$$

The first summand of the right-hand side of (4.39) can be estimated with the uniform essential bound of J_ε and the Hölder and Young inequalities by

$$(J_\varepsilon \hat{f}_\varepsilon, \nabla \hat{u}_\varepsilon)_{L^2((0,t)\times\Omega_\varepsilon)} \leqslant C\big(\|\hat{f}_\varepsilon\|^2_{L^2((0,t)\times\Omega_\varepsilon)} + \|\hat{u}_\varepsilon\|^2_{L^2((0,t)\times\Omega_\varepsilon)}\big). \tag{4.40}$$

For the second term of the right-hand side of (4.39), we use additionally the boundedness of g, the Hölder and Young inequalities as well as the uniform trace inequality of Lemma 1.28 for Γ_ε. Then, we obtain for every $\delta > 0$ a constant C_δ, which is in particular independent of ε, such that

$$
\begin{aligned}
-\sum_{k\in I_\varepsilon} &\Big(\big(\tfrac{r_{\varepsilon,k}}{R}\big)^{n-1} \varepsilon g(u, r_{\varepsilon,k}), \hat{u}_\varepsilon\Big)_{L^2((0,t)\times\Gamma_{\varepsilon,k})} \\
&\leqslant \varepsilon C \| \|g\|_{L^\infty(\mathbb{R}\times[r_{\min},r_{\max}])}\|_{L^2((0,t)\times\Gamma_\varepsilon)} \|\hat{u}_\varepsilon\|_{L^2((0,t)\times\Gamma_\varepsilon)} \\
&\leqslant \varepsilon C \| \|g\|_{L^\infty(\mathbb{R}\times[r_{\min},r_{\max}])}\|^2_{L^2((0,t)\times\Gamma_\varepsilon)} + \varepsilon \|\hat{u}_\varepsilon\|^2_{L^2((0,t)\times\Gamma_\varepsilon)} \\
&\leqslant C\|g\|^2_{L^\infty(\mathbb{R}\times[r_{\min},r_{\max}])} + C_\delta \|\hat{u}_\varepsilon\|^2_{L^2((0,t)\times\Omega_\varepsilon)} + \delta\|\nabla\hat{u}_\varepsilon\|^2_{L^2((0,t)\times\Omega_\varepsilon)}
\end{aligned}
$$

After estimating the second and third term on the left-hand side of (4.39) similarly to (4.37), we can combine it with the previous estimates. Then, after collecting all the constants and choosing δ small enough, we obtain

$$
\begin{aligned}
\|J_\varepsilon^{1/2}(t)\hat{u}_\varepsilon(t)\|_{L^2(\Omega_\varepsilon)} \leqslant C\big(&\|\hat{u}_\varepsilon\|^2_{L^2((0,t)\times\Omega_\varepsilon)} + \|\hat{f}_\varepsilon\|^2_{L^2((0,T)\times\Omega_\varepsilon)} \\
&+ \|J_\varepsilon^{1/2}(0)\hat{u}_\varepsilon^{\text{in}}\|_{L^2(\Omega_\varepsilon)} + \|g\|^2_{L^\infty(\mathbb{R}\times[r_{\min},r_{\max}])}\big).
\end{aligned}
$$

Lemma 4.5 shows that $\|\hat{f}_\varepsilon\|^2_{L^2((0,T)\times\Omega_\varepsilon)}$ is uniformly bounded and a similar argumentation yields the uniform boundedness of $\|J_\varepsilon^{1/2}(0)\hat{u}_\varepsilon^{\text{in}}\|_{L^2(\Omega_\varepsilon)}$. Then, by applying the Lemma of Gronwall, we obtain the uniform a-priori estimate (4.36).

For $\hat{f}_\varepsilon = 0$, $g = 0$ and $\hat{u}_\varepsilon^{\text{in}} = 0$ the Lemma of Gronwall yields $\hat{u}_\varepsilon = 0$. Together with the linearity of the equation, this yields the uniqueness of the solution of (4.28).

By testing (4.28) with $\varphi \in L^2((0,T;H^1(\Omega_\varepsilon))$ and employing the estimate (4.36), we obtain with similar estimates the uniform boundedness of $\|\partial_t(J_\varepsilon\hat{u}_\varepsilon)\|_{L^2(0,T;H^1(\Omega_\varepsilon))}$. Afterwards, we use (4.38) and (4.36) in order to estimate

$$
\begin{aligned}
&\int_0^T \langle \partial_t\hat{u}_\varepsilon(t), \varphi(t)\rangle_{H^1(\Omega_\varepsilon)',H^1(\Omega_\varepsilon)} \\
&\quad\leqslant \|\partial_t(J_\varepsilon\hat{u}_\varepsilon)\|_{L^2(0,T;H^1(\Omega_\varepsilon))}\|J_\varepsilon^{-1}\varphi\|_{L^2(0,T;H^1(\Omega_\varepsilon))} + \|\hat{u}_\varepsilon\|_{L^2((0,t)\times\Omega_\varepsilon)}\|\partial_t J_\varepsilon J_\varepsilon^{-1}\varphi\|_{L^2((0,t)\times\Omega_\varepsilon)} \\
&\quad\leqslant \varepsilon^{-1}C\|\varphi\|_{L^2(0,T;H^1(\Omega_\varepsilon))} + C\|\varphi\|_{L^2((0,t)\times\Omega_\varepsilon)} \leqslant \varepsilon^{-1}C\|\varphi\|_{L^2((0,t)\times\Omega_\varepsilon)},
\end{aligned}
$$

where the factor ε^{-1} arises, since $\|\nabla J_\varepsilon^{-1}\|_{L^\infty((0,T)\times\Omega_\varepsilon)} \leqslant \varepsilon^{-1}C$. Thus, we obtain (4.35). □

Lemma 4.13. *Let $\mathcal{L}_2$ be defined as in the proof of Theorem 4.6. Then, there exists a constant C_ε such that*

$$
\begin{aligned}
\|(\hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2})(t)\|^2_{L^2(\Omega_\varepsilon)} &+ \|\nabla(\hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2})\|^2_{L^2((0,t)\times\Omega_\varepsilon)} \\
\leqslant C_\varepsilon\big(&\|\hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2}\|^2_{L^2((0,t)\times\Omega_\varepsilon)} + \|u_1 - u_2\|^2_{L^2((0,t)\times\Gamma_\varepsilon)} \\
&+ \|r_{\varepsilon,1} - r_{\varepsilon,1}\|^2_{L^\infty((0,t)\times\Omega)} + \|r_{\varepsilon,1} - r_{\varepsilon,2}\|^2_{L^\infty(\Omega;L^2(0,t))}\big),
\end{aligned}
\tag{4.41}
$$

for every $t \in (0,T)$ and $(r_{\varepsilon,i}, u_i) \in R_\varepsilon \times L^2(0,T;H^\beta(\Omega_\varepsilon))$ where $\hat{u}_{\varepsilon,i} = \mathcal{L}_2(r_{\varepsilon,i}, u_i)$ for $i \in \{1,2\}$. In particular, $\mathcal{L}_2$ is continuous.

Proof. Let $r_{\varepsilon,i} \in R_\varepsilon$ and $u_i \in L^2(0,T;H^1(\Omega_\varepsilon))$ for $i \in \{1,2\}$. Then, we denote the corresponding transformation quantities by $\psi_{\varepsilon,i}$, $\Psi_{\varepsilon,i}$, $J_{\varepsilon,i}$ and $A_{\varepsilon,i}$ for $i \in \{1,2\}$ and $\hat{f}_{\varepsilon,i}(t,x) := f(t,\psi_{\varepsilon,i}(t,x))$. Now, define $\hat{u}_{\varepsilon,i} := \mathcal{L}_2(r_{\varepsilon,1},u_i)$. Moreover, we write $\delta r_\varepsilon := r_{\varepsilon,1} - r_{\varepsilon,2}$, $\delta u := u_1 - u_2$, $\delta\psi_\varepsilon := \psi_{\varepsilon,1} - \psi_{\varepsilon,2}$ and similarly for the differences of the coefficients and the products of those terms.

In order to estimate $\delta\hat{u}_\varepsilon$, we test (4.28) with $\delta\hat{u}_\varepsilon$, integrate over $(0,t)$ for $t \in (0,T)$ and subtract the resulting equations by each other, which yields

$$\begin{aligned}\int_0^t &\langle \partial_t \delta(J_\varepsilon \hat{u}_\varepsilon)(\tau), \delta\hat{u}_\varepsilon(\tau)\rangle_{H^1(\Omega_\varepsilon)',H^1(\Omega_\varepsilon)} \,\mathrm{d}\tau + (\delta(A_\varepsilon D\Psi_\varepsilon^{-\top}\nabla\hat{u}_\varepsilon), \nabla\delta\hat{u}_\varepsilon)_{L^2((0,t)\times\Omega_\varepsilon)}\\ &+ (\delta(A_\varepsilon \partial_t\psi_\varepsilon \hat{u}_\varepsilon), \nabla\delta\hat{u}_\varepsilon)_{L^2((0,t)\times\Omega_\varepsilon)} = (\delta(J_\varepsilon \hat{f}_\varepsilon), \delta\hat{u}_\varepsilon)_{L^2((0,t)\times\Omega_\varepsilon)}\\ &- \sum_{k\in I_\varepsilon} \varepsilon\Big(\big(\tfrac{r_{\varepsilon,1,k}}{R}\big)^{n-1} g(u_1, r_{\varepsilon,1,k}) - \big(\tfrac{r_{\varepsilon,2,k}}{R}\big)^{n-1} g(u_2, r_{\varepsilon,2,k}), \varphi\Big)_{L^2((0,t)\times\Gamma_{\varepsilon,k})},\end{aligned} \tag{4.42}$$

Using

$$\begin{aligned}&\int_0^t \langle \partial_t \delta(J_\varepsilon \hat{u}_\varepsilon)(\tau), \delta\hat{u}_\varepsilon(\tau)\rangle_{H^1(\Omega_\varepsilon)',H^1(\Omega_\varepsilon)} \,\mathrm{d}\tau\\ =&\|J_{\varepsilon,1}^{1/2}(t)\delta\hat{u}_\varepsilon(t)\|_{L^2(\Omega_\varepsilon)} + \tfrac{1}{2}(\partial_t J_{\varepsilon,1}\delta\hat{u}_\varepsilon, \delta\hat{u}_\varepsilon)_{L^2((0,t)\times\Omega_\varepsilon)} + \tfrac{1}{2}(\partial_t \delta J_{\varepsilon,1}\hat{u}_{\varepsilon,2}, \delta\hat{u}_\varepsilon)_{L^2((0,t)\times\Omega_\varepsilon)}\\ &+ \int_0^t \langle \partial_t \hat{u}_{\varepsilon,2}(\tau), \delta J_\varepsilon(\tau)\delta\hat{u}_\varepsilon(\tau)\rangle_{H^1(\Omega_\varepsilon)',H^1(\Omega_\varepsilon)} \,\mathrm{d}\tau,\end{aligned} \tag{4.43}$$

we can rewrite (4.42) by

$$\begin{aligned}I_1 + I_2 :=& \|J_{\varepsilon,1}^{1/2}(t)\delta\hat{u}_\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon)} + (A_{\varepsilon,1} D\Psi_{\varepsilon,1}^{-\top}\nabla\delta\hat{u}_\varepsilon, \nabla\delta\hat{u}_\varepsilon)^2_{L^2((0,t)\times\Omega_\varepsilon)}\\ =&\tfrac{1}{2}(\partial_t J_{\varepsilon,1}\delta\hat{u}_\varepsilon, \delta\hat{u}_\varepsilon)^2_{L^2((0,t)\times\Omega_\varepsilon)} - (\partial_t \delta J_\varepsilon \hat{u}_{\varepsilon,2}, \delta\hat{u}_\varepsilon)_{L^2((0,t)\times\Omega_\varepsilon)}\\ &- \int_0^t \langle \partial_t \hat{u}_{\varepsilon,2}(\tau), \delta J_\varepsilon(\tau)\delta\hat{u}_\varepsilon(\tau)\rangle_{H^1(\Omega_\varepsilon)',H^1(\Omega_\varepsilon)} \,\mathrm{d}\tau - (\delta(A_\varepsilon D\Psi_\varepsilon^{-\top})\nabla\hat{u}_{\varepsilon,2}, \nabla\delta\hat{u}_\varepsilon)_{L^2((0,t)\times\Omega_\varepsilon)}\\ &- (\delta(A_\varepsilon \partial_t\psi_\varepsilon \hat{u}_\varepsilon), \nabla\delta\hat{u}_\varepsilon)_{L^2((0,t)\times\Omega_\varepsilon)} + (\delta(J_\varepsilon \hat{f}_\varepsilon), \delta\hat{u}_\varepsilon)_{L^2((0,t)\times\Omega_\varepsilon)}\\ &- \sum_{k\in I_\varepsilon} \varepsilon\Big(\big(\tfrac{r_{\varepsilon,1,k}}{R}\big)^{n-1} g(u_1, r_{\varepsilon,1,k}) - \big(\tfrac{r_{\varepsilon,2,k}}{R}\big)^{n-1} g(u_2, r_{\varepsilon,2,k}), \delta\hat{u}_\varepsilon\Big)_{L^2((0,t)\times\Gamma_{\varepsilon,k})} =: \sum_{i=3}^{9} I_i\end{aligned} \tag{4.44}$$

In the next step, we estimate I_1 and I_2 from below and $I_3, \dots, I_9$ from above. For this, we use the uniform estimates for the coefficients of Lemma 4.3 and the uniform Lipschitz estimates of Lemma 4.4.

- I_1, I_2: With the uniform coercivity of $J_{\varepsilon,1}$ and $A_{\varepsilon,1}D\Psi_{\varepsilon,1}^{-\top}$, we obtain

$$\begin{aligned}
I_1 &= \|J_{\varepsilon,1}^{1/2}(t)\delta\hat{u}_\varepsilon(t)\|_{L^2(\Omega_\varepsilon)}^2 \geqslant c_J\|\delta\hat{u}_\varepsilon(t)\|_{L^2(\Omega_\varepsilon)}^2,\\
I_2 &= (A_{\varepsilon,1}D\Psi_{\varepsilon,1}^{-\top}\nabla\delta\hat{u}_\varepsilon, \nabla\delta\hat{u}_\varepsilon)_{L^2((0,t)\times\Omega_\varepsilon)}^2 \geqslant \alpha\|\nabla\delta\hat{u}_\varepsilon\|_{L^2((0,t)\times\Omega_\varepsilon)}^2.
\end{aligned}$$

- I_3: With the Hölder inequality and the essential boundedness of $\partial_t J_{\varepsilon,1}$, we obtain

$$I_3 = \tfrac{1}{2}(\partial_t J_{\varepsilon,1}\delta\hat{u}_\varepsilon, \delta\hat{u}_\varepsilon)_{L^2((0,t)\times\Omega_\varepsilon)} \leqslant C\|\delta\hat{u}_\varepsilon\|_{L^2((0,t)\times\Omega_\varepsilon)}^2.$$

- I_4: With the Lipschitz estimate from Lemma 4.4, the boundedness of $\hat{u}_{\varepsilon,2}$ and the Hölder and Young inequalities, we obtain a constant $C_\varepsilon >$ such that

$$\begin{aligned}
I_4 &= -(\partial_t\delta J_\varepsilon\hat{u}_{\varepsilon,2}, \delta\hat{u}_\varepsilon)_{L^2((0,t)\times\Omega_\varepsilon)}\\
&\leqslant \|\partial_t\delta J_\varepsilon\|_{L^\infty(\Omega_\varepsilon;L^2(0,t))}\|\hat{u}_{\varepsilon,2}\|_{L^\infty(0,t;L^2(\Omega_\varepsilon))}\|\delta\hat{u}_\varepsilon\|_{L^2(0,t;L^2(\Omega_\varepsilon))}\\
&\leqslant C_\varepsilon\big(\|\partial_t\delta r_\varepsilon\|_{L^\infty(\Omega;L^2(0,t))} + \|\delta r_\varepsilon\|_{L^\infty((0,t)\times\Omega)}\big)\|\delta\hat{u}_\varepsilon\|_{L^2(0,t;L^2(\Omega_\varepsilon))}\\
&\leqslant C_\varepsilon\big(\|\partial_t\delta r_\varepsilon\|_{L^\infty(\Omega;L^2(0,t))}^2 + \|\delta r_\varepsilon\|_{L^\infty((0,t)\times\Omega)}^2 + \|\delta\hat{u}_\varepsilon\|_{L^2(0,t;L^2(\Omega_\varepsilon))}^2\big).
\end{aligned}$$

- I_5: First, we use the boundedness of $\|\partial_t\hat{u}_{\varepsilon,2}\|_{L^2(0,t;H^1(\Omega_\varepsilon)')}$ in order to estimate

$$\begin{aligned}
I_5 &= -\int_0^t \langle\partial_t\hat{u}_{\varepsilon,2}(\tau), \delta J_\varepsilon(\tau)\delta\hat{u}_\varepsilon(\tau)\rangle_{H^1(\Omega_\varepsilon)',H^1(\Omega_\varepsilon)}\,\mathrm{d}\tau\\
&\leqslant \|\partial_t\hat{u}_{\varepsilon,2}\|_{L^2(0,t;H^1(\Omega_\varepsilon)')}\|\delta J_\varepsilon\delta\hat{u}_\varepsilon\|_{L^2(0,t;H^1(\Omega_\varepsilon))} \leqslant \varepsilon^{-1}C\|\delta J_\varepsilon\delta\hat{u}_\varepsilon\|_{L^2(0,t;H^1(\Omega_\varepsilon))}.
\end{aligned}$$

  Afterwards, we obtain with the Hölder and Young inequalities for every $\lambda > 0$ a constant C_λ such that

$$\begin{aligned}
\|\delta J_\varepsilon\delta\hat{u}_\varepsilon\|_{L^2(0,t;H^1(\Omega_\varepsilon))} =&\|\delta J_\varepsilon\delta\hat{u}_\varepsilon\|_{L^2(0,t;L^2(\Omega_\varepsilon))} + \|\nabla(\delta J_\varepsilon\delta\hat{u}_\varepsilon)\|_{L^2(0,t;L^2(\Omega_\varepsilon))}\\
\leqslant&\big(C_\lambda\|\delta J_\varepsilon\|_{L^\infty((0,t)\times\Omega_\varepsilon)}^2 + C\|\delta\hat{u}_\varepsilon\|_{L^2((0,t)\times\Omega_\varepsilon)}^2\\
&+ C\|\delta\nabla J_\varepsilon\|_{L^\infty((0,t)\times\Omega_\varepsilon)}^2 + \lambda\|\delta\nabla\hat{u}_\varepsilon\|_{L^2((0,t)\times\Omega_\varepsilon)}^2\big)
\end{aligned}$$

  and, then, the Lipschitz estimate from Lemma 4.4 yields

$$I_5 \leqslant C_\varepsilon C_\lambda\|\delta r_\varepsilon\|_{L^\infty((0,t)\times\Omega)}^2 + C_\varepsilon\|\delta\hat{u}_\varepsilon\|_{L^2((0,t)\times\Omega_\varepsilon)}^2 + \lambda\|\delta\nabla\hat{u}_\varepsilon\|_{L^2((0,t)\times\Omega_\varepsilon)}^2.$$

- I_6: By similar arguments as above and the boundedness of $\|\nabla\hat{u}_{\varepsilon,2}\|_{L^2((0,t)\times\Omega_\varepsilon)}$, we can infer

$$\begin{aligned}
I_6 &= -(\delta(A_\varepsilon D\Psi_\varepsilon^{-\top})\nabla\hat{u}_{\varepsilon,2}, \nabla\delta\hat{u}_\varepsilon)_{L^2((0,t)\times\Omega_\varepsilon)}\\
&\leqslant \|\delta(A_\varepsilon D\Psi_\varepsilon^{-\top})\|_{L^\infty((0,t)\times\Omega_\varepsilon)}\|\nabla\hat{u}_{\varepsilon,2}\|_{L^2((0,t)\times\Omega_\varepsilon)}\|\delta\nabla\hat{u}_\varepsilon\|_{L^2((0,t)\times\Omega_\varepsilon)}\\
&\leqslant C_\lambda\|\delta r_\varepsilon\|_{L^\infty((0,t)\times\Omega)}^2 + \lambda\|\delta\nabla\hat{u}_\varepsilon\|_{L^2((0,t)\times\Omega_\varepsilon)}^2.
\end{aligned}$$

