Dirk Blömker, Martin Hairer, Grigorios A. Pavliotis

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Dirk Blömker
Institut für Mathematik
Universität Augsburg
86135 Augsburg

Some Remarks on Stabilization by Additive Noise

Dirk Blömker[*], Martin Hairer [†], Grigorios A. Pavliotis[‡]

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Abstract

We review some results on stabilization of solutions to semilinear parabolic PDEs near a change of stability due to additive degenerate noise. Our analysis is based on the rigorous derivation of a stochastic amplitude equation for the dominant mode and on careful estimates on its solution. Furthermore, a few numerical examples which corroborate our theoretical findings are presented.

1 Introduction

Stabilization of solutions to (ordinary) stochastic differential equations (SDEs) due to multiplicative noise is a well know phenomenon that has been studied extensively in several different contexts. For example, Stratonovich multiplicative noise leads to an averaging of the noise over stable and unstable directions, as was noted by Arnold, Crauel, and Wihstutz [1] and Pardoux and Wihstutz [20, 21]. Furthermore, it has been shown that when the SDE is driven by Itô multiplicative noise the stabilization of the solution is due to the Itô-Stratonovich correction, e.g., Kwiecinska [15, 16]. For stochstic partial differential equations (SPDEs) there are several works by Caraballo, Liu, and Mao [9], Cerrai [10], Caraballo, Kloeden, Schmallfuß [8] and many others. Stabilization due to rotation has been studied by Baxendale, Hennig [2] or Crauel et.al. [11]. Results related to stabilization by multiplicative noise also presented in [17], [13].

Amplitude equations for finite dimensional truncations for SPDEs of Burgers type have been derived by Majda, Timofeyev, Vanden Eijnden [18, 19]. The amplitude equations derived by these authors have additive and/or multiplicative noise and it was observed by the authors that the noise can have a stabilizing effect. In principle, their formal calculations can be justified by using Kurtz's theorem [14]. However, this approach does not enable us to obtain error estimates, nor does it seem to be possible to generalize it to arbitrary dimensions.

The aim of this paper is to review some recent rigorous error estimates for amplitude equations and to present some analysis of the interplay between noise and nonlinearity. This is based on recent results obtained by the authors on the

*Institut für Mathematik, Universität Augsburg, 86135 Augsburg, Germany, dirk.bloemker@math.uni-augsburg.de

†Mathematics Department, The University of Warwick, CV4 7AL Coventry, United Kingdom, m.hairer@warwick.ac.uk

‡Department of Mathematics, Imperial College London, London SW7 2AZ, United Kingdom, g.pavliotis@imperial.ac.uk

stabilizing effects of additive noise on solutions to semilinear parabolic stochastic PDEs with quadratic nonlinearities [6]. This improves results of [22], where numerical experiments and formal calculations based on center manifold theory indicated that additive noise has the potential of stabilizing dominant behavior. Our proof is based on the rigorous derivation of an amplitude SDE for the dominant mode and on a careful analysis of this equation. This enables us to justify rigorously formal asymptotic expansions, and the approach is very well adapted to the infinite dimensional problem (i.e. there is no need to consider finite dimensional truncations) and leads to the derivation of sharp error estimates.

We consider two cases. First SPDEs in a scaling where the noise acts directly as additive noise on the dominant behavior. Secondly, we show in a different scaling that degenerate additive noise is transported to the dominant mode by the nonlinearity. As a result, the evolution of the dominant mode is governed by an SDE with multiplicative noise which can, potentially, stabilize the solution of this SDE.

For simplicity of presentation in this article we focus on SPDEs of Burgers-type near a change of stability. There it is well-known [4, 7, 6] that, for SPDEs of this form, the dominant modes evolve on a slow time-scale, and stable modes decay on a fast time-scale. Moreover, the evolution of the dominant modes is given by a finite dimensional SDE, the so called amplitude equation. The reduction to an amplitude equation is well-known in physics [12].

2 Numerical Example

As an example consider the following Burgers-type SPDE

$$\partial_t u = (\partial_x^2 + 1)u + \epsilon^2 u + u\partial_x u + \sigma\epsilon\xi \tag{1}$$

where $u(t,x) \in \mathbb{R}$ for $t > 0$, $x \in [0,\pi]$ subject to Dirichlet boundary conditions ($u(t,0) = u(t,\pi) = 0$)) and $\epsilon \ll 1$. Notice the different scaling of the linear term $\epsilon^2 u$ and the noise $\sigma\epsilon\xi$. For the numerical experiment we set $\epsilon = 0.1$ and we use the highly degenerate noise $\xi(t,x) = \partial_t\beta(t)\sin(2x)$ acting only on the second Fourier mode, where $\beta(t)$ is a standard one-dimensional, real-valued Brownian motion.

