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A POSTERIORI ERROR ANALYSIS OF HYBRIDIZED MIXED FINITE ELEMENT METHODS FOR SECOND ORDER ELLIPTIC BOUNDARY VALUE PROBLEMS

RONALD H. W. HOPPE*†‡, JOHANNES NEHER†, AND NATASCHA SHARMA*

Abstract. The mixed hybrid finite element approximation of second order elliptic boundary value problems by hybridized Raviart-Thomas elements of any order can be seen as a nonconforming approximation of the primal mixed formulation of the problem. In this paper, we provide a unified framework for the a posteriori error analysis in terms of residual-type a posteriori error estimators consisting of element and face (edge) residuals. This unified framework allows to establish the reliability of the error estimators on the basis of appropriate interpolation operators in $H^1, H(\mathbf{curl}), H(\mathrm{div})$ and L^2 as well as suitable reconstruction operators.

Key words. adaptive hybridized mixed finite element methods, a posteriori error analysis, elliptic boundary value problems

AMS subject classifications. 65N30, 65N50

1. Introduction. Hybridized mixed finite element methods for second order elliptic boundary value problems on polygonal or polyhedral domains provide approximations of the solution in terms of elementwise given scalar and vector valued functions and a multiplier on the set of interior edges or faces of the underlying triangulation of the domain. The multiplier is an approximate trace which satisfies a global variational equation. Once it has been computed, the elementwise given functions can be obtained via the solution of strictly local subproblems. The idea of hybridization of mixed finite element methods has been known for a long time (cf., e.g., [7, 20] and the references therein) and has recently attracted particular interest within a unified framework for hybridization of mixed and discontinuous Galerkin methods (cf., e.g., [13]-[16]). Residual a posteriori error estimators for mixed finite element methods of Raviart-Thomas type have been developed and studied, e.g., in [2, 4, 8, 19, 22] in terms of their reliability and efficiency. A convergence analysis in the sense of a guaranteed reduction of the L^2-error in the flux has been provided in [10], whereas optimality results with regard to computational complexity have been obtained in [3]. However, a posteriori error estimators for hybridized mixed methods are mostly limited to the lowest order case and have been derived on the basis of superconvergence results due to the fact that the nonconforming lifting of the approximate trace provides a better approximation of the solution than does the approximation of the primal variable (cf., e.g., [5, 22] for such estimators).

In this paper, we will provide a unified a posteriori error control of hybridized Raviart-Thomas (RT-H) methods of any order for second order elliptic boundary value problems. For nonconforming methods, such a unified framework has been provided recently in [11, 12] (cf. also [9]).
In particular, given a bounded polygonal domain Ω with boundary $\Gamma = \partial\Omega$, and a function $f \in L^2(\Omega)$ as well as a symmetric, uniformly positive definite matrix valued function $a = a(x), x \in \Omega$, and a scalar nonnegative function $d = d(x), x \in \Omega$, we

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consider the problem

$$-\nabla \cdot a\nabla u \; + \; du \;=\; f \qquad \text{in } \Omega \;, \tag{1.1a}$$

$$u \;=\; 0 \quad \text{on } \Gamma \;. \tag{1.1b}$$

Introducing the flux $\mathbf{p} := a\nabla u$ as an additional variable enables to rewrite the second order equation (1.1a) as the first order system

$$a^{-1}\mathbf{p} \;=\; \nabla u \;, \tag{1.2a}$$

$$\nabla \cdot \mathbf{p} \;-\; du \;=\; -f \;. \tag{1.2b}$$

The primal mixed formulation of (1.1a),(1.1b) requires the computation of $(\mathbf{p}, u) \in \mathbf{Q} \times V$ such that

$$a_P(\mathbf{p}, \mathbf{q}) \;+\; b_P(\mathbf{q}, u) \;=\; \ell_{P,1}(\mathbf{q}) \quad , \quad \mathbf{q} \in \mathbf{Q} \;, \tag{1.3a}$$

$$b_P(\mathbf{p}, v) \;-\; c_P(u, v) \;=\; \ell_{P,2}(v) \quad , \quad v \in V \;. \tag{1.3b}$$

Here, the function spaces $\mathbf{Q}, V$ and the bilinear forms $a_P(\cdot,\cdot), b_P(\cdot,\cdot), c_P(\cdot,\cdot)$ are given by $\mathbf{Q} := L^2(\Omega)^d, V := H_0^1(\Omega)$ and

$$a_P(\mathbf{p}, \mathbf{q}) := \int_\Omega a^{-1}\mathbf{p} \cdot \mathbf{q} dx \;,\; b_P(\mathbf{q}, u) := -\int_\Omega \mathbf{q} \cdot \nabla u dx \;,\; c_P(u, v) := \int_\Omega duv dx \;,$$

whereas the functionals $\ell_{P,1} \in \mathbf{Q}^*, \ell_{P,2} \in V^*$ are specified according to

$$\ell_{P,1}(\mathbf{q}) := 0 \quad , \quad \ell_{P,2}(v) := -\int_\Omega fv dx \;.$$

The system (1.3a),(1.3b) admits a unique solution $(\mathbf{p}, u)$ (cf., e.g., [7]).
On the other hand, in the dual mixed formulation of (1.1a),(1.1b) we are looking for $(\mathbf{p}, u) \in \mathbf{Q} \times V$ such that

$$a_D(\mathbf{p}, \mathbf{q}) \;+\; b_D(\mathbf{q}, u) \;=\; \ell_{D,1}(\mathbf{q}) \quad , \quad \mathbf{q} \in \mathbf{Q} \;, \tag{1.4a}$$

$$b_D(\mathbf{p}, v) \;-\; c_D(u, v) \;=\; \ell_{D,2}(v) \quad , \quad v \in V \;. \tag{1.4b}$$

Here, the functions spaces $\mathbf{Q}$ and V are chosen according to $\mathbf{Q} := H(\text{div}; \Omega)$ and $V := L^2(\Omega)$. For the bilinear forms and functionals we have $a_D(\cdot,\cdot) := a_P(\cdot,\cdot), c_D(\cdot,\cdot) := c_P(\cdot,\cdot)$ and $\ell_{D,i} = \ell_{P,i}, 1 \le i \le 2$, whereas $b_D(\cdot,\cdot)$ is given by

$$b_D(\mathbf{q}, v) \;:=\; \int_\Omega \nabla \cdot \mathbf{q} v dx \;.$$

The system ((1.4a),(1.4b) has a unique solution $(\mathbf{p}, u) \in \mathbf{Q} \times V \setminus (\text{Ker } B_D^* \cap \text{Ker } C_D)$, where B_D and C_D stand for the operators associated with $b_D(\cdot,\cdot)$ and $c_D(\cdot,\cdot)$, respectively.
We assume $\mathcal{T}_h$ to be a simplicial triangulation of the computational domain Ω and denote by $\mathcal{F}_h$ the set of edges or faces with $\mathcal{F}_h^\Omega$ and $\mathcal{F}_h^\Gamma$ referring to the subsets of interior edges or faces and those on the boundary, respectively. We further denote by h_T the diameter of $T \in \mathcal{T}_h$ and by h_F the length of an edge or the diameter of a face $F \in \mathcal{F}_h$. We denote by $P_{k-1}(T)$ and $P_{k-1}(F), k \in \mathbb{N}$, the sets of polynomials of order

$k-1$ on T and F, respectively. For $a, b \in \mathbb{R}_+$ we use the notation $a \lesssim b$, if their exists a constant $C > 0$ independent of the granularity of the mesh such that $a \leq Cb$, and we will write $a \approx b$, if $a \lesssim b$ and $b \lesssim a$.
The RT-H methods are based on the function spaces

$$\mathbf{Q}_h := \prod_{T \in \mathcal{T}_h} RT_{k-1}(T) \quad , \quad V_h := \prod_{T \in \mathcal{T}_h} P_{k-1}(T) \quad , \quad M_h := \prod_{F \in \mathcal{F}_h^\Omega} P_{k-1}(F) \ . \quad (1.5)$$