- I_7: We note that $\delta(A_\varepsilon \partial_t \psi_\varepsilon \hat{u}_\varepsilon) = \delta(A_\varepsilon \partial_t \psi_\varepsilon)\hat{u}_{\varepsilon,1} + (A_\varepsilon \partial_t \psi_\varepsilon)_1 \delta \hat{u}_\varepsilon$. Then, we obtain by similar arguments as above

$$\begin{aligned}I_7 &= \big(C\|\delta(A_\varepsilon \partial_t \psi_\varepsilon)\|_{L^\infty((0,t)\times\Omega_\varepsilon)} + C\|\delta\hat{u}_\varepsilon\|_{L^2((0,t)\times\Omega_\varepsilon)}\big)\|\nabla\delta\hat{u}_\varepsilon\|_{L^2((0,t)\times\Omega_\varepsilon)}\\ &\leqslant C_\varepsilon C_\lambda\big(\|\delta r_\varepsilon\|^2_{L^\infty((0,t)\times\Omega)} + \|\partial_t\delta r_\varepsilon\|^2_{L^\infty(\Omega;L^2(0,t))} + \|\delta\hat{u}_\varepsilon\|^2_{L^2((0,t)\times\Omega_\varepsilon)}\big)\\ &\quad + \lambda\|\delta\nabla\hat{u}_\varepsilon\|^2_{L^2((0,t)\times\Omega_\varepsilon)}.\end{aligned}$$

- I_8: With the Hölder inequality, we can estimate

$$\begin{aligned}I_8 &\leqslant \|\delta(J_\varepsilon \hat{f}_\varepsilon)\|_{L^2((0,t)\times\Omega_\varepsilon)}\|\delta\hat{u}_\varepsilon\|_{L^2((0,t)\times\Omega_\varepsilon)}\\ &\leqslant \big(\|\delta J_\varepsilon\|_{L^\infty((0,t)\times\Omega_\varepsilon)}\|f(\cdot,\psi_{\varepsilon,1})\|_{L^2((0,t)\times\Omega_\varepsilon)} + \|J_{\varepsilon,2}\|_{L^\infty((0,t)\times\Omega_\varepsilon)}\|\delta\hat{f}_\varepsilon\|_{L^2((0,t)\times\Omega_\varepsilon)}\big)\\ &\quad\|\delta\hat{u}_\varepsilon\|_{L^2((0,t)\times\Omega_\varepsilon)}.\end{aligned}$$

From Lemma 4.5, we get $\|f(\cdot,\psi_{\varepsilon,1})\|_{L^2((0,t)\times\Omega_\varepsilon)} \leqslant C$ and with the Lipschitz continuity of f and the boundedness of r_ε, we obtain

$$\|\delta\hat{f}_\varepsilon\|_{L^2((0,t)\times\Omega_\varepsilon)} \leqslant \|L\delta\psi_\varepsilon\|_{L^2((0,t)\times\Omega_\varepsilon)} \leqslant \varepsilon C\|\delta r_\varepsilon\|_{L^2((0,t)\times\Omega_\varepsilon)} \leqslant \varepsilon C\|\delta r_\varepsilon\|_{L^\infty((0,t)\times\Omega)}.$$

Thus, we get

$$I_8 \leqslant C\|\delta r_\varepsilon\|^2_{L^\infty((0,t)\times\Omega)} + C\|\delta\hat{u}_\varepsilon\|^2_{L^2((0,t)\times\Omega_\varepsilon)}.$$

- I_9: We employ the Lipschitz estimate on g, the uniform boundedness of r_ε and g, the Hölder and Young inequalities as well as the trace inequality to deduce

$$\begin{aligned}I_9 &= -\sum_{k\in I_\varepsilon}\Big(\Big(\tfrac{r_{\varepsilon,1,k}}{R}\Big)^{n-1} g(u_1, r_{\varepsilon,1,k}) - \Big(\tfrac{r_{\varepsilon,2,k}}{R}\Big)^{n-1} g(u_2, r_{\varepsilon,2,k}), \delta\hat{u}_\varepsilon\Big)_{L^2((0,t)\times\Gamma_\varepsilon)}\\ &\leqslant C_\varepsilon\big(\|\delta r_\varepsilon\|_{L^\infty((0,t)\times\Omega)} + \|\delta u\|_{L^2((0,t)\times\Gamma_\varepsilon)}\big)\|\delta\hat{u}_\varepsilon\|_{L^2((0,t)\times\Gamma_\varepsilon)}\\ &\leqslant C_\varepsilon\big(\|\delta r_\varepsilon\|^2_{L^\infty((0,t)\times\Omega)} + \|\delta u\|^2_{L^2((0,t)\times\Gamma_\varepsilon)} + \|\delta\hat{u}_\varepsilon\|^2_{L^2((0,t)\times\Gamma_\varepsilon)}\big)\\ &\leqslant C_\varepsilon\big(\|\delta r_\varepsilon\|^2_{L^\infty((0,t)\times\Omega)} + \|\delta u\|^2_{L^2((0,t)\times\Gamma_\varepsilon)}\big)\\ &\quad + C_\varepsilon\big(C_\lambda\|\delta\hat{u}_\varepsilon\|^2_{L^2((0,t)\times\Omega_\varepsilon)} + \lambda\|\nabla\delta\hat{u}_\varepsilon\|^2_{L^2((0,t)\times\Omega_\varepsilon)}\big).\end{aligned}$$

After combining the estimates of $I_1, \dots, I_9$ with (4.44), choosing δ small enough so that the gradient terms on the right-hand side can be absorbed by the gradient term on the left-hand side, we collect all constants and get (4.41).

The continuity of $\mathcal{L}_2$ can be inferred from (4.41) with the lemma of Gronwall. □

Lemma 4.14. *The system* (4.25) *has at most one solution.*

Proof. Let $(\hat{u}_{\varepsilon,i}, r_{\varepsilon,i}) \in L^2(0,T;H^1(\Omega_\varepsilon)) \times C^{0,1}([0,T];[r_{\min}, r_{\max}])^{|I_\varepsilon|}$ for $i \in \{1,2\}$ be two solutions of (4.25). Following the proof of Theorem 4.6, these solution $\hat{u}_{\varepsilon,i}$ solve also the fixedpoint problem $\hat{u}_{\varepsilon,i} = \mathcal{L}_2(r_{\varepsilon,i}, \hat{u}_{\varepsilon,i})$, $r_{\varepsilon,i} = \mathcal{L}_1(\hat{u}_{\varepsilon,i})$ for $\mathcal{L}_1$ and $\mathcal{L}_2$ defined as in the proof of Theorem 4.6. We define $\delta\hat{u}_\varepsilon = \hat{u}_{\varepsilon,1} - \hat{u}_{\varepsilon,2}$ and $\delta r_\varepsilon = r_{\varepsilon,1} - r_{\varepsilon,2}$.

Then, the Lipschitz estimate from Lemma 4.13, yields a constant C_ε such that for a.e. $t \in (0,T)$

$$\begin{aligned}&\|\delta\hat{u}_\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon)} + \|\nabla\delta\hat{u}_\varepsilon\|^2_{L^2((0,t)\times\Omega_\varepsilon)}\\ &\quad\leqslant C_\varepsilon\big(\|\delta\hat{u}_\varepsilon\|^2_{L^2((0,t)\times\Omega_\varepsilon)} + \|\delta\hat{u}_\varepsilon\|^2_{L^2((0,t)\times\Gamma_\varepsilon)} + \|\delta r_\varepsilon\|_{L^\infty((0,t)\times\Omega)} + \|\partial_t\delta r_\varepsilon\|_{L^\infty(\Omega;L^2(0,t))}\big).\end{aligned}$$

With Lemma 4.8, we can estimate δr_ε and $\delta\partial_t r_\varepsilon$ in terms of $\delta\hat{u}_\varepsilon$. Then, we apply the trace inequality for Γ_ε and obtain for every $\lambda > 0$ a constant $C_{\varepsilon,\lambda}$ such that

$$\begin{aligned}\|\delta\hat{u}_\varepsilon(t)\|_{L^2(\Omega_\varepsilon)} + \|\nabla\delta\hat{u}_\varepsilon\|_{L^2((0,t)\times\Omega_\varepsilon)} &\leqslant C_\varepsilon\big(\|\delta\hat{u}_\varepsilon\|^2_{L^2((0,t)\times\Omega_\varepsilon)} + \|\delta\hat{u}_\varepsilon\|^2_{L^2((0,t)\times\Gamma_\varepsilon)}\big)\\ &\leqslant C_{\varepsilon,\lambda}\|\delta\hat{u}_\varepsilon\|^2_{L^2((0,t)\times\Omega_\varepsilon)} + \lambda\|\nabla\delta\hat{u}_\varepsilon\|^2_{L^2((0,t)\times\Omega_\varepsilon)}.\end{aligned}$$

After choosing λ small, we can absorb the gradient term from the right-hand side with the gradient term from the left-hand side. Then, the Lemma of Gronwall yields

$$\|\delta\hat{u}_\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon)} + \|\nabla\hat{u}_\varepsilon\|^2_{L^2((0,t)\times\Omega_\varepsilon)} = 0.$$

for a.e. $t \in (0,T)$ and, thus, the solution is unique. □

The a-priori estimates (4.26) do not control the time-derivative $\partial_t\hat{u}_\varepsilon$ uniformly with respect to ε, which would be necessary in order to apply the Aubin–Lions lemma. Nevertheless, we can control some time shifts of $\hat{u}_\varepsilon$ uniformly with respect to ε.

Lemma 4.15. *Let $\hat{u}_\varepsilon$ be the solution of* (4.25). *Then,*

$$\int_0^{T-h} \|\hat{u}_\varepsilon(t+h) - \hat{u}_\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon)}\,\mathrm{d}t \to 0 \text{ uniformly with respect to } \varepsilon \tag{4.45}$$

for $h \to 0$, i.e. there exists a continuous monotonically increasing function $\omega : [0,T)$ with $\omega(0) = 0$ such that

$$\int_0^{T-h} \|\hat{u}_\varepsilon(t+h) - \hat{u}_\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon)} \leqslant \omega(h) \tag{4.46}$$

for every $h > 0$ and every $\varepsilon > 0$.

Proof. For a time-dependent function φ, we define $\delta_h\varphi(t) := \varphi(t+h) - \varphi(t)$ for $h > 0$. First, we note that

$$\delta_h(J_\varepsilon\hat{u}_\varepsilon) = J_\varepsilon\delta_h\hat{u}_\varepsilon + \delta_h J_\varepsilon\hat{u}_\varepsilon(\cdot + h). \tag{4.47}$$

After multiplication with $\delta_h\hat{u}_\varepsilon$, we can estimate with the uniform coercivity of J_ε and the triangle inequality

$$\begin{aligned} c_J\|\delta_h\hat{u}_\varepsilon\|^2_{L^2((0,T-h)\times\Omega_\varepsilon)} &\leqslant (J_\varepsilon\delta_h\hat{u}_\varepsilon,\delta_h u_\varepsilon)_{L^2((0,T-h)\times\Omega_\varepsilon)} \\ &\leqslant \left|(\delta_h(J_\varepsilon\hat{u}_\varepsilon),\delta_h u_\varepsilon)_{L^2((0,T-h)\times\Omega_\varepsilon)}\right| + \left|(\delta_h J_\varepsilon\hat{u}_\varepsilon(\cdot+h),\delta_h u_\varepsilon)_{L^2((0,T-h)\times\Omega_\varepsilon)}\right|. \end{aligned} \tag{4.48}$$

Since $\frac{1}{h}\|\delta_h J_\varepsilon\|_{L^\infty((0,T)\times\Omega_\varepsilon)} \leqslant \|\partial_t J_\varepsilon\|_{L^\infty((0,T)\times\Omega_\varepsilon)} \leqslant C$, we can estimate the last term on the right-hand side of (4.48) by

$$\begin{aligned} \left|(\delta_h J_\varepsilon\hat{u}_\varepsilon(\cdot+h),\delta_h u_\varepsilon)_{L^2((0,T-h)\times\Omega_\varepsilon)}\right| &\leqslant Ch\|\hat{u}_\varepsilon(\cdot+h)\|_{L^2((0,T-h)\times\Omega_\varepsilon)}\|\delta_h u_\varepsilon\|_{L^2((0,T-h)\times\Omega_\varepsilon)} \\ &\leqslant Ch\|\delta_h u_\varepsilon\|_{L^2((0,T-h)\times\Omega_\varepsilon)} \\ &\leqslant 2hC\|\hat{u}_\varepsilon\|_{L^2((0,T)\times\Omega_\varepsilon)} \leqslant hC. \end{aligned} \tag{4.49}$$

Hence, this term converges uniformly to zero and it suffices to show the uniform convergence for the first term on the right-hand side of (4.48). For this, we use the following Steklov average argument. We rewrite the first term on the left-hand side of (4.25) for $\varphi \in H^1(0,T;H^1(\Omega_\varepsilon))$ by

$$\begin{aligned} \int_0^T \langle\partial_t(J_\varepsilon\hat{u}_\varepsilon)(t),\varphi(t)\rangle_{H^1(\Omega_\varepsilon)',H^1(\Omega_\varepsilon)}\,\mathrm{d}t &= -(J_\varepsilon\hat{u}_\varepsilon,\partial_t\varphi)_{L^2((0,T)\times\Omega_\varepsilon)} \\ &\quad + (J_\varepsilon(T)\hat{u}_\varepsilon(T);\varphi(T))_{L^2(\Omega_\varepsilon)} - (J_\varepsilon(0)\hat{u}_\varepsilon(0),\varphi(0))_{L^2(\Omega_\varepsilon)}. \end{aligned}$$

Now, we assume that $\varphi \in H^1(-h,T;H^1(\Omega_\varepsilon))$ with $\varphi(-h) = \varphi(T) = 0$, test (4.25) with $\delta_{-h}\varphi$ for $\delta_{-h}\varphi(t) := \varphi(t-h) - \varphi(t)$ and use

$$\begin{aligned} (J_\varepsilon\hat{u}_\varepsilon,\partial_t\delta_{-h}\varphi)_{L^2((0,T)\times\Omega_\varepsilon)} =& (\delta_h(J_\varepsilon\hat{u}_\varepsilon),\partial_t\varphi)_{L^2((0,T-h)\times\Omega_\varepsilon)} + ((J_\varepsilon\hat{u}_\varepsilon)(\cdot+h),\partial_t\varphi)_{L^2((-h,0)\times\Omega_\varepsilon)} \\ &- (J_\varepsilon\hat{u}_\varepsilon,\partial_t\varphi)_{L^2((T-h,T)\times\Omega_\varepsilon)}. \end{aligned}$$

Then, we get

$$\begin{aligned} &(\delta_h(J_\varepsilon\hat{u}_\varepsilon),\partial_t\varphi)_{L^2((0,T-h)\times\Omega_\varepsilon)} \\ &= ((J_\varepsilon\hat{u}_\varepsilon)(\cdot+h),\partial_t\varphi)_{L^2((-h,0)\times\Omega_\varepsilon)} + (J_\varepsilon\hat{u}_\varepsilon,\partial_t\varphi)_{L^2((T-h,T)\times\Omega_\varepsilon)} \\ &\quad + (J_\varepsilon(0)\hat{u}_\varepsilon(0),\varphi(0))_{L^2(\Omega_\varepsilon)} + (J_\varepsilon(T)\hat{u}_\varepsilon(T);\varphi(T-h))_{L^2(\Omega_\varepsilon)} \\ &\quad + (A_\varepsilon D\Psi_\varepsilon^{-\top}\nabla\hat{u}_\varepsilon,\nabla\delta_{-h}\varphi)_{L^2((0,T)\times\Omega_\varepsilon)} + (A_\varepsilon\partial_t\psi_\varepsilon\hat{u}_\varepsilon,\nabla\delta_{-h}\varphi)_{L^2((0,T)\times\Omega_\varepsilon)} \\ &\quad - (J_\varepsilon\hat{f}_\varepsilon,\delta_{-h}\varphi)\delta_{-h}\varphi)_{L^2((0,T)\times\Omega_\varepsilon)} + \sum_{k\in I_\varepsilon}\left(\varepsilon\left(\tfrac{r_{\varepsilon,k}}{R}\right)^{n-1} g(\hat{u}_\varepsilon,r_{\varepsilon,k}),\delta_{-h}\varphi\right)_{L^2((0,T)\times\Gamma_{\varepsilon,k})} \\ &=: \sum_{i=1}^{8} M_i. \end{aligned} \tag{4.50}$$

Now, we choose

$$\varphi(t) = \frac{1}{h} \int_t^{t+h} \hat{u}_\varepsilon(\tau)\,\mathrm{d}\tau$$

for $t \in [-h, T]$, where we implicitly extend $\hat{u}_\varepsilon$ by 0 outside $[0, T]$. Then, we obtain

$$\partial_t \varphi(t) = \begin{cases} h^{-1}\hat{u}_\varepsilon(t+h) & \text{for } -h < t < 0, \\ h^{-1}(\hat{u}_\varepsilon(t+h) - \hat{u}_\varepsilon(t)) & \text{for } 0 < t < T-h, \\ -h^{-1}\hat{u}_\varepsilon(t) & \text{for } T-h < t < T. \end{cases} \tag{4.51}$$

for a.e. $t \in (0, T)$. Consequently, the left-hand side of (4.50) is the term that we want to estimate, i.e.

$$(\delta_h(J_\varepsilon \hat{u}_\varepsilon), \partial_t \varphi)_{L^2((0,T-h)\times\Omega_\varepsilon)} = h^{-1}(\delta_h(J_\varepsilon \hat{u}_\varepsilon), \delta_h \hat{u}_\varepsilon)_{L^2((0,T-h)\times\Omega_\varepsilon)}. \tag{4.52}$$

Now, we estimate the terms M_i for $i \in \{1, \dots, 8\}$ for this choice of φ:

- $M_1, \dots, M_4$: Since $\hat{u}_\varepsilon \in L^2(0, T; H^1(\Omega_\varepsilon)) \cap H^1(0, T; H^1(\Omega_\varepsilon)')$, we obtain that $\hat{u}_\varepsilon \in C([0, T]; L^2(\Omega_\varepsilon))$, and thus the uniform bound $\|\hat{u}_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))} \leqslant C$ holds pointwise, i.e. $\|\hat{u}_\varepsilon\|_{C([0,T];L^2(\Omega_\varepsilon))} \leqslant C$. Using additionally the uniform bound $\|J_\varepsilon\|_{L^\infty((0,T)\times\Omega_\varepsilon)} \leqslant C$, we obtain

$$\begin{aligned} M_1 &= -h^{-1}((J_\varepsilon \hat{u}_\varepsilon)(\cdot + h), \hat{u}_\varepsilon(\cdot + h))_{L^2((-h,0)\times\Omega_\varepsilon)} \leqslant h^{-1} h C \leqslant C, \\ M_2 &= -h^{-1}(J_\varepsilon \hat{u}_\varepsilon, \hat{u}_\varepsilon)_{L^2((T-h,T)\times\Omega_\varepsilon)} \leqslant h^{-1} h C \leqslant C, \\ M_3 &= \Big(J_\varepsilon(0)\hat{u}_\varepsilon(0), h^{-1} \int_0^{t+h} \hat{u}_\varepsilon(\tau)\,\mathrm{d}\tau\Big)_{L^2(\Omega_\varepsilon)} h^{-1} h C \leqslant C \\ M_4 &= \Big(J_\varepsilon(T)\hat{u}_\varepsilon(T), h^{-1} \int_{T-h}^{T} \hat{u}_\varepsilon(\tau)\,\mathrm{d}\tau\Big)_{L^2(\Omega_\varepsilon)} h^{-1} h C \leqslant C. \end{aligned}$$

- $M_5, \dots, M_7$: We show the estimate for M_5. The estimates for M_6 and M_7 follow analogously. First, we decompose M_5 into

$$\begin{aligned} M_5 =& h^{-1} \int_0^T \Big(A_\varepsilon(t) D\Psi_\varepsilon^{-\top}(t) \nabla \hat{u}_\varepsilon(t), \int_{t-h}^{t} \hat{u}_\varepsilon(\tau)\,\mathrm{d}\tau\Big)_{L^2(\Omega_\varepsilon)} \mathrm{d}t \\ &+ h^{-1} \int_0^T \Big(A_\varepsilon(t) D\Psi_\varepsilon^{-\top}(t) \nabla \hat{u}_\varepsilon(t), \int_{t}^{t+h} \hat{u}_\varepsilon(\tau)\,\mathrm{d}\tau\Big)_{L^2(\Omega_\varepsilon)} \mathrm{d}t =: M_{5,a} + M_{5,b}. \end{aligned}$$

Then, we change the order of the integration and estimate with the Hölder inequality

$$\begin{aligned} M_{5,a} &= h^{-1}\int_0^h\int_0^T (A_\varepsilon(t)D\Psi_\varepsilon^{-\top}(t)\nabla\hat{u}_\varepsilon(t), \nabla\hat{u}_\varepsilon(t-h+\tau))_{L^2(\Omega_\varepsilon)}\,\mathrm{d}t\,\mathrm{d}\tau \\ &\leqslant Ch^{-1}\int_0^h \|\nabla\hat{u}_\varepsilon\|_{L^2((0,T)\times\Omega_\varepsilon)}\|\nabla\hat{u}_\varepsilon(\cdot-h+\tau)\|_{L^2((0,T)\times\Omega_\varepsilon)}\,\mathrm{d}\tau \\ &\leqslant C\|\nabla\hat{u}_\varepsilon\|^2_{L^2((0,T)\times\Omega_\varepsilon)}. \end{aligned}$$

The same argumentation provides also a uniform bound for $M_{5,b}$.