We solve equation (1) using a spectral Galerkin method. We keep only the first four Fourier-modes. This is sufficient to provide us with an accurate solution of (1), since higher order modes are negligible [6]. Figures 1, 2, and 3 show snapshots of solutions an their first and second Fourier-modes for $\sigma = 2$ and $\sigma = 10$. The 3^{rd} and 4^{th} mode are not shown, as they are small.

3 Multiscale Analysis for the Stochastic Burgers Equation

The theory presented in [6] enables us to prove rigorously the stabilization effect that was observed in the numerical experiment. For simplicity of presentation in this article we will consider only a modified scalar Burgers SPDE.

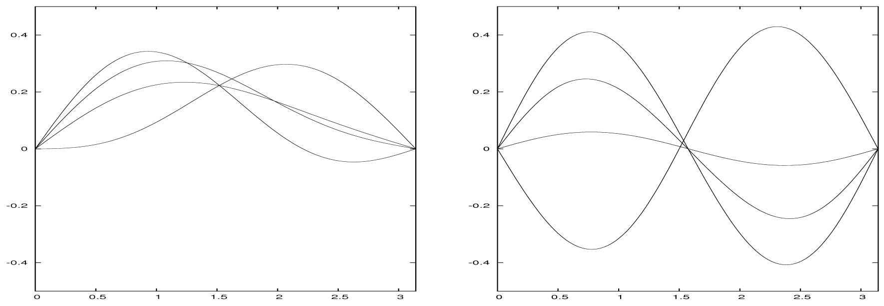

Figure 1: Snapshot of the solution of the 4-mode truncation of (1) for $\sigma = 2$ (left) and for $\sigma = 10$ (right).

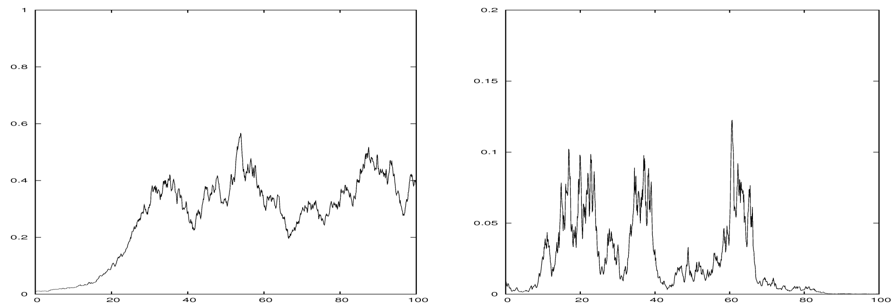

Figure 2: First Fourier mode of the solution of the 4-mode truncation of (1) for $\sigma = 2$ (left) and for $\sigma = 10$ (right) for a single typical realization. It is clearly seen that 0 is stabilized (i.e., sin destabilized) by large noise.

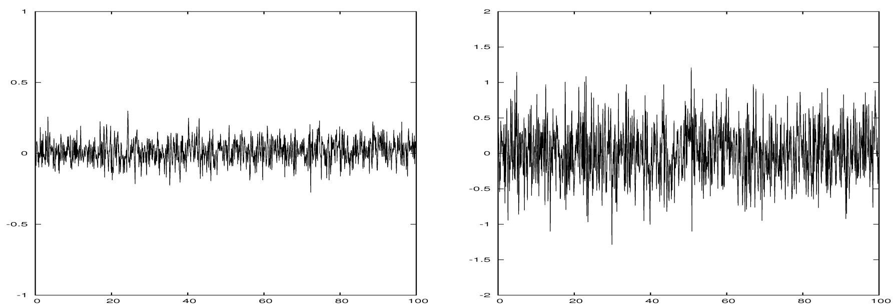

Figure 3: Second Fourier mode of the solution of the 4-mode truncation of (1) for $\sigma = 2$ (left) and for $\sigma = 10$ (right) for a single typical realization. For $\epsilon \to 0$ one can show that it converges to white noise acting on $\sin(2x)$.