Here, $RT_{k-1}(T), k \in \mathbb{N}$, stands for the Raviart-Thomas element (cf., e.g., [7])

$$RT_{k-1}(T) \ := \ P_{k-1}(T)^d \ + \ x \ P_{k-1}(T) \ .$$

For a function $v_h \in V_h$, we denote by $\{v_h\}_F := (v_h|_{T_1 \cap F} + v_h|_{T_2 \cap F})/2, F \in \mathcal{F}_h^\Omega, F = T_1 \cap T_2, T_i \in \mathcal{T}_h, 1 \leq i \leq 2$, the average of v_h on F and by $[v_h]_F := (v_h|_{T_1 \cap F} - v_h|_{T_2 \cap F})$ the jump of v_h across F. We use the same notation $\{\mathbf{q}_h\}_F$ and $[\mathbf{q}_h]_F$ for vector fields $\mathbf{q}_h \in \mathbf{Q}_h$.
We further introduce bilinear forms $a_h(\cdot,\cdot) : \mathbf{Q}_h \times \mathbf{Q}_h \to \mathbb{R}, b_h(\cdot,\cdot) : \mathbf{Q}_h \times V_h \to \mathbb{R}, c_h(\cdot,\cdot) : V_h \times V_h \to \mathbb{R}$ and $d_h(\cdot,\cdot) : M_h \times \mathbf{Q}_h \to \mathbb{R}$, where

$$a_h(\cdot,\cdot) := \sum_{T \in \mathcal{T}_h} a|_T(\cdot,\cdot) \quad , \quad c_h(\cdot,\cdot) := \sum_{T \in \mathcal{T}_h} c|_T(\cdot,\cdot) \ ,$$

and the bilinear forms b_h and d_h are given by

$$b_h(\mathbf{q}_h, v_h) \ := \ \sum_{T \in \mathcal{T}_h} (\nabla_h \cdot \mathbf{q}_h, v_h)_{0,T} \quad , \quad \mathbf{q}_h \in \mathbf{Q}_h \ , \ v_h \in V_h \ , \quad (1.6a)$$

$$d_h(\mu_h, \mathbf{q}_h) \ := \ \sum_{F \in \mathcal{F}_h^\Omega} (\mu_h, \nu_F \cdot [\mathbf{q}_h]_F)_{0,F} \quad , \quad \mu_h \in M_h \ , \ \mathbf{q}_h \in \mathbf{Q}_h \ . \quad (1.6b)$$

The RT-H methods require the computation of $(\mathbf{p}_h, u_h, \lambda_h) \in \mathbf{Q}_h \times V_h \times M_h$ such that for all $(\mathbf{q}_h, v_h, \mu_h) \in \mathbf{Q}_h \times V_h \times M_h$ there holds

$$a_h(\mathbf{p}_h, \mathbf{q}_h) \ + \ b_h(\mathbf{q}_h, u_h) - d_h(\lambda_h, \mathbf{q}_h) \ = \ 0 \ , \quad (1.7a)$$

$$b_h(\mathbf{p}_h, v_h) \ - \ c_h(u_h, v_h) \ = \ -(f, v_h)_{0,\Omega} \ , \quad (1.7b)$$

$$d_h(\mu_h, \mathbf{p}_h) \ = \ 0 \ . \quad (1.7c)$$

The system (1.7) admits a unique solution (cf., e.g., Chapter V, Theorem 1.1 in [7]).

The rest of the paper is organized as follows: In section 2 we will present a unified framework for the derivation of reliable residual-type a posteriori error estimators for the discretization errors $(\|u^P - u_h\|_{0,\Omega}^2 + \|\mathbf{p}^P - \mathbf{p}_h\|_{0,\Omega}^2)^{1/2}$, where $(\mathbf{p}^P, u^P) \in \mathbf{Q} \times V$ is the solution of the primal mixed formulation (1.3a),(1.3b), whereas $(\mathbf{p}_h, u_h, \lambda_h) \in \mathbf{Q}_h \times V_h \times M_h$ is the solution of the RT-H method (1.7a)-(1.7c). We will present the error estimators which consist of element and face residuals (edge residuals in 2D) and state their reliability (Theorem 2.3). Section 3 contains basic assumptions in terms of appropriate interpolation and reconstruction operators, whereas section 4 is devoted to the proof of the main theorem. In section 5, we illustrate the performance of the adaptive RT-H method for two representative test examples.

2. A posteriori error control within a unified framework. For a proper specification of the function spaces, the bilinear forms and the functionals, the systems (1.3a),(1.3b) and (1.4a),(1.4b) can be rewritten in a unified way according to

$$\mathcal{A}(\mathbf{p}, u) \;=\; \ell_1 \;+\; \ell_2 \;, \tag{2.1}$$

where the operator $\mathcal{A} : \mathbf{Q} \times V \to (\mathbf{Q} \times V)^*$ is given by

$$(\mathcal{A}(\mathbf{p}, u))(\mathbf{q}, v) \;:=\; a(\mathbf{p}, \mathbf{q}) + b(\mathbf{q}, u) + b(\mathbf{p}, v) - d(u, v) \;. \tag{2.2}$$

The operator $\mathcal{A}$ represents a bounded linear and bijective operator such that

$$\|(\mathbf{p}, u)\|_{\mathbf{Q}\times V} \;\leq\; C \left(\|\ell_1\|_{\mathbf{Q}^*} \;+\; \|\ell_2\|_{V^*} \right)$$

with a constant $C > 0$ depending on the data of the problem.
Let $(\tilde{\mathbf{p}}_h, \tilde{u}_h) \in \mathbf{Q} \times V$ be approximations of the solution $(\mathbf{p}, u) \in \mathbf{Q} \times V$ of (2.1) based on the computed solution $(\mathbf{p}_h, u_h, \lambda_h) \in \mathbf{Q}_h \times V_h \times M_h$ of (1.7a)-(1.7c). Then, we have

$$\|u - \tilde{u}_h\|_V^2 + \|\mathbf{p} - \tilde{\mathbf{p}}_h\|_{\mathbf{Q}}^2 \;\lesssim\; \|\mathrm{Res}_1\|_{\mathbf{Q}^*}^2 \;+\; \|\mathrm{Res}_2\|_{V^*}^2 \;, \tag{2.3}$$

where the residuals $\mathrm{Res}_1 \in \mathbf{Q}^*$ and $\mathrm{Res}_2 \in V^*$ are given by

$$\mathrm{Res}_1(\mathbf{q}) \;:=\; \ell_1(\mathbf{q}) - a(\tilde{\mathbf{p}}_h, \mathbf{q}) - b(\mathbf{q}, \tilde{u}_h) \quad , \quad \mathbf{q} \in \mathbf{Q} \;, \tag{2.4a}$$
$$\mathrm{Res}_2(v) \;:=\; \ell_2(v) - b(\tilde{\mathbf{p}}_h, v) + c(\tilde{u}_h, v) \quad , \quad v \in V \;. \tag{2.4b}$$

If $(\mathbf{p}_h, u_h, \lambda_h) \in \mathbf{Q}_h \times V_h \times M_h$ is the solution of (1.7a)-(1.7c) and we consider $(\mathbf{p}_h, u_h)$ as an approximation of the solution $(\mathbf{p}^P, u^P)$ of (1.3a),(1.3b), we will show that

$$\eta_P^2 \;:=\; \sum_{T\in\mathcal{T}_h(\Omega)} \sum_{i=1}^{3} (\eta_{P,T}^{(i)})^2 \;+\; \sum_{F\in\mathcal{F}_h(\Omega)} \sum_{i=1}^{4} (\eta_{P,F}^{(i)})^2 \;+\; \sum_{F\in\mathcal{F}_h(\Gamma)} (\eta_{P,F}^{(5)})^2 \;. \tag{2.5}$$

provides a reliable residual a posteriori error estimator. It consists of element residuals $\eta_{P,T}^{(i)}, 1 \leq i \leq 3$, and face residuals $\eta_{P,F}^{(i)}, 1 \leq i \leq 5$. For 3D problems, the element residuals are given by

$$\eta_{P,T}^{(1)} \;:=\; \|a^{-1}\mathbf{p}_h - \nabla_h u_h\|_{0,T} \quad , \quad T \in \mathcal{T}_h \;, \tag{2.6a}$$
$$\eta_{P,T}^{(2)} \;:=\; \frac{h_T}{k+1} \; \|\mathbf{curl}(a^{-1}\mathbf{p}_h)\|_{0,T} \quad , \quad T \in \mathcal{T}_h \;, \tag{2.6b}$$
$$\eta_{P,T}^{(3)} \;:=\; \frac{h_T}{k+1} \; \|f + \nabla \cdot \mathbf{p}_h - d u_h\|_{0,T} \quad , \quad T \in \mathcal{T}_h \;. \tag{2.6c}$$

REMARK 2.1. *In the 2D case, the operator* **curl** *in* (2.6b) *has to be replaced by the scalar rotational curl.*

Observing $\mathbf{curl}(a^{-1}\mathbf{p}^P) = 0$, the element residuals $\eta_{P,T}^{(1)}$ and $\eta_{P,T}^{(2)}$ are the residuals associated with (1.2a), whereas the element residual $\eta_{P,T}^{(3)}$ is the residual with respect to the equilibrium equation (1.2b).