- M_8: We split M_8 in two sums like for M_5. We show the estimate for the first summand and the second summand can be estimated analogously. We change again the order of integration and use the essential boundedness of g and the ε-scaled trace inequality Lemma 1.28 for Γ_ε

$$\begin{aligned} &\sum_{k\in I_\varepsilon}\int_0^T \left(\varepsilon\left(\tfrac{r_{\varepsilon,k}(t)}{R}\right)^{n-1} g(\hat{u}_\varepsilon(t), r_{\varepsilon,k}(t)), h^{-1}\int_{t-h}^t \hat{u}_\varepsilon(\tau)\,\mathrm{d}\tau\right)_{L^2(\Gamma_{\varepsilon,k})}\mathrm{d}t \\ &\quad\leqslant \sum_{k\in I_\varepsilon} h^{-1}\int_0^h\int_0^T \left(\varepsilon\left(\tfrac{r_{\varepsilon,k}(t)}{R}\right)^{n-1} g(\hat{u}_\varepsilon(t), r_{\varepsilon,k}(t)), \hat{u}_\varepsilon(t-h+\tau)\,\mathrm{d}\tau\right)_{L^2(\Gamma_{\varepsilon,k})}\mathrm{d}t \\ &\quad\leqslant \varepsilon\sum_{k\in I_\varepsilon} h^{-1}C\int_0^h\int_0^T\int_{\Gamma_{\varepsilon,k}} |\hat{u}_\varepsilon(t-h+\tau,x)|\,\mathrm{d}\sigma_x\,\mathrm{d}\tau\,\mathrm{d}t \\ &\quad\leqslant \varepsilon h^{-1}C\int_0^h\int_0^T\int_{\Gamma_\varepsilon} |\hat{u}_\varepsilon(t-h+\tau,x)|\,\mathrm{d}\tau\,\mathrm{d}\sigma_x\,\mathrm{d}t \\ &\quad\leqslant C\|\hat{u}_\varepsilon\|_{L^1((0,T)\times\Omega_\varepsilon)} + \varepsilon C\|\nabla\hat{u}_\varepsilon\|_{L^1((0,T)\times\Omega_\varepsilon)} \leqslant C. \end{aligned}$$

Combining the estimates of $M_1,\dots,M_8$, with (4.50) and (4.52) shows that

$$(\delta_h(J_\varepsilon\hat{u}_\varepsilon), \delta_h\hat{u}_\varepsilon)_{L^2((0,T-h)\times\Omega_\varepsilon)} \leqslant hC.$$

Together with (4.48) and (4.49), this yields the desired result. □

4.4. Derivation of the limit equations

Strong compactness for $\hat{u}_\varepsilon$

In order to derive some strong compactness result for $\hat{u}_\varepsilon$, we use the following Simon–Kolmogorow compactness result.

Lemma 4.16. *Let $F \subset L^p(0,T;B)$. F is relatively compact in $L^p(0,T;B)$ for $1 \leqslant p < \infty$ if and only if:*

- $\left\{ \int_{t_1}^{t_2} f(t)\,\mathrm{d}t \mid f \in F \right\}$ *is relatively comapct in B for all $0 < t_1 < t_2 < T$,*
- $\|f(\cdot + h) - f\|_{L^p(0,T-h;B)} \to 0$ *uniformly as $h \to 0$ for $f \in F$.*

Proof. See [Sim87, Theorem 1]. □

The following lemma translates Lemma 4.16 in the framework of our a-priori estimates and provides the strong two-scale for a subsequence of $\hat{u}_\varepsilon$.

Proposition 4.17. *Let v_ε be a bounded sequence in $L^2(0,T;H^1(\Omega_\varepsilon))$ such that*

$$\int_0^{T-h} \|v_\varepsilon(t+h) - v_\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon)}\,\mathrm{d}t \to 0 \text{ uniformly with respect to } \varepsilon \tag{4.53}$$

for $h \to 0$.

Then, there exists $v_0 \in L^2((0,T)\times\Omega)$ and a subsequence such that

$$E_\varepsilon v_\varepsilon \to v_0 \qquad\qquad \text{in } L^2((0,T)\times\Omega),$$

where E_ε denotes the extension operator from Lemma 1.24.

Proof. Let $E_\varepsilon v_\varepsilon$ be the extension of v_ε. Then,

$$\begin{aligned}\|(E_\varepsilon v_\varepsilon)(\cdot + h) - E_\varepsilon v_\varepsilon\|_{L^2((0,T-h)\times\Omega)} &= \|E_\varepsilon(v_\varepsilon(\cdot + h) - v_\varepsilon)\|_{L^2((0,T-h)\times\Omega)} \\ &\leqslant C\|v_\varepsilon(\cdot + h) - v_\varepsilon\|_{L^2((0,T-h)\times\Omega_\varepsilon)} \to 0\end{aligned}$$

converges uniformly to zero for $h \to 0$ with respect to ε. Moreover, we can estimate with the Hölder inequality, for every $0 \leqslant t_1 < t_2 < T$,

$$\begin{aligned}\Big\| \int_{t_1}^{t_2} E_\varepsilon v_\varepsilon(t)\,\mathrm{d}t \Big\|^2_{H^1(\Omega)} &= \int_\Omega \Big(\int_{t_1}^{t_2} E_\varepsilon v_\varepsilon(t,x)\,\mathrm{d}t \Big)^2 \mathrm{d}x + \int_\Omega \Big(\int_{t_1}^{t_2} \nabla E_\varepsilon v_\varepsilon(t,x)\,\mathrm{d}t \Big)^2 \mathrm{d}x \\ &\leqslant \int_\Omega \|1\|^2_{L^2(0,T)} \|E_\varepsilon v_\varepsilon(x)\|^2_{L^2(0,T)}\,\mathrm{d}t + \int_\Omega \|1\|^2_{L^2(0,T)} \|\nabla E_\varepsilon v_\varepsilon(x)\|^2_{L^2(0,T)}\,\mathrm{d}t \\ &\leqslant C\|E_\varepsilon v_\varepsilon\|^2_{L^2(0,T;H^1(\Omega_\varepsilon))} \leqslant C,\end{aligned}$$

which shows that $\int_{t_1}^{t_2} E_\varepsilon v_\varepsilon(t)\,\mathrm{d}t$ is bounded in $H^1(\Omega)$ and, therefore, relatively compact in $L^2(\Omega)$. Hence, we have verified both assumptions of Lemma 4.16 and it provides the desired result. □

Strong convergence of r_ε
In order to formulate the convergence for the radii $r_\varepsilon \in C([0,T];[r_{\min}, r_{\max}])^{|I_\varepsilon|}$, we recap the embedding in $L^\infty((0,T)\times\Omega)$ where we identify r_ε with the piecewise constant function $r_\varepsilon : [0,T]\times\Omega \to [r_{\min}, r_{\max}]$, $r_\varepsilon(t,x) := r_{\varepsilon,k}(t)$ for $x \in \varepsilon(k+Y)$ with $k \in I_\varepsilon$. Similarly, we can consider $\partial_t r_\varepsilon$ as element in $L^\infty((0,T)\times\Omega)$. This embedding allows the formulation of the convergence results in L^p-spaces.

Lemma 4.18. *Let $\hat{u}_\varepsilon$ be a bounded sequence in $L^2(0,T;H^1(\Omega_\varepsilon))$ and $\hat{u}_0 \in L^2((0,T)\times\Omega)$ such that*

$$\hat{u}_\varepsilon \xrightarrow{2,2} \hat{u}_0 \qquad \text{on } \Gamma_\varepsilon$$

and let $r_\varepsilon^{\text{in}} \in [r_{min}, r_{max}]^{|I_\varepsilon|}$ such that $r_\varepsilon^{\text{in}} \to r_0^{\text{in}}$ in $L^2(\Omega)$ for some $r_0^{\text{in}} \in L^\infty(\Omega)$ with $r_{min} \leqslant r_0 \leqslant r_{max}$ almost everywhere. Assume that r_ε satisfies

$$\begin{aligned} \partial_t r_{\varepsilon,k}(t) &= \frac{1}{c_s} \fint_{\Gamma_{\varepsilon,k}} \varepsilon g(\hat{u}_\varepsilon(t,x), r_{\varepsilon,k}(t))\,\mathrm{d}\sigma_x \qquad \text{for all } k \in I_\varepsilon, \\ r_\varepsilon(0) &= r_\varepsilon^{\text{in}} \end{aligned} \tag{4.54}$$

for a.e. $t \in (0,T)$. Then,

$$\begin{aligned} r_\varepsilon &\to r_0 && \text{in } L^\infty(0,T;L^p(\Omega)), \\ \partial_t r_\varepsilon &\to \partial_t r_0 && \text{in } L^p((0,T)\times\Omega), \end{aligned}$$

for every $p \in [1,\infty)$, where $r_0 \in L^\infty((0,T)\times\Omega)$ with $\partial_t \in L^\infty((0,T)\times\Omega)$ is the unique solution of

$$\begin{aligned} \partial_t r_0(t,x) &= \tfrac{1}{c_s} g(\hat{u}_0(t,x), r_0(t,x)), \\ r_0(0,x) &= r_0^{\text{in}}(x) \end{aligned} \tag{4.55}$$

for a.e. $(t,x) \in (0,T)\times\Omega$.

Proof. Carathéodory's existence theorem provides a unique solution $r_0 \in W^{1,1}(0,T;L^2(\Omega_\varepsilon))$ of (4.54). Due to the boundedness of g, one has $r_0, \partial_t r_0 \in L^\infty((0,T)\times\Omega_\varepsilon)$ and with assumption (4.9), we can infer that r_0 attains only values in $[r_{\min}, r_{\max}]$. In order to show the convergence of $r_\varepsilon \to r_0$, we multiply (4.54) with $r_{\varepsilon,k} - r_0$, integrate over $x \in \varepsilon k + \varepsilon Y$ and $(0,t)$ for $t \in (0,T)$ and sum over $k \in I_\varepsilon$, which gives

$$\begin{aligned} &\int_0^t \int_\Omega \partial_t r_\varepsilon(\tau,x)(r_\varepsilon - r_0)(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau \\ &\quad = \int_0^t \sum_{k\in I_\varepsilon} \int_{\varepsilon k+\varepsilon Y} \partial_t r_{\varepsilon,k}(\tau)(r_{\varepsilon,k}(\tau) - r_0(\tau,x))\,\mathrm{d}x\,\mathrm{d}\tau \end{aligned}$$

$$= \int_0^t \sum_{k\in I_\varepsilon} \int_{\varepsilon k+\varepsilon Y} \frac{1}{c_s} \fint_{\Gamma_{\varepsilon,k}} \varepsilon g(\hat{u}_\varepsilon(\tau,y), r_{\varepsilon,k}(\tau))\,\mathrm{d}\sigma_y (r_{\varepsilon,k}(\tau) - r_0(\tau,x))\,\mathrm{d}x\,\mathrm{d}t$$

$$= \int_0^t \int_\Omega \frac{1}{c_s} \fint_\Gamma g(\mathcal{T}_\varepsilon(\hat{u}_\varepsilon)(\tau,x,y), \mathcal{T}_\varepsilon(r_\varepsilon)(\tau,x))\,\mathrm{d}\sigma_y (r_\varepsilon - r_0)(\tau,x)\,\mathrm{d}x\,\mathrm{d}t. \tag{4.56}$$

Note that $\mathcal{T}_\varepsilon(r_\varepsilon)(t,x,y) = \mathcal{T}_\varepsilon(r_\varepsilon)(t,x)$ since r_ε is constant on every cell. Similarly, we multiply (4.55) with $r_{\varepsilon,k} - r_0$, integrate over $x \in \varepsilon k + \varepsilon Y$ and $(0,t)$ and sum over $k \in I_\varepsilon$, which gives

$$\int_0^t \int_\Omega \partial_t r_0(t,x)(r_\varepsilon - r_0)(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau = \int_0^t \int_\Omega \frac{1}{c_s} g(\hat{u}_0(\tau,x), r_0(\tau,x))\,\mathrm{d}\sigma_y (r_\varepsilon - r_0)(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau. \tag{4.57}$$

Subtracting (4.57) from (4.56) leads to

$$\int_0^t \int_\Omega \partial_t (r_\varepsilon - r_0)(\tau,x)(r_\varepsilon - r_0)(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau$$
$$= \int_0^t \int_\Omega \frac{1}{c_s} \fint_\Gamma g(\mathcal{T}_\varepsilon(\hat{u}_\varepsilon)(\tau,x,y), \mathcal{T}_\varepsilon(r_\varepsilon)(\tau,x)) - g(\hat{u}_0(\tau,x), r_0(\tau,x))\,\mathrm{d}\sigma_y (r_\varepsilon - r_0)(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau.$$

We rewrite the left-hand side and estimate the right-hand side with the Cauchy–Schwarz inequality, the Lipschitz continuity of g and the Young inequality. Thus, we obtain

$$\begin{aligned}
&\tfrac{1}{2}\|r_\varepsilon(t) - r_0(t)\|^2_{L^2(\Omega)} - \tfrac{1}{2}\|r_\varepsilon(0) - r_0(0)\|^2_{L^2(\Omega)} \\
&\quad\leqslant \int_0^t \int_\Omega \frac{1}{c_s} \fint_\Gamma L_g\big(|(\mathcal{T}_\varepsilon(\hat{u}_\varepsilon) - \hat{u}_0)(\tau,x,y)| + |\mathcal{T}_\varepsilon(r_\varepsilon)(\tau,x) - r_0(\tau,x)|\big)\,\mathrm{d}\sigma_y \\
&\qquad\qquad |r_\varepsilon(\tau,x) - r_0(\tau,x)|\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\leqslant C\big(\|\mathcal{T}_\varepsilon(\hat{u}_\varepsilon) - \hat{u}_0\|^2_{L^2((0,T)\times\Omega\times\Gamma)} + \|r_\varepsilon - r_0\|_{L^2((0,T)\times\Omega)}\big).
\end{aligned}$$

With the Lemma of Gronwall, we can estimate further

$$\|r_\varepsilon - r_0\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} \leqslant C\big(\|\mathcal{T}_\varepsilon(\hat{u}_\varepsilon) - \hat{u}_0\|^2_{L^2((0,T)\times\Omega\times\Gamma)} + \|r^{\mathrm{in}}_\varepsilon - r^{\mathrm{in}}_0\|^2_{L^2(\Omega)}\big) \to 0, \tag{4.58}$$

where the convergence of the first summand of the right-hand side is given by the two-scale convergence of $\hat{u}_\varepsilon$ on Γ_ε and the strong convergence of the initial values is given the Assumption 4.1.

Given the boundedness of r_ε and r_0, one gets the convergence of $r_\varepsilon \to r_0$ with the Hölder inequality with respect to the $L^\infty(0,T;L^p(\Omega))$-norm for every $p \in [1,\infty)$.

The strong convergence $\partial_t r_\varepsilon \to \partial_t r_0$ in $L^2((0,T)\times\Omega)$ can be shown like the strong convergence of $r_\varepsilon \to r_0$ by multiplying the differential equation of (4.25) and (4.55) with $\partial_t(r_{\varepsilon,k} - r_0)$. Afterwards, the uniform essential boundedness of $\partial_t r_\varepsilon$ and $\partial_t r_0$ lead to the strong convergence $\partial_t r_\varepsilon \to \partial_t r_0$ in $L^p((0,T)\times\Omega)$ for every $p \in [1,\infty)$. □

Strong convergence of ψ_ε

From the strong convergence of r_ε, we can infer the strong two-scale convergence of ψ_ε and its derivatives. We define the limit transformation mapping ψ_0 by

$$\psi_0(t,x,y) := \psi(r_0(t,x),y) \tag{4.59}$$

and the corresponding displacement mapping by

$$\breve{\psi}_0(t,x,y) := \psi_0(t,x,y) - y = \breve{\psi}(r_0(t,x),y) \tag{4.60}$$

for a.e. $x \in \Omega$ and every $(t,y) \in [0,T]\times\overline{Y}$. We recap the notation for the Jacobian matrix, its determinant and its adjugate matrix, namely,

$$\begin{aligned}&\Psi_0(t,x,y) := \partial_y\psi_0(t,x,y), \quad J_0(t,x,y) := \det(\partial_y\psi_0(t,x,y)),\\ &A_0(t,x,y) = \operatorname{Adj}(\Psi_0(t,x,y)).\end{aligned} \tag{4.61}$$

Lemma 4.19. *Let $r_\varepsilon \in C([0,T];[r_{min},r_{max}])^{|I_\varepsilon|} \subset L^\infty((0,T)\times\Omega)$ and $r_0 \in L^\infty((0,T)\times\Omega)$ such that $r_\varepsilon \to r_0$ in $L^2((0,T)\times\Omega)$. Let ψ_ε by given by (4.18) and ψ_0 by (4.59), then,*

$$\begin{aligned}&\varepsilon^{-1}(\psi_\varepsilon(t,x) - x) \xrightarrow{<\infty,<\infty} \psi_0(t,x,y) - y,\\ &\partial_x\psi_\varepsilon \xrightarrow{<\infty,<\infty} \partial_y\psi_0,\\ &\varepsilon^{-1}\partial_x\partial_x\psi_\varepsilon \xrightarrow{<\infty,<\infty} \partial_y\partial_y\psi_0.\end{aligned}$$

If additionally $r_\varepsilon \in C^{0,1}([0,T];[r_{min},r_{max}])^{|I_\varepsilon|}$ with $\|\partial_t r_{\varepsilon,k}\|_{L^\infty(0,T)} \leqslant C$ for every $k \in I_\varepsilon$ and $\partial_t r_0 \in L^\infty((0,T)\times\Omega)$ such that $\partial_t r_\varepsilon \to R_0$ in $L^2((0,T)\times\Omega)$, then

$$\begin{aligned}&\varepsilon^{-1}\partial_t\psi_\varepsilon(t,x) \xrightarrow{<\infty,<\infty} \partial_t\psi_0(t,x,y)\\ &\partial_x\partial_t\psi_\varepsilon \xrightarrow{<\infty,<\infty} \partial_y\partial_t\psi_0\\ &\varepsilon^{-1}\partial_x\partial_x\partial_t\psi_\varepsilon \xrightarrow{<\infty,<\infty} \partial_y\partial_y\partial_t\psi_0.\end{aligned}$$

Proof. With (4.19), we can rewrite

$$\begin{aligned}&\varepsilon^{-1}(\psi_\varepsilon(t,x) - x) = \varepsilon^{-1}\breve{\psi}_\varepsilon(t,x) = \breve{\psi}(r_\varepsilon(t,x), x/\varepsilon),\\ &\partial_x\psi_\varepsilon(t,x) = \partial_x\breve{\psi}_\varepsilon(t,x) + \mathbb{1} = \partial_y\breve{\psi}(r_\varepsilon(t,x), x/\varepsilon) + \mathbb{1},\\ &\varepsilon\partial_x\partial_x\psi_\varepsilon(t,x) = \varepsilon\partial_x\partial_x\breve{\psi}_\varepsilon(t,x) = \partial_y\partial_y\breve{\psi}(r_\varepsilon(t,x), x/\varepsilon)\end{aligned}$$

and similarly we rewrite with (4.60)

$$\begin{aligned}
&\breve{\psi}(r_0(t,x),y) = \breve{\psi_0}(r_0(t,x),y) = \breve{\psi_0}(t,x,y) - y, \\
&\partial_y\breve{\psi}(r_0(t,x),y) + \mathbb{1} = \partial_y\breve{\psi_0}(r_0(t,x),y) + \mathbb{1} = \partial_y\breve{\psi_0}(r_0(t,x),y) + \mathbb{1}, \\
&\partial_y\partial_y\breve{\psi}(r_0(t,x),y) = \partial_y\partial_y\breve{\psi_0}(r_0(t,x),y).
\end{aligned}$$

Thus, it suffices to show

$$\partial_{y_\alpha}\breve{\psi}(r_\varepsilon(t,x),x/\varepsilon) \xrightarrow{<\infty,<\infty} \partial_{y_\alpha}\breve{\psi_0}(r_0(t,x),y)$$

for multi indices $\alpha \in \{0,1,2\}^n$ with $|\alpha| \leqslant 2$. Using the unfolding operator, we can translate the two-scale convergence in classical L^p-convergence, namely into the strong convergence

$$\mathcal{T}_\varepsilon(\partial_{y_\alpha}\breve{\psi}(r_\varepsilon,\cdot/\varepsilon)) \to \partial_{y_\alpha}\breve{\psi_0}(r_0,\cdot) \tag{4.62}$$

in $L^p((0,T)\times\Omega\times Y)$ for every $p \in (1,\infty)$. We rewrite the left-hand side of (4.62) by

$$\mathcal{T}_\varepsilon(\partial_{y_\alpha}\breve{\psi}(r_\varepsilon,\cdot/\varepsilon))(t,x,y) = \partial_{y_\alpha}\breve{\psi}(r_\varepsilon(t,[x]_{\varepsilon,Y}+\varepsilon y),([x]_{\varepsilon,Y}+\varepsilon y)/\varepsilon)) = \partial_{y_\alpha}\breve{\psi}(r_\varepsilon(t,x),y). \tag{4.63}$$

Due to the strong convergence of r_ε, we can pass to a subsequence r_ε such that, for a.e. $(t,x) \in \Omega \times Y$, $r_\varepsilon(t,x) \to r_0(t,x)$. This pointwise convergence, can be transferred via the continuity of ∂_{y_α} and (4.63) to the pointwise convergence

$$\mathcal{T}_\varepsilon(\partial_{y_\alpha}\breve{\psi}(r_\varepsilon,\cdot/\varepsilon))(t,x,y) \to \partial_{y_\alpha}\breve{\psi_0}(r_0(t,x),y) \tag{4.64}$$

for a.e. $(t,x,y) \in (0,T)\times\Omega\times Y$. Together with the bound for $|\mathcal{T}_\varepsilon(\partial_{y_\alpha}\breve{\psi}(r_\varepsilon,\cdot/\varepsilon))(t,x,y)| \leqslant \|\partial_{y_\alpha}\breve{\psi}\|_{C([r_{\min},r_{\max}]\times Y)}$, we can apply Lebesgue's dominated convergence theorem and get (4.62), for this subsequence. Since this argument is valid for every arbitrary subsequence, we obtain the convergence for the whole sequence.