Remark 1. *However, our theory is applicable a much larger class of stochastic PDEs with quadratic nonlinearities. All of the following examples can be studied with the same methodology, if we consider them at onset of instability with the right scaling of* ν *and* σ *w.r.t.* ϵ.

$$\textit{Burgers equation: } \partial_t u = \partial_x^2 u + \nu u + u\partial_x u + \sigma\xi.$$

$$\textit{Surface Growth Model: } \partial_t h = -\partial_x^4 h - \nu\partial_x^2 h - \partial_x^2|\partial_x h|^2 + \sigma\xi.$$

See [5] and the references therein. The final example is Rayleigh Bénard Convection given by a 3D-Navier-Stokes coupled to a heat equation.

In this section we will consider the Burgers SPDE under the following scaling, where the noise scales like ϵ^2.

$$\partial_t u = (\partial_x^2 + 1)u + \nu\epsilon^2 u + \tfrac{1}{2}\partial_x u^2 + \epsilon^2\xi \tag{B}$$

where $u(t,x) \in \mathbb{R}$ for $t > 0$ and $x \in [0,\pi]$ is subject to Dirichlet boundary conditions (i.e., $u(t,0) = u(t,\pi) = 0$)). The term $\nu\epsilon^2 u$ is a linear (in)stability and the small parameter $|\nu\epsilon^2| \ll 1$ measures the distance from bifurcation. The noise process $\xi(t,x)$ is Gaussian, white in time and colored in space. The detailed description of $\xi(t,x)$ is given below. Consider the linear operator $L := -\partial_x^2 - 1$ subject to Dirichlet boundary conditions on $[0,\pi]$. It is a standard result that the eigenfunctions of this operator, $\{e_k = \sqrt{\frac{2}{\pi}} sin(kx)\}_{k=1}^{+\infty}$, form an orthonormal basis for $L^2(0,\pi)$, with corresponding eigenvalues $\lambda_k = k^2 - 1$, $k \in \mathbb{N}$.

We will refer to the first eigenfunction $e_1 = \sin(x)$, which corresponds to the zero eigenvalue $\lambda_1 = 0$, as the **dominant mode**. We will use the notation $\mathcal{N} = \text{span}\{\sin\}$ for the kernel of L. The n-th mode is given by e_n.

Assumptions 2. *The noise* $\xi(t,x) = \partial_t W(t,x)$ *is given formally as the time derivative of an infinite dimensional Wiener process* W *such that*

$$W(t,x) = \sum_{k=1}^{\infty} \sigma_k \beta_k(t) \sin(kx)$$

where $\sigma_k \in \mathbb{R}$ *with* $|\sigma_k| \leq C$ *and* $\{\beta_k\}_{k\in\mathbb{N}}$ *are independent and identically distributed standard* 1*-dimensional Brownian motions.*

We will consider two cases of noise

- White noise acting directly on $\mathcal{N}$, i.e. $\sigma_1 \neq 0$.
- Degenerate noise not acting directly on $\mathcal{N}$, i.e. $\sigma_1 = 0$.

Remark 3. *Space-time white noise is given by* $\sigma_k = 1$ $\forall k$.

Our goal is to understand how the noise affects the dynamics of the dominant modes in $\mathcal{N}$.

4 Amplitude Equation

We rewrite Equation (B) in the form

$$\partial_t u = -Lu + \nu\epsilon^2 u + B(u,u) + \epsilon^2 \partial_t W \tag{B1}$$

with $B(u,v) = \frac{1}{2}\partial_x(uv)$. Observe that the Burgers nonlinearity does not map $\mathcal{N}$ to $\mathcal{N}$. Higher order modes are involved.

We will use the ansatz $u(t,x) = \epsilon a(\epsilon^2 t)\sin(x) + \mathcal{O}(\epsilon^2)$ to derive (formally) the **amplitude equation**

$$\partial_T a = \nu a - \tfrac{1}{12}a^3 + \partial_T \beta, \tag{A}$$

where $\beta(T) = \epsilon\sigma_1\beta_1(\epsilon^{-2}T)$ is the rescaled noise in $\mathcal{N}$.