The edge residuals are of the form

$$\eta_{P,F}^{(1)} := \Big(\frac{h_F}{k+1}\Big)^{1/2} \|[\boldsymbol{\pi}_t(a^{-1}\mathbf{p}_h) - \nabla_F u_h]_F\|_{0,F} \quad , \quad F \in \mathcal{F}_h^{\Omega} , \tag{2.7a}$$

$$\eta_{P,F}^{(2)} := \Big(\frac{h_F}{k+1}\Big)^{1/2} \|\boldsymbol{\nu}_F \cdot [\mathbf{curl}(a^{-1}\mathbf{p}_h)]_F\|_{0,F} \quad , \quad F \in \mathcal{F}_h^{\Omega} , \tag{2.7b}$$

$$\eta_{P,F}^{(3)} := \Big(\frac{k+1}{h_F}\Big)^{1/2} \|\lambda_F - \{u_h\}_F\|_{0,F} \quad , \quad F \in \mathcal{F}_h^{\Omega} , \tag{2.7c}$$

$$\eta_{P,F}^{(4)} := \Big(\frac{k+1}{h_F}\Big)^{1/2} \|[u_h]_F\|_{0,F} \quad , \quad F \in \mathcal{F}_h^{\Omega} , \tag{2.7d}$$

$$\eta_{P,F}^{(5)} := \Big(\frac{k+1}{h_F}\Big)^{1/2} \|u_h\|_{0,F} \quad , \quad F \in \mathcal{F}_h^{\Gamma} . \tag{2.7e}$$

where $\boldsymbol{\pi}_t(\mathbf{q}) := \boldsymbol{\nu}_F \wedge (\mathbf{q}|_F \wedge \boldsymbol{\nu}_F)$ denotes the tangential trace components on F and ∇_F stands for the tangential gradient on F.
The face residuals $\eta_{P,F}^{(1)}, \eta_{P,F}^{(2)}$ and $\eta_{P,F}^{(4)}$ are weighted L^2-norms which measure the smoothness of the solution. Since $[\pi_t(a^{-1}\mathbf{p}^P)]_F, [\nabla_F u^P]_F \in H^{-1/2}(F)^2$ as well as $\boldsymbol{\nu}_F \cdot [\mathbf{curl}(a^{-1}\mathbf{p}_h)]_F \in H^{-1/2}(F)$, the weighted L^2-norm $(\frac{h_F}{k+1})^{1/2} \ \|\cdot\|_{0,F}$ is a discrete analogue of the $H^{-1/2}(F)$-norm. Observe that the weight in (2.7d) is different from those in (2.7a),(2.7b). With regard to the primal mixed formulation $u_h \in V_h$ is assumed to approximate $u^P \in V = H_0^1(\Omega)$ and hence, the weighted L^2-norm $(\frac{k+1}{h_F})^{1/2} \ \|\cdot\|_{0,F}$ represents a discrete analogue of the $H^{1/2}(F)$-norm. Likewise, the multiplier $\lambda_h \in M_h$ is known to approximate the trace of u^P on the interior faces $F \in \mathcal{F}_h^{\Omega}$ which explains the weighted residual (2.7c). Finally, (2.7e) stand for the weighted residuals with respect to the homogeneous Dirichlet boundary condition $u^P|_\Gamma = 0$.

REMARK 2.2. *In the 2D case,* $[\boldsymbol{\pi}_t(a^{-1}\mathbf{p}_h) - \nabla_F u_h]_F$ *and* $\boldsymbol{\nu}_F \cdot [\mathbf{curl}(a^{-1}\mathbf{p}_h)]_F$ *have to be replaced by* $[\mathbf{t}_F \cdot (a^{-1}\mathbf{p}_h) - \partial u_h/\partial s]_F$ *and* $[curl(a^{-1}\mathbf{p}_h)]_F$*, respectively, where* $\mathbf{t}_F$ *is the tangential unit vector and* $\partial/\partial s$ *denotes the tangential derivative on the edge* $F \in \mathcal{F}_h^{\Omega}$.

The main result is the reliability of η_P:

THEOREM 2.3. *Let* $(\mathbf{p}^P, u^P) \in L^2(\Omega)^3 \times H_0^1(\Omega)$ *and* $(\mathbf{p}_h, u_h, \lambda_h) \in \mathbf{Q}_h \times V_h \times M_h$ *be the solutions of* (1.3a),(1.3b) *and* (1.7a)-(1.7c)*, respectively, and let* η_P *be the residual a posteriori error estimator given by* (2.5)*. Then, there holds*

$$\|u^P - u_h\|_{0,\Omega}^2 + \|\mathbf{p}^P - \mathbf{p}_h\|_{0,\Omega}^2 \lesssim \eta_P^2 . \tag{2.8}$$

Proof. The proof of (2.8) will be given in section 4. □

3. Interpolation and reconstruction operators. Given a bounded polygonal or polyhedral domain $\Omega \subset \mathbb{R}^d$, with boundary $\Gamma = \partial\Omega$, we denote by $L^2(D), D \subseteq \Omega$, the Hilbert space of square integrable real valued functions with inner product $(\cdot,\cdot)_{0,D}$ and associated norm $\|\cdot\|_{0,D}$. The space $H^1(D)$ stands for the Sobolev space with inner product $(\cdot,\cdot)_{1,D}$ and norm $\|\cdot\|_{1,D}$, whereas $H_{0,\Sigma}^1(D), \Sigma \subset \partial D$, stands for its subspace $H_{0,\Sigma}^1(D) := \{v \in H^1(D) \mid v|_\Sigma = 0\}$. Further, we refer to $H(\text{curl}; D)$ and $H(\text{div}; D)$ as the Hilbert spaces of vector fields $\mathbf{q} \in L^2(D)^d$ such that $\mathbf{curl}(\mathbf{q})$ and $\text{div}(\mathbf{q})$ are square integrable as well, equipped with the graph norms $\|\cdot\|_{curl;D}$

and $\|\cdot\|_{div;D}$. The spaces $H(\mathbf{curl^0};D) := \{\mathbf{q} \in H(\mathbf{curl};D) \mid \mathbf{curl}\mathbf{q} = 0\}$ and $H(\mathrm{div}^0;D) := \{\mathbf{q} \in H(\mathrm{div};D) \mid \mathrm{div}(\mathbf{q}) = 0\}$ stand for the subspaces of irrotational and solenoidal vector fields, whereas $H_{0,\Sigma}(\mathbf{curl};D) := \{\mathbf{q} \in H(\mathbf{curl};D) \mid \boldsymbol{\gamma}_t(\mathbf{q})|_\Sigma = 0\}$ and $H_{0,\Sigma}(\mathrm{div};D) := \{\mathbf{q} \in H(\mathrm{div};D) \mid (\boldsymbol{\nu}_\Sigma \cdot \mathbf{q})|_\Sigma = 0\}$ stand for the subspaces with vanishing tangential trace and vanishing normal component on $\Sigma \subseteq D$, respectively.
Given a simplicial triangulation $\mathcal{T}_h$ of Ω, for $k \in \mathbb{N}$ we denote by $S_k(\Omega;\mathcal{T}_h) \subset H^1(\Omega)$ the finite element space of conforming P_k finite elements, by $Nd_{k-1}(\Omega;\mathcal{T}_h) \subset H(\mathbf{curl};\Omega)$ the curl-conforming edge element space of Nédélec's first family, by $RT_{k-1}(\Omega;\mathcal{T}_h) \subset H(\mathrm{div};\Omega)$ the Raviart-Thomas finite element space, and finally by $W_{k-1}(\Omega;\mathcal{T}_h) \subset L^2(\Omega)$ the space of elementwise polynomials of degree $k-1$. For $\Sigma \subseteq \Gamma$, we refer to $S_{k,\Sigma}(\Omega;\mathcal{T}_h), Nd_{k-1,\Sigma}(\Omega;\mathcal{T}_h)$ and $RT_{k-1,\Sigma}(\Omega;\mathcal{T}_h)$ as the subspaces of functions with vanishing trace, tangential trace and normal components on Σ.
We assume $\Pi_h^V : H^1(\Omega) \to S_k(\Omega;\mathcal{T}_h)$, $\Pi_h^{Nd} : H(\mathbf{curl};\Omega) \to Nd_{k-1}(\Omega;\mathcal{T}_h)$, and $\Pi_h^{RT} : H(\mathrm{div};\Omega) \to RT_{k-1}(\Omega;\mathcal{T}_h)$ as well as $\Pi_h^T : L^2(\Omega) \to W_{k-1}(\Omega;\mathcal{T}_h)$ to be quasi-interpolation operators such that