The convergence for the time-derivatives can be shown analogously. Namely, we rewrite

$$\begin{aligned}
&\varepsilon^{-1}\partial_t\psi_\varepsilon(t,x) = \varepsilon^{-1}\partial_t\breve{\psi_\varepsilon}(t,x) = \partial_r\breve{\psi}(r_\varepsilon(t,x),x/\varepsilon)\partial_t r_\varepsilon(t,x), \\
&\partial_x\partial_t\psi_\varepsilon(t,x) = \partial_x\partial_t\breve{\psi_\varepsilon}(t,x) = \partial_y\partial_r\breve{\psi}(r_\varepsilon(t,x),x/\varepsilon)\partial_t r_\varepsilon(t,x), \\
&\varepsilon\partial_x\partial_x\partial_t\psi_\varepsilon(t,x) = \partial_x\partial_x\partial_t\breve{\psi_\varepsilon}(t,x) = \partial_y\partial_y\partial_r\breve{\psi}(r_\varepsilon(t,x),x/\varepsilon)\partial_t r_\varepsilon(t,x)
\end{aligned}$$

and, similarly, the limit functions

$$\begin{aligned}
&\partial_r\breve{\psi}(r_0(t,x),y)\partial_t r_0(t,x) = \partial_t\breve{\psi_0}(r_0(t,x),y) = \psi_0(t,x,y) - y, \\
&\partial_y\breve{\psi}(r_0(t,x),y)\partial_t r_0(t,x) = \partial_y\breve{\psi_0}(r_0(t,x),y) = \partial_y\breve{\psi_0}(r_0(t,x),y), \\
&\partial_y\partial_y\breve{\psi}(r_0(t,x),y)\partial_t r_0(t,x) = \partial_y\partial_y\breve{\psi_0}(r_0(t,x),y).
\end{aligned}$$

Then, we can translate the convergence into classical L^p-convergence with the unfolding operator and argue via the pointwise convergence as above. □

Having the above compactness and two-scale convergence results, we can pass to the two-scale limit in (4.25), which yields the following two-scale limit problem:

Weak form of the two-scale limit problem of the reactive transport

Find $(\hat{u}_0, \hat{u}_1) \in L^2(0,T;H^1(\Omega)) \times L^2((0,T)\times\Omega; H^1_\#(Y^*)/\mathbb{R})$ and $r_0 \in L^\infty((0,T)\times\Omega)$ with $\partial_t r_0 \in L^\infty((0,T)\times\Omega)$ and $\partial_t(\Theta(r_0)\hat{u}_0) \in L^2(0,T;H^1(\Omega)')$ such that for a.e. $t \in (0,T)$

$$
\begin{aligned}
\langle \partial_t(\Theta(r_0)\hat{u}_0)(t), \varphi\rangle_{H^1(\Omega)',H^1(\Omega)} + \int_\Omega\int_{Y^*} & A_0(t,x,y) D\Psi_0^{-\top}(t,x,y) \\
(\nabla_x \hat{u}_0(t,x) + \nabla_y \hat{u}_1(t,x,y)) & \cdot (\nabla_x \varphi_0(x) + \nabla_y \varphi_1(x,y)) \,\mathrm{d}y\,\mathrm{d}x \\
&= \int_\Omega \big(\Theta(r_0(t,x)) f(t,x) - c_s \partial_t(\Theta(r_0(t,x)))\big)\varphi_0(x)\,\mathrm{d}x, \\
\partial_t r_0(t,x) &= \tfrac{1}{c_s} g(\hat{u}_0(t,x), r_0(t,x)) \\
\Theta(r_0) &= 1 - V_n(r_0), \\
(\Theta(r_0)\hat{u}_0)(0) &= \Theta(r_0^{\mathrm{in}})\hat{u}_0^{\mathrm{in}} \\
r_0(0) &= r_0^{\mathrm{in}}
\end{aligned}
\tag{4.65}
$$

for all $(\varphi_0, \varphi_1) \in L^2(0,T;H^1(\Omega)) \times L^2((0,T)\times\Omega; H^1_\#(Y^*)/\mathbb{R})$.

Since $r_0 \in L^\infty((0,T)\times\Omega)$, we get $\Theta(r_0) \in L^\infty((0,T)\times\Omega)$, and with $u_0 \in L^2((0,T)\times\Omega)$ we obtain $\Theta(r_0)u_0 \in L^2((0,T)\times\Omega) \subset L^2(0,T;H^1(\Omega)')$. Since $\partial_t(\Theta(r_0)u_0) \in L^2(0,T;H^1(\Omega)')$, we can infer $\Theta(r_0)u_0 \in C([0,T];H^1(\Omega)')$ and, thus, the initial condition $(\Theta(r_0)\hat{u}_0)(0) = \Theta(r_0^{\mathrm{in}})\hat{u}_0^{\mathrm{in}}$ of (4.65) makes sense in $H^1(\Omega)'$.

Theorem 4.20. *Let $(\hat{u}_\varepsilon, r_\varepsilon) \in L^2(0,T;H^1(\Omega_\varepsilon)) \times C^{0,1}([0,T];[r_{min}, r_{max}])^{|I_\varepsilon|}$ be the solution of (4.25). Then, for every subsequence $(\hat{u}_\varepsilon, r_\varepsilon)$ there exists a further subsequence $(\hat{u}_\varepsilon, r_\varepsilon)$ such that*

$$\hat{u}_\varepsilon \overset{2,2}{\twoheadrightarrow} \hat{u}_0, \tag{4.66}$$

$$\nabla\hat{u}_\varepsilon \overset{2,2}{\rightharpoonup} \chi_{Y^*}\nabla_x\hat{u}_0 + \nabla\hat{u}_1, \tag{4.67}$$

$$r_\varepsilon \to r_0 \quad \text{in } L^\infty(0,T;L^p(\Omega)), \tag{4.68}$$

$$\partial_t r_\varepsilon \to \partial_t r_0 \quad \text{in } L^p((0,T)\times\Omega) \tag{4.69}$$

for every $p \in [1,\infty)$, where $(\hat{u}_0, \hat{u}_1, r_0) \in L^2(0,T;H^1(\Omega)) \times L^2((0,T)\times\Omega; H^1_\#(Y^*)/\mathbb{R}) \times L^\infty((0,T)\times\Omega)$ is a solution of (4.65).

Proof. Having the uniform a-priori estimates (4.26), we obtain $\hat{u}_0, \hat{u}_1 \in L^2(0,T;H^1(\Omega)) \times$

$L^2((0,T)\times\Omega; H^1_\#(Y^*)/\mathbb{R})$ such that for a subsequence

$$\hat{u}_\varepsilon \xrightarrow{2,2} \chi_{Y^*}\hat{u}_0, \qquad \nabla\hat{u}_\varepsilon \xrightarrow{2,2} \chi_{Y^*}\nabla_x\hat{u}_0 + \nabla_x\hat{u}_1. \tag{4.70}$$

With (4.45), we can additionally control the time shift of $\hat{u}_\varepsilon$, and we can apply Proposition 4.17, which provides a function $v_0 \in L^2(0,T;L^2(\Omega))$ such that

$$E_\varepsilon\hat{u}_\varepsilon \to v_0 \text{ in } L^2((0,T)\times\Omega) \tag{4.71}$$

after passing to a further subsequence. By multiplying $E_\varepsilon\hat{u}_\varepsilon$ with $\chi_{\Omega_\varepsilon}$ and passing to the limit $\varepsilon \to 0$, we can identify $v_0 = \hat{u}_0$ and, thus, the first convergence in (4.70) is in fact strong for this subsequence.

In the next step, we transfer the strong two-scale convergence with the unfolding operator $\mathcal{T}_\varepsilon$ on the trace of $\hat{u}_\varepsilon$. The strong convergence (4.71) implies the strong two-scale convergence of $E_\varepsilon\hat{u}_\varepsilon$, and hence

$$\mathcal{T}_\varepsilon(E_\varepsilon\hat{u}_\varepsilon) \to u_0 \qquad \text{in } L^2((0,T)\times\Omega\times Y). \tag{4.72}$$

Using the identity $\mathcal{T}_\varepsilon(\nabla E_\varepsilon\hat{u}_\varepsilon) = \varepsilon^{-1}\nabla_y\mathcal{T}_\varepsilon(E_\varepsilon\hat{u}_\varepsilon)$, the isometry of $\mathcal{T}_\varepsilon$ and the uniform boundedness of $\nabla\hat{u}_\varepsilon$, we obtain

$$\begin{aligned}\|\nabla_y\mathcal{T}_\varepsilon(E_\varepsilon\hat{u}_\varepsilon)\|_{L^2((0,T)\times\Omega\times Y)} &= \varepsilon\|\mathcal{T}_\varepsilon(\nabla E_\varepsilon\hat{u}_\varepsilon)\|_{L^2((0,T)\times\Omega\times Y)} = \varepsilon\|\nabla E_\varepsilon\hat{u}_\varepsilon\|_{L^2((0,T)\times\Omega)}\\ &\leqslant \varepsilon\|\nabla\hat{u}_\varepsilon\|_{L^2((0,T)\times\Omega)} \leqslant \varepsilon C \to 0,\end{aligned}$$

Since $\hat{u}_0$ is independent of y, we can deduce with (4.72) and the trace operator for Γ

$$\begin{aligned}&\|\mathcal{T}_\varepsilon(\hat{u}_\varepsilon) - \hat{u}_0\|_{L^2((0,T)\times\Omega\times\Gamma)}\\ &\qquad\leqslant C\|\mathcal{T}_\varepsilon(\hat{u}_\varepsilon) - \hat{u}_0\|_{L^2((0,T)\times\Omega\times Y^*)} + C\|\nabla_y(\mathcal{T}_\varepsilon(\hat{u}_\varepsilon) - \hat{u}_0)\|_{L^2((0,T)\times\Omega\times Y^*)}\\ &\qquad\leqslant C\|\mathcal{T}_\varepsilon(E_\varepsilon\hat{u}_\varepsilon) - \hat{u}_0\|_{L^2((0,T)\times\Omega\times Y)} + C\|\nabla_y\mathcal{T}_\varepsilon(E_\varepsilon\hat{u}_\varepsilon)\|_{L^2((0,T)\times\Omega\times Y)} \to 0,\end{aligned} \tag{4.73}$$

which is equivalent to the strong two-scale convergence $\hat{u}_\varepsilon \xrightarrow{2,2} \hat{u}_0$ on Γ_ε.

Having the strong convergence of $\hat{u}_\varepsilon$, we can infer with Lemma 4.18 the strong convergence of r_ε and afterwards with Lemma 4.19 the strong two-scale convergence of ψ_ε and its derivatives. The convergence of ψ_ε can be transferred by Lemma 2.44 to the strong two-scale convergence of the coefficients

$$\Psi_\varepsilon \xrightarrow{<\infty,<\infty} \Psi_0, \quad \Psi_\varepsilon^{-\top} \xrightarrow{<\infty,<\infty} \Psi_0^{-\top}, \quad A_\varepsilon \xrightarrow{<\infty,<\infty} A_0, \quad J_\varepsilon \xrightarrow{<\infty,<\infty} J_0. \tag{4.74}$$

Now, we have all necessary convergences for the individual terms in order to pass to the limit $\varepsilon \to 0$ in (4.25). For this, we test (4.25) with $\varphi_0(t,x) + \varepsilon\varphi_1\big(t,x,\frac{x}{\varepsilon}\big)$ for $\varphi_0 \in C^\infty([0,T];C^\infty(\overline{\Omega}))$ and $\varphi_1 \in C^\infty([0,T];C^\infty(\overline{\Omega};C^\infty_\#(\overline{Y})))$ with $\varphi_0(T) = \varphi_1(T) = 0$ and integrate the time-derivative term by parts, which leads to

$$\begin{aligned}
&\int\limits_{\Omega_\varepsilon} J_\varepsilon(0,x)\hat{u}_\varepsilon^{\text{in}}(x)\left(\varphi_0(0,x) + \varepsilon\varphi_1\left(0,x,\tfrac{x}{\varepsilon}\right)\right)\mathrm{d}x \\
&\quad - \int\limits_0^T\int\limits_{\Omega_\varepsilon} J_\varepsilon(t,x)\hat{u}_\varepsilon(t,x)\left(\partial_t\varphi_0(t,x) + \varepsilon\partial_t\varphi_1\left(t,x,\tfrac{x}{\varepsilon}\right)\right)\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int\limits_0^T\int\limits_{\Omega_\varepsilon} A_\varepsilon(t,x)D\Psi_\varepsilon^{-\top}(t,x)\nabla\hat{u}_\varepsilon(t,x)\left(\nabla_x\varphi_0(t,x) + \varepsilon\nabla_x\varphi_1\left(t,x,\tfrac{x}{\varepsilon}\right) + \nabla_y\varphi_1\left(t,x,\tfrac{x}{\varepsilon}\right)\right)\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int\limits_0^T\int\limits_{\Omega_\varepsilon} A_\varepsilon(t,x)\partial_t\psi_\varepsilon(t,x)\hat{u}_\varepsilon(t,x)\left(\nabla_x\varphi_0(t,x) + \varepsilon\nabla_x\varphi_1\left(t,x,\tfrac{x}{\varepsilon}\right) + \nabla_y\varphi_1\left(t,x,\tfrac{x}{\varepsilon}\right)\right)\mathrm{d}x\,\mathrm{d}t \\
&= \int\limits_0^T\int\limits_{\Omega_\varepsilon} J_\varepsilon(t,x)\hat{f}_\varepsilon(t,x)\left(\varphi_0(t,x) + \varepsilon\varphi_1\left(t,x,\tfrac{x}{\varepsilon}\right)\right)\mathrm{d}x\,\mathrm{d}t \\
&\quad - \sum_{k\in I_\varepsilon}\int\limits_0^T\int\limits_{\Gamma_{\varepsilon,k}}\left(\frac{r_\varepsilon(t,x)}{R}\right)^{n-1}\varepsilon g(\hat{u}_\varepsilon(t,x), r_{\varepsilon,k}(t))\left(\nabla_x\varphi_0(t,x) + \varepsilon\varphi_1\left(t,x,\tfrac{x}{\varepsilon}\right)\right)\mathrm{d}\sigma_x\,\mathrm{d}t.
\end{aligned} \tag{4.75}$$

For all terms besides the boundary term, we can pass to the limit by standard arguments. We note that the fourth summand vanishes since it is of order ε and, therefore, we do not need the strong convergence of $\partial_t r_\varepsilon$ for the identification of the limit equations.

In order to pass to the limit in the boundary term, we rewrite it with the unfolding operator and use (4.73)

$$\begin{aligned}
&\sum_{k\in I_\varepsilon}\int\limits_0^T\int\limits_{\Gamma_{\varepsilon,k}}\left(\frac{r_\varepsilon(t,x)}{R}\right)^{n-1}\varepsilon g(\hat{u}_\varepsilon, r_{\varepsilon,k})\left(\varphi_0(t,x) + \varepsilon\varphi_1\left(t,x,\tfrac{x}{\varepsilon}\right)\right)\mathrm{d}\sigma_x\,\mathrm{d}t \\
&\quad = \int\limits_0^T\int\limits_\Omega\int\limits_\Gamma\left(\frac{r_\varepsilon(t,x)}{R}\right)^{n-1} g(\mathcal{T}_\varepsilon(\hat{u}_\varepsilon)(t,x,y), \mathcal{T}_\varepsilon(r_\varepsilon)(t,x)) \\
&\qquad\qquad \left(\mathcal{T}_\varepsilon(\varphi_0)(t,x,y) + \varepsilon\mathcal{T}_\varepsilon\left(\varphi_1\left(\cdot,\cdot,\tfrac{\cdot}{\varepsilon}\right)\right)(t,x,y)\right)\mathrm{d}\sigma_y\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \to \int\limits_0^T\int\limits_\Omega\int\limits_\Gamma\left(\frac{r_0(t,x)}{R}\right)^{n-1} g(\hat{u}_0(t,x), r_0(t,x))\varphi_0(t,x)\,\mathrm{d}\sigma_y\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{4.76}$$

With the identity $S_{n-1}(r) = \partial_r V_n(r)$ and the ordinary differential equation for r_0, we get

$$\begin{aligned}|\Gamma| \left(\tfrac{r_0(t,x)}{R}\right)^{n-1} g(\hat{u}_0(t,x), r_0(t,x)) &= S_{n-1}(r_0(t,x)) c_s \partial_t r_0(t,x) \\ &= c_s \partial_r V_n(r_0(t,x)) \partial_t r_0(t,x) \\ &= c_s \partial_t (V_n(r_0(t,x))),\end{aligned}$$

which allows us to rewrite the right-hand side of (4.76), leading to

$$\begin{aligned}&\sum_{k\in I_\varepsilon} \int_0^T \int_{\Gamma_{\varepsilon,k}} \left(\frac{r_\varepsilon(t,x)}{R}\right)^{n-1} \varepsilon g(\hat{u}_\varepsilon, r_{\varepsilon,k}) \left(\varphi_0(t,x) + \varepsilon\varphi_1\left(t,x,\tfrac{x}{\varepsilon}\right)\right) \mathrm{d}\sigma_x \, \mathrm{d}t \\ &\to \int_0^T \int_\Omega c_s \partial_t V_n(r_0(t,x)) \varphi_0(t,x) \, \mathrm{d}x \, \mathrm{d}t.\end{aligned}$$

Then, we obtain for the limit $\varepsilon \to 0$ in (4.75)

$$\begin{aligned}&\int_\Omega \int_{Y^*} J_0(0,x,y) \, \mathrm{d}y \; \hat{u}_0^{\mathrm{in}}(x)\varphi_0(0,x) - \int_0^T \int_\Omega \int_{Y^p} J_0(t,x,y) \, \mathrm{d}y \; \hat{u}_0(t,x) \partial_t \varphi_0(t,x) \, \mathrm{d}x \, \mathrm{d}t \\ &+ \int_0^T \int_\Omega \int_{Y^p} A_0(t,x,y) D\Psi_0^{-\top}(t,x,y)(\nabla_x \hat{u}_0(t,x) + \nabla_y \hat{u}_1(t,x,y)) \\ &\qquad \cdot (\nabla_x \varphi_0(t,x) + \nabla_y \varphi_1(t,x,y)) \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}t \\ &= \int_0^T \int_\Omega \int_{Y^p} J_0(t,x,y) f(t,x) \, \mathrm{d}y \; \varphi_0(t,x) - c_s \partial_t V_n(r_0(t,x)) \varphi_0(t,x) \, \mathrm{d}x \, \mathrm{d}t.\end{aligned} \tag{4.77}$$

With the identity $\int_{Y^*} J_0(t,x,y) \, \mathrm{d}y = \Theta(t,x) = 1 - V_n(r_0(t,x))$, integration by parts of the time-derivative term in (4.77) and the fundamental lemma of the calculus of variations, we get (4.65). Finally, by a density argument, (4.65) holds for all $(\varphi_0, \varphi_1) \in H^1(\Omega_\varepsilon) \times L^2(\Omega; H^1_\#(Y^*))$. ☐

The homogenised limit equations

In order to derive the homogenised limit system, we separate the micro- and macroscopic variables in (4.65). Then, we parameterise the reference cell by means of the radius, i.e. we employ the identity $\psi_0(t,x,y) = \psi(r(t,x); y)$. For this, we define the Jacobian matrix of ψ, its determinant and its adjugate matrix by

$$\Psi(r;y) := \partial_y \psi(r;y), \qquad J_\varepsilon(r;y) := \det(\Psi(r;y)), \qquad A(r;y) := \mathrm{Adj}(\Psi(r;y)) \tag{4.78}$$

for every $(r, y) \in [r_{\min}, r_{\max}] \times \overline{Y}$.