More precisely, for the formal calculation we use the ansatz

$$u(t,x) = \epsilon \underbrace{A(\epsilon^2 t)}_{\in\mathcal{N}} + \epsilon^2 \underbrace{\psi(\epsilon^2 t)}_{\perp\mathcal{N}} + \ldots$$

We use the slow time $T = \epsilon^2 t$, the projection P_c onto $\mathcal{N}$ and $P_s = I - P_c$. As $P_c B(A,A) = 0$, we derive

$$\partial_T A = \nu A + 2P_c B(A,\psi) + \partial_T P_c \tilde{W} + \mathcal{O}(\epsilon),$$

and with $\tilde{W}(T) = \epsilon W(\epsilon^{-2}T)$

$$\epsilon^2 \partial_T \psi = -L\psi + P_s B(A,A) + \epsilon\partial_T P_s \tilde{W} + \mathcal{O}(\epsilon)\ .$$

Neglecting higher order terms leads to $\psi = L^{-1}P_s B(A,A)$ and

$$\partial_T A = \nu A + 2P_c B(A,\psi) + \partial_T P_c \tilde{W}\ .$$

For the real-valued amplitude a of the dominant mode $\sin(\cdot)$ (i.e. $A(T,\cdot) = a(T)\sin(\cdot)$) we obtain Equation (A), with

$$-\tfrac{1}{12} = 2P_c B(\sin(\cdot), L^{-1}P_s B(\sin(\cdot),\sin(\cdot))).$$

This formal calculation can be made rigorous. In fact, we can prove the following theorem (see also [4, 7]).

Theorem 4. *Let u be a solution of* (B1) *and a is solution of (A). Suppose $u(0,\cdot) = \epsilon a(0)\sin(\cdot) + \epsilon^2\psi_0(\cdot)$ with $\psi_0(\cdot) \perp \sin(\cdot)$ and $a(0), \psi_0 = \mathcal{O}(1)$.*

Then for κ, T_0, $p > 0$ there is $C > 0$ such that

$$\mathbb{P}\Big(\sup_{t\in[0,T_0\epsilon^{-2}]} \|u(t,\cdot) - \epsilon a(t\epsilon^2)\sin(\cdot)\|_\infty > \epsilon^{2-\kappa}\Big) < C\epsilon^p.$$

Thus $u(t) = \epsilon a(\epsilon^2 t) + \mathcal{O}(\epsilon^{2-})$.

Remark 5. *Only the projection of the noise onto the dominant mode enters into the amplitude equation. The noise which appears in the higher modes, and under the scaling of the noise considered in Equation* (B)*, is too weak to affect the dynamics of the dominant mode.*

5 Stabilization by Additive Noise

In this section we investigate whether additive degenerate noise (i.e. noise that does not act directly to the dominant mode) can lead to stabilization of the solution of the SPDE (B). In particular, we will assume that no noise acts directly onto the dominant mode (i.e., $\sigma_1 = 0$):

$$W(t) = \sum_{k=2}^{\infty} \sigma_k \beta_k(t) \sin(k\cdot) \;, \qquad \xi(t) = \partial_t W(t)$$

Our aim is to understand how the noise interacts with the nonlinearity to produce a stabilization effect for the solution of the amplitude equation. We will consider two examples.

- Highly degenerate noise only on the second mode, i.e. $\sigma_k = 0$ for $k \neq 2$
- Near white noise, i.e. $\sigma_k = 1$ for $k \geq 2$

Consider first the case of highly degenerate noise:

$$\partial_t W(t,x) = \Phi(t,x) = \partial_t \beta_2(t) \sin(2x) \;.$$

Theorem 4 applied to this case shows that, for noise-strength of order ϵ^2, that the amplitude equation (A) becomes a deterministic equation:

$$\partial_T a = \nu a - \tfrac{1}{12} a^3 \;.$$

Hence, there is no impact of the noise on the dominant behaviour. In order to see the effect of degenerate noise, we have to consider stronger noise. To this end, we set $\sigma_\epsilon = \sigma\epsilon$ and consider the SPDE

$$\partial_t u = -Lu + \nu\epsilon^2 u + B(u,u) + \sigma\epsilon\Phi \tag{B2}$$

A formal calculation [6] yields the amplitude equation

$$da = (\nu - \tfrac{\sigma^2}{88}) a dT - \tfrac{1}{12} a^3 dT + \tfrac{\sigma}{6} a \circ d\tilde{\beta}_2, \tag{A2}$$

where the noise is interpreted in the Stratonovich sense, with $\tilde{\beta}_2(T) = \epsilon\beta_2(\epsilon^{-2}T)$.