$$\begin{array}{ccccccc} H^1(\Omega) & \longrightarrow & H(\mathbf{curl};\Omega) & \longrightarrow & H(\mathrm{div};\Omega) & \longrightarrow & L^2(\Omega) \\ \downarrow \Pi_h^V & & \downarrow \Pi_h^{Nd} & & \downarrow \Pi_h^{RT} & & \downarrow \Pi_h^T \\ S_k(\Omega;\mathcal{T}_h) & \longrightarrow & Nd_{k-1}(\Omega;\mathcal{T}_h) & \longrightarrow & RT_{k-1}(\Omega;\mathcal{T}_h) & \longrightarrow & W_{k-1}(\Omega;\mathcal{T}_h) \end{array} \quad (3.1)$$

satisfies the commuting diagram property, i.e.,

$$\nabla \Pi_h^V = \Pi_h^{Nd} \nabla \quad , \quad \mathbf{curl} \Pi_h^{Nd} = \Pi_h^{RT} \mathbf{curl} \quad , \quad \mathrm{div} \Pi_h^{RT} = \Pi_h^T \mathrm{div} \ . \quad (3.2)$$

Moreover, we make the following assumptions:

$(\mathbf{A_1})$ The operator $\Pi_h^V : H^1_{0,\Sigma}(\Omega) \to S_{k,\Sigma}(\Omega;\mathcal{T}_h), \Sigma \subseteq \Gamma$, is stable in the sense that

$$\|\Pi_h^V v\|_{0,T} \lesssim \|v\|_{0,\omega_T^V} \ , \quad (3.3a)$$
$$\|\nabla \Pi_h^V v\|_{0,T} \lesssim \|\nabla v\|_{0,\omega_T^V} \ , \quad (3.3b)$$

and has the local approximation properties

$$\|v - \Pi_h^V v\|_{0,T} \lesssim \frac{h_T}{k+1} \|v\|_{1,\omega_T^V} \ , \quad (3.4a)$$
$$\|\nabla(v - \Pi_h^V v)\|_{0,T} \lesssim \|\nabla v\|_{0,\omega_T^V} \ , \quad (3.4b)$$
$$\|v - \Pi_h^V v\|_{0,F} \lesssim \Big(\frac{h_F}{k+1}\Big)^{1/2} \|v\|_{1,\omega_F^V} \ , \quad (3.4c)$$

where $\omega_T^V, \omega_F^V \subset \bar{\Omega}$ are patches associated with $T \in \mathcal{T}_h$ and $F \in \mathcal{F}_h^\Omega \cup \mathcal{F}_h^{\Gamma\setminus\Sigma}$

$(\mathbf{A_2})$ The operator $\Pi_h^{Nd} : H_{0,\Gamma}(\mathrm{curl};\Omega) \to Nd_{k-1,\Gamma}(\Omega;\mathcal{T}_h)$, is stable in the sense that

$$\|\Pi_h^{Nd} \mathbf{q}\|_{0,T} \lesssim \|\mathbf{q}\|_{0,\omega_T^{Nd}} \ , \quad (3.5a)$$
$$\|\mathbf{curl}(\Pi_h^{Nd} \mathbf{q})\|_{0,T} \lesssim \|\mathbf{curl}(\mathbf{q})\|_{0,\omega_T^{Nd}} \ . \quad (3.5b)$$

Further, it allows for the decomposition

$$\mathbf{q} \ - \ \boldsymbol{\Pi}_h^{Nd} \mathbf{q} \ = \ \nabla \zeta \ + \ \mathbf{z} \quad , \quad \zeta \in H^1_{0,\Gamma}(\Omega) \ , \ \mathbf{z} \in H^1_{0,\Gamma}(\Omega)^3 \ .$$

This decomposition satisfies

$$\|\zeta\|_{0,T} \ \lesssim\ \frac{h_T}{k+1}\ \|\mathbf{q}\|_{0,\omega_T^{Nd}}\ , \tag{3.6a}$$

$$\|\zeta\|_{0,F} \ \lesssim\ \Big(\frac{h_F}{k+1}\Big)^{1/2}\ \|\mathbf{curl}(\mathbf{q})\|_{0,\omega_F^{Nd}}\ , \tag{3.6b}$$

$$\|\mathbf{z}\|_{0,T} \ \lesssim\ \frac{h_T}{k+1}\ \|\mathrm{curl}(\mathbf{q})\|_{0,\omega_T^{Nd}}\ . \tag{3.6c}$$

Moreover, if $\mathbf{q} \in H_{0,\Gamma}(\mathbf{curl};\Omega) \cap H_0^1(\Omega)^3$, there holds

$$\|\boldsymbol{\gamma}_t(\mathbf{q} - \Pi_h^{Nd}\mathbf{q})\|_{0,F} \ \lesssim\ \Big(\frac{h_F}{k+1}\Big)^{1/2}\ \|\mathbf{curl}(\mathbf{q})\|_{0,\omega_F^{Nd}}\ . \tag{3.7}$$

Here, $\omega_T^{Nd}, \omega_F^{Nd} \subset \bar{\Omega}$ are patches associated with $T \in \mathcal{T}_h$ and $F \in \mathcal{F}_h^{\Omega}$.

$(\mathbf{A_3})$ The operator $\Pi_h^{RT} : H_{0,\Sigma}(\mathrm{div};\Omega) \to RT_{k-1,\Sigma}(\Omega;\mathcal{T}_h), \Sigma \subseteq \Gamma$, allows for the decomposition

$$\mathbf{q} \ -\ \Pi_h^{RT}\mathbf{q} \ =\ \mathbf{curl}(\boldsymbol{\psi}) \ +\ \mathbf{z} \quad , \quad \boldsymbol{\psi} \in H_{0,\Sigma}^1(\Omega)^3 \ ,\ \mathbf{z} \in H_{0,\Sigma}^1(\Omega)^3\ .$$

This decomposition satisfies

$$\|\boldsymbol{\psi}\|_{0,T} \ \lesssim\ \frac{h_T}{k+1}\ \|\mathbf{q}\|_{0,\omega_T^{RT}}\ , \tag{3.8a}$$

$$\|\boldsymbol{\gamma}_t(\boldsymbol{\psi})\|_{0,F} \ \lesssim\ \Big(\frac{h_F}{k+1}\Big)^{1/2}\ \|\mathbf{q}\|_{div,\omega_F^{RT}}\ , \tag{3.8b}$$

$$\|\mathbf{z}\|_{0,T} \ \lesssim\ \frac{h_T}{k+1}\ \|\mathrm{div}(\mathbf{q})\|_{0,\omega_T^{RT}}\ , \tag{3.8c}$$

$$\|\boldsymbol{\nu}_F \cdot \mathbf{z}\|_{0,F} \ \lesssim\ \Big(\frac{h_F}{k+1}\Big)^{1/2}\ \|\mathbf{q}\|_{div,\omega_F^{RT}}\ . \tag{3.8d}$$

Here, $\omega_T^{RT}, \omega_F^{RT} \subset \bar{\Omega}$ are patches associated with $T \in \mathcal{T}_h$ and $F \in \mathcal{F}_h^{\Omega} \cup \mathcal{F}_h^{\Gamma\setminus\Sigma}$.