Then, we obtain the following homogenised limit system:

Effective coupled reactive transport system in the reference coordinates

$$\begin{aligned} \partial_t(\Theta(r_0)u_0) - \operatorname{div}(D^*(r_0)\nabla u_0) &= \Theta f + c_s\partial_t\Theta(r_0) \\ \partial_t r_0 &= \tfrac{1}{c_s} g(u_0, r_0), \\ \Theta(r_0) &= 1 - V_n(r_0), \\ (\Theta(r_0)u_0)(0) &= \Theta(r_0^{\text{in}})u_0^{\text{in}}, \\ r_0(0) &= r_0^{\text{in}}, \end{aligned} \tag{4.79}$$

where the homogenised coefficient $D^* : [r_{\min}, r_{\max}] \to \mathbb{R}^{n\times n}$ is given by

$$D^*_{ij}(r) := \int_{Y^*} A(r; y)D\Psi^{-\top}(r; y)(e_j + \nabla_y\hat{\zeta}_j(r; y)) \cdot e_i \,\mathrm{d}y \tag{4.80}$$

where $\hat{\zeta}_i(r; \cdot) \in H^1_\#(Y^*)$, for $[r_{\min}, r_{\max}]$, is the unique solution of

$$\begin{aligned} -\operatorname{div}_y(A(r; y)D\Psi^{-\top}(r; y)(e_j + \nabla_y\hat{\zeta}_j(r; y))) &= 0 && \text{in } Y^*, \\ (A(r; y)D\Psi^{-\top}(r; y)(e_j + \nabla_y\hat{\zeta}_j(r; y))) \cdot n &= 0 && \text{on } \Gamma, \\ y \mapsto \hat{\zeta}_j(r; y) & && Y\text{-periodic.} \end{aligned}$$

In order to formulate the limit problem in terms of the natural upscaled domains, we transform the cell problems from Y^* onto $Y_r^* := Y\backslash\overline{B_r(\mathfrak{m})}$ via $\psi(r; \cdot)$. Moreover, we define the interface for an upscaled cell with obstacle radius r by $\Gamma_r := \partial B_r(\mathfrak{m})$.

Following the arguments of Chapter 2, we transforming the solutions of the cell problems via

$$\begin{aligned} \hat{\zeta}_i(r; y) &= \zeta_i(r; \psi(r; y)) + \breve{\psi}(r; y) \cdot e_i && \text{in } Y^*, \\ \zeta_i(r; y) &= \hat{\zeta}_i(r; \psi(r; y)) + \widetilde{\psi^{-1}}(r; y) \cdot e_i && \text{in } Y_r^*. \end{aligned}$$

This leads to transformation-independent cell problems and a transformation-independent formula for the effective diffusion coefficient D^*. The complete transformation-independent homogenised limit problem is given as follows.

Effective coupled reactive transport system

$$\begin{aligned} \partial_t(\Theta(r_0)u_0) - \operatorname{div}(D^*(r_0)\nabla u_0) &= \Theta f + c_s\partial_t\Theta(r_0) \\ \partial_t r_0 &= \tfrac{1}{c_s} g(u_0, r_0), \\ \Theta(r_0) &= 1 - V_n(r_0), \\ (\Theta(r_0)u_0)(0) &= \Theta(r_0^{\text{in}})u_0^{\text{in}}, \\ r_0(0) &= r_0^{\text{in}}, \end{aligned} \tag{4.81}$$

where the homogenised coefficient $D^* : [r_{\min}, r_{\max}] \to \mathbb{R}^{n\times n}$ is given by

$$D^*_{ij}(r) = \int_{Y_r^*}\int_{Y^*} D(e_j + \nabla_y \zeta_j(r;y)) \cdot e_i \,\mathrm{d}y, \tag{4.82}$$

where $\zeta_i(r;\cdot) \in H^1_\#(Y_r^*)$, for $[r_{\min}, r_{\max}]$, is the unique solution of

$$\begin{aligned} -\operatorname{div}_y(D(e_j + \nabla_y\zeta_j(r;y))) &= 0 && \text{in } Y_r^*, \\ (e_j + \nabla_y\zeta_j(r;y))) \cdot n &= 0 && \text{on } \Gamma_r, \\ y \mapsto \zeta_j(r;y) & && Y\text{-periodic.} \end{aligned}$$

The weak form of (4.79) and (4.81) is given by:

Weak form of the effective coupled reactive transport system

Find $u_0 \in L^2(0,T;H^1(\Omega))$ and $r_0 \in L^\infty(0,T\times\Omega)$ with $\partial_t r_0 \in L^\infty(0,T\times\Omega)$ and $(\Theta(r_0)u_0) \in L^2(0,T;H^1(\Omega)')$ such that, for a.e. $t \in (0,T)$

$$\begin{aligned} \langle \partial_t(\Theta(r_0u_0)(t), \varphi)\rangle_{H^1(\Omega)';H^1(\Omega)} &+ (D^*(r_0(t))\nabla u_0(t), \nabla\varphi(t))_{L^2(\Omega)} \\ &= (\Theta(r_0(t))f(t) + c_s\partial_t\Theta(r_0(t)), \varphi(t))_{L^2(\Omega)}, \\ \partial_t r_0(t) &= \tfrac{1}{c_s} g(u_0(t), r_0(t)), \\ \Theta(r_0) &= 1 - V_n(r_0), \\ (\Theta(r_0)u_0)(0) &= \Theta(r_0^{\text{in}}){u_0}^{\text{in}}, \\ r_0(0) &= r_0^{\text{in}} \end{aligned} \tag{4.83}$$

for all $(\varphi_0, \varphi_1) \in L^2(0,T;H^1(\Omega)) \times L^2((0,T)\times\Omega; H^1_\#(Y^*))$. The initial condition $(\Theta(r_0)u_0)(0) = \Theta(r_0^{\text{in}}){u_0}^{\text{in}}$ holds in $H^1(\Omega)'$.

Theorem 4.21. *Let* $(\hat{u}_0, \hat{u}_1, r_0)$ *be the solution of* (4.65)*. Then,* (u_0, r_0)*, for* $\hat{u}_0 = u_0$*, solves* (4.83)*, where the effective diffusivity* D^* *is given by* (4.80) *or equivalently by* (4.82)*.*

Proof. Following Chapter 2, we can separate the micro- and macroscopic variables in

(4.65). Then, we use the identity $\psi_0(t,x,y) = \psi(r_0(t,x),y)$ in order to parameterise the cell problems and the effective coefficient. Finally, we transform the cell problems and the formula of the effective coefficients from the fixed reference domain Y^* to the domain Y_r^* by arguments as in Chapter 2. □

Remark 4.22. *After a substitution of r in terms of Θ one can reparameterise D^* and g such that this limit system can be formulated in terms of the local porosity and u_0 only without the internal variable r_0.*

Remark 4.23. *In the two-scale limit system* (4.65) *and the homogenised system* (4.83)*, the initial values for Θu_0 hold only in $H^1(\Omega)'$ and not in $L^2(\Omega)$. The reason for this is that $\Theta(r_0)u_0$ cannot be embedded a-priori into $C([0,T];L^2(\Omega))$.*

With the following additional assumptions, we get $\Theta(r_0)u_0 \in C([0,T];L^2(\Omega))$ and can improve the space for the initial condition. Moreover, it allows us to reformulate the initial condition in terms of u_0 only. If we assume that the initial value r_0^{in} has higher regularity, namely $r_0^{\text{in}} \in H^1(\Omega)$, the ordinary differential equation for r_0 becomes an equation in $H^1(\Omega)$. This improves the regularity of r_0 to $C([0,T];H^1(\Omega) \cap L^\infty(\Omega))$ and due to the polynomial structure of Θ and the uniform boundedness of r_0 from below, we obtain $\Theta(r_0), \Theta(r_0)^{-1} \in L^\infty(0,T;H^1(\Omega) \cap L^\infty(\Omega))$. Moreover, under the additional assumption that the initial values $\hat{u}_\varepsilon$ are uniformly essentially bounded, i.e. $\|\hat{u}_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} \leqslant C$, it can be shown that $\hat{u}_\varepsilon$ is also uniformly essentially bounded, i.e. $\|\hat{u}_\varepsilon\|_{L^\infty((0,T)\times\Omega_\varepsilon)} \leqslant C$ (see also [WP23, Theorem 5]). Therefore, the two-scale limit of $\hat{u}_\varepsilon$ is essentially bounded and we can restrict the weak forms (4.65) *and* (4.83) *to solutions u_0 that are in $L^\infty((0,T) \times \Omega)$. Having these additional regularities, we obtain $\Theta(r_0)u_0 \in L^2(0,T;H^1(\Omega))$ and with $\partial_t(\Theta(r_0)u_0) \in L^2(0,T;H^1(\Omega)')$, we get $(\Theta(r_0)u_0) \in C([0,T];L^2(\Omega))$ and consequently, $u_0 = \Theta(r_0)^{-1}\Theta(r_0)u_0 \in C([0,T];L^2(\Omega))$. Thus, we can formulate the initial condition in the space $L^2(\Omega)$ and in terms of u_0 or equivalently in terms of $\Theta(r_0)u_0$.*

Remark 4.24. *In Theorem 4.20, the convergence of $\hat{u}_\varepsilon$ and r_ε is formulated only for a subsequence. If the solution of* (4.65) *and* (4.83) *is unique, the convergence holds for the whole sequence.*

The argument that we have used to show the uniqueness in the ε-scaled problem, cannot be used for the limit system since it requires to controlling ∇J_ε in $L^\infty((0,T) \times \Omega_\varepsilon)$. This would correspond to control $\nabla\Theta$ or ∇r_0 in $L^\infty((0,T) \times \Omega)$. If the solutions have a higher regularity, the uniqueness can be shown by similar arguments as in the ε-scaled problem. Otherwise arguments as in [Ott96] may be useful.

Chapter 5.

Conclusion and outlook

We discussed the homogenisation of several processes in non-periodically perforated domains, which may evolve in time. For this, we transformed the corresponding differential equations onto a periodically perforated substitute domain. This translates the non-periodicity of the domain into a non-periodicity of additional coefficients in the equations. The resulting substitute problems in the periodic domain can be homogenised by means of two-scale convergence, which can handle the non-periodic coefficients arising from the transformation of non-periodic domains. We derived a generic framework for which the homogenisation of the substitute problem is equivalent to the homogenisation of the actual problem. For this, we employed a family of ε-scaled coordinate transformations ψ_ε and a family of cell transformations ψ_0. We showed that the coordinate transformations commute with the two-scale convergence in the sense

$$u_\varepsilon(x) \xrightarrow{2} u_0(x,y) \quad \text{if and only if} \quad u_\varepsilon(\psi_\varepsilon(x)) \xrightarrow{2} u_0(x, \psi_0(x,y)),$$

which justified the homogenisation of the transformed problem. In particular, we formulated the assumptions on ψ_ε by purely asymptotic statements and did not employ any structural assumptions, which leads to a very general setting and allows for purely compactness argument based proofs. Moreover, we provided an additional transformation result for the correctors, which arise in the homogenisation. This enables the back-transformation of the homogenisation result leading to transformation-independent homogenisation results in physically meaningful domains. We transferred this appproach also to the case of time-dependent microstructure by parameterising the transformations ψ_ε and ψ_0 over a time-interval.

We applied this transformation method to homogenise the quasi-stationary and the instationary Stokes equations in a porous medium with locally evolving cavities. For the quasi-stationary case, this led to a quasi-stationary Darcy law, i.e.

$$\begin{aligned} v &= K(f - \nabla p), \\ \operatorname{div}(v) &= -\partial_t \Theta \end{aligned}$$

with a time- and space-dependent permeability $K = K(t,x)$ and an inhomogeneous divergence condition. The permeability tensor can capture locally different microstructure and the divergence condition incorporates the local change of the porosity leading to an additional source or sink term for the pressure. For the instationary case, the homogenisation led to a Darcy law with memory

$$\begin{aligned} v &= v^{\mathrm{in}} + \int_0^t K_{\mathrm{in}}(t,s)(f - \nabla p)(s)\,\mathrm{d}s \\ \operatorname{div}(v) &= -\partial_t \Theta, \end{aligned}$$

with a permeability tensor $K_{\mathrm{in}}(t,s) = K_{\mathrm{in}}(s,t,x)$.

Moreover, we homogenised a reaction–diffusion equation in a perforated domain with free boundary. The evolution of the domain is coupled with the concentration and, thus, a-priori unknown. The result is a system for coupled reactive transport

$$\begin{aligned} \partial_t(\Theta u) - \operatorname{div}(A(\Theta)\nabla u) &= \Theta f - c_s \partial_t \Theta, \\ \partial_t \Theta &= g(u, \Theta). \end{aligned}$$

It couples the evolution of the local porosity $\Theta = \Theta(t,x)$ to the unknown concentration by means of a family of ordinary differential equations. Moreover, it adjusts the effective concentration flux via cell problems depending on the local microstructure and rescales the local change in concentration by taking into account the evolution of the porosity.

Outlook

This work can be continued in a number of ways. For instance, the Stokes flow can be coupled with the reactive transport. At the microscopic level this leads to a system consisting of an advection–reaction–diffusion equation, the Stokes equations providing a model for the fluid velocity and a model for the evolution of the pore domain. The coupling of the domain evolution with the unknown concentration can be extended as well. In particular, connected solid domains might be studied. This is not only interesting for the case where the solid domain change due to dissolution or precipitation processes but also for fluid–solid interactions where the solid domains changes due to deformation.

From a more theoretical point of view the following extension is very worth following. Here, the transformation approach is derived for the case where the microstructure is aligned along an ε-scaled grid with only a small distortion, which vanishes in the limit. Thus, the upscaled geometry inside the cells is locally different while the shape of the periodicity cell is macroscopically constant. This setting can only handle microscopic geometry changes which do not affect the cell position. In deformable porous media several processes can affect the cell alignment and, thus, change the macroscopic shape of the porous media. In order to consider the homogenisation for such problems, the framework presented in this work has to be extended. A new notion of local two-scale convergence could be helpful, which can capture locally different sizes and shapes of the reference cell. This would correspond to an extension of the asymptotic behaviour of the transformations ψ_ε taking into account also macroscopic domain evolution. The large number of further research directions shows the potential of the presented transformation method and its importance on the applications in the field of homogenisation.

Appendix A.

Time dependent–differential algebraic-equations

This appendix provides a proof of Theorem 3.33 by means of Rothe's method.

Proposition A.1. *Let the assumptions of Theorem 3.33 hold. Let $N \in \mathbb{N}$ be large enough, $k = T/N$ and $t_i = mk$ for $m \in \{0, 1, \dots, N\}$. We define*

$$a_m := a(t_m), \quad b_m := b(t_m), \quad f_{1,m} := \frac{1}{k}\int_{t_m}^{t_{m+1}} f_1(t)\,\mathrm{d}t, \quad f_{2,m} := f_2(t_m), \quad g_m := g(t_m)$$

for $m \in \{1, \dots, N\}$. Then, there exists a unique solution $v_i \in V$ for $i \in \{0, \dots, N\}$ and $p_i \in Q$ for $i \in \{1, \dots, N\}$ such that

$$\begin{aligned} v_0 &= v^{\mathrm{in}} && \text{in } V, \\ a_m\left(\frac{v_m - v_{m-1}}{k}\right) + b_m v_m + c^* p_m &= f_{1,m} + f_{2,m} && \text{in } V', \\ c v_m &= g_m && \text{in } Q' \end{aligned} \tag{A.1}$$

for all $m \in \{1, \dots, N\}$.

Moreover, there exists a constant C, which is independent of N, such that

$$\max_{m\in\{0,\dots N\}} \|v_m\|_H + k\sum_{m=1}^{N} \|v_m\|_V^2 + \sum_{m=1}^{N} \tfrac{1}{k}\|v_m - v_{v-1}\|_H^2 + \sum_{m=1}^{N} k\|p_m\|_P \leqslant C. \tag{A.2}$$

Moreover, the constant C depends only on $T, C_{V\to H}, C_a,\ C_b, C_{b^1}, C_{b^2}, C_{b^3}, C_{b^4}, \alpha, \beta, \gamma, L_a$, L_{b^1}, L_{b^3}, $\|v^{\mathrm{in}}\|_V$, $\|g\|_{H^1(0,T;Q')}$, $\|f_1\|^2_{L^2(0,T;H')}$, $\|f_2\|_{H^1(0,T;V')}$, which are given in Assumption 3.34, but not on $\|c\|_{\mathcal{L}(V,Q')}$.

Before we prove Proposition A.1 we note that the assumptions on b imply the following Gårding's inequality.

Lemma A.2. *Assume that b is given as in Theorem 3.33 and β as in Assumption 3.34(2.). Then, there exists $c_b > 0$ such that*

$$\tfrac{1}{2}\beta\|v\|_V^2 - c_b\|v\|_H^2 \leqslant b(t)(v, v) \tag{A.3}$$

for all $v \in V$.

Proof. We use the decomposition of b, the coercivity of b^1 and the Young inequality and obtain

$$\begin{aligned} b(t)(v,v) &= b^1(t)(v,v) + b^2(t)(v,v) + b^3(t)(v,v) + b^4(t)(v,v) \\ &\geqslant \|v\|_V^2 - C_{b^2}\|v\|_V\|v\|_H - C_{b^3}\|v\|_V\|v\|_H - C_{b^4}\|v\|_H\|v\|_H \\ &\geqslant \|v\|_V^2 - \tfrac{\beta}{4}\|v\|_V^2 - \tfrac{C_{b^2}^2}{\beta}\|v\|_H^2 - \tfrac{\beta}{4}\|v\|_V^2 - \tfrac{C_{b^3}^2}{\beta}\|v\|_H^2 - C_{b^4}\|v\|_H^2 \\ &\geqslant \tfrac{\beta}{2}\|v\|_V^2 - \underbrace{\left(\frac{C_{b^2}^2}{\beta} + \frac{C_{b^3}^2}{\beta} + C_{b^4}\right)}_{=:c_b}\|v\|_H^2 \end{aligned}$$

for every $t \in [0,T]$ and every $v \in V$. □

Proof of Proposition A.1. We show the existence and uniqueness of a solution $(v_m, p_m) \in V \times P$ for $m \in \{1, \ldots, N\}$. First, we choose $v_0 = v^{\mathrm{in}}$. Then, we rewrite (A.1) into the following saddle point problem:

$$\begin{aligned} \tfrac{1}{k}a_m v_m + b_m v_m + c^* p_m &= \tfrac{1}{k}a_m v_{m-1} + f_{1,m} + f_{2,m} && \text{in } V', \\ c v_m &= g_{1,m} && \text{in } Q'. \end{aligned} \tag{A.4}$$

From the coercivity estimate (3.98) for a_m and Gårding's inequality (A.3), we obtain for N big enough ($\frac{\alpha}{k} \geqslant c_b$)

$$\tfrac{1}{k}a_m(\varphi,\varphi) + b_m(\varphi,\varphi) \geqslant \tfrac{\alpha}{k}\|\varphi\|_H^2 + \tfrac{\beta}{2}\|\varphi\|_V^2 - c_b\|\varphi\|_H^2 \geqslant \tfrac{\beta}{2}\|\varphi\|_V^2 \tag{A.5}$$

for every $m \in \{1, \ldots, N\}$ and every $\varphi \in V$. This provides the coercivity of $\frac{1}{k}a_m + b_m$ in V. Moreover, we have the inf–sup condition for c and the embedding $V \subset H$ yields $\frac{1}{k}v_m + f_{1,m} + f_{2,m} \in V'$. Hence, Proposition 3.5 provides iteratively the existence and uniqueness of a solution $(u_m, p_m) \in V \times Q$ for $m \in \{1, \ldots, N\}$.