Remark 6. *It is not hard to show that, for $\nu \in (0, \sigma^2/88)$, the solution of* (A2) *converges to* 0 *almost surely. Hence, in this parameter regime we get stabilization due to additive noise in a very strong sense.*

Let us see in more detail, where the stabilizing term in (A2) comes from. The Itô to Stratonovich correction is $-\frac{\sigma^2}{72}a$, but this does not explain $-\frac{\sigma^2}{88}a$ that appears in the amplitude equation (A2).

Let us recall the formal calculation. We consider the SPDE at the slow time scale. Substituting $u(t) = \epsilon\psi(\epsilon^2 t)$ we derive from (B2)

$$\partial_T \psi = -\epsilon^{-2} L\psi + \nu\psi + \epsilon^{-1} B(\psi,\psi) + \epsilon^{-1}\partial_T \tilde{\Phi}, \tag{B2'}$$

where $\tilde{\Phi}(T) = \epsilon^{-1}\Phi(T\epsilon^{-2})$ is the rescaled noise. Let $B_k(u,v)$ denote the projection of $B(u,v)$ onto $\mathrm{span}(\sin(kx))$. Note that $\tilde{\Phi} = \tilde{\Phi}_2$. We use the following ansatz with $\psi_k \in \mathrm{span}(\sin(kx))$

$$\psi(T) = \psi_1(T) + \psi_2(T) + \epsilon\psi_3(T) + \mathcal{O}(\epsilon)$$

We obtain, using $B_n(\psi_k,\psi_l)=0$ for $n\notin\{|k-l|,k+l\}$,
$1^{\rm st}$ mode: $\partial_T\psi_1=\nu\psi_1+2\epsilon^{-1}B_1(\psi_2,\psi_1)+2B_1(\psi_2,\psi_3)+\mathcal{O}(\epsilon)$.
$2^{\rm nd}$ mode: $L\psi_2=\epsilon B_2(\psi_1,\psi_1)+\epsilon\partial_T\tilde\Phi_2+\mathcal{O}(\epsilon^2)$.
$3^{\rm rd}$ mode: $L\psi_3=2B_3(\psi_2,\psi_1)+\mathcal{O}(\epsilon)$.
There is a new contribution to the $1^{\rm st}$ mode given by

$$4\epsilon^2B_1(L^{-1}\partial_T\tilde\Phi_2,L^{-1}B_3(\partial_T\tilde\Phi_2,\psi_1))=c(\epsilon\partial_T\tilde\beta_2)^2A$$

We need now to define the term (noise)^2 that appears on the righthand side of the equation above. Instead of $\epsilon\partial_T\tilde\beta_2$ we use $Z_\epsilon(T)=\epsilon^{-1}\int_0^Te^{-3(T-s)\epsilon^{-2}}d\tilde\beta_2(s)$ in the proofs and the following averaging with error bounds (see [6]):

Lemma 7. *Suppose A is a stochastic process, such that for all $\gamma\in(0,\frac12)$, $\kappa,p,T_0>0$ there is a constant $C>0$ such that*

$$\mathbb{E}\sup_{t,s\in[0,T_0]}\frac{|A(t)-A(s)|^p}{|t-s|^{p\gamma}}\le C\epsilon^{-p\kappa}$$

then

$$\int_0^TA(s)Z_\epsilon(s)^2ds=\tfrac16\int_0^TA(s)ds+r_\epsilon(T)$$

where $\mathbb{E}\sup_{[0,T_0]}|r_\epsilon|^p\le C_{T_0,\kappa,p}\epsilon^{\frac p2-\kappa}$.

Proof. We only give a sketch. If A is the solution of (A2) then Itô's gives the result, with $r_\epsilon=\mathcal{O}(\epsilon)$. In our case A is Hölder continuous, and we have bounds on moments of Hölder quotients up to Hölder exponents less then $\frac12$. Thus it is enough to prove the lemma first for $A\equiv$ const, and then carry over using Hölder continuity of A, where we split the integral into many small parts. □