Remark 3.1. *In case $k = 1$, interpolation operators satisfying the commuting diagram property* (3.2) *as well as assumptions* $(\mathbf{A_1}), (\mathbf{A_2})$ *and* $(\mathbf{A_3})$ *have been constructed in [21] (cf. also [18]).*

We further assume the existence of appropriate reconstruction operators:

$(\mathbf{A_4})$ There exist operators $R : V_h \to V$ such that for $v_h \in V_h$ there holds

$$\sum_{T\in\mathcal{T}_h} \|\nabla_h(Rv_h - v_h)\|_{0,T}^2 \ \lesssim\ \sum_{F\in\mathcal{F}_h^{\Omega}} \frac{k+1}{h_F}\ \|[v_h]_F\|_{0,F}^2\ . \tag{3.9}$$

Remark 3.2. *For the construction of operators $R : V_h \to V$ satisfying* (3.9) *we refer to [9].*

4. A posteriori error analysis. In this section, we will use the unified framework established in section 2 as well as the assumptions made in section 3 to provide proofs of the reliability result stated in Theorem 2.3.
For $\mathbf{q} \in \mathbf{Q} := L^2(\Omega)^3$ we have the decomposition (cf. section 3.5 in [1])

$$\mathbf{q} \ =\ \nabla\varphi + \mathbf{curl}(\boldsymbol{\psi}) \ =:\ \mathbf{q}^{(1)} + \mathbf{q}^{(2)}\ , \tag{4.1a}$$

$$\varphi \in H^1(\Omega)\ ,\ \boldsymbol{\nu}_{\Gamma_N} \cdot \nabla\varphi|_{\Gamma} = 0\ , \tag{4.1b}$$

$$\boldsymbol{\psi} \in H_{0,\Gamma}(\mathbf{curl};\Omega) \cap H_{0,\Gamma}(\mathrm{div}^0;\Omega)\ ,\ \boldsymbol{\nu}_{\Gamma} \cdot \mathbf{curl}(\boldsymbol{\psi})|_{\Gamma_N} = 0\ . \tag{4.1c}$$

As an immediate consequence we have $\boldsymbol{\psi} \in H_0^1(\Omega)^3$, since $H_{0,\Gamma}(\mathbf{curl};\Omega) \cap H_{0,\Gamma}(\mathrm{div};\Omega) = H_0^1(\Omega)^3$ (cf., e.g., Theorem 2.5 in [1]). Moreover, there holds

$$\|\mathbf{q}\|_{0,\Omega}^2 \;=\; \|\nabla\varphi\|_{0,\Omega}^2 + \|\mathbf{curl}(\boldsymbol{\psi})\|_{0,\Omega}^2 \ . \tag{4.2}$$

We choose $\mathbf{q}_h \in \mathbf{Q}_h$ according to

$$\mathbf{q}_h \;:=\; \nabla(\Pi_h^V \varphi) + \mathbf{curl}(\Pi_h^{Nd}\boldsymbol{\psi}) \;=:\; \mathbf{q}_h^{(1)} + \mathbf{q}_h^{(2)} \ . \tag{4.3}$$

Hence, in view of (3.2)

$$\mathbf{q}_h^{(1)} \;:=\; \Pi_h^{Nd}\nabla\varphi \in Nd_{k-1}(\Omega;\mathcal{T}_h) \ , \tag{4.4a}$$

$$\mathbf{q}_h^{(2)} \;:=\; \Pi_h^{RT}\mathbf{curl}(\boldsymbol{\psi}) \in RT_{k-1,\Gamma}(\Omega;\mathcal{T}_h) \ . \tag{4.4b}$$

For $\tilde{\mathbf{p}}_h = \mathbf{p}_h$ we thus obtain

$$\begin{aligned} |\mathrm{Res}_1(\mathbf{q})| \;=\; & \sum_{T\in\mathcal{T}_h} \Big((a^{-1}\mathbf{p}_h - \nabla_h u_h, \mathbf{q} - \mathbf{q}_h)_{0,T} - (\mathbf{q}, \nabla_h(\tilde{u}_h - u_h))_{0,T} \Big) + \\ & + \sum_{T\in\mathcal{T}_h} \Big((a^{-1}\mathbf{p}_h, \mathbf{q}_h)_{0,T} - (\mathbf{q}_h, \nabla_h u_h)_{0,T} \Big) \ . \end{aligned} \tag{4.5}$$

PROPOSITION 4.1. *Let η_P be the residual error estimator as given by* (2.5). *Then, there holds*

$$\|Res_1\|_{\mathbf{Q}^*}^2 \;+\; \|Res_2\|_{V^*}^2 \;\lesssim\; \sum_{T\in\mathcal{T}_h} \sum_{i=1}^{2} (\eta_{P,T}^{(i)})^2 \sum_{F\in\mathcal{F}^{\bar{\Omega}}} \sum_{i=1}^{5} (\eta_{P,F}^{(i)})^2 \ . \tag{4.6}$$

The proof of (4.6) follows from the subsequent three lemmas.

LEMMA 4.2. *There holds*

$$\begin{aligned} & \sum_{T\in\mathcal{T}_h} (a^{-1}\mathbf{p}_h - \nabla_h u_h, \mathbf{q} - \mathbf{q}_h)_{0,T} \;\lesssim \\ & \lesssim\; \|\mathbf{q}\|_{0,\Omega} \, \Big(\Big(\sum_{T\in\mathcal{T}_h} \|a^{-1}\mathbf{p}_h - \nabla_h u_h\|_{0,T}^2 \Big)^{1/2} + \\ & + \Big(\sum_{T\in\mathcal{T}_h} \frac{h_T^2}{(k+1)^2} \, \|\mathbf{curl}(a^{-1}\mathbf{p}_h)\|_{0,T}^2 \Big)^{1/2} + \\ & + \Big(\sum_{F\in\mathcal{F}_h^{\Omega}} \frac{h_F}{k+1} \, \|\boldsymbol{\nu}_F \cdot [\mathbf{curl}(a^{-1}\mathbf{p}_h)]_F\|_{0,F}^2 \Big)^{1/2} + \\ & + \Big(\sum_{F\in\mathcal{F}_h^{\Omega}} \frac{h_F}{k+1} \, \|[\boldsymbol{\pi}_t(a^{-1}\mathbf{p}_h) - \nabla_F u_h]_F\|_{0,F}^2 \Big)^{1/2} \ . \end{aligned} \tag{4.7}$$

Proof. Using (4.1a)-(4.1c), (4.3), (4.4a),(4.4b) and (3.4b) it follows that

$$\begin{aligned} & (a^{-1}\mathbf{p}_h - \nabla_h u_h, \mathbf{q}^{(1)} - \mathbf{q}_h^{(1)})_{0,T} \;=\; (a^{-1}\mathbf{p}_h - \nabla_h u_h, \nabla(\varphi - \Pi_h^V\varphi))_{0,T} \;\lesssim \\ & \lesssim\; \|a^{-1}\mathbf{p}_h - \nabla_h u_h\|_{0,T} \; \|\nabla(\varphi - \Pi_h^V\varphi)\|_{0,T} \;\lesssim\; \|a^{-1}\mathbf{p}_h - \nabla_h u_h\|_{0,T} \; \|\nabla\varphi\|_{0,\omega_T^V} \ . \end{aligned}$$

Summing over all $T \in \mathcal{T}_h$, observing that the patches ω_T^V have a finite overlap and using (4.2) yields

$$\sum_{T\in\mathcal{T}_h} (a^{-1}\mathbf{p}_h - \nabla_h u_h, \mathbf{q}^{(1)} - \mathbf{q}_h^{(1)})_{0,T} \lesssim \tag{4.8}$$
$$\lesssim \|\mathbf{q}\|_{0,\Omega} \Big(\sum_{T\in\mathcal{T}_h} \|a^{-1}\mathbf{p}_h - \nabla_h u_h\|_{0,T}^2 \Big)^{1/2} .$$

On the other hand, an elementwise application of Stokes' theorem gives

$$(a^{-1}\mathbf{p}_h - \nabla_h u_h, \mathbf{q}^{(2)} - \mathbf{q}_h^{(2)})_{0,T} = (a^{-1}\mathbf{p}_h - \nabla_h u_h, \mathbf{curl}(\boldsymbol{\psi} - \Pi_h^{Nd}\boldsymbol{\psi}))_{0,T} = \tag{4.9}$$
$$= (\mathbf{curl}(a^{-1}\mathbf{p}_h), \boldsymbol{\psi} - \Pi_h^{Nd}\boldsymbol{\psi}))_{0,T} + (\boldsymbol{\pi}_t(a^{-1}\mathbf{p}_h) - \nabla_F u_h), \boldsymbol{\gamma}_t(\boldsymbol{\psi} - \Pi_h^{Nd}\boldsymbol{\psi}))_{0,\partial T} ,$$

where we have used that $\boldsymbol{\pi}_t \nabla_h u_h = \nabla_F u_h$ with ∇_F denoting the tangential gradient operator on F.
In view of Assumption $(\mathbf{A_2})$ there exist $\zeta \in H^1_{0,\Gamma}(\Omega)$ and $\mathbf{z} \in H^1_{0,\Gamma}(\Omega)^3$ such that

$$(\mathbf{curl}(a^{-1}\mathbf{p}_h), \boldsymbol{\psi} - \Pi_h^{Nd}\boldsymbol{\psi})_{0,T} = (\mathbf{curl}(a^{-1}\mathbf{p}_h), \nabla\zeta + \mathbf{z})_{0,T} .$$