In order to derive the a priori estimates, we decompose v_m along the V-orthogonal decomposition $V = V_0 \oplus V_0^{\perp}$ for $V_0 := \ker(c)$, i.e. let $v_m = z_m + w_m$ for all $m \in \{1, \ldots, N\}$ with $z_m \in V_0$ and $w_m \in V_0^{\perp}$.

In the lemmas below, we estimate z_m and w_m separately. Combining the estimate (A.17) for w_m and the estimate (A.20) for z_m yields

$$k\sum_{m=1}^{N}\|v_m\|_V^2 \leqslant C. \tag{A.6}$$

Using additionally the embedding of V in H, we obtain also the pointwise estimate for v_m from (A.16) and (A.18)

$$\|v_m\|_H \leqslant \|z_m\|_H + \|w_m\|_H \leqslant \|z_m\|_H + C_{V\to H}\|w_m\|_V \leqslant C. \tag{A.7}$$

Similarly, we estimate the discrete time derivative with (A.15) and (A.36)

$$\sum_{m=1}^{N} \tfrac{1}{k}\|v_m - v_{v-1}\|_H^2 \leqslant 2\sum_{m=1}^{N} \tfrac{1}{k}\|z_m - z_{v-1}\|_H^2 + \|w_m - w_{v-1}\|_H^2$$
$$\leqslant 2\sum_{m=1}^{N} \tfrac{1}{k}\|z_m - z_{v-1}\|_H^2 + C_{V\to H}\|w_m - w_{v-1}\|_V^2 \leqslant C.$$

Finally, we obtain the estimate for the pressure from Lemma A.10. □

Proof of Theorem 3.33. Let v_0^N and (v_m^N, p_m^N) for $m \in \{1,\dots,N\}$ be the discrete Rothe approximation given by Proposition A.1, where we stress the N-dependence since we pass to the limit $N \to \infty$. Then, we define the piecewise constant function $v^N : [0,T] \to V$ and the piecewise affine function $\bar{v}^N\colon [0,T] \to V$ by

$$\bar{v}^N(t) := \begin{cases} v_0^N & \text{for } t = 0, \\ v_m^N & \text{for } t \in (t_{m-1}, t_m], \end{cases}$$
$$v^N(t) := \begin{cases} v_0^N & \text{for } t = 0, \\ v_m^N + \frac{t-t_m}{k}(v_m^N - v_{m-1}^N) & \text{for } t \in (t_{m-1}, t_m]. \end{cases}$$

Moreover, we define the piecewise constant function q by

$$\bar{q}^N(t) := \begin{cases} q_1^N & \text{for } t = 0, \\ q_m^N & \text{for } t \in (t_{m-1}, t_m]. \end{cases}$$

In the same way, we define the piecewise constant operators $\bar{a}^N : [0,T] \to \mathcal{L}(H,H')$ and $\bar{b}^N : [0,T] \to \mathcal{L}(V,V')$, which we identify with their corresponding Nemytskii operators, i.e. $\bar{a}^N \in \mathcal{L}(L^2(0,T;H); L^2(0,T;H))$, $\bar{b}^N \in \mathcal{L}(L^2(0,T;V); L^2(0,T;V'))$. Moreover, we denote the piecewise constant extensions of the right-hand sides f_1, f_2 and g, by $\bar{f}_1^N, \bar{f}_2^N, \bar{g}^N$. Then, we can reformulate the discretised equation (A.1) into

$$\begin{aligned} \bar{a}^N \partial_t v^N + \bar{b}^N \bar{v}^N + c^* \bar{p}^N &= \bar{f}_1^N + \bar{f}_2^N && \text{in } L^2(0,T;V'), \\ c\bar{v}^N &= \bar{g}^N && \text{in } L^2(0,T;Q'). \end{aligned} \tag{A.8}$$

From the a-priori estimates of Proposition A.1, compactness arguments and standard Rothe method arguments, we obtain $v \in H^1(0,T;H)) \cap L^2(0,T;V)$ and $q \in L^2(0,T;H))$, such that for a subsequence $\bar{v}^N$ converges weakly to v in $L^2(0,T;V)$, $\partial_t v^N$ converges weakly to $\partial_t v$ in $L^2(0,T;H)$ and $\bar{q}^N$ converges weakly to q in $L^2(0,T;H)$.

In order to pass to the limit $N \to \infty$ in (A.8), we identify a and b with their corresponding Nemytskii operators. Then, we obtain

$$\int_0^T \langle \bar{b}^N(t)\bar{v}^N(t) - b(t)v(t), \varphi(t)\rangle_{V',V}\,\mathrm{d}t =$$

$$= \int_0^T \langle (\bar{b}^N(t) - b(t))\bar{v}^N(t), \varphi(t)\rangle + \langle \bar{v}^N(t) - v(t), b^*\varphi(t)\rangle \,\mathrm{d}t \to 0,$$

where the first term on the right-hand side tends to zero due to the uniform Lipschitz continuity of b and the boundedness of $\bar{v}^N$. The second term tends to zero since $b^*\varphi \in L^2(0,T;V')$. With the same argumentation, we can also pass to the limit $N \to \infty$ for the fist term in the left-hand side of (A.8). Moreover, with standard Rothe method arguments, we can pass to the limit in the other terms and can show that the initial values are fulfilled. Thus, (v,p) solves (3.96), which provides the existence of a solution of (3.96). The a-priori estimates are transferred via the limit process from the discrete Rothe approximation onto v and p. The uniqueness of the solution follows from Lemma A.11, which is shown below. □

In order to estimate the Rothe solutions in Proposition A.1, we have to control the right-hand sides by the following lemmas.

Lemma A.3. *Let g_m be defined as in Proposition A.1 for $g \in H^1(0,T;Q')$. Then,*

$$\frac{1}{k}\sum_{m=1}^{N} \|g_m - g_{m-1}\|_{Q'}^2 \leqslant \|\partial_t g\|_{L^2(0,T;Q')}^2 \tag{A.9}$$

and $\|g_m\|_{Q'} \leqslant \|g\|_{C([0,T];Q')} \leqslant C\|g\|_{H^1(0,T;Q')}$ for every $m \in \{1,\dots,N\}$ and a constant C which is independent of m and N.

Proof. With the Hölder inequality, we obtain

$$\|g_m - g_{m-1}\|_{Q'} = \|\int_{t_{m-1}}^{t_m} g(t)\,\mathrm{d}t\|_{Q'} \leqslant \int_{t_{m-1}}^{t_m} \|g(t)\|_{Q'}\,\mathrm{d}t \leqslant k^{\frac{1}{2}}\Big(\int_{t_{m-1}}^{t_m} \|\partial_t g(t)\|_{Q'}^2\,\mathrm{d}t\Big)^{\frac{1}{2}} \tag{A.10}$$

for $m \in \{1,\dots N\}$. Squaring both sides in (A.10), multiplying with $\frac{1}{k}$ and summing over $m \in \{1,\dots,N\}$ yields

$$\frac{1}{k}\sum_{m=1}^{N} \|g_m - g_{m-1}\|_{Q'}^2 \leqslant \sum_{m=1}^{N}\int_{t_{m-1}}^{t_m} \|\partial_t g(t)\|_{Q'}^2\,\mathrm{d}t = \|\partial_t g\|_{L^2(0,T;Q')}^2,$$

which shows (A.9).

The estimate on g_m can be infered from the continuous embedding of $C([0,T];Q')$ in $H^1(0,T;Q')$. □

Lemma A.4. *Let $f_{2,m}$ be defined as in Proposition A.1 for $f_2 \in H^1(0,T;V')$. Then,*

$$\frac{1}{k}\sum_{m=1}^{N} \|f_{2,m} - f_{2,m-1}\|_{V'}^2 \leqslant \|\partial_t f_2\|_{L^2(0,T;V')}^2 \tag{A.11}$$

and $\|f_{2,m}\|_{V'} \leqslant \|f_{2,m}\|_{C([0,T];Q')} \leqslant C\|f_2\|_{H^1(0,T;V')}$ *for every* $m \in \{1, \dots, N\}$ *and a constant* C *which is independent of* m *and* N.

Proof. Lemma A.4 can be shown in the same way as Lemma A.3. ☐

Lemma A.5. *Let* $f_{1,m}$ *be defined as in Proposition A.1 for* $f \in L^2(0,T;H')$*. Then,*

$$k\sum_{m=1}^{N} \|f_{1,m}\|_{H'}^2 \leqslant \|f_1\|_{L^2(0,T;H')} ^2 \tag{A.12}$$

Proof. We obtain the desired result with the Hölder inequality via

$$\begin{aligned} k\sum_{m=1}^{N} \|f_{1,m}\|_{H'}^2 &= \frac{1}{k}\sum_{m=1}^{N} \Big\| \int_{t_{m-1}}^{t_m} f_1(t)\,\mathrm{d}t \Big\|_{H'}^2 \leqslant \frac{1}{k}\sum_{m=1}^{N} \Big(\int_{t_{m-1}}^{t_m} \|f_1(t)\|_{H'}\,\mathrm{d}t \Big)^2 \\ &= \frac{1}{k}\sum_{m=1}^{N} \int_{t_{m-1}}^{t_m} \|f_1(t)\|_{H'}^2\,\mathrm{d}t\,|t_m - t_{m-1}| = \int_0^T \|f_1(t)\|_{H'}^2\,\mathrm{d}t = \|f_1\|_{L^2(0,T;H')}^2. \end{aligned}$$

☐

First, we estimate w_m, which is the part of v_m that is orthogonal to V_0, using the algebraic condition. Therefore, we have to exchange the vector spaces in the inf–sup condition, which can be done by considering only $V_0^\top$.

Lemma A.6. *Let* U, W *be Banach spaces,* $c \in \mathcal{L}(V, Q')$*. Then, the following statements are equivalent:*

- *There exists a constant* $\gamma > 0$ *such that*

$$\inf_{q \in Q\setminus\{0\}} \sup_{v \in V\setminus\{0\}} \frac{|c(v,q)|}{\|v\|_V \|q\|_Q} \geqslant \gamma. \tag{A.13}$$

- *The operator* $c : \ker(c)^\perp \to Q'$ *is an isomorphism and*

$$\|c(v)\|_{Q'} \geqslant \gamma \|v\|_V \text{ for all } v \in \ker(c)^\perp. \tag{A.14}$$

Proof. See for instance [Bra07, Lemma 4.2]. ☐

Lemma A.7. *Let* w_m *for* $m \in \{0, \dots, N\}$ *be given as in the proof of Proposition A.1. Then,*

$$k\sum_{m=1}^{N} \Big\| \frac{w_m - w_{m-1}}{k} \Big\|_V^2 \leqslant \frac{1}{\gamma^2} \|\partial_t g\|_{L^2(0,T;Q')}^2, \tag{A.15}$$

$$\|w_m\|_V^2 \leqslant \frac{1}{\gamma^2}\|g_m\|_{Q'}^2 \leqslant C\|g\|_{H^1(0,T;Q')}^2 \qquad \text{for every } m \in \{0, \dots, N\}, \tag{A.16}$$

$$k \sum_{m=1}^{N} \|w_m\|_V^2 \leqslant \frac{T}{\gamma^2} \|g\|^2_{C([0,T];Q')} \leqslant C\|g\|^2_{H^1(0,T;Q')}. \tag{A.17}$$

Proof. We observe, that $(w_m - w_{m-1}) \in V_0^\perp$ by construction. Hence Lemma A.6 yields

$$\gamma\|w_m - w_{m-1}\|_V \leqslant \|c(w_m - w_{m-1})\|_{Q'}$$

and, moreover, it holds $c(w_m - w_{m-1}) = c(v_m - v_{m-1})$. Then, we can estimate

$$\gamma^2\|w_m - w_{m-1}\|_V^2 \leqslant \|c(w_m - w_{m-1})\|^2_{Q'} = \|c(v_m - v_{m-1})\|^2_{Q'} = \|g_m - g_{m-1}\|^2_{Q'}.$$

After multiplying with $\frac{1}{k}$, summing over $m \in \{1, \dots, N\}$, we obtain with Lemma A.3

$$k \sum_{m=1}^{N} \left\| \frac{w_m - w_{m-1}}{k} \right\|_V^2 \leqslant \frac{1}{\gamma^2} k \sum_{m=1}^{N} \left\| \frac{g_m - g_{m-1}}{k} \right\|_{Q'}^2 \leqslant \frac{1}{\gamma^2} \|\partial_t g\|_{L^2(0,T;Q')},$$

which shows (A.15).

Similarly, we obtain

$$\gamma^2\|w_m\|_V^2 \leqslant \|c(w_m)\|^2_{Q'} = \|c(v_m)\|^2_{Q'} = \|g_m\|^2_{Q'} \leqslant \|g\|^2_{C([0,T];Q')}$$

and after multiplication with k, and summing over $m \in \{1, \dots, N\}$, Lemma A.3 yields

$$\gamma^2 k \sum_{m=1}^{N} \|w_m\|_V^2 \leqslant k \sum_{m=1}^{N} \|g_m\|^2_{L^2(0,T;Q')} \leqslant T\|g\|^2_{C([0,T];Q')} \leqslant C\|g\|^2_{H^1(0,T;Q')}.$$

Thus, we obtain (A.17). □

In the next step, we estimate z_m, i.e. the V-orthogonal projection of v_m to V_0 using the parabolic equation of the saddle point problem. Compared to the estimate for w_m, this estimate does not provide a uniform control of the discrete time derivative and we have to estimate the discrete time derivative afterwards.

Lemma A.8. *Let z_m for $m \in \{0, \dots, N\}$ be given by the proof of Proposition A.1. Then, there exists a constant C_Z, which depends only on $T, C_{V\to H}, C_a, C_b, \alpha, \beta, \gamma, L_a, \|v^{\mathrm{in}}\|_V$, $\|g\|_{H^1(0,T;Q')}$, $\|f_1\|_{L^2(0,T;H')}$, $\|f_2\|_{H^1(0,T;V')}$, such that*

$$\|z_m\|_H^2 \leqslant C_Z \qquad \text{for all } m \in \{1, \dots, N\}, \tag{A.18}$$

$$\sum_{i=1}^{N} \|z_m - z_{m-1}\|_H^2 \leqslant C_Z, \tag{A.19}$$

$$k \sum_{i=1}^{N} \|z_m\|_V^2 \leqslant C_Z. \tag{A.20}$$

Proof. We multiply (A.1) by k and rewrite it employing $v_m = z_m + w_m$

$$a_m(z_m - z_{m-1}) + kb_m z_m + kc^* p_m = -a_m(w_m - w_{m-1}) - kb_m w_m + kf_{1,m} + kf_{2,m}. \tag{A.21}$$

Now, we multiply (A.21) by $2z_m$ and define $\|\varphi\|^2_{m,H} := a_m(\varphi, \varphi)$. Using the symmetry of a_m, we can rewrite the resulting first term of the left-hand side of (A.21) by

$$a_m(z_m - z_{m-1}, 2z_m) = \|z_m - z_{m-1}\|^2_{m,H} + \|z_m\|^2_{m,H} - \|z_{m-1}\|^2_{m,H}.$$

We estimate the second term of the resulting left-hand side of (A.21) from below by (A.3), the third term on the left-hand side of (A.21) vanishes since $z_m \in V_0$. Moreover, we estimate the terms on the right-hand side with the continuity estimates and the Young inequality and obtain in total

$$\begin{aligned} \|z_m - z_{m-1}\|^2_{m,H} + \|z_m\|^2_{m,H} - \|z_{m-1}\|^2_{m,H} + k\beta\|z_m\|^2_V - 2kc_b\|z_m\|^2_H \\ \leqslant 2C_a k\|w_m - w_{m-1}\|_H\|z_m\|_H + 2kC_b\|w_m\|_V\|z_m\|_V \\ + 2k\|f_{1,m}\|_{H'}\|z_m\|_H + 2k\|f_{2,m}\|_{V'}\|z_m\|_V \\ \leqslant C_a \tfrac{1}{k}\|w_m - w_{m-1}\|^2_H + \tfrac{1}{2}k\|z_m\|^2_H + \tfrac{2C_b}{\beta}k\|w_m\|^2_V + \tfrac{\beta}{4}k\|z_m\|^2_V \\ + k\|f_{1,m}\|^2_{H'} + \tfrac{1}{2}k\|z_m\|^2_H + 2k\|f_{2,m}\|^2_{V'} + \tfrac{\beta}{4}k\|z_m\|^2_V. \end{aligned} \tag{A.22}$$

The coercivity estimate of a provides

$$\|z_m\|^2_H \leqslant \tfrac{1}{\alpha}\|z_m\|^2_{m,H} \tag{A.23}$$

and, thus, we can estimate (A.22) further and obtain after rearranging

$$\begin{aligned} (1 - \tfrac{2c_b+1}{\alpha}k)\|z_m\|^2_{m,H} + \|z_m - z_{m-1}\|^2_{m,H} - \|z_{m-1}\|^2_{m,H} + \frac{\beta}{2}k\|z_m\|^2_V \\ \leqslant C_a\tfrac{1}{k}\|w_m - w_{m-1}\|^2_H + \tfrac{2C_b}{\beta}k\|w_m\|^2_V + k\|f_{1,m}\|^2_{H'} + 2k\|f_{2,m}\|^2_{V'}. \end{aligned} \tag{A.24}$$

In the next step, we want to sum (A.24) over m such that the first term on the left-hand side of (A.24) for $m-1$ cancels with the third term on the left-hand side of (A.24). Therefore, we multiply (A.24) with $\lambda(k)^{M-m} \geqslant 0$, where we fix λ later, for $M \in \{1, \dots, N\}$, and obtain

$$\sum_{m=1}^{M} \lambda(k)^{M-m}(1 - \tfrac{2c_b+1}{\alpha}k)\|z_m\|^2_{m,H} + \sum_{m=1}^{M} \lambda(k)^{M-m}\|z_m - z_{m-1}\|^2_{m,H} \\ - \sum_{m=1}^{M} \lambda(k)^{M-m}\|z_{m-1}\|^2_{m,H} + \sum_{m=1}^{M} \lambda(k)^{N-m}\frac{\beta}{2}k\|z_m\|^2_V \\ \leqslant \sum_{m=1}^{M} \lambda(k)^{M-m}\Big(C_a \tfrac{1}{k}\|w_m - w_{m-1}\|^2_H + \tfrac{2C_b}{\beta}k\|w_m\|^2_V + k\|f_{1,m}\|^2_{H'} + 2k\|f_{2,m}\|^2_{V'}\Big). \tag{A.25}$$

The first and third summands on the left hand side of (A.25) can be estimated by

$$\begin{aligned} &\sum_{m=1}^{M}\lambda(k)^{M-m}(1 - \tfrac{2c_b+1}{\alpha}k)\|z_m\|^2_{m,H} - \sum_{m=1}^{M} \lambda(k)^{M-m}\|z_{m-1}\|^2_{m,H} \\ &= \sum_{m=1}^{M} \lambda(k)^{M-m}(1 - \tfrac{2c_b+1}{\alpha}k)\|z_m\|^2_{m,H} - \sum_{m=0}^{M-1} \lambda(k)^{M-(m+1)}\|z_m\|^2_{m+1,H} \\ &\leqslant (1 - \tfrac{2c_b+1}{\alpha}k)\|z_M\|^2_{M,H} - \lambda(k)^{M-1}\|z_0\|^2_{1,H} \\ &\quad + \sum_{m=1}^{M-1} \lambda(k)^{M-m}\Big((1 - \tfrac{2c_b+1}{\alpha}k)\|z_m\|^2_{m,H} - \lambda(k)^{-1}\|z_m\|^2_{m+1,H}\Big). \end{aligned} \tag{A.26}$$