Theorem 8 ([6])**.** *Let u be a continuous $H_0^1([0,\pi])$-valued solution of (B2) with $u(0)=\epsilon a(0)\sin(\cdot)+\epsilon\psi_0$, where $\psi_0\perp\sin$ and $a(0),\psi_0=\mathcal{O}(1)$. Let a be a solution of (A2) and define*

$$R(t)=e^{-Lt}\psi_0+\sigma\left(\int_0^te^{-3(t-s)}d\beta_2(s)\right)\sin(2\,\cdot),$$

then for all $\kappa,p,T_0>0$ there is a constant C such that

$$\mathbb{P}\Big(\sup_{t\in[0,T_0\epsilon^{-2}]}\|u(t)-\epsilon a(\epsilon^2t)\sin-\epsilon R(t)\|_{H^1}>\epsilon^{3/2-\kappa}\Big)\le C\epsilon^p.$$

Consider finally the case of white noise on $\mathcal{N}^\perp$, i.e.

$$W(t,x)=\sum_{k=2}^\infty\beta_k(t)\sin(kx)\ .$$

Equation (B2) becomes

$$\partial_tu=-Lu+\nu\epsilon^2u+\tfrac12\partial_xu^2+\epsilon\partial_tW\ . \tag{B3}$$

The results of [6] applied to this problems show that there exists a Brownian motion B and constants (ν_0, σ_a, σ_b) such that the amplitude equation for (B3) is

$$da=\nu_0a\,dT-\tfrac1{12}a^3dT+\sqrt{\sigma_aa^2+\sigma_b}\,dB\ . \tag{A3}$$

There are explicit formulas for all the constants that appear in this amplitude equation. We emphasize the fact that this equation has both multiplicative and additive noise. We already saw where the multiplicative noise term comes from. The additive noise arises from (noise)^2 times an independent noise.

This result relies on a martingale approximation result of a (one-dimensional) stochastic integral driven by an infinite-dimensional Brownian motion by a stochastic integral driven by the one-dimensional Brownian motion B that appears in the amplitude equation (A3). Sharp error estimates are also obtained depending on estimates for quadratic variations of the stochastic integrals.

Lemma 9. *Let $M(t)$ be a continuous martingale with quadratic variation f and let g be an arbitrary adapted increasing process with $g(0) = 0$. Then, with respect to an enlarged filtration, there exists a continuous martingale $\tilde{M}(t)$ with quadratic variation g such that, for every $\gamma < 1/2$ there exists a constant C with*

$$\begin{aligned}\mathbb{E} \sup_{t\in[0,T]} |M(t) - \tilde{M}(t)|^p \quad &\leq \quad C\big(\mathbb{E}g(T)^{2p}\big)^{1/4}\big(\mathbb{E} \sup_{t\in[0,T]} |f(t) - g(t)|^p\big)^\gamma \\ &\qquad + C\mathbb{E} \sup_{t\in[0,T]} |f(t) - g(t)|^{p/2} \ .\end{aligned}$$

Theorem 10 ([6])**.** *Suppose $\mathcal{N}$ is one-dimensional. For $\alpha \in [0, \frac{1}{2})$ let u be a continuous $H_0^\alpha([0,\pi])$-valued solution of (B3) with $u(0) = \epsilon a(0)\sin + \epsilon\psi_0$, where $\psi_0 \perp \sin$ and $a(0), \psi_0 = \mathcal{O}(1)$. Let a be a solution of (A3) and define*

$$R(t) = e^{-tL}\psi_0 + \int_0^t e^{-(t-s)L} dW(s) \ .$$

Then for all $\kappa, p, T_0 > 0$ there is a constant $C > 0$ such that

$$\mathbb{P}\left(\sup_{t\in[0,T_0\epsilon^{-2}]} \|u(t) - \epsilon a(\epsilon^2 t)\sin - \epsilon R(t)\|_{H^\alpha} > \epsilon^{\frac{5}{4}-\kappa}\right) \leq C\epsilon^p.$$

6 Conclusions and Open Problems

Some recent results on stabilization of solutions to SPDEs of Burgers type due to additive noise were presented in this paper. It was shown that the reason for stabilization is because the noise from the stable modes, is being transported, due to the nonlinearity and the scale separation, to the amplitude equation where it acts as both additive and multiplicative noise. Our theory applies to wide class of SPDEs with quadratic nonlinearities.

There are still many open questions in the theory of amplitude equations for SPDEs. As examples we mention the proof of attractivity, the approximation of moments and the approximation of the invariant measure(s) of the SPDE by the invariant measure(s) of the amplitude equation. The difficulty in obtaining these results is mainly due to the lack of nonlinear stability for our SPDE, which makes estimates on solutions for arbitrary initial conditions not easy.

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