By Green's formula we obtain

$$(\mathbf{curl}(a^{-1}\mathbf{p}_h), \nabla\zeta)_{0,T} = (\boldsymbol{\nu}_{\partial T} \cdot \mathbf{curl}(a^{-1}\mathbf{p}_h), \zeta)_{\partial T} .$$

Summing over all $T \in \mathcal{T}_h$ results in

$$\sum_{T\in\mathcal{T}_h} (\mathbf{curl}(a^{-1}\mathbf{p}_h), \boldsymbol{\psi} - \Pi_h^{Nd}\boldsymbol{\psi}))_{0,T} = \sum_{T\in\mathcal{T}_h} (\mathbf{curl}(a^{-1}\mathbf{p}_h), \mathbf{z})_{0,T} +$$
$$+ \sum_{F\in\mathcal{F}_h^\Omega} (\boldsymbol{\nu}_F \cdot [\mathbf{curl}(a^{-1}\mathbf{p}_h)]_F, \zeta)_{0,F} .$$

Using (3.5a)-(3.5b) and (4.2) we get

$$\sum_{T\in\mathcal{T}_h} (\mathbf{curl}(a^{-1}\mathbf{p}_h), \boldsymbol{\psi} - \Pi_h^{Nd}\boldsymbol{\psi}))_{0,T} \lesssim$$
$$\lesssim \sum_{T\in\mathcal{T}_h} \frac{h_T}{k+1} \|\mathbf{curl}(a^{-1}\mathbf{p}_h)\|_{0,T} \|\mathbf{curl}(\boldsymbol{\psi})\|_{0,\omega_T^{Nd}}$$
$$+ \sum_{F\in\mathcal{F}_h^\Omega} \Big(\frac{h_F}{k+1}\Big)^{1/2} \|\boldsymbol{\nu}_F \cdot [\mathbf{curl}(a^{-1}\mathbf{p}_h)]_F\|_{0,F} \|\mathbf{curl}(\boldsymbol{\psi})\|_{0,\omega_F^{Nd}} .$$

Due to the finite overlap of the patches and (4.2) we deduce

$$\sum_{T\in\mathcal{T}_h} (\mathbf{curl}(a^{-1}\mathbf{p}_h), \boldsymbol{\psi} - \Pi_h^{Nd}\boldsymbol{\psi}))_{0,T} \lesssim \tag{4.10}$$
$$\lesssim \|\mathbf{q}\|_{0,\Omega} \Big(\Big(\sum_{T\in\mathcal{T}_h} \frac{h_T^2}{(k+1)^2} \|\mathbf{curl}(a^{-1}\mathbf{p}_h)\|_{0,T}^2 \Big)^{1/2} +$$
$$+ \Big(\sum_{F\in\mathcal{F}_h^\Omega} \frac{h_F}{k+1} \|\boldsymbol{\nu}_F \cdot [\mathbf{curl}(a^{-1}\mathbf{p}_h)]_F\|_{0,F}^2 \Big)^{1/2} \Big) .$$

For the last term on the right-hand side in (4.9), by summation over all $T \in \mathcal{T}_h$ and in view of (3.6),(4.2) we obtain

$$
\begin{aligned}
&\sum_{T\in\mathcal{T}_h} (\boldsymbol{\pi}_t(a^{-1}\mathbf{p}_h) - \nabla_F u_h, \boldsymbol{\gamma}_t(\boldsymbol{\psi} - \Pi_h^{Nd}\boldsymbol{\psi}))_{0,\partial T} \ \le \\
&\le \sum_{F\in\mathcal{F}_h^\Omega} \|[\boldsymbol{\pi}_t(a^{-1}\mathbf{p}_h) - \nabla_F u_h]_F\|_{0,F} \ \|\boldsymbol{\gamma}_t(\boldsymbol{\psi} - \Pi_h^{Nd}\boldsymbol{\psi})\|_{0,F} \ \lesssim \\
&\lesssim \sum_{F\in\mathcal{F}_h^\Omega} \Big(\frac{h_F}{k+1}\Big)^{1/2} \|[\boldsymbol{\pi}_t(a^{-1}\mathbf{p}_h) - \nabla_F u_h]_F\|_{0,F} \ \|\mathbf{curl}(\boldsymbol{\psi})\|_{0,\omega_F^{Nd}} \ \lesssim \\
&\lesssim \|\mathbf{q}\|_{0,\Omega} \ \Big(\sum_{F\in\mathcal{F}_h^\Omega} \frac{h_F}{k+1} \ \|[\boldsymbol{\pi}_t(a^{-1}\mathbf{p}_h) - \nabla_F u_h)]_F\|_{0,F}^2 \Big)^{1/2} .
\end{aligned}
\tag{4.11}
$$

The assertion now follows from (4.8),(4.10) and (4.11). □

Lemma 4.3. *There holds*

$$
\sum_{T\in\mathcal{T}_h} (\mathbf{q}, \nabla_h(\tilde{u}_h - u_h))_{0,T} \ \lesssim \ \|\mathbf{q}\|_{0,\Omega} \ \Big(\sum_{F\in\mathcal{F}_h^\Omega} \frac{k+1}{h_F} \ \|[u_h]_F\|_{0,F}^2 \Big)^{1/2} . \tag{4.12}
$$

Proof. The proof is an immediate consequence of $(\mathbf{A_4})$. □

Lemma 4.4. *There holds*

$$
\begin{aligned}
&\sum_{T\in\mathcal{T}_h} (a^{-1}\mathbf{p}_h - \nabla_h u_h, \mathbf{q}_h)_{0,T} \ \lesssim \\
&\lesssim \ \|\mathbf{q}\|_{0,\Omega} \ \Big(\Big(\sum_{F\in\mathcal{F}_h^\Omega} \frac{k+1}{h_F} \ \|\lambda_F - \{u_h\}_F\|_{0,F}^2 \Big)^{1/2} + \\
&+ \ \Big(\sum_{F\in\mathcal{F}_h^\Omega} \frac{k+1}{h_F} \ \|[u_h]_F\|_{0,F}^2 \Big)^{1/2} \ + \ \Big(\sum_{F\in\mathcal{F}_h^\Gamma} \frac{k+1}{h_F} \ \|u_h\|_{0,F}^2 \Big)^{1/2} \Big) .
\end{aligned}
\tag{4.13}
$$

Proof. An elementwise application of Green's formula gives

$$
\begin{aligned}
\sum_{T\in\mathcal{T}_h} (\mathbf{q}_h^{(1)}, \nabla_h u_h)_{0,T} \ &= \ - \sum_{T\in\mathcal{T}_h} (\nabla_h \cdot \mathbf{q}_h^{(1)}, u_h)_{0,T} \ + \\
&+ \ \sum_{F\in\mathcal{F}_h^\Omega} \Big((\boldsymbol{\nu}_F \cdot \{\mathbf{q}_h^{(1)}\}_F, [u_h]_F)_{0,F} + (\boldsymbol{\nu}_F \cdot [\mathbf{q}_h^{(1)}]_F, \{u_h\}_F)_{0,F} \Big) \ + \\
&+ \ \sum_{F\in\mathcal{F}_h^\Gamma} (\boldsymbol{\nu}_F \cdot \mathbf{q}_h^{(1)}, u_h)_{0,F} \ .
\end{aligned}
$$

Since $\mathbf{q}_h^{(1)}$ is an admissible test function in (1.7a)-(1.7c), we have

$$
\sum_{T\in\mathcal{T}_h} \Big((a^{-1}\mathbf{p}_h, \mathbf{q}_h^{(1)})_{0,T} + (\nabla_h \cdot \mathbf{q}_h^{(1)}, u_h)_{0,T} \Big) \ = \ \sum_{F\in\mathcal{F}_h^\Omega} (\lambda_F, \boldsymbol{\nu}_F \cdot [\mathbf{q}^{(1)}]_F)_{0,F} \ .
$$

It follows that

$$\sum_{T\in\mathcal{T}_h} (a^{-1}\mathbf{p}_h - \nabla_h u_h, \mathbf{q}_h^{(1)})_{0,T} \;=\; \sum_{F\in\mathcal{F}_h^\Omega} (\lambda_F - \{u_h\}_F, \boldsymbol{\nu}_F\cdot[\mathbf{q}_h^{(1)}]_F)_{0,F} \; + \tag{4.14}$$
$$+ \sum_{F\in\mathcal{F}_h^\Omega} ([u_h]_F, \boldsymbol{\nu}_F\cdot\{\mathbf{q}_h^{(1)}\}_F)_{0,F} + \sum_{F\in\mathcal{F}_h^\Gamma} (u_h, \boldsymbol{\nu}_F\cdot\mathbf{q}_h^{(1)})_{0,F} \;.$$