Moreover, the Lipschitz continuity and the coercivity of a yields

$$\begin{aligned} \|z_m\|^2_{m+1,H} &= a_{m+1}(z_m, z_m) = a_m(z_m, z_m) + (a_{m+1} - a_m)(z_m, z_m) \\ &\leqslant \|z_m\|^2_{m,H} + L_a k\|z_m\|^2_H \leqslant \|z_m\|^2_{m,H} + \tfrac{L_a}{\alpha}k a_m(z_m, z_m) \\ &= \|z_m\|^2_{m,H} + \tfrac{L_a}{\alpha}k\|z_m\|^2_{m,H}. \end{aligned} \tag{A.27}$$

Thus, by choosing

$$\lambda(k) \geqslant (1 + \tfrac{L_a}{\alpha}k)(1 - \tfrac{2c_b+1}{\alpha}k)^{-1}, \tag{A.28}$$

we obtain for k small enough

$$\begin{aligned} &(1 - \tfrac{2c_b+1}{\alpha}k)\|z_m\|^2_{m,H} - \lambda(k)^{-1}\|z_m\|^2_{m+1,H} \\ &\quad \geqslant (1 - \tfrac{2c_b+1}{\alpha}k)\|z_m\|^2_{m,H} - \lambda(k)^{-1}(1 + \tfrac{L_a}{\alpha}k)\|z_m\|^2_{m,H} \geqslant 0 \end{aligned}$$

and we can estimate (A.26) further

$$\begin{aligned} &\sum_{m=1}^{M}\lambda(k)^{M-m}(1 - \tfrac{2c_b+1}{\alpha}k)\|z_m\|^2_{m,H} - \sum_{m=1}^{M} \lambda(k)^{M-m}\|z_{m-1}\|^2_{m,H} \\ &\quad \geqslant (1 - \tfrac{2c_b+1}{\alpha}k)\|z_M\|^2_{M,H} - \lambda(k)^{M-1}\|z_0\|^2_{1,H}. \end{aligned} \tag{A.29}$$

Now, we choose $\lambda(k) := 1 + \mu k$ such that (A.28) holds for k small enough, namely, we choose $\mu := \frac{L_a+2(2c_b+1)}{\alpha} + \frac{L_a}{\alpha}2\frac{2c_b+1}{\alpha}$. Then, for $k \leqslant \max\left\{1, \frac{\alpha}{2(2c_b+1)}\right\}$, we obtain with Lemma A.12 from below

$$\begin{aligned}(1+\tfrac{L_a}{\alpha}k)(1-\tfrac{2c_b+1}{\alpha}k)^{-1} &\leqslant (1+\tfrac{L_a}{\alpha}k)(1+2\tfrac{2c_b+1}{\alpha}k)\\ &\leqslant 1+\tfrac{L_a+2(2c_b+1)}{\alpha}k+\tfrac{L_a}{\alpha}2\tfrac{2c_b+1}{\alpha}k^2\\ &\leqslant 1+(\tfrac{L_a+2(2c_b+1)}{\alpha}+\tfrac{L_a}{\alpha}2\tfrac{2c_b+1}{\alpha})k \leqslant 1+\mu k.\end{aligned}$$

For $m \in \{0,\dots,M\}$, we can estimate λ^{M-m} from below by $1 \leqslant \lambda^{M-m}$ and from above by

$$\lambda^{M-m} = e^{(M-m)\ln(\lambda)} = e^{(M-m)\ln(1+\mu k)} \leqslant e^{(M-m)\mu k} \leqslant e^{N\mu k} = e^{T\mu}. \tag{A.30}$$

Combining (A.25),(A.29) and (A.30) gives

$$\begin{aligned}&(1-\tfrac{2c_b+1}{\alpha}k)\|z_M\|^2_{M,H} - e^{T\mu}\|z_0\|^2_{1,H} + \sum_{m=1}^{M}\Big(\|z_m-z_{m-1}\|^2_{m,H} + \frac{\beta}{2}k\|z_m\|^2_V\Big)\\ &\leqslant e^{T\mu}\sum_{m=1}^{M}\Big(C_a\tfrac{1}{k}\|w_m-w_{m-1}\|^2_H + \tfrac{2C_b}{\beta}k\|w_m\|^2_V + k\|f_{1,m}\|^2_{H'} + 2k\|f_{2,m}\|^2_{V'}\Big).\end{aligned} \tag{A.31}$$

Estimating $\|z_0\|^2_{1,H}$ by

$$\|z_0\|^2_{1,H} \leqslant C_a\|z_0\|^2_H \leqslant C_{V\to H}C_a\|z_0\|^2_V \leqslant C_{V\to H}C_a\|v_0\|^2_V = C_{V\to H}C_a\|v^{\mathrm{in}}\|^2_V \tag{A.32}$$

and the right hand-side of (A.31) with Lemma A.4, Lemma A.5 and Lemma A.7 yields

$$\begin{aligned}&(1-\tfrac{2c_b+1}{\alpha}k)\|z_M\|^2_{M,H} + \sum_{m=1}^{M}\Big(\|z_m-z_{m-1}\|^2_{m,H} + \frac{\beta}{2}k\|z_m\|^2_V\Big)\\ &\leqslant e^{T\mu}\Big(C_{V\to H}C_a\|v^{\mathrm{in}}\|^2_V + C^2_{V\to H}\frac{C_a}{\gamma^2}\|\partial_t g\|^2_{L^2(0,T;Q')} + \tfrac{2C_bT}{\beta\gamma^2}\|g\|^2_{C([0,T];Q')}\\ &\qquad + \|f_1\|^2_{L^2(0,T;H')} + 2T\|f_2\|^2_{C([0,T];V')}\Big).\end{aligned} \tag{A.33}$$

With the coercivity of a, we can estimate for $k \leqslant \frac{\alpha}{2(2c_b+1)}$

$$\tfrac{\alpha}{2}\|z_M\|^2_H \leqslant \tfrac{1}{2}\|z_M\|^2_{M,H} \leqslant (1-\tfrac{2c_b+1}{\alpha}k)\|z_M\|^2_{M,H}. \tag{A.34}$$

Then, (A.33)–(A.34) yield (A.18)–(A.20). □

Now, we estimate the discrete time derivative of z_m.

Lemma A.9. *Let z_m, for $m \in \{0,\dots,N\}$, be given by the proof of Proposition A.1. Then, there exists a constant C, which depends only on $T, C_{V\to H}, C_a, C_b, C_{b^1}, C_{b^2}, C_{b^3}, C_{b^4}, \alpha$, β,*

γ, L_a, L_{b^1}, L_{b^3}, $\|v^{\text{in}}\|_V$, $\|g\|_{H^1(0,T;Q')}$, $\|f_1\|^2_{L^2(0,T;H')}$, $\|f_2\|_{H^1(0,T;V')}$ *such that*

$$\|v_m\|_V^2 \leqslant C, \tag{A.35}$$

$$\sum_{m=1}^{N} \tfrac{1}{k}\|z_m - z_{m-1}\|_H^2 \leqslant C. \tag{A.36}$$

Proof. We use the decomposition $v_m - v_{m-1} = w_m - w_{m-1} + (z_m - z_{m-1})$ and multiply (A.1) with $z_m - z_{m-1}$. Then, we obtain with $z_m - z_{m-1} \in V_0$

$$\begin{aligned} &\tfrac{1}{k}a_m(z_m - z_{m-1}, z_m - z_{m-1}) + b_m(v_m, z_m - z_{m-1}) \\ &= -\tfrac{1}{k}a_m(w_m - w_{m-1}, z_m - z_{m-1}) + f_{1,m}(z_m - z_{m-1}) + f_{2,m}(z_m - z_{m-1}). \end{aligned} \tag{A.37}$$

With the coercivity of a, the estimates for f_1 and f_2 and the Young inequality, we obtain

$$\begin{aligned} &\tfrac{\alpha}{k}\|z_m - z_{m-1}\|_H^2 + b_m(v_m, z_m - z_{m-1}) \\ &\quad\leqslant \tfrac{1}{k}C_a\|w_m - w_{m-1}\|_H\|z_m - z_{m-1}\|_H + \|f_{1,m}\|_{H'}\|z_m - z_{m-1}\|_H \\ &\qquad + f_{2,m}(z_m - z_{m-1}) \\ &\quad\leqslant \tfrac{2C_a^2}{\alpha k}\|w_m - w_{m-1}\|_H^2 + \tfrac{\alpha}{8k}\|z_m - z_{m-1}\|_H^2 \\ &\qquad + \tfrac{2k}{\alpha}\|f_{1,m}\|_{H'}^2 + \tfrac{\alpha}{8k}\|z_m - z_{m-1}\|_H^2 + f_{2,m}(z_m - z_{m-1}). \end{aligned} \tag{A.38}$$

In order to estimate b_m, we decompose $b_m = b_m^1 + b_m^2 + b_m^3 + b_m^4$ analogously to (3.100)

$$\begin{aligned} b_m(v_m, z_m - z_{m-1}) =& b_m^1(v_m, z_m - z_{m-1}) + b_m^2(v_m, z_m - z_{m-1}) \\ &+ b_m^3(v_m, z_m - z_{m-1}) + b_m^4(v_m, z_m - z_{m-1}) \end{aligned} \tag{A.39}$$

We note that the estimates and the pointwise properties of b^1, b^2, b^3, b^4 from Assumption 3.34 hold for the decomposition of b_m as well. Thus, b_m^1 is coercive, continuous and symmetric and, hence, it defines a scalar product on V and a norm via

$$\|u\|_{m,V}^2 := b_m^1(u,u)$$

for $u \in V$. Using the Cauchy–Schwarz and the Young inequalities, we obtain for any $u, s \in V$

$$b_m^1(u,s) = \|u\|_{m,V}\|s\|_{m,V} \leqslant \tfrac{1}{2}\|u\|_{m,V}^2 + \tfrac{1}{2}\|s\|_{m,V}^2$$

and we can conclude

$$\begin{aligned} b_m^1(u, u - s) &= b_m^1(u,u) - b_m^1(u,s) \geqslant \|u\|_{m,V}^2 - \tfrac{1}{2}\|u\|_{m,V}^2 - \tfrac{1}{2}\|s\|_{m,V}^2 \\ &= \tfrac{1}{2}\|u\|_{m,V}^2 - \tfrac{1}{2}\|s\|_{m,V}^2. \end{aligned} \tag{A.40}$$

Employing (A.40), we can estimate

$$\begin{aligned} b^1_m(v_m, z_m - z_{m-1}) &= b^1_m(v_m, v_m - v_{m-1}) - b^1_m(v_m, w_m - w_{m-1}) \\ &= \tfrac{1}{2}\|v_m\|^2_{m,V} - \tfrac{1}{2}\|v_{m-1}\|^2_{m,V} - b^1_m(v_m, w_m - w_{m-1}). \end{aligned} \tag{A.41}$$

Using the continuity of b^2_m and b^4_m and the Young inequality, we can estimate the second and the fourth summands on the right-hand side of (A.39) by

$$\begin{aligned} -b^2_m(v_m, z_m - z_{m-1}) &\leqslant C_{b^2}\|v_m\|_V\|z_m - z_{m-1}\|_H \\ &\leqslant \tfrac{2}{\alpha}C^2_{b^2}k\|v_m\|^2_V + \tfrac{\alpha}{8k}\|z_m - z_{m-1}\|^2_H \\ -b^4_m(v_m, z_m - z_{m-1}) &\leqslant C_{b^2}\|v_m\|_H\|z_m - z_{m-1}\|_H \\ &\leqslant \tfrac{2}{\alpha}C^2_{b^4}k\|v_m\|^2_H + \tfrac{\alpha}{8k}\|z_m - z_{m-1}\|^2_H. \end{aligned} \tag{A.42}$$

Combining (A.38), (A.39), (A.41) and (A.42) yields

$$\begin{aligned} &\tfrac{\alpha}{2k}\|z_m - z_{m-1}\|^2_H + \tfrac{1}{2}\|v_m\|^2_{m,V} - \tfrac{1}{2}\|v_{m-1}\|^2_{m,V} \\ &\leqslant \tfrac{2C^2_a}{\alpha k}\|w_m - w_{m-1}\|^2_H + \tfrac{2k}{\alpha}\|f_{1,m}\|^2_{H'} + \tfrac{2}{\alpha}C^2_{b^2}k\|v_m\|^2_V + \tfrac{2}{\alpha}C^2_{b^4}k\|v_m\|^2_H \\ &\quad + b^1_m(v_m, w_m - w_{m-1}) + f_{2,m}(z_m - z_{m-1}) - b^3_m(v_m, z_m - z_{m-1}). \end{aligned} \tag{A.43}$$

With the continuity estimate for b^1 and the Young inequality, we obtain

$$\begin{aligned} b^1_m(v_m, w_m - w_{m-1}) &\leqslant C_{b^1}\|v_m\|_V\|w_m - w_{m-1}\|_V \\ &\leqslant \tfrac{C^2_{b^1}}{4}k\|v_m\|^2_V + \tfrac{1}{k}\|w_m - w_{m-1}\|^2_V. \end{aligned} \tag{A.44}$$

After inserting (A.44) in (A.43), we obtain with the continuous embedding of V into H

$$\begin{aligned} &\tfrac{\alpha}{2k}\|z_m - z_{m-1}\|^2_H + \tfrac{1}{2}\|v_m\|^2_{m,V} - \tfrac{1}{2}\|v_{m-1}\|^2_{m,V} \\ &\leqslant \left(\tfrac{2C^2_a}{\alpha}C^2_{V\to H} + 1\right)\tfrac{1}{k}\|w_m - w_{m-1}\|^2_V + \tfrac{2k}{\alpha}\|f_{1,m}\|^2_{H'} \\ &\quad + \left(\tfrac{C^2_{b^1}}{4} + \tfrac{2}{\alpha}C^2_{b^2} + \tfrac{2}{\alpha}C^2_{b^4}C^2_{V\to H}\right)k\|v_m\|^2_V \\ &\quad + f_{2,m}(z_m - z_{m-1}) - b^3_m(v_m, z_m - z_{m-1}). \end{aligned} \tag{A.45}$$

Now, we multiply (A.45) by $\lambda(k)^{M-m} \geqslant 0$, where $\lambda(k)$ is determined later, for $M \in \{1, \ldots, N\}$ and sum over $m \in \{1, \ldots, M\}$. Thus, we obtain

$$\begin{aligned} &\sum_{m=1}^{M}\lambda(k)^{M-m}\tfrac{\alpha}{2k}\|z_m - z_{m-1}\|^2_H + \tfrac{1}{2}\sum_{m=1}^{M}\lambda(k)^{M-m}\left(\|v_m\|^2_{m,V} - \|v_{m-1}\|^2_{m,V}\right) \\ &\quad\leqslant \sum_{m=1}^{M}\lambda(k)^{M-m}\left(\left(\tfrac{2C^2_a}{\alpha}C^2_{V\to H} + 1\right)\tfrac{1}{k}\|w_m - w_{m-1}\|^2_V + \tfrac{2k}{\alpha}\|f_{1,m}\|^2_{H'}\right) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{m=1}^{M}\lambda(k)^{M-m}\left(\left(\frac{C_{b^1}^2}{4}+\frac{2}{\alpha}C_{b^2}^2+\frac{2}{\alpha}C_{b^4}^2C_{V\to H}^2\right)k\|v_m\|_V^2\right)\\
&+\sum_{m=1}^{M}\lambda(k)^{M-m}f_{2,m}(z_m-z_{m-1})-\sum_{m=1}^{M}\lambda(k)^{M-m}b_m^3(v_m,z_m-z_{m-1})\\
=:&I_1+I_2+I_3+I_4. \qquad (A.46)
\end{aligned}
$$

We rewrite, the second term on the left-hand side of (A.46)

$$
\begin{aligned}
&\tfrac{1}{2}\sum_{m=1}^{M}\lambda(k)^{M-m}\left(\|v_m\|_{m,V}^2-\|v_{m-1}\|_{m,V}^2\right)\\
=&\tfrac{1}{2}\sum_{m=1}^{M}\lambda(k)^{M-m}\|v_m\|_{m,V}^2-\tfrac{1}{2}\sum_{m=0}^{M-1}\lambda(k)^{M-(m+1)}\|v_m\|_{m+1,V}^2\\
=&\tfrac{1}{2}\|v_M\|_{M,V}^2-\tfrac{1}{2}\lambda(k)^{M-1}\|v_0\|_{1,V}^2\\
&+\tfrac{1}{2}\sum_{m=1}^{M-1}\lambda(k)^{M-m}\left(\|v_m\|_{m,V}^2-\lambda(k)^{-1}\|v_m\|_{m+1,V}^2\right).
\end{aligned}
\qquad (A.47)
$$

From the Lipschitz continuity and the coercivity of b_m^1, we obtain

$$
\begin{aligned}
\|v_m\|_{m+1,V}^2&=b_{1,m+1}(v_m,v_m)=b_m^1(v_m,v_m)+(b_m^1-b_{1,m-1})(v_m,v_m)\\
&\leqslant\|v_m\|_{m,V}^2+L_{b^1}k\|v_m\|_V^2\leqslant\|v_m\|_{m,V}^2+\tfrac{L_{b^1}}{\beta}kb_m^1(v_m,v_m)\\
&=\|v_m\|_{m,V}^2+\tfrac{L_{b^1}}{\beta}k\|v_m\|_{m,V}^2.
\end{aligned}
\qquad (A.48)
$$

Now, we choose

$$\lambda(k)=1+\tfrac{L_{b^1}}{\beta}k \qquad (A.49)$$

such that we can estimate the last term on the right-hand side of (A.47) with (A.48) from below

$$\|v_m\|_{m,V}^2-\lambda(k)^{-1}\|v_m\|_{m+1,V}^2\geqslant\|v_m\|_{m,V}^2-\lambda(k)^{-1}(1+\tfrac{L_{b^1}}{\beta}k)\|v_m\|_{m,V}^2=0. \qquad (A.50)$$

Moreover, we have

$$\lambda(k)^{M-m}=e^{(M-m)\ln(\lambda(k))}\leqslant e^{(M-m)\ln\left(1+\frac{L_{b^1}}{\beta}k\right)}\leqslant e^{(M-m)\frac{L_{b^1}}{\beta}k}\leqslant e^{T\frac{L_{b^1}}{\beta}}. \qquad (A.51)$$

Using (A.50), (A.51), the coercivity and continuity of b_1, we can estimate (A.47) further

$$\begin{aligned}\tfrac{1}{2}\sum_{m=1}^{M}\lambda(k)^{M-m}\left(\|v_m\|_{m,V}^2-\|v_{m-1}\|_{m,V}^2\right)&\geqslant \tfrac{1}{2}\|v_M\|_{M,V}^2-\tfrac{1}{2}\lambda(k)^{M-1}\|v_0\|_{1,V}^2\\&\geqslant \beta\tfrac{1}{2}\|v_M\|_V^2-\tfrac{1}{2}e^{T\frac{L_{b^1}}{\beta}}C_{b^1}\|v_0\|_V^2\geqslant \beta\tfrac{1}{2}\|v_M\|_V^2-\tfrac{1}{2}e^{T\frac{L_{b^1}}{\beta}}C_{b^1}\|v^{\mathrm{in}}\|_V^2.\end{aligned}\tag{A.52}$$

Now, we estimate the terms I_1, I_2, I_3, I_4 of the right-hand side of (A.46). Using (A.51), Lemma A.7 and Lemma A.5, we can estimate I_1 by

$$I_1\leqslant e^{T\frac{L_{b^1}}{\beta}}\left(\left(\tfrac{2C_a^2}{\alpha}C_{V\to H}^2+1\right)\tfrac{1}{\gamma^2}\|\partial_t g\|_{L^2(0,T;Q')}^2+\tfrac{2}{\alpha}\|f_1\|_{L^2(0,T;H')}^2\right)\tag{A.53}$$

and with (A.51), Lemma A.7 and Lemma A.8, we obtain

$$I_2\leqslant e^{T\frac{L_{b^1}}{\beta}}\left(\tfrac{C_{b^1}^2}{4}+\tfrac{2}{\alpha}C_{b^2}^2+\tfrac{2}{\alpha}C_{b^4}^2C_{V\to H}^2\right)(C_Z+\tfrac{T}{\gamma^2}\|g\|_{C([0,T];Q')}).\tag{A.54}$$