For $F\in\mathcal{F}_h^\Omega$ such that $F = T_1\cap T_2, T_i\in\mathcal{T}_h, 1\le i\le 2$, a scaling argument and (3.3b) show

$$\|\boldsymbol{\nu}_F\cdot[\nabla\Pi_h^V\varphi]_F\|_{0,F} \;\lesssim\; \Big(\frac{k+1}{h_F}\Big)^{1/2} \sum_{i=1}^2 \|\nabla\Pi_h^V\varphi\|_{0,T_i} \;\lesssim\; \Big(\frac{k+1}{h_F}\Big)^{1/2} \sum_{i=1}^2 \|\nabla\varphi\|_{0,\omega_{T_i}^V} \;.$$

Likewise, for $F\in\mathcal{F}_h^\Gamma$ such that $F\subset\partial T, T\in\mathcal{T}_h$ we have

$$\|\boldsymbol{\nu}_F\cdot[\nabla\Pi_h^V\varphi]_F\|_{0,F} \;\lesssim\; \Big(\frac{k+1}{h_F}\Big)^{1/2} \|\nabla\Pi_h^V\varphi\|_{0,T} \;\lesssim\; \Big(\frac{k+1}{h_F}\Big)^{1/2} \|\nabla\varphi\|_{0,\omega_T^V} \;.$$

Hence, from (4.14) and (4.2) we deduce

$$\sum_{T\in\mathcal{T}_h} (a^{-1}\mathbf{p}_h - \nabla_h u_h, \mathbf{q}_h^{(1)})_{0,T} \;\lesssim \tag{4.15}$$
$$\lesssim\; \|\mathbf{q}\|_{0,\Omega} \Big(\Big(\sum_{F\in\mathcal{F}_h^\Omega} \frac{k+1}{h_F}\, \|\lambda_F - \{u_h\}_F\|_{0,F}^2\Big)^{1/2} +$$
$$+ \Big(\sum_{F\in\mathcal{F}_h^\Omega} \frac{k+1}{h_F}\, \|[u_h]_F\|_{0,F}^2\Big)^{1/2} + \Big(\sum_{F\in\mathcal{F}_h^\Gamma} \frac{k+1}{h_F}\, \|u_h\|_{0,F}^2\Big)^{1/2}\Big) \;.$$

On the other hand, observing $\mathbf{q}_h^{(2)}\in RT_{k-1,\Gamma}(\Omega;\mathcal{T}_h)$, Green's formula yields

$$\sum_{T\in\mathcal{T}_h} (\mathbf{q}_h^{(2)}, \nabla_h u_h)_{0,T} \;=\; -\sum_{T\in\mathcal{T}_h} (\nabla_h\cdot\mathbf{q}_h^{(2)}, u_h)_{0,T} \;+\; \sum_{F\in\mathcal{F}_h^\Omega} \Big((\boldsymbol{\nu}_F\cdot\mathbf{q}_h^{(2)}, [u_h]_F)_{0,F} \;.$$

Since $\mathbf{q}_h^{(2)}$ also is admissible in (1.7a)-(1.7c), we have

$$\sum_{T\in\mathcal{T}_h} \Big((a^{-1}\mathbf{p}_h, \mathbf{q}_h^{(2)})_{0,T} + (\nabla_h\cdot\mathbf{q}_h^{(2)}, u_h)_{0,T}\Big) \;=\; 0 \;,$$

whence

$$\sum_{T\in\mathcal{T}_h} (a^{-1}\mathbf{p}_h - \nabla_h u_h, \mathbf{q}_h^{(2)})_{0,T} \;=\; \sum_{F\in\mathcal{F}_h^\Omega} ([u_h]_F, \boldsymbol{\nu}_F\cdot\mathbf{q}_h^{(2)})_{0,F} \;. \tag{4.16}$$

For $F\in\mathcal{F}_h^\Omega$ such that $F = T_1\cap T_2, T_i\in\mathcal{T}_h, 1\le i\le 2$, by a scaling argument and (3.5b) we get

$$\|\boldsymbol{\nu}_F\cdot\mathbf{curl}(\Pi_h^{Nd}\boldsymbol{\psi})\|_{0,F} \;\lesssim$$
$$\lesssim\; \Big(\frac{k+1}{h_F}\Big)^{1/2} \sum_{i=1}^2 \|\mathbf{curl}(\Pi_h^{Nd}\boldsymbol{\psi})\|_{0,T_i} \;\lesssim\; \Big(\frac{k+1}{h_F}\Big)^{1/2} \sum_{i=1}^2 \|\mathbf{curl}(\boldsymbol{\psi})\|_{0,\omega_{T_i}^{Nd}} \;.$$

Summing over all $F \in \mathcal{F}_h^\Omega$ and observing (4.2), from (4.16) we deduce

$$\sum_{T \in \mathcal{T}_h} (a^{-1}\mathbf{p}_h - \nabla_h u_h, \mathbf{q}_h^{(2)})_{0,T} \lesssim \|\mathbf{q}\|_{0,\Omega} \Big(\sum_{F \in \mathcal{F}_h^\Omega} \frac{k+1}{h_F} \, \|[u_h]_F\|_{0,F}^2 \Big)^{1/2} . \qquad (4.17)$$

The assertion now follows from (4.15) and (4.17). □

We will now estimate the residual Res_2 associated with the equilibrium equation. Since $V_h^c \subset \text{Ker Res}_2$, for $v \in V$ and $v_h = \Pi_h^V v$ we have

$$\begin{aligned} |\text{Res}_2(v)| &= \sum_{T \in \mathcal{T}_h} (f, v - v_h)_{0,T} - \\ &- \sum_{T \in \mathcal{T}_h} \Big((\mathbf{p}_h, \nabla \cdot (v - v_h))_{0,T} + (du_h, v - v_h)_{0,T} \Big) - \sum_{T \in \mathcal{T}_h} (d(\tilde{u}_h - u_h), v)_{0,T} . \end{aligned}$$

Proposition 4.5. *There holds*

$$\|Res_2\|_{V^*}^2 \lesssim \sum_{T \in \mathcal{T}_h} (\eta_{P,T}^{(3)})^2 + \sum_{F \in \mathcal{F}_h^\Omega} (\eta_{P,F}^{(4)})^2 . \qquad (4.18)$$

Proof. Observing $\mathbf{p}_h \in RT_{k-1}(\Omega; \mathcal{T}_h)$, by an elementwise application of Green's formula we find

$$\sum_{T \in \mathcal{T}_h} (\mathbf{p}_h, \nabla(v - v_h))_{0,T} = - \sum_{T \in \mathcal{T}_h} (\nabla \cdot \mathbf{p}_h, v - v_h)_{0,T}$$

and thus obtain

$$|\text{Res}_2(v)| = \sum_{T \in \mathcal{T}_h} (f + \nabla \cdot \mathbf{p}_h - du_h, v - v_h)_{0,T} - \sum_{T \in \mathcal{T}_h} (d(\tilde{u}_h - u_h), v)_{0,T} . \quad (4.19)$$

In view of (3.4a) we get

$$\begin{aligned} &\sum_{T \in \mathcal{T}_h} (f + \nabla \cdot \mathbf{p}_h - du_h, v - v_h)_{0,T} \le \qquad (4.20) \\ &\le \sum_{T \in \mathcal{T}_h} \|f + \nabla \cdot \mathbf{p}_h - du_h\|_{0,T} \, \|v - v_h\|_{0,T} \lesssim \\ &\lesssim \sum_{T \in \mathcal{T}_h} \frac{h_T}{k+1} \, \|f + \nabla \cdot \mathbf{p}_h - du_h\|_{0,T} \, |v|_{1,\omega_T^V} \lesssim \\ &\lesssim |v|_{1,\Omega} \Big(\sum_{T \in \mathcal{T}_h} \frac{h_T^2}{(k+1)^2} \, \|f + \nabla \cdot \mathbf{p}_h - du_h\|_{0,T}^2 \Big)^{1/2} , \end{aligned}$$

whereas assumption $(\mathbf{A_4})$ implies

$$\sum_{T \in \mathcal{T}_h} (d(\tilde{u}_h - u_h), v)_{0,T} \lesssim |v|_{1,\Omega} \Big(\sum_{F \in \mathcal{F}_h^\Omega} \frac{k+1}{h_F} \, \|[u_h]_F\|_{0,F}^2 \Big)^{1/2} . \qquad (4.21)$$

The assertion follows from (4.19)-(4.21). □

Proof of Theorem 2.3. In view of the Poincaré-Friedrichs inequality for functions in V, (1.7), Proposition 4.1 and Proposition 4.5 show

$$\|u-\tilde{u}_h\|_{0,\Omega}^2 + \|\mathbf{p}-\mathbf{p}_h\|_{\mathbf{Q}}^2 \;\lesssim\; \|u-\tilde{u}_h\|_V^2 + \|\mathbf{p}-\mathbf{p}_h\|_{\mathbf{Q}}^2 \;\lesssim\; \eta_P^2 \; . \tag{4.22}$$

On the other hand, we have

$$\|u-u_h\|_{0,\Omega}^2 \;\le\; 2\Big(\|u-\tilde{u}_h\|_{0,\Omega}^2 + \|\tilde{u}_h-u_h\|_{0,\Omega}^2\Big) \; . \tag{4.23}$$

The Poincaré-Friedrichs inequality for broken H^1 functions (cf., e.g., [6]) tells us

$$\|\tilde{u}_h-u_h\|_{0,\Omega}^2 \;\lesssim\; \|\nabla_h(\tilde{u}_h-u_h)\|_{0,\Omega}^2 + \sum_{F\in\mathcal{F}_h^\Omega} \frac{k+1}{h_F}\;\|[u_h]_F\|_{0,F}^2 \; . \tag{4.24}$$

The first term on the right-hand side in (4.24) can be further estimated by $(\mathbf{A_4})$ according to

$$\|\nabla_h(\tilde{u}_h-u_h)\|_{0,\Omega}^2 \;\lesssim\; \sum_{F\in\mathcal{F}_h^\Omega} \frac{k+1}{h_F}\;\|[u_h]_F\|_{0,F}^2 \; . \tag{4.25}$$

Using (4.22)-(4.25) gives the assertion. □

5. Numerical results. In this section, we provide a documentation of numerical results for two representative 2D test examples based on a mixed hybrid discretization by means of the lowest order broken Raviart-Thomas spaces, i.e., $k = 1$. The implementation of the adaptive mixed hybrid approximation has been done according to the standard cycle

$$\text{SOLVE} \implies \text{ESTIMATE} \implies \text{MARK} \implies \text{REFINE}$$

used for adaptive finite element methods. Here, SOLVE stands for the numerical solution of the discretized problem by solving a global system for the Lagrange multiplier, followed by local solves for the primal and dual variable. The next step ESTIMATE is devoted to the computation of the residual error estimator η_P according to (2.5). The following step MARK deals with the selection of elements and edges for refinement. Here, we use the so-called bulk criterion from the convergence analysis of adaptive finite elements, also known as Dörfler marking (cf., e.g., [17]). Given a universal constant $0 < \Theta < 1$, we select a set $\mathcal{M}_h^{(1)}$ of elements and a set $\mathcal{M}_h^{(2)}$ of edges such that

$$\Theta\; \eta_P \;\le\; \Big(\sum_{T\in\mathcal{M}_h^{(1)}} \sum_{i=1}^{3} (\eta_{P,T}^{(i)})^2 + \sum_{E\in\mathcal{M}_h^{(2)}} \sum_{i=1}^{5} (\eta_{P,E}^{(i)})^2\Big)^{1/2} \; . \tag{5.1}$$

The bulk criterion is realized by a greedy algorithm. The final step REFINE is concerned with the technical realization of the refinement which is taken care of by bisection.

Example 1: L-shaped domain
We consider Poisson's equation on the L-shaped domain $\Omega = (-1,+1)^2 \setminus (0,+1) \times (-1,0)$ where the right-hand side and the boundary data are chosen such that $u^*(r,\varphi)$

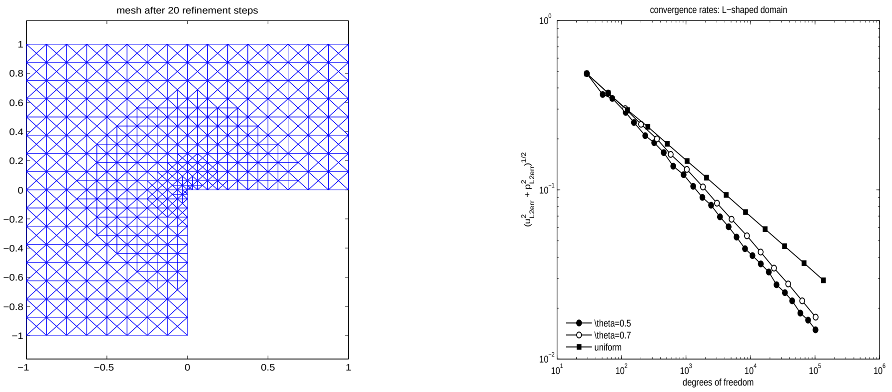

FIG. 5.1. *Mesh after 20 refinement steps (left) and convergence history (right).*

$= r^{2/3} \sin \frac{2\varphi}{3}$ is the exact solution (in polar coordinates). The solution has a singularity at the origin and belongs to $H^{1+2/3-\varepsilon}(\Omega)$ for any $\varepsilon > 0$.

Figure 5.1 (left) shows the adaptively refined mesh after 20 refinement steps of the adaptive algorithm for $\Theta = 0.5$ in the bulk criterion (5.1). As expected, we observe a pronounced refinement in a vicinity of the reentrant corner. Figure 5.1 (right) reflects the convergence history of the adaptive algorithm. On a logarithmic scale, the error $(\|u^R - u_h\|_{0,\Omega}^2 + \|\mathbf{p}^P - \mathbf{p}_h\|_{0,\Omega}^2)^{1/2}$ is displayed as a function of the total number of degrees of freedom for uniform refinement (line marked by filled squares), and for adaptive refinement with $\Theta = 0.7$ (marked by circles) and $\Theta = 0.5$ (marked by filled circles), the latter one being close to optimality regarding the regularity of the exact solution.

Example 2: Slit domain

We consider Poisson's equation on a hexagon with corners $(\pm 1, 0)$, $(\pm\frac{1}{2}, \frac{\sqrt{3}}{2})$, $(\pm\frac{1}{2}, -\frac{\sqrt{3}}{2})$ and a slit along $y = 0$ and $x > 0$. The data of the problem are chosen such that $u^*(r,\varphi) = r^{1/4} \sin(\frac{1}{4}\varphi))$ is the exact solution (in polar coordinates). The solution has a singularity at the origin and belongs to $H^{3/2}(\Omega)$.

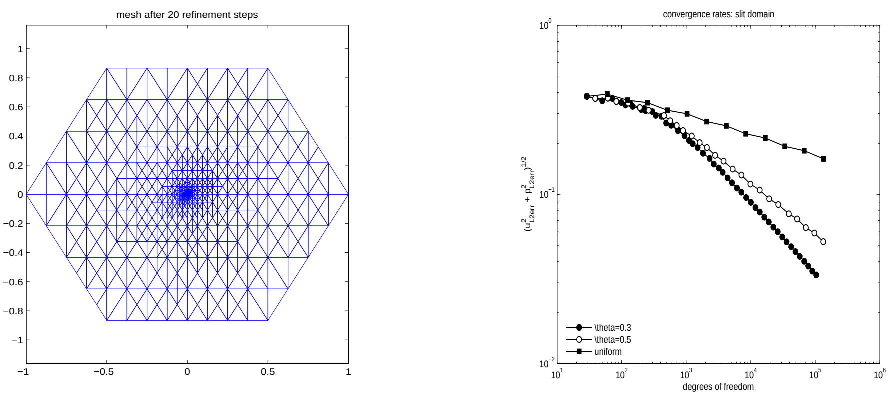

FIG. 5.2. *Mesh after 20 refinement steps (left) and convergence history (right).*

As in the first example, Figure 5.2 (left) displays the adaptively refined mesh after 20 refinement steps of the adaptive algorithm for $\Theta = 0.5$ in the bulk criterion (5.1). The refinement is strictly confined to a neighborhood of the origin. On a logarithmic scale, Figure 5.1 (right) shows the error $(\|u^R - u_h\|_{0,\Omega}^2 + \|\mathbf{p}^P - \mathbf{p}_h\|_{0,\Omega}^2)^{1/2}$ as a function of the total number of degrees of freedom. The legend is similar as in Example 1 (uniform refinement marked by filled squares, adaptive refinement for $\Theta = 0.5$ marked by circles and for $\Theta = 0.3$ marked by filled circles). The result for $\Theta = 0.3$ is close to optimality with respect to the regularity of the exact solution. The dependence on the parameter Θ in the bulk criterion (5.1) is more pronounced than in the first example which is due to the nature of the singularity.

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