Furthermore, we obtain

$$\begin{aligned}I_3=&f_{2,M}(z_M)+\sum_{m=1}^{M-1}\lambda(k)^{M-(m+1)}(\lambda(k)-1)f_{2,m}(z_m)\\&+\sum_{m=1}^{M-1}\lambda(k)^{M-(m+1)}\left(f_{2,m}(z_m)-f_{2,m+1}(z_m)\right)+\lambda(k)^{M-1}f_{2,1}(z_0)\\\leqslant&\|f_{2,M}\|_{V'}\|z_M\|_V+e^{T\frac{L_{b^1}}{\beta}}\sum_{m=1}^{M-1}k\tfrac{L_{b^1}}{\beta}\|f_{2,m}\|_{V'}\|z_m\|_V\\&+e^{T\frac{L_{b^1}}{\beta}}\sum_{m=1}^{M-1}\|f_{2,m}-f_{2,m-1}\|_{V'}\|z_m\|_V+e^{T\frac{L_{b^1}}{\beta}}\|f_{2,1}\|_{V'}\|z_0\|_V.\end{aligned}\tag{A.55}$$

Then, applying the Young inequality, the estimate $\|z_M\|_V\leqslant\|v_M\|_V$, Lemma A.4 and Lemma A.8 give

$$\begin{aligned}I_3\leqslant&\tfrac{2}{\beta}\|f_{2,M}\|_{V'}^2+\tfrac{\beta}{8}\|z_M\|_V^2+\sum_{m=1}^{M-1}k\tfrac{L_{b^1}^2}{\beta^2}\|f_{2,m}\|_{V'}^2+k\|z_m\|_V^2\\&+\sum_{m=1}^{M-1}e^{2T\frac{L_{b^1}}{\beta}}\tfrac{1}{4k}\|f_{2,m}-f_{2,m-1}\|_{V'}^2+k\|z_m\|_V^2+\tfrac{1}{4}e^{2T\frac{L_{b^1}}{\beta}}\|f_{2,1}\|_{V'}^2+\|z_0\|_V^2\\\leqslant&\tfrac{2}{\beta}\|f_2\|_{C([0,T];V')}+\tfrac{\beta}{8}\|v_M\|_V^2+T\tfrac{L_{b^1}^2}{\beta^2}\|f_2\|_{C([0,T];V')}+C_Z\\&+e^{2T\frac{L_{b^1}}{\beta}}\tfrac{1}{4}\|\partial_t f_2\|_{L^2(0,T;V')}^2+C_Z+\tfrac{1}{2}e^{2T\frac{L_{b^1}}{\beta}}\|f_2\|_{C([0,T];V')}+\|z_0\|_V^2\end{aligned}$$

$$\leqslant (\tfrac{2}{\beta} + T\tfrac{L_{b1}^2}{\beta^2} + \tfrac{1}{4}e^{2T\frac{L_{b1}}{\beta}})\|f_2\|_{C([0,T];V')} + \tfrac{\beta}{8}\|v_M\|_V^2 + 2C_Z$$
$$+ e^{2T\frac{L_{b1}}{\beta}}\tfrac{1}{2}\|\partial_t f_2\|_{L^2(0,T;V')}^2 + \|v^{\text{in}}\|_V^2. \tag{A.56}$$

In the next step, we estimate

$$\begin{aligned} I_4 =& - b_{3,M}(v_M, z_M) + \lambda(k)^{M-1}b_{3,1}(v_1, z_0) \\ & - \sum_{m=1}^{M-1} \lambda(k)^{M-(m+1)}\big(\lambda(k)b_m^3(v_m, z_m) - b_{3,m+1}(v_{m+1}, z_m)\big) \\ :=& J_1 + J_2 + J_3. \end{aligned} \tag{A.57}$$

Using the continuity of b_m^3, the Lipschitz continuity of b^3, the Young inequality, the estimate $\|z_M\|_V \leqslant \|v_M\|_V$, Lemma A.7 and Lemma A.8 gives

$$\begin{aligned} J_1 &\leqslant C_{b^3}\|v_M\|_H\|z_M\|_V \leqslant C_{b^3}^2\tfrac{1}{\beta}(\|w_M\|_H + \|z_M\|_H)^2 + \beta\tfrac{1}{4}\|z_M\|_V^2 \\ &\leqslant \tfrac{4}{\beta}C_{b^3}^2C_{V\to H}^2\tfrac{1}{\gamma^2}\|g\|_{C([0,T];Q')}^2 + \tfrac{4}{\beta}C_{b^3}^2C_Z + \tfrac{\beta}{8}\|v_M\|_V^2 \\ J_2 &\leqslant e^{T\frac{L_{b1}}{\beta}}C_{b^3}\|v_1\|_H\|z_0\|_V \leqslant e^{2T\frac{L_{b1}}{\beta}}C_{b^3}^2\tfrac{1}{2}(\|z_1\|_H + \|w_1\|_H)^2 + \tfrac{1}{2}\|v_0\|_V^2 \\ &\leqslant e^{2T\frac{L_{b1}}{\beta}}C_{b^3}^2C_Z + e^{2T\frac{L_{b1}}{\beta}}C_{b^3}^2C_{V\to H}^2\tfrac{1}{\gamma}\|g\|_{C([0,T];Q')}^2 + \tfrac{1}{2}\|v^{\text{in}}\|_V^2. \end{aligned} \tag{A.58}$$

In order to estimate J_3, we estimate first

$$\begin{aligned} &-(\lambda(k)b_m^3(v_m, z_m) - b_{3,m+1}(v_{m+1}, z_m)) \\ =& - (\lambda(k)b_m^3(v_m, z_m) - b_m^3(v_m, z_m)) - (b_m^3(v_m - v_{m+1}, z_m)) \\ & - (b_m^3 - b_{3,m+1})(v_{m+1}, z_m) \\ \leqslant& \tfrac{L_{b1}}{\beta}C_{b^3}k\|v_m\|_H\|z_m\|_V + L_{b^3}k\|v_{m+1}\|_H\|z_m\|_V \\ & + C_{b^3}(\|z_m - z_{m+1}\|_H + \|w_m - w_{m+1}\|_H)\|z_m\|_V \\ \leqslant& \tfrac{L_{b1}^2}{2\beta^2}C_{b^3}^2C_{V\to H}^2k\|v_m\|_V^2 + \tfrac{1}{2}k\|z_m\|_V^2 + L_{b^3}^2\tfrac{1}{2}C_{V\to H}^2k\|v_{m+1}\|_V^2 + \tfrac{1}{2}k\|z_m\|_V^2 \\ & + \tfrac{\alpha}{4k}\|z_m - z_{m+1}\|_H^2 + \tfrac{\alpha}{4k}C_{V\to H}^2\|w_m - w_{m+1}\|_V^2 + \tfrac{2}{\alpha}C_{b^3}^2k\|z_m\|_V^2. \end{aligned} \tag{A.59}$$

Having this, we can estimate J_3 with Lemma A.7 and Lemma A.8

$$\begin{aligned} J_3 \leqslant& e^{T\frac{L_{b1}}{\beta}}\sum_{m=1}^{M-1}\Big(\tfrac{L_{b1}^2}{2\beta^2}C_{b^3}^2C_{V\to H}^2k(\|z_m\|_V^2 + \|w_m\|_V^2) + (1 + \tfrac{2}{\alpha}C_{b^3}^2)k\|z_m\|_V^2 \\ & \qquad + L_{b^3}^2\tfrac{1}{2}C_{V\to H}^2k(\|z_{m+1}\|_V^2 + \|w_{m+1}\|_V^2) + \tfrac{\alpha}{4k}C_{V\to H}^2\|w_m - w_{m+1}\|_V^2\Big) \\ & + \sum_{m=1}^{M-1}\lambda(k)^{M-(m+1)}\tfrac{\alpha}{4k}\|z_m - z_{m+1}\|_H^2 \end{aligned}$$

$$
\begin{aligned}
&\leqslant e^{T\frac{L_{b^1}}{\beta}}\Big(\tfrac{L_{b^1}^2}{\beta^2}C_{b^3}^2C_{V\to H}^2(C_Z+\tfrac{T}{\gamma^2}\|g\|^2_{C([0,T];Q')})+(1+\tfrac{2}{\alpha}C_{b^3}^2)C_Z\\
&\qquad\qquad +L_{b^3}^2\tfrac{1}{2}C_{V\to H}^2(C_Z+\tfrac{T}{\gamma^2}\|g\|^2_{C([0,T];Q')})+\tfrac{\alpha}{4}C_{V\to H}^2\tfrac{1}{\gamma^2}\|\partial_t g\|^2_{L^2(0,T;Q')}\Big)\\
&\quad+\sum_{m=2}^{M}\lambda(k)^{M-m}\tfrac{\alpha}{4k}\|z_{m-1}-z_m\|_H^2.
\end{aligned}\tag{A.60}
$$

Estimating the left-hand side of (A.46) by (A.52) and the right-hand side by (A.53), (A.54), (A.56), (A.57), (A.58), and (A.60), we obtain after collecting all the constants and using $1\leqslant\lambda(k)$

$$
\tfrac{\beta}{4}\|v_M\|_V^2+\sum_{m=1}^{M}\lambda(k)^{M-m}\tfrac{\alpha}{4k}\|z_m-z_{m-1}\|_H^2\leqslant C,
$$

where C depends on $T, C_{V\to H}, C_a, C_b, C_{b^1}, C_{b^2}, C_{b^3}, C_{b^4}, \alpha, \beta, \gamma, L_a, L_{b^1}, L_{b^3}$, $\|v^{\mathrm{in}}\|_V$, $\|g\|_{C([0,T];Q')}$, $\|\partial_t g\|_{L^2(0,T;Q')}$, $\|f_1\|^2_{L^2(0,T;H')}$, $\|\partial_t f_2\|_{L^2(0,T;V')}$. This shows (A.35)–(A.36). $\square$

Lemma A.10. *Let p_m be given as in Proposition A.1. Then,*

$$
\sum_{m=1}^{M}k\|p_m\|_P\leqslant C \tag{A.61}
$$

for a constant C, which depends only on $T, C_{V\to H}, C_a, C_b, C_{b^1}, C_{b^2}, C_{b^3}, C_{b^4}, \alpha, \beta, \gamma$, L_a, L_{b^1}, L_{b^3}, $\|v^{\mathrm{in}}\|_V$, $\|g\|_{H^1(0,T;Q')}$, $\|f_1\|^2_{L^2(0,T;H')}$, $\|f_2\|_{H^1(0,T;V')}$.

Proof. Using (A.1), we can estimate

$$
\begin{aligned}
\|c^*p_m\|_{V'}&=\|-\tfrac{1}{k}a_m\,(v_m-v_{m-1})+b_m v_m+f_{1,m}+f_{2,m}\|_{V'}\\
&\leqslant C_aC_{V\to H}\tfrac{1}{k}\|v_m-v_{m-1}\|_H+C_b\|v_m\|_V+C_{V\to H}\|f_{2,m}\|_{H'}+\|f_{2,m}\|_{V'}.
\end{aligned}\tag{A.62}
$$

From the inf–sup estimate of c, we obtain $\gamma\|p_m\|_Q\leqslant\|c^*p_m\|_{V'}$, which yields

$$
\gamma\|p_m\|_Q\leqslant C_aC_{V\to H}\tfrac{1}{k}\|v_m-v_{m-1}\|_H+C_b\|v_m\|_V+C_{V\to H}\|f_{2,m}\|_{H'}+\|f_{2,m}\|_{V'}.
$$

After multiplication by k and summing over $m\in\{0,\dots,N\}$, we obtain

$$
\begin{aligned}
\gamma\sum_{m=1}^{M}k\|p_m\|_P&\leqslant\sum_{m=1}^{M}kC_aC_{V\to H}\tfrac{1}{k}\big(\|v_m-v_{m-1}\|_H\big)+C_bk\|v_m\|_V\\
&\quad+\sum_{m=1}^{M}C_{V\to H}k\|f_{1,m}\|_{H'}+k\|f_{2,m}\|_{V'}.
\end{aligned}\tag{A.63}
$$

Then, we obtain (A.61) with the estimates from Lemma A.5, Lemma A.4, Lemma A.7 and Lemma A.9. $\square$

Lemma A.11. *The solution of Theorem 3.33 is unique.*

Proof. Due to the linearity of (3.96), we can assume that $f_1, f_2, g, v^{\mathrm{in}} = 0$ and it suffices to show that every solution v, p of (3.96) is already 0. We decompose $v = w + z$ as above and obtain similarly as in Lemma A.7 that $w = 0$. Then, we multiply (3.96) by $z(t)$ and get

$$a(t)(\partial_t z(t), z(t)) + b(t)(z(t), z(t)) = -a(t)(\partial_t w(t), z(t)) - b(t)(w(t), z(t)) = 0.$$

Since $a \in C^{0,1}([0,T]; \mathcal{L}(H, H'))$, we get $t \mapsto a(t)(v, w) \in C^{0,1}([0,T]) = W^{1,\infty}(0,T)$ for every fixed $v, w \in H$ and it holds $|\partial_t a(t)(v,w)| \leqslant L_a \|v\|_H \|v\|_W$ for a.e. $t \in (0,T)$ and all $v, w \in H$. Hence, a is family of regular operators in the sense of [Sho97, Chapter III.3]. Then, $a'(t) \in \mathcal{L}(H, H')$ can be defined by $a'(t)v := \partial_t(a(t)(v, \cdot)$ and we obtain from [Sho97, Chapter III. Proposition 3.2]

$$\partial_t(a(t)(z(t), z(t))) = 2a(t)(\partial_t z(t), z(t)) + a'(t)(z(t), z(t))$$

almost everywhere, which leads to

$$\tfrac{1}{2}\partial_t(a(t)(z(t), z(t))) + b(t)(z(t), z(t)) = \tfrac{1}{2}a'(t)(z(t), z(t)).$$

Integrating over $(0,t)$ yields

$$\tfrac{1}{2}(a(t)(z(t), z(t))) - \tfrac{1}{2}(a(0)(z(0), z(0))) + \int_0^t b(\tau)(z(\tau), z(\tau))\,\mathrm{d}\tau = \int_0^t \tfrac{1}{2}a'(\tau)(z(\tau), z(\tau))\,\mathrm{d}\tau.$$

Due to the zero initial values, the coercivity of a, the Gårding's inequality for $b(t)$ and the estimate for $a'(t)$ with the Lipschitz constant of a, we obtain

$$\tfrac{1}{2}\alpha\|z(t)\|_H^2 + \tfrac{1}{2}\beta\|z\|_{L^2((0,t);V)}^2 \leqslant \left(c_b + \tfrac{1}{2}L_a\right)\|z\|_{L^2((0,t);H)}^2.$$

Then, the Lemma of Gronwall shows that $z = 0$ and, therefore $v = 0$. Then, we obtain with the inf–sup estimate

$$\gamma\|p(t)\|_Q \leqslant \|c^* p(t)\|_{V'} = \| - a(t)\partial_t v(t) - b(t)v(t)\|_{V'} = 0,$$

which shows $p = 0$. □

Lemma A.12. *Let $b > a > 0$. Then,*

$$(1 - ax)^{-1} \leqslant (1 + bx) \tag{A.64}$$

for all $x \in [0, \frac{b-a}{ab}]$

Proof. We note that $0 \leqslant x \leqslant \frac{b-a}{ab}$ yield $(a-b)x + abx^2 \leqslant 0$ and

$$1 - (1+bx)(1-ax) \leqslant 0.$$

Moreover, we have $1 - ax \geqslant 1 - a\frac{b-a}{ab} = \frac{1}{b} > 0$ and, therefore,

$$(1-ax)^{-1} - (1+bx) = \frac{1-(1+bx)(1-ax)}{1-ax} \leqslant 0$$

which provides the desired result. □

Notation

$\mathbb{1}$	identity matrix
$\det(A)$	determinant
$\mathrm{Adj}(A)$	adjugate matrix
$\mathrm{tr}(A)$	trace
$Y^\perp$	orthogonal complement
$\lvert x\rvert$	Euclidean norm in $\mathbb{R}^n$ or Forbenius norm in $\mathbb{R}^{n\times n}$
$\lvert x\rvert_\infty$	maximum norm
$\mathrm{dist}(A,B)$	distance
$B_r(x)$	ball with radius r around x
$\mathrm{int}(A)$	interior
$\overline{A}$	closure
∂A	boundary
e_i	Euclidean unit vector in $\mathbb{R}^n$
$\mathrm{supp}(f)$	support of a function
C	generic constant
C_ε	generic constant depending on ε
c	generic constant used for bounds from below
$C(\Omega)$	continuous functions
$C^{m,\alpha}(\Omega)$	Hölder continuous functions
$C^{1,\alpha}(\Omega)$	Lipschitz continuous functions
$C^k(\overline{\Omega})$	k-times continuously differentiable functions
$C_0^\infty(\Omega)$	infinitely differentiable functions with compact support
$D(\Omega)$	the set $C_0^\infty(\Omega)$ with a suitable topology (see [Alt16])
$C_\#(Y)$	Y-periodic continuous functions
$C_\#^\infty(Y)$	Y-periodic infinitely differentiable functions
$L^p(\Omega)$	Lebesgue space
$L^p(\Omega,B)$	Lebesgue-Bochner space
$W^{k,p}(\Omega)$	Sobolev space
$H^k(\Omega)$	$= W^{k,2}(\Omega)$
$W_0^{1,p}(\Omega)$	functions with zero trace on $\partial\Omega$
$W_\Gamma^{1,p}(\Omega)$	functions with zero trace on Γ
$W_\#^{1,p}(Y)$	Y-periodic functions
$\mathcal{L}(V,W)$	space of linear and continuous operators
V'	dual space
$x\cdot y$	Euclidean inner product in $\mathbb{R}^n$
$A:B$	Euclidean inner product in $\mathbb{R}^{n\times n}$
(x,y)	inner product
$\langle x',x\rangle_{X',X}$	dual pairing
$⨍_U$	average integral $\left(=\frac{1}{\lvert U\rvert}\int_U\right)$
c^*	adjoint operator

Notation for derivatives:
Let $U \subset \mathbb{R}^n$ and $u\colon U \to \mathbb{R}$, $v\colon U \to \mathbb{R}^n$ and $A\colon U \to \mathbb{R}^{n\times n}$, $x \in U$ and $i, j, k \in \{1, \dots, n\}$:

$$
\begin{aligned}
&\partial_x u(x) \in \mathbb{R}^{1\times n}, \quad \partial_x u(x)_{1i} := \partial_{x_i} u(x),\\
&\nabla u(x) := (\partial_x u(x))^\top \in \mathbb{R}^n,\\
&\partial_x v(x) \in \mathbb{R}^{n\times n}, \quad \partial_x v(x)_{ji} := \partial_{x_i} v_j(x),\\
&\nabla v(x) := (\partial_x v(x))^\top \in \mathbb{R}^{n\times n},\\
&\partial_x A(x) \in \mathbb{R}^{(n\times n)\times n}, \quad \partial_x A(x)_{jki} := \partial_{x_i} A_{jk}(x),\\
&\nabla A(x) := (\partial_x A(x))^\top \in \mathbb{R}^{n\times(n\times n)}, \quad \nabla A(x)_{ijk} = \partial_x A(x)_{jki},\\
&\operatorname{div}(v(x)) := \sum_{i=1}^n \partial_{x_i} v_i = \operatorname{tr}(\partial_x v),\\
&\operatorname{div}(A(x)) \in \mathbb{R}^n, \quad \operatorname{div}(A(x))_j := \operatorname{div}((A_{ij})_{i=1}^n(x)) = \sum_{i=1}^n \partial_{x_i} A_{ij}\\
&\Delta u(x) := \operatorname{div}(\nabla u)\\
&\Delta v(x) := \operatorname{div}(\nabla v(x)) \in \mathbb{R}^n, \quad (\Delta v(x))_i = \Delta v_i(x)
\end{aligned}
$$

In particular, these notations for the derivatives lead to the following Leibniz rules

$$
\begin{aligned}
\partial_x(uv) &= v\partial_x u + u\partial_x v,\\
\partial_x(uA) &= A\partial_x u + u\partial_x A,\\
\partial_x(Av) &= v^\top \partial_x A + A\partial_x v,\\
\operatorname{div}(uv) &= u\operatorname{div}(v) + \nabla u \cdot v,\\
\operatorname{div}(uA) &= u\operatorname{div}(A) + A^\top : \nabla u,\\
\operatorname{div}(Av) &= \operatorname{div}(A)\cdot v + A : \nabla v.
\end{aligned}
$$

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